

Minimax Optimal Two-Sample Testing under Local Differential Privacy

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Abstract

We explore the trade-off between privacy and statistical utility in private two-sample testing under local differential privacy (LDP) for both multinomial and continuous data. We begin by addressing the multinomial case, where we introduce private permutation tests using practical privacy mechanisms such as Laplace, discrete Laplace, and Google’s RAPPOR. We then extend our multinomial approach to continuous data via binning and study its uniform separation rates under LDP over Hölder and Besov smoothness classes. The proposed tests for both discrete and continuous cases rigorously control the type I error for any finite sample size, strictly adhere to LDP constraints, and achieve minimax separation rates under LDP. The attained minimax rates reveal inherent privacy-utility trade-offs that are unavoidable in private testing. To address scenarios with unknown smoothness parameters in density testing, we propose an adaptive test based on a Bonferroni-type approach that ensures robust performance without prior knowledge of the smoothness parameters. We validate our theoretical findings with extensive numerical experiments and demonstrate the practical relevance and effectiveness of our proposed methods.

Keywords: local differential privacy, minimax optimality, permutation tests, U-statistics, two-sample testing

1 Introduction

Large-scale internet services such as Netflix and Amazon collect sensitive data from massive user bases, allowing companies to conduct cost-effective randomized experiments by assigning users to two different user interfaces or marketing campaigns. By testing whether the resulting two independent sets of samples originate from the same distribution—a procedure known as A/B testing or two-sample testing—companies can statistically assess the impact of new interfaces or campaigns on various user behaviors. However, the sensitivity of detailed personal data raises substantial privacy concerns in data analysis. Since privacy protection inherently conceals some of the information in the data and in turn compromises statistical utility, it is crucial to characterize and balance the

trade-off between statistical utility and data privacy. Differential privacy (DP; [Dwork et al. \(2006\)](#)) provides a rigorous framework for this trade-off, defining data privacy as a mathematical concept that supports such balancing.

We briefly review two notions of DP: central and local DP. The central DP constraint assumes that a trusted data curator (or distributor) has access to the entire original dataset and calculates a noisy statistical result. This centralized approach requires that the probability of any event remains essentially the same when a single data entry is arbitrarily perturbed. Under this constraint, one cannot reliably extract any individual-level information from the noisy statistical result. In contrast, data owners do not place trust in the curator in the local DP constraint (LDP; [Kasiviswanathan et al. \(2011\)](#)). Instead, each data owner independently reports a noisy version of their data. This more stringent local constraint inevitably impairs statistical utility more than the central constraint. However, it provides a higher level of privacy and separates the data curator from the responsibility for disclosure risk.

These benefits have driven widespread adoption of the LDP framework in the internet-scale deployment of data analysis under privacy, such as in Microsoft ([Ding et al., 2017](#)), Google ([Erilingsson et al., 2014](#)), Apple ([Apple Differential Privacy Team, 2017](#)), and Uber ([Near, 2018](#)). With the massive user bases, these companies require stringent privacy protections, while also having the capacity to obtain large samples which allows statistically meaningful analyses within strong privacy constraints. Various implementations of LDP in industrial applications has naturally spurred a substantial body of work ([Acharya et al., 2021d](#); [Duchi et al., 2018](#); [Lam-Weil et al., 2022](#); [Lalanne et al., 2023](#); [Cai et al., 2021](#)), examining the intrinsic trade-off between privacy and statistical utility in various settings. This paper contributes to this body of work by exploring this trade-off in the context of the two-sample testing problem.

The two-sample testing problem, which originates from the classical two-sample t -test ([Student, 1908](#)), has gained renewed interest in recent years due to the emergence of high-dimensional and complex data. Notably, several novel methodological approaches have been developed including kernel-based tests ([Gretton et al., 2009, 2012](#)), distance-based tests ([Szekely and Rizzo, 2004](#); [Székely and Rizzo, 2005](#)) and regression/classification-based tests ([Kim et al., 2019, 2021](#)), which have demonstrated promising capabilities in dealing with modern datasets. On the theoretical front, researchers have explored the fundamental limit of this problem through the lens of minimax analysis both in the statistics (e.g., [Arias-Castro et al., 2018](#); [Kim et al., 2022a](#); [Schrab et al., 2023](#)) and computer science (e.g., [Batu et al., 2000](#); [Chan et al., 2014a](#); [Diakonikolas and Kane, 2016](#); [Goldreich and Ron, 2011](#)) literature. In addition to the methodological and theoretical advancements, the topic has found contemporary applications such as in education research ([Rabin et al., 2019](#)), network traffic analysis ([Kohout and Pevný, 2018](#)), and audio segmentation ([Harchaoui et al., 2009](#)). Despite its long history and fundamental roles in practice, most of the existing work on two-sample testing has focused on non-private settings with a few exceptions. Exceptions include private versions of multinomial tests ([Acharya et al., 2018](#); [Aliakbarpour et al., 2019, 2018](#)), traditional non-parametric tests (e.g., Mann-Whitney and Wilcoxon signed-rank tests) [Couch et al. \(2019\)](#); [Task and Clifton \(2016\)](#), and kernel tests [Raj et al. \(2020\)](#); [Kim and Schrab \(2023\)](#) under central DP. The literature on two-sample testing under LDP is even more scarce. Notable contributions in this area include [Ding et al. \(2018\)](#) and [Waudby-Smith et al. \(2022\)](#), both of which are mainly concerned with detecting differences in location. In this context, our primary focus is on developing two-sample tests for general alternatives under LDP and shedding new light on the fundamental limits of the two-sample problem under LDP.

In the following subsection, we begin with a review of related work and then discuss our techniques and contributions.

1.1 Related prior work

Private hypothesis testing has been extensively studied in the statistics and computer science literature. Among various studies on this topic, we briefly review those closely related to our work. Initially motivated by the privacy attack on genome-wide association study (GWAS) (Homer et al., 2008), the early work on private testing mainly concentrates on private versions of chi-square tests and explores their asymptotic properties (Gaboardi et al., 2016; Gaboardi and Rogers, 2018; Johnson and Shmatikov, 2013; Rogers and Kifer, 2017; Uhler et al., 2013; Vu and Slavkovic, 2009; Wang et al., 2015; Yu et al., 2014). In contrast, a recent line of work in computer science is concerned with non-asymptotic properties of private tests designed for multinomial datasets, and studies optimal sample complexities of testing problems from a minimax perspective. This line of work has been initiated by Cai et al. (2017) for central DP and Sheffet (2018) for LDP, and continued by Acharya et al. (2018); Aliakbarpour et al. (2019, 2018) for central DP and Acharya et al. (2020, 2021a) for LDP, respectively. The optimal sample complexity is usually achieved through a systematic analysis of both its upper and lower bounds. For the upper bound analysis, the prior works extend non-private multinomial tests such as in (Acharya et al., 2015; Chan et al., 2014a; Diakonikolas et al., 2018; Diakonikolas and Kane, 2016; Goldreich and Ron, 2011; Valiant and Valiant, 2014) to corresponding private counterparts by incorporating randomization mechanisms. On the other hand, the lower bound analysis relies on information-theoretic techniques, such as Le Cam’s method, while treating the DP requirement as information constraints (for detailed discussions, see e.g., Acharya et al. (2020); Duchi et al. (2018)).

There are also a few recent papers from the statistics community that explore univariate goodness-of-fit testing under LDP. Specifically, Dubois et al. (2023) propose minimax optimal goodness-of-fit tests for Hölder densities under LDP in both non-interactive and interactive scenarios. Lam-Weil et al. (2022) also consider goodness-of-fit testing under LDP, and develop minimax optimal tests for multinomials and for continuous densities over Besov balls. Our work builds on their framework and extends the focus from goodness-of-fit testing to two-sample testing for both (i) multinomials and (ii) multivariate Hölder and Besov densities. It is worth highlighting that previous works (Lam-Weil et al., 2022; Dubois et al., 2023) rely solely on the Laplace mechanism (Dwork and Roth, 2013) to establish their theoretical results. In contrast, we explore various LDP mechanisms that achieve similar optimality properties and empirically demonstrate that the Laplace mechanism can underperform in practical scenarios. Specifically, we delve into the Google’s RAPPOR (Gaboardi and Rogers (2018); Erlingsson et al. (2014); Acharya et al. (2019)), generalized randomized response (Gaboardi and Rogers, 2018), and (discrete) Laplace mechanisms (Ghosh et al., 2009), and illustrate their theoretical and empirical performance.

1.2 Techniques and results

Previous work on hypothesis testing under local differential privacy has primarily focused on goodness-of-fit testing (Dubois et al., 2023; Lam-Weil et al., 2022). We instead target a broader and arguably more challenging settings of two-sample testing of multinomials and multivariate densities. In particular, we provide testing methods that are both practically reliable and theoretically optimal. Our practical reliability stems from both the privatization mechanism and the testing procedure. For privatization, one of our methods leverages Google’s RAPPOR, a widely adopted open-source privacy mechanism which has demonstrated effectiveness through years of large-scale deployment in Chrome browser. Although prior works (Duchi et al., 2013; Acharya et al., 2019, 2021c) adopt RAPPOR and analyze its statistical performance under the minimax framework, their focuses are limited to ℓ_1 separation and multinomial data. Our work expands upon this by establishing minimax optimality for both multivariate continuous and multinomial data under ℓ_2 separation, marking the first result

of its kind. For the testing procedure, our methods rigorously control the type I error in all scenarios. At the heart of achieving the blend of practicality and theory lies the permutation test. Achieving practical reliability poses a significant challenge, especially in calibrating the critical value within the non-asymptotic regime while accounting for the randomization effects introduced by local differential privacy. Under a composite null hypothesis of two-sample testing, critical values cannot be determined through Monte-Carlo-approximated population quantile of test statistics, unlike in goodness-of-fit testing. The critical values obtained through concentration inequalities, on the other hand, usually depend on unspecified constants and thus are not reliable in practice. By employing a permutation procedure, our testing methods guarantee type I error control at any sample size and with sufficiently large number of permutations (which does not depend on the sample size). The permutation procedure also facilitates theoretical analysis of power, leading to minimax upper bound analyses. In particular, it enables us to leverage the technique of [Kim et al. \(2022a\)](#), namely the two moments method therein. This technique allows us to avoid directly analyzing the permutation distribution under LDP, and provides a sufficient condition for type II error control based solely on the first two moments of the test statistic. Equipped with this tool, we analyze our test statistics, which are U-statistics derived from perturbed data with a carefully selected perturbation level. A bulk of our technical effort is dedicated to bounding the moments related to the U-statistic in the presence of this data perturbation.

By obtaining matching information-theoretic lower bounds, we establish the optimality of our methods and gain insight into the fundamental trade-off between statistical power and privacy. For the lower bound analysis, we leverage a recently developed technique by [Lam-Weil et al. \(2022\)](#). This technique builds upon Ingster’s method ([Ingster, 1993](#)), a classical approach for deriving minimax separation rates in testing problems, and adapts it to incorporate the LDP constraint. At the heart of Ingster’s method is bounding the chi-square divergence between a simple null distribution and a mixture of alternative distributions. The key idea behind obtaining a tight lower bound under LDP is to construct a mixture distribution in Ingster’s method using the singular values and singular vectors of the privacy mechanism. Such construction naturally imposes extra restrictions caused by the LDP constraint, enabling us to achieve a tight lower bound under LDP. Our technical effort lies in extending the univariate result of [Lam-Weil et al. \(2022\)](#) to more general settings, including the multivariate Hölder ball and Besov ball.

Summary of our contributions. We highlight our contributions and contrast them with prior work as follows. We also refer readers to Table 1, which summarizes the non-private and private minimax separation rates for two-sample testing, derived from both prior work and our findings.

- **Optimal multinomial testing under LDP (Theorem 3):** We start by developing a private two-sample test for multinomials, and present minimax separation rates in terms of the ℓ_2 distance under LDP. The prior work [Acharya et al. \(2018\)](#); [Aliakbarpour et al. \(2019, 2018\)](#) for private two-sample testing generally focuses on central DP and imposes conditions such as equal sample sizes and Poisson sampling that may not be practically relevant. In contrast, our approach does not rely on such unnecessary conditions, and obtain optimality under more practical settings. We also highlight that our upper bound result is established using three distinct LDP mechanisms—namely Laplace, discrete Laplace and RAPPOR mechanisms—which diversifies the toolkit in practice. As mentioned earlier, this is in contrast to the prior work [Lam-Weil et al. \(2022\)](#); [Dubois et al. \(2023\)](#), which mainly focuses on the Laplace mechanism. Moreover, we show that the use of generalized randomized response mechanism ([Gaboardi and Rogers, 2018](#)) can lead to suboptimal power in Appendix H.

Table 1: Non-private and private minimax rates for two-sample multinomial and density testing in ℓ_2 and \mathbb{L}_2 separation where n_1 denotes the minimum sample size and α denotes the privacy level. For multinomial testing, k stands for the number of categories, and for density testing, s represents the smoothness parameter. The rates exhibit elbow effects—phase transitions at specific levels of privacy. See Section 3 and 4 for details.

	Non-private rate	Private rate under LDP
Testing for multinomials in ℓ_2 separation	$n_1^{-1/2}$ (Chan et al. (2014b); Kim et al. (2022a))	$\frac{k^{1/4}}{(n_1\alpha^2)^{1/2}} \vee n_1^{-1/2}$ (Theorem 3)
Testing for Hölder and Besov densities in \mathbb{L}_2 separation	$n_1^{\frac{-2s}{4s+d}}$ (Arias-Castro et al. (2018))	$(n_1\alpha^2)^{\frac{-2s}{4s+3d}} \vee n_1^{\frac{-2s}{4s+d}}$ (Theorem 11)

- **Optimal density testing under LDP (Theorem 11):** We next consider the two-sample problem for continuous data and derive optimal \mathbb{L}_2 separation rates under LDP, by leveraging the prior work of Lam-Weil et al. (2022); Kim et al. (2022a). In particular, we examine both Hölder and Besov smoothness classes, and show that the proposed private test is optimal for both classes with the finite-sample validity. In contrast to the prior work (Ding et al., 2018; Waudby-Smith et al., 2022), primarily focused on location alternatives, our private test is sensitive against a broad range of nonparametric alternatives. Moreover, we highlight that our method controls the type I error in any finite sample sizes, and exactly satisfies the LDP condition, distinguishing it from the prior work of Raj et al. (2020).
- **Adaptive density testing under LDP (Theorem 12):** Similar to other nonparametric methods for density testing, the optimality of the proposed density test relies on the knowledge of the underlying smoothness parameter, which is typically unknown. To tackle this issue, we introduce a Bonferroni-type approach that adapts to the unknown smoothness parameter at the expense of extra logarithmic factors in the separation rate. The proof of the adaptation result leverages the exponential inequality of the permuted U-statistic (Kim et al., 2022a). This technique leads to an improvement of the adaptive technique used in Lam-Weil et al. (2022), which resorts to a simple upper bound for the variance of the U-statistic along with Chebyshev’s inequality.
- **Numerical validation (Section 5):** Lastly, we assess the empirical performance of the proposed tests under various scenarios and showcase the trade-off between privacy and utility through numerical simulations. It is important to emphasize that previous research on private testing has primarily centered on theoretical optimality, often lacking empirical validation of their findings. We address this gap by complementing theoretical justifications with empirical evaluation, thereby enhancing practical relevance. Since no previous methods exist for two-sample testing under LDP, we create baseline methods by extending one-sample LDP χ^2 -tests (Gaboardi and Rogers, 2018) to the two-sample problem (Appendix G.1), and compare their empirical performance with our main proposals.

