Investigating diffraction patterns using the Discrete Fourier Transform

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Abstract

The Discrete Fourier Transform (DFT) was used to visualise the diffraction pattern resulting from various configurations of single and double slits, in both 1- and 2-dimensions. It was also used to verify the convolution theorem for the case of the single slit.

1 Introduction

Real light sources are generally composed of a superposition of different frequencies. These frequencies can be seperated into a fourier series, and in the limit where the seperation between the frequencies becomes very small, a new continuous function can be generated to express the frequency distribution. This is the Fourier Transform and it can be used to describe the diffraction pattern of light incident on a slit. The one dimensional Discrete Fourier Transform (DFT) is given by:

$$F(u) = \frac{1}{2N} \sum_{-N}^{N-1} f(x) e^{-\frac{\pi i x u}{N}} \tag{1}$$

This can be seperated into real and imaginary parts:

$$\mathbb{R}\{F(u)\} = \frac{1}{2N} \sum_{-N}^{N-1} \mathbb{R}\{f(x)\} \cos\frac{\pi x u}{N} + \mathbb{I}\{f(x)\} \sin\frac{\pi x u}{N} \tag{2}$$

$$\mathbb{I}\{F(u)\} = -\frac{1}{2N} \sum_{-N}^{N-1} \mathbb{R}\{f(x)\} \sin\frac{\pi x u}{N} + \mathbb{I}\{f(x)\} \cos\frac{\pi x u}{N} \tag{3}$$

Similarly, the two dimensional DFT is given by:

$$F(u,v) = \frac{1}{4NM} \sum_{x=-N}^{N-1} \sum_{y=-M}^{M-1} f(x,y) e^{-\pi i (\frac{xu}{N} + \frac{yv}{M})} \tag{4}$$

This can be separated to give:

$$\mathbb{R}\{F(u,v)\} = \frac{1}{4NM} \sum_{x=-N}^{N-1} \sum_{y=-M}^{M-1} \mathbb{R}\{f(x,y)\} \cos\left(\frac{\pi x u}{N} + \frac{\pi y v}{M}\right) + \mathbb{I}\{f(x,y)\} \sin\left(\frac{\pi x u}{N} + \frac{\pi y v}{M}\right) \quad (5)$$

$$\mathbb{I}\{F(u,v)\} = \frac{1}{4NM} \sum_{x=-N}^{N-1} \sum_{y=-M}^{M-1} \mathbb{I}\{f(x,y)\} \cos\left(\frac{\pi x u}{N} + \frac{\pi y v}{M}\right) - \mathbb{R}\{f(x,y)\} \sin\left(\frac{\pi x u}{N} + \frac{\pi y v}{M}\right) \quad (6)$$

In both cases, the total amplitude of the DFT can be calculated by combining the real and imaginary components at each point.

$$F(u) = \sqrt{\left(\mathbb{R}\{F(u)\}\right)^2 + \left(\mathbb{I}\{F(u)\}\right)^2} \quad (7)$$

$$F(u,v) = \sqrt{\left(\mathbb{R}\{F(u,v)\}\right)^2 + \left(\mathbb{I}\{F(u,v)\}\right)^2} \quad (8)$$

The intensity distribution is given by the square of the total amplitude.

Convolution is a mathematical operation which is related to the Fourier Transform by the convolution theorem. The convolution of two functions f(x) and g(x) is defined as:

$$h(X) = f(x) * g(x) = \int_{-\infty}^{\infty} f(x)g(X-x)dx \quad (9)$$

And a normalized discrete convolution is given by:

