The Rising Sea - Exercise Solutions

Jonathan Komada Eriksen

Contents

1 Category theory

1.1 Categories and Functors

Exercise 1.1.A. A category in which each morphism is an isomorphism is called a **groupoid**.

a) A perverse definition of a **group** is a groupoid with one object. Make sense of this. Similarily a **monoid** is a category with one object. Make sense of this.

b) Describe a groupoid that is not a group.

Solution. For a) this, is just realising that the morphisms form the group objects, composition the group operation (and for groups, inverses are given, since the morphisms are isomorphisms)
for b) consider the category with two elements A, B, with $\mathrm{Hom}(A, A)$ and $\mathrm{Hom}(B, B)$ containing only the identity, and $\mathrm{Hom}(A, B)$ consisting of one isomorphism. (Of course this is not a group: there is no way to compose id_A and id_B). □

Exercise 1.1.B. Show that the invertible elements in $\mathrm{Hom}(A, A)$ form a group. What are the automorphism groups in the categories of sets and vectorspaces over k? Show that two isomophic objects have isomorphic automorphism groups.

Solution. The first part is essentially the previous exercise. The automorphism groups in the category of sets are bijections $A \to A$, and in vector spaces, it is invertible linear transformations $V \to V$.
For the last part, just notice that the isomorphism $\varphi : A \to B$ induces an isomorphism

$$\begin{aligned} \mathrm{Aut}(B, B) &\to \mathrm{Aut}(A, A) \\ \psi &\to \varphi^{-1} \circ \psi \circ \varphi \end{aligned}$$

with the obvious inverse. □

Exercise 1.1.C. Show that the dobbel-dual $\vee\vee$ is naturally isomorphic to the identity functor on the category of finite dimensional vectorspaces over k.

Solution. Define $m_V : V \to V^{\vee\vee} = \mathrm{Hom}(\mathrm{Hom}(V, k), k)$ as $m_V(v)(\varphi) := g(v)$. This is a natural transformation: The square

$$\begin{array}{ccc} V & \xrightarrow{h} & W \\ \downarrow{\scriptstyle m_V} & & \downarrow{\scriptstyle m_W} \\ V^{\vee\vee} & \xrightarrow{h^{\vee\vee}} & W^{\vee\vee} \end{array}$$

commutes, since

$$h^{\vee\vee}(m_V(v))(\psi) := m_V(v)(\psi \circ h) = \psi(h(v)) = m_W(h(v))$$

This part is true, even for infinite dimensional vector-spaces, however to show that m_V is invertible, one needs finite dimensionality: This is the standard proof that $\dim V = \dim V^\vee$, where one shows that basis $e_1, \ldots e_n$ of V induces a basis $e^1, \ldots e^n$ of $V^\vee$ defined by $e^i(e_j) = \delta_{i,j}$.
□

Exercise 1.1.D. Show that the category of vector-spaces k^n under linear maps (matrices) is equivalent to the category of finite-dimensional vector spaces over k.

Solution. This is just defining the inverse functor by picking a basis of every vector space (this also sends linear transformations to matrices by looking at the action on the chosen bases).
As the following "asside" points out: The argument above generalizes to show that a fully faithful functor is an equivalence iff. it is essentially surjective (the natural transformations are just given by picking an isomorphic opject in the image of F for each object in the target category). $\square$

1.2 Universal properties determine an object up to unique isomorphism

Exercise 1.2.A. Show that two initial objects are uniquely isomorphic. Show that two final objects are uniquely isomorphic.

Solution. Let I_1, I_2 be the initial objects. There unique morphisms $I_1 \to I_2 \to I_1$, which must be the unique morphism $\mathrm{id}_1 : I_1 \to I_1$ (and the other way around).
The same argument goes for the final objects (alternatively, the final object is the initial object in the opposite category). $\square$

Exercise 1.2.B. What are the initial and final objects in the category of Sets, Rings and topological spaces respectively, if they exist?

Solution. Sets: $\varnothing$, and $\{*\}$.
Rings: $\mathbb{Z}$, and the trivial ring (if you allow $0 = 1$).
Topological spaces: Same as sets. $\square$

Exercise 1.2.C. Show that $A \to S^{-1}A$ is an injection iff. S contains no zero-divisors.

Solution. If s contains a zero-divisor, say $s \in S$ satisfies $as = 0$, then $a/1 = 0/1$, since $s(a - 0) = 0$.
In the other direction, $a/1 = 0$ implies that $s(a - 0) = 0$ for some $s \in S$, so $a = 0$. $\square$

Exercise 1.2.D. Verify that $A \to S^{-1}A$ satisfies the following universal property: It is the initial object in the category of B-algebras (i.e. ring maps $A \to B$), where every element of S is invertible in B.

Solution. Let $\varphi : A \to B$ be an object in the mentioned category. This defines a map $S^{-1}A \to B$ by $\varphi'(a/s) = \varphi(a)/\varphi(s)$. This is a well defined map of rings: If b/t is another representative, then there exists s' such that $s'(at - bs) = 0$. But then $\varphi(a)\varphi(t) - \varphi(b)\varphi(s) = 0$, implying that $\varphi(b)/\varphi(t) = \varphi(a)/\varphi(s)$.
Further, the diagram

$$\begin{tikzcd}
S^{-1}A \arrow[rr, "\varphi'"] & & B \\
& A \arrow[ul] \arrow[ur, "\varphi"] &
\end{tikzcd}$$

obviously commutes, so it is a valid map of A-algebras. Clearly, this is also the only map making the above diagram commute (notice that the image of a/s is already defined by φ, since s is invertible in B.) $\square$

Exercise 1.2.E. Show that $\varphi : M \to S^{-1}M$ exists, by constructing something satisfying the universal property.

Solution. Let $S^{-1}M$ consist of elements of the form $m/s, m \in M, s \in S$, under the equivalence relation $m_1/s_1 = m_2/s_2$ if there exists $s \in S$ such that $s(m_1s_2 - m_2s_1) \sim 0$. The additive structure is defined by $m_1/s_1 2 m_2/s_2 = (m_1s_2 + m_2s_1)/s_1s_2$. This is well defined: $m_1/s_1 + m_2/s_2 - (m_1/s_1 + m_2'/s_2') = (m_1s_2 + m_2s_1)/s_1s_2 - (m_1s_2' + m_2's_1)/s_1s_2' = (m_1s_2s_2' + m_2s_1s_2' - m_1s_2's_2 - m_2's_1s_2)/s_1s_2s_2' = (m_2s_2' + m_2's_2)s_1/s_1s_2s_2' \sim 0$. Similarily, scalar multiplication (by $S^{-1}A$) can be defined by $a_1/s_1 \cdot m_2/s_2 = (a_1m_2)/(s_1s_2)$ (proving well-definedness is again tedious but trivial). This satisfies the universal property (repeat the argument from Exercise 1.2.D). $\square$

Exercise 1.2.F.
a) Show that localization commutes with finite products, or equivalently, finite direct sums.

b) Show that localization commutes with arbitrary direct sums

c) show that localization does NOT commute with arbitrary direct products in general.

Solution. We do a) and b) together: Finite direct sums are isomorphic to direct products, so doing b) is in fact enough. Now proving b) can boil down to the following (admittedly quite ugly) diagram:

$$\begin{tikzcd}
 & S^{-1}(\bigoplus_n M_n) \arrow[d, dotted, bend left] & \\
 & \bigoplus_n S^{-1}M_n \arrow[u, dotted, bend left] & \\
S^{-1}M_1 \arrow[ur] \arrow[uur, dotted] & S^{-1}M_2 \arrow[u] \arrow[uu, dotted, bend right] & \dots \\
M_1 \arrow[u] \arrow[dr] & M_2 \arrow[u] \arrow[d] & \dots \\
 & \bigoplus_n M_n \arrow[uuuu, bend left] \arrow[uuu, dotted, bend right] &
\end{tikzcd}$$

First, we prove that the map

$$S^{-1}(\bigoplus_n M_n) \to \bigoplus_n S^{-1}M_n.$$

exists by universal properties. First, there exists a unique map $\bigoplus_n M_n \to \bigoplus_n S^{-1}M_n$ by the universal property of the direct sum (colimit). But, since S acts invertibly on $\bigoplus_n S^{-1}M_n$, the map we wanted is uniquely defined by the universal property of localization.
Next, there also exists a uniquely defined map

$$\bigoplus_n S^{-1}M_n \to S^{-1}(\bigoplus_n M_n)$$

To see this, notice first the maps $S^{-1}M_n \to S^{-1}(\bigoplus_n M_n)$, uniquely defined by the universal property of licalization (over maps from M_n). These maps give the map we wanted, from the universal property of direct sums again.
Since all maps where uniquely defined, the composition of the two maps still satisfies the universal properties, and thus they must be the identity, and hence the two objects are isomorphic.
To show c), follow the hint and simply notice that $(1/p, 1/p^2, 1/p^3, \dots) \in \prod_n \mathbb{Q} = \prod_n \mathbb{Z}_{(0)}$ is not in the image of the uniquely defined map

$$(\mathbb{Z} \setminus \{0\})^{-1} \prod_n \mathbb{Z} \longrightarrow \prod_n \mathbb{Z}_{(0)}.$$

$\square$

Exercise 1.2.G. Show that $\mathbb{Z}/(10) \otimes \mathbb{Z}/(12) \cong \mathbb{Z}/(2)$.

Solution. The usual proof that $\mathbb{Z}/(n)\otimes\mathbb{Z}/(m) \cong \mathbb{Z}/(\gcd(m,n))$ can be shown by: The tensor product is cyclic, generated by $1\otimes 1$ (use $a\otimes b = ab(1\otimes 1)$). But then both $n(1\otimes 1) = m(1\otimes 1) = 0$, and the result follows. $\square$

Exercise 1.2.H. Show that $- \otimes_A N$ gives a covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$. Show that it is right-exact.

Solution. For $f : M_1 \to M_2$, we can define $f \otimes_A N : M_1 \otimes N \to M_2 \otimes N$ by $f(\sum_i m_i \otimes n_i) = \sum f(m_i) \otimes n_i$.
If f is surjective, then $f \otimes N$ is still clearly surjective. Thus, given

$$M' \xrightarrow{g} M \xrightarrow{f} M'' \to 0$$

we need to show that $\operatorname{im} g \otimes N = \ker f \otimes N$. We certainly have $\operatorname{im} g \otimes N \subseteq \ker f \otimes N$, since they compose to zero, and the result now follows by surjecting onto the kernel of f. $\square$

Exercise 1.2.I. Show that $(M \otimes N, \iota : M \times N \to M \otimes N)$ satisfying the universal property of the tensor product is unique up to unique isomorphism.

Solution. Given two such pairs, use the universal properties to construct maps between them. Composing them must yield the identity by the universal property. $\square$

Exercise 1.2.J. Show that the construction of the tensor product of modules over A satisfies the universal property.

Solution. First, the fact that the map $\iota((m,n)) = m \otimes n$ is A-bilinear is because we quotient out by $(m_1 + m_2, n) = (m_1, n) + (m_2, n)$ (and same in n), and $a(m,n) = (am, n) = (m, an)$.
Next, given an A-bilinear map $f : M \times N \to T$, we can to construct the A-module homomorphism $M \otimes N \to T$ given by $\sum m_i \otimes n_i = \sum f(m_i, n_i)$, which is indeed the unique map making the diagram commute. $\square$

Exercise 1.2.K. a) If M is an A module, and $A \to B$ is a ring-homomorphism, give $B \otimes_A M$ the structure of a B-module. Show that this defines a functor $\mathrm{Mod}_A \to \mathrm{Mod}_B$.

b) If further $A \to C$ is another ring-homomorphism, show that $B \otimes_A C$ has a natural ring-structure.

Solution. For a), $B \otimes_A M$ is already an A-module, since B can be seen as a B-module. The B-module structure is defined by $b(b' \otimes m) = bb' \otimes m$. This is functorial in B; given $f : M \to n$, define $B \otimes_A f : B \otimes M \to B \otimes N$ by sending $b \otimes m \to b \otimes f(m)$. Clearly, this is also a homomorphism of B-modules.
For b) we can define $(b_1 \otimes c_1)(b_2 \otimes c_2) = b_1 b_2 \otimes c_1 c_2$, and extend linearly. $\square$

Exercise 1.2.L. If S is a multiplicative subset of A, and M is an A-module, describe a natural isomorphism $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$.

Solution. Notice that the universal property satisfyied by $S^{-1}A \otimes_A M$ is the same as that of localizations: An A-bilinear map $S^{-1}A \times M \to T$ is the same as a map of A-modules $M \to T$, such that every element of S acts invertibly on T.
This natural isomorphism $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$ is given by factoring the map $a/s \otimes m \to (am)/s$. $\square$

Exercise 1.2.M. Show that the tensor product commutes with arbitrary direct sums.

Solution. This can be done using almost exactly the same argument as in Exercise 1.2.F. Explicitly, the map is given by

$$M \otimes (\bigoplus_i N_i) \to \bigoplus_i M \otimes N_i$$
$$\sum_i m_i \otimes (\bigoplus_j n_j) \to \bigoplus_j \sum_i (m_i \otimes n_j).$$

(Recall that, by definition of the direct sum, only finitely many n_j are non-zero). The map is surjective, because every generator of the right hand side is clearly in the image. Injectivity is much trickier, and probably best proven using the canonical maps that occurs, similarly to Exercise 1.2.F. $\square$

Exercise 1.2.N. Show that in the category of sets,

$$X \times_Z Y = \{(x, y) \in X \times Y \mid \alpha(x) = \beta(y)\},$$

where $\alpha : X \to Z, \beta : Y \to Z$.

Solution. We need to show that it satisfies the universal property: Let $\varphi_X : T \to X, \varphi_Y : T \to Y$ be set maps such that $\alpha \circ \varphi_X = \beta \circ \varphi_Y$. Define the map $T \to X \times_Z Y$ (defined as in the exercise) by $t \to (\varphi_X(t), \varphi_Y(t))$. This is well defined, and makes everything commute. It is also the only such map, since $T \to X \times_Z Y \to X$ is is given by φ_X, and similarily with Y. $\square$

Exercise 1.2.O. If X is a topological space show that the fibered products always exists in the category of open sets of X.

Solution. It is given by the intersection. $\square$

Exercise 1.2.P. If Z is the final object in a category $\mathscr{C}$, show that $X \times_Z Y = X \times Y$.

Solution. The universal property is the same: The additional requirement that the maps to Z should commute becomes mute, because of the universal property of the final object. $\square$

Exercise 1.2.Q. If the two smaller squares in the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

commutes, show that so does the larger square.

Solution.

$$\begin{aligned} U \to W \to Y \to Z &= U \to W \to X \to Z \\ &= U \to V \to X \to Z \end{aligned}$$

$\square$

Exercise 1.2.R. Given morphisms $X_1 \to Y, X_2 \to Y$, and $Y \to Z$, show that there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$, assuming both fibered products exists.

Solution. The map exists by the universal property of $X_1 \times_Z X_2$ (the two maps $X_1 \times_Y X_2 \to Z$ both factor through Y, and commutes there by definition). $\square$

Exercise 1.2.S. Suppose we are given morphisms $X_1, X_2 \to Y$, and $Y \to Z$. Show that the following diagram cummutes:

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

Solution. See Exercise 1.3.B. $\square$

Exercise 1.2.T. Show that the coproduct in the category of sets is the disjoint union.

