

The Rising Sea - Exercise Solutions

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Contents

1	Category theory	1
1.1	Categories and Functors	1
1.2	Universal properties determine an object up to unique isomorphism	3
1.3	Limits and colimits	10
1.4	Adjoint	13
1.5	Abelian categories	16
1.6	*Spectral sequences	24
2	Sheaves	24
2.1	The sheaf of smooth functions	24
2.2	Definition of sheaf and presheaf	25
2.3	Morphisms of presheaves and sheaves	27
2.4	Properties determined at the level of stalks and sheafification	31
2.5	Recovering sheaves from a “sheaf on a base”	37
2.6	Sheaves of abelian groups, and \mathcal{O}_X -modules, form abelian categories	39

1 Category theory

1.1 Categories and Functors

Exercise 1.1.A. A category in which each morphism is an isomorphism is called a **groupoid**.

- a) A perverse definition of a **group** is a groupoid with one object. Make sense of this. Similarly a **monoid** is a category with one object. Make sense of this.
- b) Describe a groupoid that is not a group.

Solution. For a) this, is just realising that the morphisms form the group objects, composition the group operation (and for groups, inverses are given, since the morphisms are isomorphisms)
for b) consider the category with two elements A, B , with $\text{Hom}(A, A)$ and $\text{Hom}(B, B)$ containing only the identity, and $\text{Hom}(A, B)$ consisting of one isomorphism. (Of course this is not a group: there is no way to compose id_A and id_B). \square

Exercise 1.1.B. Show that the invertible elements in $\text{Hom}(A, A)$ form a group. What are the automorphism groups in the categories of sets and vector spaces over k ? Show that two isomorphic objects have isomorphic automorphism groups.

Solution. The first part is essentially the previous exercise. The automorphism groups in the category of sets are bijections $A \rightarrow A$, and in vector spaces, it is invertible linear transformations $V \rightarrow V$.

For the last part, just notice that the isomorphism $\varphi : A \rightarrow B$ induces an isomorphism

$$\begin{aligned} \text{Aut}(B, B) &\rightarrow \text{Aut}(A, A) \\ \psi &\rightarrow \varphi^{-1} \circ \psi \circ \varphi \end{aligned}$$

with the obvious inverse. □

Exercise 1.1.C. Show that the double-dual $\vee\vee$ is naturally isomorphic to the identity functor on the category of finite dimensional vector spaces over k .

Solution. Define $m_V : V \rightarrow V^{\vee\vee} = \text{Hom}(\text{Hom}(V, k), k)$ as $m_V(v)(\varphi) := \varphi(v)$. This is a natural transformation: The square

$$\begin{array}{ccc} V & \xrightarrow{h} & W \\ \downarrow m_V & & \downarrow m_W \\ V^{\vee\vee} & \xrightarrow{h^{\vee\vee}} & W^{\vee\vee} \end{array}$$

commutes, since

$$h^{\vee\vee}(m_V(v))(\psi) := m_V(v)(\psi \circ h) = \psi(h(v)) = m_W(h(v))$$

This part is true, even for infinite dimensional vector-spaces, however to show that m_V is invertible, one needs finite dimensionality: This is the standard proof that $\dim V = \dim V^{\vee}$, where one shows that basis e_1, \dots, e_n of V induces a basis e^1, \dots, e^n of V^{\vee} defined by $e^i(e_j) = \delta_{i,j}$. □

Exercise 1.1.D. Show that the category of vector-spaces k^n under linear maps (matrices) is equivalent to the category of finite-dimensional vector spaces over k .

Solution. This is just defining the inverse functor by picking a basis of every vector space (this also sends linear transformations to matrices by looking at the action on the chosen bases). As the following “asside” points out: The argument above generalizes to show that a fully faithful functor is an equivalence iff. it is essentially surjective (the natural transformations are just given by picking an isomorphic object in the image of F for each object in the target category). □

1.2 Universal properties determine an object up to unique isomorphism

Exercise 1.2.A. Show that two initial objects are uniquely isomorphic. Show that two final objects are uniquely isomorphic.

Solution. Let I_1, I_2 be the initial objects. There unique morphisms $I_1 \rightarrow I_2 \rightarrow I_1$, which must be the unique morphism $\text{id}_1 : I_1 \rightarrow I_1$ (and the other way around). The same argument goes for the final objects (alternatively, the final object is the initial object in the opposite category). \square

Exercise 1.2.B. What are the initial and final objects in the category of Sets, Rings and topological spaces respectively, if they exist?

Solution. Sets: \emptyset , and $\{*\}$.
Rings: \mathbb{Z} , and the trivial ring (if you allow $0 = 1$).
Topological spaces: Same as sets. \square

Exercise 1.2.C. Show that $A \rightarrow S^{-1}A$ is an injection iff. S contains no zero-divisors.

Solution. If s contains a zero-divisor, say $s \in S$ satisfies $as = 0$, then $a/1 = 0/1$, since $s(a - 0) = 0$. In the other direction, $a/1 = 0$ implies that $s(a - 0) = 0$ for some $s \in S$, so $a = 0$. \square

Exercise 1.2.D. Verify that $A \rightarrow S^{-1}A$ satisfies the following universal property: It is the initial object in the category of B -algebras (i.e. ring maps $A \rightarrow B$), where every element of S is invertible in B .

Solution. Let $\varphi : A \rightarrow B$ be an object in the mentioned category. This defines a map $S^{-1}A \rightarrow B$ by $\varphi'(a/s) = \varphi(a)/\varphi(s)$. This is a well defined map of rings: If b/t is another representative, then there exists s' such that $s'(at - bs) = 0$. But then $\varphi(a)\varphi(t) - \varphi(b)\varphi(s) = 0$, implying that $\varphi(b)/\varphi(t) = \varphi(a)/\varphi(s)$. Further, the diagram

$$\begin{array}{ccc} S^{-1}A & \xrightarrow{\varphi'} & B \\ & \nwarrow \varphi \nearrow & \\ & A & \end{array}$$

obviously commutes, so it is a valid map of A -algebras. Clearly, this is also the only map making the above diagram commute (notice that the image of a/s is already defined by φ , since s is invertible in B .) \square

Exercise 1.2.E. Show that $\varphi : M \rightarrow S^{-1}M$ exists, by constructing something satisfying the universal property.

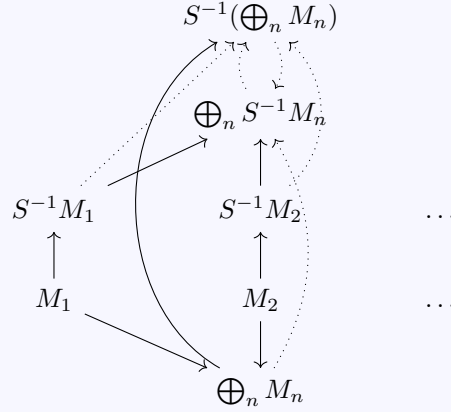
Solution. Let $S^{-1}M$ consist of elements of the form $m/s, m \in M, s \in S$, under the equivalence relation $m_1/s_1 = m_2/s_2$ if there exists $s \in S$ such that $s(m_1s_2 - m_2s_1) \sim 0$. The additive structure is defined by $m_1/s_1 + m_2/s_2 = (m_1s_2 + m_2s_1)/s_1s_2$. This is well defined: $m_1/s_1 + m_2/s_2 - (m_1/s_1 + m'_2/s'_2) = (m_1s_2 + m_2s_1)/s_1s_2 - (m_1s'_2 + m'_2s_1)/s_1s'_2 = (m_1s_2s'_2 + m_2s_1s'_2 - m_1s'_2s_2 - m'_2s_1s_2)/s_1s_2s'_2 = (m_2s'_2 + m'_2s_2)s_1/s_1s_2s'_2 \sim 0$. Similarly, scalar multiplication (by $S^{-1}A$) can be defined by $a_1/s_1 \cdot m_2/s_2 = (a_1m_2)/(s_1s_2)$ (proving well-definedness is again tedious but trivial). This satisfies the universal property (repeat the argument from Exercise 1.2.D). □

Exercise 1.2.F. a) Show that localization commutes with finite products, or equivalently, finite direct sums.

b) Show that localization commutes with arbitrary direct sums

c) show that localization does NOT commute with arbitrary direct products in general.

Solution. We do a) and b) together: Finite direct sums are isomorphic to direct products, so doing b) is in fact enough. Now proving b) can boil down to the following (admittedly quite ugly) diagram:



First, we prove that the map

$$S^{-1}\left(\bigoplus_n M_n\right) \rightarrow \bigoplus_n S^{-1} M_n.$$

exists by universal properties. First, there exists a unique map $\bigoplus_n M_n \rightarrow \bigoplus_n S^{-1} M_n$ by the universal property of the direct sum (colimit). But, since S acts invertibly on $\bigoplus_n S^{-1} M_n$, the map we wanted is uniquely defined by the universal property of localization.

Next, there also exists a uniquely defined map

$$\bigoplus_n S^{-1} M_n \rightarrow S^{-1}\left(\bigoplus_n M_n\right)$$

To see this, notice first the maps $S^{-1} M_n \rightarrow S^{-1}(\bigoplus_n M_n)$, uniquely defined by the universal property of localization (over maps from M_n). These maps give the map we wanted, from the universal property of direct sums again.

Since all maps were uniquely defined, the composition of the two maps still satisfies the universal properties, and thus they must be the identity, and hence the two objects are isomorphic. To show c), follow the hint and simply notice that $(1/p, 1/p^2, 1/p^3, \dots) \in \prod_n \mathbb{Q} = \prod_n \mathbb{Z}_{(0)}$ is not in the image of the uniquely defined map

$$(\mathbb{Z} \setminus \{0\})^{-1} \prod_n \mathbb{Z} \longrightarrow \prod_n \mathbb{Z}_{(0)}.$$

□

Exercise 1.2.G. Show that $\mathbb{Z}/(10) \otimes \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$.

Solution. The usual proof that $\mathbb{Z}/(n) \otimes \mathbb{Z}/(m) \cong \mathbb{Z}/(\gcd(m, n))$ can be shown by: The tensor product is cyclic, generated by $1 \otimes 1$ (use $a \otimes b = ab(1 \otimes 1)$). But then both $n(1 \otimes 1) = m(1 \otimes 1) = 0$, and the result follows. \square

Exercise 1.2.H. Show that $- \otimes_A N$ gives a covariant functor $\text{Mod}_A \rightarrow \text{Mod}_A$. Show that it is right-exact.

Solution. For $f : M_1 \rightarrow M_2$, we can define $f \otimes_A N : M_1 \otimes N \rightarrow M_2 \otimes N$ by $f(\sum_i m_i \otimes n_i) = \sum f(m_i) \otimes n_i$.
If f is surjective, then $f \otimes N$ is still clearly surjective. Thus, given

$$M' \xrightarrow{g} M \xrightarrow{f} M'' \rightarrow 0$$

we need to show that $\text{im } g \otimes N = \ker f \otimes N$. We certainly have $\text{im } g \otimes N \subseteq \ker f \otimes N$, since they compose to zero, and the result now follows by surjecting onto the kernel of f . \square

Exercise 1.2.I. Show that $(M \otimes N, \iota : M \times N \rightarrow M \otimes N)$ satisfying the universal property of the tensor product is unique up to unique isomorphism.

Solution. Given two such pairs, use the universal properties to construct maps between them. Composing them must yield the identity by the universal property. \square

Exercise 1.2.J. Show that the construction of the tensor product of modules over A satisfies the universal property.

Solution. First, the fact that the map $\iota((m, n)) = m \otimes n$ is A -bilinear is because we quotient out by $(m_1 + m_2, n) = (m_1, n) + (m_2, n)$ (and same in n), and $a(m, n) = (am, n) = (m, an)$.
Next, given an A -bilinear map $f : M \times N \rightarrow T$, we can construct the A -module homomorphism $M \otimes N \rightarrow T$ given by $\sum m_i \otimes n_i \mapsto \sum f(m_i, n_i)$, which is indeed the unique map making the diagram commute. \square

Exercise 1.2.K. a) If M is an A module, and $A \rightarrow B$ is a ring-homomorphism, give $B \otimes_A M$ the structure of a B -module. Show that this defines a functor $\text{Mod}_A \rightarrow \text{Mod}_B$.

b) If further $A \rightarrow C$ is another ring-homomorphism, show that $B \otimes_A C$ has a natural ring-structure.

Solution. For a), $B \otimes_A M$ is already an A -module, since B can be seen as a B -module. The B -module structure is defined by $b(b' \otimes m) = bb' \otimes m$. This is functorial in B ; given $f : M \rightarrow N$, define $B \otimes_A f : B \otimes M \rightarrow B \otimes N$ by sending $b \otimes m \rightarrow b \otimes f(m)$. Clearly, this is also a homomorphism of B -modules.

For b) we can define $(b_1 \otimes c_1)(b_2 \otimes c_2) = b_1 b_2 \otimes c_1 c_2$, and extend linearly. \square

Exercise 1.2.L. If S is a multiplicative subset of A , and M is an A -module, describe a natural isomorphism $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$.

Solution. Notice that the universal property satisfied by $S^{-1}A \otimes_A M$ is the same as that of localizations: An A -bilinear map $S^{-1}A \times M \rightarrow T$ is the same as a map of A -modules $M \rightarrow T$, such that every element of S acts invertibly on T . This natural isomorphism $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$ is given by factoring the map $a/s \otimes m \rightarrow (am)/s$. \square

Exercise 1.2.M. Show that the tensor product commutes with arbitrary direct sums.

Solution. This can be done using almost exactly the same argument as in Exercise 1.2.F. Explicitly, the map is given by

$$\begin{aligned} M \otimes \left(\bigoplus_i N_i \right) &\rightarrow \bigoplus_i M \otimes N_i \\ \sum_i m_i \otimes \left(\bigoplus_j n_j \right) &\rightarrow \bigoplus_j \sum_i (m_i \otimes n_j). \end{aligned}$$

(Recall that, by definition of the direct sum, only finitely many n_j are non-zero). The map is surjective, because every generator of the right hand side is clearly in the image. Injectivity is much trickier, and probably best proven using the canonical maps that occurs, similarly to Exercise 1.2.F. \square

Exercise 1.2.N. Show that in the category of sets,

$$X \times_Z Y = \{(x, y) \in X \times Y \mid \alpha(x) = \beta(y)\},$$

where $\alpha : X \rightarrow Z, \beta : Y \rightarrow Z$.