1.3 Notation

Throughout this paper, real numbers are represented by lowercase, non-bold letters, such as a , while vectors in \mathbb{R}^d for $d \geq 2$ are written in boldface lowercase, such as \mathbf{a} . Constant vectors are

denoted using bold numerals, such as $\mathbf{1}$ and $\mathbf{0}$. The i th element of \mathbf{a} is denoted by a_i . For indexed vector such as \mathbf{a}_j , its i th element is denoted as a_{ji} . Unless otherwise specified, random variables are written in uppercase non-bold (e.g., X), while random vectors use bold uppercase (e.g., \mathbf{X}). The i th element of \mathbf{X} and \mathbf{X}_j are denoted as X_i and X_{ji} , respectively. The set of non-negative integers is denoted by $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. For positive integers u and v , $[u]$ represents $\{1, \dots, u\}$, and $[u]^v$ denotes its Cartesian product taken v times. A set of u elements indexed by i is written as $\{a_i\}_{i \in [u]} := \{a_1, \dots, a_u\}$. For any real $s > 0$, $\lfloor s \rfloor$ denotes the largest integer strictly smaller than s . For $a, b \in \mathbb{R}$, we define $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. Given $\mathbf{w} \in \mathbb{R}^d$ and $p \geq 1$, its ℓ_p -norm is defined by $\|\mathbf{w}\|_p := (\sum_{i=1}^d |w_i|^p)^{1/p}$. Similarly, the \mathbb{L}_p -norm of a function supported on $[0, 1]^d$ is defined as follows:

$$\|f\|_{\mathbb{L}_p} := \left(\int_{[0,1]^d} |f(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}, \quad 1 \leq p < \infty.$$

For any $\mathbf{u} \in \mathbb{R}^d$ and $\mathbf{v} \in \mathbb{R}^d$, with $u_j \leq v_j$ for $j = 1, \dots, d$, a hyperrectangle $[\mathbf{u}, \mathbf{v}]$ is defined as follows:

$$[\mathbf{u}, \mathbf{v}] := \prod_{j=1}^d [u_j, v_j].$$

Given the privacy parameter $\alpha > 0$, we write $z_\alpha := e^{2\alpha} - e^{-2\alpha} = 2 \sinh(2\alpha)$. Throughout, a constant that only depends on parameters $\theta_1, \theta_2, \dots$ is denoted as $C(\theta_1, \theta_2, \dots)$. The indicator function $\mathbf{1}(\mathcal{A})$ takes value 1 if the event \mathcal{A} is true and 0 otherwise.

1.4 Outline of the paper

The remainder of the paper is organized as follows: Section 2 introduces the necessary background on LDP, the minimax framework, and the permutation procedure. Section 3 illustrates the minimax analysis for multinomial testing under LDP and the optimal permutation testing procedure. Building on this result, Section 4 presents the minimax analysis for multivariate two-sample density testing under LDP and an optimal permutation testing procedure. Finally, Section 5 presents numerical validation of the procedure proposed in Sections 3 and 4. All proofs and additional simulation results are deferred to the appendix.

2 Background

This section introduces the notion of local differential privacy and two-sample testing under LDP. We then explain the minimax framework for two-sample testing under LDP, along with the permutation test procedure considered throughout this paper.

2.1 Two-sample testing under local differential privacy constraint

Let \mathcal{P} denote the space of pairs of distributions of interest—multinomial distributions taking values in $[k]$ or continuous distributions taking values in $[0, 1]^d$. Given a pair of distributions $(P_Y, P_Z) \in \mathcal{P}$, for each $i \in [n_1]$, the i th data owner draws Y_i from P_Y independently from the others. Similarly, for each $j \in [n_2]$, the j th data owner independently draws Z_j from P_Z . We allow the sample sizes n_1 and n_2 to differ, and assume $n_1 \leq n_2$, without loss of generality, throughout this paper. We denote the pooled sample size as $n := n_1 + n_2$. Under the LDP constraint, each owner releases only a randomized transformation of their raw sample as follows:

Definition 1 (Local differential privacy) *Given a privacy level $\alpha > 0$, let X_i and \tilde{X}_i be random elements mapped to measurable spaces $(\mathcal{X}, \mathcal{F})$ and $(\tilde{\mathcal{X}}_i, \tilde{\mathcal{F}}_i)$, respectively, for each $i \in [n]$. Then \tilde{X}_i is an α -local differentially private (α -LDP) view of X_i if there exists a bivariate function $Q_i(\cdot | \cdot)$ on $\tilde{\mathcal{F}}_i \times \mathcal{F}$ such that:*

1. *For any $x \in \mathcal{X}$, $Q_i(\cdot | x)$ is a conditional distribution of \tilde{X}_i given $X_i = x$,*
2. *For any $A \in \tilde{\mathcal{F}}_i$, $x \mapsto Q_i(A | x)$ is a measurable function on \mathcal{X} , and*
3. *For any $x, x' \in \mathcal{X}$ and $A \in \tilde{\mathcal{F}}_i$, $Q_i(A | x) \leq e^\alpha Q_i(A | x')$.*

Let \mathcal{Q}_α be the set of joint distributions whose marginals satisfy the above properties. Then $Q \in \mathcal{Q}_\alpha$ is called an α -LDP mechanism (or channel) associated with $\{X_i\}_{i=1}^n$.

The curator, aware of the privacy level α and the mechanism Q , only receives the α -LDP views $\{\tilde{X}_i\}_{i \in [n]}$, consisting of $\{\tilde{Y}_i\}_{i \in [n_1]}$ and $\{\tilde{Z}_j\}_{j \in [n_2]}$, and uses them to decide whether $H_0 : P_Y = P_Z$ or $H_1 : P_Y \neq P_Z$. The definition of LDP above is non-interactive in a sense that the i th conditional distribution Q_i is assumed to be independent of other private views $\tilde{X}_1, \dots, \tilde{X}_{i-1}, \tilde{X}_{i+1}, \dots, \tilde{X}_n$. It has been pointed out that allowing Q_i to be interactive with private views $\{\tilde{X}_{i'}\}_{i' \in [n] \setminus i}$ can yield more efficient statistical procedures (e.g., [Acharya et al. \(2021b\)](#); [Berrett and Butucea \(2020\)](#); [Kasiviswanathan et al. \(2011\)](#)). The non-interactive approach, however, requires less communication between data owners and a curator and thus is more suitable for large-scale statistical inference than the interactive counterpart ([Berrett et al., 2021](#); [Joseph et al., 2019](#)). Thereby, we focus on the non-interactive privacy mechanisms throughout this paper.

Unless otherwise specified, all statements regarding expectations and variances in this paper are with respect to the distributions of the α -LDP views defined as:

$$P_{\tilde{Y}_i}(A) := \int_{\tilde{\mathcal{X}}_i} Q_i(A | y) P_Y(dy) \text{ and } P_{\tilde{Z}_j}(A) := \int_{\tilde{\mathcal{X}}_j} Q_i(A | z) P_Z(dz),$$

for $i \in [n_1]$ and $j \in [n_2]$ where the raw samples are marginalized out.

2.2 Non-private and LDP minimax framework for two-sample testing

Let \mathcal{P}_0 and \mathcal{P}_1 denote the collection of null distributions, and the collection of alternative distributions, respectively. In the minimax framework, we focus on a subset of \mathcal{P}_1 , denoted as $\mathcal{P}_1(\rho_{n_1, n_2})$, where ρ_{n_1, n_2} indicates a minimum separation between $\mathcal{P}_1(\rho_{n_1, n_2})$.

For a given privacy mechanism Q , the curator evaluates a test function $\Delta_Q : \tilde{\mathcal{X}}^{(n_1+n_2)} \mapsto \{0, 1\}$ on the α -LDP views and rejects H_0 if $\Delta_Q = 1$ and accepts H_0 otherwise. Our objective is to design the private test Δ_Q that controls the type I and II errors uniformly over distributions in \mathcal{P}_0 and $\mathcal{P}_1(\rho_{n_1, n_2})$, respectively. In particular, for fixed $\gamma, \beta \in (0, 1)$, we aim for the private test Δ_Q to satisfy the following conditions:

$$\begin{aligned} \text{Type I error: } & \sup_{(P_Y, P_Z) \in \mathcal{P}_0} \mathbb{E}[\Delta_Q] \leq \gamma, \text{ and} \\ \text{Type II error: } & \sup_{(P_Y, P_Z) \in \mathcal{P}_1(\rho_{n_1, n_2})} \mathbb{E}[1 - \Delta_Q] \leq \beta. \end{aligned} \tag{1}$$

Let Φ_γ^α be the set of α -LDP level γ tests, which take α -LDP views as inputs and control the type I error as in (1). The quality of a test $\Delta_{Q, \gamma} \in \Phi_\gamma^\alpha$ is assessed by its uniform separation rate, which quantifies the proximity between two hypotheses that can still be successfully distinguished by the

test. In more technical terms, the uniform separation rate $\tilde{\rho}_{n_1, n_2}(\Delta_{Q, \gamma})$ is the smallest separation ρ_{n_1, n_2} which accomplishes the type II error control as in (1), i.e.,

$$\tilde{\rho}_{n_1, n_2}(\Delta_{Q, \gamma}) := \inf \left\{ \rho_{n_1, n_2} > 0 : \sup_{(P_Y, P_Z) \in \mathcal{P}_1(\rho_{n_1, n_2})} \mathbb{E}[1 - \Delta_{Q, \gamma}] \leq \beta \right\}. \quad (2)$$

Since different α -LDP mechanism Q may assume different $\tilde{\mathcal{X}}$, a test $\Delta_{Q, \gamma}$ depends on a particular LDP mechanism Q . Therefore, an optimal private level γ test can be described as a pair of an α -LDP mechanism and a test function, designed to achieve the minimal uniform separation rate (2). We define this minimal uniform separation rate as the optimal testing rate under LDP.

Definition 2 (α -LDP non-asymptotic minimax testing rate) For a fixed $\alpha > 0$, the α -LDP non-asymptotic minimax rate of testing is defined as

$$\rho_{n_1, n_2, \alpha}^* := \inf_{Q \in \mathcal{Q}_\alpha} \inf_{\Delta_{Q, \gamma} \in \Phi_\gamma^\alpha} \tilde{\rho}_{n_1, n_2}(\Delta_{Q, \gamma}). \quad (3)$$

Our main interest is to figure out the price to pay for privacy by comparing the private minimax testing rate (3) with the non-private (unconstrained) rate $\rho_{n_1, n_2}^* := \inf_{\Delta_\gamma \in \Phi_\gamma} \tilde{\rho}_{n_1, n_2}(\Delta_\gamma)$, where Φ_γ denotes the set of all level γ tests without privacy constraints. Since it is mostly infeasible to obtain a test that achieves the exact minimax risk in a nonparametric setting, we follow the convention (Ingster, 1994, 2000; Baraud, 2002) and focus on minimax *rate* optimality. In particular, we say that a test is minimax rate optimal if its uniform separation rate is upper bounded by the minimax testing rate up to some constant.

2.3 Permutation testing procedure

The permutation test is a simple yet powerful method to calibrate a test statistic, yielding a valid γ -level test under exchangeability of the samples, i.e., when the distribution of the permuted samples is the same as the distribution of the original samples. Note that some α -LDP mechanisms might violate exchangeability even when the raw samples are i.i.d. In order to guarantee exchangeability, throughout this paper, we only consider α -LDP mechanisms with identical marginals. Below, we briefly explain the permutation test in the two-sample setting.

Let \mathbb{X}_n denote the pooled sample $\{\xi_1, \dots, \xi_n\} := \{Y_1, \dots, Y_{n_1}, Z_1, \dots, Z_{n_2}\}$. Let π_n be the set of all possible permutations of $[n]$, and denote its cardinality as $|\pi_n|$. Given a permutation $\pi := (\pi_1, \dots, \pi_n)$ sampled from a uniform distribution over π_n , the permuted version of \mathbb{X}_n is denoted as $\mathbb{X}_n^\pi := \{\xi_{\pi_1}, \dots, \xi_{\pi_n}\}$. For a two-sample statistic $T_n(\mathbb{X}_n)$, its permutation distribution function conditional on \mathbb{X}_n is $F_T(t) := \sum_{\pi \in \pi_n} \mathbb{1}(T(\mathbb{X}_n^\pi) \leq t) / |\pi_n|$. The permutation testing procedure rejects the null hypothesis if $T(\mathbb{X}_n) > \inf\{t : F_T(t) \geq 1 - \gamma\}$. If the exchangeability assumption of $\mathbb{X}_n \stackrel{d}{=} \mathbb{X}_n^\pi$ for any $\pi \in \pi_n$ is satisfied under H_0 , the resulting test controls the type I error non-asymptotically (see e.g., (Ramdas et al., 2023)).

In practice, however, it is computationally infeasible to consider all $|\pi_n|$ permutations. Therefore, it is now a standard practice to consider a Monte Carlo-based (MC-based) permutation test which uses much smaller number of permutations. To explain, for a given $B > 0$, let π_1, \dots, π_B be B independent random permutations sampled from uniform distribution over π_n . These permutations are utilized to calculate the following MC-based permutation p-value:

$$\hat{p}_B := \frac{1}{B+1} \left[1 + \sum_{b=1}^B \mathbb{1}\{T_n^{\pi_b} \geq T_n(\mathbb{X}_n)\} \right], \quad (4)$$

which controls the type I error non-asymptotically under exchangeability for any B and any test statistic (for details, see e.g., [Hemerik and Goeman \(2018\)](#); [Ramdas et al. \(2023\)](#)). Regarding type II error, the MC-based permutation test using the U-statistic in (9) achieves power comparable to the full permutation test if B is sufficiently large—independent of sample size and data dimension, and much smaller than $|\pi_n|$ (see Proposition I.1 of [Kim et al. \(2022a\)](#) for details).

3 Two-sample testing for multinomials under LDP constraint

Having introduced the background, we now proceed to present the main results of this paper. In Section 3.1, we consider the problem of comparing two multinomial distributions, and establish the corresponding minimax rate under the LDP constraint. The upper bound for the minimax rate is attained by the LDP permutation tests that we propose in Section 3.2. These tests serve as crucial components in establishing the minimax rate for continuous data, as discussed in Section 4.

