Transformasi Linear

Definisi : Misal U dan V ruang vektor. Suatu fungsi $T : U \to V$ disebut transformasi linear hanya jika :

1. $T(u_1 + u_2) = T(u_1) + T(u_2)$, $\forall u_1, u_2$ di U
2. $T(\alpha u_1) = \alpha T(u_1)$, $\forall u_1$ di U , $\alpha \in R$.

contoh :

1. $T : R^3 \to R^2$

   $(x, y, z) \mapsto (x + y, \ y + z)$

   maka :

   - untuk $(1, 2, 3)$ di R^3,

     $T(1, 2, 3) = (1+2, \ 2+3) = (3, 5) \in R^2$.

   - ambil sebarang 2 vektor di R^3, misal (a_1, a_2, a_3) dan (b_1, b_2, b_3) .

     $T(a_1, a_2, a_3) = (a_1 + a_2, \ a_2 + a_3)$

     $T(b_1, b_2, b_3) = (b_1 + b_2, \ b_2 + b_3)$

     $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3)$

     $$\begin{aligned} T(a_1, a_2, a_3) + T(b_1, b_2, b_3) &= (a_1 + a_2, a_2 + a_3) + (b_1 + b_2, b_2 + b_3) \\ &= (a_1 + a_2 + b_1 + b_2, \ a_2 + a_3 + b_2 + b_3) \end{aligned}$$

     $$\begin{aligned} T[(a_1, a_2, a_3) + (b_1, b_2, b_3)] &= T(a_1 + b_1, \ a_2 + b_2, \ a_3 + b_3) \\ &= (a_1 + b_1 + a_2 + b_2 \ , \ a_2 + b_2 + a_3 + b_3) \\ &= (a_1 + a_2 + b_1 + b_2 \ , \ a_2 + a_3 + b_2 + b_3) \\ &= T(a_1, a_2, a_3) + T(b_1, b_2, b_3) \end{aligned}$$

   - $\alpha(a_1, a_2, a_3) = (\alpha a_1, \ \alpha a_2, \ \alpha a_3)$

     $$\begin{aligned} T(\alpha a_1, \alpha a_2, \alpha a_3) &= (\alpha a_1 + \alpha a_2 \ , \ \alpha a_2 + \alpha a_3) \\ &= \alpha(a_1 + a_2 \ , \ a_2 + a_3) \\ &= \alpha T(a_1, a_2, a_3) \end{aligned}$$

   $\therefore$ T bersifat linear .

2. $T : P_2 \to R$,
$p(x) \mapsto \int_0^1 p(x)\, dx$

3. $T : P_2 \to P_3$
$(a_0 + a_1 x + a_2 x^2) \longmapsto (a_0 + a_1) + (a_0 + a_2) x + (a_1 + a_2) x^2 + (a_2) x^3$.

4. $T : R^2 \to R^3$
$(x, y) \mapsto (x - y, x + y, 2x + y)$

Contoh yang tidak bersifat linear

5. $T : R^2 \to R^3$
$(x, y) \mapsto (x - y, x + y - 1, 2x + y)$

6. $T : R^3 \to R^2$
$(x, y, z) \mapsto (xy + z, 2x + y^2)$

7. $T : M_{2 \times 2} \to R$
$A \mapsto |A|$

8. $T : P_3 \to P_2$
$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \longmapsto a_1 a_2 + a_0 a_1 x + a_0 a_2 x^2$

Misal $T : U \to V$ Transformasi linear,

- kernel T adalah $\{ u \in U \mid T(u) = 0 \in V \}$
  kernel $T \equiv \ker(T) \equiv \mathrm{Int}(T)$

- Range T adalah $\{ T(u) \in V \}$
  Range $T \equiv R(T) \equiv$ Jangkauan (T).

