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This note contains the basic definitions of sheaves and schemes. Its mainly for me to look up. “Proofs” where they are written should be understood as “Proof idea”, not actual proofs.

# 1 Preliminary definitions

## 1.1 Affine Varieties

This sections covers some of the very basics of affine varieties. They mainly serve as motivating examples.

In this first subsection,  $k$  is assumed to be algebraically closed. Importantly, we will not assume that for the rest of the note.

### Definition 1.1: Algebraic set

An **algebraic set** is a set of the form

$$Z(S) = \{x \in \mathbb{A}^n(k) \mid f(x) = 0, \forall f \in S\}$$

where  $S \subset k[x_1, \dots, x_n]$  (or equivalently,  $\langle S \rangle$ ).

Inclusions of ideals turns into flipped inclusions of algebraic sets. A bit more thoroughly, we have

**Proposition 1.1.1.** *For ideals  $\mathfrak{a}, \mathfrak{b}$ , and  $(\mathfrak{a}_i)_{i \in I}$ , we have*

- $\mathfrak{a} \subset \mathfrak{b} \Rightarrow Z(\mathfrak{a}) \supset Z(\mathfrak{b})$ .
- $Z(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} Z(\mathfrak{a}_i)$
- $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$
- $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$

*Proof.* All of these are immediate from the definition. □

Thus, we can define the Zariski topology as the topology (on  $\mathbb{A}^n(k)$ ), where the basis of closed sets are given by  $Z(\mathfrak{a})$  where  $\mathfrak{a}$  varies over the ideals of  $k[x_1, \dots, x_n]$ .

In fact the above proposition is made even stronger by the famous

**Theorem 1.1.2** (Hilbert’s Nullstellensatz). *Let  $k$  be algebraically closed. Then there is a inclusion-reversing bijection*

$$\begin{aligned} \{\text{radical } \mathfrak{a} \subset k[x_1, \dots, x_n]\} &\leftrightarrow \{\text{closed sets } X \subset \mathbb{A}^n(k)\} \\ \mathfrak{a} &\mapsto Z(\mathfrak{a}) \\ I(X) &\leftarrow X \end{aligned}$$

where  $I(X) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0, \forall x \in X\}$ .

*Proof.* The inclusion-reversion was already from proposition 1.1.1, so bijection is shown by showing  $Z(I(X)) = X$  and  $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ . The first is straight from the definition (remember  $X$  is closed!), and showing  $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$  is also trivial. In fact, these three inclusions hold even for non-algebraically closed fields. The last is a bit more involved, and involves showing the **weak nullstellensatz**, saying that every maximal ideal of  $k[x_1, \dots, x_n]$  is of the form  $(x_1 - a_1, \dots, x_n - a_n)$ . This obviously only holds for algebraically closed fields.  $\square$

Definitions for affine varieties varies a bit, but typically (and I prefer this too), they should be irreducible, so we should probably define

#### Definition 1.2: Irreducible

A topological space  $X$  is **irreducible** if it cannot be written as a union of two proper closed subsets.

Which in turn allows

#### Definition 1.3: Affine Variety

An **affine variety** is an irreducible algebraic set.

This proposition shows what irreducible (and hence affine varieties) means in the algebraic sense.

**Proposition 1.1.3.** *An algebraic set is irreducible if and only if  $I(X)$  is a prime ideal.*

*Proof.* Obvious from definition.  $\square$

In general, any algebraic set can be written as a finite union of irreducible components. This is straight from the **Lasker-Noether**-theorem (essentially just translated using previous definitions).

The most important invariant of an affine variety is

#### Definition 1.4: Coordinate ring and Function Field

Given an affine variety  $X \subset \mathbb{A}^n(k)$ , we define its **coordinate ring** to be

$$A(X) = k[x_1, \dots, x_n]/I(X).$$

Further, since (by definition), this is an integral domain, we can also define its **function field** to be

$$k(X) = \text{frac} A(X).$$

. The coordinate ring allows us to directly translate many notions between geometry and algebra.

- Radical ideals of  $A(X)$  correspond to closed subsets of  $X$
- Prime ideals of  $A(X)$  correspond to irreducible algebraic sets of  $X$  (subvarieties).
- Maximal ideals of  $A(X)$  correspond to points of  $X$
- We **define** the dimension of  $X$  to be the Krull dimension of  $A(X)$ .

Actually, you can define the “dimension” of any topological space  $X$  as the longest chain of irreducible closed proper subsets of  $X$ , but its not used a lot outside of geometry apparently.

Another thing thats worth pointing out is that for any non-constant  $f \in k[x_1, \dots, x_n]$ , we have that  $\dim Z(f) = \dim k[x_1, \dots, x_n]/\langle f \rangle = n - 1$  by **Krull’s principal ideal theorem**.

We now study regular and rational functions in more detail.

#### Definition 1.5: Regular function

A function  $f \in k(X)$  is said to be regular at  $P \in X$  if  $f$  can be written as  $f = a/b$  with  $a, b \in A(X)$ , and  $b(P) \neq 0$ .

