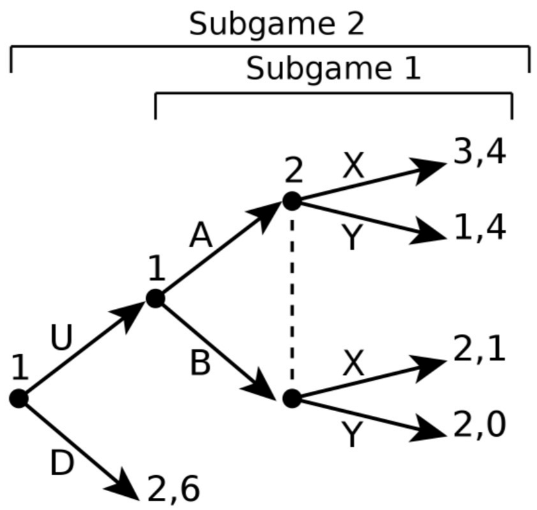
6 Phase 1: Facility Placement

6.1 Subgame Perfect Equilibria

Since our 2-Phase game is dynamic, we make use of the concept of subgame perfect equilibria. This describes strategy profiles, which represent a Nash Equilibrium for every subgame. A strategy profile is thus only an SPE if there are both no improving moves for customers and no improving moves for facilities. The following example of the book "Strategy : an introduction to game theory" illustrates the concept.



1 \ 2	X	Y
UA	3,4	1,4
UB	2,1	2,0
DA	2,6	2,6
DB	2,6	2,6

Player 1 can choose between starting with strategy U or strategy D and whether he would react with X or Y depending on Player 2 choosing A or B. As we can see in the utility matrix for Subgame 2 to the right $(UA, X), (DA, Y), (DB, Y)$ are all nash equilibria. Subgame 1 on the other hand has the unique nash equilibrium (A, X). Thus, for the entire game (DA, Y) and (DB, Y) are not subgame perfect equilibria, because they contain the strategies (A, Y) and (B, Y), both no equilibria for Subgame 1. The Nash equilibrium (UA, X) is subgame perfect because it contain the subgame Nash equilibrium (A, X) as part of its strategy.

6.2 Finding Welfare Optimum

6.3 Price of Anarchy

6.4 Welfare Optimum Stability

6.5 Strategy Space

6.5.1 Mini-Graph Strategy

For the most limited case, all facilities can only select one possible location. In that case, any customer equilibrium for this placement is a two-phase equilibrium, so it is not possible to block equilibria with this restriction. This does no longer hold in general for two or more possible locations. Let the first possible location B cover exactly all black customers, the second W all white customers. We consider three facilities and ignore the rest, since their choices are independent (if at least one of the three facilities selects B then one of those serves all black customers, the same holds for W). The following matrix defines the customer strategy. The columns f_1, f_2, f_3 each denote which strategy (B or W) the respective facility uses. The column w denotes which facility we assign all white customers to, b which facility we assign all black customers to. Every row additionally has an index in column *idx* and the index of a state to which there is an improving move in column *nxt*.

idx	f_1	f_2	f_3	w	b	nxt
1	B	B	B	f_1	$\varnothing$	3
2	B	B	W	f_1	f_2	1
3	B	W	B	f_3	f_2	7
4	B	W	W	f_1	f_2	3
5	W	B	B	f_2	f_1	6
6	W	B	W	f_2	f_3	2
7	W	W	B	f_3	f_1	5
8	W	W	W	$\varnothing$	f_2	4

As we can see, this customer strategy enables an improving move for each facility placement. Since we now know, that for some credible customer strategies there is no SPE, the question arises whether for any underlying graph an SPE exists.

Lemma 8. *If the number of facilities is at least twice the number of facility strategies, then there always exists an SPE.*

Proof.
Consider a facility strategy vector v in which for every strategy there are at least two facilities using this strategy. We start with the first strategy and assign the black customers to one of the two facilities and the white customers to the other one. We then iterate through the remaining strategies in the same way, only

considering customers which have not been assigned yet. This is in customer equilibrium because the distribution is monochromatic. For any facility strategy vector, with a single facility deviating from the previously described one, we take any welfare optimizing placement as basis. We then shift any customers of the deviant to the 2 other facilities f_i, f_j using the same strategy. If this is not possible because the facilities serve the same color of customers, then we first shift all customers in f_i to f_j. Since the optimum has to be monochromatic, and we maintain the monochromacy while preventing any singular deviation from v (utility will change to or remain 0), together with this constructed customer strategy we have an SPE. □