3.1 Private minimax rates for two-sample multinomial testing

The problem of interest is formulated as follows. Let $\mathcal{P}_{\text{multi}}^{(k)}$ be the family of pairs of multinomial probability vectors with k categories and $(\mathbf{p}_Y, \mathbf{p}_Z) \in \mathcal{P}_{\text{multi}}^{(k)}$ be the probability vectors of the unknown data-generating multinomial distributions. The curator obtains two sets of α -LDP views $\{\tilde{Y}_i\}_{i \in [n_1]}$ and $\{\tilde{Z}_j\}_{j \in [n_2]}$ as described in Section 2.1 and decides whether $(\mathbf{p}_Y, \mathbf{p}_Z)$ came from $\mathcal{P}_{0,\text{multi}} := \{(\mathbf{p}_Y, \mathbf{p}_Z) \in \mathcal{P}_{\text{multi}}^{(k)} : \mathbf{p}_Y = \mathbf{p}_Z\}$ or

$$\mathcal{P}_{1,\text{multi}}(\rho_{n_1,n_2}) := \{(\mathbf{p}_Y, \mathbf{p}_Z) \in \mathcal{P}_{\text{multi}}^{(k)} : \|\mathbf{p}_Y - \mathbf{p}_Z\|_2 \geq \rho_{n_1,n_2}\}. \quad (5)$$

Let us fix the type I error γ and the type II error β such that $2\gamma + \beta < 1$. This constraint arises from Ingster’s minimax lower bounding method, as similarly considered in [Lam-Weil et al. \(2022\)](#). The main result of this section, stated below, establishes a lower bound as well as an upper bound for the minimax separation rate for this multinomial problem under LDP in terms of the ℓ_2 distance.

Theorem 3 (Minimax rate for two-sample multinomial testing under LDP) *There exist positive constants $C_\ell(\gamma, \beta)$ and $C_u(\gamma, \beta)$ such that the α -LDP minimax testing rate $\rho_{n_1,n_2,\alpha}^*$ over the class of alternatives $\mathcal{P}_{1,\text{multi}}$ in (5) is bounded as*

$$C_\ell(\gamma, \beta) \left[\left(\frac{k^{1/4}}{(n_1 z_\alpha^2)^{1/2}} \wedge (k \log k)^{-1/2} \right) \vee \frac{1}{n_1^{1/2}} \right] \leq \rho_{n_1,n_2,\alpha}^* \leq C_u(\gamma, \beta) \left[\frac{k^{1/4}}{(n_1 \alpha^2)^{1/2}} \vee \frac{1}{n_1^{1/2}} \right], \quad (6)$$

where we recall $z_\alpha = 2\sinh(2\alpha)$.

Theorem 3 states that the private separation rate for two-sample multinomials is notably different from its non-private counterpart $n_1^{-1/2}$ with respect to the ℓ_2 distance ([Chan et al., 2014b](#); [Kim et al., 2022a](#)). In particular, in the high privacy regime, we observe additional dependence on k and α . The result also indicates that the privacy guarantee can be obtained at no additional cost in the low privacy regime where $n_1^{-1/2}$ dominates the other term.

We point out that while the lower bound and the upper bound do not exactly match, the gap is notably small. For instance, in the regime where the $(k \log k)^{-1/2}$ term is negligible, the only different terms are α and z_α . We note that these two terms are the same, up to a constant factor, as long as α is bounded. Hence, in most practical scenarios where a small value of α is of interest, the upper bound matches the lower bound.

We now discuss the proof of Theorem 3, and a detailed analysis can be found in Appendix D.

- **Lower bound.** We can obtain the lower bound almost for free by observing that the two-sample problem is more difficult than the one-sample problem. In particular, one can always turn the one-sample problem into the two-sample problem by drawing additional samples from the target distribution. Therefore, a minimax lower bound for the one-sample problem does not exceed that of the two-sample problem (see Lemma 1 of [Arias-Castro et al. \(2018\)](#) for a formal argument). Given this insight, the lower bound follows by combining lower bound results of Theorem 3.2 in ([Lam-Weil et al., 2022](#)) for the one-sample problem under LDP, as well as ([Chan et al., 2014b; Kim et al., 2022a](#)) for the two-sample problem without privacy constraints. We point out that the lower bound in ([Chan et al., 2014b; Kim et al., 2022a](#)) depends on $\max\{\|\mathbf{p}_Y\|_2, \|\mathbf{p}_Z\|_2\}$. This quantity becomes a constant at the worst case scenario and it can be thereby disregarded in the lower bound for the global minimax rate.
- **Upper bound.** To prove the upper bound, we leverage the ℓ_2 permutation test in [Kim et al. \(2022a\)](#), which achieves the non-private minimax rate. The considered test statistic is essentially a U-statistic of $\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2$ based on private views of indicator functions generated by privacy mechanisms. Given this privatized U-statistic detailed in Section 3.2, we derive the upper bound by using the two moments method of [Kim et al. \(2022a\)](#), which provides a sufficient condition for significant power of a permutation test based on the first two moments of the test statistic. The following subsection outlines the privacy mechanisms involved, introduces a specific form of the test statistic, and presents the testing procedure that achieves the upper bound.

3.2 Privacy mechanisms and testing procedure for the upper bound

This subsection is dedicated to explaining the private permutation testing procedure which achieves the upper bound stated in (6). As mentioned earlier, this procedure builds on the U-statistic ([Kim et al., 2022a](#)) estimating the squared ℓ_2 distance $\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2$. In a similar way, we deal with the U-statistic of $k\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2$, but a notable difference is that ours is based on α -LDP views generated by privacy mechanisms. Before formally introducing the test statistic, let us explain the considered LDP mechanisms: Laplace, discrete Laplace, and RAPPOR mechanisms.

- **Privacy mechanisms.** The first one is the standard Laplace mechanism, also considered in [Lam-Weil et al. \(2022\)](#) and [Berrett and Butucea \(2020\)](#), which adds an independent Laplace noise to the original data represented as indicator variables.

Definition 4 (Laplace mechanism for multinomial data: LapU) *Consider a pooled raw multinomial sample $\{X_i\}_{i \in [n]}$ with k categories. Fix the privacy level $\alpha > 0$. Each data owner adds noise to their data point, with the noise variance parameterized by*

$$\sigma_\alpha := \frac{2\sqrt{2k}}{\alpha}.$$

The resulting locally privatized sample $\{\tilde{\mathbf{X}}_i\}_{i \in [n]} = \{\tilde{\mathbf{Y}}_i\}_{i \in [n_1]} \cup \{\tilde{\mathbf{Z}}_i\}_{i \in [n_2]}$ is a set of k -dimensional random vectors whose m th element \tilde{X}_{im} is defined as follows:

$$\tilde{X}_{im} := \sqrt{k} \mathbf{1}(X_i = m) + \sigma_\alpha W_{im}. \quad (7)$$

Here, $\{W_{im}\}_{i \in [n], m \in [k]} \stackrel{i.i.d.}{\sim} \text{Lap}(1/\sqrt{2})$ is independent of $\{X_i\}_{i \in [n]}$, and $\text{Lap}(1/\sqrt{2})$ denotes the centered Laplace distribution with variance one.

The second mechanism is based on discrete Laplace noise. We say that a random variable W follows a discrete Laplace distribution with parameter $\zeta \in (0, 1)$, denoted as $W \sim \text{DL}(\zeta)$, if its probability mass function satisfies

$$\mathbb{P}(W = w) = \frac{1 - \zeta}{1 + \zeta} \zeta^{|w|}, \quad \text{for all } w \in \mathbb{Z}. \quad (8)$$

The second mechanism is similar to the first one but it replaces continuous Laplace noise with discrete Laplace noise.

Definition 5 (Discrete Laplace mechanism for multinomial data: DiscLapU) *Consider a pooled raw multinomial sample $\{X_i\}_{i \in [n]}$ with k categories. Fix the privacy level $\alpha > 0$. Each data owner adds noise to their data point, with the noise distribution parametrized by*

$$\zeta_\alpha := e^{-\frac{\alpha}{2\sqrt{k}}},$$

where $\zeta_\alpha \in (0, 1)$ for any value of $\alpha > 0$ and $k \geq 2$. The resulting locally privatized sample $\{\tilde{\mathbf{X}}_i\}_{i \in [n]} = \{\tilde{\mathbf{Y}}_i\}_{i \in [n_1]} \cup \{\tilde{\mathbf{Z}}_i\}_{i \in [n_2]}$ is a set of k -dimensional random vectors whose m th element \tilde{X}_{im} is defined as follows:

$$\tilde{X}_{im} := \sqrt{k} \mathbf{1}(X_i = m) + \sigma_\alpha W_{im}.$$

Here, $\{W_{im}\}_{i \in [n], m \in [k]} \stackrel{i.i.d.}{\sim} \text{DL}(\zeta_\alpha)$.

The third mechanism that we consider privatizes multinomial data vectors by randomly flipping individual components, instead of injecting additive random noise. This mechanism, proposed by [Duchi et al. \(2013\)](#), is equivalent to Google's basic one-time RAPPOR (randomized aggregatable privacy-preserving ordinal response) ([Erlingsson et al., 2014](#)).

Definition 6 (Basic one-time RAPPOR mechanism for multinomial data: RAPPOR) *Consider a pooled raw multinomial sample $\{X_i\}_{i \in [n]}$ with k categories. Fix the privacy level $\alpha > 0$. Each data owner perturbs their data point, resulting in the privatized sample $\{\tilde{\mathbf{X}}_i\}_{i \in [n]} = \{\tilde{\mathbf{Y}}_i\}_{i \in [n_1]} \cup \{\tilde{\mathbf{Z}}_i\}_{i \in [n_2]}$; a set of k -dimensional random vectors whose m th element \tilde{X}_{im} is defined as follows:*

$$\tilde{X}_{im} := \begin{cases} \mathbf{1}(X_i = m) & \text{with probability } \frac{e^{\alpha/2}}{e^{\alpha/2} + 1}, \\ \mathbf{1}(X_i \neq m) & \text{with probability } \frac{1}{e^{\alpha/2} + 1}. \end{cases}$$

The next lemma proves the α -LDP guarantee for Laplace, discrete Laplace, and RAPPOR mechanisms. The proof for Laplace and discrete Laplace mechanisms can be found in Appendix C, whereas the proof for RAPPOR mechanism is provided in Section 3.2 of [Duchi et al. \(2013\)](#).

Lemma 7 (LDP guarantee) *The random vectors $\{\tilde{\mathbf{X}}_i\}_{i \in [n]}$ generated by any of LapU mechanism (Definition 4), DiscLapU mechanism (Definition 5) and RAPPOR (Definition 6) are α -LDP views of $\{X_i\}_{i \in [n]}$.*

As defined below, our test statistic for multinomial testing builds on the α -LDP views $\{\tilde{\mathbf{X}}_i\}_{i \in [n]}$ from one of the $\{\text{LapU}, \text{DiscLapU}, \text{RAPPOR}\}$ mechanisms. All of these mechanisms rigorously maintain local differential privacy and return a test that achieves the same separation rate as in Theorem 3.

- **Testing procedure.** Given α -LDP views from one of the $\{\text{LapU}, \text{DiscLapU}, \text{RAPPOR}\}$ mechanisms, we use the following U-statistic:

$$U_{n_1, n_2} := \frac{1}{n_1(n_1 - 1)} \sum_{1 \leq i_1 \neq i_2 \leq n_1} \tilde{\mathbf{Y}}_{i_1}^\top \tilde{\mathbf{Y}}_{i_2} + \frac{1}{n_2(n_2 - 1)} \sum_{1 \leq j_1 \neq j_2 \leq n_2} \tilde{\mathbf{Z}}_{j_1}^\top \tilde{\mathbf{Z}}_{j_2} - \frac{2}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \tilde{\mathbf{Y}}_i^\top \tilde{\mathbf{Z}}_j. \quad (9)$$

To carry out a test, the test statistic is calibrated by the permutation procedure described in Section 2.3 with the pooled α -LDP views $\{\tilde{\mathbf{X}}_i\}_{i \in [n]}$. Specifically, we reject the null when the p -value based on the test statistic (9) is smaller than or equal to significance level γ . The type I error of the resulting test is controlled at γ as $\{\tilde{\mathbf{X}}_i\}_{i \in [n]}$ are i.i.d. random vectors under the null hypothesis. Our technical effort lies in studying the type II error guarantee of the proposed test, and in turn proving the upper bound in Theorem 3. We refer to Appendix D for details.

The statistic (9) based on either Laplace or discrete Laplace mechanism is an unbiased estimator of a scaled and squared ℓ_2 distance between probability vectors:

$$\mathbb{E}[U_{n_1, n_2}] = k \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \quad (10)$$

On the other hand, the test statistic based on the **RAPPOR** mechanism does not maintain the unbiasedness property, with its expectation shrinking to zero as α decreases (see Lemma B.2). It therefore requires a more careful and, indeed, more challenging analysis compared to the other two mechanisms.

Despite the fact that all three mechanisms ensure the minimax separation rate, their finite-sample power performance may differ in various scenarios as illustrated in Section 5. In particular, our numerical studies demonstrate that tests based on Laplace or discrete Laplace mechanisms tend to underperform compared to those based on the **RAPPOR** mechanism. This underperformance is partly because the private views from Laplace and discrete Laplace mechanism can take extreme values due to their unbounded support, whereas those from **RAPPOR** mechanism are always bounded. Accordingly, we advocate for using the **RAPPOR** mechanism over the Laplace and discrete Laplace mechanisms, even though they present the same theoretical guarantee in terms of the separation rate.

Another LDP mechanism for multinomial testing suggested by Gaboardi et al. (2016) is the generalized randomized response mechanism (**GenRR**). In contrast to **RAPPOR** mechanism, which destroys the structure of one-hot vectors, the generalized randomized response mechanism maintains the one-hot vector format while randomly altering the position of the non-zero component. It turns out, however, that the test based on the generalized randomized response mechanism is notoriously suboptimal in terms of the separation rate, as we show in Appendix H. Demonstrating this negative result requires a careful asymptotic analysis of the U-statistic, which we want to highlight as our technical contribution.

4 Two-sample testing for Hölder and Besov densities under LDP

In this section, we switch gears to testing for equality between two multivariate densities under the LDP constraint. To this end, we consider two classes of smooth densities, namely the Hölder ball

and Besov ball, and establish the minimax rate in terms of the \mathbb{L}_2 distance for each class. Especially, we derive the upper bound for the minimax rate by building on the multinomial permutation test introduced in Section 3.2 with a careful discretization scheme. We also introduce an aggregated test, which is adaptive to the unknown smoothness parameter.

4.1 Hölder and Besov smoothness classes

We start by formally defining the Hölder ball and Besov ball. The Hölder ball extends the notion of Lipschitz continuity and can be thought of as functions with bounded fractional derivatives. The following definition of the Hölder ball rephrases the one stated in Section 2.1 of [Arias-Castro et al. \(2018\)](#).

Definition 8 (Hölder ball) *The Hölder ball with smoothness parameter $s > 0$ and radius $R > 0$, denoted as $\mathcal{B}_{d,s}^H(R)$, is the class of functions $f : [0, 1]^d \mapsto \mathbb{R}$ satisfying the following conditions:*

1. $|f^{(\lfloor s \rfloor)}(\mathbf{x}) - f^{(\lfloor s \rfloor)}(\mathbf{x}')| \leq R \|\mathbf{x} - \mathbf{x}'\|^{s - \lfloor s \rfloor}$, for all $\mathbf{x}, \mathbf{x}' \in [0, 1]^d$.
2. $\|f^{(s')}\|_\infty \leq R$ for each $s' \in [\lfloor s \rfloor]$,

where $f^{(\lfloor s \rfloor)}$ denotes the $\lfloor s \rfloor$ -order derivative of f .

