6 Phase 2: Customers Choice

6.1 Existence of Equilibria

6.1.1 Definitions

Definition 1. *Let the team function, $team : F \rightarrow \{0, 1\}$ denote, for each football customer, whether team 1 or 0 is his favorite team.*

Definition 2. *Let $X = \{s : s \in S \ \wedge F(s) = 1\}$ be the set of customers of team X, respectively we define Y for team Y.*

Definition 3. *Let $customers : B \rightarrow 2^S$ be the function returning the set of customers a facility is serving.*

Definition 4. *For a facility b let $r_X^b = \frac{|customers(b) \cap X|}{|customers(b)|}$ be the relative amount of customers of team 1 in the i-th facility, r_Y^b respectively for team 0.*

Definition 5. *Let $r_X^* = \max_{b \in B} r_X^b$ be the greatest ratio for customers of team Y, respectively r_Y^* for team X.*

Definition 6. *Let $\tilde{r}_X = \min b \in B r_X^b$ be the smallest ratio for customers of team Y, respectively r_Y^* for team X.*

6.1.2 Potentials

Let

$$\Phi_1(f_1, .., f_n) = sort(\{r_x^{f_i}\})$$

be the ordered facility ratios for team X (descending) with arbitrary tie breaking.

$$\Phi_2 = \sum_{i=1}^{m} \max(|customers(f_i) \cap X|, |customers(f_i) \cap Y|)$$

be the sum of numbers of customers in the majority for each facility.

$$\Phi_3 = |\{f_i \in B | r_x^{f_i} = r_y^{f_i}\}|$$

be the number of facilities with customer parity.

6.1.3 Equlibria with thresholds below or equal $\frac{1}{2}$

Lemma 1. *For any two unequal customer ratios, the difference between the two is at least $\frac{1}{n^2}$.*

Proof. Let $\frac{r}{s} \neq \frac{p}{q}$ be any two unequal facility ratios at any point in time. W.l.o.g $\frac{r}{s} > \frac{p}{q}$.

$$\frac{r}{s} - \frac{p}{q} = \frac{r \cdot q - p \cdot s}{q \cdot s} \geq \frac{1}{n^2}$$

□

Lemma 2. *Inside an improving response cycle, a move can only take place if the customer is in a strict minority* ($\frac{p}{q} < \frac{1}{2}$) *and wishes to enter a facility in a strict minority*($\frac{r}{s} < \frac{1}{2}$).

Proof. We consider two cases of improving moves and show that all moves either increase Φ_2 or have no effect on Φ_2, thus the cycle can only contain moves which do not increase Φ_2.
Suppose v moves from a facility with X_1 customers of the same team, Y_1 customers of the opposing team to a facility with X_2, Y_2 customers respectively. Since v does an improving move, his satisfying threshold is not met, and thus it must be that $X_1 < Y_1$, then the following cases can occur.

1. Case: $X_2 \geq Y_2$
   A move from the minority to the majority will increase Φ_2 by 1. Since the number of majority customers in the previous facility is preserved and the number of majority customers in the target facility is increased, by 1.

2. Case: $X_2 < Y_2$
   Both majorities are unchanged, so Φ_2 is unchanged.

□

Definition 7. *A customer is in a strong minority if he is in a strict minority, and like-minded customers in the same facility remain in a strict minority after his improving move.*

Lemma 3. *Inside an improving response cycle, the only valid moves are from a strong minority to a strong minority.*

Proof. By Lemma 2 we already know, that any valid move can only be from a strict minority to another strict minority. We now show, that the only valid moves are those which do not increase the number of parity facilities in Φ_3, since no valid moves decreases the potential. Suppose v moves from a facility with X_1 customers of the same team, Y_1 customers of the opposing team to a facility with X_2, Y_2 customers respectively.

1. Case: $X_2 < Y_2 - 1$ Both minorities are preserved with the move and Φ_3 remains unchanged.

2. Case: $X_2 = Y_2 - 1$ The minority v came from is preserved and the move into the other facility establishes parity, so Φ_3 is increased by 1.