Solution. We need to show that it satisfies the universal property: Let $\varphi_X : T \rightarrow X, \varphi_Y : T \rightarrow Y$ be set maps such that $\alpha \circ \varphi_X = \beta \circ \varphi_Y$. Define the map $T \rightarrow X \times_Z Y$ (defined as in the exercise) by $t \rightarrow (\varphi_X(t), \varphi_Y(t))$. This is well defined, and makes everything commute. It is also the only such map, since $T \rightarrow X \times_Z Y \rightarrow X$ is given by φ_X , and similarly with Y . \square

Exercise 1.2.O. If X is a topological space show that the fibered products always exists in the category of open sets of X .

Solution. It is given by the intersection. \square

Exercise 1.2.P. If Z is the final object in a category \mathcal{C} , show that $X \times_Z Y = X \times Y$.

Solution. The universal property is the same: The additional requirement that the maps to Z should commute becomes mute, because of the universal property of the final object. \square

Exercise 1.2.Q. If the two smaller squares in the diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

commutes, show that so does the larger square.

Solution.

$$\begin{aligned} U \rightarrow W \rightarrow Y \rightarrow Z &= U \rightarrow W \rightarrow X \rightarrow Z \\ &= U \rightarrow V \rightarrow X \rightarrow Z \end{aligned}$$

\square

Exercise 1.2.R. Given morphisms $X_1 \rightarrow Y, X_2 \rightarrow Y$, and $Y \rightarrow Z$, show that there is a natural morphism $X_1 \times_Y X_2 \rightarrow X_1 \times_Z X_2$, assuming both fibered products exists.

Solution. The map exists by the universal property of $X_1 \times_Z X_2$ (the two maps $X_1 \times_Y X_2 \rightarrow Z$ both factor through Y , and commute there by definition). \square

Exercise 1.2.S. Suppose we are given morphisms $X_1, X_2 \rightarrow Y$, and $Y \rightarrow Z$. Show that the following diagram commutes:

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_Z Y \end{array}$$

Solution. See Exercise 1.3.B. \square

Exercise 1.2.T. Show that the coproduct in the category of sets is the disjoint union.

Solution. Given any set T with maps $f_i : X_i \rightarrow T$, the map satisfying the universal property is

$$\coprod_i X_i \rightarrow T$$

$$(a, i) \rightarrow f_i(a)$$

□

Exercise 1.2.U. Suppose $A \rightarrow B$ and $A \rightarrow C$ are ring morphisms. Recall that $B \otimes_A C$ has a ring-structure. Show that there is a natural morphism $B \rightarrow B \otimes_A C$ (and similarly for C), making $B \otimes_A C$ the fibered coproduct on rings.

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \otimes_A C \end{array}$$

Solution. The morphisms are given by $b \rightarrow b \otimes 1_C$ and $c \rightarrow 1_B \otimes c$. Assume now that R is a ring, with morphisms from B and C such that everything commutes; we must show the existence and uniqueness of h in the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \otimes_A C \end{array} \quad \begin{array}{c} \searrow g \\ \downarrow h \\ \nearrow f \end{array} \quad \begin{array}{c} \\ \\ R \end{array}$$

The uniqueness and existence of the h as a map of A -modules is the universal property of the tensor product: There are bilinear A -module maps $B \times C \rightarrow B \otimes_A C$ and $B \times C \rightarrow R$, where the last map is defined by $(b, c) \rightarrow f(b)g(c)$ (bilinearity is immediate, since the maps to begin with are ring-morphisms, and multiplication in R is distributive, while the A -module structure comes from commuting relation, making the maps into A -algebra maps), hence h is uniquely defined as a map of A -modules. We need to show that h also preserves the ring-structure: to see this, notice that $h(b \otimes c) = f(b)g(c)$, thus we have $h(1_B \otimes 1_C) = f(1_B)g(1_C) = 1_R 1_R = 1_R$, and similarly $h(b_1 b_2 \otimes c_1 c_2) = f(b_1 b_2)g(c_1 c_2) = f(b_1)f(b_2)g(c_1)g(c_2) = f(b_1)g(c_1)f(b_2)g(c_2) = h(b_1 \otimes c_1)h(b_2 \otimes c_2)$. □

Exercise 1.2.V. Show that the composition of two monomorphisms is a monomorphism.

Solution. Let $\iota_1 : A \rightarrow B$ and $\iota_2 : B \rightarrow C$ be monomorphisms. Then $(\iota_2 \iota_1)f_1 = (\iota_2 \iota_1)f_2$ implies that $\iota_1 f_1 = \iota_1 f_2$ (ι_2 is mono), and hence, $f_1 = f_2$. □

Exercise 1.2.W. Prove that a morphism $\pi : X \rightarrow Y$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced diagonal $\delta_X : X \rightarrow X \times_Y X$ is an isomorphism. We may then take this as the definition of monomorphisms.

Solution. If π is mono, X (together with the identity maps) satisfies the universal property of the fibered product over Y (this is just a translation of the property of being mono). Thus X is canonically isomorphic to $X \times_Y X$, but the diagonal map is uniquely defined, and must thus give the isomorphism. Conversely, assume that the diagonal map induces an isomorphism. Then X also satisfies the universal property of the fibered product, which translates to the property of being mono, since either composition $X \xrightarrow{\delta_X} X \times_Y X \rightarrow X$ is the identity. \square

Exercise 1.2.X. Show that if $Y \rightarrow Z$ is mono, then the morphism $X_1 \times_Y X_2 \rightarrow X_1 \times_Z X_2$ (see Exercise 1.2.S) is an isomorphism.

Solution. This is easy: $X_1 \times_Z X_2$ satisfies the universal property of the fibered product over Y : since $Y \rightarrow Z$ is mono, any two maps being equal at Z were already equal at Y . Alternatively, we can use that the square in Exercise 1.2.S is a fibered product (see Exercise 1.3.B), and that $Y \rightarrow Y \times_Z Y$ is an isomorphism. \square

1.3 Limits and colimits

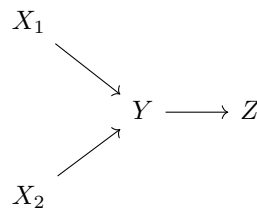
Exercise 1.3.A. Suppose that the partially ordered set \mathcal{I} has an initial object e . Show that the limit of any diagram indexed by \mathcal{I} exists.

Solution. Let $F : \mathcal{I} \rightarrow \mathcal{C}$ be the diagram. Then it is easy to see that

$$\varprojlim_{i \in \mathcal{I}} F(i) = F(e).$$

\square

Exercise 1.3.B. Solve Exercise 1.2.S by identifying both $X_1 \times_Y X_2$ and $Y \times_{Y \times_Z Y} X_1 \times_Z X_2$ as the limit of the diagram



Solution. First, showing what the exercise asks solves Exercise 1.2.S: the diagram commutes, since $X_1 \times_Y X_2$ would then be the fibered product, and hence commutes by definition. Now, $X_1 \times_Y X_2$ is clearly the limit of the diagram in question (the extra Z does not add anything, since maps to Z will already be defined by maps to Y). To see that $W := Y \times_{Y \times_Z Y} X_1 \times_Z X_2$ is also the limit of this diagram, consider

$$\begin{array}{ccccc}
 & & X_1 & & \\
 & \nearrow & & \searrow & \\
 X_1 \times_Z X_2 & & Y \times_Z Y & \xleftarrow{\quad} & Y \longrightarrow Z \\
 & \searrow & & \nearrow & \\
 & & X_2 & &
 \end{array}$$

We can now unravel the definition of the universal property of W : Suppose we are given an object T with maps $\alpha : T \rightarrow Y$, and $\beta : T \rightarrow X_1 \times_Z X_2$ such that the maps to $Y \times_Z Y$ commutes. Giving β is the same as giving maps to X_1 and X_2 that become equal at Z . Now the fact that this should commute with α at $Y \times_Z Y$ says that the map should be equal already at Y (this uses the fact that the diagonal map $Y \rightarrow Y \times_Z Y$ is a monomorphism; this is clearly true, it is even a split monomorphism by definition). \square

Exercise 1.3.C. Show that in the category of sets,

$$\varprojlim_{\mathcal{I}} A_i = \{(a_i)_i \in \prod_{i \in \mathcal{I}} A_i \mid F(m)(a_j) = a_k, \forall m \in \text{Hom}_{\mathcal{I}}(i, j)\}$$

along with the obvious projection maps.

Solution. Assuming we are given an object T , and maps $f_i : T \rightarrow A_i$ such that everything commutes, we construct the map $T \rightarrow \varprojlim_{\mathcal{I}} A_i$ by $t \rightarrow (f_i(t))$. This really lands in $\varprojlim_{\mathcal{I}} A_i$, since everything commutes, and further it is clearly the only such maps, as can be seen by composing with the projections. \square

Exercise 1.3.D. a) Interpret the statement $\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$

b) Interpret the union of some subsets of a given sets as a colimit (dually, intersections can be interpreted as a limit).

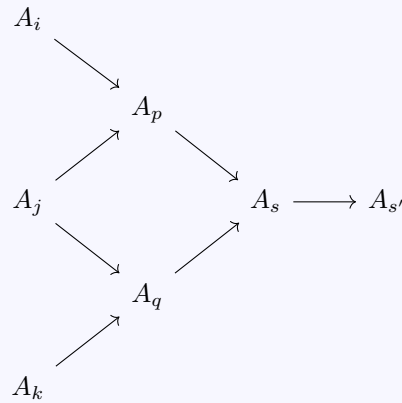
Solution. For a), the positive natural numbers form a poset under divisibility. Take this as the index set and map to sets (or ab. groups) by $F(n) \rightarrow \frac{1}{n}\mathbb{Z}$. If $n \mid m$, we get a corresponding map (inclusion) by $\frac{1}{n}\mathbb{Z} \rightarrow \frac{1}{m}\mathbb{Z}$. The colimit is clearly the rational numbers. For b), take the category to be the category of subsets, and morphisms inclusion. The coproduct (which is indeed a colimit) is here the union. \square

Exercise 1.3.E. Supposed \mathcal{I} is filtered. Show that any diagram in the category of sets indexed by \mathcal{I} , has the colimit

$$\varinjlim_{\mathcal{I}} A_i = \{(a_i, i) \in \prod_{i \in \mathcal{I}} A_i\} / \sim$$

with the obvious maps to it, where \sim is the equivalence relation defined by $(a_i, i) \sim (a_j, j)$ if and only if there are $F(m) : A_i \rightarrow A_k$ and $F(m') : A_j \rightarrow A_k$ such that $F(m)(a_i) = F(m')(a_j)$.

Solution. First, we show that it is an equivalence relation: The symmetric and reflexive properties are obvious. For transitivity, assume that there are $F(m) : A_i \rightarrow A_p, F(m') : A_j \rightarrow A_p$, and $F(n) : A_j \rightarrow A_q, F(n') : A_k \rightarrow A_q$, so that $F(m)(a_i) = F(m')(a_j), F(n)(a_j) = F(n')(a_k)$. Since \mathcal{I} is filtered, there now exists s, s' and arrows $F(t) : A_s \rightarrow A_p, F(t') : A_q \rightarrow A_s, F(t'') : A_s \rightarrow A_{s'}$, so that the two paths $A_j \rightarrow A_{s'}$ in



are equal, showing the reflexive property.

Now that we know that this object is well defined, it is easy to show that it is also the colimit. Given another object T and maps $f_i : A_i \rightarrow T$ making everything commute, we can set the map $\varinjlim_{\mathcal{I}} A_i \rightarrow T$ by $(a_i, i) \rightarrow f_i(a_i)$. Clearly, this is also the only map making everything commute. \square

Exercise 1.3.F. Verify that that the colimit in the category of A -modules is the set as defined in Exercise 1.3.E, and where addition is defined by taking two arrows $F(u) : M_i \rightarrow M_k, F(v) : M_j \rightarrow M_k$, and setting $(m_i, i) + (m_j, j) = (F(u)(m_i) + F(v)(m_j), k)$, and multiplication by A is defined in the obvious way.

Solution. Replacing (m_i, i) by a different representative, and or $F(u)$ with a different arrow can be shown to give the same result, because they eventually become equal (so use again the filtered requirements to find an index and maps in which everything becomes equal). \square

Exercise 1.3.G. Generalize Exercise 1.3.D a) to interpret localization of an integral domain as a colimit over a filtered set.

Asside: Can you make it work for general commutative rings (i.e. not necessarily integral domains?).

Solution. We do the case without requiring the integral domain hypothesis right away: We form a filtered set of S in the obvious way, with arrows corresponding to $s \mid s'$, i.e. there exists some $a \in A$ such that $s' = cs$ (this is filtered, because s, t both have arrows to st). Thus the limit $\varinjlim_S A_s$ (recall A_s contain elements of the form a/s^n) exists, with maps $A_s \rightarrow A_{s'}$ defined by $a/s^n \rightarrow c^n a/s'^n$. Now there are canonical isomorphisms $\varinjlim_S A_s \cong S^{-1}A$: The map $S^{-1}A \rightarrow \varinjlim_S A_s$ exists because there is an obvious map $A \rightarrow \varinjlim_S A_s$, and every element of s is invertible in $\varinjlim_S A_s$, this we can apply the universal property of $S^{-1}A$. $\varinjlim_S A_s \rightarrow S^{-1}A$ exists because there are obvious maps $A_s \rightarrow S^{-1}A$ making everything commute. Now by the uniqueness property, these maps are mutually inverse, and hence these objects are canonically isomorphic. When A is an integral domain, we may replace A_s by (the A -module) $\frac{1}{s}A \subset K$, where K is the field of fractions of A , and the same argument works. \square

Exercise 1.3.H. Suppose you are given a diagram of A -modules indexed by $F : \mathcal{I} \rightarrow \text{Mod}A$, where we let $M_i := F(i)$. Show that the colimit is $\bigoplus_{i \in \mathcal{I}} M_i$ modulo the relations $m_i - F(n)(m_i)$ for every $n : i \rightarrow j$ in \mathcal{I} .