Note that unless  $A(X)$  is a UFD, the representation  $f = a/b$  is not unique! Hence, we might have to choose different representations for different points. Thus, while  $f = a/b$  is very obviously regular on all of  $D(b)$ , it might actually be regular on an even larger set.

Note that if  $f$  is regular on  $U \subset X$ ,  $f$  is really a “function” in the traditional sense,

$$\begin{aligned} f : U &\rightarrow k \\ f(P) &= a(P)/b(P), \end{aligned}$$

where this value can be shown to be independent of representation.

For any open set  $U \subset X$ , we define

$$\mathcal{O}_X(U) = \{f \in k(X) \mid f \text{ is regular on } U\},$$

i.e. functions without poles on  $U$ . For open sets of the form  $D(f)$ , it is immediately clear that

$$\mathcal{O}_X(D(f)) = A(X)_f.$$

The **local ring** at  $P \in X$  is more or less the same definition

$$\mathcal{O}_{X,P} = \{f \in k(X) \mid f \text{ is regular at } P\}.$$

Similarly to before, it is clear that for  $\mathfrak{m}_P$ , the maximal ideal corresponding to  $P$ , we have

$$\mathcal{O}_{X,P} = A(X)_{\mathfrak{m}_P},$$

which is clearly a local ring.

**Example 1.1.4.** *I see this example everywhere, so its probably worth writing down. Let  $U = \mathbb{A}^2(k) - \{(0,0)\}$ . Then  $\mathcal{O}_{\mathbb{A}^2(k)}(U)$  is simply the restriction of a polynomial function (i.e. it doesn't contain anything more than  $A(\mathbb{A}^2(k))$ ). To show this, write  $f \in \mathcal{O}_{\mathbb{A}^2(k)}(U)$  as  $f = a/b$ . This representation is unique since  $k[x, y]$  is a UFD, and hence  $f$  is not regular on  $Z(b)$ . But there exists no non-constant  $b$  so that  $Z(b) \subset \{(0,0)\}$ , since  $Z(b)$  is a variety of dimension 1 (again by Krull's principal ideal theorem).*

Next, we define morphisms. We need one preliminary definition, which will anyway be very useful.

**Definition 1.6: Pullback**

Let  $X, Y$  be affine varieties, and let  $U \subset X, V \subset Y$  be open sets. Given any continuous map  $f : U \rightarrow Y$ , and any  $g \in \mathcal{O}_Y(V)$ , we define the **pullback** to be

$$f^\#(g) = g \circ f.$$

This allows us to define

**Definition 1.7: Morphism of varieties**

A **morphism** is a continuous map  $f : U \rightarrow Y$  such that the pullback sends regular functions to regular functions, i.e.

$$f^\# : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(f^{-1}(V))$$

is well defined.

This definition sounds a bit complicated (later we will see that this is really the definition of a morphism of schemes, and that this condition just says that the function induces a map of sheaves). However, for affine varieties, we have the following much simpler equivalent definition coming from this

**Theorem 1.1.5.** *Let  $X$  be an affine variety, and let  $U \subset X$  be an open set. The morphisms*

$$f : U \rightarrow \mathbb{A}^n(k)$$

*are exactly the maps of the form  $f(P) = (f_1(P), \dots, f_n(P))$  for  $f_i \in \mathcal{O}_X(U)$ .*

*Proof.* Direct from pulling back the coordinate functions on  $\mathbb{A}^n(k)$ . □

**Corollary 1.1.6.** *Let  $X \subset \mathbb{A}^n(k), Y \subset \mathbb{A}^m(k)$  be affine varieties. Then any morphism  $f : X \rightarrow Y$  is the restriction of a morphism of the form*

$$\begin{aligned} h : \mathbb{A}^n(k) &\rightarrow \mathbb{A}^m(k) \\ h(P) &= (h_1(P), \dots, h_m(P)), \end{aligned}$$

*where  $h_i \in A(\mathbb{A}^n(k)) = k[x_1, \dots, x_n]$ .*

Thus, maps of varieties and of finitely generated  $k$ -algebras are really the same thing. To summarise this whole part, we have a (contravariant) equivalence of categories between affine varieties and finitely generated  $k$ -algebras (that are integral domains, though this can be dropped if you drop the irreducibility condition on affine varieties I think?), where the objects, i.e. the varieties are sent to their coordinate rings, and morphisms are sent to the pullback homomorphism.

## 1.2 The Spectrum

We want to generalise the stuff in the last section from finitely generated  $k$ -algebras to general rings  $A$ . The biggest change we need is to consider the spectrum of the ring, i.e.

$$\operatorname{Spec} A = \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ prime ideal}\}$$

Most of the stuff in the previous section, i.e. the Zariski topology and surrounding results transfer directly; you know this so I won't bother writing it down.

One thing that is worth pointing out is that we still want to consider elements of  $A$  as functions. In general, these functions take values in *different fields*, unlike in the variety case.

