7 Phase 1: Facility Placement

7.1 Subgame Perfect Equilibria

Since our 2-Phase game is dynamic, we make use of the concept of subgame perfect equilibria. This describes strategy profiles, which represent a Nash Equilibrium for every subgame. A strategy profile is thus only an SPE if there are both no improving moves for customers and no improving moves for facilities.

7.2 Strategy Space

7.2.1 Mini-Graph Strategy

For the most limited case, all facilities can only select one possible location. In that case, any customer equilibrium for this placement is a two-phase equilibrium, so it is not possible to block equilibria with this restriction. This does no longer hold in general for two or more possible locations. Let the first possible location V cover exactly all red customers, the second W all white customers. We define a facility having black priority as that any black customer will always enter this facility when he has the chance, analogous for white. We consider three facilities and ignore the rest, since their choices are independent (if at least one of the three facilities selects V then one of those will have priority on V, the same holds for W). The following matrix demonstrates where we assign each customer according to his color (w,b), depending on the strategies of f_1,f_2 and f_3.

idx	f_1	f_2	f_3	w	b	nxt
1	V	V	V	f_1	$\varnothing$	3
2	V	V	W	f_1	f_2	1
3	V	W	V	f_3	f_2	7
4	V	W	W	f_1	f_2	3
5	W	V	V	f_2	f_1	6
6	W	V	W	f_2	f_3	2
7	W	W	V	f_3	f_1	5
8	W	W	W	$\varnothing$	f_2	4

As we can see for this strategy for each facility placement, there is an improving facility move, nxt being the index of the state after an improving move. Since we now know, that for some credible customer strategies there is no SPE, the question arises whether for any underlying graph an SPE exists.

Lemma 8. *If the number of facilities is at least twice the number of facility strategies, then there always exists an SPE.*

Proof.
Consider a facility strategy vector v in which for every strategy there are at least two facilities using this strategy. We start with the first strategy and assign the black customers to one of the two facilities and the white customers to the other one. We then iterate through the remaining strategies in the same way, only considering customers which have not been assigned yet. For any facility strategy vector, with a single facility deviating from the previously described one, we take any welfare optimizing placement as basis. We then shift any customers of the deviant to the 2 other facilities f_i, f_j using the same strategy. If this is not possible because the facilities serve the same color of customers, then we first shift all customers in f_i to f_j. Since the optimum has to be monochromatic, and we maintain the monochromacy while preventing any singular deviation from v (utility will change to or remain 0), together with this constructed customer strategy we have an SPE. □

Theorem 7. *With 2 strategies, there always exists a Subgame Perfect Equilibrium.*

Proof.
Let X, Y be the two possible facility strategies.
Case 1: The number of facilities is 1
This case is trivial since for any fixed customer strategy the facility can select the facility strategy which maximizes its utility without any competing facility.
Case 2:The number of facilities is 2
First we observe, that in any equilibrium the two facilities always split up their intersecting customers monochromatically. Suppose this was not true, then after the customers are distributed, one of both facilities has a black ratio greater-equal the other. This would however enable an improving move for a black customer in the other facility, which is a contradiction to the assumption of the distribution being in equilibrium. The same holds analogously for white. Thus, the only option for how the customers distribute themselves is between which one of both facilities gets the black and which the white intersection customers. Suppose now that, (X, X) together with a customer strategy, of the form defined by the matrix below, is not an SPE.
Let b_i, w_i be the number of black and respectively white customers in strategy i and $maj_i = \max\{b_i, w_i\}, min_i = \min\{b_i, w_i\}, i \in \{X, Y\}$

$strategy$	$f_1 - utility$	$f_2 - utility$
X, X	min_1	maj_1
X, Y	u_1	u_2
Y, Y	maj_2	min_2
Y, X	u_2	u_1

The cycle starting at (X, X) is w.l.o.g.:
$(X, X) \rightarrow (X, Y) \rightarrow (Y, Y) \rightarrow (Y, X) \rightarrow (X, X)$

$$1 : (X, X) \rightarrow (X, Y) \implies maj_1 < u_2$$
$$2 : (X, Y) \rightarrow (Y, Y) \implies u_1 < maj_2$$

$$3 : (Y, Y) \to (Y, X) \implies min_2 < u_1$$

$$4 : (Y, X) \to (X, X) \implies u_2 < min_1$$

1 and 4 imply $maj_1 < min_1 \lightning$ It follows that no such cycle can exist.
Case 3: The number of facilities is 3

$strategy$	$f_1 - utility$	$f_2 - utility$	$f_3 - utility$
XXY	min_x	maj_x	u_y
XXX	min_x	maj_x	0
XYY	u_x	min_y	maj_y
YXY	min_y	u_x	maj_y
YYX	min_y	maj_y	u_x
YYY	min_y	maj_y	0
YXX	u_y	min_x	maj_x
XYX	min_x	u_y	maj_x