Solution. Assume as always, we are given an A -module T and homomorphisms $f_i : M_i \rightarrow T$ such that everything commutes. Start by forming the direct sum $\bigoplus_{i \in \mathcal{I}} M_i$. There of course exists a unique map $\bigoplus_{i \in \mathcal{I}} M_i \rightarrow T$ which commutes with the f_i (universal property of the coproduct), thus we are done if we show that the map factors through the submodule generated by the relations: The fact that the f_i also commute with $F(n)$ for every n shows that every element of the form $m_i - F(n)(m_i)$ is in the kernel of the map, thus the map $\varinjlim_{\mathcal{I}} M_i \rightarrow T$ is well-defined. \square

1.4 Adjoints

Exercise 1.4.A. Write down what this diagram (meaning: the “naturality” in the definition of an adjoint pair for a morphism $g : B \rightarrow B'$) should be.

Solution.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{B}}(F(A), B) & \xrightarrow{g_*} & \text{Hom}_{\mathcal{B}}(F(A), B') \\ \downarrow \tau_{A,B} & & \downarrow \tau_{A,B'} \\ \text{Hom}_{\mathcal{A}}(A, G(B)) & \xrightarrow{G(g_*)} & \text{Hom}_{\mathcal{A}}(A, G(B')) \end{array}$$

\square

Exercise 1.4.B. Show that the maps $\tau_{A,B}$ has the following properties. For each A , there is a map $\eta_A : A \rightarrow G(F(A))$ so that for any $g : F(A) \rightarrow B$, the corresponding $\tau_{A,B}(g) : A \rightarrow G(B)$ is given

by the composition

$$A \xrightarrow{\eta_A} G(F(A)) \xrightarrow{G(g)} G(B).$$

Similarly, there is a map $\epsilon_B : F(G(B)) \rightarrow B$ such that for any $f : A \rightarrow G(B)$, the corresponding $\tau_{A,B}^{-1}(f) : F(A) \rightarrow B$ is given by

$$F(A) \xrightarrow{F(f)} F(G(B)) \xrightarrow{\epsilon_B} B$$

Solution. Start with $\text{id}_{F(A)} : F(A) \rightarrow F(A)$. Then set $\eta_A := \tau_{A,F(A)}(\text{id}_{F(A)}) \in \text{Hom}_A(A, GF(A))$. Now the definition of $\tau_{A,B}(g)$ follows directly from naturality applied to the diagram

$$\begin{array}{ccc} \text{Hom}_B(F(A), F(A)) & \xrightarrow{g_*} & \text{Hom}_B(F(A), B) \\ \downarrow \tau_{A,A} & & \downarrow \tau_{A,B'} \\ \text{Hom}_A(A, GF(A)) & \xrightarrow{G(g)_*} & \text{Hom}_A(A, G(B)) \end{array}$$

and similarly for the counit ϵ_A for functions $f : A \rightarrow G(B)$. □

Exercise 1.4.C. Suppose M, N, P are A -modules. Describe a bijection $\text{Hom}_A(M \otimes_A N, P) \leftrightarrow \text{Hom}_A(M, \text{Hom}_A(N, P))$.

Solution. Let $f : M \rightarrow \text{Hom}_A(N, P)$. Say $f(m) = g$. We may consider f as a function $f' : M \times N \rightarrow P$ by $f'(m, n) = f(m)(n)$. Since this is bilinear (both f and $f(m)$ are homomorphisms), there exists a unique map $h : M \otimes N \rightarrow P$ corresponding to f . Correspondingly, given a map $g : M \otimes N \rightarrow P$, define the map $g' : M \rightarrow \text{Hom}_A(N, P)$ by $g'(m) = g(m \otimes -)$. It is clear that these maps are mutually inverse bijections. □

Exercise 1.4.D. Show that $- \otimes_A N$ and $\text{Hom}_A(N, -)$ are adjoint functors.

Solution. By the previous exercise, we only need to show naturality. That is, given $g : P \rightarrow P'$, we need to show that the square

$$\begin{array}{ccc} \text{Hom}_A(M \otimes N, P) & \xrightarrow{g_*} & \text{Hom}_A(M \otimes N, P') \\ \downarrow \tau_{M,P} & & \downarrow \tau_{M,P'} \\ \text{Hom}_A(M, \text{Hom}_A(N, P)) & \xrightarrow{g_{**}} & \text{Hom}_A(M, \text{Hom}_A(N, P')) \end{array}$$

commutes. But this is obvious by using the definition of $\tau_{M,P}$ from the previous exercise. □

Exercise 1.4.E. Suppose $\varphi : B \rightarrow A$ is a morphism of rings. If M is an A -module, you can create a B -module M_B by considering it as a B module (scalar multiplication given by $b \cdot m = \varphi(b)m$). This gives a functor $\cdot_B : \text{Mod}_A \rightarrow \text{Mod}_B$. Show that this functor is right-adjoint to $- \otimes_B A$.

Solution. Assume we are given a homomorphism of A -modules $f : N \otimes_B A \rightarrow M$, for some B -module N . This gives a homomorphism of B -modules $\tau_{N,M}(f) : N \rightarrow M_B$ by $\tau_{N,M}(f)(n) = f(n \otimes_B 1_A)$.

Conversely, given $g : N \rightarrow M_B$, we can define $\tau_{N,M}^{-1}(g) : N \otimes_B A \rightarrow M$ by $\tau_{N,M}^{-1}(g)(n \otimes_B a) = ag(n)$. These maps are mutual inverses, and similarly to before, naturality is seen simply by inspecting the maps. \square

Exercise 1.4.F. Show that if an abelian semigroup is already a group, then the identity morphism is the groupification.

Solution. The identity morphism clearly satisfies the universal property: For every map $f : S \rightarrow G$, the unique map making the diagram commute is simply f . \square

Exercise 1.4.G. Construct the “groupification functor” H from the category of nonempty abelian semigroups to the category of abelian groups. Show that this is left adjoint to the forgetfull functor F from the category of abelian groups to the category of abelian semigroups.

Solution. We follow the construction: let S be an abelian semigroup, and form $H(S)$, where the elements are pairs $(a, b) \in S \times S$, under the equivalence relation $(a, b) \sim (c, d)$ if there exists $e \in S$ such that $a + d + e = b + c + e$. This becomes a group under the obvious addition $(a, b) + (c, d) = (a + c, b + d)$: The identity are all elements of the form (a, a) , since $(a + b, a + c) \sim (b, c)$ since $a + b + b = a + c + c$. Addition is obviously associative, and inverses are given by $(a, b)^{-1} = (b, a)$. The canonical map $S \rightarrow H(S)$ is given by $b \rightarrow (a + b, a)$ for any $a \in S$.

To show the universal property, let $f : S \rightarrow G$ be a morphism, where G is a group. We form $f' : H(S) \rightarrow G$ by $f'((a, b)) = f(a) - f(b)$. This makes the triangle commute, and is the only map to do so (since it is completely defined by f).

Finally, we show the adjointness: Given $h \in \text{Hom}(S, G)$, $\tau_{S,G}^{-1}(h) \in \text{Hom}(H(S), G)$ is simply given by h' from the universal property (this is actually the same as what $H(h)$ would be in the case where G is already a group). Other way, given $g : H(S) \rightarrow G$, $\tau_{S,G}(g)(b) = g((b + a, a))$. Now checking that the square commutes for some $\varphi : G \rightarrow G'$ (notice, there is nothing to check), and the same applies to $\psi : S \rightarrow S'$. \square

Exercise 1.4.H. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$ -modules are a full subcategory of A -modules. Then, the localization functor $\text{Mod } A \rightarrow \text{Mod } S^{-1}A$ can be interpreted as a left adjoint to the forgetfull functor.

Solution. This exercise is almost verbatim the same arguments as the previous exercise; notice the similarity between the universal properties of localization/groupification. \square

1.5 Abelian categories

Exercise 1.5.A. Describe the exact sequences

$$0 \longrightarrow \operatorname{im} f^i \longrightarrow A^{i+1} \longrightarrow \operatorname{coker} f^i \longrightarrow 0$$

$$0 \longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{coker} f^{i-1} \longrightarrow \operatorname{im} f^i \longrightarrow 0$$

Solution. The first sequence is literally the definition of $\operatorname{im} f^i$, as the kernel of $\operatorname{coker} f^i$:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \searrow & & & & \\
 & & \operatorname{im} f^i & & & & \\
 & & \searrow & & & & \\
 & & A^i & \xrightarrow{f^i} & A^{i+1} & & \\
 & & \searrow & & \searrow & & \\
 & & & & 0 & & \operatorname{coker} f^i \\
 & & & & & & \searrow \\
 & & & & & & 0
 \end{array}$$

For the second map, applying the snake lemma to the top two rows shows that the bottom row is exact

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \operatorname{im} f^{i-1} & \longrightarrow & \operatorname{im} f^{i-1} & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \ker f^i & \longrightarrow & A^i & \longrightarrow & \operatorname{im} f^i \longrightarrow 0 \\
 & & \vdots & & \vdots & & \vdots \\
 0 & \longrightarrow & H^i(A^\bullet) & \longrightarrow & \operatorname{coker} f^{i-1} & \longrightarrow & \operatorname{im} f^i \longrightarrow 0
 \end{array}$$

□

Exercise 1.5.B. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of k -vectorspaces. Define $h^i(A^\bullet) := \dim H^i(A^\bullet)$. Show that $\sum (-1)^i \dim A^i = \sum (-1)^i h^i(A^\bullet)$. In particular, if A^\bullet is exact, then $\sum (-1)^i \dim A^i = 0$.

Solution. For each i , we have $\dim A^i = \dim \ker d^i + \dim \operatorname{im} d^i$. Similarly, we have $h^i(A^\bullet) = \dim \ker d^i - \dim \operatorname{im} d^{i-1}$. Thus

$$\begin{aligned} \sum_{i=0}^n (-1)^i \dim A^i &= \sum_{i=0}^n (-1)^i (\dim \ker d^i + \dim \operatorname{im} d^i) \\ &= \dim \ker d^0 + \dim \operatorname{im} d^n + \sum_{i=1}^n (-1)^i (\dim \ker d^i - \dim \operatorname{im} d^{i-1}) \\ &= 0 + 0 + \sum_{i=0}^n h^i(A^\bullet). \end{aligned}$$

In particular, if the sequence is exact, then all the $h^i(A^\bullet) = 0$, and the result follows. \square

Exercise 1.5.C. Suppose \mathcal{C} is an abelian category. Define the category $\operatorname{Com}_{\mathcal{C}}$ of complexes over \mathcal{C} , where morphisms $x^\bullet : A^\bullet \rightarrow B^\bullet$ are morphisms (x^i) making the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} \longrightarrow \dots \\ & & \downarrow x^{i-1} & & \downarrow x^i & & \downarrow x^{i+1} \\ \dots & \longrightarrow & B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1} \longrightarrow \dots \end{array}$$

commute. Show that this is an abelian category.

Solution. First, we show its additive:

Ad1 : $\text{Hom}(A, B)$ are abelian groups: This follows from just defining $(x^i) + (y^i) = (x^i + y^i)$ (everything still commutes, and distributes over other morphisms).

Ad2 : $\text{Com}_{\mathcal{C}}$ has a zero object: This is just the complex of all zeroes (where the zeroes are the zero in \mathcal{C}); this is initial and final, because the zeroes are initial and final in \mathcal{C} .

Ad3 : It has products of two objects: Forming $A^\bullet \times B^\bullet$ by taking products componentwise. Again, this satisfies the universal property of products, because it does so componentwise.

And further, it is abelian:

Ab1 : Every map has a kernel and cokernel: This comes down to showing that the diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \ker x^{i-1} & \longrightarrow & \ker x^i & \longrightarrow & \ker x^{i+1} \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} \longrightarrow \dots \\
 & & \downarrow x^{i-1} & & \downarrow x^i & & \downarrow x^{i+1} \\
 \dots & \longrightarrow & B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1} \longrightarrow \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & \text{coker } x^{i-1} & \longrightarrow & \text{coker } x^i & \longrightarrow & \text{coker } x^{i+1} \longrightarrow \dots
 \end{array}$$

commutes. Each square comes from the commutative diagram

$$\begin{array}{ccc}
 \ker x & \xrightarrow{h} & \ker x' \\
 \downarrow & & \downarrow \\
 A & \longrightarrow & A' \\
 \downarrow x & & \downarrow x' \\
 B & \longrightarrow & B'
 \end{array}$$

where h exists by the universal property of $\ker x'$ (and $\ker x \rightarrow B'$ is zero since it factors through B). The argument for the cokernel is analogous. Finally, one shows that the complexes are in fact complexes, by the commuting squares, and the fact that \ker / coker is mono / epi.

Ab2 : Every monomorphism is the kernel of its cokernel: This is true, since it is true in \mathcal{C} to begin with (so we get isomorphisms component-wise).

Ab3 : Every epimorphism is the cokernel of its kernel. Same argument

□

Exercise 1.5.D. Show that a morphism $\varphi : A^\bullet \rightarrow B^\bullet$ induces a map in homology $H^i(A^\bullet) \rightarrow$

$H^i(B^\bullet)$. Furthermore show that H^i is a covariant functor $\text{Com}_{\mathcal{C}} \rightarrow \mathcal{C}$.

Solution. This comes down to diagram chasing. We can do it for the special case of \mathcal{C} a module category. Take any element $x \in \ker d_A^i \subset A^i$. By the commutativity of the squares, $\varphi(x) \in \ker d_B^i$. Now let x' be another element so that $x - x' \in \text{im } d_A^{i-1}$. Set y so that $d_A^{i-1}(y) = x - x'$. Then $d_B^{i-1}(\varphi(y)) = \varphi(x - x') = \varphi(x) - \varphi(x')$, so φ indeed induces a well-defined map in homology.

This is functorial: Clearly, identity morphisms are sent to identity morphisms, and the fact that $H^i(\varphi \circ \psi) = H^i(\varphi) \circ H^i(\psi)$ is simply by inspection. \square

Exercise 1.5.E. Show two homotopic maps give the same map on homology.