Solution. Given any set T with maps $f_i : X_i \to T$, the map satisfying the universal property is

$$\coprod_i X_i \to T$$
$$(a,i) \to f_i(a)$$

$\square$

Exercise 1.2.U. Suppose $A \to B$ and $A \to C$ are ring morphisms. Recall that $B \otimes_A C$ has a ring-structure. Show that there is a natural morphism $B \to B \otimes_A C$ (and similarly for C), making $B \otimes_A C$ the fibered coproduct on rings.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & B \otimes_A C \end{tikzcd}$$

Solution. The morphisms are given by $b \to b \otimes 1_C$ and $c \to 1_B \otimes c$. Assume now that R is a ring, with morphisms from B and C such that everything commutes; we must show the existance and uniqueness of h in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr, bend left, "g"] & \\ B \arrow[r] \arrow[drr, bend right, "f"'] & B \otimes_A C \arrow[dr, dotted, "h"] & \\ & & R \end{tikzcd}$$

The uniqueness and existence of the h as a map of A-modules is the universal property of the tensor product: There are bilinear A-module maps $B \times C \to B \otimes_A C$ and $B \times C \to R$, where the last map is defined by $(b,c) \to f(b)g(c)$ (bilinearity is immidiate, since the maps to begin with where ring-morphisms, and multiplication in R is distributive, while the A-module structure comes from commuting relation, making the maps into A-algebra maps), hence h is uniquely defined as a map of A-modules. We need to show that h also preserves the ring-structure: to see this, notice that $h(b \otimes c) = f(b)g(c)$, thus we have $h(1_B \otimes 1_C) = f(1_B)g(1_C) = 1_R 1_R = 1_R$, and similarly $h(b_1 b_2 \otimes c_1 c_2) = f(b_1 b_2)g(c_1 c_2) = f(b_1)f(b_2)g(c_1)g(c_2) = f(b_1)g(c_1)f(b_2)g(c_2) = h(b_1 \otimes c_1)h(b_2 \otimes c_2)$. $\square$

Exercise 1.2.V. Show that the composition of two monomorphisms is a monomorphism.

Solution. Let $\iota_1 : A \to B$ and $\iota_2 : B \to C$ be monomorphisms. Then $(\iota_2 \iota_1) f_1 = (\iota_2 \iota_1) f_2$ implies that $\iota_1 f_1 = \iota_1 f_2$ (ι_2 is mono), and hence, $f_1 = f_2$. $\square$

Exercise 1.2.W. Prove that a morphism $\pi : X \to Y$ is a monomorphism if and only if the fibered product $X \times_Y X$ exists, and the induced diagonal $\delta_X : X \to X \times_Y X$ is an isomorphism. We may then take this as the definition of monomorphisms.

Solution. If π is mono, X (together with the identity maps) satisfies the universal property of the fibered product over Y (this is just a translation of the property of being mono). Thus X is canonically isomorphic to $X \times_Y X$, but the diagonal map is uniquely defined, and must thus give the isomorphism.
Conversely, assume that the diagonal map induces an isomorphism. Then X also satisfies the universal property of the fibered product, which translates to the property of being mono, since either composition $X \xrightarrow{\delta_X} X \times_Y X \to X$ is the identity. $\square$

Exercise 1.2.X. Show that if $Y \to Z$ is mono, then the morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ (see Exercise 1.2.S) is an isomorphism.

Solution. This is easy: $X_1 \times_Z X_2$ satisfies the universal property of the fibered product over Y: since $Y \to Z$ is mono, any two maps being equal at Z were already equal at Y.
Alternatively, we can use that the square in Exercise 1.2.S is a fibered product (see Exercise 1.3.B), and that $Y \to Y \times_Z Y$ is an isomorphism. $\square$

1.3 Limits and colimits

Exercise 1.3.A. Suppose that the partially ordered set $\mathscr{I}$ has an initial object e. Show that the limit of any diagram indexed by $\mathscr{I}$ exists.

Solution. Let $F : \mathscr{I} \to \mathscr{C}$ be the diagram. Then it is easy to see that

$$\varprojlim_{i \in \mathscr{I}} F(i) = F(e).$$

$\square$

Exercise 1.3.B. Solve Exercise 1.2.S by identifying both $X_1 \times_Y X_2$ and $Y \times_{Y \times_Z Y} X_1 \times_Z X_2$ as the limit of the diagram

$$\begin{tikzcd}
X_1 \arrow[dr] & & \\
& Y \arrow[r] & Z \\
X_2 \arrow[ur] & &
\end{tikzcd}$$

Solution. First, showing what the execise asks solves Exercise 1.2.S: the diagram commutes, since $X_1 \times_Y X_2$ would then be the fibered product, and hence commutes by definition.
Now, $X_1 \times_Y X_2$ is clearly the limit of the diagram in question (the extra Z does not add anything, since maps to Z will already be defined by maps to Y). To see that $W := Y \times_{Y \times_Z Y} X_1 \times_Z X_2$ is also the limit of this diagram, consider

$$\begin{tikzcd}
& X_1 \arrow[dr] & & \\
X_1 \times_Z X_2 \arrow[ur] \arrow[dr] & Y \times_Z Y \arrow[r, bend left] \arrow[r, bend right] & Y \arrow[l, dashed] \arrow[r] & Z \\
& X_2 \arrow[ur] & &
\end{tikzcd}$$

We can now unravel the definition of the universal property of W: Suppose we are given an object T with maps $\alpha : T \to Y$, and $\beta : T \to X_1 \times_Z X_2$ such that the maps to $Y \times_Z Y$ commutes. Giving β is the same as giving maps to X_1 and X_2 that become equal at Z. Now the fact that this should commute with α at $Y \times_Z Y$ says that the map should be equal already at Y (this uses the fact that the diagonal map $Y \to Y \times_Z Y$ is a monomorphism; this is clearly true, it is even a split monomorphism by definition). $\square$

Exercise 1.3.C. Show that in the category of sets,

$$\varprojlim_{\mathcal{I}} A_i = \{(a_i)_i \in \prod_{i \in \mathcal{I}} A_i \mid F(m)(a_j) = a_k, \forall m \in \mathrm{Hom}_{\mathcal{I}}(i, j)\}$$

along with the obvious projection maps.

Solution. Assuming we are given an object T, and maps $f_i : T \to A_i$ such that everything commutes, we construct the map $T \to \varprojlim_{\mathcal{I}}$ by $t \to (f_i(t))$. This really lands in $\varprojlim_{\mathcal{I}}$, since everything commutes, and further it is clearly the only such maps, as can be seen by composing with the projections. $\square$

Exercise 1.3.D. a) Interpret the statement $\mathbb{Q} = \varinjlim \frac{1}{n}\mathbb{Z}$

b) Interpret the union of some subsets of a given sets as a colimit (dually, intersections can be interpreted as a limit).

Solution. For a), the positive natural numbers form a poset under divisibility. Take this as the index set and map to sets (or ab. groups) by $F(n) \to \frac{1}{n}\mathbb{Z}$. If $n \mid m$, we get a corresponding map (inclusion) by $\frac{1}{n}\mathbb{Z} \to \frac{1}{m}\mathbb{Z}$. The colimit is clearly the rational numbers.
For b), take the category to be the category of subsets, and morphisms inclusion. The coproduct (which is indeed a colimit) is here the union. $\square$

Exercise 1.3.E. Supposed $\mathcal{I}$ is filtered. Show that any diagram in the category of sets indexed by $\mathcal{I}$, has the colimit

$$\varinjlim_{\mathcal{I}} A_i = \{(a_i, i) \in \coprod_{i \in \mathcal{I}} A_i\} / \sim$$

with the obvious maps to it, where $\sim$ is the equivalence relation defined by $(a_i, i) \sim (a_j, j)$ if and only if there are $F(m) : A_i \to A_k$ and $F(m') : A_j \to A_k$ such that $F(m)(a_i) = F(m)(a_j)$.

Solution. First, we show that it is an equivalence relation: The symmetric and reflexive properties are obvious. For transitivity, assume that there are $F(m) : A_i \to A_p, F(m') : A_j \to A_p$, and $F(n) : A_j \to A_q, F(n') : A_k \to A_q$, so that $F(m)(a_i) = F(m')(a_j), F(n)(a_j) = F(n')(a_k)$. Since $\mathcal{I}$ is filtered, there now exists s, s' and arrows $F(t) : A_s \to A_s, F(t') : A_q \to A_s, F(t'') : A_s \to A_{s'}$, so that the two paths $A_j \to A_{s'}$ in

$$\begin{tikzcd}
A_i \arrow[dr] & & & \\
& A_p \arrow[dr] & & \\
A_j \arrow[ur] \arrow[dr] & & A_s \arrow[r] & A_{s'} \\
& A_q \arrow[ur] & & \\
A_k \arrow[ur] & & &
\end{tikzcd}$$

are equal, showing the reflexive property.
Now that we know that this object is well defined, it is easy to show that it is also the colimit. Given another object T and maps $f_i : A_i \to T$ making everything commute, we can set the map $\varinjlim_{\mathcal{I}} A_i \to T$ by $(a_i, i) \to f_i(a_i)$. Clearly, this is also the only map making everything commute. $\square$

Exercise 1.3.F. Verify that that the colimit in the category of A-modules is the set as defined in Exercise 1.3.E, and where addition is defined by taking two arrows $F(u) : M_i \to M_k, F(v) : M_j \to M_k$, and setting $(m_i, i) + (m_j, j) = (F(u)(m_i) + F(v)(m_j), k)$, and multiplication by A is defined in the obvious way.

Solution. Replacing (m_i, i) by a different representative, and or $F(u)$ with a different arrow can be shown to give the same result, because they eventually become equal (so use again the filtered requirements to find an index and maps in which everything becomes equal). $\square$

Exercise 1.3.G. Generalize Exercise 1.3.D a) to interpret localization of an integral domain as a colimit over a filtered set.

Asside: Can you make it work for general commutative rings (i.e. not necessarily integral domains?).

Solution. We do the case without requiring the integral domain hypothesis right away: We form a filtered set of S in the obvious way, with arrows corresponding to $s \mid s'$, i.e. there exists some $a \in A$ such that $s' = cs$ (this is filtered, because s, t both have arrows to st). Thus the limit $\varinjlim_S A_s$ (recall A_s contain elements of the form a/s^n) exists, with maps $A_s \to A_{s'}$ defined by $a/s^n \to c^n a/s'^n$. Now there are canonical isomorphisms $\varinjlim_S A_s \cong S^{-1}A$:
The map $S^{-1}A \to \varinjlim_S A_s$ exists because there is an obvious map $A \to \varinjlim_S A_s$, and every element of s is invertible in $\varinjlim_S A_s$, this we can apply the universal property of $S^{-1}A$.
$\varinjlim_S A_s \to S^{-1}A$ exists because there are obvious maps $A_s \to S^{-1}A$ making everything commute.
Now by the uniqueness property, these maps are mutually inverse, and hence these objects are canonically isomorphic.
When A is an integral domain, we may replace A_s by (the A-module) $\frac{1}{s}A \subset K$, where K is the field of fractions of A, and the same argument works. □

Exercise 1.3.H. Suppose you are given a diagram of A-modules indexed by $F : \mathcal{I} \to \mathrm{Mod}A$, where we let $M_i := F(i)$. Show that the colimit is $\bigoplus_{i\in\mathcal{I}} M_i$ modulo the relations $m_i - F(n)(m_i)$ for every $n : i \to j$ in $\mathcal{I}$.

Solution. Assume as always, we are given an A-module T and homomorphisms $f_i : M_i \to T$ such that everything commutes. Start by forming the direct sum $\bigoplus_{i\in\mathcal{I}} M_i$. There of course exists a unique map $\bigoplus_{i\in\mathcal{I}} M_i \to T$ which commutes with the f_i (universal property of the coproduct), thus we are done if we show that the map factors through the submodule generated by the relations: The fact that the f_i also commute with $F(n)$ for every n shows that every element of the form $m_i - F(n)(m_i)$ is in the kernel of the map, thus the map $\varinjlim_{\mathcal{I}} M_i \to T$ is well-defined. □

1.4 Adjoints

Exercise 1.4.A. Write down what this diagram (meaning: the "naturality" in the definition of an adjoint pair for a morphism $g : B \to B'$) should be.

Solution.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}(F(A), B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{B}}(F(A), B') \\ \downarrow{\scriptstyle \tau_{A,B}} & & \downarrow{\scriptstyle \tau_{A,B'}} \\ \mathrm{Hom}_{\mathcal{A}}(A, G(B)) & \xrightarrow{G(g_*)} & \mathrm{Hom}_{\mathcal{A}}(A, G(B')) \end{array}$$

□

Exercise 1.4.B. Show that the maps $\tau_{A,B}$ has the following properties. For each A, there is a map $\eta_A : A \to G(F(A))$ so that for any $g : F(A) \to B$, the corresponding $\tau_{A,B}(g) : A \to G(B)$ is given by the composition

$$A \xrightarrow{\eta_A} G(F(A)) \xrightarrow{G(g)} G(B).$$

Similarily, there is a map $\epsilon_B : F(G(B)) \to B$ such that for any $f : A \to G(B)$, the corresponding $\tau_{A,B}^{-1}(f) : F(A) \to B$ is given by

$$F(A) \xrightarrow{F(f)} F(G(B)) \xrightarrow{\epsilon_B} B$$

Solution. Start with $\mathrm{id}_{F(A)} : F(A) \to F(A)$. Then set $\eta_A := \tau_{A,F(A)}(\mathrm{id}_{F(A)}) \in \mathrm{Hom}_{\mathcal{A}}(A, GF(A))$. Now the definition of $\tau_{A,B}(g)$ follows directly from naturality applied to the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}(F(A), F(A)) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{B}}(F(A), B) \\ \downarrow{\tau_{A,A}} & & \downarrow{\tau_{A,B'}} \\ \mathrm{Hom}_{\mathcal{A}}(A, GF(A)) & \xrightarrow{G(g)_*} & \mathrm{Hom}_{\mathcal{A}}(A, G(B)) \end{array}$$

and similarly for the counit ϵ_A for functions $f : A \to G(B)$. □

Exercise 1.4.C. Suppose M, N, P are A-modules. Describe a bijection $\mathrm{Hom}_A(M \otimes_A N, P) \leftrightarrow \mathrm{Hom}_A(M, \mathrm{Hom}(N, P))$.

Solution. Let $f : M \to \mathrm{Hom}_A(N, P)$. Say $f(m) = g$ We may consider f as a function $f' : M \times N \to P$ by $f'(m, n) = f(m)(n)$. Since this is bilinear (both f and $f(m)$ are homomorphisms), there exists a unique map $h : M \otimes N \to P$ corresponding to f.
Corresondingly, given a map $g : M \otimes N \to P$, define the map $g' : M \to \mathrm{Hom}_A(N, P)$ by $g'(m) = g(m \otimes -)$. It is clear that these maps are mutually inverse bijections. □

Exercise 1.4.D. Show that $- \otimes_A N$ and $\mathrm{Hom}_A(N, -)$ are adjoint functors.

Solution. By the previous exercise, we only need to show naturality. That is, given $g : P \to P'$, we need to show that the square

$$\begin{array}{ccc} \mathrm{Hom}_A(M \otimes N, P) & \xrightarrow{g_*} & \mathrm{Hom}_A(M \otimes N, P') \\ \downarrow{\tau_{M,P}} & & \downarrow{\tau_{M,P'}} \\ \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) & \xrightarrow{g_{**}} & \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P')) \end{array}$$

commutes. But this is obvious by using the definition of $\tau_{M,P}$ from the previous exercise. □

Exercise 1.4.E. Suppose $\varphi : B \to A$ is a morphism of rings. If M is an A-module, you can create a B-module M_B by considering it as a B module (scalar multiplication given by $b \cdot m = \varphi(b)m$). This gives a functor $\cdot_B : \mathrm{Mod}_A \to \mathrm{Mod}_B$. Show that this functor is right-adjoint to $- \otimes_B A$.

Solution. Assume we are given a homomorphism of A-modules $f : N \otimes_B A \to M$, for some B-module N. This gives a homomorphism of B-modules $\tau_{N,M}(f) : N \to M_B$ by $\tau_{N,M}(f)(n) = f(n \otimes_B 1_A)$.
Conversely, given $g : N \to M_B$, we can define $\tau_{N,M}^{-1}(g) : N \otimes_B A \to M$ by $\tau_{N,M}^{-1}(g)(n \otimes_B a) = ag(n)$. These maps are mutual inverses, and similarily to before, naturality is seen simply by inspecting the maps. □

Exercise 1.4.F. Show that if an abelian semigroup is already a group, then the identity morphism is the groupification.

Solution. The identity morphism clearly satisfies the universal property: For every map $f : S \to G$, the unique map making the diagram commute is simply f. □

Exercise 1.4.G. Construct the "groupification functor" H from the category of nonempty abelian semigroups to the category of abelian groups. Show that this is left adjoint to the forgetfull functor F from the category of abelian groups to the category of abelian semigroups.