□

Theorem 1. *Every sequence of improving moves ends in a pure nash equilibrium.*

Proof. Suppose, for any sequence of improving moves, there exists an improving response cycle. Consider a point in time t at which the facility of a customer v of team Y, who performs an improving move in the cycle at some point, has the lowest value in Φ_1. Note that Φ_1 is sorted by the ratios for team X, so a low value is good for v. If there is no such v then we only move customers of team X and it can be shown that Φ_1 is a lexicographic potential function in that case, which contradicts the existence of an improving response cycle. In order for v to do his move, the ratio he is in has to increase (decrease for him), since else we would further improve his situation and have not been looking at the correct point in time t. An increase can only happen by a customer w of team X entering or a customer x of team Y leaving. The second case can not happen, since him leaving would make x happier than v and he does a move inside the cycle, so we should have been looking at x not v. We differentiate 3 cases for the point in time t.

1. v is in a parity facility
   No valid move goes from or to a parity facility according to Lemma 3, which contradicts our assumption, that v will make a move at some point.

2. v is in a strict minority
   Since v is in a strict minority, all X customers in the facility are in a strict majority and thus their satisfaction threshold is met. Any Y customer entering or leaving would yield a facility with an even lower ratio. In order for v to leave the facility without decreasing the ratio, a X customer

must enter. Since no valid move is possible between a strict minority and a strict majority, this can never happen.

3. v is in a strict majority
Therefore, the only valid moves are moves of customers of team X inside and outside the facility. This will however never result in a parity facility by Lemma 3, so there will never be an incentive for v to move out, which again contradicts our assumption that v will make a move.

□

6.1.4 Equilibrium for blind customers with arbitrary thresholds

It can be shown that if the only information a customer can obtain is the reduced ratio of each facility, instead of the distribution for each, then Φ_1 is a lexicographic potential function. This is leads to a customer not knowing its impact that his move will bring.

Theorem 2. *Every sequence of improving moves in this scenario ends in a pure Nash equilibrium.*

Proof. For this, we show by induction that every entry of Φ_1 obtains and maintains its maximum value at some point.
Let's assume that i=1
We consider four cases for a move from f_i to $f_j, i < j$:

1. $f_i \neq f_1 \wedge f_j \neq f_1$ Such a move may only replace f_1 in the next step by an even higher ratio at the first index, or not affect it at all. Thus, the entry is not reduced.

2. the improving move is either a X customer entering f_1 or a Y customer leaving f_1. In both cases, the ratio for f_1 is increased.

3. the improving move is a X customer leaving f_1. Let $r_x^{f_j} = \frac{r}{s}, r_x^{f_1} = \frac{p}{q}$, then it follows that
$$\frac{p}{q} < \frac{r}{s}$$
. Thus, f_j with a ratio of $\frac{r+1}{s+1}$ is the new first entry, which has increased.

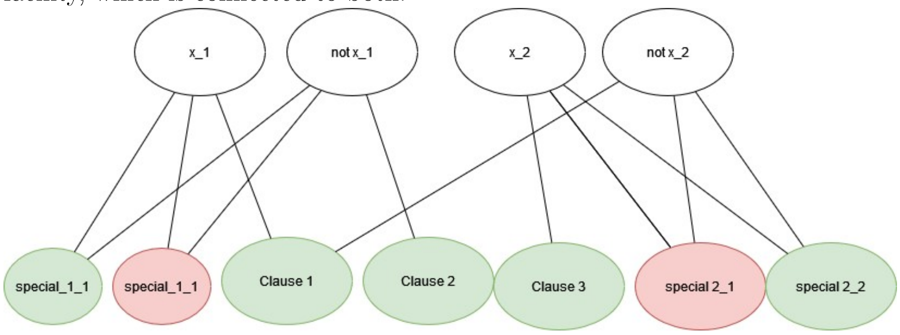
4. the improving move is a Y customer entering f_1.
$$1 - \frac{p}{q} > 1 - \frac{r}{s}$$
. Thus, $\frac{r}{s} > \frac{p}{q}$ and f_j with a ratio of $\frac{r}{s-1}$ is the new first entry, which has increased.

