Lecture 9 slide 79:

prove that the laplacian in $\mathbb{R}^2$ is the local difference between a point and its average neighbour value.

<u>proof</u>: Consider $f: \mathbb{R}^2 \mapsto \mathbb{R}^2$

$$\bar{f} = \frac{1}{4h^2} \int_{-h}^{h}\int_{-h}^{h} f(x,y)\, dx\, dy \quad \text{——} ① \leftarrow$$ this is the mean of f in square region (h is small)

Apply Taylor expansion and ignore the cross terms and high order terms:

$$f(x,y) \simeq f(0,0) + x\frac{\partial f(0)}{\partial x} + y\frac{\partial f(0)}{\partial y} + x^2\frac{\partial^2 f(0)}{\partial x^2} + y^2\frac{\partial^2 f(0)}{\partial y^2} \cdots \quad ②$$

plug ② in ①

$$\bar{f} = \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} f(0,0) + x\frac{\partial f(0)}{\partial x} + y\frac{\partial f(0)}{\partial y} + x^2\frac{\partial^2 f(0)}{\partial x^2} - y^2\frac{\partial^2 f(0)}{\partial y^2} + \cdots \quad ④$$

Compute the integration for each of the terms:

$$\int_{-h}^{h}\int_{-h}^{h} f(0,0)\, dx\, dy = \int_{-h}^{h} x f(0,0)\Big|_{-h}^{h} dy = \int_{-h}^{h} 2h f(0,0)\, dy = \underline{\underline{4h^2 f(0,0)}}$$

$$\int_{-h}^{h}\int_{-h}^{h} x\frac{\partial f(0)}{\partial x}\, dx\, dy = \int_{-h}^{h} \frac{x^2}{2}\frac{\partial f(0)}{\partial x}\Big|_{-h}^{h} dy = \int_{-h}^{h} 0\, dy = 0$$

$$\int_{-h}^{h}\int_{-h}^{h} x\frac{\partial f(0)}{\partial y}\, dx\, dy = 0 \leftarrow \text{same as the above.}$$

$$\int_{-h}^{h}\int_{-h}^{h} x^2\frac{\partial^2 f(0)}{\partial x^2}\, dx\, dy = \int_{-h}^{h} \frac{x^3}{3}\frac{\partial^2 f(0)}{\partial x^2}\Big|_{-h}^{h} dy = \int_{-h}^{h} \frac{2h^3}{3}\frac{\partial^2 f(0)}{\partial x^2}\, dy$$

$$= \frac{2h^3}{3}\int_{-h}^{h} \frac{\partial^2 f(0)}{\partial x^2}\, dy = \frac{2h^3}{3}\left[x\frac{\partial^2 f(0)}{\partial x^2}\right]_{-h}^{h} = \frac{2h^3}{3}\left(2h\frac{\partial^2 f(0)}{\partial x^2}\right)$$

$$= \frac{4h^4}{3} \frac{\partial^2 f(0)}{\partial x^2}$$

$$\int_{-h}^{h}\int_{-h}^{h} y^2 \frac{\partial^2 f(0)}{\partial y^2}\, dx\,dy = \int_{-h}^{h}\int_{-h}^{h} y^2 \frac{\partial^2 f(0)}{\partial y^2}\, dy\,dx = \frac{4h^4}{3} \frac{\partial^2 f(0)}{\partial y^2}$$

Plug these solutions in (*)

$$\bar{f} = \frac{1}{4h^2}\left[4h^2 f(0,0) + \frac{4h^4}{3}\left(\frac{\partial^2 f(0)}{\partial x^2} + \frac{\partial^2 f(0)}{\partial y^2}\right)\right]$$

We know that $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

$$\Rightarrow \bar{f} \cong f(0,0) + \frac{h^2}{3} \Delta f$$

$$\Rightarrow \Delta f = 3\left(\frac{\bar{f} - f(0,0)}{h^3}\right)$$

← Which is a difference between the value of the function and its mean in a small range (neighbours).

$\square$.