Solución II Examen II Semestre 2024

1. [3 pts] Calcule $\overline{(i+3)} \cdot \dfrac{5i}{2-i}$.

$$\overline{(i+3)} \cdot \frac{5i}{2-i} = -i+3 \cdot \frac{5i}{2-i}$$

$$= \frac{-i \cdot 5i + 3 \cdot 5i}{2-i}$$

$$= \frac{-5i^2 + 15i}{2-i} \cdot \frac{2+i}{2+i}$$

$$= \frac{(5+15i)(2+i)}{2^2 - i^2}$$

$$= \frac{10 + 5i + 30i + 15i^2}{5}$$

$$= \frac{10 + 35i - 15}{5}$$

$$= \frac{-5 + 35i}{5}$$

$$= -1 + 7i$$

2. Sean $x = 3\,\mathrm{cis}(110°)$, $y = 2\,\mathrm{cis}(85°)$, $z = xy$.

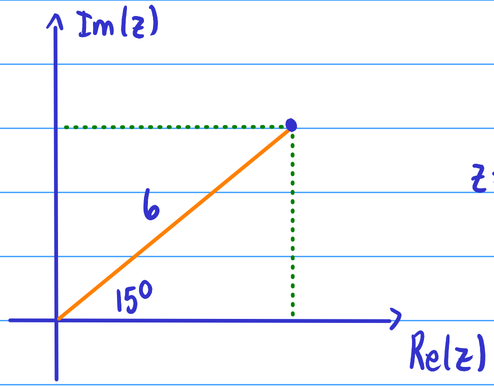
(a) **[2 pts]** Calcule z en forma rectangular.

$$z = xy \Rightarrow z = 3\,\mathrm{Cis}(110°) \cdot 2\,\mathrm{Cis}(85°)$$
$$\Rightarrow z = 6\,\mathrm{Cis}(110° + 85°)$$
$$\Rightarrow z = 6\,\mathrm{Cis}(195°)$$
$$\Rightarrow z = 6\left[\mathrm{Cos}(195°) + i\,\mathrm{Sen}(195°)\right]$$
$$\Rightarrow z = 6\,\mathrm{Cos}(195°) + 6\,\mathrm{Sen}(195°)\,i$$
$$\Rightarrow z = 6\left(\frac{-\sqrt{6}-\sqrt{2}}{4}\right) + 6\cdot\left(\frac{-\sqrt{6}+\sqrt{2}}{4}\right)i$$
$$\Rightarrow z = \frac{3}{2}\left(-\sqrt{6}-\sqrt{2}\right) + \frac{3}{2}\left(-\sqrt{6}+\sqrt{2}\right)i$$

(b) **[1 pto]** Encuentre el argumento principal de z.

Note que $z = 6\,\mathrm{Cis}(195°)$, entonces $\theta = 195°$, que se encuentra en el tercer cuadrante, por tanto el argumento principal sería $\theta = 195° - 180° \Rightarrow \theta = 15°$

(c) **[1 pto]** Represente z en el plano complejo.



$z = (6, 15°)$

Se grafica en polar, pero pudo hacerse en rectangular.

3. **[3 pts]** Considere el polinomio $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$. Si se sabe que $P(i) = 0$, factorice completamente en $\mathbb{C}$ el polinomio $P(x)$.

1	-3	4	-4	3	-1	i
↓	i	$-3i-1$	$3i+3$	$-3-i$	1	
1	$-3+i$	$-3i+3$	$3i-1$	$-i$	0	

→ $[x-i]$

1	$-3+i$	$-3i+3$	$3i-1$	$-i$	$-i$
↓	$-i$	$3i$	$-3i$	i	
1	-3	3	-1	0	

→ $[x--i]$

por Teorema Ceros conjugados

1	-3	3	-1	1
↓	1	-2	1	
1	-2	1	0	

→ $[x-1]$

1	-2	1	1
↓	1	-1	
1	-1	0	

→ $[x-1]$

1	-1	1
↓	1	
1	0	

→ $[x-1]$

Así $P(x) = (x-i)(x+i)(x-1)^3$

4. [4 pts] Escriba en la forma $a + ib$ el número complejo $(2+i)^{1-i}$.