We now assume that the first i entries obtain and maintain their maximum value at some point. $i \rightarrow i + 1$
Analogous to the case for $i = 1$ any move to f_i will either increase its ratio or yield a new facility with a higher ratio that will overtake its place(no facility above can be further increased). Since this is bounded up by 1 and each increment is at least $\frac{1}{n^2}$ by Lemma 1 this has to end after at most n^2 steps. $\square$

6.2 Finding the Social Welfare Optimum is NP hard

6.2.1 Threshold 1

We show that deciding if every customer can be satisfied with a given threshold is already NP hard, by reducing from 3SAT.
Construction:
Given a 3SAT instance in CNF we build an instance with threshold 1. First, we introduce one facility for each literal (so one for x and $\neg x$) and one facility for each clause. Each clause is then connected to the 3 respective literals. Every clause facility is then colored white. For each pair x $\neg x$ we introduce a new white facility, which is connected to both. Similarly, we introduce a new white facility, which is connected to both.



Correctness:
$(\rightarrow)$ Given a satisfying assignment for the 3SAT instance, we leave every literal-facility empty if the literal is false and make it open for entering if the literal is true. Since the assignment is satisfying, each clause-customer can enter an open literal-facility. The black special-customers can then enter the empty facilities, of which one always has to exist, since a valid assignment has either $x_i = False$ or $\neg x_i = False$. The same applies for the white special-customers. In conclusion every customer can enter a facility with a satisfaction of 1.
$(\leftarrow)$ Assume that we can solve the instance, such that every customer has a satisfaction of 1. In order for the special-customers to be fully satisfied, either x or $\neg x$ must contain no clause-customers. The interpretation being that the non-empty one of those is True. Since a literal-facility can only cover those clause-customers which the literal can also set to true in the SAT formula and since every clause-customer is inside a monochromatic literal-facility we can deduce the variable assignment, by setting it to true if the negated facility is monochromatic black and else setting it to false.

6.2.2 Arbitrary Threshold

The former proof can be extended to arbitrary satisfaction thresholds t above 0. For any variable x_i, let the literal x_i occur in r many clauses and let the literal $\neg x_i$ occur in s many clauses, w.l.o.g. $r \geq s$. In contrast to before, we now introduce $\lceil \frac{(r+s)}{t} \rceil + 1$ black special vertices which we connect to both literals.

This forces a solution, in which everyone's satisfaction threshold is met, to be a monochromatic solution.
($\rightarrow$)
This direction remains the same, since when there is a satisfying variable assignment for the SAT formula we can assign the black customers to the literals, which are False. Since everyone can be satisfied by 1 the satisfaction thresholds are all met.
($\leftarrow$)
If the solution is monochromatic, we can deduce an assignment as before. Suppose that there is a literal-facility x_i, which is not monochromatic. Since the black special vertices for x_i have to split up between x_i and $\neg x_i$ w.l.o.g. there are at least as many black customers in x_i than in $\neg x_i$. Let x_i have $g \leq r$ white customers it serves. In that case, their utility is at most $\frac{g}{g+\frac{r}{t}} \leq \frac{r}{r+\frac{r}{t}} = \frac{1}{1+\frac{1}{t}} < \frac{1}{\frac{1}{t}} = t$. Thus the satisfaction threshold cannot be met, and we arrive at a contradiction to our assumption that every customer has their satisfaction threshold met.

