Variational Autoencoders (VAEs)

- Decoder : $P(x \mid z, \theta)$ (e.g. $N(\mu^{\theta}(z) ; \Sigma^{\theta}(z))$)
- Encoder : $q(z \mid x, \phi)$ (e.g. $N(\mu^{\phi}(x) ; \Sigma^{\phi}(x))$)

- likelihood of X given θ

$$\log \mathbb{P}(X \mid \theta) = \log \int \mathbb{P}(x \mid z, \theta) \, d\mathbb{P}(z \mid \theta)$$

$$= \log \int \frac{\mathbb{P}(x \mid z, \theta) \, \mathbb{P}(z \mid \theta)}{\mathbb{Q}(z \mid x, \phi)} \, d\mathbb{Q}(z \mid x, \phi)$$

Jensen's inequality

$$\geq \int \log \left(\frac{\mathbb{P}(x \mid z, \theta) \, \mathbb{P}(z \mid \theta)}{\mathbb{Q}(z \mid x, \phi)} \right) d\mathbb{Q}(z \mid x, \phi)$$

$$= \underbrace{\mathbb{E}^{\mathbb{Q}_\phi} \left[\log \mathbb{P}(x \mid z, \theta) \right]}_{\text{reconstruction loss}} \underbrace{- \mathbb{E}^{\mathbb{Q}_\phi} \left[\log \left(\frac{\mathbb{Q}(z \mid x, \phi)}{\mathbb{P}(z \mid \theta)} \right) \right]}_{\substack{\text{Kullback-Leibler distance} \\ \text{acts as a regularizer}}}$$

NB: The quantities of interest in the ELBO depend heavily on the distributions $\mathbb{P}, \mathbb{Q}$.

Optimization procedure

Let ϕ, Θ be artificial neural networks (ANNs) for respectively the encoder and decoder.

1. Process x through ANN ϕ
   ↳ gives $\mu^{\phi}(x)$ and $\Sigma^{\phi}(x)$.

2. Sample z from $q_{\phi}(\cdot \mid x)$
   ↳ gives z's drawn from the encoder.

3. Process z through ANN Θ
   ↳ gives $\mu^{\Theta}(z)$ and $\Sigma^{\Theta}(z)$.

4. Compute Monte-Carlo estimates of ELBO

$$\log \mathbb{P}(x \mid z, \Theta(z)) - \log\left(\frac{\mathbb{Q}(z \mid x, \phi(x))}{\mathbb{P}(z \mid \Theta(z))}\right).$$

5. Maximize the ELBO with respect to ϕ, Θ.
   ↳ updates both ANNs

6. Repeat until convergence.