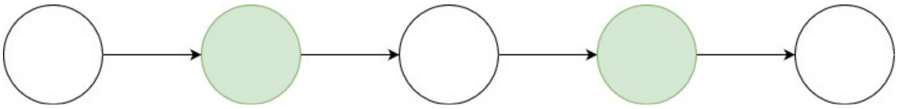
As soon as two facilities share a strategy, one facility
Suppose neither XXY nor YYX is in equilibrium. XXY can only have improving moves to XYX or YXY.
(1) $XXY \to XYY$ implies $maj_X < min_Y$
(2) $XXY \to YXY$ implies $min_X < min_Y$
(3) $YYX \to XYX$ implies $min_Y < min_X$
(4) $YYX \to YXX$ implies $maj_Y < min_X$
(1) and (3) implies $maj_X < min_Y < min_X \leq maj_X \lightning$
(1) and (4) implies $maj_X + maj_Y < min_Y + min_X \leq maj_Y + maj_X \lightning$.
(2) and (3) implies $min_X < min_Y < min_X \lightning$
(2) and (4) implies $maj_Y < min_X < min_Y \leq maj_Y \lightning$
Since we assumed that both strategy vectors are not in equilibrium, but any pairing of the possible implications yields a contradiction, we conclude that the premise must be false and XXY or YYX is in equilibrium.
Case 4: The number of facilities is at least 4, See Lemma 8.

□

7.2.2 Path-Graph Strategy

Lemma 9. *If the number of facilities in a path graph is at least as big as the number of facility strategies, then there always exists an SPE.*

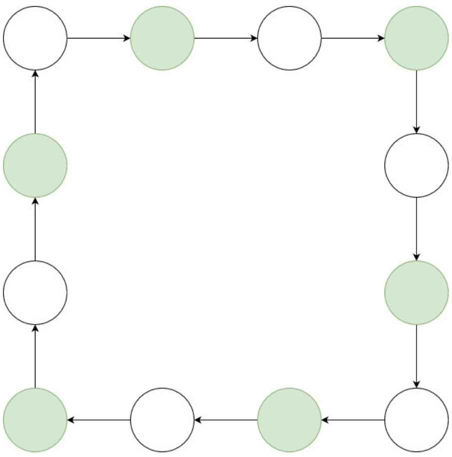
Proof. We split the path into parts of two and assign two facilities to the right of the two vertices. If there is a remainder of 1 we leave this location empty and assign the remaining facility to another one where there are already 2 others located. For this distribution, we choose any customer equilibrium. Suppose any facility changes its location to a vertex that is not one of both ends. In that case there is always a location in range, in which there are two facilities, which can claim all customers of the deviant, setting his utility to 0. This is a contradiction since we assumed that the facility has an improving move, but since any facility utility is greater or equal to 0 this can not be the case. Suppose now that a facility moves to one of the ends. If the end has a distance greater than 1 to the old position, then there are still two facilities in range as before, and we can set the new utility of the deviant to 0. If the distance is 1 then we can apply the same customer distribution as before the move, such that the utility of the deviant does not change.



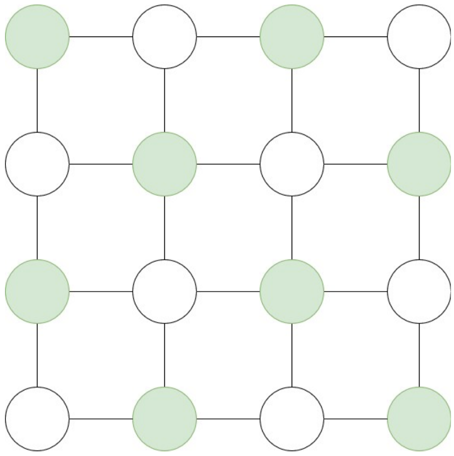
□

The same argument holds on cycles and grids.
Small-Path

7.2.3 Cycle-Graph Strategy



7.2.4 Grid-Graph Strategy



7.2.5 Further strategies

For this analysis, I consider that the underlying graph for the possible facility placements enables all possible strategies.

Lemma 10. *If one strategy X includes all white customers and there are more than two facilities,then there is always an SPE. The same holds for black.*

Proof. Consider a facility strategy vector with at least three facilities f_1, f_2, f_3 sharing X. We assign all white customers to f_1 and distribute the remaining black customers in the following way: We successively assign the strategy with most open black customers to the facility and assign all those customers to the facility. If f_1 deviates we assign all white customers to f_2 and the rest in the same way as before. Thus the new utility of f_1 is unchanged at $\frac{n}{2}$, n being the number of customers and no improving move would exist. If any other facility f_i deviates, it may only deviate to a strategy which is not used yet, since in every other case we do not reassign any customers. This however contradicts that we assigned to f_i the strategy with most open black customers at the time. Since the number of open black customers per strategy only further decreases, the old choice remains better. □

We define a power, graph as a bipartite graph where there is one location node for each customer subset. Because of the power set structure, we call this graph a power-graph. Since all strategies are allowed Lemma 10 proves the SPE existence.

Conjecture 1. *If two strategies X, Y together include all white customers, there is always an SPE. The same holds for black.*