Solution. The definition of homotopy says that the maps f and g (when applied to elements in $\ker d$) differ by an element in the image. \square

Exercise 1.5.F. Suppose F is an exact functor. Show that applying F to an exact sequence preserves exactness. For example, if F is covariant, and $A' \rightarrow A \rightarrow A''$ is exact, then $F(A') \rightarrow F(A) \rightarrow F(A'')$ is also exact.

Solution. This is actually a bit trickier than it sounds. In general, we want to show that if

$$\dots \rightarrow A^{i-1} \rightarrow A^i \rightarrow A^{i+1} \rightarrow \dots$$

is exact, then so is

$$\dots \rightarrow F(A^{i-1}) \rightarrow F(A^i) \rightarrow F(A^{i+1}) \rightarrow \dots$$

this comes down to showing that first, left-exactness preserves kernels: Since left exactness means that

$$0 \rightarrow F(\ker f) \rightarrow F(A) \xrightarrow{F(f)} F(B)$$

is exact, and since every monomorphism is the kernel of its cokernel in an abelian category, we get $F(\ker f) = \ker F(f)$. Similarly, right-exactness preserves cokernels.

Thus, we can finally say that F preserves images, since $\text{im } f = \ker(\text{coker } f)$ by definition. Thus, $\text{im } F(d^i) = F(\text{im } d^i) = F(\ker d^{i+1}) = \ker F(d^{i+1})$, and exactness is preserved. \square

Exercise 1.5.G. Suppose A is a ring, $S \subset A$ is a multiplicative subset, and M is an A -module.

- Show that localization of A -modules is an exact covariant functor.
- Show that $- \otimes_A M$ is a right-exact covariant functor $\text{Mod}_A \rightarrow \text{Mod}_A$.
- Show that $\text{Hom}(M, -)$ is a left-exact covariant functor $\text{Mod}_A \rightarrow \text{Mod}_A$. If \mathcal{C} is an abelian category, show that $\text{Hom}_{\mathcal{C}}(C, -)$ is a left-exact covariant functor $\mathcal{C} \rightarrow \text{Ab}$.
- Show that $\text{Hom}(-, M)$ is a left-exact covariant functor $\text{Mod}_A \rightarrow \text{Mod}_A$. If \mathcal{C} is an abelian category, show that $\text{Hom}_{\mathcal{C}}(-, C)$ is a left-exact covariant functor $\mathcal{C} \rightarrow \text{Ab}$.

Proof. b) was done in Exercise 1.2.H.

a) by b), the localization functor (which can be phrased as $- \otimes S^{-1}A$) is right-exact, covariant. Thus, given

$$0 \rightarrow M' \xrightarrow{f} M \rightarrow M'' \rightarrow 0$$

we know that

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \rightarrow S^{-1}M'' \rightarrow 0$$

is exact. We need to show that the first map is injective. Assume $S^{-1}f(m/s) = 0$. That means there exists $s' \in S$ such that $s'f(m) = 0$. But every element of S acts invertibly on $S^{-1}M$, so this implies $f(m) = 0$, which in turn implies $m = 0$, since f was injective to begin with.

c) We show the second part first: By the def. of an additive category, $\text{Hom}_{\mathcal{C}}(C, -)$ is a covariant functor to the category of abelian groups. Now if

$$0 \rightarrow D' \xrightarrow{f} D \rightarrow D''$$

is exact, then f is mono, and the definition of f being mono says that $f \circ g = 0$ implies $g = 0$, so $\text{Hom}_{\mathcal{C}}(C, -)$ is left-exact. Now the first statement is just realising that $\text{Hom}_A(M, -)$ carries an A -module structure by $(a \cdot f)(m) = af(m) = f(am)$.

d) Is the same, but with arrows reversed (and doing f^* instead of f_* , using properties of epimorphisms). \square

Exercise 1.5.H. Suppose M is a finitely presented A -module, i.e. there is an exact-sequence

$$A^{\oplus q} \rightarrow A^{\oplus p} \rightarrow M \rightarrow 0$$

Describe an isomorphism

$$S^{-1}\text{Hom}_A(M, N) \simeq \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

Solution. Applying $\text{Hom}_A(-, N)$ to the sequence gives

$$0 \rightarrow \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(A^{\oplus p}, N) \rightarrow \text{Hom}_A(A^{\oplus q}, N)$$

Now since Hom commutes with direct sums, and $\text{Hom}_R(R, L) = L$ for any ring R , and R -module L , we get

$$0 \rightarrow \text{Hom}_A(M, N) \rightarrow N^{\oplus p} \rightarrow N^{\oplus q}$$

and finally, applying localization with S gives

$$0 \rightarrow S^{-1}\text{Hom}_A(M, N) \rightarrow S^{-1}N^{\oplus p} \rightarrow S^{-1}N^{\oplus q}$$

On the other hand, we can start by applying localization, and then $\text{Hom}_{S^{-1}A}(-, S^{-1}N)$, and identifying $\text{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N) = N$, which results in

$$0 \rightarrow \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \rightarrow S^{-1}N^{\oplus p} \rightarrow S^{-1}N^{\oplus q}$$

which shows the desired isomorphism.

The isomorphism is given by $f/s \rightarrow (f/s)'(m/s') = S^{-1}f(m)/(ss')$. \square

Exercise 1.5.I (The FHHF Theorem). Suppose $F : \mathcal{A} \rightarrow \mathcal{B}$ is a covariant functor of abelian categories, and C^\bullet is a complex in \mathcal{A} .

- a) F right-exact yields $FH^\bullet \rightarrow H^\bullet F$: If F is right-exact, describe a natural morphism $FH^\bullet \rightarrow H^\bullet F$.
- b) F left-exact yields $FH^\bullet \leftarrow H^\bullet F$: If F is left-exact, describe a natural morphism $H^\bullet F \rightarrow FH^\bullet$.
- c) F exact yields $FH^\bullet \simeq H^\bullet F$: If F is exact, show that the two morphisms in a) and b) are inverses, and thus isomorphisms.

Solution. a) As we have established before, right-exactness preserves cokernels. Further, since for all $f : A \rightarrow B$, we have

$$0 \rightarrow \operatorname{im} f \rightarrow B \rightarrow \operatorname{coker} f \rightarrow 0$$

exact, we get that

$$F(\operatorname{im} f) \rightarrow F(B) \rightarrow \operatorname{coker} F(f) \rightarrow 0$$

is exact, and thus there is an epimorphism $F(\operatorname{im} f) \twoheadrightarrow \operatorname{im} F(f)$ by the universal property of the kernel (which is played by $\operatorname{im} f$ here).

Now, applying the two functors in the two orders to the second exact sequence in Exercise 1.5.A gives the following commutative diagram

$$\begin{array}{ccccccc} F(H^i(C^\bullet)) & \longrightarrow & F(\operatorname{coker} d^{i-1}) & \longrightarrow & F(\operatorname{im} d^i) & \longrightarrow & 0 \\ \downarrow !h & & \downarrow \sim & & \downarrow & & \\ 0 & \longrightarrow & H^i(F(C^\bullet)) & \longrightarrow & \operatorname{coker} F(d^{i-1}) & \longrightarrow & \operatorname{im} F(d^i) \longrightarrow 0 \end{array}$$

where the unique induced map h exists by the universal property of kernels. b) The argument is quite similar; start with the exact sequence

$$0 \rightarrow \ker f \rightarrow A \rightarrow \operatorname{im} f \rightarrow 0$$

which after applying the left-exact F looks like

$$0 \rightarrow F(\ker f) \rightarrow F(A) \rightarrow F(\operatorname{im} f)$$

which now gives a morphism $\operatorname{im} F(f) \rightarrow F(\operatorname{im} f)$, by the universal property of cokernels. In this way, we can construct the map by the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{im} F(d^{i-1}) & \longrightarrow & \ker F(d^i) & \longrightarrow & H^i(F(C^\bullet)) \longrightarrow 0 \\ & & \downarrow & & \downarrow \sim & & \downarrow !h \\ 0 & \longrightarrow & F(\operatorname{im} d^{i-1}) & \longrightarrow & F(\ker d^{i-1}) & \longrightarrow & F(H^i(C^\bullet)) \end{array}$$

c) Its easy to see that h and h' are isomorphisms when F is exact, and they must be mutually inverse, since they were defined with universal properties. \square

Exercise 1.5.J (Kernels commute with limits). Suppose \mathcal{C} is an abelian category, and let $a : \mathcal{I} \rightarrow \mathcal{C}$ and $b : \mathcal{I} \rightarrow \mathcal{C}$ be two diagrams in \mathcal{C} indexed by \mathcal{I} . Write $A_i := a(i)$ and $B_i := b(i)$ for objects in the respective diagrams. Let $h_i : A_i \rightarrow B_i$ be maps commuting with the maps in the diagram (i.e. $h : a \rightarrow b$ is a natural transformation). Then $\ker h_i$ form another diagram in \mathcal{C} indexed by \mathcal{I} . Describe a canonical isomorphism

$$\varprojlim_{\mathcal{I}} \ker h_i \xrightarrow{\sim} \ker(\varprojlim_{\mathcal{I}} A_i \rightarrow \varprojlim_{\mathcal{I}} B_i)$$

assuming all limits exists.

Solution. The composition $h_i \circ \pi_i : \varprojlim_{\mathcal{I}} A_i \rightarrow A_i \rightarrow B_i$ shows that there exists a unique map $\varprojlim_{\mathcal{I}} A_i \rightarrow \varprojlim_{\mathcal{I}} B_i$. Now let the kernel of this map be denoted by K . Similarly, we may take the kernel of each h_i , which induces a new diagram in \mathcal{C} indexed by \mathcal{I} . The maps $\varprojlim_{\mathcal{I}} \ker h_i \rightarrow \ker h_i \rightarrow A_i$ gives a unique map $\varprojlim_{\mathcal{I}} \ker h_i \rightarrow \varprojlim_{\mathcal{I}} A_i$, and since composing with the h_i obviously becomes 0, we get a canonical map $\varprojlim_{\mathcal{I}} \ker h_i \rightarrow K$. However, the maps $K \rightarrow \varprojlim_{\mathcal{I}} A_i \rightarrow A_i \rightarrow B_i$ are zero, we get induced maps $K \rightarrow \ker h_i$, which in turn gives a universal map $K \rightarrow \varprojlim_{\mathcal{I}} \ker h_i$. These maps are mutually inverse isomorphisms. \square

Exercise 1.5.K. Make sense of the statement that “limits commutes with limits” in a general category, and prove it.

Solution. We use two indexing categories \mathcal{I} and \mathcal{J} . Diagrams in \mathcal{C} indexed by \mathcal{I} form a new category, namely the category of functors $\mathcal{I} \rightarrow \mathcal{C}$ (denoted $\mathcal{C}^{\mathcal{I}}$). Now consider a diagram $\mathcal{J} \rightarrow \mathcal{C}^{\mathcal{I}}$. First, notice that maybe surprisingly, this data is symmetric in \mathcal{I} and \mathcal{J} . That is, this data gives for each $i \in \mathcal{I}$, a diagram $\mathcal{J} \rightarrow \mathcal{C}$, and morphisms in \mathcal{I} gives natural transformations in $\mathcal{C}^{\mathcal{J}}$. Thus, more conceptually, we may consider such a double diagram a single

$$F : \mathcal{I} \times \mathcal{J} \rightarrow \mathcal{C}$$

where $F(-, j)$ and $F(i, -)$ gives diagrams from \mathcal{I} and \mathcal{J} respectively. The now two ways of forming the double limit obviously commute, since they are both equal to the single limit $\varprojlim_{(i,j) \in \mathcal{I} \times \mathcal{J}} A_{ij}$. \square

Exercise 1.5.L. Show that in $\text{Mod} A$, colimits over filtered index categories are exact.

Solution. The previous exercise also shows that colimits commute with colimits, thus colimits preserves cokernels, i.e. colimits are right-exact.

Thus, given

$$0 \rightarrow A_i \xrightarrow{f_i} B_i \rightarrow C_i \rightarrow 0,$$

that additinally commutes with every map from \mathcal{J} , we can apply the colimit, and get

$$\varinjlim_{\mathcal{J}} A_i \xrightarrow{f} \varinjlim_{\mathcal{J}} B_i \rightarrow \varinjlim_{\mathcal{J}} C_i \rightarrow 0$$

we must show that the first map f is injective. Let $f(a) = 0$. This means that for some n , $f_n(a_n) = 0 \in B_n$ (since colimits over filtered index categories satisfies 0 in the limit if and only if 0 at some index). But f_n is injective, and thus $a_n = 0 \in A_n$, which in turn implies that $a = 0$ to begin with. \square

Exercise 1.5.M. Show that filtered colimits commute with homology in $\text{Mod} A$.

Solution. By the previous exercise, filtered colimits are exact, and thus it follows from the FHHF-theorem (Exercise 1.5.I). \square

Exercise 1.5.N. Suppose

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & A_0 & \longrightarrow & B_0 & \longrightarrow & C_0 \longrightarrow 0 \end{array}$$

is an inverse system of exact sequences of modules over a ring, such that the maps $A_{n+1} \rightarrow A_n$ are surjective. Show that the limit

$$0 \rightarrow \varprojlim_n A_n \rightarrow \varprojlim_n B_n \rightarrow \varprojlim_n C_n \rightarrow 0$$

is exact.

Solution. First, all maps are defined by the universal property of limits. Limits are left-exact, thus the sequence

$$0 \rightarrow \varprojlim_n A_n \rightarrow \varprojlim_n B_n \rightarrow \varprojlim_n C_n$$

is certainly exact. We must show that the last map is also exact: Assume $(c_n)_n \in \varprojlim_n C_n$ is an element. We may consider $(b_n)_n$, where $f_n(b_n) = c_n$ (since the individual maps are surjective). However, $(b_n)_n$ may not be a valid element of $\varprojlim_n B_n$. We correct this piecewise: For each non-zero $d_{n+1}(b_{n+1}) - b_n$, we know that $f_n(d_{n+1}(b_{n+1}) - b_n) = 0$, thus there exists $a_n \in A_n$ with $g_n(a_n) = d_{n+1}(b_{n+1}) - b_n$. Since the left row is surjective, we can lift to a_{n+1} , and set $b'_{n+1} := b_{n+1} - g_{n+1}a_{n+1}$. Of course, this is still a valid lift of c_n , but now, $d_{n+1}(b'_{n+1}) - b_n = 0$. Doing this for all n gives a valid preimage of $(c_n)_n$, showing that the last map is surjective. \square

1.6 *Spectral sequences

2 Sheaves

2.1 The sheaf of smooth functions

In the first two exercises, we are considering a manifold $X \subseteq \mathbb{R}^n$, and \mathcal{O}_X , the sheaf of smooth functions on X .