The Besov ball, on the other hand, measures the smoothness of a function by capturing its abrupt oscillations through wavelets. In this respect, it can address spatially inhomogeneous functions whose smoothness can vary substantially across their domain. To elaborate, we consider an orthonormal wavelet basis of $\mathbb{L}_2([0, 1]^d)$ at a fixed prime resolution level $J \in \mathbb{N}_0$. We denote this basis as $\bar{\Phi}_J \cup (\cup_{j \geq J} \bar{\Psi}_j)$, which are classified into two distinct categories. For $\phi \in \bar{\Phi}_J$, the scaling coefficient $\theta_\phi(f) := \int_{[0,1]^d} f(\mathbf{x})\phi(\mathbf{x}) d\mathbf{x}$ detects an overall trend of f . On the other hand, for $\psi \in \bar{\Psi}_j$ with its resolution level $j \geq J$, the wavelet coefficient $\theta_\psi(f) := \int_{[0,1]^d} f(\mathbf{x})\psi(\mathbf{x}) d\mathbf{x}$ captures abrupt oscillations from the general trend. As J and j increase, the coefficients capture more detailed behaviors. Among many existing types of wavelet basis, this paper focuses on Haar multivariate wavelet basis. It is useful because projection of densities into a subset of this basis is equivalent to discretization of densities used in our proposed test, and thus the discretization error can be captured by the wavelet coefficients. This basis is constructed by taking tensor products of many rescaled and shifted versions of two basic univariate functions (see Appendix B.3 for the details).

The Besov ball can be defined through the magnitudes of wavelet coefficients. The following definition paraphrases the one presented in Section 3 of [Tang and Yang \(2023\)](#).

Definition 9 (Besov seminorm and Besov ball) *Fix a smoothness parameter $s > 0$, a microscopic parameter $1 \leq q \leq \infty$ and a wavelet basis $\bar{\Phi}_0 \cup (\cup_{j \geq 0} \bar{\Psi}_j)$. The Besov seminorm of $f \in \mathbb{L}_2([0, 1]^d)$ is defined using the sequences of its wavelet coefficients $(\theta_\psi(f))_{\psi \in \bar{\Psi}_j}$ as follows:*

$$\|f\|_{s,2,q} := \begin{cases} \left[\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{\psi \in \bar{\Psi}_j} |\theta_\psi(f)|^2 \right)^{\frac{q}{2}} \right]^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \sup_{j \in \mathbb{N}_0} 2^{js} \left(\sum_{\psi \in \bar{\Psi}_j} |\theta_\psi(f)|^2 \right)^{\frac{1}{2}}, & q = \infty. \end{cases}$$

For a radius $R > 0$, we define the Besov ball $\mathcal{B}_{d,s,q}^B(R)$ as

$$\mathcal{B}_{d,s,q}^B(R) := \{f \in \mathbb{L}_2([0, 1]^d) : \|f\|_{s,2,q} \leq R\}.$$

Neither Definition 8 nor Definition 9 specifically pertains to density functions. We build our models by defining the class of distributions whose density functions lie within these smooth function classes.

Definition 10 (Smooth distribution pair classes) Let $\mathcal{P}_{d,s}^{\text{H},2}(R)$ denote the set of pairs of distributions $(P_{\mathbf{Y}}, P_{\mathbf{Z}})$ that satisfy the following conditions:

1. Both $P_{\mathbf{Y}}$ and $P_{\mathbf{Z}}$ have density functions $f_{\mathbf{Y}}$ and $f_{\mathbf{Z}}$, respectively, with the \mathbb{L}_2 norms bounded by R .
2. The difference of these densities $(f_{\mathbf{Y}} - f_{\mathbf{Z}})$ lies in $\mathcal{B}_{d,s}^{\text{H}}(R)$.

Similarly, let $\mathcal{P}_{d,s,q}^{\text{B},2}(R)$ denote the set of pairs of distributions that satisfy the two conditions above, with $\mathcal{B}_{d,s}^{\text{H}}(R)$ replaced by $\mathcal{B}_{d,s,q}^{\text{B}}(R)$.

The superscript 2 indicates that the set contains pairs of distributions, distinguishing it from the set of single distributions used in Appendix E.2. With regard to $\mathcal{B}_{d,s,q}^{\text{B}}(R)$, extending the analysis of (Lam-Weil et al., 2022) into a multivariate setting, we focus on the Besov ball with multivariate Haar wavelet basis and smoothness parameter $s < 1$. The details of the basis functions are provided in Appendix B.3. We note that there is no restriction on the microscopic parameter q .

4.2 Private minimax testing rates for two-sample density testing

Building on the smooth distribution classes defined in Definition 10, we formally state the two problems of interest and present the minimax testing rate that applies to both problems. In this subsection, we assume that the smoothness parameter s for Hölder ball and Besov ball is known. We deal with the case of unknown s in Section 4.4. The density testing problems are defined as follows. Assume that the data-generating distributions $(P_{\mathbf{Y}}, P_{\mathbf{Z}})$ is contained in a class \mathcal{P} . The curator utilizes two sets of α -LDP views $\{\tilde{\mathbf{Y}}_i\}_{i \in [n_1]}$ and $\{\tilde{\mathbf{Z}}_j\}_{j \in [n_2]}$, privatized as described in Section 2.1, to decide whether $(P_{\mathbf{Y}}, P_{\mathbf{Z}})$ came from

$$\mathcal{P}_0 := \{(P_{\mathbf{Y}}, P_{\mathbf{Z}}) \in \mathcal{P} : f_{\mathbf{Y}} = f_{\mathbf{Z}}\} \quad \text{or} \quad \mathcal{P}_1(\rho_{n_1, n_2}) := \{(P_{\mathbf{Y}}, P_{\mathbf{Z}}) \in \mathcal{P} : \|f_{\mathbf{Y}} - f_{\mathbf{Z}}\|_{\mathbb{L}_2} \geq \rho_{n_1, n_2}\}, \quad (11)$$

where $f_{\mathbf{Y}}$ and $f_{\mathbf{Z}}$ are densities of $P_{\mathbf{Y}}$ and $P_{\mathbf{Z}}$, respectively. We consider the testing problems of $\mathcal{P} = \mathcal{P}_{d,s}^{\text{H},2}(R)$ and $\mathcal{P} = \mathcal{P}_{d,s,q}^{\text{B},2}(R)$. We would like to emphasize that, although we focus on the class of either \mathcal{P} being Hölder or Besov smooth null distributions, the permutation-based test presented in Section 4.3 guarantees type I error control over a broader class of null distributions, extending beyond those defined over Hölder or Besov balls.

Let us fix the type I error $\gamma \in (0, 1)$ and the type II error $\beta \in (0, 1)$ such that $2\gamma + \beta < 1$ as in Section 3.1, and assume further that $n_1 \alpha^2 \geq 1$, similarly considered in Lam-Weil et al. (2022). The main result of this section, stated below, establishes a lower bound as well as an upper bound for the minimax separation rate for two-sample density testing under the LDP constraint.

Theorem 11 (Minimax rates for two-sample density testing under LDP) Assume $n_1 \alpha^2 \geq 1$. 1. For the testing problem of distinguishing between the null and alternatives as in (11), where the distribution class \mathcal{P} is $\mathcal{P}_{d,s}^{\text{H},2}(R)$, there exist positive constants $C_\ell(\gamma, \beta, R, s) \equiv C_\ell$ and $C_u(\gamma, \beta, R, s, d) \equiv C_u$ such that the α -LDP minimax testing rate $\rho_{n_1, n_2, \alpha}^*$ is bounded as

$$C_\ell \left[\frac{(n_1 z_\alpha^2)^{\frac{-2s}{4s+3d}}}{\sqrt{\log(n_1 z_\alpha^2)}} \vee n_1^{\frac{-2s}{4s+d}} \right] \leq \rho_{n_1, n_2, \alpha}^* \leq C_u \left[(n_1 \alpha^2)^{\frac{-2s}{4s+3d}} \vee n_1^{\frac{-2s}{4s+d}} \right], \quad (12)$$

where we recall $z_\alpha = 2\sinh(2\alpha)$. Similarly, for the testing problem of distinguishing between the null and alternatives as in (11), where the distribution class \mathcal{P} is $\mathcal{P}_{d,s,q}^{\text{B},2}(R)$, the minimax testing is also bounded as (12).

Theorem 11 indicates that the private minimax separation rate for two-sample multivariate Hölder and Besov densities is noticeably different from its non-private counterpart $n_1^{-2s/(4s+d)}$ with respect to the \mathbb{L}_2 distance (Arias-Castro et al., 2018; Kim et al., 2022a). We point out that in the high privacy regime, a polynomial degradation on the minimax rate is observed, and this degradation becomes worse as the data dimension d increases. The result also implies the privacy guarantee can be secured at no additional charge in the low privacy regime where $n_1^{-2s/(4s+d)}$ dominates the other term.

The bounds (12) are tight up to a logarithmic factor in the denominator of the lower bound, which can be omitted when $n_1 z_\alpha^2 \geq 1$. As already noted in Theorem 3, α and z_α are the same, up to a constant factor, as long as α is bounded. Hence, in most practical scenarios where a small value of α is of interest, the upper bound matches the lower bound.

We now discuss the proof of Theorem 11, and a detailed analysis can be found in Appendix E.

- **Lower bound.** To obtain the lower bound, we once again use the observation that two-sample testing is harder than goodness-of-fit testing. Based on this observation, we employ the lower bound result for goodness-of-fit testing, mirroring the approach employed in Theorem 3 for multinomial testing. We then extend the strategy presented in Lam-Weil et al. (2022) from the univariate case to the multivariate case. The same proof strategy also applies to the multivariate Hölder ball with only minor modifications, and details can be found in Appendix E.2.
- **Upper bound.** To prove the upper bound, we leverage the private multinomial permutation test in Section 3.2, which achieves the private minimax rate. For that purpose, we divide each side of the support $[0, 1]^d$ into κ equally-sized subintervals, effectively transforming the initially continuous observations into multinomial observations. The detailed procedure is outlined in Section 4.3. We then apply the same privacy mechanism and permutation testing procedure as outlined in Section 3.2. We exploit the Hölder and Besov smoothness conditions to find the optimal number of bins κ that effectively controls the discretization error and thus leads to a tight upper bound.

4.3 Minimax optimal privacy mechanism and permutation testing procedure

Our proposed density test applies our proposed private multinomial test to data discretized by binning the density support into a certain sample-size dependent number of bins, defined as follows:

$$\kappa := \begin{cases} \lfloor (n_1^{2/(4s+d)}) \wedge (n_1 \alpha^2)^{2/(4s+3d)} \rfloor & \text{if } (f_{\mathbf{Y}}, f_{\mathbf{Z}}) \in \mathcal{P}_{d,s}^{\text{H},2}(R), \\ \sup\{2^J : J \in \mathbb{N}_0 \text{ and } 2^J \leq (n_1^{2/(4s+d)}) \wedge (n_1 \alpha^2)^{2/(4s+3d)}\} & \text{if } (f_{\mathbf{Y}}, f_{\mathbf{Z}}) \in \mathcal{P}_{d,s,q}^{\text{B},2}(R). \end{cases} \quad (13)$$

Let $\{B_1, \dots, B_{\kappa^d}\}$ be an enumeration of d -dimensional hypercubes whose length is set to $1/\kappa$. Each data owner bins their raw sample using the equal-sized binning function $D_\kappa : [0, 1]^d \mapsto [\kappa^d]$, such that $D_\kappa(\mathbf{x}) = m$ if and only if $\mathbf{x} \in B_m$. Then, based on the discretized data $\{D_\kappa(\mathbf{Y}_1), \dots, D_\kappa(\mathbf{Y}_{n_1})\}$ and $\{D_\kappa(\mathbf{Z}_1), \dots, D_\kappa(\mathbf{Z}_{n_2})\}$, we carry out our proposed private multinomial test in Section 3.2; the permutation test with test statistic (9). The resulting test achieves the tight upper bound in (12).

We now outline how this test maintains the privacy and testing error guarantees. Since the discretized data remain i.i.d. under the null, the permutation procedure maintains the type I error at γ . As for the privacy guarantee, the reduced distinguishability between samples due to discretization, combined with any of $\{\text{LapU}, \text{DiscLapU}, \text{RAPPOR}\}$, guarantees α -LDP. Regarding the type II error, first note that the test compares two multinomial distributions with κ^d categories. The corresponding probability vectors, $\mathbf{p}_{\mathbf{Y}}$ and $\mathbf{p}_{\mathbf{Z}}$, are defined as $\mathbf{p}_{\mathbf{Y}}(m) := \int_{B_m} f_{\mathbf{Y}}(\mathbf{t}) \, d\mathbf{t}$ and

$\mathbf{p}\mathbf{z}(m) := \int_{B_m} f_{\mathbf{z}}(\mathbf{t}) \, d\mathbf{t}$ for $m \in [\kappa^d]$. The boldface subscripts in $\mathbf{p}\mathbf{Y}$ and $\mathbf{p}\mathbf{z}$ indicate that these vectors correspond to multivariate data, distinguishing them from those in Section 3. Thus the test statistic (9) estimates $\kappa^d \|\mathbf{p}\mathbf{Y} - \mathbf{p}\mathbf{z}\|_2^2$ instead of $\|f_{\mathbf{Y}} - f_{\mathbf{z}}\|_{\mathbb{L}_2}^2$, introducing an approximation error that depends on the number of bins κ . This error is controlled using smoothness conditions on densities and techniques from (Arias-Castro et al., 2018; Lam-Weil et al., 2022) (see Appendix B.4 for details). We show that when the number of bins κ is chosen as in (13) and the \mathbb{L}_2 distance between $f_{\mathbf{Y}}$ and $f_{\mathbf{z}}$ exceeds the threshold in (12), the multinomial test controls the type II error to be at most β . A detailed proof is provided in Appendix E.1.

We note that our choice of κ in (13) depends on the smoothness parameter s , typically unknown. In the next subsection, we introduce an adaptive procedure that accommodates this unknown s without significant loss of power.

4.4 Adaptive procedure for private two-sample density testing

To achieve the tight upper bound in Theorem 11 via our proposed test in Section 4.3, the number of bins must be determined based on the unknown smoothness parameter s . To circumvent this requirement, this section introduces a multiscale permutation testing procedure that adapts to the unknown s . Following Ingster (2000), it aggregates test results from different bin numbers using a Bonferroni-corrected significance level. This procedure does not significantly sacrifice power compared to the one relying on the true value of s .