Solution. We follow the construction: let S be an abelian semigroup, and form $H(S)$, where the elements are pairs $(a, b) \in S \times S$, under the equivalence relation $(a, b) \sim (c, d)$ if there exists $e \in S$ such that $a + d + e = b + c + e$. This becomes a group under the obvious addition $(a, b) + (c, d) = (a + c, b + d)$: The identity are all elements of the form (a, a), since $(a + b, a + c) \sim (b, c)$ since $a + b + b = a + c + c$. Addition is obviously associative, and inverses are given by $(a, b)^{-1} = (b, a)$. The canonical map $S \to H(S)$ is given by $b \to (a + b, a)$ for any $a \in S$.
To show the universal property, let $f : S \to G$ be a morphism, where G is a group. We form $f' : H(S) \to G$ by $f((a, b)) = f(a) - f(b)$. This makes the triangle commute, and is the only map to do so (since it is completely defined by f).
Finally, we show the adjointness: Given $h \in \mathrm{Hom}(S, G)$, $\tau_{S,G}^{-1}(h) \in \mathrm{Hom}(H(S), G)$ is simply given by h' from the universal property (this is actually the same as what $H(h)$ would be in the case where G is already a group). Other way, given $g : H(S) \to G$, $\tau_{S,G}(g)(b) = g((b + a, a))$. Now checking that the square commutes for some $\varphi : G \to G'$ (notice, there is nothing to check), and the same applies to $\psi : S \to S'$. □

Exercise 1.4.H. Suppose A is a ring, and S is a multiplicative subset. Then $S^{-1}A$-modules are a full subcategory of A-modules. Then, the localization functor $\mathrm{Mod}A \to \mathrm{Mod}S^{-1}A$ can be interpreted as a left-adjoint to the forgetfull functor.

Solution. This exercise is almost verbatim the same arguments as the previous exercise; notice the similarity between the universal properties of localization/groupificiation. □

1.5 Abelian categories

Exercise 1.5.A. Describe the exact sequences

$$0 \longrightarrow \operatorname{im} f^i \longrightarrow A^{i+1} \longrightarrow \operatorname{coker} f^i \longrightarrow 0$$

$$0 \longrightarrow H^i(A^\bullet) \longrightarrow \operatorname{coker} f^{i-1} \longrightarrow \operatorname{im} f^i \longrightarrow 0$$

Solution. The first sequence is literally the definition of $\operatorname{im} f^i$, as the kernel of $\operatorname{coker} f^i$:

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
& \operatorname{im} f^i \arrow[dr] & & & \\
& A^i \arrow[r, "f^i"] \arrow[drr, "0"'] & A^{i+1} \arrow[dr] & & \\
& & & \operatorname{coker} f^i \arrow[dr] & \\
& & & & 0
\end{tikzcd}$$

For the second map, applying the snake lemma to the top two rows shows that the bottom row is exact

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} f^{i-1} \arrow[r] \arrow[d] & \operatorname{im} f^{i-1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker f^i \arrow[r] \arrow[d, dotted] & A^i \arrow[r] \arrow[d, dotted] & \operatorname{im} f^i \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & H^i(A^\bullet) \arrow[r] & \operatorname{coker} f^{i-1} \arrow[r] & \operatorname{im} f^i \arrow[r] & 0
\end{tikzcd}$$

□

Exercise 1.5.B. Suppose

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of k-vectorspaces. Define $h^i(A^\bullet) := \dim H^i(A^\bullet)$. Show that $\sum(-1)^i \dim A^i = \sum(-1)^i h^i(A^\bullet)$. In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$.

Solution. For each i, we have $\dim A^i = \dim\ker d^i + \dim\operatorname{im} d^i$. Similarily, we have $h^i(A^\bullet) = \dim\ker d^i - \dim\operatorname{im} d^{i-1}$. Thus

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i \dim A^i &= \sum_{i=0}^{n}(-1)^i(\dim\ker d^i + \dim\operatorname{im} d^i)\\ &= \dim\ker d^0 + \dim\operatorname{im} d^n + \sum_{i=1}^{n}(-1)^i(\dim\ker d^i - \dim\operatorname{im} d^{i-1})\\ &= 0 + 0 + \sum_{i=0}^{n} h^i(A^\bullet).\end{aligned}$$

In partiuclar, if the sequence is exact, then all the $h^i(A^\bullet) = 0$, and the result follows. $\square$

Exercise 1.5.C. Suppose $\mathscr{C}$ is an abelian category. Define the category $\mathrm{Com}_{\mathscr{C}}$ of complexes over $\mathscr{C}$, where morphisms $x^\bullet : A^\bullet \to B^\bullet$ are morphisms (x^i) making the diagram

$$\begin{tikzcd} \dots \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "x^{i-1}"] & A^i \arrow[r] \arrow[d, "x^i"] & A^{i+1} \arrow[r] \arrow[d, "x^{i+1}"] & \dots \\ \dots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \dots \end{tikzcd}$$

commute. Show that this is an abelian category.

Solution. First, we show its additive:

Ad1 : $\mathrm{Hom}(A, B)$ are abelian groups: This follows from just defining $(x^i) + (y^i) = (x^i + y^i)$ (everything still commutes, and distributes over other morphisms).

Ad2 : $\mathrm{Com}_{\mathscr{C}}$ has a zero object: This is just the complex of all zeroes (where the zeroes are the zero in $\mathscr{C}$); this is initial and final, because the zeroes are initial and final in $\mathscr{C}$.

Ad3 : It has products of two objects: Forming $A^\bullet \times B^\bullet$ by taking products componentwise. Again, this satisfies the universal property of products, because it does so componentwise.

And further, it is abelian:

Ab1 : Every map has a kernel and cokernel: This comes down to showing that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & \ker x^{i-1} \arrow[r] \arrow[d] & \ker x^i \arrow[r] \arrow[d] & \ker x^{i+1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & A^{i-1} \arrow[r] \arrow[d, "x^{i-1}"] & A^i \arrow[r] \arrow[d, "x^i"] & A^{i+1} \arrow[r] \arrow[d, "x^{i+1}"] & \dots \\
\dots \arrow[r] & B^{i-1} \arrow[r] \arrow[d] & B^i \arrow[r] \arrow[d] & B^{i+1} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \operatorname{coker} x^{i-1} \arrow[r] & \operatorname{coker} x^i \arrow[r] & \operatorname{coker} x^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

commutes. Each square comes from the commutative diagram

$$\begin{tikzcd}
\ker x \arrow[r, dotted, "h"] \arrow[d] & \ker x' \arrow[d] \\
A \arrow[r] \arrow[d, "x"] & A' \arrow[d, "x'"] \\
B \arrow[r] & B'
\end{tikzcd}$$

where h exists by the universal property of $\ker x'$ (and $\ker x \to B'$ is zero since it factors through B). The argument for the cokernel is analogous. Finally, one shows that the complexes are in fact complexes, by the commuting squares, and the fact that ker / coker is mono / epi.

Ab2 : Every monomorphism is the kernel of its cokernel: This is true, since it is true in $\mathcal{C}$ to begin with (so we get isomorphisms component-wise).

Ab3 : Every epimorphism is the cokernel of its kernel. Same argument

$\square$

Exercise 1.5.D. Show that a morphism $\varphi : A^\bullet \to B^\bullet$ induces a map in homology $H^i(A^\bullet) \to$

$H^i(B^\bullet)$. Furthermore show that H^i is a coveriant functor $\mathrm{Com}_{\mathscr{C}} \to \mathscr{C}$.

Solution. This comes down to diagram chasing. We can do it for the special case of $\mathscr{C}$ a module category. Take any element $x \in \ker d_A^i \subset A^i$. By the commutativity of the squares, $\varphi(x) \in \ker d_B^i$. Now let x' be another element so that $x - x' \in \operatorname{im} d_A^{i-1}$. Set y so that $d_A^{i-1}(y) = x - x'$. Then $d_B^{i-1}(\varphi(y)) = \varphi(x - x') = \varphi(x) - \varphi(x')$, so φ indeed induces a well-defined map in homology.
This is functorial: Clearly, identity morphisms are sent to identity morphisms, and the fact that $H^i(\varphi \circ \psi) = H^i(\varphi) \circ H^i(\psi)$ is simply by inspection. □

Exercise 1.5.E. Show two homotopic maps give the same map on homology.

Solution. The definition of homotopy says that the maps f and g (when applied to elements in $\ker d$) differ by an element in the image. □

Exercise 1.5.F. Suppose F is an exact functor. Show that applying F to an exact sequence preserves exactness. For example, if F is covariant, and $A' \to A \to A''$ is exact, then $F(A') \to F(A) \to F(A'')$ is also exact.

Solution. This is actually a bit trickier than it sounds. In general, we want to show that if

$$\cdots \to A^{i-1} \to A^i \to A^{i+1} \to \dots$$

is exact, then so is

$$\cdots \to F(A^{i-1}) \to F(A^i) \to F(A^{i+1}) \to \dots$$

this comes down to showing that first, left-exactness preserves kernels: Since left exact-ness means that

$$0 \to F(\ker f) \to F(A) \xrightarrow{F(f)} F(B)$$

is exact, and since every monomorphism is the kernel of its cokernel in an abelian category, we get $F(\ker f) = \ker F(f)$. Similarily, right-exactness preserves cokernels.
Thus, we can finally say that F preserves images, since $\operatorname{im} f = \ker(\operatorname{coker} f)$ by definition. Thus, $\operatorname{im} F(d^i) = F(\operatorname{im} d^i) = F(\ker d^{i+1}) = \ker F(d^{i+1})$, and exactness is preserved. □

Exercise 1.5.G. Suppose A is a ring, $S \subset A$ is a multiplicative subset, and M is an A-module.

a) Show that localization of A-modules is an exact covariant functor.

b) Show that $- \otimes_A M$ is a right-exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$.

c) Show that $\mathrm{Hom}(M, -)$ is a left-exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$. If $\mathscr{C}$ is an abelian category, show that $\mathrm{Hom}_{\mathscr{C}}(C, -)$ is a left-exact covariant functor $\mathscr{C} \to \mathrm{Ab}$.

d) Show that $\mathrm{Hom}(-, M)$ is a left-exact covariant functor $\mathrm{Mod}_A \to \mathrm{Mod}_A$. If $\mathscr{C}$ is an abelian category, show that $\mathrm{Hom}_{\mathscr{C}}(-, C)$ is a left-exact covariant functor $\mathscr{C} \to \mathrm{Ab}$.

Proof. b) was done in Exercise 1.2.H.

a) by b), the localization functor (which can be phrased as $- \otimes S^{-1}A$) is right-exact, covariant. Thus, given

$$0 \to M' \xrightarrow{f} M \to M'' \to 0$$

we know that

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \to S^{-1}M'' \to 0$$

is exact. We need to show that the first map is injective. Assume $S^{-1}f(m/s) = 0$. That means there exists $s' \in S$ such that $s'f(m) = 0$. But every element of s acts invertibly on $S^{-1}M$, so this implies $f(m) = 0$, which in turn implies $m = 0$, since f was injective to begin with.

c) We show the second part first: By the def. of an additive category, $\mathrm{Hom}_{\mathscr{C}}(C, -)$ is a covariant functor to the category of abelian groups. Now if

$$0 \to D' \xrightarrow{f} D \to D''$$

is exact, then f is mono, and the definition of f being mono says that $f \circ g = 0$ implies $g = 0$, so $\mathrm{Hom}_{\mathscr{C}}(C, -)$ is left-exact. Now the first statement is just realising that $\mathrm{Hom}_A(M, -)$ carries an A-module structure by $(a \cdot f)(m) = af(m) = f(am)$.

d) Is the same, but with arrows reversed (and doing f^* instead of f_*, using properties of epimorphisms). $\square$

Exercise 1.5.H. Suppose M is a finitely presented A-module, i.e. there is an exact-sequence

$$A^{\oplus q} \to A^{\oplus p} \to M \to 0$$

Describe an isomorphism

$$S^{-1}\mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

Solution. Applying $\mathrm{Hom}_A(-, N)$ to the sequence gives

$$0 \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(A^{\oplus p}, N) \to \mathrm{Hom}_A(A^{\oplus q}, N)$$

Now since Hom commutes with direct sums, and $\mathrm{Hom}_R(R, L) = L$ for any ring R, and R-module L, we get

$$0 \to \mathrm{Hom}_A(M, N) \to N^{\oplus p} \to N^{\oplus q}$$

and finally, applying localization with S gives

$$0 \to S^{-1}\mathrm{Hom}_A(M, N) \to S^{-1}N^{\oplus p} \to S^{-1}N^{\oplus q}$$

One the other hand, we can start by applying localization, and then $\mathrm{Hom}_{S^{-1}A}(-, S^{-1}N)$, and identifying $\mathrm{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N) = N$, which results in

$$0 \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to S^{-1}N^{\oplus p} \to S^{-1}N^{\oplus q}$$

which shows the desired isomorphism.
The isomorphism is given by $f/s \to (f/s)'(m/s') = S^{-1}f(m)/(ss')$. $\square$

Exercise 1.5.I (The FHHF Theorem)**.** Suppose $F : \mathscr{A} \to \mathscr{B}$ is a covariant functor of abelian categories, and $C^\bullet$ is a complex in $\mathscr{A}$.

a) F right-exact yields $FH^\bullet \to H^\bullet F$: If F is right-exact, describe a natural morphism $FH^\bullet \to H^\bullet F$.

b) F left-exact yields $FH^\bullet \leftarrow H^\bullet F$: If F is left-exact, describe a natural morphism $H^\bullet F \to FH^\bullet$.

c) F exact yields $FH^\bullet \simeq H^\bullet F$: If F is exact, show that the two morphisms in a) and b) are inverses, and thus isomorphisms.

Solution. a) As we have established before, right-exactness preserves cokernels. Further, since for all $f : A \to B$, we have

$$0 \to \operatorname{im} f \to B \to \operatorname{coker} f \to 0$$

exact, we get that

$$F(\operatorname{im} f) \to F(B) \to \operatorname{coker} F(f) \to 0$$

is exact, and thus there is an epimorphism $F(\operatorname{im} f) \twoheadrightarrow \operatorname{im} F(f)$ by the universal property of the kernel (which is played by $\operatorname{im} f$ here).
Now, applying the two functors in the two orders to the second exact sequence in Exercise 1.5.A gives the following commutative diagram

$$\begin{tikzcd}
 & F(H^i(C^\bullet)) \arrow[r] \arrow[d, dotted, "!h"] & F(\operatorname{coker} d^{i-1}) \arrow[r] \arrow[d, "\sim"] & F(\operatorname{im} d^i) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & H^i(F(C^\bullet)) \arrow[r] & \operatorname{coker} F(d^{i-1}) \arrow[r] & \operatorname{im} F(d^i) \arrow[r] & 0
\end{tikzcd}$$

where the unique induced map h exists by the universal property of kernels. b) The argument is quite similar; start with the exact sequence

$$0 \to \ker f \to A \to \operatorname{im} f \to 0$$

which after applying the left-exact F looks like

$$0 \to F(\ker f) \to F(A) \to F(\operatorname{im} f)$$

which now gives a morphism $\operatorname{im} F(f) \to F(\operatorname{im} f)$, by the universal property of cokernels. In this way, we can construct the map by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} F(d^{i-1}) \arrow[r] \arrow[d] & \ker F(d^i) \arrow[r] \arrow[d, "\sim"] & H^i(F(C^\bullet)) \arrow[r] \arrow[d, dotted, "!h"] & 0 \\
0 \arrow[r] & F(\operatorname{im} d^{i-1}) \arrow[r] & F(\ker d^{i-1}) \arrow[r] & F(H^i(C^\bullet)) &
\end{tikzcd}$$

c) Its easy to see that h and h' are isomorphisms when F is exact, and they must be mutually inverse, since they were defined with universal properties. $\square$

Exercise 1.5.J (Kernels commute with limits)**.** Suppose $\mathscr{C}$ is an abelian category, and let $a : \mathscr{I} \to \mathscr{C}$ and $b : \mathscr{I} \to \mathscr{C}$ be two diagrams in $\mathscr{C}$ indexed by $\mathscr{I}$. Write $A_i := a(i)$ and $B_i := b(i)$ for objects in the respective diagrams. Let $h_i : A_i \to B_i$ be maps commuting with the maps in the diagram (i.e. $h : a \to b$ is a natural transformation). Then $\ker h_i$ form another diagram in $\mathscr{C}$ indexed by $\mathscr{I}$. Describe a canonical isomorphism

$$\varprojlim_{\mathscr{I}} \ker h_i \xleftrightarrow{\sim} \ker(\varprojlim_{\mathscr{I}} A_i \to \varprojlim_{\mathscr{I}} B_i)$$

assuming all limits exists.