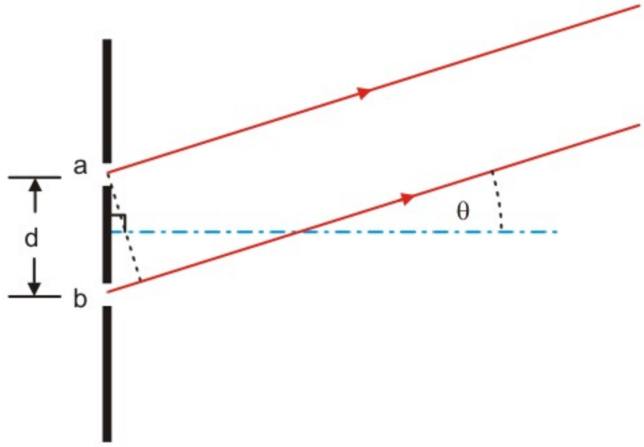
$$h(X) = f(x) * g(x) = \frac{1}{2N} \sum_{x=-N}^{N} f(x)g(X-x) \quad (10)$$

The convolution theorem states that:

$$H(u) = F(u)G(u) \quad (11)$$

where, as before, F(u) represents the Fourier transform of f(x), and likewise for H and G.

For a double slit arrangement, our ray diagram will look like this:



From this diagram we can see that the path difference between the two rays is given by $d \sin\theta$. Constructive interference therefore occurs when $n\lambda = d \sin\theta$. Assuming that $D >> d$ (where D is the distance to the screen) and by applying the small angle approximation, we find that the fringe spacing is given by:

$$s = \frac{D\lambda}{d} \tag{12}$$

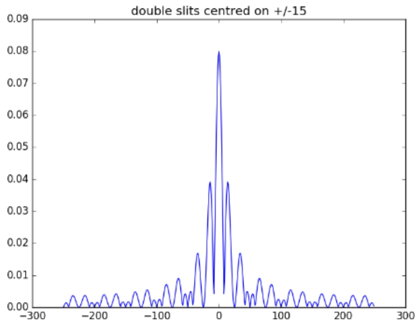
2 Methods

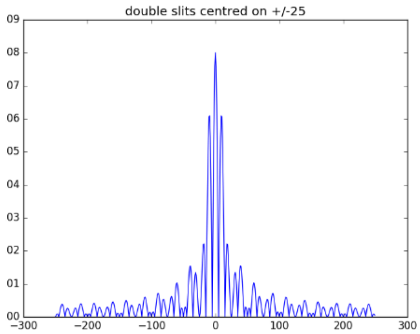
A Fraunhofer diffraction pattern can be visualised by calculating the Discrete Fourier Transform of the slit and plotting the result. For the one dimensional case, the real, imaginary and total amplitudes of the DFT can be computed using equations 2, 3 and 7, respectively. The intensity pattern is given by the square of the total part. Here, the function f(x) represents the slit, so $\mathbb{I}\{f(x)\}$ is set to 0, and $\mathbb{R}\{f(x)\}$ represents the height of the slit at position x. In order to verify the convolution theorem, the convolution of a single slit with itself should be calculated. The Fourier transforms of both the slit and its convolution can then be obtained. The square of the transform of the slit can then be compared to the transform of the convolution and they should be found to be equivalent.

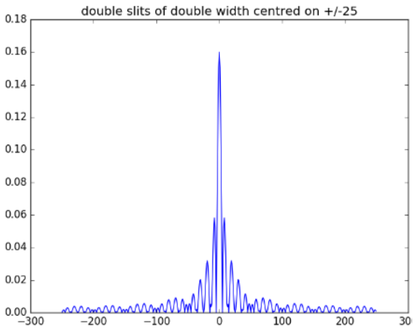
3 Results

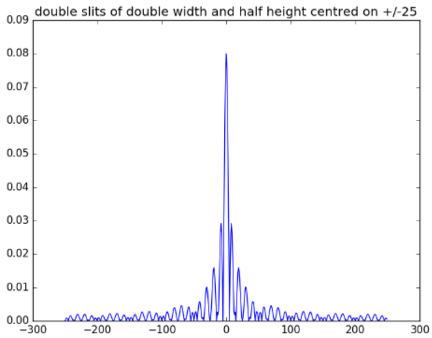
3.1 2 slits in 1 dimension

The following graphs show the total amplitudes of the DFTs of various double slit arrangements. Unless otherwise stated, $height = 1$ and $width = 20$.