$\left.\begin{array}{l} \text{• } \ker(T) \text{ merupakan sub ruang dari } U \\ \text{• } R(T) \text{ merupakan sub ruang dari } V \end{array}\right\}$ buktikan !

contoh :

1. $T : \mathbb{R}^2 \to \mathbb{R}^3$

$(x,y) \mapsto (x-y\ ,x+y\ ,2x+y)$

$\Rightarrow$ $\ker(T) = \{(x,y) \mid T(x,y) = (0,0,0)\}$

$T(x,y) = (x-y;\ x+y\ ,\ 2x+y) = (0,0,0)$

$\xrightarrow{SPL}$ $\begin{aligned} x-y &= 0 \\ x+y &= 0 \\ 2x+y &= 0 \end{aligned}$ $\xrightarrow{\text{solusi}}$ $\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$

$\ker(T) = \{(x,y) = (0,0)\}$. $\ker(T)$ dibangun oleh vektor $(0,0)$, satu vektor yang tidak bebas linier di $\mathbb{R}^2$. Jadi $\ker(T)$ tidak punya basis.

$\cdot)$ $R(T) = \{(x-y,\ x+y\ ,\ 2x+y\}$

$= \{(x,x,2x) + (-y,y,y)\}$

$= \{x(1,1,2) + y(-1,1,1)\}$

$= \text{lin}\,[(1,1,2),\ (-1,1,1)]$

$R(T)$ dibangun oleh $\{(1,1,2),(-1,1,1)\}$ dua vektor yang bebas linier di $\mathbb{R}^3$. Jadi basis $R(T)$: $\{(1,1,2),(-1,1,1)\}$ dan $\dim(R(T)) = 2$

2. $T : P_2 \to \mathbb{R}$

$p(x) \mapsto \int_0^1 p(x)\,dx$

$\cdot)$ $\ker(T) = \{p(x) \mid \int_0^1 p(x)\,dx = 0\}$

$\int_0^1 p(x)\,dx = \int_0^1 (a_0 + a_1x + a_2x^2)\,dx$

$= a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 \Big|_0^1$

$= a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0 \to a_0 = -\frac{a_1}{2} - \frac{a_2}{3}$.

$Ker(T) = \{ p(x) = a_0 + a_1 x + a_2 x^2 \mid a_0 = -\frac{a_1}{2} - \frac{a_2}{3} \}$

$= \{(-\frac{a_1}{2} - \frac{a_2}{3}) + a_1 x + a_2 x^2 = a_1(-\frac{1}{2} + x) + a_2(-\frac{1}{3} + x^2)\}$

$= lin[-\frac{1}{2} + x \; ; -\frac{1}{3} + x^2]$

Ker (T) dibangun oleh $-\frac{1}{2}+x$ dan $-\frac{1}{3}+x^2$ dua vektor di P_2 yang bebas linier. Jadi basis Ker (T) : $\{-\frac{1}{2}+x \,,\, -\frac{1}{3}+x^2\}$

dim (Ker (T)) = 2 .

-) $R(T) = \{ T(p(x)) = \int_0^1 (a_0 + a_1 x + a_2 x^2)\, dx \}$

$= \{ a_0 + \frac{a_1}{2} + \frac{a_2}{3} \mid a_0, a_1, a_2$ bil. real $\}$

maka setiap bil. real a selalu ada a_0, a_1, a_2 $\ni$ $a = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$. Jadi $R(T) = R$. basis $R = \{1\}$.

dim (R (T)) = 1 .

3. Misal diambil basis $E_3 = \{(1,0,0), (0,1,0), (0,0,1)\}$ untuk R^3 dan $E_2 = \{(1,0), (0,1)\}$ untuk R^2. Maka $T: R^3 \to R^2$ yang didefinisikan dg ketentuan $T(1,0,0) = (1,2)$ dan $T(0,1,0) = (-1,1)$ dan $T(0,0,1) = (0,-3)$, adalah juga bersifat linier.

Dalam hal ini, $T(x,y,z) = T(x(1,0,0) + y(0,1,0) + z(0,0,1))$

$= x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$

$= x(1,2) + y(-1,1) + z(0,-3)$

$= (x-y \,,\, 2x+y-3z)$.