Lemma 9. *Let f_1, f_2 be two facilities with strategies, s_1, s_2 and let the set of customers which both f_1 and f_2 can serve be C. Any strategy pair in customer-equilibrium must assign only white customers in C to the first facility and only black customers to the second facility or vice versa.*

Proof. Suppose this was not true and w.l.o.g. f_1 serves both black and white customers, having a black ratio of $\frac{p}{q}$ and f_2 of $\frac{r}{s}$, w.l.o.g. $\frac{p}{q} \leq \frac{r}{s}$. This however enables an improving move for a black customer in f_1 to f_2, which is a contradiction to the assumption of the distribution being in customer-equilibrium. The same holds analogously for white. □

Theorem 7. *With 2 strategies, there always exists a Subgame Perfect Equilibrium.*

Proof.
Let X, Y be the two possible facility strategies.
Case 1: The number of facilities is 1
This case is trivial since for any fixed customer strategy the facility can select the facility strategy which maximizes its utility without any competing facility.
Case 2:The number of facilities is 2
By Lemma 9 the only option for how the customers distribute themselves is between which one of both facilities gets the black and which the white intersection customers. Suppose now that, (X, X) together with a customer strategy, of the form defined by the matrix below, is not an SPE.The column *strategy* with entry A, B denotes that f_1 chooses strategy A and f_2 chooses strategy B. The columns $f_j - utility$ are min_i/maj_i when all customers of the minority/majority color of strategy i are assigned to f_j $i, j \in \{X, Y\}$. u_i is the utility of f_j in the case that f_j chooses strategy i and f_k the other strategy.
Let b_i, w_i be the number of black and respectively white customers in the range of strategy i and $maj_i = \max\{b_i, w_i\}, min_i = \min\{b_i, w_i\}$

$strategy$	$f_1 - utility$	$f_2 - utility$
X, X	min_X	maj_X
X, Y	u_X	u_Y
Y, Y	maj_Y	min_Y
Y, X	u_Y	u_X

The smallest cycle starting at (X, X) is w.l.o.g.:

$(X, X) \to (X, Y) \to (Y, Y) \to (Y, X) \to (X, X)$

$$1 : (X, X) \to (X, Y) \implies maj_X < u_2$$
$$2 : (X, Y) \to (Y, Y) \implies u_1 < maj_Y$$
$$3 : (Y, Y) \to (Y, X) \implies min_Y < u_1$$
$$4 : (Y, X) \to (X, X) \implies u_2 < min_X$$

1 and 4 imply $maj_X < min_X$ ↯ It follows that no such cycle can exist. No smaller cycle can exist, since it must have even length and a cycle has size 3 at least. A bigger cycle on the other hand would revisit a strategy vector, contradicting it being a smallest cycle.
The alternative cycle $(X, X) \to (Y, X) \to (Y, Y) \to (X, Y) \to (X, X)$ gives the symmetric contradiction $maj_Y < min_Y$.

Case 3: The number of facilities is 3

strategy	$f_1 - utility$	$f_2 - utility$	$f_3 - utility$
XXY	min_x	maj_x	u_y
XXX	min_x	maj_x	0
XYY	u_x	min_y	maj_y
YXY	min_y	u_x	maj_y
YYX	min_y	maj_y	u_x
YYY	min_y	maj_y	0
YXX	u_y	min_x	maj_x
XYX	min_x	u_y	maj_x

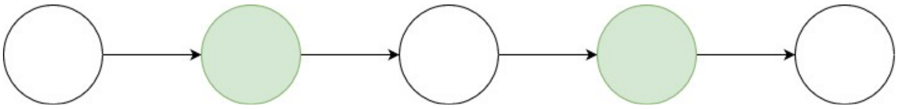
As soon as two facilities share a strategy, one facility
Suppose neither XXY nor YYX is in equilibrium. XXY can only have improving moves to XYX or YXY.
(1) $XXY \to XYY$ implies $maj_X < min_Y$
(2) $XXY \to YXY$ implies $min_X < min_Y$
(3) $YYX \to XYX$ implies $min_Y < min_X$
(4) $YYX \to YXX$ implies $maj_Y < min_X$
(1) and (3) implies $maj_X < min_Y < min_X \leq maj_X$ ↯
(1) and (4) implies $maj_X + maj_Y < min_Y + min_X \leq maj_Y + maj_X$ ↯.
(2) and (3) implies $min_X < min_Y < min_X$ ↯
(2) and (4) implies $maj_Y < min_X < min_Y \leq maj_Y$ ↯
Since we assumed that both strategy vectors are not in equilibrium, but any pairing of the possible implications yields a contradiction, we conclude that the premise must be false and XXY or YYX is in equilibrium.
Case 4: The number of facilities is at least 4, See Lemma 8.