6.2.3 Instances in P

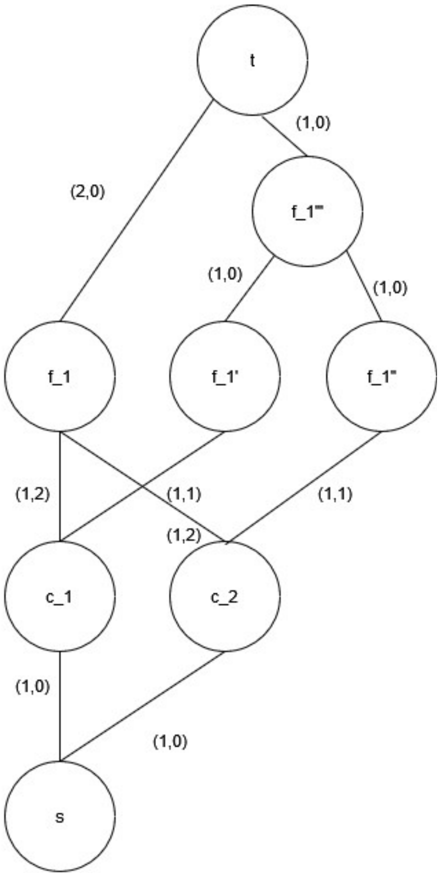
As we can see in the previous proof, where the highest facility degree is 3, this is already enough to show NP hardness. This can not be extended to at most degree 2 for each facility, since this is solvable in polynomial time Assigning as many customers to facilities with degree 1 as possible is optimal, since in any optimal solution we can apply this assignment and the result would only improve. Moving a customer would give him a satisfaction of 1. If in the facility he left, there is another customer, then he will also have a satisfaction of 1. After assigning the customer, we can delete him from the instance and reiterate until there are only facilities with degree 2 exactly. If a facility has two customers of the same color in range, then we can also simply assign the customers there by the same argument as before. The remaining core of the instance is facilities with degree exactly 2 and a mixed color. For an assignment with k customers served alone, the social welfare is

$$k + \frac{n-k}{2} = \frac{n}{2} + \frac{k}{2}$$

Since n is the same for any assignment, and the welfare is only dependent on k, an optimal assignment maximizes the number of customers that are served alone. We can calculate this the following way using cost flows. Every facility is split into 4 vertices as pictured below. The two customers connected to a facility each have two facility vertices they can push flow to, the first clone costing 2 per flow, the second clone costing 1. The fourth facility clone in the layer above prevents two customers from both pushing their flow of 1 with cost 1 to the sink. We then calculate the MinCostMaxFlow on this instance. The cost of a MaxFlow with k customers seved alone is 1 for a facility with one customer and 3 for a facility with two customers in total

$$k + 3 \cdot (n-k) = 3 \cdot n - 2 \cdot k$$

. This is also only dependent on k and is minimized when k is maximized.



6.3 Price of Anarchy

Consider a facility f with n_f customers in any distribution of customers to facilities.

Lemma 4. *If the majority of customers in f has a utility of 1 and the minority of customers of less than 1, then the sum of utilities in this facility is at-least $n_f \cdot (1 - \frac{t}{2})$.*

Proof. Let the minority ratio be $t - \epsilon$ with $\epsilon > 0$ since their utility is not 1. The sum of utilities for the minority is then by the definition of the utility function

$$(t - \epsilon) \cdot n_f \cdot \frac{t - \epsilon}{t} = \frac{t^2 \cdot n_f - 2 \cdot t \cdot \epsilon \cdot n_f + \epsilon^2 \cdot n_f}{t}$$

and for the majority customers it is

$$(1 - t + \epsilon) \cdot n_f \cdot 1 = n_f - n_f \cdot t + \epsilon \cdot n_f$$

summed together we get

$$\frac{t^2 \cdot n_f - 2 \cdot t \cdot \epsilon \cdot n_f + \epsilon^2 \cdot n_f + n_f \cdot t - n_f \cdot t^2 + \epsilon \cdot n_f \cdot t}{t}$$

To find the minimum for this expression we derive with respect to ϵ and set it to zero.

$$\frac{-2 \cdot t \cdot n_f + 2 \cdot \epsilon \cdot n_f + n_f \cdot t}{t} = 0$$

Since t as well as n_f are strictly above 0 we can multiply first by t and then divide by n_f

$$-2 \cdot t + 2 \cdot \epsilon + t = 0 \iff \epsilon = \frac{t}{2}$$

Since this extremum is unique and the second derivative is $\frac{2 \cdot n_f}{t} > 0$ we know that this is the unique minimum. Entering this value for epsilon in the sum of utilities, yields

$$(\frac{t}{2})^2 \cdot \frac{n_f}{t} + (1 - \frac{t}{2})^2 \cdot n_f = n_f \cdot (\frac{t}{4} + 1 - t + \frac{t}{4}) = n_f \cdot (1 - \frac{t}{2})$$

□

Lemma 5. *If all customers in f have a utility of less than 1, then the sum of utilities in this facility is at-least $n_f \cdot \frac{1}{2 \cdot t}$.*

Proof. The utility sum for the minority is the same as in Lemma 4.

