DSO 699: Statistics Theory
Special Topics in Data Sciences and Operations

Week 6

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September 28, 2023

Announcements

- HW1 is due today (please email your work if not already)
- Solutions to HW1 will be posted on BB tomorrow.
- Midterm exam next week.
  - ✓ in-class exam
  - ✓ you can have access to slides
  - ✓ no internet
  - ✓ **Show up on time**

Recap from the previous class

We talked about:

- False discovery rate (FDR), false discovery proportion (FDP) and false discovery exceedance
- Benjamini-Hochberg procedure
  - ✓ for independent p-values
  - ✓ for arbitrarily dependent p-values
- Storey's procedure (improving BH by estimating the fraction of nulls)

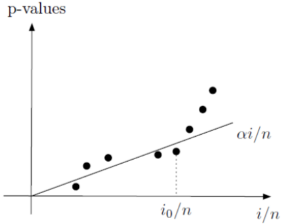
Outline for today

1. Online control of false discovery rate: problem formulation
2. Examples/Applications
3. Methods for online control of FDR
4. Some particular rules: LOND and LORD

Online control of false discovery rate: problem formulation

Benjamini-Hochberg (BH) procedure

1. Sort p-values $p_{(1)} \leq p_{(2)} \leq \ldots \leq p_{(n)}$
2. Let $i_0 := \max\{i \in [n] : \; p_{(i)} \leq \frac{i}{n}\alpha\}$
3. Reject all p-values $p_{(i)}$ for $i \leq i_0$ (or $p_j \leq p_{(i_0)}$)



BH cutoff is shown by dashed line.

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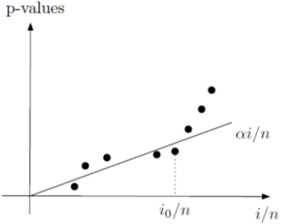

BH cutoff is shown by dashed line.

Some observations:

- BH cutoff depends on all p-values
- i_0 : total $\#$ of rejections, cutoff:= $\frac{\alpha i_0}{n}$

Some concerns about the BH procedure

- Cutoff is a function of α, n and *all* p-values
- If we get a new test(p-values), need to rerun the procedure
- Computational complexity aside, we may need to alter outcomes of previous test!

Offline procedures: requires to know all p-values upfront!

Online setting

Null hypotheses:

$$H_{0,1}, H_{0,2}, \ldots, H_{0,M}$$

Sequence of p-values: one at each time

$$p_1, p_2, p_3, \ldots$$

Ground truth:

$$\theta_1, \theta_2, \theta_3, \ldots [H_{0,i} : \theta_i = 0]$$

Test ouput ($\boldsymbol{p}_1^t = (p_1, \ldots, p_t)$):

$$T_1(\boldsymbol{p}_1^1), T_2(\boldsymbol{p}_1^2), T_3(\boldsymbol{p}_1^3), \cdots \in \{0, 1\}$$

[Foster, Stine, 2007]

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$$T_1(p_1),\, T_2(p_2;\, T_1),\, T_3(p_3;\, T_1,\, T_2), \cdots \in \{0, 1\}$$

[Foster, Stine, 2007]

Online control of FDR

- $V(n) \equiv$ False discoveries up to time n
- $R(n) \equiv$ Total number of discoveries up to time n

$$\mathrm{FDR}(n) \equiv \mathbb{E}\Big\{\frac{V(n)}{\max(R(n),1)}\Big\}$$

Want $\mathrm{FDR}(n) \leq \alpha$ for all n, θ

Examples/Applications

A/B testing

- Assume (!) that I am the CTO of a big web company
- $\approx$ 1000 data scientists
- $\approx$ 1000 *'brilliant ideas'* per day
  - ✓ Users are more likely to click on the first search result
  - ✓ Users are more likely to on top right ads
  - ✓ Users are more engaged with page layout A
- How to avoid wasting company resources?

Compute 'significance level' from data!