Exercise 2.1.A. Show that the only maximal ideal of \mathcal{O}_p is \mathfrak{m}_p .

Solution. Let $(f, U) \in \mathcal{O}_p \setminus \mathfrak{m}_p$ be a germ at p . Since the roots of f are distinct, there exists an open set $V \subset U$ such that f has no roots in V . Then $(1/f, V) \in \mathcal{O}_p$ ($1/f$ is still smooth on regions where f has no roots) is an inverse of (f, U) . \square

Exercise 2.1.B. Prove that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a module over $\mathcal{O}_p/\mathfrak{m}_p \simeq \mathbb{R}$. Prove that it is naturally the cotangent space to the differentiable or analytic manifold at p .

Solution. The module part straight-forward: We simply define $[f][g] = [fg]$, where $f \in \mathcal{O}_p, g \in \mathfrak{m}_p$ are representatives. This is well-defined: assume $f' = f + h$ for some $h \in \mathfrak{m}_p$. Then $[f'][g] = [fg + hg] = [fg]$, since $hg \in \mathfrak{m}_p^2$. For the map $\mathfrak{m}_p \rightarrow T_p^*X$, send $f_p \rightarrow df_p$, where $df_p : T_pX \rightarrow \mathbb{R}$ is the differential $v \rightarrow v(f)$, thought of as “taking the derivative of f in the direction of v ”. This is linear, and the kernel is \mathfrak{m}_p^2 : If $f \in \mathfrak{m}_p^2$, then $f = \sum_i a_i b_i$ for $a_i, b_i \in \mathfrak{m}_p$, but then $df = \sum a_i db_i + b_i da_i$, which is still 0 at p . Conversely, if $df_p = 0$, then the Taylor expansion around p shows that f has no linear part, so $f_p \in \mathfrak{m}_p^2$. \square

2.2 Definition of sheaf and presheaf

Exercise 2.2.A. Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets.

Solution. This is just checking both definitions. $\text{Res}_{U,U} = \text{id}_{\mathcal{F}(U)}$ is one of the functor axioms, and the compatibility of the restriction maps is just functoriality. \square

Exercise 2.2.B. Show that the following are presheaves on \mathbb{C} (with the classical topology), but not sheaves: a) bounded functions, b) holomorphic functions admitting a holomorphic square root.

Solution. It is easy to see that both a) and b) satisfy the presheaf axioms. Further, they both clearly satisfy the identity axiom (functions are defined by what they do on points). However, neither satisfies gluing: a) gluing together infinitely many bounded functions may not stay bounded, and b) there is famously no well-defined square root on all of \mathbb{C} (so you are unable to glue together $f(z) = z$ defined on patches). \square

Exercise 2.2.C. The identity and gluability axiom may be interpreted as saying that $\mathcal{F}(\bigcup_{i \in I} U_i)$ is a certain limit. What is that limit?

Solution. This is the interpretation as an equalizer (since an equalizer is a limit): We have two arrows

$$\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

where the arrows are given by the two restriction maps $\mathcal{F}(U_i) \rightarrow \mathcal{F}(U_i \cap U_j)$ and $\mathcal{F}(U_j) \rightarrow \mathcal{F}(U_i \cap U_j)$. Now saying

$$\mathcal{F}(U) = \varprojlim \left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right)$$

is then equivalent to the sheaf axioms: The unique map $\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i)$ is identity axiom (this map must clearly be injective), and the gluing axiom is saying that elements in $\mathcal{F}(U_i)$ that agree on the intersections correspond to an element of $\mathcal{F}(U)$. \square

Exercise 2.2.D. a) Verify that smooth functions or continuous functions or real-analytic functions, or plain real-valued functions on a manifold are indeed sheaves.

b) Show that real-valued continuous functions on (open sets of) a topological space X form a sheaf.

Solution. a) There is not much to show here, all the properties are stable under restriction, so they are all obviously presheaves. That they all satisfy identity is also immediate. Gluing is maybe the only part that requires some thinking, but the fact that all these properties can be checked locally means that they can all be glued.
b) Again, there is not really anything to check here, as functions can be defined on points. \square

Exercise 2.2.E (Constant sheaf). Let $\mathcal{F}(U)$ be the set of maps $U \rightarrow S$ that are locally constant, i.e. for each point $p \in U$, there is an open neighborhood $p \in V$ such that $\mathcal{F}(V)$ is constant. (An alternative description is endowing S with the discrete topology, and defining $\mathcal{F}(U)$ to be all continuous maps $U \rightarrow S$.)

Solution. First notice that each $f \in \mathcal{F}(U)$ is forced to take constant values in S on connected components (and the value of x is given by the connected component). As usual, presheaf axioms and identity are clear. Gluing is also clear by the description just given: given $f \in \mathcal{F}(U), g \in \mathcal{F}(V)$, defined $h \in \mathcal{F}(U \cup V)$ by what it does on each connected component of $U \cup V$ (given by either f, g or both if they are both defined). \square

Exercise 2.2.F. Suppose Y is a topological space. Show that “continuous maps to Y ” forms a sheaf of sets on X , i.e. $\mathcal{F}(U) = \text{continuous maps } U \rightarrow Y$ is a sheaf.

Solution. Presheaf axioms, and identity is just checking. We may obviously glue morphisms as well; we just need to check that it is continuous. Let $f : U \rightarrow Y$ be a morphism defined by continuous morphisms on an open cover $f_i : U_i \rightarrow Y$. Let $V \subseteq Y$ be open. Now $f^{-1}(V) = \bigcup f_i^{-1}(V)$, thus open. \square

Exercise 2.2.G. This is a fancier version of the previous exercise:

- a) Suppose we are given a continuous map $\mu : Y \rightarrow X$. Show that “sections of μ ” form a sheaf, i.e. $\mathcal{F}(U) = \text{continuous maps } s : U \rightarrow Y \text{ such that } \mu \circ s = \text{id}_U$.
- b) Suppose that Y is a topological group. Show that continuous maps to Y form a sheaf of groups.

Solution. a) In the previous exercise, we showed that continuous maps to Y form a sheaf. So we only need to restrict to sections of μ and see what happens: Restriction maps are still okay (so it is a presheaf), and so is both identity and gluing.
b) We may indeed given $f, g \in \mathcal{F}(U)$ form $fg(x) = f(x)g(x)$. Since the group operation is continuous, this is still a continuous function defined on U . The fact that the restrictions are also group-homomorphisms is obvious (restricting, then multiplying is the same as multiplying the functions, then restricting). \square

Exercise 2.2.H. Suppose $\pi : X \rightarrow Y$ is a continuous map, and that \mathcal{F} is a (pre)sheaf on X . Then define a presheaf $\pi_*\mathcal{F}$ on Y by $\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$. Show that $\pi_*\mathcal{F}$ is a (pre)sheaf on Y .

Solution. First, assume \mathcal{F} is a presheaf. Viewed as a functor, it is then clear that $\pi_*(\mathcal{F})$ is a presheaf, as it is the composition

$$\text{Open}(Y) \rightarrow \text{Open}(X) \xrightarrow{\mathcal{F}} \mathcal{C}$$

where the first map is given by inverse images of π .

Assuming that \mathcal{F} is a sheaf, then $\pi_*\mathcal{F}$ is also a sheaf: let $\{V_i\}$ be an open cover of V . For identity, assume that $s, t \in \pi_*\mathcal{F}(V)$, and that $s|_{V_i} = t|_{V_i}$ for all i . By definition, s and t are elements of $\mathcal{F}(\pi^{-1}(V))$, and similarly with the restrictions. Now, since $\pi^{-1}(V) = \pi^{-1}(\bigcup V_i) = \bigcup \pi^{-1}(V_i)$, it follows that $s = t$.

Gluing can be shown similarly, using $\pi^{-1}(V_i \cap V_j) = \pi^{-1}(V_i) \cap \pi^{-1}(V_j)$. \square

Exercise 2.2.I. Suppose $\pi : X \rightarrow Y$ is a continuous map, and \mathcal{F} is a sheaf of sets on X . If $\pi(p) = q$, describe the natural map of stalks $(\pi_*\mathcal{F})_q \rightarrow \mathcal{F}_p$.

Solution. Using explicit representatives: If $s_q \in (\pi_*\mathcal{F})_q$, that means there exists some open U containing q , such that $s \in \pi_*\mathcal{F}(U)$ is a representative of s_q . But then, by definition of π_* , $s \in \mathcal{F}(\pi^{-1}(U))$, and since $p \in \pi^{-1}(U)$, $(s, \pi^{-1}(U))$ can be seen as an element in \mathcal{F}_p . Assume now that (s', V) is another representative of s_q . Then there exists some $W \subseteq U \cap V$ containing q , where $s|_W = s'|_W$. But then this just says that $s|_{\pi^{-1}(W)} = s'|_{\pi^{-1}(W)}$, and thus, they would map to the same element of \mathcal{F}_p .

Using the universal property: For each open set U containing q , there is an obvious map $\pi_*\mathcal{F}(U) \rightarrow \mathcal{F}_p$ given by sending $s \rightarrow (s, \pi^{-1}(U))$. These commute with the restriction maps (by definition they are the same on the smaller set), thus the universal property of the colimit gives a map $(\pi_*\mathcal{F})_q \rightarrow \mathcal{F}_p$. \square

Exercise 2.2.J. If (X, \mathcal{O}_X) is a ringed space, and \mathcal{F} is an \mathcal{O}_X -module, describe how for each $p \in X$, \mathcal{F}_p is a $\mathcal{O}_{X,p}$ -module.

Solution. Assume $(x, U) \in \mathcal{O}_{X,p}$, and $(a, U) \in \mathcal{F}_p$ (by passing to the intersection, we can assume that two representatives are defined on the same open set). By definition, there is a well-defined action $x \star a \in \mathcal{F}(U)$. Now since the square (2.2.13.1) in the book commutes, this action is well defined, regardless of representative, both of x_p and a_p . \square

2.3 Morphisms of presheaves and sheaves

Exercise 2.3.A. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves on X , and $p \in X$, describe an induced morphism of stalks $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$.

Solution. For each U containing p , we have the map

$$\begin{aligned}\mathcal{F}(U) &\rightarrow \mathcal{G}_p \\ s &\rightarrow (\varphi(U)(s), U),\end{aligned}$$

and the naturality of φ , it commutes with restriction maps, thus by the universal property of the colimit, we get a map $\mathcal{F}_p \rightarrow \mathcal{G}_p$. \square

Exercise 2.3.B. Suppose $\pi : X \rightarrow Y$ is a continuous map of topological spaces. Show that the pushforward gives a functor $\pi_* : \text{Sets}_X \rightarrow \text{Sets}_Y$.

Solution. We have already shown that the pushforward sends objects to objects, thus we need to show functoriality. Assume $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on X . Now, the pushforward induces the map

$$\begin{aligned}\pi_*\varphi(U) : \pi_*\mathcal{F}(U) &\rightarrow \pi_*\mathcal{G}(U) \\ s &\rightarrow \varphi(U)(s)\end{aligned}$$

which makes sense since indeed, if $s \in \mathcal{F}(\pi^{-1}(U))$, then $\varphi(s) \in \mathcal{G}(\pi^{-1}(U)) = (\pi_*\mathcal{G})(U)$. The fact that these maps indeed play well with restriction also follows from the fact that φ does. It is clear that sending φ to $\pi_*\varphi$ is functorial: both preserving identity and associativity is obvious. \square

Exercise 2.3.C (Sheaf Hom). Suppose \mathcal{F} and \mathcal{G} are two sheaves of sets on X (in fact, it will suffice that \mathcal{F} is a presheaf). Let $\mathcal{HOM}(\mathcal{F}, \mathcal{G})$ be the collection of data

$$\mathcal{HOM}(\mathcal{F}, \mathcal{G})(U) := \text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Show that this is a sheaf of sets on X .

Solution. First, we show its a presheaf:

If $V \subseteq U$, then there is an obvious map $\mathcal{HOM}(\mathcal{F}, \mathcal{G})(U) \rightarrow \mathcal{HOM}(\mathcal{F}, \mathcal{G})(V)$ by just restricting the morphisms to V . This clearly preserves identity, and is associative, thus it is a presheaf.

We now go on to show its a sheaf: assume that $\{U_i\}$ is an open cover of the open set $U \subseteq X$. Assume we are given $\varphi_i \in \mathcal{HOM}(\mathcal{F}, \mathcal{G})(U_i)$ that agree on overlaps $U_i \cap U_j$. Define a new $\varphi \in \mathcal{HOM}(\mathcal{F}, \mathcal{G})(U)$ as follows: For any $V \subseteq U$, and $s \in \mathcal{F}|_U(V)$, set

$$\varphi(V)(s) = s'$$

where the section $s' \in \mathcal{G}|_U(V)$ is obtained by gluing $\varphi_i(V \cap U_i)(s|_{V \cap U_i})$ (this gluing is possible because φ_i agree on overlaps, and \mathcal{G} is a sheaf).

To show uniqueness, assume now that $\psi : \mathcal{F}|_U \rightarrow \mathcal{G}|_U$ is another morphism of sheaves, such that $\psi|_{U_i} = \varphi_i$ for all i . Then $\psi(V)(s) = \varphi(V)(s)$ for all V and s , since they agree on an open cover $\{V \cap U_i\}$ and \mathcal{G} is a sheaf (hence glues uniquely). \square

- Exercise 2.3.D.** a) If \mathcal{F} is a sheaf of sets on X , then show that $\mathcal{HOM}(\underline{\{p\}}, \mathcal{F}) \cong \mathcal{F}$, where $\underline{\{p\}}$ is the constant sheaf on X with values in $\{p\}$.
- b) If \mathcal{F} is a sheaf of abelian groups on X , then show that $\mathcal{HOM}_{\text{Ab}_X}(\underline{\mathbb{Z}}, \mathcal{F}) \cong \mathcal{F}$ (an isomorphism of sheaves of abelian groups).
- c) If \mathcal{F} is an \mathcal{O}_X -module, then show that $\mathcal{HOM}_{\text{Mod}_{\mathcal{O}_X}}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$ (an isomorphism of \mathcal{O}_X -modules).