Denote the number of the test for the Bonferroni-type procedure as:

$$\mathcal{N} := \left\lceil \left\{ \frac{2}{d} \log_2 \left(\frac{n_1}{\log \log n_1} \right) \right\} \wedge \left\{ \frac{2}{3d} \log_2 \left(\frac{n_1 \alpha^2}{(\log n_1)^2 \log \log n_1} \right) \right\} \right\rceil. \quad (14)$$

For each $t \in [\mathcal{N}]$, let $\Delta_{\gamma/\mathcal{N}}^t$ denote the test function of our proposed method in Section 4.3, using 2^t bins, significance level γ/\mathcal{N} , and (α/\mathcal{N}) -LDP guarantee. The α -LDP adaptive test is formally defined as follows:

$$\Delta_{\gamma}^{\text{adapt}} := \max_{t \in [\mathcal{N}]} \Delta_{\gamma/\mathcal{N}}^t. \quad (15)$$

This adaptive procedure queries \mathcal{N} number of (α/\mathcal{N}) -LDP views per observation, each using a different number of bins for discretization. By the composition theorem of differential privacy (McSherry and Talwar, 2007), releasing \mathcal{N} number of (α/\mathcal{N}) -LDP views satisfies the LDP constraint with a privacy level of $\mathcal{N} \times \alpha/\mathcal{N} = \alpha$. The type I error is at most γ by the union bound, and for the type II error, Theorem 12 states that the adaptive procedure achieves the same testing rate up to logarithmic factors.

Theorem 12 (Minimax upper bound for the adaptive private testing procedure) *Consider the problem and conditions stated in Theorem 11. Further assume that $n_1 \asymp n_2$, $\gamma \leq e^{-1}$, $n_1 > e^e$ and $\alpha \leq n_1 \leq n_2$. Then there exists a positive constant $C_u \equiv C(s, d, R, \gamma, \beta)$ such that the condition*

$$\rho_{n_1, n_2} \geq C_u \left[\left(\frac{n}{\log \log n_1} \right)^{\frac{-2s}{4s+d}} \vee \left(\frac{n_1 \alpha^2}{(\log^2 n_1) \log \log n_1} \right)^{\frac{-2s}{4s+3d}} \right]$$

implies that the testing errors of the adaptive test $\Delta_{\gamma}^{\text{adapt}}$ in (15) are uniformly bounded as in (1).

Comparing Theorems 12 and 11, in the high privacy regime, we find that the adaptive procedure incurs an additional cost of $((\log^2 n_1) \log \log n_1)^{2s/(4s+3d)}$. In the low privacy regime, the additional

cost $(\log \log n_1)^{2s/(4s+d)}$ matches the adaptivity cost for non-private testing rates found in [Fromont and Laurent \(2006\)](#) and [Kim et al. \(2022a\)](#). Whether these additional logarithmic factors are necessary or can be improved upon remains an open question for future work. We briefly discuss the proof of Theorem 12; a detailed analysis is provided in Appendix F.

- **Type I error control.** The overall type I error is at most γ by the union bound, though this approach usually is conservative in practice. Introducing an additional layer of calibration, as in [Schrab et al. \(2023\)](#), could mitigate this issue but would increase the noise level of privacy mechanisms. Developing an adaptive test with precise type I error control and robust power guarantee is an interesting avenue for future research.
- **Type II error control.** Since the significance level now depends on d and n_1 , we use a refined version of the two moments method presented in [Kim et al. \(2022a\)](#). It improves the dependence on the significance level γ from $\sqrt{1/\gamma}$ of Theorem 21 to $\log(1/\gamma)$, at the cost of an additional requirement of $n_1 \asymp n_2$ on the sample sizes. At the heart of this refinement is an exponential concentration inequality for permuted U-statistics. This technique allows us to improve the adaptivity result of [Lam-Weil et al. \(2022\)](#) in their Theorem 5.2, replacing logarithmic factors with iterated logarithmic factors.

5 Numerical results

In this section, we conduct a series of simulation studies to illustrate the finite sample performance of our proposed tests. Specifically, we investigate the privacy-utility trade-offs by varying the privacy level parameter α . These simulation studies aim to confirm our theoretical results regarding these trade-offs, and also to demonstrate the rate at which the power diminishes as the privacy parameter α decreases in practical scenarios.

It is worth pointing out that there is no baseline method available in the literature for the problem we tackle. Therefore, we create the baseline methods by extending the LDP goodness-of-fit tests ([Gaboardi and Rogers, 2018](#)) into a two-sample setting. The first method combines the generalized randomized response (**GenRR**; ([Gaboardi and Rogers, 2018](#))) privacy mechanism and the classical chi-square statistic (**Chi**). The second method combines the generalized randomized response mechanism with the ℓ_2 -type U-statistic in (9). The third method combines **RAPPOR** privacy mechanism and projected chi-square statistic (**ProjChi**; ([Gaboardi and Rogers, 2018](#))). A formal description and asymptotic properties of these extensions can be found in Appendix G.1. From now on, we refer to the LDP two-sample testing methods as “privacy mechanism+test statistic”, for example, **RAPPOR+ProjChi** or **GenRR+Chi** for ease of reference.

Recall that the proposed method for density testing, defined as a multinomial test applied to equal-sized binned data, requires the original data to lie within the unit hypercube $[0, 1]^d$. In order to apply the multinomial test to continuous data with larger and potentially unbounded domains, we transform the data through a map $T : \mathbb{R}^d \mapsto [0, 1]^d$, which is applied on a component-wise basis. A specific transformation that we focus on in this simulation is given as:

$$T(\mathbf{x}) = (\Phi(x_1), \Phi(x_2), \dots, \Phi(x_d)), \quad (16)$$

where $\Phi(x) := (2\pi)^{-1/2} \int_{-\infty}^x e^{-t^2/2} dt$ is the standard normal cumulative distribution function. We then apply our procedure in Section 4.3 to these transformed observations through the map in (16). Since the minimax optimal κ value in (13) depends on the unknown smoothness parameter s , the value of κ must be determined to balance two effects: a large κ reduces discretization error but potentially weakens the signal and becomes vulnerable to the impact of added noise due to the

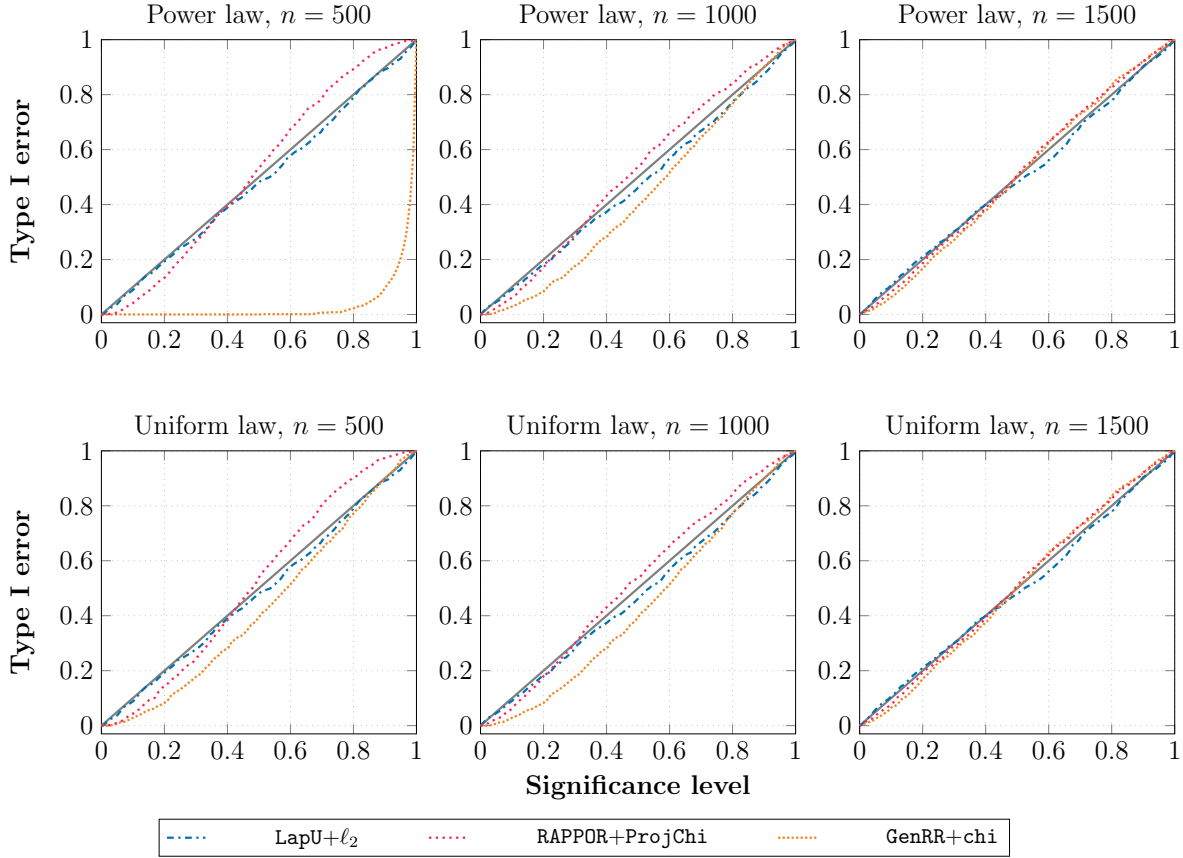


Figure 1: Comparison of type I error control between a permutation-calibrated method ($\text{LapU} + \ell_2$) and methods calibrated through asymptotic chi-square null distribution ($\text{RAPPOR} + \text{ProjChi}$ and $\text{GenRR} + \text{Chi}$), across configurations of uniform (top row) and power-law (bottom row) null distributions as described in (17). The gray solid lines represent the $y = x$ line, indicating perfect type I error control.

privacy mechanism. Our simulation results indicate that $\kappa = 4$ is a reasonable choice in most of the scenarios considered in this section, and thus we stick with the equal-sized binning scheme with $\kappa = 4$ for density testing.

In all simulation scenarios, we consider equal sample sizes, denoted as $n := n_1 = n_2$, and fix the significance level at $\gamma = 0.05$. We estimate the power by independently repeating the test 2000 times, and calculating the rejection ratio of the null hypothesis. The permutation procedure employs the Monte Carlo p -value given in (4) with $B = 999$. The code that supports the findings of this study is openly available in GitHub at <https://github.com/Jong-Min-Moon/optimal-local-dp-two-sample.git>.

- **Type I error control.** First, we compare the type I error rates of three methods for multinomial testing: $\text{LapU} + \ell_2$, $\text{GenRR} + \text{Chi}$, and $\text{RAPPOR} + \text{ProjChi}$, to highlight the advantages of the permutation approach. We consider two null distributions, where for $m \in [k]$, the m th elements of \mathbf{p}_Y and \mathbf{p}_Z are defined as:

$$(a) \text{ power law: } p_{Ym} = p_{Zm} \propto 1/m \quad \text{and} \quad (b) \text{ uniform law: } p_{Ym} = p_{Zm} = 1/k. \quad (17)$$

We set $k = 500$ and $\alpha = 0.1$, and investigate how the type I error rate varies with sample sizes $n \in \{500, 1000, 1500\}$. The results, displayed in Figure 1, indicate that the type I

error of the permutation test, $\text{LapU}+\ell_2$, is well controlled at any sample size and significance level (up to a small numerical error) as expected. In contrast, we see that the asymptotic tests, namely $\text{RAPPOR}+\text{ProjChi}$ and $\text{GenRR}+\text{Chi}$, have the size significantly deviated from the straight baseline, especially when the sample size is small or moderate. This indicates that the resulting asymptotic test can be either conservative or anti-conservative depending on the significance level.

- **Simulation settings for power comparison in multinomial testing.** We next compare the power of our proposed methods ($\text{RAPPOR}+\ell_2$, $\text{LapU}+\ell_2$, and $\text{DiscLapU}+\ell_2$) with baseline methods ($\text{RAPPOR}+\text{ProjChi}$, $\text{GenRR}+\ell_2$, and $\text{GenRR}+\text{Chi}$) for distinguishing between two multinomial distributions. As observed in Figure 1, both $\text{GenRR}+\text{Chi}$ and $\text{RAPPOR}+\text{ProjChi}$ can be miscalibrated when their thresholds are determined by the asymptotic null distributions. In order to ensure a fair power comparison, $\text{GenRR}+\text{Chi}$ and $\text{RAPPOR}+\text{ProjChi}$ were calibrated using permutation procedures in this power simulation. We aim to illustrate how the testing power varies with changes in the number of categories k and the privacy parameter α . The analysis is conducted under a perturbed uniform distribution scenario, where for $m \in [k]$, the m th elements of \mathbf{p}_Y and \mathbf{p}_Z are defined as:

$$p_{Ym} = \frac{1}{k} + (-1)^m \eta \quad \text{and} \quad p_{Zm} = \frac{1}{k} + (-1)^{(m+1)} \eta, \quad \text{for } m \in [k]. \quad (18)$$

The simulation considers the following combinations of three parameters, namely the number of categories k , the perturbation size η and the privacy parameter α :

$$(k, \eta, \alpha) \in \{(4, 0.04), (40, 0.015), (400, 0.002)\} \times \{2, 1, 0.5\}.$$

The simulation results for this setting are provided in Figure 2.

- **Simulation settings for power comparison in density testing.** We also evaluate the density testing power of the same methods used in the simulations for multinomial testing. We consider two scenarios where two density functions differ in their location parameters or scale parameters. Since the results for scale difference show trend similar to that of location difference, we present these results in Appendix G.3. Specifically, we analyze scenarios involving mean and covariance differences between two d -dimensional Gaussian distributions $P_Y = \mathcal{N}(\boldsymbol{\mu}_Y, \Sigma_Y)$ and $P_Z = \mathcal{N}(\boldsymbol{\mu}_Z, \Sigma_Z)$. Let $\mathbf{1}_d := (1, \dots, 1)^\top \in \mathbb{R}^d$, $\mathbf{0}_d := (0, \dots, 0)^\top \in \mathbb{R}^d$, $\mathbf{J}_d := \mathbf{1}_d \mathbf{1}_d^\top \in \mathbb{R}^{d \times d}$, and \mathbf{I}_d denote the identity matrix in $\mathbb{R}^{d \times d}$. We set the mean vectors and covariance matrices of the Gaussian distributions as:

$$\boldsymbol{\mu}_Y = 0.5 \times \mathbf{1}_d, \quad \boldsymbol{\mu}_Z = -\boldsymbol{\mu}_Y, \quad \text{and} \quad \Sigma_Y = \Sigma_Z = 0.5 \times \mathbf{J}_d + 0.5 \times \mathbf{I}_d. \quad (19)$$

In simulations, the dimension of the original data is chosen as $d \in \{3, 4, 5\}$, and after binning through the map T in (16) with $\kappa = 4$, the number of categories becomes $k \in \{64, 256, 1024\}$.

