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The goal of this note is to prove the fundamental Riemann-Roch theorem.

1 Čech Cohomology

Recall that given a sheaf $\mathcal{F}$ on X, there is a functor of global sections $\Gamma(X, \mathcal{F})$. Moreover this functor is left-exact, i.e. given

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

we get the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H}).$$

We thus know that there exists a unique derived functor $H^\bullet$ (the category of sheaves have enough projectives (or injectives?)) giving an exact sequence

$$0 \to H^0(F) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(F) \to H^1(\mathcal{G}) \to \dots.$$

One way to construct this derived functor explicitly is through Čech cohomology. Čech cohomology is defined relative to a finite open covering $\{U_0, \dots U_n\}$ of X, but the general theory of derived functors says that it is independent of the covering.

For simplicity, assume that $\mathcal{F}$ is a sheaf of abelian groups (the case of $\mathcal{O}_X$ modules is essentially no different). We start by constructing the complex of abelian groups $A^\bullet(\mathcal{F})$.

Define $U_{i_0 i_1 \dots i_s} = U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_s}$, and set

$$A^r(\mathcal{F}) = \prod_{i_0 < \dots < i_r} \mathcal{F}(U_{i_0 \cdots r})$$

for $r \le n$ (the number of open sets in the covering), and $A^r(\mathcal{F}) = 0$ for $r > n$. The elements of $A^r(\mathcal{F})$ are called **cochains**. Each cochain simply consists of a section over each intersection.

Next, the maps in the complex (the differentials) are defined as

$$\begin{aligned} d^{r+1} : A^r(\mathcal{F}) &\to A^{r+1}(\mathcal{F}), \\ (d^{r+1}(s))_{i_0 \dots i_{r+1}} &= \sum_{k=0}^{r} (-1)^k (s)_{i_0 \dots \hat{i_k} \dots i_r} \mid_{U_{i_0 \dots i_{r+1}}}, \end{aligned}$$

and one can check that $d^{r+1} \circ d^r = 0$ for all r. Thus this is truly a chain complex of groups, and as usual, we define $H^r(\mathcal{F}) = \ker d^{r+1} / \operatorname{im} d^r$ (elements of $\ker d^{r+1}$ are referred to as **cochains** while elements in $\operatorname{im} d^r$ are **coboundaries**).

Proposition 1.1. *Letting $X, \mathcal{F}$ and $H^\bullet$ be as before, we have $H^0(\mathcal{F}) = \Gamma(X, \mathcal{F})$.*

Proof. We unravel the definition of the elements of $H^0(\mathcal{F})$ to make this statement trivial. First, a cochain s in $A^0(\mathcal{F})$ is simply a section over each open set U_i in a chosen open covering of X. For s to be a cocycle, it needs to satisfy

$$\begin{aligned} 0 = (0)_{i_0 i_1} &= (d^1(s))_{i_0 i_1} \\ &= s_{i_0} \mid_{U_{i_0 i_1}} - s_{i_1} \mid_{U_{i_0 i_1}} \end{aligned}$$

i.e. all the sections must agree on the intersections. By the sheaf axiom, the $(s)_i$ can thus be glued together to a global section s. □

One can thus prove that Čech cohomology is indeed the derived functor of $\Gamma(X, \mathfrak{F})$ under a few assumptions (X should be separated, $\mathcal{F}$ quasi-coherent, and the open covering should be an affine open covering).

Given that the Čech cohomology is defined using a finite covering, it is obvious that it vanishes for high enough index.

1.1 Euler-Poincaré Characteristic

We define the Euler-Poincaré characteristic.