Solution. For a), first note that the constant sheaf $\underline{\{p\}}$ really is the same as the constant presheaf with values in $\{p\}$; usually a constant sheaf \underline{A} would associate $\underline{A}(U)$ with continuous functions $U \rightarrow A$ (where A is given the discrete topology), but in this case, with the one-element set, there is only one such function anyway, sending everything to p . So to solve the exercise, we first construct a morphism of sheaves $\mathcal{HOM}(\underline{\{p\}}, \mathcal{F}) \rightarrow \mathcal{F}$ by sending for each open U , and section $\varphi : \underline{\{p\}}|_U \rightarrow \mathcal{F}|_U$ to $\varphi(U)(s_p)$, where s_p is the unique section in $\underline{\{p\}}(U)$ (notice that $\varphi(s_p)$ completely determines φ because for any $V \subseteq U$, $\varphi(V)(s_p|_V) = \varphi(U)(s_p)|_V$). Conversely, any $s \in \mathcal{F}(U)$ determines such a morphism of sheaves, thus for any U , this is an isomorphism between $\mathcal{HOM}(\underline{\{p\}}, \mathcal{F})(U)$ and $\mathcal{F}(U)$, and it is compatible with restrictions, so this is an isomorphism of sheaves of sets.

For b) and c), it is obviously the same idea, except there is an additional difficulty; the constant sheaf is not the constant presheaf. However, we need only use the universal property; namely $\mathcal{HOM}_{\text{Ab}_X}(\underline{\mathbb{Z}}^{\text{pre}}, \mathcal{F}) \cong \mathcal{F}$ by essentially the same argument, sending φ to $\varphi(1)$. Now this is again isomorphic to $\mathcal{HOM}_{\text{Ab}_X}(\underline{\mathbb{Z}}, \mathcal{F})$ by the universal property of sheafification. The same argument goes through for c). \square

Exercise 2.3.E. Show that $\ker_{\text{pre}} \varphi$ is a presheaf.

Solution. It's clear that $\ker_{\text{pre}} \varphi$ assigns opens to abelian groups, thus we need only check the restriction maps. The restriction maps are given by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker_{\text{pre}} \varphi(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ & & \downarrow !h & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker_{\text{pre}} \varphi(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \end{array}$$

where the map h exists by universal property of the kernel. (These maps also make it clear that the composition of these restriction maps behave as expected). \square

Exercise 2.3.F. Show that the presheaf cokernel satisfies the universal property of cokernels.

Solution. The universal property it should satisfy is that if $\psi : \mathcal{G} \rightarrow \mathcal{H}$ is a morphism of presheaves such that $\psi \circ \varphi = 0$, then there exists a unique morphism $\eta : \text{coker}_{\text{pre}} \varphi \rightarrow \mathcal{H}$ making everything commute.

By definition, for every open U , $(\text{coker}_{\text{pre}} \varphi)(U) = \text{coker } \varphi(U)$. Thus, there exists a unique map $(\text{coker}_{\text{pre}} \varphi)(U) \rightarrow \mathcal{H}(U)$. We only need to check that these maps commute with restriction. But this follows since in the following diagram

$$\begin{array}{ccccc} \mathcal{G}(U) & \longrightarrow & (\text{coker}_{\text{pre}} \varphi)(U) & \longrightarrow & \mathcal{H}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}(V) & \longrightarrow & (\text{coker}_{\text{pre}} \varphi)(V) & \longrightarrow & \mathcal{H}(V) \end{array}$$

the outer and the left square commutes. Thus, the right square commutes when precomposed with $\mathcal{G}(U) \rightarrow (\text{coker}_{\text{pre}} \varphi)(U)$, but this is an epimorphism, and thus we are done. \square

Exercise 2.3.G. Show (or observe) that for a topological space X with open set U , $\mathcal{F} \rightarrow \mathcal{F}(U)$ gives a functor from presheaves of abelian groups on X to abelian groups. Then show that this functor is exact.

Solution. That it is a functor is indeed obvious; presheaf maps $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ are sent to $\alpha(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$.

Notice also that by definition,

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

is an exact sequence of presheaves if for every open V ,

$$0 \rightarrow \mathcal{F}(V) \rightarrow \mathcal{G}(V) \rightarrow \mathcal{H}(V) \rightarrow 0$$

is exact. This says that the functor is exact.

(More generally, if one has two categories, one can form the category of functors $\mathcal{J} \rightarrow \mathcal{C}$, denoted $\mathcal{C}^{\mathcal{J}}$, at least if \mathcal{J} is a small category. Then for any $j \in \mathcal{J}$, the “evaluation functor”, sending $\mathcal{F} \rightarrow \mathcal{F}(j)$ is an exact functor from $\mathcal{C}^{\mathcal{J}} \rightarrow \mathcal{C}$.) \square

Exercise 2.3.H. Show that a sequence of presheaves $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \cdots \rightarrow \mathcal{F}_n \rightarrow 0$ is exact if and only if $0 \rightarrow \mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U) \rightarrow \cdots \rightarrow \mathcal{F}_n(U) \rightarrow 0$ is exact for all U .

Solution. The top sequence being exact means, by definition, that the bottom sequence is exact for all U . Conversely, the evaluation functor is exact by the previous exercise. \square

Exercise 2.3.I. Suppose $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves. Show that the presheaf kernel $\text{ker}_{\text{pre}} \varphi$ is in fact a sheaf. Show that it satisfies the universal property of kernels.

Solution. We need to check that it satisfies the gluing and identity axiom. Take an open U , and an open cover $\{U_i\}_i$. Given sections $s_i \in (\ker \varphi)(U_i)$ that agree on intersections, we can push them through to $\mathcal{F}(U_i)$, where they of course still agree on intersections. These glue together uniquely to $s \in \mathcal{F}(U)$. However, s is really in $\ker \varphi(U)$, because the s_i become 0 in $\mathcal{G}(U_i)$, and thus glue uniquely to 0. Thus, the s_i glue together to $s \in (\ker \varphi)(U)$, in a unique way.

Notice that if we try the same argument for the cokernel, it does not work at all.

For the second statement, we have already shown that it satisfies the universal property of a kernel in the category of presheaves. This is already enough, since the category of sheaves is a full subcategory of the category of presheaves. \square

Exercise 2.3.J. Let X be \mathbb{C} with the classical topology, let \mathcal{O}_X be the sheaf of holomorphic functions, and let \mathcal{F} be the presheaf of functions admitting a holomorphic logarithm. Describe an exact sequence of presheaves on X :

$$0 \rightarrow \mathbb{Z} \xrightarrow{\varphi} \mathcal{O}_X \xrightarrow{\psi} \mathcal{F} \rightarrow 0$$

where $\mathbb{Z} \rightarrow \mathcal{O}_X$ is the natural inclusion, and $\mathcal{O}_X \rightarrow \mathcal{F}$ is given by $f \rightarrow e^{2\pi i f}$. Show that \mathcal{F} is not a sheaf.

Solution. Notice that, by definition, $\mathcal{F}(U) := \{g \in \mathcal{O}_X(U) \mid \exists f \in \mathcal{O}_X(U), e^{2\pi i f} = g\}$. Thus, $\psi(U)$ is obviously surjective, and exactness comes down to showing that $\text{im } \varphi = \ker \psi$. It's clear that $\text{im } \varphi \subseteq \ker \psi$. Assume now that $f \in \ker \psi(U)$, i.e. that $e^{2\pi i f(z)} = 1$ for all $z \in U$. That means that $f(z) \in \mathbb{Z}$ for all $z \in U$, and since f is holomorphic, this again implies that f is a locally constant integer, i.e. an element of $\text{im } \varphi(U)$.

However, \mathcal{F} is famously not a sheaf: For instance, consider the open cover of $U = \mathbb{C} \setminus \{0\}$ given by $U_1 = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $U_2 = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The function $g(z) = z$ are part of both $\mathcal{F}(U_i)$ (i.e. there is a holomorphic branch of the logarithm function on U_i), but famously do not glue to $\mathcal{F}(U)$ (i.e. there is no single-valued logarithm function on all of \mathbb{C}^\times). \square

2.4 Properties determined at the level of stalks and sheafification

Exercise 2.4.A. Prove that a section of a (separated pre)sheaf of sets is determined by its germs, i.e. the natural map

$$\mathcal{F}(U) \rightarrow \prod_{p \in U} \mathcal{F}_p$$

is injective.

Solution. Given a section $s \in \mathcal{F}(U)$ the natural map sends s to the equivalence class of (s, U) at p for all p . Assume now that $s' \in \mathcal{F}(U)$ is another section satisfying $(s', U) \sim_p (s, U)$ at each point p . This means that for each point p , we can find an open V such that $s|_V = s'|_V$. Obviously, the V form an open cover, so the identity axiom says that $s = s'$. \square

Exercise 2.4.B. Prove that any choice of compatible germs for a sheaf of sets \mathcal{F} over U is the image of a section \mathcal{F} over U .

Solution. The definition of a compatible germs $(s_p)_p$, says that we can form a cover $\{U_i\}_i$ over U , and choose sections s_i such that (s_i, U_i) is a valid representative of s_p for all $p \in U_i$. Notice that this implies that the s_i agree on intersections: let $V = U_i \cap U_j$. Then $(s_i|_V, V) \simeq_p (s_j|_V, V)$ for all $p \in V$, but this implies that $s_i = s_j$ by the previous exercise. Thus, we may glue the s_i by the gluability axiom. \square

Exercise 2.4.C. If φ_1 and φ_2 are morphisms from a presheaf of sets \mathcal{F} to sheaf of sets \mathcal{G} that induce the same maps on each stalk, show that $\varphi_1 = \varphi_2$.

Solution. By the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_i} & \mathcal{G}(U) \\ \downarrow & & \downarrow \iota \\ \prod_{p \in U} \mathcal{F}_p & \longrightarrow & \prod_{p \in U} \mathcal{G}_p \end{array}$$

$\varphi_1 \circ \iota = \varphi_2 \circ \iota$ (since the down-right path is equal by the hypothesis). But ι is mono, so this implies that $\varphi_1 = \varphi_2$. \square

Exercise 2.4.D. Show that a morphism of sheaves of sets is an isomorphism if and only if it induces an isomorphism of all stalks.

Solution. Passing to stalks is functorial, so an isomorphism of sheaves induces an isomorphism at all stalks. The tricky part is the converse: Assuming that the induced map on stalks is an isomorphism, the commuting square looks like

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_{p \in U} \mathcal{F}_p & \xrightarrow{\sim} & \prod_{p \in U} \mathcal{G}_p \end{array}$$

(since \mathcal{F} is also assumed to be a sheaf), which shows that the map $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ must be injective for all U . To get surjectivity, assume we are given $s \in \mathcal{G}$. Passing to stalks, and using the inverse map on stalks, notice that this gives germs $(t_p)_p \in \prod \mathcal{F}_p$. If we can show that these are compatible germs, we can apply Exercise 2.4.B to get a corresponding section $t \in \mathcal{F}(U)$, and it is obvious that $\varphi(U)(t) = s$ (again using the commuting square).

It remains to show that being compatible is stable under isomorphism of stalks. By construction, have that $\varphi_p(t_p) = s_p$ for all p , meaning that for each p , there is an open U_p , and section $f_p \in \mathcal{F}(U_p)$ such that $(\varphi(U_p)(f_p), U_p)$ is a valid representative of s_p , i.e. after possibly shrinking U_p , we get $(\varphi(U_p)(f_p) = s|_{U_p})$ for all p . Now looking at intersections, these f_p must all agree: Let $V = U_p \cap U_q$, then $\varphi(V)(f_p|_V) = \varphi(V)(f_q|_V) = s|_V$. But we already proved injectivity, so this implies that $f_p|_V = f_q|_V$. \square

Exercise 2.4.E. a) Show that Exercise 2.4.A is false for general presheaves.

b) Show that Exercise 2.4.C is false for general presheaves.

c) Show that Exercise 2.4.D is false for general presheaves.

Solution. Let $X = \{p, q\}$ be a two point sets with the discrete topology, and let \mathcal{F} be the sheaf of sets defined by $\mathcal{F}(\{p\}) = \mathcal{F}(\{q\}) = \{1\}$ and $\mathcal{F}(X) = \{1, 2\}$.

For a), it is obvious that there is no injective function $\{1, 2\} \rightarrow \{1\} \times \{1\}$.

For b), consider morphisms of sheaves $\varphi : \mathcal{F} \rightarrow \mathcal{F}$. There is only one possible map on the stalks, but clearly many possible maps $\mathcal{F}(X) \rightarrow \mathcal{F}(X)$.

For c), consider $\varphi : \mathcal{F} \rightarrow \mathcal{F}$ given by $\varphi(X)(b) = 1$ for $b \in \{1, 2\}$. □

Exercise 2.4.F. Show that sheafification is unique up to unique isomorphism, assuming it exists. Show that if \mathcal{F} is already a sheaf, then the sheafification is $\text{id} : \mathcal{F} \rightarrow \mathcal{F}$.

Solution. This is direct from the universal property. □

Exercise 2.4.G. Assume for now that sheafification exists. Use the universal property to show that for any morphism of presheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, we get a natural induced morphism of sheaves $\varphi^{\text{sh}} : \mathcal{F}^{\text{sh}} \rightarrow \mathcal{G}^{\text{sh}}$. Show that sheafification is a functor from presheaves on X to sheaves on X .

Solution. The universal property of sheafification ensures a map $\varphi^{\text{sh}} : \mathcal{F}^{\text{sh}} \rightarrow \mathcal{G}^{\text{sh}}$ so that $\varphi^{\text{sh}} \circ \text{sh}_{\mathcal{F}} = \text{sh}_{\mathcal{G}} \circ \varphi$. This is well-defined, by the universal property of $\text{sh}_{\mathcal{F}}$ (and also \mathcal{G} in the sense that this morphism is also unique up to unique isomorphism).