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Example

Idea: *Users click more on the first search result than on the second*

Null H_0: Users are equaly likely to click on first and second

Data:

- n events
- n_1 clicks on the *first* result
- $n_2 = n - n_1$ clicks on the *second* result

p-value

$$H_0 \quad \Rightarrow \quad z \equiv \frac{n_1 - n_2}{\sqrt{n}} \approx \mathrm{N}(0,1) \quad \Rightarrow \quad p = 1 - \Phi(z) \sim \mathrm{Uniform}([0,1])$$

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Company policy

Collect p-values every day, and run BH

Problems:

- Centralized
- Controls end-of-day FDR
  Not end-of-year FDR

$\rightarrow$ Online FDR control

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Anomaly detection

Another example of online FDR control is for anomaly detection in streaming, real–time applications.

NYC taxi dataset:

- records counts of NYC taxi passengers every 30 minutes from July 1, 2014 to January 31, 2015
- There were five known anomaly in this period:
  (the NYC marathon, Thanksgiving, Christmas, New Years day and a snow storm)

[Numenta Anomaly Benchmark (NAB) repository]

NYC taxi dataset

Figure 5: NYC Taxi passenger count time series from July 1st 2014 to Jan 31st 2015. Blue lines are Loess smoothed time series indicating the overall trend change.

[Gang, Sun, Wang 2020]

Formulating anomaly detection as hypothesis testing

Consider a sequence of hypotheses:

$$H_{0,t} : \text{no anomaly at time } t, \qquad H_{1,t} : \text{otherwise.}$$

Hypotheses arrive sequentially!

Constructing p-values:

1. Use **S**easonal-**T**rend decomposition using **L**oess (STL) to write:

$$\text{taxis count} = \underbrace{\text{trend} + \text{seasonal}}_{bias} + \underbrace{\text{remainder}}_{\text{new effects}}$$

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data_stl <- stlplus(data$Value, n.p =12, s.window = 13,
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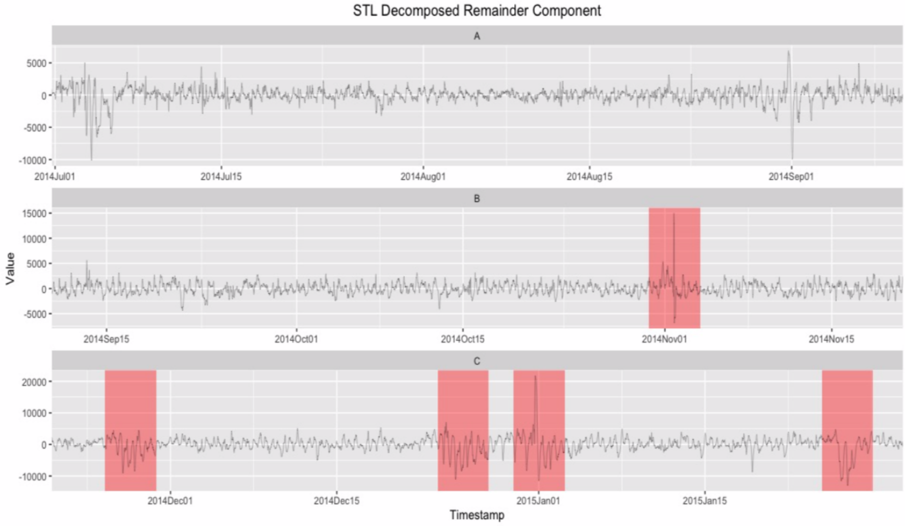
Remainder component

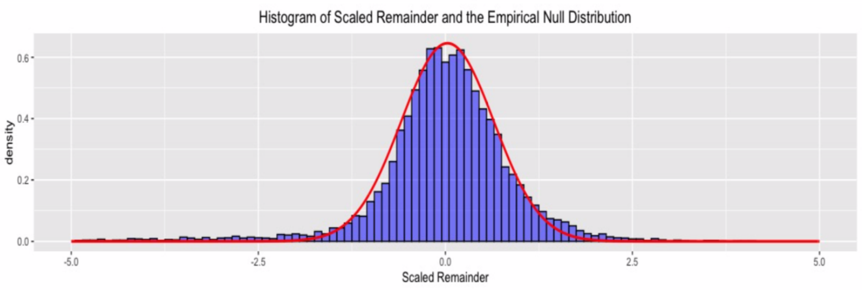


[Gang, Sun, Wang 2020]

Formulating anomaly detection as hypothesis testing (cont'd)

Constructing p-values:

2. Looking at the distribution of the remainder:



Model it as a mixture distribution:

$$r_j \sim \pi_0 N(\mu_0, \sigma_0^2) + (1 - \pi_0) N(\mu_j, \sigma_j^2), \quad 1 \leq j \leq n$$

estimate π_0, μ_0, σ_0^2 to obtain the null distribution $F_0 := N(\mu_0, \sigma_0^2)$.