Solution. The composition $h_i \circ \pi_i : \varprojlim_{\mathscr{I}} A_i \to A_i \to B_i$ shows that there exists a unique map $\varprojlim_{\mathscr{I}} A_i \to \varprojlim_{\mathscr{I}} B_i$. Now let the kernel of this map be denoted by K.
Similarily, we may take the kernel of each h_i, which induces a new diagram in $\mathscr{C}$ indexed by $\mathscr{I}$. The maps $\varprojlim_{\mathscr{I}} \ker h_i \to \ker h_i \to A_i$ gives a unique map $\varprojlim_{\mathscr{I}} \ker h_i \to \varprojlim_{\mathscr{I}} A_i$, and since composing with the h_i obviously becomes 0, we get a canonical map $\varprojlim_{\mathscr{I}} \ker h_i \to K$.
However, the maps $K \to \varprojlim_{\mathscr{I}} A_i \to A_i \to B_i$ are zero, we get induced maps $K \to \ker h_i$, which in turn gives a universal map $K \to \varprojlim_{\mathscr{I}} \ker h_i$. These maps are mutually inverse isomorphisms.
$\square$

Exercise 1.5.K. Make sense of the statement that "limits commutes with limits" in a general category, and prove it.

Solution. We use two indexing categories $\mathscr{I}$ and $\mathscr{J}$. Diagrams in $\mathscr{C}$ indexed by $\mathscr{I}$ form a new category, namely the categroy of functors $\mathscr{I} \to \mathscr{C}$ (denoted $\mathscr{C}^{\mathscr{I}}$). Now consider a diagram $\mathscr{J} \to \mathscr{C}^{\mathscr{I}}$. First, notice that maybe surprisingly, this data is symmetric in $\mathscr{I}$ and $\mathscr{J}$. That is, this data gives for each $i \in I$, a diagram $\mathscr{J} \to \mathscr{C}$, and morphisms in I gives natural transformations in $\mathscr{C}^{\mathscr{J}}$. Thus, more conceptually, we may consider such a double diagram a single

$$F : \mathscr{I} \times \mathscr{J} \to \mathscr{C}$$

where $F(-, j)$ and $F(i, -)$ gives diagrams from $\mathscr{I}$ and $\mathscr{J}$ respectively. The now two ways of forming the double limit obviously commute, since they are both equal to the single limit $\varprojlim_{(i,j)\in\mathscr{I}\times\mathscr{J}} A_{ij}$. $\square$

Exercise 1.5.L. Show that in Mod$A$, colimits over filtered index categories are exact.

Solution. The previous exercise also shows that colimits commute with colimits, thus colimits preserves cokernels, i.e. colimits are right-exact.
Thus, given

$$0 \to A_i \xrightarrow{f_i} B_i \to C_i \to 0,$$

that additinally commutes with every map from $\mathscr{I}$, we can apply the colimit, and get

$$\varinjlim_{\mathscr{I}} A_i \xrightarrow{f} \varinjlim_{\mathscr{I}} B_i \to \varinjlim_{\mathscr{I}} C_i \to 0$$

we must show that the first map f is injective. Let $f(a) = 0$. This means that for some n, $f_n(a_n) = 0 \in B_n$ (since colimits over filtered index categories satisfies 0 in the limit if and only if 0 at some index). But f_n is injective, and thus $a_n = 0 \in A_n$, which in turn implies that $a = 0$ to begin with. $\square$

Exercise 1.5.M. Show that filtered colimits commute with homology in Mod$A$.

Solution. By the previous exercise, filtered colimits are exact, and thus it follows from the FHHF-theorem (Exercise 1.5.I). $\square$

Exercise 1.5.N. Suppose

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_0 \arrow[r] & B_0 \arrow[r] & C_0 \arrow[r] & 0
\end{tikzcd}$$

is an inverse system of exact sequences of modules over a ring, such that the maps $A_{n+1} \to A_n$ are surjective. Show that the limit

$$0 \to \varprojlim_n A_n \to \varprojlim_n B_n \to \varprojlim_n C_n \to 0$$

is exact.

Solution. First, all maps are defined by the universal property of limits.
Limits are left-exact, thus the sequence

$$0 \to \varprojlim_n A_n \to \varprojlim_n B_n \to \varprojlim_n C_n$$

is certainly exact. We must show that the last map is also exact: Assume $(c_n)_n \in \varprojlim_n C_n$ is an element. We may consider $(b_n)_n$, where $f_n(b_n) = c_n$ (since the individual maps are surjective). However, $(b_n)_n$ may not be a valid element of $\varprojlim_n B_n$. We correct this piecewise: For each non-zero $d_{n+1}(b_{n+1}) - b_n$, we know that $f_n(d_{n+1}(b_{n+1}) - b_n) = 0$, thus there exists $a_n \in A_n$ with $g_n(a_n) = d_{n+1}(b_{n+1}) - b_n$. Since the left row is surjective, we can lift to a_{n+1}, and set $b'_{n+1} := b_{n+1} - g_{n+1}a_{n+1}$. Of course, this is still a valid lift of c_n, but now, $d_{n+1}(b'_{n+1}) - b_n = 0$. Doing this for all n gives a valid preimage of $(c_n)_n$, showing that the last map is surjective. $\square$

1.6 *Spectral sequences

2 Sheaves

2.1 The sheaf of smooth functions

In the first two exercises, we are considering a manifold $X \subseteq \mathbb{R}^n$, and $\mathcal{O}_X$, the sheaf of smooth functions on X.

Exercise 2.1.A. Show that the only maximal ideal of $\mathcal{O}_p$ is $\mathfrak{m}_p$.

Solution. Let $(f, U) \in \mathcal{O}_p \setminus \mathfrak{m}_p$ be a germ at p. Since the roots of f are distinct, there exists an open set $p \in V \subset U$ such that f has no roots in V. Then $(1/f, V) \in \mathcal{O}_p$ ($1/f$ is still smooth on regions where f has no roots) is an inverse of (f, U). $\square$

Exercise 2.1.B. Prove that $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a module over $\mathcal{O}_p/\mathfrak{m}_p \simeq \mathbb{R}$. Prove that it is naturally the cotangent space to the differentiable or analytic manifold at p.

Solution. The module part straight-forward: We simply define $[f][g] = [fg]$, where $f \in \mathcal{O}_p, g \in \mathfrak{m}_p$ are representatives. This is well-defined: assume $f' = f + h$ for some $h \in \mathfrak{m}_p$. Then $[f'][g] = [fg + hg] = [fg]$, since $hg \in \mathfrak{m}_p^2$.
For the map $\mathfrak{m}_p \to T_p^* X$, send $f_p \to df_p$, where $df_p : T_pX \to \mathbb{R}$ is the differential $v \to v(f)$, thought of ass "taking the derivative of f in the direction of v". This is linear, and the kernel is $\mathfrak{m}_p^2$: If $f \in \mathfrak{m}_p^2$, then $f = \sum_i a_i b_i$ for $a_i, b_i \in \mathfrak{m}_p$, but then $df = \sum a_i db_i + b_i da_i$, which is still 0 at p. Conversely, if $df_p = 0$, then the Taylor expansion around p shows that f has no linear part, so $f_p \in \mathfrak{m}_p^2$. $\square$

2.2 Definition of sheaf and presheaf

Exercise 2.2.A. Verify that the data of a presheaf is precisely the data of a contraviariant functor from the category of open sets of X to the category of sets.

Solution. This is just checking both definitions. $\mathrm{Res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$ is one of the functor axioms, and the compatibility of the restriction maps is just functoriality. $\square$

Exercise 2.2.B. Show that the following are presheaves on $\mathbb{C}$ (with the classical topology), but not sheaves: a) bounded functions, b) holomorphic functions admitting a holomorphic square root.

Solution. It is easy to see that both a) and b) satisfy the presheaf axioms. Further, they both clearly satisfy the identity axiom (functions are defined by what they do on points). However, neither satisfies gluing: a) gluing together infinitely many bounded functions may not stay bounded, and b) there is famously no well-defined square root on all of $\mathbb{C}$ (so you are unable to glue together $f(z) = z$ defined on patches). $\square$

Exercise 2.2.C. The identity and gluability axiom may be interpreted as saying that $\mathscr{F}(\bigcup_{i\in I} U_i)$ is a certain limit. What is that limit?

Solution. This is the interpretation as an equalizer (since an equalizer is a limit): We have two arrows

$$\prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

where the arrows are given by the two restriction maps $\mathscr{F}(U_i) \to \mathscr{F}(U_i \cap U_j)$ and $\mathscr{F}(U_j) \to \mathscr{F}(U_i \cap U_j)$. Now saying

$$\mathscr{F}(U) = \varprojlim(\prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j))$$

is then equivalent to the sheaf axioms: The unique map $F(U) \to \prod_i \mathscr{F}(U_i)$ is identity axiom (this map must clearly be injective), and the gluing axiom is saying that a elements in $\mathscr{F}(U_i)$ that agree on the intersections correspond to an element of $\mathscr{F}(U)$. $\square$

Exercise 2.2.D. a) Verify that smooth functions or continuous functions or real-analytic functions, or plain real-valued functions on a manifold are indeed sheaves.

b) Show that real-valued continuous functions on (open sets of) a topological space X form a sheaf.

Solution. a) There is not much to show here, all the properties are stable under restriction, so they are all obviously presheaves. That they all satisfy identity is also immidiate. Gluing is maybe the only part that requires some thinking, but the fact that all these properties can be checked locally means that they can all be glued.
b) Again, there is not really anything to check here, as functions can be defined on points. □

Exercise 2.2.E (Constant sheaf)**.** Let $\mathscr{F}(U)$ be the set of maps $U \to S$. that are locally constant, i.e. for each point $p \in U$, there is an open neighborhood $p \in V$ such that $\mathscr{F}(V)$ is constant. (An alternative description is endowing S with the discrete topology, and defining $\mathscr{F}(U)$ to be all continuous maps $U \to S$.)

Solution. First notice that each $f \in \mathscr{F}(U)$ is forced to take constant values in S on connected components (and the value of x is given by the connected component). As usual, presheaf axioms and identity are clear. Gluing is also clear by the description just given: given $f \in \mathscr{F}(U), g \in \mathscr{F}(V)$, defined $h \in F(U \cup V)$ by what it does on each connected component of $U \cup V$ (given by either f, g or both if they are both defined). □

Exercise 2.2.F. Suppose Y is a topological space. Show that "continuous maps to Y" forms a sheaf of sets on X, i.e. $\mathscr{F}(U) =$ continuous maps $U \to Y$ is a sheaf.

Solution. Presheaf axioms, and identity is just checking. We may obviously glue morphisms as well; we just need to check that it is continuous. Let $f : U \to Y$ be a morphism defined by continuous morphisms on an open cover $f_i : U_i \to Y$. Let $V \subseteq Y$ be open. Now $f^{-1}(V) = \bigcup f_i^{-1}(V)$, thus open. □

Exercise 2.2.G. This is a fancier version of the previous exercise:

a) Suppose we are given a continuous map $\mu : Y \to X$. Show that "sections of μ" form a sheaf, i.e. $\mathscr{F}(U) =$ continuous maps $s : U \to Y$ such that $\mu \circ s = \mathrm{id}_U$.

b) Suppose that Y is a topological group. Show that continuous maps to Y form a sheaf of groups.

Solution. a) In the previous exercise, we showed that continuous maps to Y form a sheaf. So we only need to restrict to sections of μ and see what happens: Restriction maps are still okay (so it is a presheaf), and so is both identity and gluing.
b) We may indeed given $f, g \in \mathscr{F}(U)$ form $fg(x) = f(x)g(x)$. Since the group operation is continuous, this is still a continuous function defined on U. The fact that the restrictions are also group-homomorphisms is obvious (restricting, then multiplying is the same as multiplying the functions, then restricting). □

Exercise 2.2.H. Suppose $\pi : X \to Y$ is a continuous map, and that $\mathscr{F}$ is a (pre)sheaf on X. Then define a presheaf $\pi_*\mathscr{F}$ on Y by $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$. Show that $\pi_*\mathscr{F}$ is a (pre)sheaf on Y.

Solution. First, assume $\mathscr{F}$ is a presheaf. Viewed as a functor, it is then clear that Then $\pi_*(\mathscr{F})$ is a presheaf, as it is the composition

$$\mathrm{Open}(Y) \to \mathrm{Open}(X) \xrightarrow{\mathscr{F}} \mathscr{C}$$

where the first map is given by inverse images of π.
Assuming that $\mathscr{F}$ is a sheaf, then $\pi_*\mathscr{F}$ is also a sheaf: let $\{V_i\}$ be an open cover of V. For identity, assume that $s, t \in \pi_*\mathscr{F}(V)$, and that $s\mid_{V_i} = u \mid_{V_i}$ for all i. By definition, s and t are elements of $\mathscr{F}(\pi^{-1}(V))$, and similarly with the restrictions. Now, since $\pi^{-1}(V) = \pi^{-1}(\bigcup V_i) = \bigcup \pi^{-1}(V_i)$, it follows that $s = t$.
Gluing can be shown similarily, using $\pi^{-1}(V_i \cap V_j) = \pi^{-1}(V_i) \cap \pi^{-1}(V_j)$. □

Exercise 2.2.I. Suppose $\pi : X \to Y$ is a continuous map, and $\mathscr{F}$ is a sheaf of sets on X. If $\pi(p) = q$, describe the natural map of stalks $(\pi_*\mathscr{F})_q \to \mathscr{F}_p$.

Solution. Using explicit representatives: If $s_q \in (\pi_*\mathscr{F})_q$, that means there exists some open U containing q, such that $s \in \pi_*\mathscr{F}(U)$ is a representative of s_q. But then, by definition of π_*, $s \in \mathscr{F}(\pi^{-1}(U))$, and since $p \in \pi^{-1}(U)$, $(s, \pi^{-1}(U))$ can be seen as an element in $\mathscr{F}_p$. Assume now that (s', V) is another representative of s_q. Then there exists som $W \subseteq U \cap V$ contiaining q, where $s\mid_W = s'\mid_W$. But then this just says that $s\mid_{\pi^{-1}(W)} = s'\mid_{\pi^{-1}(W)}$, and thus, they would map to the same element of $\mathscr{F}_p$.
Using the universal property: For each open set U containing q, there is an obvious map $\pi_*\mathscr{F}(U) \to \mathscr{F}_p$ given by sending $s \to (s, \pi^{-1}(U))$. These commute with the restriction maps (by definition they are the same on the smaller set), thus the universal property of the colimit gives a map $(\pi_*\mathscr{F})_q \to \mathscr{F}_p$. □

Exercise 2.2.J. If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathscr{F}$ is an $\mathcal{O}_X$-module, describe how for each $p \in X$, $\mathscr{F}_p$ is a $\mathcal{O}_{X,p}$-module.

Solution. Assume $(x, U) \in \mathcal{O}_{X,p}$, and $(a, U) \in \mathscr{F}_p$ (by passing to the intersection, we can assume that two representatives are defined on the same open set). By definition, there is a well-defined action $x \star a \in \mathscr{F}(U)$. Now since the square (2.2.13.1) in the book commutes, this action is well defined, regardless of representative, both of x_p and a_p. □

2.3 Morphisms of presheaves and sheaves

Exercise 2.3.A. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on X, and $p \in X$, describe an induced morphism of stalks $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$.

Solution. For each U containing p, we have the map

$$\begin{aligned}\mathscr{F}(U) &\to \mathscr{G}_p \\ s &\to (\varphi(U)(s), U),\end{aligned}$$

and the naturality of φ, it commutes with restriction maps, thus by the universal property of the colimit, we get a map $\mathscr{F}_p \to \mathscr{G}_p$. $\square$

Exercise 2.3.B. Suppose $\pi : X \to Y$ is a continuous map of topological spaces. Show that the pushforward gives a functor $\pi_* : \mathrm{Sets}_X \to \mathrm{Sets}_Y$.