This is also functorial: Composition is preserved, since if $\psi : \mathcal{G} \rightarrow \mathcal{H}$ is another morphism of sheaves, then

$$\psi^{\text{sh}} \circ \varphi^{\text{sh}} \circ \text{sh}_{\mathcal{F}} = \psi^{\text{sh}} \circ \text{sh}_{\mathcal{G}} \circ \varphi = \text{sh}_{\mathcal{H}} \circ \psi \circ \varphi,$$

so by the universal property, $(\psi \circ \varphi)^{\text{sh}} = \psi^{\text{sh}} \circ \varphi^{\text{sh}}$.

Identities are sent to identities. To see this, note that the sheafification of $\text{id}_{\mathcal{F}}$ is the morphism satisfying

$$(\text{id}_{\mathcal{F}})^{\text{sh}} \circ \text{sh}_{\mathcal{F}} = \text{sh}_{\mathcal{F}} \circ \text{id}_{\mathcal{F}} = \text{sh}_{\mathcal{F}}$$

and clearly, $\text{id}_{\mathcal{F}^{\text{sh}}}$ satisfies this. □

Exercise 2.4.H. Show that \mathcal{F}^{sh} (using the tautological restriction maps) forms a sheaf.

Solution. Recall, \mathcal{F}^{sh} is defined by letting the functions on an open be precisely the compatible germs:

$$\mathcal{F}^{\text{sh}}(U) := \{(f_p \in \mathcal{F}_p)_{p \in U} \mid \forall p \in U, \text{ there exists an open neighborhood } V \subseteq U \text{ of } p, \text{ and } s \in \mathcal{F}(V) \text{ such that } s_q = f_q \text{ for all } q \in V.\}$$

The restriction maps (which now really deserves the name now) obviously make this into a presheaf on X .

We now show that it is a sheaf. Given an open cover $\{U_i\}_i$ of U , and sections $(f_p^i \in \mathcal{F}_p)_{p \in U_i} \in \mathcal{F}^{\text{sh}}(U_i)$ that agree on intersections, it is clear that we may form the element $(f_p \in \mathcal{F}_p)_{p \in U} \in \mathcal{F}^{\text{sh}}(U)$ by $f_p = f_p^i$ for any i with $p \in U_i$. These are again compatible germs, since for each $p \in U$, assume $p \in U_i$, then $f_p = f_p^i$ which is represented by an element $s \in \mathcal{F}(V \cap U_i)$ that also satisfies $s_q = f_q^i$ for all $q \in V \cap U_i$, thus we have gluability.

The identity axiom is obvious, since a section of compatible germs are determined by the underlying germs. \square

Exercise 2.4.I. Describe a natural map of presheaves $\text{sh} : \mathcal{F} \rightarrow \mathcal{F}^{\text{sh}}$.

Solution. Send $s \in \mathcal{F}(U)$ to $(s_p \in \mathcal{F}_p)_{p \in U} \in \mathcal{F}^{\text{sh}}(U)$. \square

Exercise 2.4.J. Show that the map sh satisfies the universal property of sheafification.

Solution. Assume that $\mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves, and that \mathcal{G} is a sheaf. Define φ^{sh} by sending $\varphi^{\text{sh}}(U)(s)$ to the element obtained by gluing the compatible sections $(\varphi_p(s_p) \in \mathcal{G}_p)_{p \in U}$ (or, if you don't wanna glue, we can work with \mathcal{G}^{sh} directly, which we know is canonically isomorphic to \mathcal{G}). It is clear that $\varphi^{\text{sh}} \circ \text{sh} = \varphi$ (for instance, by noting that the map of stalks is the same, and applying Exercise 2.4.C). Assume now that ψ is another morphism with $\psi \circ \text{sh} = \varphi$. Then φ^{sh} and ψ again give the same map on stalks; by Exercise 2.4.C they are the same map. \square

Exercise 2.4.K. Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X .

Solution. We need to show the existence of a natural bijection $\text{Hom}_{\text{PSh}}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\text{Sh}}(\mathcal{F}^{\text{sh}}, \mathcal{G})$. This bijection, we've already mentioned many times, sending φ to φ^{sh} . Naturality says that for any $\psi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$, the square

$$\begin{array}{ccc} \text{Hom}_{\text{Sh}}(\mathcal{F}_2^{\text{sh}}, \mathcal{G}) & \xrightarrow{\sim} & \text{Hom}_{\text{PSh}}(\mathcal{F}_2, \mathcal{G}) \\ \psi^{\text{sh}*} \downarrow & & \downarrow \psi^* \\ \text{Hom}_{\text{Sh}}(\mathcal{F}_1^{\text{sh}}, \mathcal{G}) & \xrightarrow{\sim} & \text{Hom}_{\text{PSh}}(\mathcal{F}_1, \mathcal{G}) \end{array}$$

should commute, which it does: Starting with f in the upper left corner results in $f \circ \psi^{\text{sh}} \circ \text{sh}_{\mathcal{F}_1} = f \circ \text{sh}_{\mathcal{F}_2} \circ \psi$ anyway (equality here is really just the universal property of sheafification). The similar square for any morphism $\psi' : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ also commutes. \square

Exercise 2.4.L. Show that $\mathcal{F} \rightarrow \mathcal{F}^{\text{sh}}$ induces an isomorphism of stalks.

Solution. The induced map on stalks works by sending a representative $s_x = (s, U) \in \mathcal{F}_x$ to the representative $((s_p)_{p \in U}, U) \in \mathcal{F}_x^{\text{sh}}$. We show that this is a bijection: First we show injectivity. Assume that $t_x = (t, V) \in \mathcal{F}_x$ is sent to the same germ in $\mathcal{F}_x^{\text{sh}}$. By definition, this means that for all $q \in U \cap V$, there exists an open neighborhood $W \subseteq U \cap V$ such that $(t, W) = (s, W)$ in \mathcal{F}_q , i.e. that $t|_W = s|_W$ for some open neighborhood $W' \subseteq W$ of q . But x is in this intersection, so clearly $t_x = s_x$. Next, we show surjectivity. Assume now that $t_x = ((t_p)_{p \in U}, U) \in \mathcal{F}_x^{\text{sh}}$. By definition of a compatible germ, this means there exists some open neighborhood $V \subseteq U$ of x , and $f \in \mathcal{F}(V)$ with $f_q = t_q$ for all $q \in V$. But then clearly, $\varphi_x(f_x) = t_x$. \square

Exercise 2.4.M. Suppose $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves of sets on a topological space X . Show that the following are equivalent.

- a) φ is a monomorphism in the category of sheaves.
- b) φ is injective on the level of stalks: $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$ is injective for all $p \in X$.
- c) φ is injective on the level of open sets: $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is injective for all open $U \subseteq X$.

Solution. We start with $b) \Rightarrow a)$:

Assume φ is injective on the level of stalks. Assume now that $\varphi \circ \psi_1 = \varphi \circ \psi_2$ for morphisms $\psi_i : \mathcal{H} \rightarrow \mathcal{F}$. Then, for every open U , we get a diagram like

$$\begin{array}{ccccc} \mathcal{H}(U) & \xrightarrow[\psi_2(U)]{\psi_1(U)} & \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ & & \downarrow & & \downarrow \\ & & \prod_{p \in U} \mathcal{F}_p & \longrightarrow & \prod_{p \in U} \mathcal{G}_p \end{array}$$

The assumption is that the top row is equal. Then, the morphisms are still equal when composing with $\mathcal{G}(U) \hookrightarrow \prod_{p \in U} \mathcal{G}_p$. Now using the commuting square, and that the other two morphisms are mono (this is the hypothesis $b)$), we get that $\psi_1(U) = \psi_2(U)$ for all U .

Next, $a) \Rightarrow c)$:

Let $U \subseteq X$ be open. Consider the “indicator sheaf” \mathcal{I} with $\mathcal{I}(V) = \{1\}$ if $V \subseteq U$, and $\mathcal{I}(V) = \emptyset$ otherwise. Consider now morphisms $\psi : \mathcal{I} \rightarrow \mathcal{F}$, and notice that such a morphism is uniquely defined by what it sends the unique element of $\mathcal{I}(U)$ to (and any choice of element also gives such a morphism). Assume now that there exists some open $U \subseteq X$, for which $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is not injective, say $\varphi(U)(x) = \varphi(U)(y)$. Then we may form the morphisms $\psi_x : \mathcal{I} \rightarrow \mathcal{F}$ and ψ_y by sending 1 to x, y respectively. Notice now, for all open $W \subseteq X$, the composition $\mathcal{I}(W) \rightarrow \mathcal{F}(W) \rightarrow \mathcal{G}(W)$ will be equal, implying that $\varphi \circ \psi_x = \varphi \circ \psi_y$. But using the hypothesis, this implies that $\psi_x = \psi_y$, which in turn implies that $x = y$.

Finally, $c) \Rightarrow b)$ can be shown directly:

Assume that $\varphi_p(s_p) = \varphi_p(t_p)$ for germs $s_p, t_p \in \mathcal{F}_p$. This means that there exists some V , and sections $s, t \in \mathcal{F}(V)$ such that $\varphi(V)(s)|_W = \varphi(V)(t)|_W$ for some open neighborhood $W \subseteq V$ of p . But this means that $\varphi(W)(s|_W) = \varphi(W)(t|_W)$. Since $\varphi(U)$ was assumed injective for all U , this implies that $s|_W = t|_W$, which in turn implies that $s_p = t_p$. \square

Exercise 2.4.N. Continuing the notation of the previous exercise, show that the following are equivalent.

- a) φ is an epimorphism in the category of sheaves.
- b) φ is surjective on the level of stalks: $\varphi_p : \mathcal{F}_p \rightarrow \mathcal{G}_p$ is surjective for all $p \in X$.

Solution. We first prove $b) \Rightarrow a)$:

Assume $\psi_i : \mathcal{G} \rightarrow \mathcal{H}$ are morphisms for $i = 1, 2$, and assume that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. For any U , consider the corresponding map of stalks

$$\prod_{p \in U} \mathcal{F}_p \rightarrow \prod_{p \in U} \mathcal{G}_p \rightarrow \prod_{p \in U} \mathcal{H}_p$$

Since the first part of this map is epi, we see that the map on stalks induced by the φ_i are equal. But by Exercise 2.4.C, this implies that the φ_i themselves are equal.

Next, $a) \Rightarrow b)$:

Passing to stalks is an exact functor, so epimorphisms are sent to epimorphisms (notice that this could've been used to prove $b) \rightarrow a)$ in the previous exercise too). This can also be proven directly with the skyscraper sheaf: Assume that φ is not surjective on stalks, i.e. there exists a $p \in X$ and $t \in \mathcal{G}_p$ not in the image of φ_p . Let $\Delta_p(\{1, 2\})$ be the skyscraper sheaf at p ; $\Delta_p(V) = \{1\}$ if $p \notin V$, and $\Delta_p(V) = \{1, 2\}$ if $p \in V$. Now define two maps $\mathcal{G} \rightarrow \Delta_p(\{1, 2\})$ by $\psi_1(U)(x) = 1$ for all U and $x \in \mathcal{F}(U)$, and $\psi_2(U)(y) = 2$ if $y_p = t$, and $\psi_2(U)(y) = 1$ otherwise (notice, this is really enough to define the map!). Now, clearly $\psi_1 \circ \varphi = \psi_2 \circ \varphi$ (since they are always 1, since t is never hit on the level of stalks), which contradicts the fact that φ was epi.

□

Exercise 2.4.O. Show that $\exp : \mathcal{O}_X \rightarrow \mathcal{O}_X^*$ describes \mathcal{O}_X^* as a quotient sheaf of \mathcal{O}_X . Find an open set on which this map is not surjective.

Solution. Let $s_p = (s, U) \in \mathcal{O}_{X,p}^*$. Possibly by shrinking U , we may find logarithm $t \in \mathcal{O}_{X,p}(U)$ satisfying $\exp(t) = s$, so the induced map on stalks is surjective.

However, on $U = \mathbb{C} \setminus \{0\}$, the function $f(z) = z$ is obviously in $\mathcal{O}_X^*(U)$, but admits no single logarithm. □

2.5 Recovering sheaves from a “sheaf on a base”

Exercise 2.5.A. How can you recover a sheaf \mathcal{F} from the partial information on a base of the topology on X .

Solution. Form again the “candidate” stalk \mathcal{F}'_p as the colimit over $\mathcal{F}(B_i)$, where B_i are the open sets of the base containing p . There is an obvious map $\mathcal{F}'_p \rightarrow \mathcal{F}_p$, just by sending a representative (s, B) to the class in \mathcal{F}_p . However, the other way around, we also have a map: Given $(t, U) \in \mathcal{F}_p$, take any basis element $B \subseteq U$ containing p (which exists since U is a union of basis elements). $(t|_B, B) = (t, U)$, and it also defines an element of \mathcal{F}'_p . These maps are obviously mutually inverse.

From the stalks, we can recover \mathcal{F} by using the sheaffication (i.e. defining $\mathcal{F}(U)$ to be the compatible germs). □

Exercise 2.5.B. Verify that $F(B) \rightarrow \mathcal{F}(B)$ is an isomorphism (here F denotes a sheaf on a base, and \mathcal{F} the corresponding sheaf of compatible germs).

Solution. We construct a map $\mathcal{F}(B) \rightarrow F(B)$. Given compatible germs $(s_p)_{p \in B}$, let $\{B_i\}$ be the open cover (of basis sets!) for which the compatible germs have corresponding sections (in $F(B_i)$). By definition, these sections agree on intersections, thus they glue to a section on $F(B)$. If we were to choose a different open cover $\{C_i\}$, then for each point, there would be a smaller set $D_i \subseteq B_i \cap C_i$ on which the representatives agree. Thus the identity axiom ensures that the two gluings must be the same.
This map is obviously an inverse to the natural map $F(B) \rightarrow \mathcal{F}(B)$. □

Exercise 2.5.C (Morphisms of sheaves correspond to morphisms of sheaves on a base). a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.
b) Show that a morphism of sheaves on the base gives a morphism of the induced sheaves.