### Definition 1.8: Residue field

Let  $\mathfrak{p} \in \operatorname{Spec} A$ . The **residue field** at  $\mathfrak{p}$ , denoted  $\kappa(\mathfrak{p})$  is defined as the field of fractions of  $A/\mathfrak{p}$ .

Elements  $f \in A$  is then understood to take value in  $\kappa(\mathfrak{p})$  at  $\mathfrak{p}$ , and this value is simply the value of  $f \bmod \mathfrak{p}$ . Consider affine varieties, to see how this generalises polynomial functions taking values at maximal ideals.

Thus, the zero set of  $f \in A$  can again be defined as

$$V(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p})\},$$

where  $f(\mathfrak{p})$  really means  $f \bmod \mathfrak{p}$ . Of course, this is just the same as the “usual” definition, primes containing  $f$ .

Of course, points need not be closed in the zariski topology. From the bijection between radical ideals and closed sets of  $\operatorname{Spec} A$ , we get the

**Corollary 1.2.1.** *A point  $\mathfrak{p} \in \operatorname{Spec} A$  is closed if and only if  $\mathfrak{p}$  is maximal.*

Points which fail to be closed in the biggest way, have the following

### Definition 1.9: Generic point

A point  $\mathfrak{p} \in \operatorname{Spec} A$  is called a **generic point** if  $\overline{\{\mathfrak{p}\}} = \operatorname{Spec} A$ .

(Of course the definition above works for any topological space). The arch-typical example of a generic point is of course  $(0) \subseteq \mathbb{Z}$ .

We now discuss some more typical topological notions to study  $\text{Spec } A$ . Recall that a topological space is called **compact** if every open cover admits a finite subcover. For some weird historical reason, this property is called **quasi-compact** in AG...

Recall also that we have the so-called **distinguished open sets**  $D(f)$  for  $f \in A$

$$D(f) = \{\mathfrak{p} \in \text{Spec } A \mid f(\mathfrak{p}) \neq 0\} = \text{Spec } A - V(f).$$

These form an (open) basis of the Zariski topology. For any ring  $A$ , we know that if  $(D(f_i))_{i \in I}$  form an open cover of  $A$ , we know that  $\bigcap_{i \in I} V(f_i) = \emptyset$ , which again implies that the  $f_i$  generate the unit ideal, i.e. we can write  $1 = \alpha_1 f_1 + \dots + \alpha_r f_r$ , which in turn means that a finite sub-cover will do.

Immediately from this, we get

**Proposition 1.2.2.**  *$\text{Spec } A$  is quasi-compact.*

*Proof.* Given an open cover, cover each open set by a distinguished open set (since these form a basis). By the above, finitely many distinguished opens will suffice, and hence also finitely many of the original open cover.  $\square$

Compare the following notion with the definition of **irreducible** given before.

#### Definition 1.10: Connected

A topological space  $X$  is called **connected** if it cannot be written as the disjoint union of two proper open (or closed) subsets.

In the above definition, we can of course consider either open or closed sets by taking their complements.

It's immediately clear that irreducible implies connected (if it cannot be written as union of two proper closed subsets, it certainly cannot be written as a disjoint union), however the converse is not true. From before we already have that  $\text{Spec } A$  is irreducible if and only if the nilradical  $\sqrt{(0)}$  is prime (combine proposition 1.1.3 and the general correspondence between closed sets and radical ideals). For connected, we have the following

**Proposition 1.2.3.**  *$\text{Spec } A$  is disconnected if and only if  $A \simeq A_1 \times A_2$  for non-trivial rings  $A_1, A_2$ .*

*Proof.* See later with sheaves?  $\square$

The reason we consider prime ideals instead of only maximal ideals is because the inverse image of a prime ideal under a homomorphism is again a prime ideal (not true for maximal ideals). Thus a homomorphism  $\varphi : \text{Spec } A \rightarrow \text{Spec } B$  induces a **continuous** map

$$\begin{aligned} f : \text{Spec } B &\rightarrow \text{Spec } A \\ f(\mathfrak{p}) &= \varphi^{-1}(\mathfrak{p}) \end{aligned}$$

About this induced map, we have the following

**Proposition 1.2.4.** *Let  $f, \varphi$  be as before. Then*

- *for  $\mathfrak{a} \subset A$ , we have  $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a})B)$*
- *for  $g \in A$ , we have  $f^{-1}(D(g)) = D(\varphi(g))$*
- *for  $\mathfrak{b} \subset B$ , we have  $\overline{f(V(\mathfrak{b}))} = V(\varphi^{-1}(\mathfrak{b}))$*

*Proof.* All of these are really direct from unraveling the definitions (though it is a bit mind boggling with all the inverse of inverse maps).  $\square$

We also have

**Proposition 1.2.5.** *Let  $\varphi : A \rightarrow A/\mathfrak{a}$  be the projection map. The induced map of spectra induces a homeomorphism*

$$f : \operatorname{Spec} A/\mathfrak{a} \xrightarrow{\sim} V(\mathfrak{a}) \subseteq \operatorname{Spec} A$$

*Proof.* Just check definitions (continuous bijection is immediate, homeomorphic can also be showed by checking that the map is closed).  $\square$

As a special case, we get that  $\operatorname{Spec} A_{\text{red}} \simeq \operatorname{Spec} A$ , where  $A_{\text{red}} = A/\sqrt{(0)}$  is the reduction of  $A$ .