Solution. We have already shown that the pushforward sends objects to objects, thus we need to show functoriality. Assume $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on X. Now, the pushforward induces the map

$$\begin{aligned}\pi_*\varphi(U) : \pi_*\mathscr{F}(U) &\to \pi_*\mathscr{G}(U) \\ s &\to \varphi(U)(s)\end{aligned}$$

which makes sense since indeed, if $s \in \mathscr{F}(\pi^{-1}(U))$, then $\varphi(s) \in \mathscr{G}(\pi^{-1}(U)) = (\pi_*\mathscr{G})(U)$. The fact that these maps indeed play well with restriction also follows from the fact that φ does.
It is clear that sending φ to $\pi_*\varphi$ is functorial: both preserving identity and associativity is obvious. $\square$

Exercise 2.3.C (Sheaf Hom)**.** Suppose $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of sets on X (in fact, it will suffice that $\mathscr{F}$ is a presheaf). Let $\mathcal{HOM}(\mathscr{F}, \mathscr{G})$ be the collection of data

$$\mathcal{HOM}(\mathscr{F}, \mathscr{G})(U) := \mathrm{Hom}(\mathscr{F}\mid_U, \mathscr{G}\mid_U).$$

Show that this is a sheaf of sets on X.

Solution. First, we show its a presheaf:
If $V \subseteq U$, then there is an obvious map $\mathcal{HOM}(\mathscr{F}, \mathscr{G})(U) \to \mathcal{HOM}(\mathscr{F}, \mathscr{G})(V)$ by just restricting the morphisms to V. This clearly preserves identity, and is associative, thus it is a presheaf.
We now go on to show its a sheaf: assume that $\{U_i\}$ is an open cover of the open set $U \subseteq X$. Assume we are given $\varphi_i \in \mathcal{HOM}(\mathscr{F}, \mathscr{G})(U_i)$ that agree on overlaps $U_i \cap U_j$. Define a new $\varphi \in \mathcal{HOM}(\mathscr{F}, \mathscr{G})(U)$ as follows: For any $V \subseteq U$, and $s \in \mathscr{F}|_U(V)$, set

$$\varphi(V)(s) = s'$$

where the section $s' \in \mathscr{G}|_U(V)$ is obtained by gluing $\varphi_i(V \cap U_i)(s|_{V \cap U_i})$ (this gluing is possible because φ_i agree on overlaps, and $\mathscr{G}$ is a sheaf).
To show uniqueness, assume now that $\psi : \mathscr{F}|_U \to \mathscr{G}|_U$ is another morphism of sheaves, such that $\psi|_{U_i} = \varphi_i$ for all i. Then $\psi(V)(s) = \varphi(V)(s)$ for all V and s, since they agree on an open cover $\{V \cap U_i\}$ and $\mathscr{G}$ is a sheaf (hence glues uniquely).

$\square$

Exercise 2.3.D. a) If $\mathscr{F}$ is a sheaf of sets on X, then show that $\mathcal{HOM}(\underline{\{p\}}, \mathscr{F}) \cong \mathscr{F}$, where $\underline{\{p\}}$ is the constant sheaf on X with values in $\{p\}$.

b) If $\mathscr{F}$ is a sheaf of abelian groups on X, then show that $\mathcal{HOM}_{\mathrm{Ab}_X}(\underline{\mathbb{Z}}, \mathscr{F}) \cong \mathscr{F}$ (an isomorphism of sheaves of abelian groups).

c) If $\mathscr{F}$ is an $\mathscr{O}_X$-module, then show that $\mathcal{HOM}_{\mathrm{Mod}_{\mathscr{O}_X}}(\mathscr{O}_X, \mathscr{F}) \cong \mathscr{F}$ (an isomorphism of $\mathscr{O}_X$-modules).

Solution. For a), first note that the constant sheaf $\underline{\{p\}}$ really is the same as the constant presheaf with values in $\{p\}$; usually a constant sheaf $\underline{A}$ would associalt $\underline{A}(U)$ with continuous functions $U \to A$ (where A is given the discrete topology), but in this case, with the one-element set, there is only one such function anyway, sending everything to p. So to solve the exercise, we first construct a morphism of sheaves $\mathcal{HOM}(\underline{\{p\}}, \mathscr{F}) \to \mathscr{F}$ by sending for each open U, and section $\varphi : \underline{\{p\}}|_U \to \mathscr{F}|_U$ to $\varphi(U)(s_p)$, where s_p is the unique section in $\underline{\{p\}}(U)$ (notice that $\varphi(s_p)$ completely determines φ because for any $V \subseteq U$, $\varphi(V)(s_p|_V) = \varphi(U)(s_p)|_V$). Conversely, any $s \in \mathscr{F}(U)$ determines such a morphism of sheaves, thus for any U, this is an isomorphism between $\mathcal{HOM}(\underline{\{p\}}, \mathscr{F})(U)$ and $\mathscr{F}(U)$, and it is compatible with restrictions, so this is an isomorphism of sheaves of sets.
For b) and c), it is obviously the same idea, except there is an additional difficulty; the constant sheaf is not the constant presheaf. However, we need only use the universal property; namely $\mathcal{HOM}_{\mathrm{Ab}_X}(\underline{\mathbb{Z}}^{\mathrm{pre}}, \mathscr{F}) \cong \mathscr{F}$ by essentially the same argument, sending φ to $\varphi(1)$. Now this is again isomorphic to $\mathcal{HOM}_{\mathrm{Ab}_X}(\underline{\mathbb{Z}}, \mathscr{F})$ by the universal property of sheafification. The same argument goes through for c). □

Exercise 2.3.E. Show that $\ker_{\mathrm{pre}} \varphi$ is a presheaf.

Solution. Its clear that $\ker_{\mathrm{pre}} \varphi$ assigns opens to abelian groups, thus we need only check the restriction maps. The restriction maps are given by the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker_{\mathrm{pre}} \varphi(U) \arrow[r] \arrow[d, "!h"] & \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
0 \arrow[r] & \ker_{\mathrm{pre}} \varphi(V) \arrow[r] & \mathscr{F}(V) \arrow[r] & \mathscr{G}(V)
\end{tikzcd}$$

where the map h exists by universal property of the kernel. (These maps also make it clear that the composition of these restriction maps behave as expected). □

Exercise 2.3.F. Show that the presheaf cokernel satisfies the universal property of cokernels.

Solution. The universal property it should satisfy is that if $\psi : \mathscr{G} \to \mathscr{H}$ is a morphism of presheaves such that $\psi \circ \varphi = 0$, then there exists a unique morphism $\eta : \operatorname{coker}_{\mathrm{pre}} \varphi \to \mathscr{H}$ making everything commute.
By definition, for every open U, $(\operatorname{coker}_{\mathrm{pre}} \varphi)(U) = \operatorname{coker} \varphi(U)$. Thus, there exists a unique map $(\operatorname{coker}_{\mathrm{pre}} \varphi)(U) \to \mathscr{H}(U)$. We only need to check that these maps commute with restriction. But this follows since in the following diagram

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r] \arrow[d] & (\operatorname{coker}_{\mathrm{pre}} \varphi)(U) \arrow[r] \arrow[d] & \mathscr{H}(U) \arrow[d] \\
\mathscr{G}(V) \arrow[r] & (\operatorname{coker}_{\mathrm{pre}} \varphi)(V) \arrow[r] & \mathscr{H}(V)
\end{tikzcd}$$

the outer and the left squeare commutes. Thus, the right square commutes when precomposed with $\mathscr{G}(U) \to (\operatorname{coker}_{\mathrm{pre}} \varphi)(U)$, but this is an epimorphism, and thus we are done. $\square$

Exercise 2.3.G. Show (or observe) that for a topological space X with open set U, $\mathscr{F} \to \mathscr{F}(U)$ gives a functor from presheaves of abelian groups on X to abelian groups. Then show that this functor is exact.

Solution. That it is a functor is indeed obvious; presheaf maps $\alpha : \mathscr{F} \to \mathscr{G}$ are sent to $\alpha(U) : \mathscr{F}(U) \to \mathscr{G}(U)$.
Notice also that by definition,

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is an exact sequence of presheaves if for every open V,

$$0 \to \mathscr{F}(V) \to \mathscr{G}(V) \to \mathscr{H}(V) \to 0$$

is exact. This says that the functor is exact.
(More generally, if one has two categories, one can form the category of functors $\mathscr{J} \to \mathscr{C}$, denoted $\mathscr{C}^{\mathscr{J}}$, at least if $\mathscr{J}$ is a small category. Then for any $j \in \mathscr{J}$, the "evaluation functor", sending $\mathscr{F} \to \mathscr{F}(j)$ is an exact functor from $\mathscr{C}^{\mathscr{J}} \to \mathscr{C}$.) $\square$

Exercise 2.3.H. Show that a sequence of presheaves $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \cdots \to \mathscr{F}_n \to 0$ is exact if and only if $0 \to \mathscr{F}_1(U) \to \mathscr{F}_2(U) \to \cdots \to \mathscr{F}_n(U) \to 0$ is exact for all U.

Solution. The top sequence being exact means, by definition, that the bottom sequence is exact for all U. Conversely, the evaluation functor is exact by the previous exercise. $\square$

Exercise 2.3.I. Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. Show that the presheaf kernel $\ker_{\mathrm{pre}} \varphi$ is in fact a sheaf. Show that it satisfies the universal property of kernels.

Solution. We need to check that it satisfies the gluing and identity axiom. Take an open U, and an open cover $\{U_i\}_i$. Given sections $s_i \in (\ker \varphi)(U_i)$ that agree on intersections, we can push them through to $\mathscr{F}(U_i)$, where they of course still agree on intersections. These glue together uniquely to $s \in \mathscr{F}(U)$. However, s is really in $\ker \varphi(U)$, because the s_i become 0 in $\mathscr{G}(U_i)$, and thus glue uniquely to 0. Thus, the s_i glue together to $s \in (\ker \varphi)(U)$, in a unique way.
Notice that if we try the same argument for the cokernel, it does not work at all.
For the second statement, we have already shown that it satisfies the universal property of a kernel in the category of presheaves. This is already enough, since the category of sheaves is a full subcategory of the category of presheaves. $\square$

Exercise 2.3.J. Let X be $\mathbb{C}$ with the classical topology, let $\mathcal{O}_X$ be the sheaf of holomorphic functions, and let $\mathscr{F}$ be the presheaf of functions admitting a holomorphic logarithm. Describe an exact sequence of presheaves on X:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\varphi} \mathcal{O}_X \xrightarrow{\psi} \mathscr{F} \to 0$$

where $\underline{\mathbb{Z}} \to \mathcal{O}_X$ is the natural inclusion, and $\mathcal{O}_X \to \mathscr{F}$is given by $f \to e^{2\pi i f}$. Show that $\mathscr{F}$ is not a sheaf.

Solution. Notice that, by definition, $\mathscr{F}(U) := \{g \in \mathcal{O}_X(U) \mid \exists f \in \mathcal{O}_X(U), e^{2\pi i f} = g\}$. Thus, $\psi(U)$ is obviously surjective, and exactness comes down to showing that $\operatorname{im} \varphi = \ker \psi$. Its clear that $\operatorname{im} \varphi \subseteq \ker \psi$. Assume now that $f \in \ker \psi(U)$, i.e. that $e^{2\pi i f(z)} = 1$ for all $z \in U$. That means that $f(z) \in \mathbb{Z}$ for all $z \in U$, and since f is holomorphic, this again implies that f is a locally constant integer, i.e. an element of $\operatorname{im} \varphi(U)$.
However, $\mathscr{F}$ is famously not a sheaf: For instance, consider the open cover of $U = \mathbb{C} \setminus \{0\}$ given by $U_1 = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $U_2 = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The function $g(z) = z$ are part of both $\mathscr{F}(U_i)$ (i.e. there is a holomorphic branch of the logarithm function on U_i), but famously do not glue to $\mathscr{F}(U)$ (i.e. there is no single-valued logarithm function on all of $\mathbb{C}^\times$). $\square$

2.4 Properties determined at the level of stalks and sheafification

Exercise 2.4.A. Prove that a section of a (separated pre)sheaf of sets is determined by its germs, i.e. the natural map

$$\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p$$

is injective.

Solution. Given a section $s \in \mathscr{F}(U)$ the natural map sends s to the equivalence class of (s, U) at p for all p. Assume now that $s' \in \mathscr{F}(U)$ is another section satisfying $(s', U) \sim_p (s, U)$ at each point p. This means that for each point p, we can find an open V such that $s|_V = s'|_V$. Obviously, the V form an open cover, so the identity axiom says that $s = s'$. $\square$

Exercise 2.4.B. Prove that any choice of compatible germs for a sheaf of sets $\mathscr{F}$ over U is the image of a section $\mathscr{F}$ over U.

Solution. The definition of a compatible germs $(s_p)_p$, says that we can form a cover $\{U_i\}_i$ over U, and choose sections s_i such that (s_i, U_i) is a valid representative of s_p for all $p \in U_i$. Notice that this implies that the s_i agree on intersections: let $V = U_i \cap U_j$. Then $(s_i|_V, V) \simeq_p (s_j|_V, V)$ for all $p \in V$, but this implies that $s_i = s_j$ by the previous exercise. Thus, we may glue the s_i by the gluability axiom. □

Exercise 2.4.C. If φ_1 and φ_2 are morphisms from a presheaf of sets $\mathscr{F}$ to sheaf of sets $\mathscr{G}$ that induce the same maps on each stalk, show that $\varphi_1 = \varphi_2$.

Solution. By the commutativity of the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi_i"] \arrow[d] & \mathscr{G}(U) \arrow[d, hook, "\iota"] \\ \prod_{p \in U} \mathscr{F}_p \arrow[r] & \prod_{p \in U} \mathscr{G}_p \end{tikzcd}$$

$\varphi_1 \circ \iota = \varphi_2 \circ \iota$ (since the down-right path is equal by the hypothesis). But ι is mono, so this implies that $\varphi_1 = \varphi_2$. □

Exercise 2.4.D. Show that a morphism of sheaves of sets is an isomorphism if and only if it induces an isomorphism of all stalks.

Solution. Passing to stalks is functorial, so an isomorphism of sheaves induces an isomorphism at all stalks. The tricky part is the converse:
Assuming that the induced map on stalks is an isomorphism, the commuting square looks like

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\ \prod_{p \in U} \mathscr{F}_p \arrow[r, "\sim"] & \prod_{p \in U} \mathscr{G}_p \end{tikzcd}$$

(since $\mathscr{F}$ is also assumed to be a sheaf), which shows that the map $\mathscr{F}(U) \to \mathscr{G}(U)$ must be injective for all U. To get surjectivity, assume we are given $s \in \mathscr{G}$. Passing to stalks, and using the inverse map on stalks, notice that this gives germs $(t_p)_p \in \prod \mathscr{F}_p$. If we can show that these are compatible gemrs, we can apply Exercise 2.4.B to get a corresponding section $t \in \mathscr{F}(U)$, and it is obvious that $\varphi(U)(t) = s$ (again using the commuting square).
It remains to show that being compatible is stable under isomorphism of stalks. By construction, have that $\varphi_p(t_p) = s_p$ for all p, meaning that for each p, there is an open U_p, and section $f_p \in \mathscr{F}(U_p)$ such that $(\varphi(U_p)(f_p), U_p)$ is a valid representative of s_p, i.e. after possibly shrinking U_p, we get $(\varphi(U_p)(f_p) = s|_{U_p}$ for all p. Now looking at intersections, these f_p must all agree: Let $V = U_p \cap U_q$, then $\varphi(V)(f_p|V) = \varphi(V)(f_q|V) = s|_V$. But we already proved injectivity, so this implies that $f_p|_V = f_q|_V$. □

Exercise 2.4.E. a) Show that Exercise 2.4.A is false for general presheaves.

b) Show that Exercise 2.4.C is false for general presheaves.

c) Show that Exercise 2.4.D is false for general presheaves.