Proposition 1.2. *Let X be an algebraic variety over a field K, and $\mathcal{F}$ will be an $\mathcal{O}_X$-module. Then $H^r(\mathcal{F})$ have a natural structure of K-vectorspaces.*

Proof. This is again trivial by unraveling the definitions. Since $K \subset \mathcal{O}_X(U)$ for all U, $\mathcal{F}(U)$ are in particular K-vector spaces, and thus, so are $A^(r)(\mathcal{F})$ for all r. The result then follows by noting that the differentials are K-linear. □

In the case of the proposition above, we denote

$$h^r(\mathcal{F}) := \dim_K(H^r(\mathcal{F})).$$

Definition 1.3. *Let $\mathcal{F}$ be a coherent sheaf over a projective variety X. The Euler-Poincaré characteristic of $\mathcal{F}$ is*

$$\chi(\mathcal{F}) = \sum_{k=0}^{\infty}(-1)^k h^k(\mathcal{F}).$$

We already saw that $h^k(\mathcal{F}) = 0$ for large enough k. Thus, if we can prove that $H^k(\mathcal{F})$ is always finite dimensional, we will have proven that $\chi(\mathcal{F})$ is finite. This will be the goal of the last part of this section.

To prove this, we first study the cohomology of $\mathcal{O}_{\mathbb{P}^n}(d)$.

Theorem 1.4. *Let $n \geq 1$ and $d \geq 0$ both be integers. Let S_d denote the K-vectorspace of homogenous polynomials of degree d in $n+1$ variables (this is a vectorspace of dimension $\binom{n+d}{n}$). The following hold:*

1. $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = S_d$.

2. $H^k(\mathcal{O}_{\mathbb{P}^n}(d)) = 0$ *for all* $0 < k < n$ *and all* d.

3. $H^n(\mathcal{O}_{\mathbb{P}^n}(d)) \cong H^0(\mathcal{O}_{\mathbb{P}^n}(-d-n-1))^*$ *for all d (where V^* denotes the dual vectorspace)*

Proof. See the proof in Perrin (which is quite concrete, so probably a good idea to actually look thoroughly at it). □

Using this, one proves that h^i are indeed finite.

Theorem 1.5. *Let X be a projective algebraic variety over a field K, and $\mathcal{F}$ a coherent sheaf on X. Then $H^k(\mathcal{F})$ are finite dimensional K-vectorspaces for all k.*

Proof. Again only a sketch, but the idea is essentialy to provide an exact sequence

$$0 \to \mathcal{N} \to \mathcal{L} \to \mathcal{F} \to 0,$$

where $\mathcal{L} = \bigoplus \mathcal{O}_{\mathbb{P}^n}(-d_i)$, and doing induction on k (in the reverse direction!). □

1.2 Bezout's Theorem

Before continuing with Riemann-Roch, we pause to prove Bezout's theorem, since we have essentially done all the work.

2 Divisors on curves

From this point onwards, C will be a projective curve defined over K.

The definition of a divisor is simply an element of the free abelian group generated by the points of C.

Definition 2.1. *A (Weil)* ***divisor*** *on C is a formal sum*

$$\sum_{P \in C} n_P P,$$

where $n_P = 0$ for all but finitely many n_P. The ***degree*** *of a divisor D is*

$$\deg D = \sum_{P \in C} n_P.$$

A divisor D is ***effective****, denoted*

$$D \geq 0,$$

if all the associated $n_P \geq 0$. Similarily, we write

$$D \geq E$$

if the divisor $D - E$ is effective.

Given an element of $f \in K(C)$, we can associate the divisor

$$\operatorname{div}(f) = Z(f) - P(f)$$

where $Z(f)$ and $P(f)$ denotes the zeroes and poles of f, with multiplicity. Such a divisor is called a **principal divisor**. A bit more succinctly, we have

$$\operatorname{div}(f) = \sum_{P \in C} \operatorname{ord}_P(f) P,$$

where $\operatorname{ord}_P(f)$ denotes the valuation of f in the field of fractions of $\mathcal{O}_{C,P}$ (I guess this requires that C is smooth to make sense).

Fact 2.2. *Let $f \in H^0(\mathcal{O}_C)$ (i.e. a global section). Then f is constant.*

The fact above shows that for a function $f \in K(C)$, we have

$$f \in H^0(\mathcal{O}_C) \Leftrightarrow \operatorname{div} f = 0$$

(since the constant functions are global sections).