For maps  $\varphi : A \rightarrow B$ , the proposition above shows that the induced map gives a homeomorphism between  $\operatorname{Spec} B$  and  $V(\ker \varphi)$ . We can also study injective map, and get the following proposition.

**Proposition 1.2.6.** *Let  $\varphi, f$  be as before. Then  $f(\operatorname{Spec} B)$  is dense in  $\operatorname{Spec} A$  if and only if  $\ker \varphi \subseteq \sqrt{(0)}$ .*

*Proof.*  $f(\operatorname{Spec} B)$  being dense, is equivalent to

$$\overline{f(\operatorname{Spec} B)} = \overline{f(V(0))} = V(\varphi^{-1}(0)) = \operatorname{Spec} A$$

where the first equality is obvious and the second is from proposition 1.2.4. Thus we have that  $f(\operatorname{Spec} B)$  is dense if and only if  $\varphi^{-1}(0) = \ker \varphi \subseteq \sqrt{(0)}$ .  $\square$

**Corollary 1.2.7.** *If  $\varphi$  is injective, then  $f(\operatorname{Spec} B)$  is dense in  $\operatorname{Spec} A$ .*

This allows us to say something nice about localisation maps.

**Proposition 1.2.8.** *Let  $\varphi : A \rightarrow S^{-1}A$  be a localisation map (so  $S$  is a multiplicatively closed subset of  $A$ ). Then the induced map of spectra is a homeomorphism between  $\operatorname{Spec} S^{-1}A$  and  $D = \{\mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset\}$  (a set which is dense in  $A$ ).*

*Proof.* Pretty immediate, at least if you recall some results about localisation from commutative algebra.  $\square$

**Corollary 1.2.9.** *For any  $f \in A$ , there is a canonical homeomorphism*

$$\operatorname{Spec} A_f \simeq D(f).$$

The last thing to study is the fibers of the induced map. Actually, this section is a hot mess in the book, so eh... Anyway, for closed points, the preimage is easy by proposition 1.2.4, but you can even describe it accurately in the general case.

There is a good exercise, which gives the following proposition, which is taken directly from the answer to <https://math.stackexchange.com/questions/106043/rings-whose-spectrum-is-ha>

**Proposition 1.2.10.** *The following are equivalent:*

- $A$  is zero-dimensional
- $\operatorname{Spec} A$  has all points closed
- $\operatorname{Spec} A$  is Hausdorff

*If  $A$  is noetherian, the above is also equivalent to all of the following:*

- $A$  is artinian.
- The Zariski topology on  $\operatorname{Spec} A$  is the discrete topology
- $\operatorname{Spec} A$  is finite, and the Zariski topology on  $\operatorname{Spec} A$  is the discrete topology.

*Proof.* Should prove this. □

### 1.3 Sheaves

A sheaf should be seen as a categorical generalisation of the notion of “functions on a space”.

#### Definition 1.11: Presheaf

A **presheaf**  $\mathcal{F}$  (on  $\mathcal{C}$ ) of abelian groups (rings, modules or whatever) is simply a contravariant functor from a category  $\mathcal{C}$  to abelian groups (rings, modules or whatever).

Typically  $\mathcal{C}$  will be the category of open sets of a topological space  $X$  (in which case we call  $\mathcal{F}$  a presheaf on  $X$ ). But later, when might define “Grothendieck topologies”, and sites, in which case  $\mathcal{C}$  can really be any category.

#### Definition 1.12: Sheaf

A **sheaf**  $\mathcal{F}$  is a presheaf such that for all  $U \subseteq X$  and open coverings  $\{U_i\}$  of  $U$ , the sequence

$$0 \rightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact.

In the definition above, the map  $\alpha$  is the obvious map given by component-wise restriction (i.e.  $\alpha(s) = \prod_i s|_{U_i}$ ), while  $\beta$  is given by the difference map, i.e.

$$\beta((s_i)_i) = (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})_{i,j}.$$

This is equivalent to the usual definition: Locality is equivalent to exactness at  $\mathcal{F}(U)$ , while gluing is exactness at  $\prod_i \mathcal{F}(U_i)$ . This definition generalises better to Grothendieck sites.

If you want to be really general,  $\beta$  can be replaced by two arrows (both restriction), and the sequence is no longer exact, but should rather state that  $\mathcal{F}(U)$  is the equalizer of these two maps. This might be necessary if your sheaf takes values in a category for which the operator  $-$  makes no sense (e.g., category of sets). If your category does not even have products, then just run away.

Note that morphisms of (pre)sheaves are simply natural transformations between them as functors.