Solution. Let $X = \{p, q\}$ be a two point sets with the discrete topology, and let $\mathscr{F}$ be the sheaf of sets defined by $\mathscr{F}(\{p\}) = \mathscr{F}(\{q\}) = \{1\}$ and $\mathscr{F}(X) = \{1, 2\}$.
For a), it is obvious that there is no injective function $\{1, 2\} \to \{1\} \times \{1\}$.
For b), consider morphisms of sheaves $\varphi : \mathscr{F} \to \mathscr{F}$. There is only one possible map on the stalks, but clearly many possible maps $\mathscr{F}(X) \to \mathscr{F}(X)$.
For c), consider $\varphi : \mathscr{F} \to \mathscr{F}$ given by $\varphi(X)(b) = 1$ for $b \in \{1, 2\}$. □

Exercise 2.4.F. Show that sheafification is unique up to unique isomorphism, assuming it exists. Show that if $\mathscr{F}$ is already a sheaf, then the sheafification is $\mathrm{id} : \mathscr{F} \to \mathscr{F}$.

Solution. This is direct from the universal property. □

Exercise 2.4.G. Assume for now that sheafification exists. Use the universal property to show that for any morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$, we get a natural induced morphism of sheaves $\varphi^{\mathrm{sh}} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}^{\mathrm{sh}}$. Show that sheafification is a functor from presheaves on X to sheaves on X.

Solution. The universal property of sheafification ensures a map $\varphi^{\mathrm{sh}} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}^{\mathrm{sh}}$ so that $\varphi^{\mathrm{sh}} \circ \mathrm{sh}_{\mathscr{F}} = \mathrm{sh}_{\mathscr{G}} \circ \varphi$. This is well-defined, by the universal property of $\mathrm{sh}_{\mathscr{F}}$ (and also $\mathscr{G}$ in the sense that this morphism is also unique up to unique isomorphism).
This is also functorial: Composition is preserved, since if $\psi : \mathscr{G} \to \mathscr{H}$ is another morphism of sheaves, then

$$\psi^{\mathrm{sh}} \circ \varphi^{\mathrm{sh}} \circ \mathrm{sh}_{\mathscr{F}} = \psi^{\mathrm{sh}} \circ \mathrm{sh}_{\mathscr{G}} \circ \varphi = \mathrm{sh}_{\mathscr{H}} \circ \psi \circ \varphi,$$

so by the universal property, $(\psi \circ \varphi)^{\mathrm{sh}} = \psi^{\mathrm{sh}} \circ \varphi^{\mathrm{sh}}$.
Identities are sent to identities. To see this, note that the sheafification of $\mathrm{id}_{\mathscr{F}}$ is the morphism satisfying

$$(\mathrm{id}_{\mathscr{F}})^{\mathrm{sh}} \circ \mathrm{sh}_{\mathscr{F}} = \mathrm{sh}_{\mathscr{F}} \circ \mathrm{id}_{\mathscr{F}} = \mathrm{sh}_{\mathscr{F}}$$

and clearly, $\mathrm{id}_{\mathscr{F}^{\mathrm{sh}}}$ satisfies this. □

Exercise 2.4.H. Show that $\mathscr{F}^{\mathrm{sh}}$ (using the tautological restriction maps) forms a sheaf.

Solution. Recall, $\mathscr{F}^{\text{sh}}$ is defined by letting the functions on an open be precicely the compatible germs:

$$\mathscr{F}^{\text{sh}}(U) := \{(f_p \in \mathscr{F}_p)_{p\in U} \mid \forall p \in U, \text{there exists an open neighborhood } V \subseteq U \text{ of } p, \text{ and } s \in \mathscr{F}(V) \text{ such that } s_q = f_q \text{ for all } q \in V.\}$$

The restriction maps (which note really deserves the name now) obviously make this into a presheaf on X.
We now show that it is a sheaf. Given an open cover $\{U_i\}_i$ of U, and sections $(f_p^i \in \mathscr{F}_p)_{p\in U_i} \in \mathscr{F}^{\text{sh}}(U_i)$ that agree on intersections, it is clear that we may form the element $(f_p \in \mathscr{F}_p)_{p\in U} \in \mathscr{F}^{\text{sh}}(U)$ by $f_p = f_p^i$ for any i with $p \in U_i$. These are again compatible germs, since for each $p \in U$, assume $p \in U_i$, then $f_p = f_p^i$ which is represented by an element $s \in \mathscr{F}(V \cap U_i)$ that also satisfies $s_q = f_q^i$ for all $q \in V \cap U_i$, thus we have gluability.
The identity axiom is obvious, since a section of compatible germs are determined by the underlying germs. □

Exercise 2.4.I. Describe a natural map of presheaves $\text{sh} : \mathscr{F} \to \mathscr{F}^{\text{sh}}$.

Solution. Send $s \in \mathscr{F}(U)$ to $(s_p \in \mathscr{F}_p)_{p\in U} \in \mathscr{F}^{\text{sh}}(U)$. □

Exercise 2.4.J. Show that the map sh satisfies the universal property of sheafification.

Solution. Assume that $\mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, and that $\mathscr{G}$ is a sheaf. Define φ^{sh} by sending $\varphi^{\text{sh}}(U)(s)$ to the element obtained by gluing the compatible sections $(\varphi_p(s_p) \in \mathscr{G}_p)_{p\in U}$ (or, if you don't wanna glue, we can work with $\mathscr{G}^{\text{sh}}$ directly, which we know is canonically isomorphic to $\mathscr{G}$). It is clear that $\varphi^{\text{sh}} \circ \text{sh} = \varphi$ (for instance, by noting that the map of stalks is the same, and applying Exercise 2.4.C). Assume now that ψ is another morphism with $\psi \circ \text{sh} = \varphi$. Then φ^{sh} and ψ again give the same map on stalks; by Exercise 2.4.C they are the same map. □

Exercise 2.4.K. Show that the sheafification functor is left-adjoint to the forgetful functor from sheaves on X to presheaves on X.

Solution. We need to show the existance of a natural bijection $\mathrm{Hom}_{\mathrm{PSh}}(\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{Sh}}(\mathscr{F}^{\mathrm{sh}}, \mathscr{G})$. This bijection, we've already mentioned many times, sending φ to φ^{sh}. Naturality says that for any $\psi : \mathscr{F}_1 \to \mathscr{F}_2$, the square

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Sh}}(\mathscr{F}_2^{\mathrm{sh}}, \mathscr{G}) \arrow[r, "\sim"] \arrow[d, "\psi^{\mathrm{sh}*}"'] & \mathrm{Hom}_{\mathrm{PSh}}(\mathscr{F}_2, \mathscr{G}) \arrow[d, "\psi^*"] \\ \mathrm{Hom}_{\mathrm{Sh}}(\mathscr{F}_1^{\mathrm{sh}}, \mathscr{G}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{PSh}}(\mathscr{F}_1, \mathscr{G}) \end{tikzcd}$$

should commute, which it does: Starting with f in the upper left corner results in $f \circ \psi^{\mathrm{sh}} \circ \mathrm{sh}_{\mathscr{F}_1} = f \circ \mathrm{sh}_{\mathscr{F}_2} \circ \psi$ anyway (equality here is really just the universal property of sheafification).
The similar square for any morphism $\psi' : \mathscr{G}_1 \to \mathscr{G}_2$ also commutes. □

Exercise 2.4.L. Show that $\mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ induces an isomorphism of stalks.

Solution. The induced map on stalks works by sending a representative $s_x = (s, U) \in \mathscr{F}_x$ to the representative $((s_p)_{p \in U}, U) \in \mathscr{F}_x^{\mathrm{sh}}$. We show that this is a bijection:
First we show injectivity. Assume that $t_x = (t, V) \in \mathscr{F}_x$ is sent to the same germ in $\mathscr{F}_x^{\mathrm{sh}}$. By definition, this means that for all $q \in U \cap V$, there exists an open neighborhood $W \subseteq U \cap V$ such that $(t, W) = (s, W)$ in $\mathscr{F}_q$, i.e. that $t \mid W' = s \mid W'$ for some open neighborhood $W' \subseteq W$ of q. But x is in this intersection, so clearly $t_x = s_x$.
Next, we show surjectivity. Assume now that $t_x = ((t_p)_{p \in U}, U) \in \mathscr{F}_x^{\mathrm{sh}}$. By definition of a compatible germ, this means there exists some open neighborhood $V \subseteq U$ of x, and $f \in \mathscr{F}(V)$ with $f_q = t_q$ for all $q \in V$. But then clearly, $\varphi_x(f_x) = t_x$.
□

Exercise 2.4.M. Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. Show that the following are equivalent.

a) φ is a monomorphism in the category of sheaves.

b) φ is injective on the level of stalks: $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is injective for all $p \in X$.

c) φ is injective on the level of open sets: $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open $U \subseteq X$.

Solution. We start with $b) \Rightarrow a)$:
Assume φ is injective on the level of stalks. Assume now that $\varphi \circ \psi_1 = \varphi \circ \psi_2$ for morphisms $\psi_i : \mathscr{H} \to \mathscr{F}$. Then, for every open U, we get a diagram like

$$\begin{tikzcd} \mathscr{H}(U) \arrow[r, shift left, "\psi_1(U)"] \arrow[r, shift right, "\psi_2(U)"'] & \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, hook] & \mathscr{G}(U) \arrow[d, hook] \\ & \prod_{p\in U} \mathscr{F}_p \arrow[r, hook] & \prod_{p\in U} \mathscr{G}_p \end{tikzcd}$$

The assumption is that the top row is equal. Then, the morphisms are still equal when composing with $\mathscr{G}(U) \hookrightarrow \prod_{p\in U} \mathscr{G}_p$. Now using the commuting square, and that the other two morphisms are mono (this is the hypothesis b)), we get that $\psi_1(U) = \psi_2(U)$ for all U.
Next, $a) \Rightarrow c)$:
Let $U \subseteq X$ be open. Consider the "indicator sheaf" $\mathscr{I}$ with $\mathscr{I}(V) = \{1\}$ if $V \subseteq U$, and $\mathscr{I}(V) = \varnothing$ otherwise. Consider now morphisms $\psi : \mathscr{I} \to \mathscr{F}$, and notice that such a morphism is uniquely defined by what it sends the unique element of $\mathscr{I}(U)$ to (and any choice of element also gives such a morphism). Assume now that there exists some open $U \subseteq X$, for which $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not injective, say $\varphi(U)(x) = \varphi(U)(y)$. Then we may form the morphisms $\psi_x : \mathscr{I} \to \mathscr{F}$ and ψ_y by sending 1 to x, y respectively. Notice now, for all open $W \subseteq X$, the composition $\mathscr{I}(W) \to \mathscr{F}(W) \to \mathscr{G}(W)$ will be equal, implying that $\varphi \circ \psi_x = \varphi \circ \psi_y$. But using the hypothesis, this implies that $\psi_x = \psi_y$, which in turn implies that $x = y$.
Finally, $c) \Rightarrow b)$ can be shown directly:
Assume that $\varphi_p(s_p) = \varphi_p(t_p)$ for germs $s_p, t_p \in \mathscr{F}_p$. This means that there exists some V, and sections $s, t \in \mathscr{F}(V)$ such that $\varphi(V)(s)|_W = \varphi(V)(t)|_W$ for some open neighborhood $W \subseteq V$ of p. But this means that $\varphi(W)(s|_W) = \varphi(W)(t|_W)$. Since $\varphi(U)$ was assumed injective for all U, this implies that $s|_W = t|_W$, which in turn implies that $s_p = t_p$. □

Exercise 2.4.N. Continuing the notation of the previous exercise, show that the following are equivalent.

a) φ is an epimorphism in the category of sheaves.

b) φ is surjective on the level of stalks: $\varphi_p : \mathscr{F}_p \to \mathscr{F}_p$ is surjective for all $p \in X$.

Solution. We first prove $b) \Rightarrow a)$:
Assume $\psi_i : \mathscr{G} \to \mathscr{H}$ are morphisms for $i = 1, 2$, and assume that $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. For any U, consider the corresponding map of stalks

$$\prod_{p \in U} \mathscr{F}_p \to \prod_{p \in U} \mathscr{G}_p \to \prod_{p \in U} \mathscr{H}_p$$

Since the first part of this map is epi, we see that the map on stalks induced by the φ_i are equal. But by Exercise 2.4.C, this implies that the φ_i themselves are equal.
Next, $a) \Rightarrow b)$:
Passing to stalks is an exact functor, so epimorphisms are sent to epimorphisms (notice that this couldve been used to prove $b) \to a)$ in the previous exercise too). This can also be proven directly with the skyscraper sheaf: Assume that φ is not surjective on stalks, i.e. there exists a $p \in X$ and $t \in \mathscr{G}_p$ not in the image of φ_p. Let $\Delta_p(\{1, 2\})$ be the skyscraper sheaf at p; $\Delta_p(V) = \{1\}$ if $p \notin V$, and $\Delta_p(V) = \{1, 2\}$ if $p \in V$. Now define two maps $\mathscr{G} \to \Delta_p(\{1, 2\})$ by $\psi_1(U)(x) = 1$ for all U and $x \in \mathscr{F}(U)$, and $\psi_2(U)(y) = 2$ if $y_p = t$, and $\psi_2(U)(y) = 1$ otherwise (notice, this is really enough to define the map!). Now, clearly $\psi_1 \circ \varphi = \psi_2 \circ \varphi$ (since they are always 1, since t is never hit on the level of stalks), which contradicts the fact that φ was epi. □

Exercise 2.4.O. Show that $\exp : \mathscr{O}_X \to \mathscr{O}_X^*$ describes $\mathscr{O}_X^*$ as a quotient sheaf of $\mathscr{O}_X$. Find an open set on which this map is not surjective.

Solution. Let $s_p = (s, U) \in \mathscr{O}_{X,p}^*$. Possibly by shrinking U, we may find logaritm $t \in \mathscr{O}_{X,p}(U)$ satisfying $\exp(t) = s$, so the induced map on stalks is surjective.
However, on $U = \mathbb{C} \setminus \{0\}$, the function $f(z) = z$ is obviously in $\mathscr{O}_X^*(U)$, but admits no single logarithm. □

2.5 Recovering sheaves from a "sheaf on a base"

Exercise 2.5.A. How can you recover a sheaf $\mathscr{F}$ from the partial information on a base of the topology on X.

Solution. Form again the "candidate" stalk $\mathscr{F}_p'$ as the colimit over $\mathscr{F}(B_i)$, where B_i are the open sets of the base containing p. There is an obvious map $\mathscr{F}_p' \to \mathscr{F}_p$, just by sending a representative (s, B) to the class in $\mathscr{F}_p$. However, the other way around, we also have a map: Given $(t, U) \in \mathscr{F}_p$, take any basis element $B \subseteq U$ containing p (which exists since U is a union of basis elements). $(t|_B, B) = (t, U)$, and it also defines an element of $\mathscr{F}_p'$. These maps are obviously mutually inverse.
From the stalks, we can recover $\mathscr{F}$ by using the sheafification (i.e. defining $\mathscr{F}(U)$ to be the compatible germs). □

Exercise 2.5.B. Verify that $F(B) \to \mathscr{F}(B)$ is an isomorphism (here F denotes a sheaf on a base, and $\mathscr{F}$ the corresponding sheaf of compatible germs.

Solution. We construct a map $\mathscr{F}(B) \to F(B)$. Given compatible germs $(s_p)_{p \in B}$, let $\{B_i\}$ be the open cover (of basis sets!) for which the compatible germs have corresponding sections (in $F(B_i)$). By definition, these sections agree on intersections, thus they glue to a section on $F(B)$. If we were to choose a different open cover $\{C_i\}$, then for each point, there would be a smaller set $D_i \subseteq B_i \cap C_i$ on which the representatives agree. Thus the identity axiom ensures that the two gluings must be the same.
This map is obviously an inverse to the natural map $F(B) \to \mathscr{F}(B)$. $\square$

Exercise 2.5.C (Morphisms of sheaves correspond to morphisms of sheaves on a base)**.**

a) Verify that a morphism of sheaves is determined by the induced morphism of sheaves on the base.

b) Show that a morphism of sheaves on the base gives a morphism of the induced sheaves.