$$(t - \epsilon) \cdot n_f \cdot \frac{t - \epsilon}{t} = \frac{t^2 - 2 \cdot t \cdot \epsilon + \epsilon^2}{t} \cdot n_f$$

The utility sum for the majority is, then

$$(1 - t + \epsilon) \cdot n_f \cdot \frac{1 - t + \epsilon}{t} = \frac{1 + 2 \cdot \epsilon + \epsilon^2 - 2 \cdot t - 2 \cdot \epsilon \cdot t + t^2}{t} \cdot n_f$$

summed together we get

$$\frac{t^2 - 2 \cdot t \cdot \epsilon + \epsilon^2 + 1 + 2 \cdot \epsilon + \epsilon^2 - 2 \cdot t - 2 \cdot \epsilon \cdot t + t^2}{t} \cdot n_f$$

To find the minimum for this expression we derive with respect to ϵ and set it to zero.

$$\frac{-2 \cdot t + 2 \cdot \epsilon + 2 + 2 \cdot \epsilon - 2 \cdot t}{t} \cdot n_f = 0$$

Since t as well as n_f are strictly above 0 we can multiply first by t and then divide by n_f

$$-2 \cdot t + 2 \cdot \epsilon + 2 + 2 \cdot \epsilon - 2 \cdot t = -4 \cdot t + 4\epsilon + 2 = 0 \iff \epsilon = t - \frac{1}{2}$$

Since this extremum is unique and since the second derivative is $\frac{4 \cdot n_f}{t} > 0$ we know that this is the unique minimum. Entering this value for ϵ in the sum of utilities, yields

$$\frac{n_f}{4 \cdot t} + \frac{n_f}{4 \cdot t} = n_f \cdot (\frac{1}{2 \cdot t})$$

□

Theorem 3. *The Price of Anarchy for all t is exactly $\frac{1}{1-\frac{t}{2}}$.*

Proof. Consider a facility f with n_f customers in any distribution of n customers to facilities. We now consider three cases.

Case 1: All customers have utility 1
In that case, the sum of utilities is n_f.

Case 2: Only the majority customers have utility 1
In that case, by Lemma 4 the sum of utilities is at-least $n_f \cdot (1 - \frac{t}{2})$.

Case 3: All customers have utility of less than 1
In that case, by Lemma 5 the sum of utilities is at-least $n_f \cdot (\frac{1}{2 \cdot t})$.

There is no other case because the majority customers always have greater or equal utility. Case 2 gives the lowest bound because $\frac{1}{2t} \geq 1 - \frac{t}{2}$ for $t > 0$.

$$\frac{1}{2t} \geq 1 - \frac{t}{2} \iff 1 \geq 2t - t^2 \iff t^2 - 2t + 1 \geq 0 \iff (t-1)^2 \geq 0$$

For $t = 0$ any placement is optimal, and thus the Price of Anarchy would be $1 = \frac{1}{1-\frac{0}{2}}$. Using that the welfare optimum is at most n (all agents having utility 1) and assuming the worst case for all facilities, the Price of Anarchy is upper bounded by $\frac{n}{n \cdot (1-\frac{t}{2})} = \frac{1}{1-\frac{t}{2}}$. This bound is sharp because one can construct a graph with n' vertices, where the worst case is achieved for every facility. We construct two separate, fully connected graphs with two facilities each. For the first graph, we assign $\frac{t}{2} \cdot \frac{n'}{4}$ black customers and $1 - \frac{t}{2} \cdot \frac{n'}{4}$ white customers to each facility. For the second graph, we assign $\frac{t}{2} \cdot \frac{n'}{4}$ white customers and $1 - \frac{t}{2} \cdot \frac{n'}{4}$ black customers to each facility. This instance is in equilibrium because there are no edges between the two sub-graphs and the two facilities in each sub-graph have the same ratio. On the other hand it would be possible to assign all black customers to the first facility and all white customers to the second facility, for each graph with a total utility sum of n'. The size of n' has to be chosen such that $\frac{t}{2} \cdot \frac{n'}{4}$ and $(1 - \frac{t}{2} \cdot \frac{n'}{4})$ are integer values. Since we are only interested in ratios, one can assume t to be a rational number and scale accordingly.,