- **Simulation results for power comparisons.** The simulation results in this section are illustrated in Figures 2 and 3. We first consistently observe in all of the figures that the power tends to decrease as the privacy parameter α decreases (i.e., stronger privacy guarantee). These trade-offs are all predictable from the minimax rates in (6) and (12). We next highlight the differences in trends related to the number of categories. For multinomial distributions with a small number of categories ($k = 4$), the generalized randomized response, a natural extension of the classical mechanism, outperforms all other methods. Following the generalized

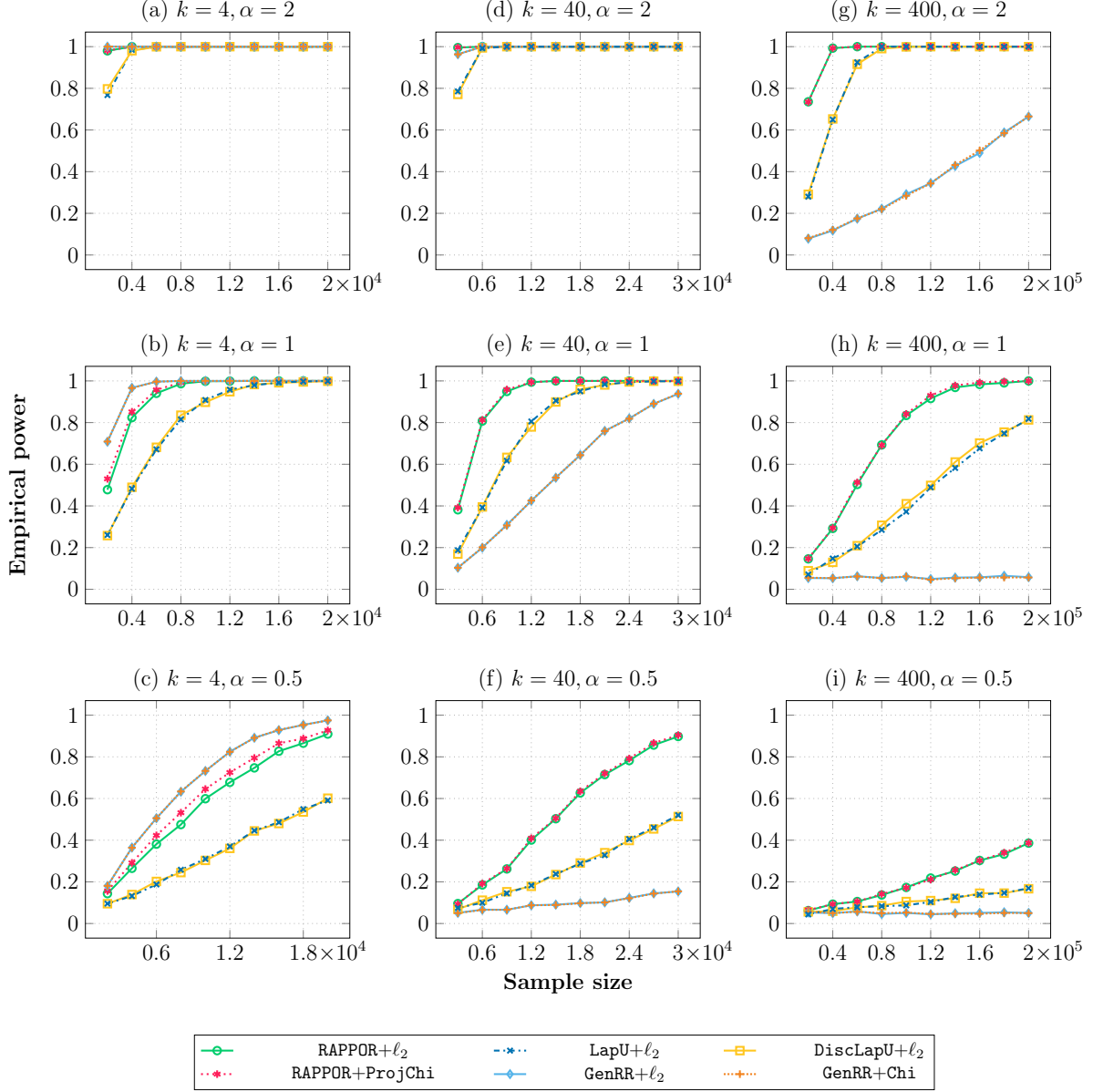


Figure 2: Comparison of the testing power between our proposed methods (first row in the legend) and baseline methods (second row in the legend) under the perturbed uniform alternatives (18). To ensure a fair comparison, all methods are calibrated using permutation procedures at level $\gamma = 0.05$.

randomized response mechanism is the RAPPOR mechanism, while Laplace-noise-based mechanisms performing the least. However, in scenarios with a larger number of categories, or in density testing scenarios (which also correspond to a large number of categories), the testing power of the generalized randomized response diminishes, and RAPPOR emerges as the method with the highest power. The suboptimal performance of the generalized randomized response in high-dimensional settings, as theoretically explored in Appendix H and numerically observed by Gaboardi and Rogers (2018), aligns with a simple intuition: since the generalized randomized response modifies data by shifting it from one category to another, the difference

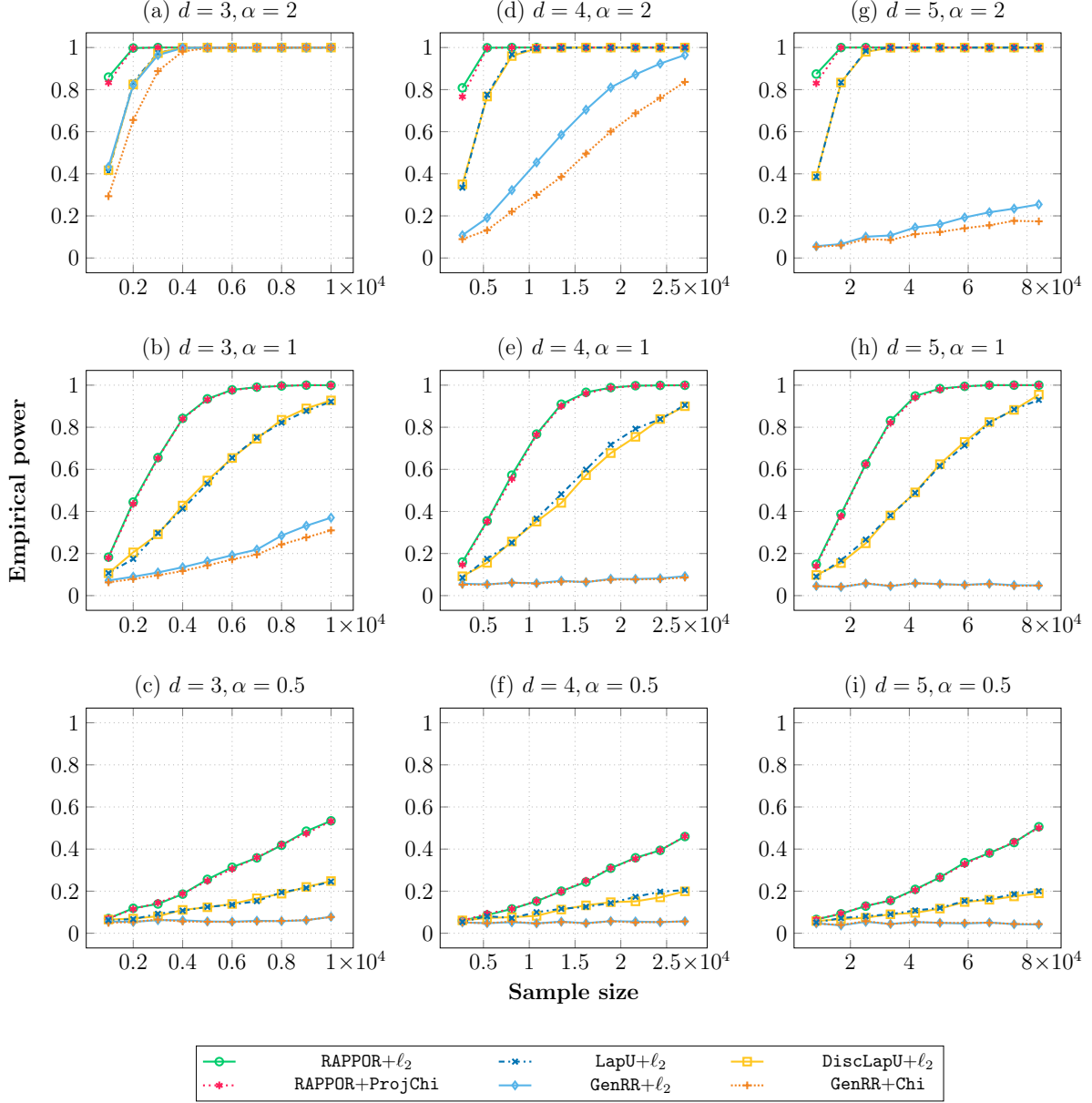


Figure 3: Comparison of the density testing power between our proposed methods (first row in the legend) and baseline methods (second row in the legend) under the location alternatives in (19). To ensure a fair comparison, all methods are calibrated using permutation procedures at level $\gamma = 0.05$.

between the original sample and its corresponding α -LDP view becomes more pronounced as the number of categories increases. These simulation results prove the superiority of RAPPOR over the other mechanisms, especially in multinomial testing with large k and density testing, and we therefore recommend using RAPPOR in practical applications. Within the tests built upon RAPPOR, we observe in Figures 2 and 3 that RAPPOR+ProjChi and RAPPOR+ ℓ_2 perform comparably to each other, and in some cases, RAPPOR+ProjChi is slightly more powerful. It is also possible that the difference between RAPPOR+ProjChi and RAPPOR+ ℓ_2 is more pronounced in some settings. For example, if the signal is large in terms of chi-square divergence

but relatively small in terms of the ℓ_2 distance, we would expect **RAPPOR+ProjChi** to perform better than **RAPPOR**+ ℓ_2 . Conversely, if the signal is large in the ℓ_2 distance, the opposite holds true; we present the numerical results that confirm this in Appendix G.2. This suggests that when practitioners have insights into the nature of the deviation between two distributions, selecting a statistic that aligns with that specific deviation might be more effective. We leave it as future work to conduct more extensive simulations in diverse settings and using various types of test statistics.

6 Discussion

In this work, we studied minimax separation rates for two-sample testing under LDP constraint. Moving beyond the univariate Besov ball $\mathcal{B}_{d,s,q}^B(R)$ with $q = \infty$ considered in Lam-Weil et al. (2022), our work encompasses a larger Besov class of densities in a multivariate setting without restriction on q . We also considered the Hölder class and extended the non-private results of Kim et al. (2022a) to a locally private setting. By noting the equivalence between the binning approach in Kim et al. (2022a) and the projection approach in Lam-Weil et al. (2022), we proposed an integrated private testing framework that provides an optimal test for a large class of smooth densities. We proved our results using three distinct LDP mechanisms, thereby extending the toolkit available for practitioners. Additionally, an adaptive test is introduced that retains optimality up to a log factor without the knowledge of the smoothness parameter. Echoing prior work (Aliakbarpour et al., 2018, 2019; Cai et al., 2017; Lam-Weil et al., 2022), our results reaffirm that there exists an inevitable trade-off between data privacy and statistical efficiency that data analysts should bear in mind.

Our paper leaves several open questions for future investigation. Throughout the paper, we focused on equal-sized binning scheme that returns minimax optimal procedures. However, this framework may be problematic in high-dimensional settings as many bins would be empty. To address this issue, one can develop a data-adaptive binning scheme and improve the high-dimensional performance. In terms of smoothness classes, future work can be dedicated to extending our minimax result to a more general Besov class and other smoothness classes. Another interesting direction of future work is to develop optimal tests of conditional independence under privacy constraints, building on the recent work of Neykov et al. (2021); Kim et al. (2022b). Finally, one can attempt to improve the $\log^2 n_1 \log \log n_1$ cost for adaptivity or find the matching lower bound. We leave all of these interesting but challenging problems for future work.

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All acknowledgements go at the end of the paper before appendices and references. Moreover, you are required to declare funding (financial activities supporting the submitted work) and competing interests (related financial activities outside the submitted work). More information about this disclosure can be found on the JMLR website.

Appendix A. Overview of Appendix

This supplementary material provides the technical proofs deferred in the main text, along with some additional results of interest. The content is organized as follows:

- Appendix B presents the technical lemmas, constructions, and calculations utilized in the main proofs of our results.
- Appendix C provides the proof of the α -LDP guarantees for our proposed privacy mechanisms.
- Appendix D proves the minimax rate result for multinomial testing, as presented in Theorem 3.
- Appendix E proves the minimax rate result for density testing, as presented in Theorem 11.
- Appendix F establishes the minimax upper bound for adaptive density testing, as shown in Theorem 12.
- Appendix G provides additional information on the numerical studies presented in Section 5.
- Appendix H discusses the asymptotic suboptimality of the **GenRR**+ ℓ_2 multinomial test mentioned in Section 5.

Appendix B. Preliminary results

This section presents the technical lemmas, constructions, and calculations used in the main proofs.

B.1 First two moments of discrete Laplace noise

We analyze the first two moments of a discrete Laplace noise random variable to establish the upper bound for the separation rate of our private test in Theorem 3. The next lemma shows that the discrete noise in discrete Laplace mechanism (Definition 5) has mean zero and variance at most $8k/\alpha^2$, which matches the variance of the continuous noise in Laplace mechanism (Definition 4) with the same privacy guarantee.

Lemma 13 *If W follows $\text{DL}(\zeta_\alpha)$ defined in (8) and Definition 5, we have*

$$\mathbb{E}(W) = 0 \quad \text{and} \quad \text{Var}(W) \leq \frac{8k}{\alpha^2}.$$

Proof *From Proposition 2.2 of Inusah and Kozubowski (2006), we have*

$$\mathbb{E}(W) = 0 \quad \text{and} \quad \text{Var}(W) = \frac{2\zeta_\alpha}{(1 - \zeta_\alpha)^2}.$$

Therefore, it suffices to show that

$$\frac{2\zeta_\alpha}{(1 - \zeta_\alpha)^2} \leq \frac{8k}{\alpha^2}.$$

For notational convenience, let $v := \alpha/\sqrt{4k} > 0$ for $\alpha > 0$, so that we have $\zeta_\alpha = \exp(-\alpha/\sqrt{4k}) = \exp(-v)$. The proof then reduces to showing that

$$v^2 \leq \frac{(1 - \zeta_\alpha)^2}{\zeta_\alpha} = (e^{v/2} - e^{-v/2})^2.$$

Since $v > 0$, both sides of the above inequality is positive. Therefore it suffices to show that

$$v \leq e^{v/2} - e^{-v/2}. \quad (20)$$

Note that Taylor expansions of $\exp(v/2)$ and $\exp(-v/2)$ have the same even order terms, and their odd order terms have the same absolute values with opposite signs. Therefore, it holds that

$$(e^{v/2} - e^{-v/2}) = 2 \cdot \sum_{n=0}^{\infty} \frac{(v/2)^{2n+1}}{(2n+1)!} = 2 \cdot \left((v/2) + \frac{(v/2)^3}{3!} + \frac{(v/2)^5}{5!} + \frac{(v/2)^7}{7!} + \dots \right).$$

Since $v > 0$, all the terms of above are positive. Thus, the condition (20) is satisfied. This concludes the proof of Lemma 13. \blacksquare

B.2 Summary statistics of RAPPOR mechanism

This section provides calculations for the summary statistics of the RAPPOR private views, which are used in Appendix D.1.2. First, Lemma 14 presents the exact calculation of the expectation of the U-statistic in (9), as well as the variance and covariance of the entries of a RAPPOR private view.