-) $T(1,2,3) = T(1(1,0,0) + 2(0,1,0) + 3(0,0,1))$

$= T(1,0,0) + 2T(0,1,0) + 3T(0,0,1)$

$= (1,2) + 2(-1,1) + 3\cdot(0,-3)$

$= (-1,-5)$.

Vektor koordinat terhadap suatu basis

Misal $A = \{a_1, a_2, a_3, \cdots, a_n\}$ basis dari ruang vektor V.
maka setiap vektor u di V dapat dinyatakan sebagai kombinasi linear dari A. Atau $u = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$.
selanjutnya vektor $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ disebut koordinat u relatif terhadap basis A untuk V. Notasi : $[u]_A = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$

Contoh :

1. $E_3 = \{(1,0,0), (0,1,0), (0,0,1)\}$ adalah basis Euclides untuk R^3.
   $(-1,3,5)$ salah satu vektor di R^3.

   $(-1,3,5) = -1(1,0,0) + 3(0,1,0) + 5(0,0,1)$.
   maka koordinat $(-1,3,5)$ relatif terhadap basis E_3 adalah .
   $[(-1,3,5)]_{E_3} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$.

2. $A = \{(1,1,1), (1,1,0), (1,0,0)\}$ juga basis R^3 (cek !).
   $(-1,3,5) = \alpha_1(1,1,1) + \alpha_2(1,1,0) + \alpha_3(1,0,0)$.

   $\xrightarrow{SPL}$
   $$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= -1 \rightarrow \alpha_3 = -4 \\ \alpha_1 + \alpha_2 &= 3 \rightarrow \alpha_2 = -2 \\ \alpha_1 &= 5 \end{aligned}$$

   maka koordinat $(-1,3,5)$ relatif terhadap basis A adalah
   $[(-1,3,5)]_A = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix}$.

3. $B = \{1, 2+x, -x+x^2\}$ basis P_2 (Cek !!)

$p(x) = 1 + x + x^2$ suatu vektor di P_2.

$$p(x) = 1 + x + x^2 = \alpha_1(1) + \alpha_2(2+x) + \alpha_3(-x+x^2)$$

$\xrightarrow{SPL}$

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= 1 \rightarrow \alpha_1 = -3 \\ \alpha_2 - \alpha_3 &= 1 \rightarrow \alpha_2 = 2 \\ \alpha_3 &= 1 \end{aligned}$$

maka koordinat $p(x) = 1 + x + x^2$ relatif terhadap basis B :

$$[1+x+x^2]_B = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

Contoh.

1. Diketahui $C = \{(-1,1,3), (2,1,0), (-3,-2,1)\} \subset R^3$.
   Jika C bisa menjadi basis R^3, tentukan $[(1,2,3)]_C$.

   P. karena $\begin{vmatrix} -1 & 2 & -3 \\ 1 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = -1 - 12 + 9 - 2 = -6 \neq 0$

   maka C bebas linier di R^3 dan bisa menjadi basis R^3.

   $$(1,2,3) = \alpha_1(-1,1,3) + \alpha_2(2,1,0) + \alpha_3(-3,-2,1)$$

   $\xrightarrow{SPL}$ $\left[\begin{array}{ccc|c} -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 2 \\ 3 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \\ b_2 + b_1 \\ b_3 + 3b_1 \end{array} \left[\begin{array}{ccc|c} -1 & 2 & -3 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & 6 & -8 & 6 \end{array}\right] b_3 - 2b_2$

   $$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 2 & 0 \end{array}\right] \begin{array}{l} \rightarrow -\alpha_1 + 2\alpha_2 - 3\alpha_3 = 1 \rightarrow \alpha_1 = 1 \\ \rightarrow 3\alpha_2 - 5\alpha_3 = 3 \rightarrow \alpha_2 = 1 \\ \rightarrow 2\alpha_3 = 0 \rightarrow \alpha_3 = 0 \end{array}$$

   Jadi. $[(1,2,3)]_C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

2. Misal diambil $D = \{(1,1,0), (0,1,1), (1,0,1)\}$ sebagai basis R^3, dan $T: R^3 \to R^2$ didefinisikan dg : $T(1,1,0) = (-1,1)$, $T(0,1,1) = (1,1)$ dan $T(1,0,1) = (0,2)$. Tentukan ker (T), R(T) beserta basis dan dimensinya.