[Jin& Cai 2007]

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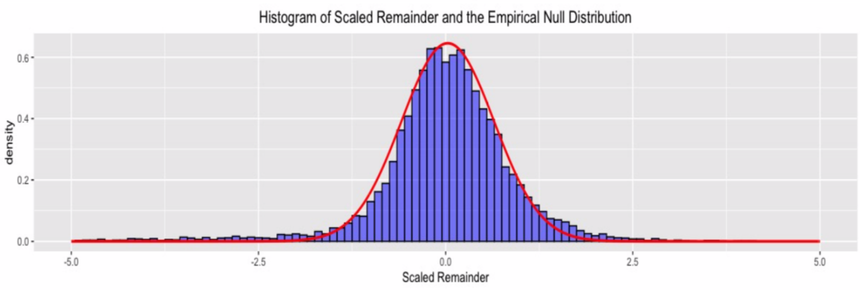

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Formulating anomaly detection as hypothesis testing (cont'd)

Constructing p-values:

3. Construct p-values as $p_i = 2F_0(-|r_i|)$

Of course, all these steps should be carried out using historical data and not the future data!

Formulating anomaly detection as hypothesis testing (cont'd)

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Of course, all these steps should be carried out using historical data and not the future data!

Summary.

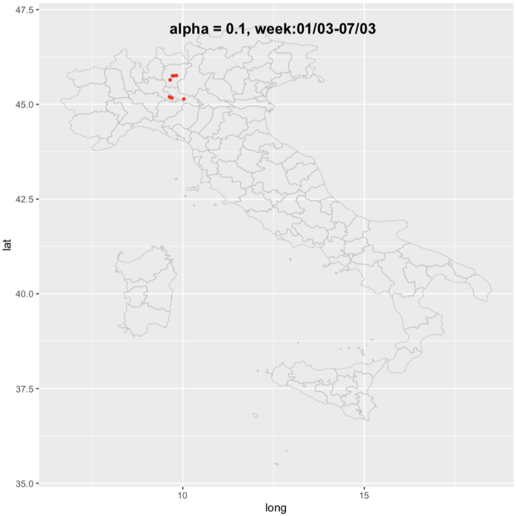
- We have a stream of p-values, $p_1, p_2, \ldots, \ldots$.
- Small $p_i \Rightarrow$ evidence for anomaly
- We formulated anomaly detection as an online hypotheses testing problem!

Another example: COVID outbreak detection

It is another example of anomaly detection:

- Working with death rates and would like to detect anomalies
- Seasonal and trend components should be removed
  (eg., Winters have higher death rates)
- Would like to declare emergency if there is significant jump in death rates.
- Any false rejections may trigger an alert system and cause unnecessary lockdowns.
- Another example of online hypotheses testing

An illustration



Performance of LORD algorithm on Italy's death rates. Neighborhoods with significant outbreaks are shown by red marks.

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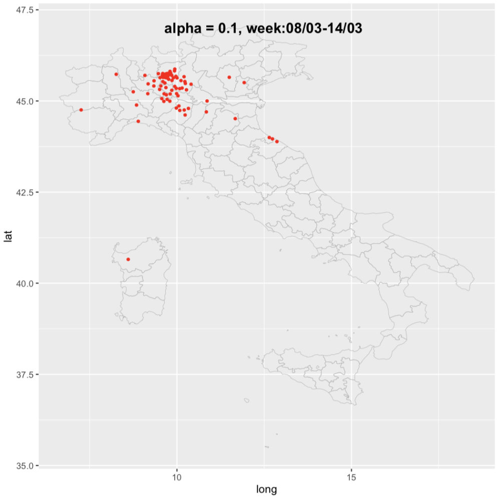