Solution. For a), write $U = \bigcup B_i$ as a union of base open sets. Then, for $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$, we can compute $\varphi(U)(s)$ by gluing $\varphi(B_i)(s|_{B_i})$, which must agree with $\varphi(U)(s)$ by naturality and the fact that $\mathscr{G}$ is a sheaf (identity axiom).
For b) the morphism of sheaves on the base already determines the map on stalks, which in term determines the morphism of sheaves (Exercise 2.4.C again!). $\square$

Exercise 2.5.D. Suppose a morphism of sheaves of sets $F \to G$ on a base $\{B_i\}$ is surjective for all B_i. Show that the corresponding morphism of sheaves is surjective.

Solution. The corresponding map on stalks is then obviously surjective, which by Exercise 2.4.N means that the corresponding morphism of sheaves is epi. $\square$

Exercise 2.5.E (Gluing sheaves)**.** Suppose $X = \bigcup U_i$ is an open cover of X, and we have sheaves $\mathscr{F}_i$ on U_i along with isomorphisms

$$\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$$

that agree on triple overlaps, i.e.

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

on $U_i \cap U_j \cap U_k$. Show that these sheaves can be glued together into a sheaf $\mathscr{F}$ on X, with isomorphisms $\mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$, and the isomorphisms over $U_i \cap U_j$ are the obvious ones.

Solution. We construct the sheaf $\mathscr{F}$ defined on the basis of open sets contained in some U_i (this is clearly a basis: any open V is the union $V = \bigcup V \cap U_i$). This is then given by $\mathscr{F}(B) = \mathscr{F}_i(B)$ where $B \subseteq U_i$. This is well defined, due to the isomorphisms φ_{ij} giving $\mathscr{F}_i(B) \simeq \mathscr{F}_j(B)$ whenever $B \subseteq U_i \cap U_j$, and the cocycle condition ensures that this isomorphism is consistent, i.e. if $B \in U_i \cap U_j \cap U_k$, the identification $\varphi_{ik}(B) : \mathscr{F}_i(B) \simeq \mathscr{F}_k(B)$ and $\varphi_{jk}(B) \circ \varphi_{ij}(B) : \mathscr{F}_i(B) \simeq \mathscr{F}_k(B)$ agree.
The sheaf properties (on the basis) now follow from the fact that the $\mathscr{F}_i$ are sheaves. □

2.6 Sheaves of abelian groups, and $\mathscr{O}_X$-modules, form abelian categories

Exercise 2.6.A. Show that the stalk of the kernel is the kernel of the stalks: for all $p \in X$, there is a natural isomorphism

$$(\ker(\mathscr{F} \to \mathscr{G}))_p \simeq \ker(\mathscr{F}_p \to \mathscr{G}_p).$$

Solution. This is just a restatement of the fact that passing to stalks is a (left) exact functor (it is a filtered colimit by definition). □

Exercise 2.6.B. Show that the stalk of the cokernel is naturally isomorphic to the cokernel of the stalk.

Solution. First, the fact that passing to stalks is a (right) exact functor, shows that this is true for the cokernel presheaf. But, this is also true for the sheafification, because the sheafification has naturally isomorphic stalks to the presheaf you start with. □

Exercise 2.6.C. Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of abelian groups. Show that the image sheaf $\operatorname{im} \varphi$ is the sheafification of the image presheaf. Show that the stalk of the image is the image of the stalk.

Solution. [Actually, this argument is circular I now realize, should fix].
The easiest way of proving this is actually the other way around. The fact that the stalk of the image is the image of the stalk follows directly from the fact that passing to stalks is exact, and that sheafification does not change the stalks.
Now we can prove the first statement. Let $\mathscr{H}$ denote the image presheaf (i.e. $\ker(\mathscr{G} \to \operatorname{coker}_{\mathrm{pre}} \varphi)$). We want to show that the $\mathscr{G}$-compatible germs and the $\mathscr{H}$-compatible germs of

$$\prod_{x \in X} \varphi_x(\mathscr{F}_x),$$

because the former describes the image sheaf by the first statement. So let $\mu \in \prod_{x \in X} \varphi_x(\mathscr{F}_x)$. If μ is $\mathscr{H}$ compatible, then it is obviously $\mathscr{G}$-compatible (since we always have $\mathscr{H}(U) \subseteq \mathscr{G}(U)$. Now, assume that μ is $\mathscr{G}$-compatible, i.e. that there exists an open cover U_x, and $g \in \mathscr{G}(U_x)$ such that $g_y = \mu_y$ for all $y \in U_x$. But $\mu_x \in \varphi_x(\mathscr{F}_x)$, thus there exists some open V containing x such that $f \in \varphi_V(V)$ satisfies $f_x = \mu_x$. But since f and g have the same germ at x, there must exist some open W such that $f|_W = g|_W$, and thus $f \in \varphi_W(W)$ by naturality. Picking such opens W for all x shows that μ is $\mathscr{H}$-compatible. □

Exercise 2.6.D. Suppose $\alpha : \mathscr{F} \to \mathscr{G}$ and $\beta : \mathscr{G} \to \mathscr{H}$ are two morphisms of sheaves of abelian groups on X. Show that

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is exact if and only if for all $p \in X$,

$$\mathscr{F}_p \xrightarrow{\alpha_p} \mathscr{G}_p \xrightarrow{\beta_p} \mathscr{H}_p$$

is exact.

Solution. As always, passing to stalks is exact, and hence all we need to prove is the "up" direction of the implication. So assuming the bottom sequence is exact for all $p \in X$, we need to check that $\operatorname{im} \alpha \simeq \ker \beta$. But this isomorphism comes for free by using the definition of sections as compatible stalks. In a little more detail: we can define the image as

$$(\operatorname{im} \alpha)(U) = \{(s_p)_p \in \prod_{p \in U} \mathscr{G}_p \mid (s_p)_p \text{ compatible, and } s_p \in \operatorname{im} \alpha_p \text{ for all } p \in U\}.$$

Similarily, we can define

$$(\ker \beta)(U) = \{(s_p)_p \in \prod_{p \in U} \mathscr{G}_p \mid (s_p)_p \text{ compatible, and } s_p \in \ker \beta_p \text{ for all } p \in U\},$$

and thus its obvious that exactness at all stalks imply exactness of sheaves. □

Exercise 2.6.E. Show that taking the stalk of a sheaf of abelian groups is an exact functor. More precisely, if X is a topological space and $p \in X$ is a point, show that taking the stalk at p defines an exact functor $Ab_X \to Ab$.

Solution. We have already used this a lot, as it was proven in Exercise 1.5.L (recall, a stalk is really a filtered colimit). □

Exercise 2.6.F. Check that the exponential exact sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\times 2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \to 1$$

is indeed an exact sequence of sheaves of abelian groups ($X = \mathbb{C}$ with the classical topologi, and $\mathscr{O}_X$ is the sheaf of holomorphic functions).

Solution. We check this at a stalk $p \in X$: $\underline{\mathbb{Z}}_p = \mathbb{Z}$, $\mathscr{O}_{X,p}$ is the abelian group of germs of holomorphic functions at p under addition, while $\mathscr{O}_{X,p}^*$ is the abelian group of germs of holomorphic functions, nonzero at p, under multiplication.
Exactness at $\underline{\mathbb{Z}}_p$ is obvious. Similarily, exactness at $\mathscr{O}_{X,p}^*$ is clear, as every nowhere vanishing holomorphic function locally admits a holomorphic logarithm. Finally, exactness at $\mathscr{O}_{X,p}$ is shown as follows: Assume $f_p \in \mathscr{O}_{X,p}$ satisfies $\exp(f_p) = 1$. This means that there is a local neighborhood U of p, with f holomorphic on U, and $\exp(f) = 1$ on U, i.e. $\exp(f(z)) = 1$ for all $z \in U$. This already shows that $f = 2n\pi i$, and thus we are done. □

Exercise 2.6.G (Left-exactness of the functor of "sections over U")**.** Supppose $U \subseteq X$ is an open set, and $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ is an exact sequence of sheaves of abelian groups. Show that

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$$

is exact. Show that the section functor need not be exact.

Solution. The fast proof is that the forgetfull functor to presheaves is a right-adjoint (to the sheafification functor), and thus preserves all limits. Hence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is an exact sequence of presheaves, and by Exercise 2.3.G the result follows.
A direct proof: The key is that the sequence is exact for all stalks. Let $s \in \mathscr{F}(U)$, and assume $\alpha(U)(s) = 0$. Then $\alpha_p(s_p) = 0$ for all $p \in U$, implying that $s_p = 0$ for all $p \in U$, in turn implying that $s = 0$. This shows exactness at $\mathscr{F}(U)$. Next, assume $s \in \mathscr{G}(U)$ satisfies $\beta(U)(s) = 0$. Again, this implies that $s_p \in \ker \beta_p = \operatorname{im} \alpha_p$ for all $p \in U$. Thus, for each $p \in U$, we may choose opens V_p, and $t \in \mathscr{F}(V_p)$ such that $\alpha(V_p)(t) = s|_{V_p}$. To glue these t, we must check that they agree on intersections. So for all $p, q \in U$, we have

$$\alpha(V_p \cap V_q)(t_p|_{V_p \cap V_q}) = s|_{V_p \cap V_q} = \alpha(V_p \cap V_q)(t_q|_{V_p \cap V_q})$$

which implies that t_p, t_q agree on the intersection, because $\alpha(V_p \cap V_q)$ is injective! This last part is what fails for proving exactness at $\mathscr{H}(U)$.
As an example, consider indeed the example from the last exercise, and take $U = \mathbb{C} \setminus \{0\}$. As mentioned before, $f(z) = z$ does not admit a holomorphic logarithm on this set. □

Exercise 2.6.H. Suppose $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is an exact sequence of abelian groups on X. If $\pi : X \to Y$ is a continuous map, show that

$$0 \to \pi_*\mathscr{F} \to \pi_*\mathscr{G} \to \pi_*\mathscr{H}$$

is exact.

Solution. The abstract non-sense proof again goes through: π_* is right adjoint to π^{-1}. □

Exercise 2.6.I. Suppose $\mathscr{F}$ is a sheaf of abelian groups on a topological space X. Show that $\mathcal{HOM}(\mathscr{F}, -)$ is a left-exact covariant functor $Ab_X \to Ab_X$. Show that $\mathcal{HOM}(-, \mathscr{F})$ is a left-exact contravariant functor.

Solution. We do the proof for $\mathcal{HOM}(\mathscr{F}, -)$. First we check functoriality: This follows defining for a morphism of sheaves $\varphi : \mathscr{G} \to \mathscr{G}'$, the morphism $\psi : \mathcal{HOM}(\mathscr{F}, \mathscr{G}) \to \mathcal{HOM}(\mathscr{F}, \mathscr{G}')$ by

$$\begin{gathered} \psi(U) : \mathcal{HOM}(\mathscr{F}|_U, \mathscr{G}|_U) \to \mathcal{HOM}(\mathscr{F}|_U, \mathscr{G}'|_U) \\ \alpha \to \varphi|_U \circ \alpha \end{gathered}$$

It is clear that this is functorial. To prove the exercise, note that it will suffice to prove

$$\mathcal{HOM}(\mathscr{F}, \mathscr{G})_x \cong \mathrm{Hom}_{Ab}(\mathscr{F}_x, \mathscr{G}_x)$$

since exactness can be checked at the level of stalks (and the usual hom-functor is left exact). The easiest way to prove this is to remember that $\mathcal{HOM}(\mathscr{F}, \mathscr{G})$ is the sheafification of the presheaf which is defined by $U \to \mathrm{Hom}(\mathscr{F}(U), \mathscr{G}(U))$. Now, computing the stalks of this presheaf we have done before, and the result follows.
The proof for $\mathcal{HOM}(-, \mathscr{F})$ is of course analogous. □

Exercise 2.6.J. Show that if $(X, \mathcal{O}_X)$ is a ringed space, then $\mathcal{O}_X$-modules form an abelian category.

Solution. This is mainly checking all the execises for the previous subsection for $\mathcal{O}_X$-modules instead of abelian groups. □

Exercise 2.6.K (Tensor product of $\mathcal{O}_X$-modules)**.** There are two parts:

a) Suppose $\mathcal{O}_X$ is a sheaf of rings on X. Define (categorically) what we should mean by tensor product of two $\mathcal{O}_X$-modules. Give an explicit construction, and show that it satisfies your categorical definition.

b) Show that the tensor product of stalks is the stalk of the tensor product.

Solution. There is a cool idea to define it by the representable functor

$$\mathcal{HOM}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \cong \mathcal{HOM}(\mathscr{F}, \mathcal{HOM}(\mathscr{G}, \mathscr{H})),$$

which would probably be the categorically most pleasing definition.
An explicit description can be given by first defining the presheaf $U \to \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$. This is indeed a presheaf: For $V \subseteq U$, restrictions $\rho_{\mathscr{F}}$ and $\rho_{\mathscr{G}}$ give a map $\mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U) \to \mathscr{F}(V) \otimes_{\mathcal{O}_X(V)} \mathscr{G}(V)$ by $f \otimes g \to \rho_{\mathscr{F}}(f) \otimes \rho_{\mathscr{G}}(g)$. Define $\mathscr{F} \otimes \mathscr{G}$ as the sheafification of this presheaf.
We will now prove b) before finishing a): We compute the stalks by the presheaf, so $(\mathscr{F} \otimes \mathscr{G})_x := \varinjlim_{x \in U} \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$. The map to $\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x$ is as usual obtained by taking any section and sending it to its representative in the stalk. As usual, this is obviously surjective, but we need to show injectivity: To show this, note that if f is zero in $\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x$, then that, by definition, means that there exists some V so that $f|_V$ is zero, and thus f was zero in the limit already.
Now look at any map $\mathscr{F} \otimes \mathscr{G} \to \mathscr{H}$. On all stalks, we get

$$\begin{aligned}
\mathcal{HOM}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H})_x &\cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(((\mathscr{F} \otimes \mathscr{G})_x, \mathscr{H}_x) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X,x}}((\mathscr{F}_x \otimes \mathscr{G}_x, \mathscr{H}_x) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{G}_x, \mathscr{H}_x)) \\
&\cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathcal{HOM}(\mathscr{G}, \mathscr{H})_x) \\
&\cong \mathcal{HOM}(\mathscr{F}, \mathcal{HOM}(\mathscr{G}, \mathscr{H}))_x
\end{aligned}$$

thus the final result is that $\mathcal{HOM}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \cong \mathcal{HOM}(\mathscr{F}, \mathcal{HOM}(\mathscr{G}, \mathscr{H}))$ which is what we wanted to show. □

2.7 The inverse image sheaf

Exercise 2.7.A. Show that (for a continuous map $\pi : X \to Y$, and a sheaf $\mathscr{G}$ on Y)

$$\pi_{\text{pre}}^{-1}\mathscr{G}(U) = \varinjlim_{V \supset \pi(U) \mathscr{G}} (U)$$

defines a presheaf on X. Show that it needn't form a sheaf.

Solution. Let $U' \subset U$. Then, note that for each $V \supset \pi(U)$, there are maps $\mathscr{G}(V) \to \varinjlim_{V' \supset \pi(U')} \mathscr{G}(V')$ (since $V \supset \pi(U) \supset \pi(U')$). By the universal property of the direct limit, there is thus a uniquely defined map between the colimits, functioning as the restriction maps. The associativity and identity of these restriction maps is obvious.
For the second part, let $X = \{0, 1\}, Y = \{\star\}$, both with the discrete topology. Now $\mathscr{G}(\star) = \mathbb{Z}$. Now, clearly $\pi_{\text{pre}}^{-1}\mathscr{G}(U) = \mathbb{Z}$ for all opens $U \subseteq X$, and furhter, all the restriction maps are just identities. But now it is immidiately clear the gluing fails, taking say $n_1 \in \pi_{\text{pre}}^{-1}\mathscr{G}(\{0\})$, and $n_1 \neq n_2 \in \pi_{\text{pre}}^{-1}\mathscr{G}(\{1\})$. □

Exercise 2.7.B ((π^{-1}, π_*) are adjoint)**.** If $\pi : X \to Y$ is a continuous map, and $\mathscr{F}$ is a sheaf on X, and $\mathscr{G}$ is a sheaf on Y, describe a bijection

$$\mathrm{Hom}_X(\pi^{-1}\mathscr{G}, \mathscr{F}) \leftrightarrow \mathrm{Hom}_Y(\mathscr{G}, \pi_*\mathscr{F})$$

Observe that your bijection is "natural".