Proposition 2.3. *Let $f \in K(C)$. Then $\deg \operatorname{div}(f) = 0$, i.e. the number of zeroes and poles of f are equal.*

Proof. Perrin, Proposition 2.7. $\square$

As we have seen, the divisors on C are a free abelian group, and it is clear that the principal divisors are a subgroup. Thus, we can make the following definition.

Definition 2.4. *The* ***divisor class group*** *of C is the quotient of the divisors on C modulo the principal divisors, denoted $\operatorname{Pic}(C)$. By the proposition above, the degree map from the divisors on C to the integers factors through $\operatorname{Pic}(C)$, hence we also have a degree map $\operatorname{Pic}(C) \to \mathbb{Z}$. The kernel of this map is denoted $\operatorname{Pic}^0(C)$, also called the* ***Jacobian*** *of C.*

In a later note on abelian varieties we will define the picard group more general. Had we been a bit more general, the following example works.

Example 2.5. *Let $\mathfrak{O}_K$ be the ring of integers in a numberfield, and let $X = \operatorname{Spec} \mathfrak{O}_K$. A Weil divisor on X is something of the form*

$$\sum_{(0) \neq \mathfrak{p} \in \operatorname{Spec} \mathfrak{O}_K} n_P \mathfrak{p}$$

(we are ignoring the ideal (0), since it is not of co-dimension 1 which is a requirement in the more general case of schemes), hence it is obvious how to relate this to fractional ideals of $\mathfrak{O}_K$ (the free group of fractional ideals is usually written multiplicatively for obvious reasons, but oh well). Further, it is clear that the principal divisors are exactly the principal ideals ($K(X) = K$ in this case), thus the divisor class group of X coincides with the usual class group of $\mathfrak{O}_K$.

2.1 Invertible Sheaf Associated to a Divisor

For a projective curve, it is not very hard to associate an invertible sheaf to a given divisor. Given a divisor D on C, and open set $U \subset C$, we define the sheaf $\mathcal{L}(D)$ by

$$\mathcal{L}(D)(U) = \{f \in K(C) \mid v_P(f) + n_P \geq 0, \forall P \in U\},$$

i.e. $\mathcal{L}(D)$ consists of rational functions whose zeroes and poles are controlled by D. In particular, the global sections

$$H^0(\mathcal{L}(D)) = \{f \in K(C) \mid \operatorname{div}(f) + D \geq 0\},$$

are of special importance. For instance, the Riemann-Roch theorem is really saying something about the dimesnion $h^0(\mathfrak{L}(D))$.

Fact 2.6. *If $D \sim D'$, i.e. $D = D' + \operatorname{div}(g)$, then there is $\mathcal{L}(D) \cong \mathcal{L}(D')$ as sheaves given by $f \to f/g$.*

One can show that

$$\mathcal{L}(D + D') = \mathcal{L}(D) \otimes_{\mathcal{O}_C} \mathcal{L}(D'),$$

and thus, it is clear that all of the $\mathcal{L}(D)$ are truly invertible (since $\mathcal{L}(0) = \mathcal{O}_C$). In fact, the divisor class group we defined earlier is really only canonically isomorphic to the picard group $\operatorname{Pic}(C)$ for projective curves C, where the picard group is defined as the set of invertible sheaves up to isomorphism, with the tensor product as the group law.

3 Riemann-Roch

We define the genus of a curve.

Definition 3.1. *Let C be a projective curve. The (arithmetic) **genus** of C is*

$$p_a(C) = 1 - \chi(\mathcal{O}_C) = h^1(\mathcal{O}_C).$$

One nice thing about the Euler-Poincaré characteristic is that it respects exact sequences.

Lemma 3.2. *Let X be a projective variety and let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be an exact sequence of sheaves on X. Then $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$.

Proof. This is surprisingly easy to see. From the exact sequence, we get an exact equence in cohomology

$$0 \to H^0(\mathcal{H}) \to H^0(\mathcal{G}) \to H^0(\mathcal{F}) \to H^1(\mathcal{H}) \to \cdots \to 0.$$

For any long-exact sequence of vectorspaces

$$0 \to E_1 \to \cdots \to E_n \to 0,$$

it is easy to show that

$$\sum (-1)^i \dim E_i = 0,$$

from which the result follows by reordering the terms in $\chi(\mathcal{H}) - \chi(\mathcal{G}) + \chi(\mathcal{F})$. □

The following is really Riemann's theorem, now known as a weaker form of Riemann-Roch, but we'll be satisfied with this for now.