□

6.4 Welfare Optimum Stability

Lemma 6. *An improving customer c move from a facility f_1 where he is in the minority to another facility f_2, where his color is in the minority, before the move, increases the social welfare.*

Proof. Let $\frac{p}{q}$ be the unreduced ratio for c in f_1 and $\frac{r}{s}$ the unreduced ratio in f_2 before the move. After the move, there are 3 types of changes in the social welfare.

$$\frac{r+1}{s+1} - \frac{p}{q}$$

is the change for c, which is the difference between his new and previous ratio. The decrease in utility for customers in f_1 is the increase in utility for the other group of customers, since the sum of two color-ratios is 1.

$$(q-2p+1) \cdot \left(\frac{q-p}{q-1} - \frac{q-p}{q} \right)$$

is the net change for customers in f_1, since using the surplus equals deficit argument, there are $p-1$ minority customers left and $q-p$ many majority customers, so there is a net surplus of $\frac{q-p}{q-1} - \frac{q-p}{q}$ for $q-p-(p-1) = q-2p+1$ many majority customers.

$$-(s-2r) \cdot \left(\frac{s-r}{s} - \frac{s-r}{s+1} \right)$$

is the net change for customers in f_2. Analogously, for the majority in f_2 we have a net deficit of $(\frac{s-r}{s} - \frac{s-r}{s+1})$ for $(s-r-r)$ many majority customers. In total we get a change in welfare of

$$\left(\frac{r+1}{s+1} - \frac{p}{q} \right) + (q-2p+1) \cdot \left(\frac{q-p}{q-1} - \frac{q-p}{q} \right) - (s-2r) \cdot (\frac{s-r}{s} - \frac{s-r}{s+1})$$

Suppose now that there exists a valid (p,q,r,s) assignment such that the expression is negative. An assignment is valid if

$$\left(\frac{1}{2} > \frac{r}{s} \geq \frac{p}{q} \right), q > p \geq 1, s \geq r \geq 1, p, q, r, s \in \mathcal{N}$$

Since it is an improving move, it holds that $\frac{p}{q} \leq \frac{r}{s}$ and $(k \cdot p, k \cdot q, r, s), k \geq 1$ is also a valid assignment, because of $\frac{k \cdot p}{k \cdot q} = \frac{p}{q}$. The same holds for $(p, q, k \cdot r, k \cdot s)$. The change in welfare with $\frac{p}{q}$ and $\frac{r}{s}$ scaled by k then becomes

$$\Delta_1 := (\frac{k \cdot r + 1}{k \cdot s + 1} - \frac{k \cdot p}{k \cdot q})$$

$$\Delta_2 := (k \cdot q - 2 \cdot k \cdot p + 1) \cdot (\frac{k \cdot q - k \cdot p}{k \cdot q - 1} - \frac{k \cdot q - k \cdot p}{k \cdot q})$$

$$\Delta_3 \coloneqq -(k \cdot s - 2 \cdot k \cdot r) * (\frac{k \cdot s - k \cdot r}{k \cdot s} - \frac{k \cdot s - k \cdot r}{k \cdot s + 1})$$
$$\Delta \coloneqq \Delta_1 + \Delta_2 + \Delta_3$$

Taking the partial derivative with respect to k yields that it is non-positive for all valid assignments since $q \geq 2$ (else c would be alone and have no improving move)

$$\frac{\partial \Delta}{\partial k} = -\frac{2(p-q)^2}{q(-1+kq)^2} - \frac{2(r-s)^2}{s(1+ks)^2}$$

Since an increase in k further decreases the change in welfare or does not change its value we can analyze how far this value can be further minimized by taking the limit for k towards positive infinity.

$$\lim_{k \to \infty} \Delta = \frac{2p^2}{q^2} - \frac{2r^2}{s^2} + \frac{4r}{s} - \frac{4p}{q} = \frac{2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2}{q^2s^2}$$

Since we assumed that our assignment gave a negative social welfare, and we only further decreased this, we arrive at the desired contradiction.

$$\frac{2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2}{q^2s^2} < 0 \implies 2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2 < 0$$

$$2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2 = 2(ps - qr)(qr - 2qs + ps) \geq 0 \lightning$$