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Methods for online control of FDR

Trivial approach

- $\mathrm{FD}(n) \equiv$ False discoveries up to time n
- $\mathrm{D}(n) \equiv$ Total number of discoveries up to time n

$$\mathrm{FDR}(n) \equiv \mathbb{E}\left\{ \frac{\mathrm{FD}(n)}{\max(\mathrm{D}(n), 1)} \right\}$$

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Bonferroni:

- Choose $\beta_i \in [0, 1]$, $\sum_{i=1}^{\infty} \beta_i \le \alpha$
- Set

$$T_i = \begin{cases} 1 & \text{if } p_i \le \beta_i, \\ 0 & \text{otherwise.} \end{cases}$$

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$$T_i = \begin{cases} 1 & \text{if } p_i \le \beta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed

$$\mathrm{FDR}(n) \le \mathbb{E}\{V(n)\} \le \sum_{i:\,\theta_i=0} \mathbb{P}(p_i \le \beta_i) = \sum_{i:\,\theta_i=0} \beta_i \le \alpha$$

Very conservative!

A better approach?

What rules are we allowed to use?

Online rule:

- Compare p-value p_i with the threshold α_i:

$$T_i = \mathbb{I}(p_i \leq \alpha_i)$$

- thresholds α_i can only be functions of *previous* outcomes

$$\alpha_i = f(T_1, T_2, \ldots, T_{i-1}).$$

How to choose rule f to ensure $\mathrm{FDR}(n) \leq \alpha$ for all n, θ?

Generalized Alpha-investing

A game between statistician and Nature

- Initial alpha-wealth to "invest" in the tests

$$W(0) = \alpha$$

- At step j Statistician pays ϕ_j; if rejects H_j earns ψ_j.

$$W(j) - W(j-1) = -\phi_j + T_j \psi_j \,.$$

- Stops when the wealth becomes negative ($W(j) < 0$).

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$$\phi_j = \phi_j(T_1, \ldots, T_{j-1}) \qquad \psi_j = \psi_j(T_1, \ldots, T_{j-1})$$

Assumptions on ϕ_j and ψ_j

We want the followings hold for all j:

- Don't bet the money you don't have!

$$\phi_j \leq W(j-1)$$

- At each step, the wealth changes by at most α:

$$\psi_j \leq \phi_j + \alpha$$

- An upper bound on the threshold α_j:

$$\psi_j \leq \frac{\phi_j}{\alpha_j} + \alpha - 1$$

- For all j, if $W(j-1) = 0$, then $\alpha_j = 0$.

A theorem

Definition (monotonicity of a rule)

For $x, y \in \{0,1\}^n$, we write $x \preceq y$ if $x_j \leq y_j$ for all $j \in [n]$.
We say that an online rule is monotone if for all j, α_j is monotone-decreasing w.r.t to this partial ordering:

$$x \preceq y \Rightarrow \alpha_j(x) \leq \alpha_j(y) \quad \forall j.$$

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Theorem [Javanmard, Montanari 2015]

Suppose the null p-values are jointly independent, and independent from the non-null p-values. Them, for any monotone generalized alpha investing rule

$$\sup_{n,\theta} \mathrm{FDR}(n) \le \alpha \,.$$

Some particular rules: LOND and LORD

A simple rule

LOND (Levels based On Number of Discoveries)

- Choose non-increasing sequence of $\gamma_j \in [0,1]$, such that $\sum_{j=1}^{\infty} \gamma_j = 1$.
- $R(j-1) \equiv$ Number of rejection by time $j-1$
- Start with wealth $W(0) = \alpha$ and set

$$
\begin{aligned}
&(\text{Bet}) \quad\ \ \phi_j = \alpha\gamma_j(R(j-1)+1) \\
&(\text{Reward})\ \psi_j = \alpha
\end{aligned}
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- Significance levels are set as $\alpha_j = \phi_j$

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LOND is a generalized alpha investing rule for any such sequence of $\{\gamma_i\}_{i\in\mathbb{N}}$

(*Why?*)

• An inherent positive feedback (more discoveries at the beginning would lead to more future discoveries!)

Another online rule

LORD (Levels based On Recent Discovery)

- Choose non-increasing sequence of $\gamma_j \in [0,1]$, such that $\sum_{j=1}^{\infty} \gamma_j = 1$.
- $\tau_j \equiv$ Time of the last discovery before j
- Set

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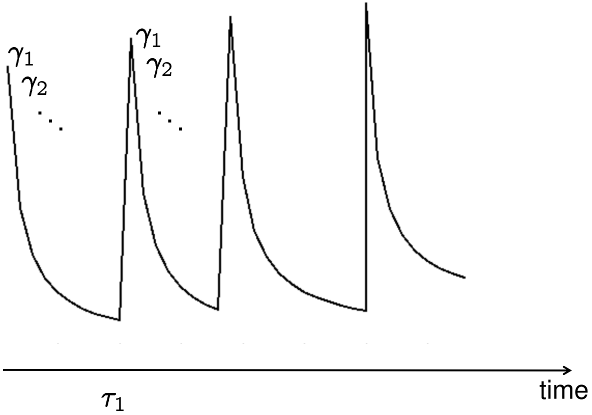
An illustration of LORD

- trend of test levels α_j



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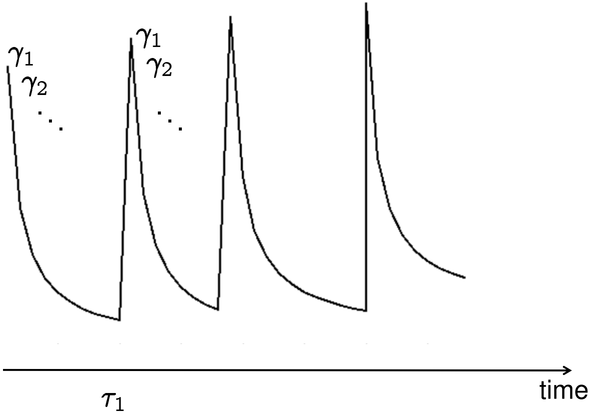
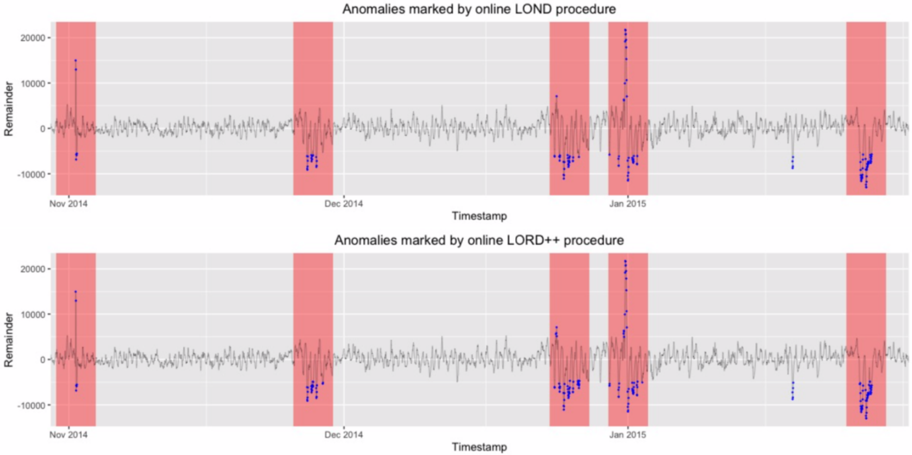


- Each discovery increases chance for further discoveries.
- This special structure helps to have batched discoveries (why?)

Back to NYC taxi example

Performance of LOND and LORD++



Anomaly points detected by LOND and LORD++ algorithms. Both detect the most anomaly points within the labeled window marked by red rectangles. Nominal significance level chosen as 0.0001.

Recap

We talked about

- Online false discovery rate
  (deciding on tests which arrive sequentially without knowledge about the future tests/p-values, no possibility to retract your previous decisions)
- Applications
  - ✓ A/B testing
  - ✓ Anomaly detection
- Methodology (Generalized alpha investing rules)
  - ✓ LOND
  - ✓ LORD

Next week

- In-class midterm exam

FIGHT ON!