**Example 1.3.1.** *A nice example of a presheaf which is not a sheaf, is the sheaf of holomorphic functions on  $X = \mathbb{C} \setminus \{0\}$  which has a square root. The function  $f(z) = z$  is obviously holomorphic, and for each point  $x \in X$  it has a neighbourhood in which the square root function is defined, but there is no square root function on all of  $X$ , so these functions do not glue.*

A big point in geometry is that the locally ringed spaces (key objects which will be defined later, see Definition 3.3) are very local in nature.

#### Definition 1.13: Stalks and germs

Let  $\mathcal{F}$  be a sheaf on  $X$ . The **stalk** at  $x \in X$  is defined as

$$\mathcal{F}_x := \varinjlim_{x \in U \subset X} \mathcal{F}(U).$$

Let  $x \in U$ . Of course, there is a surjective map

$$\mathcal{F}(U) \rightarrow \mathcal{F}_x,$$

and given a section  $s \in \mathcal{F}(U)$ , we denote the image by  $s_x$  and refer to it as the **germ** of  $s$  at  $x$ .

The stalks  $\mathcal{F}_x$  can be understood as equivalence classes of elements  $(U, s)$  where  $s \in \mathcal{F}(U)$  and  $x \in U$ , where  $(U, s) \sim (V, t)$  if there exists some open  $W \subset U \cap V$  such that  $s$  and  $t$  agree on  $W$ . Thus it is clear that two sections  $s = t$  if and only if  $s_x = t_x$  for all  $x$  (this follows from locality).

Passing to stalks is functorial.

**Proposition 1.3.2.** *Let  $\mathcal{F}, \mathcal{G}$  be presheaves on  $X$ , and let  $\alpha : \mathcal{F} \rightarrow \mathcal{G}$  be a morphism of presheaves (a natural transformation). Then, for any  $x \in X$  there is an induced map on stalks*

$$\begin{aligned}\alpha_x : \mathcal{F}_x &\rightarrow \mathcal{G}_x \\ \alpha_x(s_x) &= (\alpha(U)(s))_x\end{aligned}$$

where  $(U, s)$  is any representative of  $s_x$ .

Thus, passing to stalks defines a functor from the category of sheaves on  $X$  to the category of Abelian groups (rings, modules or whatever the sheaves take values in).

Next, we look at some induced sheafs by maps.

**Definition 1.14: Pushforward sheaf and inverse image sheaf**

Let  $(X, \mathcal{F})$  be a topological space with some sheaf, and same with  $(Y, \mathcal{G})$ , and consider a continuous map

$$f : X \rightarrow Y.$$

The **pushforward sheaf** is the sheaf on  $Y$  defined as

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)),$$

while the **inverse image sheaf** is the sheaf on  $X$  defined as

$$f^{-1}\mathcal{G}(U) = \lim_{f(U) \subseteq W} \mathcal{G}(U).$$

**Example 1.3.3.** *A typical statement is that “the stalk at  $x$  is simply the inverse image sheaf of the inclusion map  $x \hookrightarrow X$ ”.*

*Though, one thing that is a bit confusing about this statement is that it is slightly inaccurate. Because the stalk  $\mathcal{F}_x$  is really an object in whatever category  $\mathcal{F}$  takes values in, while the inverse image sheaf is another sheaf. But the space  $\{x\}$  only has one open set (apart from the empty set of course), and hence is entirely determined by its global sections. So to be very pedantic, the accurate statement is that “the stalk at  $x$  is simply the **global sections** of the inverse image sheaf of the inclusion map”.*

The inverse image sheaf has two important properties:

**Proposition 1.3.4.** *Let  $(X, \mathcal{F}), (Y, \mathcal{G})$  and  $f$  be as before. The inverse image sheaf satisfies*

$$f^{-1}\mathcal{G}_x = \mathcal{G}_{f(x)}.$$

*In particular, if  $X \subseteq Y$  is an open subset, and  $f$  is the inclusion of  $X$  into  $Y$ , then*

$$f^{-1}\mathcal{G} = \mathcal{G} \mid_X$$

*Proof.* Direct from definitions. □

## 2 Sites

We define Grothendieck topologies, and we give a summary of the main ones.

### 3 Basic definitions related to Schemes

In this chapter we will define schemes, and their morphisms. A scheme is a “locally ringed space”, which locally is isomorphic to an “affine scheme”. Thus a natural place to start is by studying affine schemes (which are essential to understand how schemes work in general).

#### 3.1 Affine schemes

Let  $A$  be a ring. We already know how to construct a topological space out of  $A$ , namely  $\text{Spec } A$ . Now we need to construct the structure sheaf.

Recall that on  $\text{Spec } A$ , a basis is given by the distinguished open sets  $D(f)$  where  $f$  runs over the elements of  $A$ . It's pretty obvious that a sheaf can be defined on a basis (i.e. a sheaf is uniquely determined by what it does to a basis), though slightly tedious to prove, so we have skipped it.