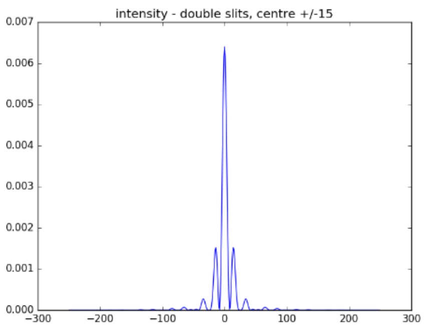


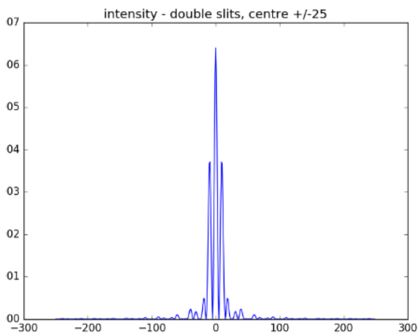


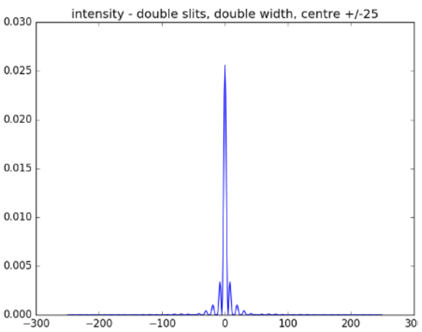


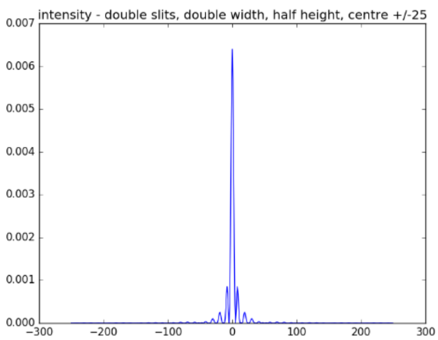


The corresponding intensity patterns are:



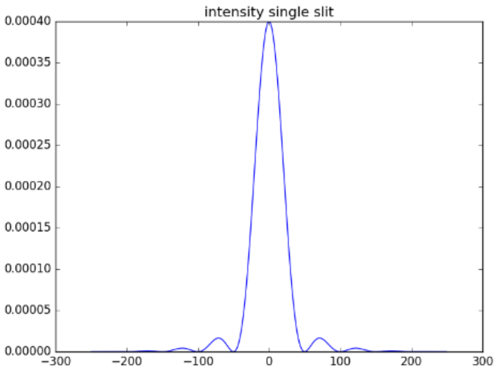


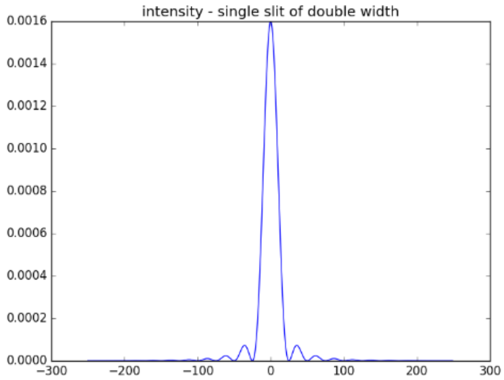




Comparing the intensity distributions for slits centred on +/-15 and +/-25, we can see that slits positioned farther apart give a narrower interference pattern - the first order minima occur at a smaller displacement from the central maximum. Equation 9 tells us that this is to be expected since fringe spacing is inversely proportional to slit separation. For a larger slit separation, there is a larger phase difference between rays for any given angle. Therefore the condition for destructive interference is acheived at a smaller angular displacement.

A double slit system can be formed by the convolution of a single slit and two delta functions. The convolution theorem then implies that the intensity pattern for a double slit system is enveloped by the single slit intensity pattern. Shown below is a comparison of the intensity patterns for a single slit, centred on the origin, of $height = 1$ and either $width = 10$ or $width = 20$.