Solution. For a), write $U = \bigcup B_i$ as a union of base open sets. Then, for $\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$, we can compute $\varphi(U)(s)$ by gluing $\varphi(B_i)(s|_{B_i})$, which must agree with $\varphi(U)(s)$ by naturality and the fact that \mathcal{G} is a sheaf (identity axiom).
For b) the morphism of sheaves on the base already determines the map on stalks, which in turn determines the morphism of sheaves (Exercise 2.4.C again!). □

Exercise 2.5.D. Suppose a morphism of sheaves of sets $F \rightarrow G$ on a base $\{B_i\}$ is surjective for all B_i . Show that the corresponding morphism of sheaves is surjective.

Solution. The corresponding map on stalks is then obviously surjective, which by Exercise 2.4.N means that the corresponding morphism of sheaves is epi. □

Exercise 2.5.E (Gluing sheaves). Suppose $X = \bigcup U_i$ is an open cover of X , and we have sheaves \mathcal{F}_i on U_i along with isomorphisms

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \rightarrow \mathcal{F}_j|_{U_i \cap U_j}$$

that agree on triple overlaps, i.e.

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

on $U_i \cap U_j \cap U_k$. Show that these sheaves can be glued together into a sheaf \mathcal{F} on X , with isomorphisms $\mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones.

Solution. We construct the sheaf \mathcal{F} defined on the basis of open sets contained in some U_i (this is clearly a basis: any open V is the union $V = \bigcup V \cap U_i$). This is then given by $\mathcal{F}(B) = \mathcal{F}_i(B)$ where $B \subseteq U_i$. This is well defined, due to the isomorphisms φ_{ij} giving $\mathcal{F}_i(B) \simeq \mathcal{F}_j(B)$ whenever $B \subseteq U_i \cap U_j$, and the cocycle condition ensures that this isomorphism is consistent, i.e. if $B \subseteq U_i \cap U_j \cap U_k$, the identification $\varphi_{ik}(B) : \mathcal{F}_i(B) \simeq \mathcal{F}_k(B)$ and $\varphi_{jk}(B) \circ \varphi_{ij}(B) : \mathcal{F}_i(B) \simeq \mathcal{F}_k(B)$ agree. The sheaf properties (on the basis) now follow from the fact that the \mathcal{F}_i are sheaves. \square

2.6 Sheaves of abelian groups, and \mathcal{O}_X -modules, form abelian categories

Exercise 2.6.A. Show that the stalk of the kernel is the kernel of the stalks: for all $p \in X$, there is a natural isomorphism

$$(\ker(\mathcal{F} \rightarrow \mathcal{G}))_p \simeq \ker(\mathcal{F}_p \rightarrow \mathcal{G}_p).$$

Solution. This is just a restatement of the fact that passing to stalks is a (left) exact functor (it is a filtered colimit by definition). \square

Exercise 2.6.B. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

Solution. First, the fact that passing to stalks is a (right) exact functor, shows that this is true for the cokernel presheaf. But, this is also true for the sheafification, because the sheafification has naturally isomorphic stalks to the presheaf you start with. \square

Exercise 2.6.C. Suppose $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\text{im } \varphi$ is the sheafification of the image presheaf. Show that the stalk of the image is the image of the stalk.

Solution. [Actually, this argument is circular I now realize, should fix].

The easiest way of proving this is actually the other way around. The fact that the stalk of the image is the image of the stalk follows directly from the fact that passing to stalks is exact, and that sheafification does not change the stalks.

Now we can prove the first statement. Let \mathcal{H} denote the image presheaf (i.e. $\ker(\mathcal{G} \rightarrow \text{coker}_{\text{pre}} \varphi)$). We want to show that the \mathcal{G} -compatible germs and the \mathcal{H} -compatible germs of

$$\prod_{x \in X} \varphi_x(\mathcal{F}_x),$$

because the former describes the image sheaf by the first statement. So let $\mu \in \prod_{x \in X} \varphi_x(\mathcal{F}_x)$. If μ is \mathcal{H} compatible, then it is obviously \mathcal{G} -compatible (since we always have $\mathcal{H}(U) \subseteq \mathcal{G}(U)$). Now, assume that μ is \mathcal{G} -compatible, i.e. that there exists an open cover U_x , and $g \in \mathcal{G}(U_x)$ such that $g_y = \mu_y$ for all $y \in U_x$. But $\mu_x \in \varphi_x(\mathcal{F}_x)$, thus there exists some open V containing x such that $f \in \varphi_V(V)$ satisfies $f_x = \mu_x$. But since f and g have the same germ at x , there must exist some open W such that $f|_W = g|_W$, and thus $f \in \varphi_W(W)$ by naturality. Picking such opens W for all x shows that μ is \mathcal{H} -compatible. \square

Exercise 2.6.D. Suppose $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ and $\beta : \mathcal{G} \rightarrow \mathcal{H}$ are two morphisms of sheaves of abelian groups on X . Show that

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is exact if and only if for all $p \in X$,

$$\mathcal{F}_p \xrightarrow{\alpha_p} \mathcal{G}_p \xrightarrow{\beta_p} \mathcal{H}_p$$

is exact.

Solution. As always, passing to stalks is exact, and hence all we need to prove is the “up” direction of the implication. So assuming the bottom sequence is exact for all $p \in X$, we need to check that $\text{im } \alpha \simeq \ker \beta$. But this isomorphism comes for free by using the definition of sections as compatible stalks. In a little more detail: we can define the image as

$$(\text{im } \alpha)(U) = \{(s_p)_p \in \prod_{p \in U} \mathcal{G}_p \mid (s_p)_p \text{ compatible, and } s_p \in \text{im } \alpha_p \text{ for all } p \in U\}.$$

Similarly, we can define

$$(\ker \beta)(U) = \{(s_p)_p \in \prod_{p \in U} \mathcal{G}_p \mid (s_p)_p \text{ compatible, and } s_p \in \ker \beta_p \text{ for all } p \in U\},$$

and thus it's obvious that exactness at all stalks imply exactness of sheaves. \square

Exercise 2.6.E. Show that taking the stalk of a sheaf of abelian groups is an exact functor. More precisely, if X is a topological space and $p \in X$ is a point, show that taking the stalk at p defines an exact functor $Ab_X \rightarrow Ab$.

Solution. We have already used this a lot, as it was proven in Exercise 1.5.L (recall, a stalk is really a filtered colimit). \square

Exercise 2.6.F. Check that the exponential exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \rightarrow 1$$

is indeed an exact sequence of sheaves of abelian groups ($X = \mathbb{C}$ with the classical topology, and \mathcal{O}_X is the sheaf of holomorphic functions).

Solution. We check this at a stalk $p \in X$: $\mathbb{Z}_p = \mathbb{Z}$, $\mathcal{O}_{X,p}$ is the abelian group of germs of holomorphic functions at p under addition, while $\mathcal{O}_{X,p}^*$ is the abelian group of germs of holomorphic functions, nonzero at p , under multiplication.

Exactness at \mathbb{Z}_p is obvious. Similarly, exactness at $\mathcal{O}_{X,p}^*$ is clear, as every nowhere vanishing holomorphic function locally admits a holomorphic logarithm. Finally, exactness at $\mathcal{O}_{X,p}$ is shown as follows: Assume $f_p \in \mathcal{O}_{X,p}$ satisfies $\exp(f_p) = 1$. This means that there is a local neighborhood U of p , with f holomorphic on U , and $\exp(f) = 1$ on U , i.e. $\exp(f(z)) = 1$ for all $z \in U$. This already shows that $f = 2n\pi i$, and thus we are done. \square

Exercise 2.6.G (Left-exactness of the functor of “sections over U ”). Suppose $U \subseteq X$ is an open set, and $0 \rightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \rightarrow 0$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U)$$

is exact. Show that the section functor need not be exact.

Solution. The fast proof is that the forgetful functor to presheaves is a right-adjoint (to the sheafification functor), and thus preserves all limits. Hence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$ is an exact sequence of presheaves, and by Exercise 2.3.G the result follows.

A direct proof: The key is that the sequence is exact for all stalks. Let $s \in \mathcal{F}(U)$, and assume $\alpha(U)(s) = 0$. Then $\alpha_p(s_p) = 0$ for all $p \in U$, implying that $s_p = 0$ for all $p \in U$, in turn implying that $s = 0$. This shows exactness at $\mathcal{F}(U)$. Next, assume $s \in \mathcal{G}(U)$ satisfies $\beta(U)(s) = 0$. Again, this implies that $s_p \in \ker \beta_p = \operatorname{im} \alpha_p$ for all $p \in U$. Thus, for each $p \in U$, we may choose opens V_p , and $t \in \mathcal{F}(V_p)$ such that $\alpha(V_p)(t) = s|_{V_p}$. To glue these t , we must check that they agree on intersections. So for all $p, q \in U$, we have

$$\alpha(V_p \cap V_q)(t_p|_{V_p \cap V_q}) = s|_{V_p \cap V_q} = \alpha(V_p \cap V_q)(t_q|_{V_p \cap V_q})$$

which implies that t_p, t_q agree on the intersection, because $\alpha(V_p \cap V_q)$ is injective! This last part is what fails for proving exactness at $\mathcal{H}(U)$.

As an example, consider indeed the example from the last exercise, and take $U = \mathbb{C} \setminus \{0\}$. As mentioned before, $f(z) = z$ does not admit a holomorphic logarithm on this set. \square

Exercise 2.6.H. Suppose $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$ is an exact sequence of abelian groups on X . If $\pi : X \rightarrow Y$ is a continuous map, show that

$$0 \rightarrow \pi_* \mathcal{F} \rightarrow \pi_* \mathcal{G} \rightarrow \pi_* \mathcal{H}$$

is exact.

Solution. The abstract non-sense proof again goes through: π_* is right adjoint to π^{-1} . \square

Exercise 2.6.I. Suppose \mathcal{F} is a sheaf of abelian groups on a topological space X . Show that $\mathcal{HOM}(\mathcal{F}, -)$ is a left-exact covariant functor $Ab_X \rightarrow Ab_X$. Show that $\mathcal{HOM}(-, \mathcal{F})$ is a left-exact contravariant functor.

Solution. We do the proof for $\mathcal{HOM}(\mathcal{F}, -)$. First we check functoriality: This follows defining for a morphism of sheaves $\varphi : \mathcal{G} \rightarrow \mathcal{G}'$, the morphism $\psi : \mathcal{HOM}(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{HOM}(\mathcal{F}, \mathcal{G}')$ by

$$\begin{aligned} \psi(U) : \mathcal{HOM}(\mathcal{F}|_U, \mathcal{G}|_U) &\rightarrow \mathcal{HOM}(\mathcal{F}|_U, \mathcal{G}'|_U) \\ \alpha &\rightarrow \varphi|_U \circ \alpha \end{aligned}$$

It is clear that this is functorial. To prove the exercise, note that it will suffice to prove

$$\mathcal{HOM}(\mathcal{F}, \mathcal{G})_x \cong \text{Hom}_{Ab}(\mathcal{F}_x, \mathcal{G}_x)$$

since exactness can be checked at the level of stalks (and the usual hom-functor is left exact). The easiest way to prove this is to remember that $\mathcal{HOM}(\mathcal{F}, \mathcal{G})$ is the sheafification of the presheaf which is defined by $U \rightarrow \text{Hom}(\mathcal{F}(U), \mathcal{G}(U))$. Now, computing the stalks of this presheaf we have done before, and the result follows.

The proof for $\mathcal{HOM}(-, \mathcal{F})$ is of course analogous. \square

Exercise 2.6.J. Show that if (X, \mathcal{O}_X) is a ringed space, then \mathcal{O}_X -modules form an abelian category.

Solution. This is mainly checking all the exercises for the previous subsection for \mathcal{O}_X -modules instead of abelian groups. \square

Exercise 2.6.K (Tensor product of \mathcal{O}_X -modules). There are two parts:

- Suppose \mathcal{O}_X is a sheaf of rings on X . Define (categorically) what we should mean by tensor product of two \mathcal{O}_X -modules. Give an explicit construction, and show that it satisfies your categorical definition.
- Show that the tensor product of stalks is the stalk of the tensor product.

Solution. There is a cool idea to define it by the representable functor

$$\mathcal{HOM}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathcal{HOM}(\mathcal{F}, \mathcal{HOM}(\mathcal{G}, \mathcal{H})),$$

which would probably be the categorically most pleasing definition.

An explicit description can be given by first defining the presheaf $U \rightarrow \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. This is indeed a presheaf: For $V \subseteq U$, restrictions $\rho_{\mathcal{F}}$ and $\rho_{\mathcal{G}}$ give a map $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \rightarrow \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{G}(V)$ by $f \otimes g \rightarrow \rho_{\mathcal{F}}(f) \otimes \rho_{\mathcal{G}}(g)$. Define $\mathcal{F} \otimes \mathcal{G}$ as the sheafification of this presheaf.

We will now prove b) before finishing a): We compute the stalks by the presheaf, so $(\mathcal{F} \otimes \mathcal{G})_x := \varinjlim_{U \ni x} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. The map to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ is as usual obtained by taking any section and sending it to its representative in the stalk. As usual, this is obviously surjective, but we need to show injectivity: To show this, note that if f is zero in $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$, then that, by definition, means that there exists some V so that $f|_V$ is zero, and thus f was zero in the limit already.

Now look at any map $\mathcal{F} \otimes \mathcal{G} \rightarrow \mathcal{H}$. On all stalks, we get

$$\begin{aligned} \mathcal{HOM}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H})_x &\cong \text{Hom}_{\mathcal{O}_{X,x}}((\mathcal{F} \otimes \mathcal{G})_x, \mathcal{H}_x) \\ &\cong \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x, \mathcal{H}_x) \\ &\cong \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{H}_x)) \\ &\cong \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{HOM}(\mathcal{G}, \mathcal{H})_x) \\ &\cong \mathcal{HOM}(\mathcal{F}, \mathcal{HOM}(\mathcal{G}, \mathcal{H}))_x \end{aligned}$$

thus the final result is that $\mathcal{HOM}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathcal{HOM}(\mathcal{F}, \mathcal{HOM}(\mathcal{G}, \mathcal{H}))$ which is what we wanted to show. \square