##### Definition 3.1: Structure sheaf

The **structure sheaf**  $\mathcal{O}_{\text{Spec } A}$  on  $\text{Spec } A$  is given (on the basis of distinguished open sets) by

$$\mathcal{O}_{\text{Spec } A}(D(f)) = A_f.$$

For  $D(g) \subseteq D(f)$ , the restriction map is given by the localization map  $A_f \rightarrow A_g$ .

The fact that this is a sheaf is really something that should be proved, but we omit the proof, as it is straight forward, since it just comes down to proving the exactness of the sequence

$$0 \rightarrow A_f \rightarrow \prod_i A_{f_i} \rightarrow \prod_{i,j} A_{f_i f_j}$$

and using general properties of sheaves defined on a basis. The stalks of  $\mathcal{O}_{\text{Spec } A, x}$  are also easily described by the following

**Proposition 3.1.1.** *Let  $x \in \text{Spec } A$  be a point corresponding to the prime ideal  $\mathfrak{p}$ . Then*

$$\mathcal{O}_{\text{Spec } A, x} = A_{\mathfrak{p}}.$$

*Proof.* The stalk may be computed by distinguished open sets. Thus

$$\mathcal{O}_{\text{Spec } A, x} = \varinjlim_{x \in D(f)} \mathcal{O}_{\text{Spec } A}(D(f)) = \varinjlim_{f \notin \mathfrak{p}} A_f$$

which is visibly  $A_{\mathfrak{p}}$ . □

The fact that the stalks of the structure sheaf is a local ring partially motivates the definition in the next section of locally ringed spaces. Even though we haven't seen the definition yet, we still put the following definition here:

### Definition 3.2: Affine scheme

An **affine scheme** is a locally ringed space (Definition 3.3) which is isomorphic (also really needs Definition 3.4) to  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$  for some ring  $A$ .

The previous definition is kind of premature, but future me will have to forgive me (I think its suitable here). The fact that  $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$  even is a locally ringed space is presumably something that should be proved, but it is completely obvious, more or less by construction.

## 3.2 Ringed Spaces

As mentioned in the intro, schemes are locally ringed spaces with a certain extra property. Thus, we need a working understanding of locally ringed spaces.

### Definition 3.3: Locally ringed space

A locally ringed space  $(X, \mathcal{O}_X)$  is a topological space  $X$ , together with a sheaf of rings  $\mathcal{O}_X$ , such that the stalk at  $x$ ,  $\mathcal{O}_{X,x}$  are local rings for all  $x \in X$ .

Next we define what we mean by morphisms of locally ringed spaces.

### Definition 3.4: Morphisms of locally ringed spaces

Let  $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$  be locally ringed spaces. A **morphism of locally ringed spaces** is a pair

$$(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y),$$

where  $f$  is a continuous map and

$$f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$$

is a morphism of sheaves respecting the local nature, i.e. for every  $x \in X$ , the map

$$f_x^\# : \mathcal{O}_{Y, f(x)} \rightarrow (f_* \mathcal{O}_X)_{f(x)} \cong \mathcal{O}_{X, x}$$

is a local map of rings, i.e.  $f_x^\#(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$ .

Two especially important morphisms are open and closed immersions.

### Definition 3.5: Open and closed immersion

Let

$$(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$$

be a morphism of locally ringed spaces. We say that  $f$  is an **open immersion** if  $f$  is a homeomorphism to an open subset of  $Y$ , and  $f^\# \mathcal{O}_Y \cong f_* \mathcal{O}_X$  is an isomorphism. Similarly, we say that  $f$  is a **closed immersion** if  $f$  is a homeomorphism to a closed subset of  $Y$  and  $f^\#$  is surjective.

Both properties on the sheaves can be replaced by saying that  $f_x^\#$  should be isomorphisms (resp. surjections) for all  $x \in X$ .

It should be pointed out that I think it is reasonable to argue that closed immersions are “more important” than open immersions.

**Example 3.2.1.** *An example of an open immersion is of course taking any subscheme, for instance  $\mathbb{A} \setminus \{0\} \hookrightarrow \mathbb{A}$ . An example of a closed immersion would be any affine variety sitting inside affine space.*

## 3.3 Schemes

Finally, we can define a scheme. In fact, we already did it in the beginning (A locally ringed space, which is locally affine), but we can now make this accurate.

### Definition 3.6: Scheme

A **scheme**  $(X, \mathcal{O}_X)$  is a locally ringed space, for which there exists an open cover  $\{U_i\}_{i \in I}$  of  $X$  such that  $(U_i, \mathcal{O}_X|_{U_i})$  is an affine scheme.

We immediately define open and closed subschemes. Closed subschemes are a bit more involved, because restricting a sheaf to a closed subset does not make sense, unless the set is also open.

### Definition 3.7: Open subscheme

Let  $(X, \mathcal{O}_X)$  be a scheme, and let  $U \subset X$  be an open subset. We call the scheme  $(U, \mathcal{O}_X|_U)$  an **open subscheme** of  $X$ . If  $(U, \mathcal{O}_X|_U)$  is an affine scheme, we call it an affine open subset.

**Definition 3.8: Closed subscheme**

Let  $(X, \mathcal{O}_X)$  be a scheme, let  $Z \subset X$  be a closed subset, and let  $(Z, \mathcal{O}_Z)$  be a scheme. Then  $(Z, \mathcal{O}_Z)$  is a **closed subscheme** of  $X$  if there is a closed immersion

$$(j, j^\#) : (Z, \mathcal{O}_Z) \rightarrow (X, \mathcal{O}_X),$$

where  $j : Z \hookrightarrow X$  is the canonical injection.

We have already defined residue fields (Definition 1.8) for affine schemes, but this is easily extended to general locally ringed spaces as

$$\kappa(x) = \mathcal{O}_{X,x} / \mathfrak{m}_x.$$

However, for schemes these again have a simpler description (of course):

**Proposition 3.3.1.** *Let  $(X, \mathcal{O}_X)$  be a scheme, and let  $x \in X$  be contained in an affine open  $U = \text{Spec } A$ , such that  $x$  corresponds to  $\mathfrak{p} \subset A$ . Then*

$$\mathcal{O}_{X,x} = A_{\mathfrak{p}}, \quad \mathfrak{m}_x = \mathfrak{p}A_{\mathfrak{p}},$$

and thus  $\kappa(x) = A_{\mathfrak{p}} / \mathfrak{p}A_{\mathfrak{p}}$ .

We do not have to give a new definition for morphisms of schemes; they are simply morphisms of locally ringed spaces.

**3.4 Affine Schemes - Revisited**

We study affine schemes a bit more, with the goal of understanding that the category of rings is fully faithfully mapped into a subcategory of schemes, and the image are precisely the affine schemes.

**Proposition 3.4.1.** *Let  $\varphi : A \rightarrow B$  be a ring-homomorphism. Then there exists a morphism of schemes*

$$(f_\varphi, f_\varphi^\#) : \text{Spec } B \rightarrow \text{Spec } A$$

such that  $f_\varphi^\#(\text{Spec } A) = \varphi$ .

*Proof.* The continuous map  $f_\varphi$  is standard from commutative algebra. The map of sheaves  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  comes from the fact that  $\varphi$  induces maps

$$\begin{aligned} A_f &\rightarrow B_{\varphi(f)} \\ a/f^n &\rightarrow \varphi(a)/\varphi(f)^n \end{aligned}$$

for all  $f \in A$ . To see that the locality condition is satisfied, simply notice that if  $a$  (as in the inline above) is in  $\mathfrak{q} \in \text{Spec } A$ , then  $\varphi(a) \in \varphi^{-1}(\mathfrak{q}) \in \text{Spec } B$ , thus  $\mathfrak{m}_{\mathfrak{q}}$  is sent to  $\mathfrak{m}_{\varphi^{-1}(\mathfrak{q})}$ .  $\square$

When looking at affine schemes, this association is bijective.

**Proposition 3.4.2.** *The association  $\varphi \rightarrow f_\varphi$  induces a bijection*

$$\mathrm{Hom}(A, B) \leftrightarrow \mathrm{Hom}(\mathrm{Spec} A, \mathrm{Spec} B)$$

*which shows that the category of commutative rings with a unit is (equivalent to) a full subcategory of the category of schemes.*

*Proof.* One can show that the association is functorial, and explicitly construct an inverse.  $\square$

We summarize in

**Theorem 3.4.3.** *The two functors*

$$\begin{aligned} \mathrm{Spec} : \mathrm{Rings} &\rightarrow \mathrm{AffSchemes} \\ A &\rightarrow \mathrm{Spec} A \end{aligned}$$

*and*

$$\begin{aligned} \Gamma : \mathrm{AffSchemes} &\rightarrow \mathrm{Rings} \\ (X, \mathcal{O}_X) &\rightarrow \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X) \end{aligned}$$

*Are mutually inverse contravariant equivalence of categories.*

We didn't define the global sections functor  $\Gamma$  yet (it gives nothing new, just emphasizes the functoriality), but we will keep using the notation different places too since it is standard (the fact that it is functorial is by definition).

Two quick results on closed immersions of affine schemes before moving on. They essentially follow from the equivalence of categories, but explicitly they say that all taking quotients in the category of rings corresponds to closed immersions of affine schemes, and that all closed immersions of affine schemes arise this way:

**Lemma 3.4.4.** *Let  $A$  be a ring and  $I$  an ideal. The canonical projection  $\pi : A \rightarrow A/I$  induces a closed immersion*

$$(f_\pi, f_\pi^\#) : \mathrm{Spec} A/I \rightarrow \mathrm{Spec} A,$$

*where the image of  $f_\pi$  is  $V(I)$ .*

*Proof.* Unravel definitions.  $\square$

**Proposition 3.4.5.** *Let  $X = \mathrm{Spec} A$  be an affine scheme, and let  $j : Z \rightarrow X$  be a closed immersion. Then  $Z$  is affine and isomorphic to  $\mathrm{Spec} A/I$  for some ideal  $I$ .*

*Proof.* See Liu (Proposition 3.20).  $\square$

### 3.5 Relative Schemes

We now define relative schemes. Relative schemes are absolutely essential for working with schemes over non-algebraically closed fields, which is one of our main motivations (the language of varieties becomes a bit messed up when not working over algebraically closed fields).