Sean $z = 2+i$ y $w = 1-i$

Así $z^w = e^{w\ln(z)}$

$= e^{(1-i)\ln(2+i)}$

⊛ Para $z = 2+i$

$|z| = \sqrt{2^2+1^2}$ $\theta = \arctan\left(\frac{1}{2}\right)$ Así $z = \sqrt{5}\,\mathrm{Cis}\left(\arctan\left(\frac{1}{2}\right)\right)$

$= \sqrt{5}$

Así $\mathrm{Ln}(2+i) = \mathrm{Ln}\left[\sqrt{5}\,\mathrm{Cis}\left(\arctan\left(\frac{1}{2}\right)\right)\right]$

$= \mathrm{Ln}(\sqrt{5}) + \arctan\left(\frac{1}{2}\right)\cdot i$

Ahora

$$e^{(1-i)\cdot\left(\mathrm{Ln}(\sqrt{5}) + \arctan\left(\frac{1}{2}\right)\cdot i\right)} = e^{\mathrm{Ln}(\sqrt{5}) + \arctan\left(\frac{1}{2}\right)\cdot i - \mathrm{Ln}(\sqrt{5})i - \arctan\left(\frac{1}{2}\right)\cdot i\cdot i}$$

$$= e^{\mathrm{Ln}(\sqrt{5}) + \arctan\left(\frac{1}{2}\right)\cdot i - \mathrm{Ln}(\sqrt{5})i + \arctan\left(\frac{1}{2}\right)}$$

$$= e^{\left[\mathrm{Ln}(\sqrt{5}) + \arctan\left(\frac{1}{2}\right)\right] + \left[\arctan\left(\frac{1}{2}\right) - \mathrm{Ln}(\sqrt{5})\right]i}$$

$$= e^{\left[\mathrm{Ln}(\sqrt{5}) + \arctan\left(\frac{1}{2}\right)\right]} \cdot e^{\left[\arctan\left(\frac{1}{2}\right) - \mathrm{Ln}(\sqrt{5})\right]i}$$

$$= e^{\left[\mathrm{Ln}(\sqrt{5}) + \arctan\left(\frac{1}{2}\right)\right]} \cdot \mathrm{Cis}\left(\arctan\left(\frac{1}{2}\right) - \mathrm{Ln}(\sqrt{5})\right)$$

$= 3{,}350259315 - 1{,}189150222\,i$

$= 3{,}350259315 + i\cdot -1{,}189150222$

5.) [3 pts] Sea a una constante real no nula y sean

$$A = \begin{pmatrix} 1 & -1 \\ \frac{a+3}{2} & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & a \\ 2 & -1 & 0 \end{pmatrix} \text{ y } C = \begin{pmatrix} 0 & 3 \\ -2 & 0 \\ 0 & -1 \end{pmatrix}$$

Compruebe que $A + \frac{1}{2}(BC)^T = I_2$.

Note que $BC = \begin{pmatrix} -1 & 0 & a \\ 2 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3 \\ -2 & 0 \\ 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -3-a \\ 2 & 6 \end{pmatrix}$$

Ahora $A + \frac{1}{2}(BC)^T = I_2 \Rightarrow \begin{pmatrix} 1 & -1 \\ \frac{a+3}{2} & -2 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 0 & 2 \\ -3-a & 6 \end{pmatrix} = I_2$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ \frac{a+3}{2} & -2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ \frac{-a-3}{2} & 3 \end{pmatrix} = I_2$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