Solution. There is also probably a nice proof by passing to stalks, but here I will do a more direct proof:
Let $\varphi : \pi^{-1}\mathscr{G} \to \mathscr{F}$. By the left-adjointness of sheafification, this is the same as a map $\pi^{-1}_{\text{pre}}\mathscr{G} \to \mathscr{F}$. Take some open $U \subseteq Y$. Now there is a map $\mathscr{G}(U) \to \pi^{-1}_{\text{pre}}\mathscr{G}(\pi^{-1}(U))$ (since $U \supset \pi \circ \pi^{-1}(U)$). Now compose with $\varphi|_{\pi^{-1}(U)}$, lands you in $\mathscr{F}(\pi^{-1}(U))$ which is $\pi_*\mathscr{F}(U)$ by definition. Everything here is compatible with restriction maps, thus we have obtained our morphism $\varphi' : \mathscr{G} \to \pi_*\mathscr{F}$ by again sheafifying.
Conversely, let $\psi : \mathscr{G} \to \pi_*\mathscr{F}$. This time, take a $V \subseteq X$. For each $W \supseteq \pi(V)$, we have maps $\mathscr{G}(W) \to \pi_*\mathscr{F}(W)$. We can further map $\pi_*\mathscr{F}(W) = \mathscr{F}(\pi^{-1}(W)) \to \mathscr{F}(V)$ with restriction. Thus, by the universal property of colimit + sheafification, we get maps $\pi^{-1}\mathscr{G}(V) \to \mathscr{F}(V)$.
Chasing everything around will show that these are mutually inverse maps. Further, the bijection is natural, essentially since everything is defined with universal properties. $\square$

Exercise 2.7.C. Show that the stalks of $\pi^{-1}\mathscr{G}$ are the same as the stalks of $\mathscr{G}$.

Solution. Let $p \in X$ be a point. Since stalks can be checked on presheaves, observe that $\varinjlim_{p \in U} \pi^{-1}\mathscr{G}(U) = \varinjlim_{p \in U} \left(\varinjlim_{V \supset \pi(U)} \mathscr{G}(V)\right) = \varinjlim_{\pi(p) \in W} \mathscr{G}(W) = \mathscr{G}_p$. The middle equality here is a bit subtle: it works because the indexing sets are cofinal. $\square$

Exercise 2.7.D. If U is an open subset of Y, $i : U \to Y$ is the inclusion, and $\mathscr{G}$ is a sheaf on Y, show that $i^{-1}\mathscr{G}$ is naturally isomorphic to $\mathscr{G}|_U$.

Solution. This is obvious, since for any open $V \subset U$, $i(V)$ is also open, and thus $i^{-1}\mathscr{G}(V) = \mathscr{G}(i(V))$ (i.e. $i(V)$ is always the final object in the direct limit). $\square$

Exercise 2.7.E. Show that π^{-1} is an exact functor from sheaves of abelian groups on Y to sheaves of abelian groups on X.

Solution. There are two proofs of this, both of which makes it immidiately obvious (as provided in the hints) by using previous exercise. Either pass to stalks, which are literally the same, and thus exactness holds, Or use that filtered colimits are exact. $\square$

Exercise 2.7.F ($\star\star$)**.** We could have defined the push-pull map in a "dual way" starting with the identity $\alpha_*\mathscr{F} \to \alpha_*\mathscr{F}$ on Z, and then using adjointness of (α^{-1}, α_*), and continuing from there. Why does this give the same definition of the push-pull map.

Solution. Hmm. □

Exercise 2.7.G. Show that $\operatorname{Supp} s$ is a closed subset of X.

Solution. We show that the compliment of $\operatorname{Supp} s$ is open: Take any point $p \in X \setminus \operatorname{Supp} s$. By definition, $s_p = 0$, which means that there exists an open set $p \in U$, with $s|_U = 0$. Now for any $q \in U$, we also have $s_q = 0$, and thus $U \subseteq (X \setminus \operatorname{Supp} s)$. □

Exercise 2.7.H. Two parts:

a) Suppse $Z \subseteq Y$ is a closed subset, and $i : Z \to Y$ is the inclusion. If $\mathscr{F}$ is a sheaf of groups on Z, then show that the stalk $(i_* \mathscr{F})_q$ is the one-element group if $q \notin Z$, and $\mathscr{F}_q$ if $q \in Z$.

b) Suppose $\operatorname{Supp} \mathscr{G} \subseteq Z$ where Z is closed. Show that the natural map $\mathscr{G} \to i_* i^{-1} \mathscr{G}$ is an isomorphism.

Solution. a) First, take $q \in Y \setminus Z$. Since $Y \setminus Z$ is open, $i_* \mathscr{F}_p$ is the trivial group, since $\mathscr{F}(i^{-1}(Y \setminus Z)) = \mathscr{F}(\varnothing)$ is already zero. On the other hand, for $p \in Z$, we have $i_* \mathscr{F}_p = \varinjlim_{p \in U} i_* \mathscr{F}(U) = \varinjlim_{p \in U} \mathscr{F}(Z \cap U) = \mathscr{F}_p$ by definition of the subspace topology.
b) look at the stalks (we have shown i^{-1} preserves stalks, while i_* follows from a)). □

3 Towards affine schemes: the underlying set, and topological space

3.1 Towards affine schemes

Exercise 3.1.A. Something with manifolds (I'll skip these, but include them for the numbering)

Exercise 3.1.B. Something with manifolds (I'll skip these, but include them for the numbering)

3.2 The underlying set of an affine scheme

Exercise 3.2.A. a) Describe the set $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$. The ring $k[\epsilon]/(\epsilon^2)$ is called the ring of **dual numbers**.

b) Describe the set $\operatorname{Spec} k[x]_{(x)}$.

Solution. 1) Importantly, (0) is not in the set, since $k[\epsilon]/(\epsilon^2)$ is not an integral domain. Further, note that any element of the form $a+b\epsilon$ with $a \neq 0$ is invertible by $(a+b\epsilon)(a^{-1}-ba^{-2}\epsilon) = 1 - aba^{-2}\epsilon + ba^{-1}\epsilon + c\epsilon^2 = 1$. Thus, the only prime ideal is of the form (ϵ). 2) Essentially the same argument works to show that the only prime ideals are (0) and (x) ($k[x]_{(x)}$ is a local integral domain, of dimension 1). □

Exercise 3.2.B. Show that for the last type of prime, of the form $(x^2 + ax + b)$, the quotient $\mathbb{R}[x]/(x^2 + ax + b)$ is always isomorphic to $\mathbb{C}$.

Solution. This can be shown with a very standard proof that any degree-2 extension field of $\mathbb{R}$ is algebraically closed, and that all algebraic closures are isomorphic. $\square$

Exercise 3.2.C. Describe the set $\mathbb{A}^1_{\mathbb{Q}}$.

Solution. Since $\mathbb{Q}[x]$ is a PID, we know that the prime ideals correspond to irreducible $f(x) \in \mathbb{Q}[x]$, and the generic point (0). So we have a point for each $n \in \mathbb{Q}$, but also for $\alpha \in \overline{\mathbb{Q}}$, we get points, although they are glued together by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (so one point for each conjugacy class in $\mathbb{Q}$). This description works for arbitrary fields of course. $\square$

Exercise 3.2.D. If k is a field, show that $\operatorname{Spec} k[x]$ has infinitely many points.

Solution. Since $k[x]$ is a PID, we can look at irreducibles. Assume there is some finite list $f_1, \dots, f_n \in k[x]$ of irreducible polynomials, and consider $g = (\prod_i f_i) + 1$. Clearly, $g \equiv 1 \pmod{f_i}$ for all i, and g is not a unit itself (it has degree > 1), and must therefore uniquely factor into irreducible polynomials, which were not in the list tobegin with. $\square$

Exercise 3.2.E. Show that we have identified all the prime ideals of $\mathbb{C}[x, y]$.

Solution. We follow the hint: principal prime ideals, correspond to irreducible polynomias. Assume therefore that $\mathfrak{p}$ is non-principal. Indeed, there must exist $f(x, y), g(x, y) \in \mathfrak{p}$ without any common factor: otherwise, that factor would be a principal generator of $\mathfrak{p}$. Now, consider $f_x(y), g_x(y) \in \mathbb{C}(x)[y]$. They are coprime, and thus they generate all of $\mathbb{C}(x)$, which implies that $h(x) \in (f(x, y), g(x, y)) \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, one of the linear factors $x - a$ of $h(x)$ must be in $\mathfrak{p}$ already (by the definition of prime ideals). The argument is completely symmetric in x, y, thus we also get $y - b \in \mathfrak{p}$. Since $(x - a, y - b)$ is already maximal, it must be all of $\mathfrak{p}$ (incidentally, this also shows that the non-principal prime ideals in this case are maximal). $\square$

Exercise 3.2.F. Show that the Nullstellensatz implies the weak Nullstellensatz.

Solution. Take any maximal ideal $\mathfrak{m} \subseteq k[x_1, \dots, x_n$. By the Nullstellensatz, the residue field of $\mathfrak{m}$ is a finite extension of k, hence k itself. From this, the result immidiately follows. $\square$

Exercise 3.2.G. Any integral domain A which is a finite k-algebra (i.e. a k-algebra that is a finite-dimensional vector space over k) must be a field.

Solution. We follow the hint, and look at $- \cdot x : A \to A$ for $0 \neq x \in A$. Since A is a domain, multiplication by x is injective, and by the k-algebra structure, it is also k-linear. Does $- \cdot x$ is an injective linear transformation between a finite-dimensional k-vector space A to itself, and must therefor also be invertible. Thus x itself is also invertible, and A must be a field. $\square$

Exercise 3.2.H. Describe the maximal ideals of $\mathbb{Q}[x, y]$ corresponding to $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Do the same for the maximal ideal corresponding to $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$.

Solution. A first guess is something related to $(x^2 - 2, y^2 - 2)$, but this would be wrong! Notice that this ideal is not maximal (its residue field is $\mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2})$. The correct ideals would be $(x - y, x^2 - 2)$ and $(x + y, x^2 - 2)$ respectively. $\square$

Exercise 3.2.I. Consider the map of sets $\Phi : \mathbb{C}^2 \to \mathbb{A}^2_{\mathbb{Q}}$ defined as follows: (z_1, z_2) is sent to the prime ideal of $\mathbb{Q}[x, y]$ consisting of polynomials vanishing at (z_1, z_2).

a) What is the image of (π, π^2)?

b) Show that Φ is surjective.

Solution. 1) Famously, π is transendental, thus satisfies no polynomial. However, we clearly have $x^2 - y \in \Phi(\pi, \pi^2)$. And $(x^2 - y)$ is already prime (with residue ring $\mathbb{Q}[x]$), thus we conclude that $\Phi(\pi, \pi^2) = (x^2 - y)$. 2) Take any prime ideal $\mathfrak{p} \subset \mathbb{Q}[x, y]$. Let A be the residue ring (which is an integral domain), and let K be the field of fractions of A. Since K is a finite extension of $\mathbb{Q}$, it embeds into $\mathbb{C}$. So we have

$$\mathbb{Q}[x, y] \to A \hookrightarrow K \hookrightarrow \mathbb{C},$$

And set z_1, z_2 to be the image of x, y respectively. Then, evaluating $g \in \mathbb{Q}[x, y]$ at (z_1, z_2) is, by construction, exactly the image through this composite map, and the kernel is clearly $\mathfrak{p}$. Thus, $\Phi((z_1, z_2)) = \mathfrak{p}$, which shows that Φ is surjective. $\square$

Exercise 3.2.J. Suppose A is a ring, and I and an ideal of A. Let $\Phi : A \to A/I$. Show that Φ^{-1} gives an inclusion-preserving bijection between prime ideals of A/I and prime ideals of A containing I. Thus, we can picture $\operatorname{Spec} A/I$ as a subset of $\operatorname{Spec} A$.

Solution. For any prime ideal $\mathfrak{p} \subset A/I$, $\Phi^{-1}(\mathfrak{p})$ is a prime ideal of A (works for any ring map), containing I (since $I = \ker \Phi$). Conversely, any prime ideal $\mathfrak{q}$ of A containing I maps to a prime ideal of A/I (it is still prime, because $(A/I)/(\mathfrak{q}/I) \cong A/\mathfrak{q}$). These maps are clearly mutually inverse, and inclusion-preserving. $\square$

Exercise 3.2.K. Suppose S is a multiplicative subset of A. Describe an order-preserving bijection between the prime ideals of $S^{-1}A$ with the prime ideals of A that don't meet the multiplicative set S.

Solution. Now we look at $A \to S^{-1}A$ given by $a \to a/1$ (recall that this is not actually necessarily injective, if A is not a domain). The prime ideals of $S^{-1}A$ pull back to prime ideals of A, and indeed they do not intersect with S (because then they would map to all of $S^{-1}A$, since elements of S become invertible). Similarily, for a prime ideal $\mathfrak{p}$ of A that do not intersect S, we have $S^{-1}(A/\mathfrak{p}) = (S^{-1}A)/S^{-1}\mathfrak{p})$, by the universal property of localization, thus $S^{-1}\mathfrak{p}$ is still prime. Again, these are inclusion-preserving mutually inverse maps. $\square$

Exercise 3.2.L. Show that the rings

$$(\mathbb{C}[x,y]/(xy))_x \xrightarrow{\sim} \mathbb{C}[x]_x$$

are isomorphic.

Solution. We obtain the map by sending $f(x,y)/x^{-n} \to f(x,0)/x^{-n}$. This is clearly a ring-map, and (xy) is in the kernel of the map, since $(x0)g(x,0) = 0$. However, $y \in (\mathbb{C}[x,y]/(xy))_x$ is already 0, since $xy = 0$, and thus the map is also injective. $\square$

Exercise 3.2.M. If $\Phi : B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, show that $\Phi^{-1}(\mathfrak{p})$ is a prime ideal of B.

Proof. Set $\mathfrak{q} := \Phi^{-1}(\mathfrak{p})$. Its closed under addition, since $x, y \in \mathfrak{q}$ implies that $\Phi(x+y) = \Phi(x)+\Phi(y) \in \mathfrak{p}$. Similarily, with multiplication by $b \in B$, we have $\Phi(bx) = b\Phi(x) \in \mathfrak{p}$, so $\Phi^{-1}(\mathfrak{p})$ is clearly an ideal. Now it is prime, because $xy \in \mathfrak{q}$ implies that $\Phi(xy) = \Phi(x)\Phi(y) \in \mathfrak{p}$, which implies that $\Phi(x)$ or $\Phi(y)$ is in $\mathfrak{p}$ since $\mathfrak{p}$ is prime. $\square$

Exercise 3.2.N. Let B be a ring.

a) Suppose $I \subset B$ is an ideal. Show that the map $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is the inclusion of an earlier exercise.

b) Suppose $S \subseteq B$ is a multiplicative subset. Show that the map $\operatorname{Spec} S^{-1}B \to B$ is the inclusion of an earlier exercise.

Proof. We already solved those exercises using those inclusions. $\square$

Exercise 3.2.O. Consider the map of complex manifolds sending $\mathbb{C} \to \mathbb{C}$ via $x \to y = x^2$. Interpret the corresponding maps of rings given by $\mathbb{C}[y] \to \mathbb{C}[x], y \to x^2$. Verify that the preimage (the fiber) above the point $a \in \mathbb{C}$ is the point(s) $\pm\sqrt{a} \in \mathbb{C}$, using the definition given above.

Solution. The double back and forth is a bit confusing here: We have a map of rings $\varphi : \mathbb{C}[y] \to \mathbb{C}[x]$, which gives a map of spectra $\operatorname{Spec}\mathbb{C}[x] \to \operatorname{Spec}\mathbb{C}[y]$, and given $(y-a) \in \operatorname{Spec}\mathbb{C}[y]$, we are asked to find the preimage. Well, the preimage is given by prime ideals $\mathfrak{q} \subset \mathbb{C}[x]$ for which $\varphi^{-1}(\mathfrak{q}) \supset (y-a)$. But this is simply requiring $\mathfrak{q} \supset \varphi((y-a)) = (x^2-a) = (x-\sqrt{a})(x+\sqrt{a})$, and thus, we find $(x \pm \sqrt{a})$ as the preimages. $\square$