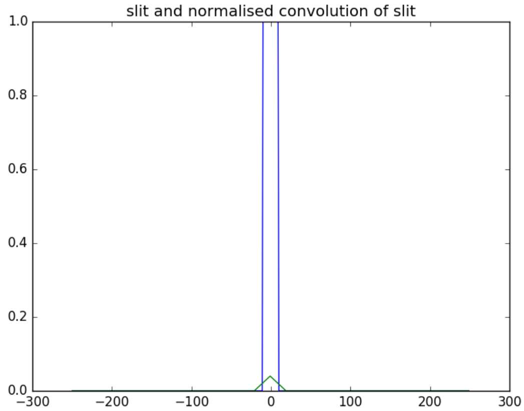


We can see that doubling the width of a single slit gives a narrower intensity distribution. Doubling the width of both slits in a two slit system results in a narrower modulation of the intensity, as expected.

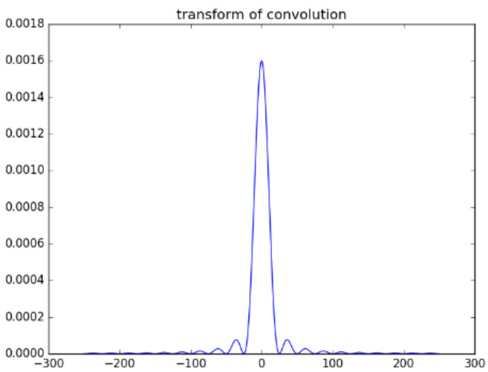
The height of the slit affects only the amount of light passing through. Halving the height quarters the amplitude of the resulting intensity pattern.

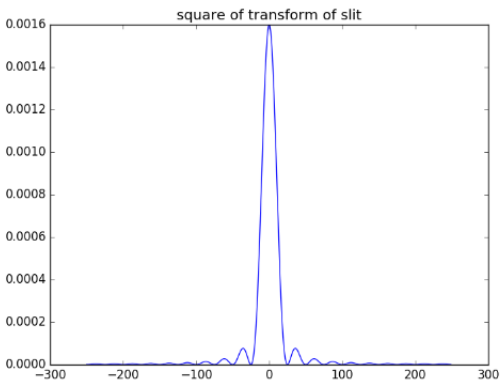
3.2 Proof of convolution theorem

Now take a single slit centred on the origin with $width = 20$ and $height = 1$. If we convolve this slit with itself and normalize the result, we obtain the following figure, where the blue line represents the original slit and the green line represents the normalized convolution.



Applying both sides of the convolution theorem we obtain the following results:

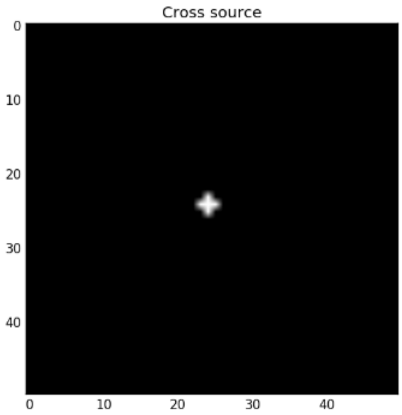


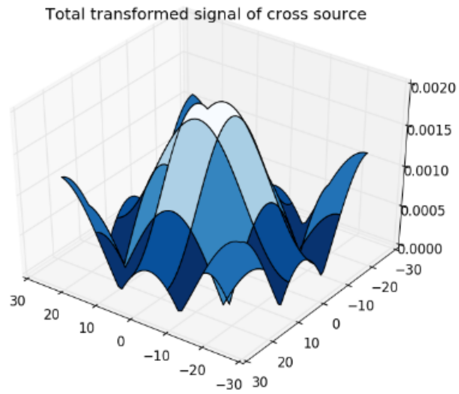


The previous two graphs are identical, thus verifying the convolution theorem.

3.3 Slits in 2 dimensions

A two dimensional, cross-shaped slit is shown below alongside its transform.





And likewise, for a square slit: