Volume-law scaling for the entanglement entropy in spin-1/2 chains

To cite this article: G Vitagliano *et al* 2010 *New J. Phys.* **12** 113049

View the article online for updates and enhancements.

New Journal of Physics

Volume-law scaling for the entanglement entropy in spin-1/2 chains

G Vitagliano[1]**, A Riera**[2,4] **and J I Latorre**[3]

[1] Dipartimento di Fisica dell'Università di Pisa, Pisa, Italy
[2] Institute of Physics and Astronomy, University of Potsdam, 14476 Potsdam, Germany
[3] Department d'Estructura i Constituents de la Matèria, Universitat de Barcelona, 647 Diagonal, 08028 Barcelona, Spain
E-mail: arnauriera@gmail.com

Abstract. Entanglement entropy obeys area law scaling for typical physical quantum systems. This may naively be argued to follow from the locality of interactions. We show that this is not the case by constructing an explicit simple spin chain Hamiltonian with nearest-neighbor interactions that presents an entanglement volume scaling law. This non-translational model is contrived to have couplings that force the accumulation of singlet bonds across the half-chain. This configuration of the couplings is suggested by real-space renormalization group arguments. Computation of the entanglement entropy is performed by mapping the system to free fermions and diagonalizing numerically its correlation matrix. An analytical relationship between the entanglement entropy and the Frobenius norm of the correlation matrix is also established. Our result is complementary to the known relationship between non-translational invariant, nearest-neighbor interacting Hamiltonians and quantum Merlin–Arthur (QMA)-complete problems.

[4] Author to whom any correspondence should be addressed.

IOP Institute of Physics Φ DEUTSCHE PHYSIKALISCHE GESELLSCHAFT

Contents

1. Introduction

Ground states of relevant physical Hamiltonians carry quantum correlations, which decrease with distance. For instance, a two-point correlation function is expected to fall off exponentially with the separation of points in the presence of a mass gap or algebraically at critical points. This amount of quantum correlation corresponds to an *area-law* scaling of entanglement entropy. To be precise, the entanglement entropy, defined by

$$S(\rho_A) = -\text{tr}\,(\rho_A \log \rho_A),\tag{1}$$

where ρ_A is the density matrix of the region of space A considered, scales with the boundary of A.

In contradistinction to the above situation, random quantum states are known to carry volume-law entropy [1]. Therefore, typical Hamiltonians produce ground states that are not generic. Indeed, relevant physics corresponds to a small corner of the total Hilbert space of a quantum system. This observation is crucial to understand the recent effort to simulate quantum states with tensor networks [2]–[8]. Such approximations are able to accommodate area law scaling for the entropy.

It is then important to understand precisely what are the properties that a Hamiltonian must obey so as to produce a ground state that only displays area law entanglement. A first heuristic approach suggests that entanglement decreases at large distances because interactions are local. That is, a local degree of freedom interacts with its neighbor and gets entangled with it. This second degree of freedom interacts in turn with a further one. This sequence of interactions would eventually entangle far separated degrees of freedom, although the strength of interactions would only manage to get the standard correlations we find in Nature. On the other hand, it is unclear whether interactions can be contrived to achieved larger entangled

states. The role of translational invariance is then critical. Some results for one-dimensional (1D) systems are well established. In 1D, if a system obeys local interactions and is gapped, area law always emerges [9]. On the other hand, if the system is at a critical point and therefore gapless, a logarithmic divergence is encountered. This logarithmic scaling of the entanglement entropy is explained by conformal field theory [10]–[14].

In [15, 16], infinite translational invariant fermionic systems of any spatial dimension with arbitrary interactions are considered. For such systems, it is shown that the entropy of a finite region typically scales with the area of the boundary times a logarithmic correction.

Although there has recently been further progress on this topic in higher dimensions [17]–[19], the necessary and sufficient conditions for an area law have not yet been defined. An explicit example of a system where area law is violated is presented in [20]. It is shown that a 1D non-translational invariant system composed of 12-level local quantum particles with nearest-neighbor interactions presents a ground state that carries a volume-law scaling of entanglement. In particular, it is proven that the problem of approximating the ground state energy of such a system is QMA-complete. This precise example shows that a quantum computer cannot simulate any 1D system, and, moreover, that there exist 1D systems that take an exponential time to relax to their ground states at any temperature, making them suitable candidates for being 1D spin glasses.

The issue addressed in this work is to examine how simple can a quantum system be to give a highly entangled ground state. In particular, we show that a simple spin-1/2 model with nearest-neighbor interactions with a suitable fine-tuning of its coupling constants can have a ground state with volume-law scaling for the entanglement entropy. Our proposal is based on translational symmetry breaking; hence, this means that the area-law violation cannot be maintained for any bipartition of the system. Nevertheless, it will be shown that the average of the entanglement entropy over all possible positions of the block fulfills a volume law. Our results are presented in the following way. First we review the real-space renormalization group (RG) technique, which brings fundamental insights into how to build Hamiltonian with a highly entangled ground state. We then turn to solving the proposed Hamiltonian using its exact diagonalization, where the final step can be taken in perturbation theory or numerically. We illustrate the real-space renormalization idea in the appendix.

2. Real-space renormalization group (RG)

2.1. Introduction to real-space RG approach

The real-space RG approach was introduced in [21] by generalizing the work presented in [22]. It is a method suited for finding the effective low-energy Hamiltonian and the ground state of random spin chains. The couplings have to satisfy the hypothesis of strong disorder, the logarithm of its probability distribution is wide. Under such conditions, the ground state of the system can be very well approximated by a product state of singlets whose spins are arbitrarily distant.

Let us review the real-space RG method for the inhomogeneous XX model case

$$H_{\mathrm{XX}} = \frac{1}{2} \sum_{i=1}^{N} J_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right). \tag{2}$$

First, we find the strongest bond $J_i \gg J_{i+1}, J_{i-1}$ and diagonalize it independently of the rest of the chain. According to the previous Hamiltonian, this leads to a singlet between spins i and
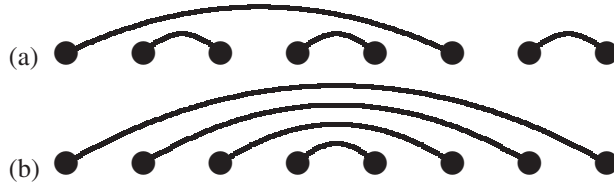
Figure 1. Diagram of a random singlet phase (a) and the concentric singlet phase (b). Each spin forms a singlet pair with another spin indicated by the bond lines.

$i+1$ (see the appendix). Therefore, the ground state at zeroth order in perturbation theory for the couplings J_{i-1} and J_{i+1} is

$$|\psi^{(0)}\rangle = |\psi_{x<i}\rangle|\psi_-\rangle|\psi_{x>i+1}\rangle, \tag{3}$$

where $|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{i,i+1} - |10\rangle_{i,i+1})$ is a singlet state between spins i and $i+1$, and $|\psi_{x<i}\rangle$ and $|\psi_{x>i}\rangle$ correspond to the states of the rest of the system.

In order to compute corrections to the ground state at higher orders, we use perturbation theory as shown in appendix. This leads to an effective interaction between the distant spins $i-1$ and $i+2$ with an effective coupling

$$\tilde{J}_{i-1,i+2} = \frac{J_{i-1}J_{i+1}}{2J_i}. \tag{4}$$

In summary, real-space RG integrates out two spins and reduces the Hamiltonian energy scale. Note that this new effective low-energy Hamiltonian couples spins $i-1$ and $i+2$; therefore, it has non-local interactions as seen from the original Hamiltonian. Iterating this procedure for an XX model with random couplings, it is seen that the ground state can be described by a random singlet phase, i.e. each spin forms a singlet pair with another one (see figure 1(a)). Most pairs involve nearby spins, but some of them produce long distance correlations.

In [23], real-space RG was used to show that, for random spin chains where the ground state is a random singlet phase, entanglement entropy scales logarithmically at the critical point as in the homogeneous case. That is,

$$S_L \sim \frac{\tilde{c}}{3}\log_2 L, \tag{5}$$

where $\tilde{c} = c\ln 2$ is an effective central charge proportional to the central charge for the same model but without disorder c. This analytical result was later checked numerically in [24]–[26].

2.2. Area-law violation for the entanglement entropy

Let us now tune the couplings J_i of our XX model in such a way that the entanglement entropy of the ground state of the system scales with the volume of the block of spins. An easy way to achieve this is by generating a ground state with a concentric singlet phase as shown in figure 1(b). We see that the system is in a product state of distant singlets between the positions $N/2 - (i-1)$ and $N/2 + i$ for $1 \leqslant i \leqslant N/2$. It is trivial to see that the entanglement entropy of this configuration would scale with the size of the block, since it merely corresponds to the number of bonds cut by the bipartition (see figure 2(a)).
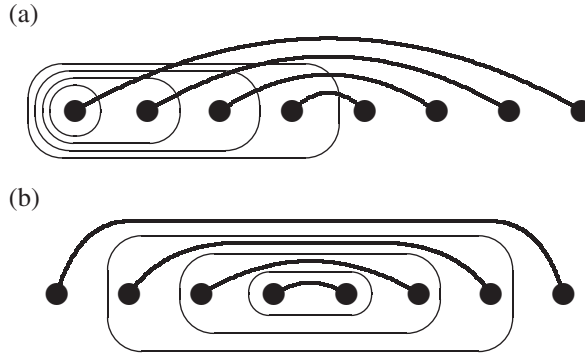
(a)

(b)

Figure 2. Diagram of the entanglement entropy scaling for the concentric singlet phase. The entanglement entropy grows maximally if we take blocks at one extreme (a) and is zero if the blocks are centered at the middle of the chain. This is an explicit example illustrating that in non-translationally invariant systems, the entanglement entropy depends on the position of the block.

Let us note that the entanglement entropy for concentric blocks would be 0 as shown in figure 2(b). As the translational invariance of the system is broken, the entanglement entropy of a block depends not only on the size of it but also on its position.

To measure how entangled is a state for non-translationally invariant systems, it is useful to introduce the average entanglement entropy over all possible positions of the block, that is,

$$\bar{S}_L = \frac{1}{N-L} \sum_{i=1}^{N-L} S_L(i),\tag{6}$$

where $S_L(i)$ is the entanglement entropy of the block of size L from the ith spin to the $(i+L)$th one.

According to the previous definition, the average entanglement entropy of the concentric singlet phase reads

$$\bar{S}_L = \left(1 - \frac{L}{2(N-L)}\right)L.\tag{7}$$

Although for the concentric singlet phase the average entropy loses its linear behavior for large blocks, $L \sim (1 - \frac{1}{\sqrt{3}})N$, it always fulfills the condition $\bar{S}_L \geqslant \frac{1}{2}L$. Thus, the concentric singlet phase represents a simple and explicit example of area-law violation of scaling of the entanglement.

The aim of this work is to tune the coupling constants J_i of the XX model such that the concentric singlet phase becomes the ground state of the system and thereby obtain an explicit example of a Hamiltonian with nearest-neighbor interactions of spins that violate the area-law scaling of entanglement.

Due to the symmetry of the state we pretend to generate, let us consider an XX chain of N spins where the central coupling between spins $N/2$ and $N/2+1$ is J_0 and the rest of them are chosen as follows:

$$J_{N/2+i,N/2+i+1} = J_{N/2-i,N/2-i+1} \equiv J_i,\tag{8}$$

where $1 \leqslant i \leqslant \frac{N}{2} - 1$ and the coupling $J_{N/2\pm i}$ connects the spins $N/2 \pm i$ and $N/2 \pm i + 1$.
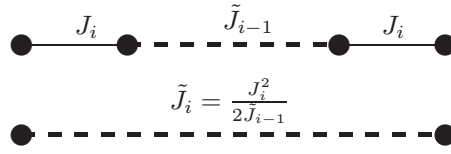
Figure 3. Diagram of the formation of an effective coupling $\tilde{J}_i$ if the condition $\tilde{J}_{i-1} \gg J_i$ is fulfilled.

We are going to use real-space RG ideas to find out at which values we have to tune the coupling constants such that the concentric singlet phase becomes the ground state of the system. If $J_0 \gg J_1$, in the low-energy limit, an effective interaction between the spins $N/2 - 1$ and $N/2 + 2$ appears. We label this effective coupling as $\tilde{J}_1$ and, according to equation (4), it reads

$$\tilde{J}_1 = \frac{J_1^2}{2J_0}. \tag{9}$$

Then, if $\tilde{J}_1 \gg J_2$, the effective low-energy Hamiltonian will have an effective bond between the spins $N/2 - 2$ and $N/2 + 3$. We would like to proceed in this way in order to generate iteratively the concentric singlet phase.

Thus, if the condition $\tilde{J}_i \gg J_{i+1}$ is fulfilled in general, where $\tilde{J}_i$ is defined by

$$\tilde{J}_i = \frac{J_i^2}{2\tilde{J}_{i-1}}, \tag{10}$$

we expect that the ground state of the system is the concentric singlet phase (see figure 3).

Specifically, if we impose that $J_i = \epsilon \tilde{J}_{i-1}$ for any i, such that it is always possible to apply equation (10), we see that the couplings J_i must decay very rapidly:

$$J_i = \epsilon \left(\frac{\epsilon^2}{2} \right)^{i-1} J_0. \tag{11}$$

In general, we are going to study chains with couplings that decay,

$$J_i = \epsilon^{\alpha(i)}, \tag{12}$$

where $\alpha(i)$ is a function that is monotonically increasing. If $\alpha(i) \sim i^2$, we would have a Gaussian decay.

Although according to real-space RG the ground state of our proposal should verify a volume law for the entanglement entropy for a fast enough decay of the coupling constants, the RG method has not been proven to give correctly the entanglement entropy. Thus, in order to show that our model exhibits a volume law for the entanglement entropy, an explicit computation of it is performed in the following sections.

3. The solution of a spin model and its entanglement entropy

Let us consider a finite spin chain with nearest-neighbor couplings J_i^x and J_i^y and an arbitrary transverse magnetic field λ_i in each spin. This system is described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{i=1}^{N} \left(J_i^x \sigma_i^x \sigma_{i+1}^x + J_i^y \sigma_i^y \sigma_{i+1}^y \right) - \sum_{i=1}^{N} \lambda_i \sigma_i^z, \tag{13}$$

where L is the size of the system and $\sigma_i^{x,z}$ are Pauli matrices at site i. The XX model presented earlier is a particular case of this Hamiltonian (13) for $J_i^x = J_i^y$ and $\lambda_i = 0 \; \forall i$.

3.1. The Jordan–Wigner transformation

An essential technique for the solution of H is the well-known mapping to spinless fermions by means of the Jordan–Wigner transformation. First, we express the spin operators $\sigma_i^{x,y,z}$ in terms of fermion creation (annihilation) operators $c_i^\dagger$ (c_i): $c_i^\dagger = a_i^+ \exp[\pi i \sum_j^{i-1} a_j^+ a_j^-]$ and $c_i = \exp[\pi i \sum_j^{i-1} a_j^+ a_j^-] a_i^-$, where $a_j^\pm = (\sigma_j^x \pm i\sigma_j^y)/2$. Doing this, H can be rewritten in quadratic form in fermion operators

$$H = \sum_{i,j=1}^N A_{ij} c_i^\dagger c_j + \frac{1}{2} \sum_{i,j=1}^N B_{ij}(c_i^\dagger c_j^\dagger + \text{h.c.}), \tag{14}$$

where matrices A and B are defined by

$$A_{ij} = 2\lambda_i \delta_{i,j} + (J_i^x + J_i^y)\delta_{i+1,j} + (J_j^x + J_j^y)\delta_{i,j+1},$$

$$B_{ij} = (J_i^x - J_i^y)\delta_{i+1,j} - (J_j^x - J_j^y)\delta_{i,j+1}, \tag{15}$$

with $1 \leqslant i, j \leqslant N$.

3.2. The Bogoliubov transformation

In a second step, the Hamiltonian is diagonalized using a Bogoliubov transformation

$$\eta_k = \sum_{i=1}^N \left(\frac{1}{2} \left(\Phi_k(i) + \Psi_k(i) \right) c_i + \frac{1}{2} \left(\Phi_k(i) - \Psi_k(i) \right) c_i^\dagger \right), \tag{16}$$

where Φ_k and Ψ_k are real and normalized vectors, $\sum_i^N \Phi_k^2(i) = \sum_i^N \Psi_k^2(i) = 1$, so that we have

$$H = \sum_{k=1}^N \Lambda_k(\eta_k^\dagger \eta_k - 1/2). \tag{17}$$

The fermionic excitation energies Λ_k and the components of the vectors Φ_k and Ψ_k are obtained from the solutions of the following equations:

$$(A - B)\Phi_k = \Lambda_k \Psi_k, \tag{18}$$

$$(A + B)\Psi_k = \Lambda_k \Phi_k. \tag{19}$$

It is easy to transform them into an eigenvalue problem,

$$(A + B)(A - B)\Phi_k = \Lambda_k^2 \Phi_k, \tag{20}$$

$$(A - B)(A + B)\Psi_k = \Lambda_k^2 \Psi_k, \tag{21}$$

from which Λ_k, Φ_k and Ψ_k can be determined.

IOP Institute of Physics ⟨Φ⟩ DEUTSCHE PHYSIKALISCHE GESELLSCHAFT

3.3. Ground state

In equations (19) and (18), we realize that transforming Φ_k into $-\Phi_k$ (or Ψ_k into $-\Psi_k$), Λ_k is changed to $-\Lambda_k$. This allows us to restrict ourselves to the sector corresponding to $\Lambda_k \geqslant 0$, $k = 1, 2, \ldots, N$. Thus, considering equation (17) and the fact that all Λ_k are positive, the ground state is a state $|\text{GS}\rangle$ that verifies

$$\eta_k|\text{GS}\rangle = 0 \quad \forall k. \tag{22}$$

In practice, what we do to restrict ourselves to the sector of positive Λ_k is to determine Φ_k and Λ_k by solving equation (20), and calculate $\Psi_k = \frac{1}{\Lambda_k}(A - B)\Phi_k$.

3.4. Computation of Von-Neumann entropy

Following [11, 27, 28], the reduced density matrix $\rho_L = \text{tr}_{N-L}|\text{GS}\rangle\langle\text{GS}|$ of the ground state of a block of L sites in a system of free fermions can be written as

$$\rho_L = \kappa e^{-\mathcal{H}}, \tag{23}$$

where κ is a normalization constant and $\mathcal{H}$ a free fermion Hamiltonian.

   Let us very briefly justify why the density matrix must have this structure. First, note that the Hamiltonian defined by equation (14) has Slater determinants as eigenstates. Thus, according to the Wick theorem, any correlation function of the ground state (or any other eigenstate) can be expressed in terms of correlators of couples of creation and annihilation operators. For instance,

$$\langle c_n^\dagger c_m^\dagger c_k c_l \rangle = \langle c_n^\dagger c_l \rangle \langle c_m^\dagger c_k \rangle - \langle c_n^\dagger c_k \rangle \langle c_m^\dagger c_l \rangle + \langle c_n^\dagger c_m^\dagger \rangle \langle c_k c_l \rangle. \tag{24}$$

If all these indices belong to a subsystem of L sites, the reduced density matrix ρ_L must reproduce the expectation values of the correlation functions,

$$\text{tr}(\rho_L c_n^\dagger c_m^\dagger c_k c_l) = \text{tr}(\rho_L c_n^\dagger c_l)\,\text{tr}(\rho_L c_m^\dagger c_k) - \text{tr}(\rho_L c_n^\dagger c_k)\,\text{tr}(\rho_L c_m^\dagger c_l) + \text{tr}(\rho_L c_n^\dagger c_m^\dagger)\,\text{tr}(\rho_L c_k c_l). \tag{25}$$

This is possible only if ρ_L is the exponential of an operator $\mathcal{H}$ that also contains creation and annihilation processes,

$$\mathcal{H} = \sum_{i,j=1}^{L} \tilde{A}_{ij} c_i^\dagger c_j + \frac{1}{2} \sum_{i,j=1}^{L} \tilde{B}_{ij} (c_i^\dagger c_j^\dagger - \text{h.c.}). \tag{26}$$

   We can diagonalize this Hamiltonian $\mathcal{H}$ by means of another Bogoliubov transformation

$$\xi_k = \sum_{i=1}^{L} \left(\frac{1}{2}(v_k(i) + u_k(i)) c_i + \frac{1}{2}(v_k(i) - u_k(i)) c_i^\dagger \right), \tag{27}$$

where $v_k(i)$ and $u_k(i)$ are real and normalized. Then, the Hamiltonian reads

$$\mathcal{H} = \sum_{k=1}^{L} \epsilon_k \xi_k^\dagger \xi_k, \tag{28}$$

where $\xi_k^\dagger$ and ξ_k are the creation and annihilation operators of some fermionic modes. In terms of these modes, the density matrix ρ_L is uncorrelated and can simply be expressed as

$$\rho_L = \otimes_{k=1}^{L} \tilde{\rho}_k, \tag{29}$$

IOP Institute of Physics ⊕ DEUTSCHE PHYSIKALISCHE GESELLSCHAFT

where

$$\tilde{\rho}_k = \frac{1}{1 + e^{-\epsilon_k}} \begin{pmatrix} e^{-\epsilon_k} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\nu_k}{2} & 0 \\ 0 & \frac{1-\nu_k}{2} \end{pmatrix}. \tag{30}$$

In the previous equation, the new parameters ν_k have been introduced in order to ensure the normalization of $\tilde{\rho}_k$, $\mathrm{tr}(\tilde{\rho}_k) = 1$. This way of expressing $\tilde{\rho}_k$ will be useful next.

Thus, the entanglement entropy of the density matrix ρ_L is merely the sum of binary entropies

$$S(L) = \sum_{k=1}^{L} S(\tilde{\rho}_k) = \sum_{k=1}^{L} H\left(\frac{1+\nu_k}{2}\right), \tag{31}$$

where $H(p) \equiv -p\log_2 p - (1-p)\log_2(1-p)$ is the binary Shannon entropy.

In order to determine the spectrum of $\tilde{\rho}_k$, let us consider the correlation matrix

$$G_{m,n} \equiv \langle \mathrm{GS}|(c_n^\dagger - c_n)(c_m^\dagger + c_m)|\mathrm{GS}\rangle. \tag{32}$$

Note that the matrix G can be computed using the vectors Φ_k and Ψ_k,

$$G_{m,n} = -\sum_{k=1}^{N} \Psi_k(m)\Phi_k(n), \tag{33}$$

where the correlations $\langle \eta_k^\dagger \eta_q \rangle = \delta_{kq}$ and $\langle \eta_k \eta_q \rangle = 0$ have been considered.

In the subspace of L spins, G is completely determined by the reduced density matrix. To avoid any confusion, let us define $T(L) \equiv G(1:L, 1:L)$ as the $L \times L$ upper-left submatrix of the correlation matrix G. Then, T can be expressed in terms of the expected values $\langle \xi_k^\dagger \xi_q \rangle$,

$$\begin{aligned} T(L)_{i,j} &= \sum_{k,q=1}^{L} u_k(i) v_q(j) \left(\langle \xi_k^\dagger \xi_q \rangle - \langle \xi_k \xi_q^\dagger \rangle \right) \\ &= \sum_{k=1}^{L} u_k(i) v_k(j) \nu_k, \end{aligned} \tag{34}$$

where the indices i and j run from 1 to L. This equation leads to the relations

$$T u_q = \nu_q v_q, \tag{35}$$

$$T^T v_q = \nu_q u_q, \tag{36}$$

which can be translated into the eigenvalue problem

$$T^T T u_q = \nu_q^2 u_q, \tag{37}$$

$$T T^T v_q = \nu_q^2 v_q. \tag{38}$$

Once the variables ν_q are computed, we can determine the entanglement entropy by means of equation (31).

3.5. Summary of the calculation

To sum up, let us enumerate the steps we have to follow in order to calculate the entanglement entropy of a block L.

1. Write down the matrices A and B in terms of the couplings of the Hamiltonian (13) according to equations (15).
2. Determine Λ_k, Φ_k and Ψ_k by solving the eigenvalue problem from equation (20).
3. Calculate the correlation matrix G defined in equation (32).
4. Take the submatrix $T(L)$ to determine the eigenvalues ν_k from equation (38).
5. Compute the entanglement entropy by means of equation (31).

4. Entanglement entropy and the Frobenius norm of $T(L)$

We would like to tune the coupling constants of the Hamiltonian (13) such that the scaling of the entanglement entropy of its ground state violates the area law. The entanglement entropy depends only on the variables ν_k. Then, it is convenient to separate the Shannon entropy of the probabilities $\frac{1 \pm \nu_k}{2}$ into

$$H\left(\frac{1 + \nu_k}{2}\right) = 1 - h(\nu_k), \tag{39}$$

where $h(x) = \frac{1}{2}((1 - x)\log_2(1 - x) + (1 + x)\log_2(1 + x))$ is a positive function for $0 \leqslant x \leqslant 1$. In this interval, it can be shown that the function $h(x)$ is tightly bounded by

$$\frac{1}{2\ln 2}x^2 \leqslant h(x) \leqslant x^2. \tag{40}$$

This, together with equation (38), implies that entanglement entropy of a block L is bounded by

$$L - \|T(L)\|_F^2 \leqslant S(L) \leqslant L - \frac{1}{2\ln 2}\|T(L)\|_F^2, \tag{41}$$

where $\|T\|_F \equiv \sqrt{\mathrm{tr}\,(TT^T)} = \sqrt{\sum_{k=1}^{L} \nu_k^2}$ is the Frobenius norm of the matrix $T(L)$. Note that equation (41) introduces a relation between the scaling of the entanglement entropy and the scaling of the matrix T. In particular, it implies that the scaling of the entanglement entropy will verify a volume law if and only if

$$\|T(L)\|_F^2 = \alpha L, \tag{42}$$

with $0 \leqslant \alpha < 1$ and $L < N/2$.

Let us also mention that the upper bound in equation (41) corresponds to the expansion of the entanglement entropy at $\nu_k \sim 0$ for any k. That is, if the system is close to being maximally entangled, the expression for the upper bound gives the entanglement entropy.

Due to the definition of $T(L)$ in equation (34) and to its relation with the entanglement entropy, the norm of the $T(L)$ matrix must fulfill some requirements:

- $\|T(L)\|_F$ is a monotonically increasing function of L.
- $\|T(0)\|_F = 0$ and $\|T(N)\|_F = \sqrt{N}$.
- $\|T_L\|_F^2 \leqslant L$. The equality holds when the state is a product state.
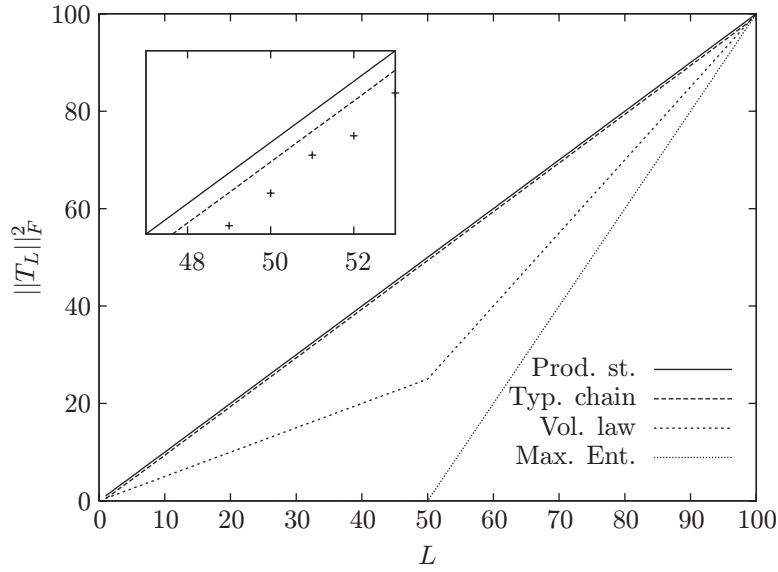
Figure 4. Plot of the square of the norm of the $T(L)$ matrix, defined in equation (34), with respect to L in a system of $N = 100$ spins for several states: a product state (Prod. st.), a maximally entangled state (Max. Ent.), the ground state of a typical XY chain (Typ. chain) and a state with a volume law for the entanglement entropy (Vol. law). The norm of the T matrix for a typical system has been computed by performing the average over 10000 realizations of a random sampling of the couplings of an XY chain. The probability distribution followed by the couplings has been chosen as uniform. In the inner plot, a zoom of the $\|T(L)\|_F$ for the typical chain is presented. The single points in that correspond to the most entangled configuration found along the sampling.

- $\|T_L\|_F^2 \geqslant 0$ if $L \leqslant N/2$, and $\|T_L\|_F^2 \geqslant 2L - N$ if $L \geqslant N/2$. The equality holds when the state is maximally entangled.

In figure 4, a plot of the square of the norm of $T(L)$ for a chain in different states is presented for a system of 100 spins. As has been mentioned, the two extreme cases of a product state (Prod. st.) and a maximally entangled state (Max. Ent.) delimit the area where $\|T(L)\|_F^2$ curves can be. Moreover, the square of the norm of $T(L)$ of a typical chain (Typ. chain) and of a chain with a volume law for the entanglement entropy (Vol. law) have also been plotted. The entropy of a typical chain has been computed by performing the average over 10 000 random samples of the couplings for the XY model and a uniform probability distribution. In the inner plot, a zoom of the $\|T(L)\|_F$ for the typical chain is presented. It is possible to appreciate that although a typical chain is slightly entangled, this is not strictly zero. The single points of this inner plot correspond to the most entangled instance found along the sampling.

The conclusion that must be drawn from figure 4 is that typical spin chains are slightly entangled. Thus, in order to get a volume law for the entanglement entropy, a very particular configuration of the coupling constants must be performed. With this aim, the previous real-space RG ideas can be exploited.
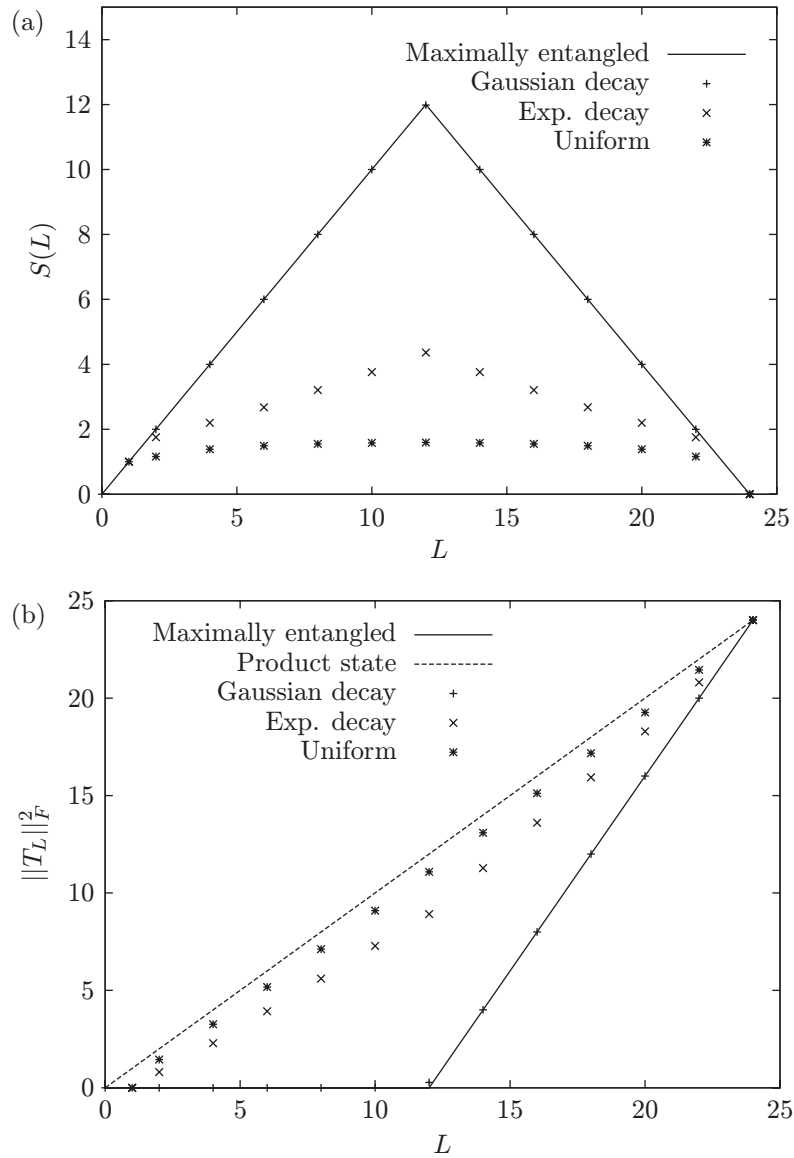
Figure 5. (a) Scaling of the entanglement entropy of a block of L contiguous spins for the ground state of an XX model with $N = 24$ spins and open boundary conditions. Three coupling distributions are considered: couplings that decay Gaussian from the strongest central bond $J_n = e^{-2n^2}$, couplings that decay exponentially $J_n = e^{-n}$ and the uniform case $J_n = 1$. The magnetic field is set to zero. (b) Scaling of the square of the norm of the $T(L)$ matrix for the same three cases.

5. Numerical results

We can follow the steps described in section 3.5 in order to calculate the entanglement entropy of the XX chain presented in section 2.2 and check if the entanglement entropy grows linearly with the size of the block.

This XX model is characterized by having the strongest bond in the middle of the chain, J_0, while the value of the rest of bonds J_n decreases rapidly with the distance n to the central one. In particular, we have studied two different kinds of decay for the coupling constants J_n: (i) Gaussian decay, $J_n \sim e^{-2n^2}$, (ii) exponential decay $J_n \sim e^{-n}$. Let us note that due to the rapid decay of the coupling constants and the finite precision of the computer, we can only consider small systems.

In figure 5(a), the entanglement entropy is plotted for the Gaussian, exponential and uniform cases. For the Gaussian configuration of the couplings, the assumptions of strong disorder of the real-space RG are fulfilled and the ground state of the system becomes maximally entangled.

In the exponentially decaying case, although the entanglement entropy also scales linearly, its slope is less than one. Thus, we observe a volume law, but the entropy is not maximal. In figure 5(b), the scaling of the square of the norm of the $T(L)$ matrix is presented for the same three cases. This allows us to see the same picture from the $\|T(L)\|_F$ perspective. Note that in the exponential case, the Frobenius norm of $T(L)$ increases linearly with L, while in the Gaussian one, it is saturated to zero.

6. Conclusions

We have constructed a 1D system composed by spin-1/2 particles with nearest-neighbor interactions with an entanglement entropy of the ground state that scales with the volume of the size of the block.

This result further confirms that violations of area law scaling for entanglement entropy are possible for local interacting Hamiltonians. Furthermore, such behavior is found possible for spin-1/2 degrees of freedom. The price to be paid for violating area law scaling is the breaking of the translational symmetry of the system. Indeed, in [29, 30], it is shown that, although translation-invariant 1D states give rise to arbitrary fast sublinear entropy growth, they cannot support a linear scaling.

It is also interesting to point out how our proposal escapes from the general results presented in [9]–[14]. In [10]–[14], where logarithmic scaling of the entanglement entropy is explained by conformal field theory, translational invariance is assumed. In [9], an area law for the entanglement entropy in gapped 1D systems has been established. Nevertheless, our proposal is gapless when N tends to infinity.

Let us finally recall that two-local Hamiltonian problems have been proven to provide QMA-complete problems. To be more precise, the problem of finding out whether the ground state energy of a two-local Hamiltonian is larger than a or smaller than b, where $|a - b| > \mathcal{O}(1/n)$, is QMA-complete. We may further argue that an efficient classical simulation of such a problem is likely to be impossible; otherwise we could solve any NP-complete problem by just simulating quantum mechanics on a classical computer. The obstacle to obtaining faithful simulations of quantum mechanical systems is in turn related to the amount of entanglement found in the system. Thus, exponentially large entanglement should be found in some 1D quantum systems. Our results are somehow completing this idea. Even spin-1/2 chains can produce highly entangled states if couplings are adequately tuned.

Acknowledgments

Appendix. Real-space RG in an XX model of four spins

First we consider a simple XX model with only four spins and couplings $\{\lambda, 1, \lambda\}$. We can rewrite the Hamiltonian of the system as a perturbation theory problem,

$$H = H_0 + \lambda V, \tag{A.1}$$

where

$$H_0 = \sigma_2^X \sigma_3^X + \sigma_2^Y \sigma_3^Y \tag{A.2}$$

and

$$V = \sigma_1^X \sigma_2^X + \sigma_1^Y \sigma_2^Y + \sigma_3^X \sigma_4^X + \sigma_3^Y \sigma_4^Y. \tag{A.3}$$

The eigenstates of H_0 are

$$
\begin{aligned}
|\psi_+\rangle &= \frac{1}{\sqrt{2}} \left(|01\rangle_{23} + |10\rangle_{23}\right), \\
|\psi_0\rangle &= |00\rangle_{23}, \\
|\psi_1\rangle &= |11\rangle_{23}, \\
|\psi_-\rangle &= \frac{1}{\sqrt{2}} \left(|01\rangle_{23} - |10\rangle_{23}\right)
\end{aligned}
\tag{A.4}
$$

with eigenvalues $+2$, 0, 0 and -2, respectively. We are interested in studying what happens to the ground state of the Hamiltonian H when the perturbation λV is introduced. The ground state of H_0 is degenerate and forms a subspace of dimension 4. In particular, we choose the set of vectors $\{|m\rangle\} = \{|0\rangle_1|\psi_-\rangle_{23}|0\rangle_4, |0\rangle_1|\psi_-\rangle_{23}|1\rangle_4, |1\rangle_1|\psi_-\rangle_{23}|0\rangle_4, |1\rangle_1|\psi_-\rangle_{23}|1\rangle_4\}$ as a basis.

We expect that the perturbation removes the degeneracy in the sense that there will be four perturbed eigenstates all with different energies. Let us call them $\{|l\rangle\}$. As λ goes to zero, $|l\rangle$ tends to $|l^{(0)}\rangle$, which are eigenstates of H_0, but which in general will not coincide with $|m\rangle$.

According to perturbation theory, let us expand the eigenstates and the eigenvalues of H in powers of λ,

$$|l\rangle = |l^{(0)}\rangle + \lambda|l^{(1)}\rangle + \lambda^2|l^{(2)}\rangle + O(\lambda^3) \tag{A.5}$$

and

$$E_l = E_{\text{GS}}^{(0)} + \lambda E_l^{(1)} + \lambda^2 E_l^{(2)} + O(\lambda^3). \tag{A.6}$$

Note that the zero-order term in the energy expansion does not depend on l, since the ground state of the non-perturbed Hamiltonian is degenerate. Substituting the previous expansions into the Schrödinger equation $(H_0 + \lambda V)|l\rangle = E_l|l\rangle$ and equating the coefficient of various powers of λ, we obtain a set of equations that will allow us to find the corrections to the perturbed eigenstates and eigenvalues.

At zero order in λ we recover the trivial non-perturbed Schrödinger equation. If we collect terms of order λ, we obtain

$$(E_D^0 - H_0)|l^{(1)}\rangle = (V - E_l^{(1)})|l^{(0)}\rangle. \tag{A.7}$$

In order to calculate the first correction to the energy, we project the previous equation (A.7) to the degenerate ground state subspace

$$\sum_{m'=1}^{4} V_{m,m'}\langle m'|l^{(0)}\rangle = E_l^{(1)}\langle m|l^{(0)}\rangle, \tag{A.8}$$

where $V_{m,m'} \equiv \langle m|V|m'\rangle$ is the projection of the interaction to this subspace. In our particular case, the matrix elements $V_{m,m'} = 0$ for all m and m'; hence, $E_l^{(1)} = 0 \,\forall\, l$. This means that the degeneration is not broken at first order in λ and forces us to consider the second order,

$$(E_D^0 - H_0)|l^{(2)}\rangle = (V - E_l^{(1)})|l^{(1)}\rangle - E_l^{(2)}|l^{(0)}\rangle. \tag{A.9}$$

We proceed as we did previously and project this equation to the degenerate ground state subspace,

$$\langle m|V - E_l^{(1)}|l^{(1)}\rangle = E_l^{(2)}\langle m|l^{(0)}\rangle. \tag{A.10}$$

From equation (A.7) we can compute the first order correction to the eigenstates $|l\rangle$,

$$|l^{(1)}\rangle = \sum_{k\notin GS} \frac{\langle k^{(0)}|V|l^{(0)}\rangle}{E_{GS}^{(0)} - E_k^{(0)}}, \tag{A.11}$$

where $|k^{(0)}\rangle$ are the H_0 eigenstates that do not belong to GS. Now, we substitute this into (A.9) and get an equation for the second-order correction to the energies and the states $|l^{(0)}\rangle$

$$\sum_{m',k} \frac{\langle m|V|k^{(0)}\rangle\langle k^{(0)}|V|m\rangle}{E_{GS}^{(0)} - E_k^{(0)}}\alpha_m^l = E_l^{(2)}\alpha_m^l, \tag{A.12}$$

where α_m^l are the coefficients of $|l^{(0)}\rangle \equiv \sum_m \alpha_m^l|m\rangle$ expressed in terms of the basis $|m\rangle$. Note that equation (A.12) is a diagonalization problem. For our particular Hamiltonian, it takes the form

$$2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = E_l^{(2)} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}. \tag{A.13}$$

Now the degeneration is completely broken and the perturbed ground state becomes

$$|GS\rangle = |\psi_-\rangle_{14}|\psi_-\rangle_{23} - \lambda\frac{1}{\sqrt{2}}(|1001\rangle + |0110\rangle) + O(\lambda^2), \tag{A.14}$$

with

$$E_{GS} = -2 + O(\lambda^3). \tag{A.15}$$

We can obtain an effective Hamiltonian by projecting the original one into the subspace of lower energy formed by $\{|l^0\rangle\}$,

$$H_{\text{eff}} = PHP^\dagger = -2 + \frac{\lambda^2}{2}\left(2 + \sigma_1^X\sigma_4^X + \sigma_1^Y\sigma_4^Y\right), \tag{A.16}$$

where $P = \sum_{l^{(0)}} |l^{(0)}\rangle\langle l^{(0)}|$ and $|l^0\rangle \in \{|\psi_-\rangle_{14}|\psi_-\rangle_{23}, |\psi_0\rangle_{14}|\psi_-\rangle_{23}, |\psi_1\rangle_{14}|\psi_-\rangle_{23}, |\psi_+\rangle_{14}|\psi_-\rangle_{23}\}$.

IOP Institute of Physics Φ DEUTSCHE PHYSIKALISCHE GESELLSCHAFT

References

[1] Page D N 1993 *Phys. Rev. Lett.* **71** 1291 (arXiv:gr-qc/9305007)

[2] White S R 1992 *Phys. Rev. Lett.* **69** 2863

[3] Vidal G 2003 *Phys. Rev. Lett.* **91** 147902 (arXiv:quant-ph/0301063)

[4] Nishino T, Okunishi K, Hieida Y, Maeshima N and Akutsu Y 2000 *Nucl. Phys.* B **575** 504 (arXiv:cond-mat/0001083)

[5] Pérez-García D, Verstraete F, Wolf M M and Cirac J I 2007 *Quantum Inf. Comput.* **7** 401 (arXiv:quant-ph/0608197)

[6] Vidal G 2008 *Phys. Rev. Lett.* **101** 110501 (arXiv:quant-ph/0610099)

[7] Verstraete F and Cirac J I 2004 (arXiv:cond-mat/0407066)

[8] Verstraete F, Cirac J I and Murg V 2008 *Adv. Phys.* **57** 143 (arXiv:0907.2796)

[9] Hastings M B 2007 *J. Stat. Mech.: Theor. Exp.* **57** P08024 (arXiv:0705.2024)

[10] Vidal G, Latorre J I, Rico E and Kitaev A 2003 *Phys. Rev. Lett.* **90** 227902 (arXiv:quant-ph/0211074)

[11] Latorre J I, Rico E and Vidal G 2004 *Quantum Inf. Comput.* **4** 48 (arXiv:quant-ph/0304098)

[12] Calabrese P and Cardy J L 2004 *J. Stat. Mech.* **0406** P002 (arXiv:hep-th/0405152)

[13] Calabrese P and Cardy J 2009 *J. Phys. A: Math. Theor.* **42** 504005 (arXiv:0905.4013)

[14] Latorre J I and Riera A 2009 *J. Phys. A: Math. Theor.* **42** 504002 (arXiv:0906.1499)

[15] Wolf M M 2006 *Phys. Rev. Lett.* **96** 010404 (arXiv:quant-ph/0503219)

[16] Farkas S and Zimborás Z 2007 *J. Math. Phys.* **48** 102110 (arXiv:0706.1805)

[17] Riera A and Latorre J I 2006 *Phys. Rev.* A **74** 052326 (arXiv:quant-ph/0605112)

[18] Amico L, Fazio R, Osterloh A and Vedral V 2008 *Rev. Mod. Phys.* **80** 517 (arXiv:quant-ph/0703044)

[19] Eisert J, Cramer M and Plenio M B 2010 *Rev. Mod. Phys.* **82** 277 (arXiv:0808.3773)

[20] Aharonov D, Gottesman D, Irani S and Kempe J 2009 *Comm. Math. Phys.* **287** 41 (arXiv:0705.4077)

[21] Fisher D S 1994 *Phys. Rev.* B **50** 3799

[22] Dasgupta C and Ma S-K 1980 *Phys. Rev.* B **22** 1305

[23] Refael G and Moore J E 2004 *Phys. Rev. Lett.* **93** 260602 (arXiv:cond-mat/0406737)

[24] Laflorencie N 2005 *Phys. Rev.* B **72** 140408 (arXiv:cond-mat/0504446)

[25] Chiara D G, Montangero S, Calabrese P and Fazio R 2006 *J. Stat. Mech.* **0603** P001 (arXiv:cond-mat/0512586)

[26] Binosi D, Chiara D G, Montangero S and Recati A 2007 *Phys. Rev.* B **76** 140405 (arXiv:0707.0266)

[27] Chung M-C and Peschel I 2001 *Phys. Rev.* B **64** 064412 (arXiv:cond-mat/0103301)

[28] Peschel I 2003 *Math. Gen.* **36** L205 (arXiv:cond-mat/0212631)

[29] Fannes M, Haegeman B and Mosonyi M 2003 *J. Math. Phys.* **44** 6005 (arXiv:math-ph/0306055)

[30] Farkas S and Zimborás Z 2005 *J. Math. Phys.* **46** 123301 (arXiv:math-ph/0507022)