Lemma 14 (Summary statistics of RAPPOR private views) *Let $\{Y_i\}_{i \in [n_1]}$ be i.i.d. multinomial sample with k categories. Let $\{\tilde{\mathbf{Y}}_i\}_{i \in [n_1]}$ represent the corresponding α -LDP views obtained through the RAPPOR mechanism, where for each $m \in [k]$, the m th entry is distributed as $\tilde{Y}_{im} \sim \text{Ber}(\alpha_{\text{bf}} \mathbf{1}(Y_i = m) + \delta_{\text{bf}})$ with*

$$\alpha_{\text{bf}} := \frac{e^{\alpha/2} - 1}{e^{\alpha/2} + 1} \quad \text{and} \quad \delta_{\text{bf}} := \frac{1}{e^{\alpha/2} + 1}.$$

Then we have $\mathbb{E}[U_{n_1, n_2}] = \alpha_{\text{bf}}^2 \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2$. For each $m \in [k]$, we have

$$\text{Var}(\tilde{Y}_{1m}) = (\alpha_{\text{bf}} p_{Ym} + \delta_{\text{bf}})(1 - \alpha_{\text{bf}} p_{Ym} - \delta_{\text{bf}}).$$

For $m, m' \in [k]$ such that $m \neq m'$, we have

$$\text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) = -\alpha_{\text{bf}}^2 p_{Ym} p_{Ym'}.$$

Proof The expectation is verified as follows:

$$\begin{aligned} \mathbb{E}[U_{n_1, n_2}] &= \mathbb{E}[(\tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Z}}_1)^\top (\tilde{\mathbf{Y}}_2 - \tilde{\mathbf{Z}}_2)] \\ &= \mathbb{E}[\tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Z}}_1]^\top \mathbb{E}[\tilde{\mathbf{Y}}_2 - \tilde{\mathbf{Z}}_2] \\ &= (\alpha_{\text{bf}} \mathbf{p}_Y - \alpha_{\text{bf}} \mathbf{p}_Z)^\top (\alpha_{\text{bf}} \mathbf{p}_Y - \alpha_{\text{bf}} \mathbf{p}_Z) \\ &= \alpha_{\text{bf}}^2 \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \end{aligned}$$

The variance is verified as follows:

$$\text{Var}(\tilde{Y}_{1m}) = \mathbb{E}[\tilde{Y}_{1m}^2] - \mathbb{E}[\tilde{Y}_{1m}]^2 \stackrel{(a)}{=} \mathbb{E}[\tilde{Y}_{1m}] - \mathbb{E}[\tilde{Y}_{1m}]^2 = (\alpha_{\text{bf}} p_{Ym} + \delta_{\text{bf}})(1 - \alpha_{\text{bf}} p_{Ym} - \delta_{\text{bf}}), \quad (21)$$

where step (a) utilizes the fact that $\tilde{Y}_{1m}^2 = \tilde{Y}_{1m}$ since \tilde{Y}_{1m} is either 0 or 1. Finally, for $m, m' \in [k]$ such that $m \neq m'$, the covariance is calculated as:

$$\text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) = \mathbb{E}[\tilde{Y}_{1m} \tilde{Y}_{1m'}] - \mathbb{E}[\tilde{Y}_{1m}] \mathbb{E}[\tilde{Y}_{1m'}]$$

$$\begin{aligned}
&\stackrel{(a)}{=} \mathbb{P}[\tilde{Y}_{1m} = 1, \tilde{Y}_{1m'} = 1] - \mathbb{E}[\tilde{Y}_{1m}]\mathbb{E}[\tilde{Y}_{1m'}] \\
&\stackrel{(b)}{=} (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}})(\alpha_{\text{bf}}p_{Ym'} + \delta_{\text{bf}}) - \alpha_{\text{bf}}^2 p_{Ym} p_{Ym'} - (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}})(\alpha_{\text{bf}}p_{Ym'} + \delta_{\text{bf}}) \\
&= -\alpha_{\text{bf}}^2 p_{Ym} p_{Ym'},
\end{aligned}$$

where step (a) uses the fact that each entry is 0 or 1, and step (b) is from Fact 1 of [Acharya et al. \(2019\)](#), which holds only when $m \neq m'$. \blacksquare

We next give upper bounds for the sum of entrywise variances and covariances of an **RAPPOR** α -LDP view.

Lemma 15 *For any $m, m' \in [k]$ such that $m' \neq m$, the following inequalities hold:*

$$\sum_{m=1}^k \text{Var}(\tilde{Y}_{1m})(p_{Ym} - p_{Zm})^2 \leq \alpha_{\text{bf}} b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}} \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2, \quad (22)$$

$$\sum_{m=1}^k \text{Var}(\tilde{Y}_{1m})^2 \leq 2\alpha_{\text{bf}}^2 b + 2\delta_{\text{bf}}^2 k, \text{ and} \quad (23)$$

$$\sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'})^2 \leq \alpha_{\text{bf}}^2 b, \quad (24)$$

where we recall that $b = \max\{\|\mathbf{p}_Y\|_2^2, \|\mathbf{p}_Z\|_2^2\}$. The same type of inequalities also hold for \tilde{Z}_{1m} and $\tilde{Z}_{1m'}$.

Proof

For (22), we have

$$\begin{aligned}
\sum_{m=1}^k \text{Var}(\tilde{Y}_{1m})(p_{Ym} - p_{Zm})^2 &\stackrel{(a)}{=} \sum_{m=1}^k (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}}) \{1 - (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}})\} (p_{Ym} - p_{Zm})^2 \\
&\stackrel{(b)}{\leq} \sum_{m=1}^k (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}}) (p_{Ym} - p_{Zm})^2 \\
&\stackrel{(c)}{\leq} \alpha_{\text{bf}} b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_4^2 + \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \\
&\stackrel{(d)}{\leq} \alpha_{\text{bf}} b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2,
\end{aligned}$$

where step (a) uses Lemma 14, step (b) uses the fact that $0 < \alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}} < 1$ for any $m \in [k]$, step (c) uses the Cauchy–Schwarz inequality, and step (d) uses the monotonicity of the ℓ_p norm, specifically $\ell_4 \leq \ell_2$.

Next, for (23), we have

$$\begin{aligned}
\sum_{m=1}^k \text{Var}(\tilde{Y}_{1m})^2 &\leq \sum_{m=1}^k (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}})^2 \{1 - (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}})\}^2 \\
&\stackrel{(a)}{\leq} \sum_{m=1}^k (\alpha_{\text{bf}}p_{Ym} + \delta_{\text{bf}})^2 \\
&\stackrel{(b)}{\leq} \sum_{m=1}^k (2\alpha_{\text{bf}}^2 p_{Ym}^2 + 2\delta_{\text{bf}}^2)
\end{aligned}$$

$$\begin{aligned}
&= 2\alpha_{\text{bf}}^2 \|\mathbf{p}_Y\|_2^2 + 2\delta_{\text{bf}}^2 k \\
&\leq 2\alpha_{\text{bf}}^2 b + 2\delta_{\text{bf}}^2 k,
\end{aligned}$$

where step (a) uses the fact that $0 < \alpha_{\text{bf}} p_{Ym} + \delta_{\text{bf}} < 1$ for any $m \in [k]$, and step (b) uses the inequality $(a + b)^2 \leq 2a^2 + 2b^2$. Finally, for (24), we have

$$\begin{aligned}
\sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'})^2 &\stackrel{(a)}{=} \sum_{1 \leq m \neq m' \leq k} \alpha_{\text{bf}}^4 p_{Ym}^2 p_{Ym'}^2 \\
&\stackrel{(b)}{\leq} \alpha_{\text{bf}}^2 \sum_{m=1}^k p_{Ym}^2 \sum_{m'=1}^k p_{Ym'}^2 \\
&= \alpha_{\text{bf}}^2 \|\mathbf{p}_Y\|_2^4 \\
&\stackrel{(c)}{\leq} \alpha_{\text{bf}}^2 \|\mathbf{p}_Y\|_2^2 \\
&\leq \alpha_{\text{bf}}^2 b,
\end{aligned}$$

where step (a) uses Lemma 14, step (b) uses the fact that $0 < \alpha_{\text{bf}} < 1$ for any $\alpha > 0$, and step (c) uses the fact that $\|\mathbf{p}_Y\|_2^2 \leq 1$. \blacksquare

B.3 Construction of multivariate Haar wavelet basis

We outline the construction of multivariate Haar wavelets, following Giné and Nickl (2015), Section 4.3.6, and Autin et al. (2010), Section 2. For $f : [0, 1] \rightarrow \mathbb{R}$ and integers u, v , define the re-scaled and shifted version as $f_{u,v}(x) := 2^{u/2} f(2^u x - v)$. Define the univariate Haar scaling and wavelet functions from $[0, 1]$ to \mathbb{R} as

$$\phi(x) := \mathbb{1}(0 \leq x < 1) \quad \text{and} \quad \psi(x) := \mathbb{1}(0 \leq x < 1/2) - \mathbb{1}(1/2 \leq x < 1),$$

respectively. A multivariate Haar wavelet basis is indexed by its prime resolution level $J \in \mathbb{N}_0$, with each element characterized by up to three (multi) indices that range over:

1. Re-scaling: $\mathbb{N} \setminus [J - 1]$,
2. Shifting: $\Lambda(j) := \{0, 1, \dots, (2^j - 1)\}^d$, for an integer $j \geq J$,
3. On-off: $\mathcal{I} := \{0, 1\}^d \setminus \mathbf{0}$,

respectively. For any $\mathbf{k} = (k_1, \dots, k_d) \in \Lambda(J)$, let $\phi_{J,\mathbf{k}} : [0, 1]^d \rightarrow \mathbb{R}$ be a tensor product of the re-scaled and shifted ϕ 's, evaluated at $\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d$:

$$\phi_{J,\mathbf{k}}(\mathbf{x}) := \prod_{p=1}^d \phi_{J,k_p}(x_p).$$

For a resolution level $j \geq J$, and for any $\boldsymbol{\ell} = (\ell_1, \dots, \ell_d) \in \Lambda(j)$, let $\psi_{j,\boldsymbol{\ell}}^\epsilon : [0, 1]^d \rightarrow \mathbb{R}$ be a mixed tensor product of re-scaled and shifted ϕ 's and ψ 's, evaluated at $\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d$:

$$\psi_{j,\boldsymbol{\ell}}^\epsilon(\mathbf{x}) := \prod_{p=1}^d \{\phi_{j,\ell_p}(x_p)\}^{1-\epsilon_p} \{\psi_{j,\ell_p}(x_p)\}^{\epsilon_p}.$$

The Haar multivariate wavelet basis at prime resolution level J is defined as

$$\Phi_J \cup \left(\bigcup_{j \geq J} \Psi_j \right), \quad \text{where } \Phi_J := \{\phi_{J,\mathbf{k}}\}_{\mathbf{k} \in \Lambda(J)} \text{ and } \Psi_j := \{\psi_{j,\boldsymbol{\ell}}^\epsilon\}_{\boldsymbol{\ell} \in \Lambda(j), \epsilon \in \mathcal{I}}. \quad (25)$$

B.4 Density discretization error analysis

This section analyzes the error arising from comparing discretized multinomial probability vectors instead of the original multivariate densities. Given the number of bins κ , let B_1, \dots, B_{κ^d} enumerate d -dimensional hypercubes with side length $1/\kappa$. For a density $f_{\mathbf{Y}}$, its step function approximation is defined as:

$$\hat{f}_{\mathbf{Y}}(\mathbf{y}) := \sum_{m \in [\kappa^d]} \mathbb{1}(\mathbf{y} \in B_m) \kappa^d \int_{B_m} f_{\mathbf{Y}}(\mathbf{t}) d\mathbf{t}. \quad (26)$$

Similarly, define $\hat{f}_{\mathbf{Z}}$ from $f_{\mathbf{Z}}$. Then it holds that $\|\mathbf{p}_{\mathbf{Y}} - \mathbf{p}_{\mathbf{Z}}\|_2^2 = \kappa^{-d} \|\hat{f}_{\mathbf{Y}} - \hat{f}_{\mathbf{Z}}\|_{\mathbb{L}_2}^2$. We analyze the difference between $\|\hat{f}_{\mathbf{Y}} - \hat{f}_{\mathbf{Z}}\|_{\mathbb{L}_2}$ and $\|f_{\mathbf{Y}} - f_{\mathbf{Z}}\|_{\mathbb{L}_2}$ when $(f_{\mathbf{Y}} - f_{\mathbf{Z}})$ lies in $\mathcal{B}_{d,s}^{\mathbf{H}}(R)$ or $\mathcal{B}_{d,s,\infty}^{\mathbf{B}}(R)$. For both cases, the discretization error scales as κ^{-s} , where we recall that s is the smoothness parameter. For the Hölder density case, the analysis is a simple application of Lemma 7.2 of [Arias-Castro et al. \(2018\)](#), rephrased below:

Lemma 16 *For a function $h \in \mathcal{B}_{d,s}^{\mathbf{H}}(R)$, let \hat{h} be its step function approximation as in (26), with κ bins. Then there exist positive constants C_1 and C_2 depending only on d, s and R , but not on h , such that*

$$\|\hat{h}_{\kappa}\|_{\mathbb{L}_2} \geq C_1 \|h\|_{\mathbb{L}_2} - C_2 \kappa^{-s}.$$

Substituting $(f_{\mathbf{Y}} - f_{\mathbf{Z}}) \in \mathcal{B}_{d,s}^{\mathbf{H}}(R)$ into h in Lemma 16, the discretization error is characterized as follows:

$$\|\mathbf{p}_{\mathbf{Y}} - \mathbf{p}_{\mathbf{Z}}\|_2 \geq \kappa^{-d/2} (C_1 \|f_{\mathbf{Y}} - f_{\mathbf{Z}}\|_{\mathbb{L}_2} - C_2 \kappa^{-s}). \quad (27)$$

For the Besov density case, we derive a similar lemma as follows:

Lemma 17 *Define $\mathcal{B}_{d,s,\infty}^{\mathbf{B}}(R)$ using Haar multivariate wavelet basis at prime resolution level J constructed in Appendix B.3. For a function $(f_{\mathbf{Y}} - f_{\mathbf{Z}}) \in \mathcal{B}_{d,s,\infty}^{\mathbf{B}}(R)$, the following error bound holds:*

$$\|\mathbf{p}_{\mathbf{Y}} - \mathbf{p}_{\mathbf{Z}}\|_2 \geq \kappa^{-d/2} (\|f_{\mathbf{Y}} - f_{\mathbf{Z}}\|_{\mathbb{L}_2} - R \kappa^{-s}), \quad (28)$$

where $\kappa = 2^J$ and $\mathbf{p}_{\mathbf{Y}}$ and $\mathbf{p}_{\mathbf{Z}}$ are binned with side length $1/\kappa$.