6. [3 pts] Considere matrices A, B y C de orden $n \times n$ y $A^T - 2B$ invertible, tales que

$$(AX^T)^T = C^T + 2(XB)$$

Si se sabe que C es una matriz simétrica, use las operaciones con matrices y sus propiedades para demostrar que

$$X = C(A^T - 2B)^{-1}$$

Note que $(AX^T)^T = C^T + 2(XB) \Rightarrow (X^T)^T A^T = C^T + 2XB$

$$\Rightarrow XA^T = C^T + 2XB$$

$$\Rightarrow XA^T - 2XB = C^T$$

$$\Rightarrow X(A^T - 2B) = C^T$$

$$\Rightarrow X = C^T(A^T - 2B)^{-1}$$

$$\Rightarrow X = C(A^T - 2B)^{-1}$$ pues C es simétrica

7. [5 pts] Use el método de Gauss-Jordan para determinar el conjunto solución del sistema:

$$\begin{cases} x+y+z=6 \\ 2x-y+3z=14 \\ 2x+2y+2z=12 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 3 & 14 \\ 2 & 2 & 2 & 12 \end{array}\right) \begin{array}{c} -2F_1+\tilde{F_2} \\ \sim \\ -2F_1+\tilde{F_3} \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} -\frac{1}{3}\tilde{F_2} \\ \sim \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} -F_2+\tilde{F_1} \\ \sim \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 4/3 & 20/3 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{Así} \begin{cases} x+\frac{4}{3}z=\frac{20}{3} \\ y-\frac{1}{3}z=\frac{-2}{3} \end{cases} \Rightarrow \begin{cases} x=\frac{20}{3}-\frac{4}{3}z \\ y=\frac{-2}{3}+\frac{1}{3}z \end{cases}, \quad \text{con } z\in\mathbb{R}$$

$$S=\left\{\left(\frac{20}{3}-\frac{4}{3}z,\ \frac{-2}{3}+\frac{1}{3}z,\ z\right),\ \text{con } z\in\mathbb{R}\right\}$$

8. [4 pts] Considere la matriz $A = \begin{pmatrix} -7 & 5 & 4 \\ 2 & -1 & -1 \\ 4 & -3 & -2 \end{pmatrix}$ y determine A^{-1}.

$$\left(\begin{array}{ccc|ccc} -7 & 5 & 4 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ 4 & -3 & -2 & 0 & 0 & 1 \end{array}\right) \underset{\sim}{-\tfrac{1}{7}\tilde{F_1}} \left(\begin{array}{ccc|ccc} 1 & -5/7 & -4/7 & -1/7 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ 4 & -3 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} -2F_1+\tilde{F_2} \\ \sim \\ -4F_1+\tilde{F_3} \end{array} \left(\begin{array}{ccc|ccc} 1 & -5/7 & -4/7 & -1/7 & 0 & 0 \\ 0 & 3/7 & 1/7 & 2/7 & 1 & 0 \\ 0 & -1/7 & 2/7 & 4/7 & 0 & 1 \end{array}\right) \underset{\sim}{\tfrac{7}{3}\tilde{F_2}} \left(\begin{array}{ccc|ccc} 1 & -5/7 & -4/7 & -1/7 & 0 & 0 \\ 0 & 1 & 1/3 & 2/3 & 7/3 & 0 \\ 0 & -1/7 & 2/7 & 4/7 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \tfrac{5}{7}F_2+\tilde{F_1} \\ \sim \\ \tfrac{1}{7}F_2+\tilde{F_3} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 1/3 & 5/3 & 0 \\ 0 & 1 & 1/3 & 2/3 & 7/3 & 0 \\ 0 & 0 & 1/3 & 2/3 & 1/3 & 1 \end{array}\right) \underset{\sim}{3\tilde{F_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 1/3 & 5/3 & 0 \\ 0 & 1 & 1/3 & 2/3 & 7/3 & 0 \\ 0 & 0 & 1 & 2 & 1 & 3 \end{array}\right)$$

$$\begin{array}{c} \tfrac{1}{3}F_3+\tilde{F_1} \\ \sim \\ -\tfrac{1}{3}F_3+\tilde{F_2} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 & 1 & 3 \end{array}\right)$$

Así $\quad A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$