This is a contradiction because of the impact blindness $\frac{p}{q} \leq \frac{r}{s} \implies ps \leq rq$ and also $qs > ps, qs \geq qr$, so both factors are smaller equal zero. $\square$

Lemma 7. *An improving customer c move from a facility f_1 where he is in the majority to another majority facility f_2 for his color increases the social welfare.*

Proof. Let $\frac{p}{q}$ be the unreduced ratio for c in f_1 and $\frac{r}{s}$ the unreduced ratio in f_2 before the move. After the move, there are 3 types of changes in the social welfare.

$$\frac{r+1}{s+1} - \frac{p}{q}$$

is the change for c, which is the difference between his new and previous ratio. The decrease in utility for customers in f_1 is the increase in utility for the other group of customers, since the sum of two color-ratios is 1.

$$-(2p - q - 1) \cdot \left(\frac{p}{q} - \frac{p-1}{q-1}\right)$$

is the net change for customers in f_1, since using the surplus equals deficit argument, there are $p - 1$ majority customers left and $q - p$ many majority customers, so there is a net deficit of $\frac{p}{q} - \frac{p-1}{q-1}$ for $p - 1 - (q - p) = 2p - q - 1$ many majority customers.

$$(s - 2r) \cdot \left(\frac{r+1}{s+1} - \frac{r}{s}\right)$$

is the net change for customers in f_2. Analogously, for the majority in f_2 we have a net surplus of $(\frac{r+1}{s+1} - \frac{r}{s})$ for $(s - r - r)$ many majority customers. In total we get a change in welfare of

$$(\frac{r+1}{s+1} - \frac{p}{q}) - (2p - q - 1) \cdot (\frac{p}{q} - \frac{p-1}{q-1}) + (s - 2r) \cdot (\frac{r+1}{s+1} - \frac{r}{s})$$

Suppose now that there exists a valid (same definition as in Lemma 6)(p, q, r, s) assignment such that the expression is negative. Since it is an improving move, it holds that $\frac{p}{q} \leq \frac{r}{s}$ and $(k \cdot p, k \cdot q, r, s), k \geq 1$ is also a valid assignment, because of $\frac{k \cdot p}{k \cdot q} = \frac{p}{q}$. The same holds for $(p, q, k \cdot r, k \cdot s)$. The change in welfare with $\frac{p}{q}$ and $\frac{r}{s}$ scaled by k then becomes

$$\Delta_1 := (\frac{k \cdot r + 1}{k \cdot s + 1} - \frac{k \cdot p}{k \cdot q})$$

$$\Delta_2 := (k \cdot q - 2 \cdot k \cdot p + 1) \cdot (\frac{k \cdot q - k \cdot p}{k \cdot q - 1} - \frac{k \cdot q - k \cdot p}{k \cdot q})$$

$$\Delta_3 := -(k \cdot s - 2 \cdot k \cdot r) \cdot (\frac{k \cdot s - k \cdot r}{k \cdot s} - \frac{k \cdot s - k \cdot r}{k \cdot s + 1})$$

$$\Delta := \Delta_1 + \Delta_2 + \Delta_3$$

Taking the partial derivative with respect to k yields that it is non-positive for all valid assignments.

$$\frac{\partial \Delta}{\partial k} = -\frac{2(p-q)^2}{q(-1+kq)^2} - \frac{2(r-s)^2}{s(1+ks)^2}$$

Since an increase in k further decreases the change in welfare or does not change its value we can analyze how far this value can be further minimized by taking the limit for k towards positive infinity.

$$\lim_{k \to \infty} \Delta = \frac{2p^2}{q^2} - \frac{2r^2}{s^2} + \frac{4r}{s} - \frac{4p}{q} = \frac{2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2}{q^2s^2}$$

Since we assumed that our assignment gave a negative social welfare, and we only further decreased this, we arrive at the desired contradiction.

$$\frac{2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2}{q^2s^2} < 0 \implies 2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2 < 0$$

$$2p^2s^2 - 2r^2q^2 + 4rsq^2 - 4pqs^2 = 2(ps - qr)(qr - 2qs + ps) \geq 0 \lightning$$

This is a contradiction because of the impact blindness $\frac{p}{q} \leq \frac{r}{s} \implies ps \leq rq$ and also $qs > ps, qs \geq qr$, so both factors are smaller or equal zero. $\square$

Theorem 4. *Every possible improving customer move increases the social welfare. The Price of Stability is therefore 1.*

Proof. There are the following possible improving customer moves:

1. Minority to Minority
   See Lemma 6

2. Minority to Parity/Majority
   Since the move improves the situation for more customers in the source facility than it worsens the situations for others, as well as in the target facility, the net change is positive.