P.

i) Mencari koordinat (x,y,z) relatif terhadap basis D.

$$(x,y,z) = \alpha_1(1,1,0) + \alpha_2(0,1,1) + \alpha_3(1,0,1)$$

$$\xrightarrow{SPL} \begin{aligned} \alpha_1 + \alpha_3 &= x \\ \alpha_1 + \alpha_2 &= y \\ \alpha_2 + \alpha_3 &= z \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 1 & 1 & 0 & y \\ 0 & 1 & 1 & z \end{array}\right] \xrightarrow{b_2-b_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 1 & 1 & z \end{array}\right] b_3-b_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 1 & -1 & y-x \\ 0 & 0 & 2 & z+x-y \end{array}\right] \begin{array}{l} \to ** \\ \to * \\ \to 2\alpha_3 = z+x-y \end{array}$$

$$\alpha_3 = \tfrac{1}{2}z + \tfrac{1}{2}x - \tfrac{1}{2}y$$

$$* \quad \alpha_2 - \alpha_3 = y - x$$

$$\alpha_2 = y - x + \tfrac{1}{2}z + \tfrac{1}{2}x - \tfrac{1}{2}y$$

$$= -\tfrac{1}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}z$$

$$** \quad \alpha_1 + \alpha_3 = x$$

$$\alpha_1 = x - \tfrac{1}{2}z - \tfrac{1}{2}x + \tfrac{1}{2}y$$

$$= \tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{2}z.$$

$$(x,y,z) = (\tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{2}z)(1,1,0) + (-\tfrac{1}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}z)(0,1,1) + (\tfrac{1}{2}z + \tfrac{1}{2}x - \tfrac{1}{2}y)(1,0,1).$$

$$T(x,y,z) = (\tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{2}z)T(1,1,0) + (-\tfrac{1}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}z)T(0,1,1) + (\tfrac{1}{2}z + \tfrac{1}{2}x - \tfrac{1}{2}y)T(1,0,1)$$

$$= (\tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{2}z)(-1,1) + (-\tfrac{1}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}z)(1,1) + (\tfrac{1}{2}z + \tfrac{1}{2}x - \tfrac{1}{2}y)(0,2)$$

$$= (-x - y + z, \; x - y + z)$$

$\cdot)$ $Ker(T) = \{(x,y,z) \mid T(x,y,z) = (0,0)\}$

$T(x,y,z) = (-x-y+z\ ,\ x-y+z) = (0,0)$

SPL $\rightarrow$ $\begin{matrix} -x-y+z=0 \\ x-y+z=0 \end{matrix} \rightarrow \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{b_2+b_1} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$ $\begin{matrix} \rightarrow -x-y+z=0 \\ x=0 \\ \rightarrow -2y+2z=0 \\ y=z \end{matrix}$

$Ker(T) = \{(x,y,z) \mid x=0,\ y=z\} = \{(x,y,z) = (0,y,y)\}$

$= \{y(0,1,1)\}$ dibangun oleh 1 vektor $\{(0,1,1)\}$

basis $Ker(T) = \{(0,1,1)\}$ dim $(Ker(T)) = 1$.

$\cdot)$ $R(T) = \{T(x,y,z) = (-x-y+z\ ,\ x-y+z)\}$

$= \{(-x,x), (-y,-y), (z,z)\}$

$= \{x(-1,1), y(-1,-1), z(1,1)\}$

$= \text{lin}[(-1,1), (-1,-1), (1,1)]$

$R(T)$ dibangun oleh $\{(-1,1),(-1,-1),(1,1)\}$ tiga vektor di R^2 yang tidak bebas linier (Cek !!).

Basis $R(T) = \{(-1,1),(-1,-1)\}$ atau $\{(-1,1),(1,1)\}$ (??!!)

dim $R(T) = 2$.