Theorem 3.3. *(Riemann) Let C be a projective curve of genus g. For every divisor D, we have*

$$\chi(\mathcal{L}(D)) = \deg D + \chi(\mathcal{O}_C) = \deg D + 1 - g.$$

Proof. Write $D = E - F$, where E and F are effective divisors. We have an exact sequence of the form

$$0 \to \mathcal{L}(-F) \to \mathcal{O}_C \to \mathcal{O}_F \to 0,$$

which after tensoring with $\mathcal{O}_C(E)$ gives

$$0 \to \mathcal{L}(D) \to \mathcal{L}(E) \to \mathcal{O}_F \to 0$$

(here one uses the flatness of $\mathcal{L}(E)$). Thus

$$\chi(\mathcal{L}(E)) = \chi(\mathcal{L}(D)) + \chi(\mathcal{O}_F) = \chi(\mathcal{L}(D)) + \deg F$$

Applying this equality to $D = 0$ shows that

$$\chi(\mathcal{L}(E)) = \chi\mathcal{O}_C + \deg E,$$

thus we get that

$$\begin{aligned}
\chi(\mathcal{L}(D)) &= \chi(\mathcal{L}(E)) - \deg F \\
&= \chi\mathcal{O}_C + \deg E - \deg F \\
&= \deg D + \chi(\mathcal{O}_C).
\end{aligned}$$

□

3.1 An Application to Elliptic Curves

For this, we need the following result, which is actually a corollary of a stronger version of Riemann-Roch.

Proposition 3.4. *Let C be a projective curve, and let D be a divisor with $\deg D > 2g - 2$. Then $H^1(\mathcal{L}(D)) = 0$, and thus*

$$h^0(\mathcal{L}(D)) = \deg D + 1 - g.$$

Proof. Proven using canonical divisors and duality. □

Now, let E be an elliptic curve (i.e. smooth, projective of genus 1), and let $P \in C$ be a point. From the proposition above, we immediately see that for all $n > 0$, we have that

$$h^0(\mathcal{L}(nP)) = n.$$

Now, pick any $x \in K(C)$, such that $1, x$ is a basis of $H^0(\mathcal{L}(2P))$. Since x can also be regarded as an element of $H^0(\mathcal{L}(3P))$, there exists another $y \in K(C)$ such that $1, x, y$ is a basis of $H^0(\mathcal{L})(3P)$. Note that x has a pole of exact order 2 at P, and similarly, y has a pole of exact order 3 at P (since they form bases).

Consider now the 7 functions $1, x, y, x^2, xy, y^2, x^3$, and observe that these are all in $H^0(\mathcal{L}(6P))$. Since $\dim_K H^0(\mathcal{L}(6P)) = 6$, these functions must be linearly dependent over K, i.e.

$$b_1 y^2 + a_1 xy + a_3 y = b_2 x^3 + a_2 x^2 + a_4 x + a6,$$

for $a_i, b_i \in K$. Further, note that $b_i \neq 0$, since they are the only terms with poles at P of equal order. Thus, up to linear transformations, we can assume that $b_i = 1$. This shows that the rational map

$$\varphi = (x : y : 1)$$

is a morphism from E to the projective curve in $\mathbb{P}^2$ defined by

$$E' : Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 + a_4 X + a6,$$

i.e. for $P \neq Q \in E$, $\varphi(Q) = (x(Q) : y(Q) : 1)$, and $\varphi(P) = (0 : 1 : 0)$.

Finally, to show that this is an isomorphism, one can look at the corresponding map of function fields. To summarize...

Theorem 3.5. *Every projective curve of genus* 1 *is isomorphic to a curve in* $\mathbb{P}^2$ *of the form*

$$E : Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 + a_4 X + a6.$$