#### Definition 3.9: Scheme over $S$

Given a scheme  $S$ , a **scheme over  $S$**  (also called a  $S$ -scheme) is another scheme  $X$ , together with a morphism  $X \rightarrow S$ . If  $A$  is a ring, we similarly define a scheme over  $A$  as a scheme  $X$  with a morphism  $X \rightarrow A$ . These are often denoted  $X/S$ . A morphism of  $S$ -schemes,  $\varphi : X/S \rightarrow Y/S$  is a morphism of schemes that respects the structure maps, i.e. the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow & \swarrow \\ & S & \end{array}$$

**Example 3.5.1.** *Probably our most standard example is a  $k$ -scheme  $X/k$  for some field  $k$ . Let  $\pi : X \rightarrow \operatorname{Spec} k$  be the structure map. In this case, we have that the sheaf  $\mathcal{O}_X$  is in fact a sheaf of  $k$ -algebras. To see this, notice: the sheaf map*

$$\pi^\# : \mathcal{O}_{\operatorname{Spec} k} \rightarrow \pi_* \mathcal{O}_X$$

*is by definition (since  $\operatorname{Spec} k$  only has one point) a map  $k \rightarrow \mathcal{O}_X(X)$ , thus the global sections of  $\mathcal{O}_X$  form a  $k$ -algebra. By restriction, all  $\mathcal{O}_X(U)$  are thus also  $k$ -algebras (for any ring-homomorphism  $A \rightarrow B$ , if  $A$  is a  $k$ -algebra, then obviously, so is  $B$ ).*

*Going further, a morphism of  $k$ -schemes, is then a morphism such that the map on global sections (and hence all sections by restriction) is  $k$ -linear.*

### 3.6 Gluing Schemes

Gluing schemes would be nice to understand I think. Not only is it “the nicest way to construct projective space”, but afaik we do it with products of elliptic curves too sometimes to create abelian surfaces.

### Definition 3.10: Gluing Schemes

Let  $\{X_i\}_{i \in I}$  be a set of schemes such that for all  $i, j \in I$ , we are given an open subscheme  $X_{ij}$  of  $X_i$  and isomorphisms of schemes

$$f_{ij} : X_{ij} \rightarrow X_{ji},$$

such that  $f_{ij} = f_{ji}^{-1}$ , and  $f_{ii} = \text{id}$  for all  $i, j$ . Additionally, the isomorphism satisfy  $f_{ij}(X_{ij} \cap X_{jk}) = X_{ji} \cap X_{jk}$  and  $f_{ik} = f_{jk} \circ f_{ij}$  on  $X_{ij} \cap X_{ik}$ .

We will construct a scheme  $(X, \mathcal{O}_X)$  called the **gluing of  $X_i$  along the  $X_{ij}$** .

We construct first a topological space  $X$  by considering  $\coprod X_i$  with the disjoint union topology, and then quotienting out by the equivalence relation  $x \sim y$  if  $y = f_{ij}(x)$ . This gives canonical injections

$$g_i : X_i \hookrightarrow X,$$

and we denote the image of  $g_i$  by  $U_i$ . Let  $\mathcal{O}_X$  be the sheaf such that

$$\mathcal{O}_X(U_i) = g_{i*} \mathcal{O}_{X_i}.$$

Then  $(X, \mathcal{O}_X)$  is a scheme, and  $g_i$  are open immersions for all  $i$ .

The most important example of gluing will be the construction of projective space.

**Example 3.6.1.** Let  $k$  be a field, and take of  $\text{Spec } k[T]$ , denoted  $X_1 = \text{Spec } k[T_1]$  and  $X_2 = \text{Spec } k[T_2]$ . In both cases, we consider the compliment of the origin, i.e.

$$X_{12} = \text{Spec } k[T_1]_{T_1} = \text{Spec } k[T_1, T_1^{-1}],$$

and vice versa. We then consider the isomorphism of these open subschemes given by

$$\begin{aligned} f_{12} : X_{12} &\rightarrow X_{21} \\ f_{12}(T_1) &= T_2^{-1}, \end{aligned}$$

and again vice versa. Its very simple to see that this satisfies the criteria for the definition above, hence we can construct the gluing  $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ , called the projective line over  $k$ . It corresponds to the affine line plus an extra point (the point at infinity). However, unlike the affine line, we get very few global sections. Convince yourself that the following sequence is exact

$$0 \rightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \rightarrow \Gamma(X_1, \mathcal{O}_{X_1}) \oplus \Gamma(X_2, \mathcal{O}_{X_2}) \rightarrow \Gamma(X_{12}, \mathcal{O}_{X_{12}})$$

Finish later.

Next: Generalise to projective  $n$ -space.