3. Parity to Minority
   This is not a legal move under the assumption of impact blindness, since a parity facility has a better ratio for the moving customer, than a minority facility.

4. Parity to Parity/Majority
   Since the move improves the situation for more customers in the source facility than it worsens the situations for others, as well as in the target facility, the net change is positive.

5. Majority to Minority/Parity
   This is not a legal move under the assumption of impact blindness, since a majority-facility has a better ratio for the moving customer, than a minority-facility or parity-facility.

6. Majority to Majority
   See Lemma 7

□

Theorem 5. *Successively applying improving customer moves yields an equilibrium in polynomial time.*

Proof. Following the previous theorem in case 1 and 6 the increase in social welfare after a move is

$$(\frac{r+1}{s+1} - \frac{p}{q}) - (2p - q - 1) \cdot (\frac{p}{q} - \frac{p-1}{q-1}) + (s - 2r) \cdot (\frac{r+1}{s+1} - \frac{r}{s})$$

Since we know that it is strictly above 0 (it is zero only in the limit with $\frac{p}{q} = \frac{r}{s}$), we can lower bound the change by $\frac{1}{(s+1) \cdot s \cdot q \cdot (q-1)} \geq \frac{1}{n^4}$. In case 2 and 4 using an amortized view, we only increase the social welfare of some majority customers, while leaving all remaining customers unchanged. Thus our change improves the social welfare at least by the improvement of the moving customer $\frac{r+1}{s+1} - \frac{p}{q} \geq \frac{1}{s+1 \cdot q} \geq \frac{1}{n^2}$. Thus since the minimal improvement is $\frac{1}{n^4}$ and the total welfare can not exceed n in $O(n^5)$ steps an equilibrium will be obtained. □

Theorem 6. *The Social Welfare Optimum is not in equilibrium in general for the impact aware case.*

Proof. Let c be a customer in a facility with unreduced ratio $\frac{p}{q} = \frac{19}{1000}$ and an improving move to a facility with unreduced ratio $\frac{r}{s} = \frac{1}{100}$

$$\frac{19}{1000} = 0.019 < \frac{1+1}{101} = \frac{2}{101} \approx 0.0198$$

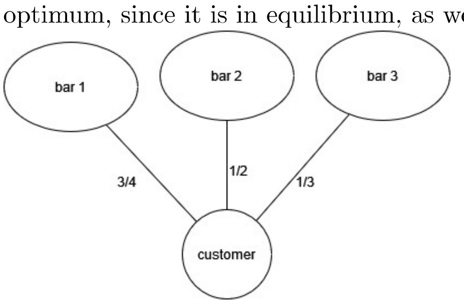
. All other customers have no further facility options. The change in welfare is as shown in Lemma 6

$$(\frac{r+1}{s+1} - \frac{p}{q}) + (q-2p+1) \cdot (\frac{q-p}{q-1} - \frac{q-p}{q}) - (s-2r) * (\frac{s-r}{s} - \frac{s-r}{s+1}) = -\frac{26427}{1868500}$$

which decreases the social welfare while ending in an equilibrium, since c is the only customer able to move and there are only 2 facilities. □

6.5 Splitting up

The relaxation such that each customer has a weight of 1 which he may split among the facilities in each neighborhood does not find any use in an equilibrium. Since as one of the neighboring facilities has the highest ratio, it only makes sense to put the entire weight on this facility. This only further increases the satisfaction. The only exception would be if all the facilities he has weight on are of ratio 1, but even then putting all the weight into one facility gives a solution of the same value. This option would also not affect the social welfare optimum, since it is in equilibrium, as we have seen in the previous section.



6.6 Implementation