□

6.5.2 Path-Graph Strategy

Lemma 10. *If the number of facilities in a path graph is at least as big as the number of facility strategies, then there always exists an SPE.*

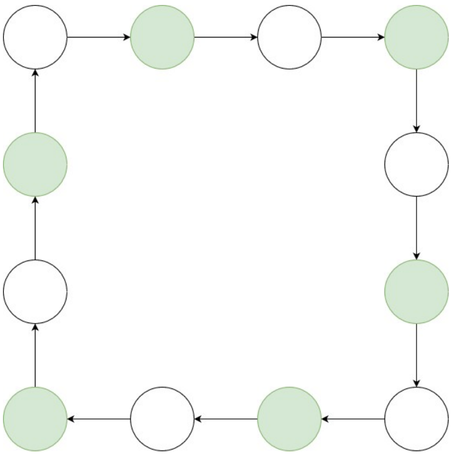
Proof. We split the path into parts of two and assign two facilities to the right of the two vertices. If there is a remainder of 1 we leave this location empty and assign the remaining facility to another one where there are already 2 others located. For this distribution, we choose any customer equilibrium. Suppose any facility changes its location to a vertex that is not one of both ends. In that case there is always a location in range, in which there are two facilities, which can claim all customers of the deviant, setting his utility to 0. This is a contradiction since we assumed that the facility has an improving move, but since any facility utility is greater or equal to 0 this can not be the case. Suppose now that a facility moves to one of the ends. If the end has a distance greater than 1 to the old position, then there are still two facilities in range as before, and we can set the new utility of the deviant to 0. If the distance is 1 then we can apply the same customer distribution as before the move, such that the utility of the deviant does not change.



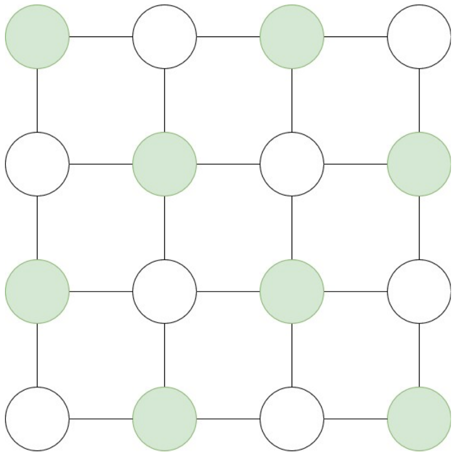
□

The same argument holds on cycles and grids.
Small-Path

6.5.3 Cycle-Graph Strategy



6.5.4 Grid-Graph Strategy



6.5.5 Further strategies

For this analysis, I consider properties of underlying graphs which lead to the existence of subgame perfect equilibria.

Lemma 11. *If one strategy X includes all white customers and there are more than two facilities,then there is always an SPE. The same holds for black.*

Proof. Consider a facility strategy vector with at least three facilities f_1, f_2, f_3 sharing X. We assign all white customers to f_1 and distribute the remaining black customers in the following way: We successively assign the strategy with most open black customers to the facility and assign all those customers to the facility. If f_1 deviates we assign all white customers to f_2 and the rest in the same way as before. Thus the new utility of f_1 is unchanged at $\frac{n}{2}$, n being the number of customers and no improving move would exist. If any other facility f_i deviates, it may only deviate to a strategy which is not used yet, since in every other case we do not reassign any customers. This however contradicts that we assigned to f_i the strategy with most open black customers at the time. Since the number of open black customers per strategy only further decreases, the old choice remains better. $\square$

We define a power, graph as a bipartite graph where there is one location node for each customer subset. Because of the power set structure, we call this graph a power-graph. Since all strategies are allowed, Lemma 10 proves the SPE existence.

Conjecture 1. *If two strategies X, Y together include all white customers, there is always an SPE. The same holds for black.*