Proof Let $\Phi_J \cup (\bigcup_{j \geq J} \Psi_j)$ denote the Haar multivariate wavelet basis that defines $\mathcal{B}_{d,s,\infty}^{\mathbf{B}}(R)$. The analysis proceeds in two steps. First, we show that the sum of squared coefficients from projecting $(f_{\mathbf{Y}} - f_{\mathbf{Z}})$ onto $\text{span}(\Phi_J)$ equals the scaled ℓ_2 distance between the probability vectors:

$$\begin{aligned} \sum_{\phi \in \Phi_J} \theta_{\phi}^2(f_{\mathbf{Y}} - f_{\mathbf{Z}}) &= \sum_{(k_1, \dots, k_d) \in \Lambda(J)} \left(\int_{[0,1]^d} (f_{\mathbf{Y}}(\mathbf{x}) - f_{\mathbf{Z}}(\mathbf{x})) \prod_{p=1}^d \phi_{J,k_p}(x_p) d\mathbf{x} \right)^2 \\ &\stackrel{(a)}{=} \kappa^d \sum_{(k_1, \dots, k_d) \in \Lambda(J)} \left(\int_{[0,1]^d} (f_{\mathbf{Y}}(\mathbf{x}) - f_{\mathbf{Z}}(\mathbf{x})) \prod_{p=1}^d \mathbb{1}\left(\frac{k_p}{\kappa} \leq x_p < \frac{k_p+1}{\kappa}\right) d\mathbf{x} \right)^2 \\ &= \kappa^d \sum_{m \in [\kappa^d]} \left(\int_{B_m} (f_{\mathbf{Y}}(\mathbf{x}) - f_{\mathbf{Z}}(\mathbf{x})) d\mathbf{x} \right)^2 \\ &= \kappa^d \|\mathbf{p}_{\mathbf{Y}} - \mathbf{p}_{\mathbf{Z}}\|_2^2, \end{aligned} \quad (29)$$

where step (a) uses the definition $\phi_{J,\mathbf{k}}(\mathbf{x}) = \prod_{p=1}^d \phi_{J,k_p}(x_p)$, with $\phi : [0, 1] \rightarrow \{0, 1\}$ defined as $\phi(x) = \mathbf{1}(0 \leq x < 1)$, and $\phi_{J,k_p}(x_p) = 2^{J/2} f(2^J x_p - k_p)$ indicates its re-scaled and shifted version. See Appendix B.3 for details. Since $\Phi_J \cup (\bigcup_{j \geq J} \Psi_j)$ forms an orthonormal basis of $\mathbb{L}_2([0, 1]^d)$, the approximation error is given by the sum of squared projection coefficients of $(f_{\mathbf{Y}} - f_{\mathbf{Z}})$ onto $\text{span}(\bigcup_{j \geq J} \Psi_j)$. Bounding this term is the second step of our analysis, which begins by noting that from Definition 9, for any $j \in \mathbb{N}_0$, the sum of squared wavelet coefficients is bounded as

$$\sum_{\psi \in \Psi_j} \theta_{\psi}^2(f_{\mathbf{Y}} - f_{\mathbf{Z}}) \leq 2^{-2js} R^2. \quad (30)$$

Then the approximation error is bounded as follows:

$$\begin{aligned} \|f_{\mathbf{Y}} - f_{\mathbf{Z}}\|_{\mathbb{L}_2}^2 - \kappa^d \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 &= \sum_{j=J}^{\infty} \sum_{\psi \in \Psi_j} \theta_{\psi}^2(f_{\mathbf{Y}} - f_{\mathbf{Z}}) \\ &\stackrel{(a)}{\leq} \sum_{j=J}^{\infty} 2^{-2js} R^2 \\ &\stackrel{(b)}{=} R^2 \frac{2^{-2Js}}{1 - 2^{-2s}} \leq R^2 2^{-2Js} = R^2 \kappa^{-2s}, \end{aligned}$$

where step (a) utilizes (30), and step (b) uses an infinite geometric series. Applying $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ to the inequality above, we get

$$\|f_{\mathbf{Y}} - f_{\mathbf{Z}}\|_{\mathbb{L}_2} = \kappa^{d/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2 + R 2^{-Js} \kappa^{d/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2 + R \kappa^{-s}.$$

■

B.5 Gaussian approximation of one-sample U-statistic

The theorems presented here are used to establish a negative result for our statistic in (9) when applied with the generalized randomized response mechanism. First, Theorem 18 restates a result from Kim (2020), which demonstrates the asymptotic normality of a one-sample U-statistic, which is a special case of our statistic in (9) with either $n_1 = \infty$ or $n_2 = \infty$ under the uniform null hypothesis.

Theorem 18 [Corollary 3.3 of Kim (2020)] *Consider a multinomial goodness-of-fit test where the null hypothesis is a discrete uniform distribution with k categories, whose probability vector is denoted as $\boldsymbol{\pi}_0$. Let $\{\mathbf{Y}_i\}_{i=1}^n$ be a random sample represented in one-hot vector form drawn from $\boldsymbol{\pi}_0$. Define the U-statistic as:*

$$U_I := \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} (\mathbf{Y}_i - \boldsymbol{\pi}_0)^\top (\mathbf{Y}_j - \boldsymbol{\pi}_0). \quad (31)$$

If $n/\sqrt{k} \rightarrow \infty$, then the test statistic U_I has the following asymptotic normality:

$$\sqrt{\binom{n}{2}} \frac{U_I}{\sqrt{(1 - 1/k)/k}} \xrightarrow{d} \mathcal{N}(0, 1).$$

The next theorem studies the limiting distribution of U_I under the alternative hypothesis.

Theorem 19 [Theorem 3.3 of Kim (2020)] Consider the same testing problem as Theorem 18. Let $\Sigma := \text{Cov}(\mathbf{Y}_1)$. Under the alternative where the data is generated from a multinomial distribution with a probability vector $\boldsymbol{\pi} \neq \boldsymbol{\pi}_0$, suppose that the following conditions hold as $n, k \rightarrow \infty$:

$$C1. \frac{\text{tr}(\Sigma^4)}{\{\text{tr}(\Sigma^2)\}^2} \rightarrow 0.$$

$$C2. \frac{\mathbb{E}[\{(\mathbf{Y}_1 - \boldsymbol{\pi}_0)^\top (\mathbf{Y}_2 - \boldsymbol{\pi}_0)\}^4] + n_1 \mathbb{E}[\{(\mathbf{Y}_1 - \mathbf{p}_Z)^\top (\mathbf{Y}_2 - \mathbf{p}_Z)\}^2 \{(\mathbf{Y}_1 - \mathbf{p}_Z)^\top (\mathbf{Y}_3 - \mathbf{p}_Z)\}^2]}{n^2 \{\text{tr}(\Sigma^2)\}^2} \rightarrow 0.$$

$$C3. (\boldsymbol{\pi} - \boldsymbol{\pi}_0)^\top \Sigma (\boldsymbol{\pi} - \boldsymbol{\pi}_0) < \infty.$$

Then the test statistic U_I in (31) has the following asymptotic normality:

$$\sqrt{\frac{n(n-1)}{2}} \frac{U_I - \|\boldsymbol{\pi} - \boldsymbol{\pi}_0\|_2^2}{\sqrt{\text{tr}(\Sigma^2) + 2(n-1)(\boldsymbol{\pi} - \boldsymbol{\pi}_0)^\top \Sigma (\boldsymbol{\pi} - \boldsymbol{\pi}_0)}} \xrightarrow{d} \mathcal{N}(0, 1). \quad (32)$$

Appendix C. Proof of Lemma 7

This section provides privacy guarantee proof for the mechanisms proposed in Section 3.2. The privacy proof for the RAPPOR mechanism is given in Section 3.2 of Duchi et al. (2013). The privacy proof for the Laplace mechanism is a minor modification of the proof of Lemma 4.2 from Lam-Weil et al. (2022), adjusting the domain of the raw samples. Thus, we only present the privacy proof for the discrete Laplace mechanism, defined in Definition 5. Its outline resembles the proof of Lemma 4.2 from Lam-Weil et al. (2022).

We note that the conditional distribution of the m th entry of its α -LDP view, denoted as \tilde{X}_{im} , is a discrete Laplace distribution with parameter $\zeta_\alpha = \exp(-\alpha/(2\sqrt{k}))$, shifted by $\sqrt{k}\mathbf{1}(X_i = m)$. Note again that for $m \neq m'$, \tilde{X}_{im} and $\tilde{X}_{im'}$ are independent. With slight abuse of notation, we denote the conditional density function of $\tilde{\mathbf{X}}_i$ as $q_i(\cdot | \cdot)$, which can be written as

$$q_i(\tilde{\mathbf{x}} | x) = \prod_{m=1}^k \frac{1 - \zeta_\alpha}{1 + \zeta_\alpha} \zeta_\alpha^{|\tilde{x}_m - \sqrt{k}\mathbf{1}(x=m)|}.$$

Then for all $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_k)^\top \in \mathbb{R}^k$ and all $x, x' \in [k]$, we have

$$\begin{aligned} \frac{q_i(\tilde{\mathbf{x}} | x)}{q_i(\tilde{\mathbf{x}} | x')} &= \prod_{m=1}^k \zeta_\alpha^{|\tilde{x}_m - \sqrt{k}\mathbf{1}(x=m)| - |\tilde{x}_m - \sqrt{k}\mathbf{1}(x'=m)|} \\ &= \zeta_\alpha^{\sum_{m=1}^k |\tilde{x}_m - \sqrt{k}\mathbf{1}(x=m)| - |\tilde{x}_m - \sqrt{k}\mathbf{1}(x'=m)|} \\ &\stackrel{(a)}{\leq} (\zeta_\alpha^{-1})^{\sum_{m=1}^k \left| |\tilde{x}_m - \sqrt{k}\mathbf{1}(x=m)| - |\tilde{x}_m - \sqrt{k}\mathbf{1}(x'=m)| \right|} \\ &\stackrel{(b)}{\leq} (\zeta_\alpha^{-1})^{\sum_{m=1}^k \{ \sqrt{k}\mathbf{1}(x=m) + \sqrt{k}\mathbf{1}(x'=m) \}} \\ &\stackrel{(c)}{\leq} (\zeta_\alpha^{-1})^{2\sqrt{k}} = e^\alpha, \end{aligned}$$

where step (a) uses the fact that $\zeta_\alpha^x \leq \zeta_\alpha^{-|x|}$ for all $x \in \mathbb{R}$ since $\zeta_\alpha \in (0, 1)$, step (b) uses the reverse triangle inequality, and step (c) holds since $\mathbf{1}(x = m) \neq 0$ for only a single value of m . Using the inequality above, for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$, and for any $x, x' \in [k]$, we have:

$$\frac{Q_i(A | x)}{Q_i(A | x')} = \frac{\int_A q_i(\tilde{\mathbf{x}} | x) d\tilde{\mathbf{x}}}{\int_A q_i(\tilde{\mathbf{x}} | x') d\tilde{\mathbf{x}}} \leq \frac{\int_A q_i(\tilde{\mathbf{x}} | x') e^\alpha d\tilde{\mathbf{x}}}{\int_A q_i(\tilde{\mathbf{x}} | x) e^{-\alpha} d\tilde{\mathbf{x}}} = \frac{Q_i(A | x')}{Q_i(A | x)} e^{2\alpha}.$$

This completes the proof of the guarantee for the discrete Laplace mechanism.

Appendix D. Proof of Theorem 3

In this section, we prove the minimax rate result for multinomial testing presented in Theorem 3, first focusing on the upper bound result followed by the lower bound result.

D.1 Upper bound

This section demonstrates that the permutation test with the U-statistic proposed in (9), in conjunction with one of the mechanisms $\{\text{LapU}, \text{DiscLapU}, \text{RAPPOR}\}$, achieves a tight upper bound presented in Theorem 3. Since the permutation procedure controls the type I error, it suffices to prove that the condition in Theorem 3 guarantees type II error control. Let

$$b := \max\{\|\mathbf{p}_Y\|_2^2, \|\mathbf{p}_Z\|_2^2\}. \quad (33)$$

The control of the type II error is then rephrased in the following lemma.

Lemma 20 *Assume the settings of Theorem 3. For each of the mechanisms $\{\text{LapU}, \text{DiscLapU}, \text{RAPPOR}\}$, there exists a constant $C_u(\gamma, \beta)$ such that the type II error of the permutation test of size γ with U-statistic in (9) is uniformly controlled by β over $\mathcal{P}_{1,\text{multi}}$ if*

$$\rho_{n_1, n_2} \geq C_u(\gamma, \beta) \left(\frac{k^{1/4}}{(n_1 \alpha^2)^{1/2}} \vee \frac{b^{1/4}}{n_1^{1/2}} \right). \quad (34)$$

Since $b \leq 1$, Lemma 20 proves the upper bound result of Theorem 3. To prove Lemma 20, we leverage the two moments method (Theorem 4.1 of Kim et al. (2022a)), rephrased in Lemma 21. This method states that if the test statistic's expectation is sufficiently larger than a variance proxy, then uniform type II error control is achieved for the permutation test with a two-sample U-statistic of order 2. Its key advantage is that the condition bypasses the randomness arising from permutations. It lets us avoid the complex analysis typically required for permuted statistics.

To proceed, consider the following kernel:

$$h_{ts}(\mathbf{y}_1, \mathbf{y}_2; \mathbf{z}_1, \mathbf{z}_2) = \mathbf{y}_1^\top \mathbf{y}_2 + \mathbf{z}_1^\top \mathbf{z}_2 - \mathbf{z}_1^\top \mathbf{y}_2 - \mathbf{z}_2^\top \mathbf{y}_1, \quad (35)$$

which defines our two-sample U-statistic U_{n_1, n_2} in (9). Its symmetrized version, denoted \check{h}_{ts} , is defined as:

$$\check{h}_{ts}(\mathbf{y}_1, \mathbf{y}_2; \mathbf{z}_1, \mathbf{z}_2) := \frac{1}{2!2!} \sum_{1 \leq i_1 \neq i_2 \leq n_1} \sum_{1 \leq j_1 \neq j_2 \leq n_2} h_{ts}(\mathbf{y}_{i_1}, \mathbf{y}_{i_2}; \mathbf{z}_{j_1}, \mathbf{z}_{j_2}).$$

For the upper bound analysis, we use a U-statistic represented by \check{h}_{ts} , which is equivalent to our original statistic in U_{n_1, n_2} (9). Recall that under the LDP constraint, the raw samples $\{Y_i\}_{i \in [n_1]}$ and $\{Z_j\}_{j \in [n_2]}$ are generated from $P = (P_Y, P_Z)$ and then transformed into LDP-views $\{\tilde{Y}_i\}_{i \in [n_1]}$ and $\{\tilde{Z}_j\}_{j \in [n_2]}$ through an LDP mechanism Q . Let us denote the associated moments as

$$\begin{aligned} M_{Y,1}(P, Q) &:= \text{Var}_{P,Q}[\mathbb{E}\{\check{h}_{ts}(\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2; \tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2) \mid \tilde{\mathbf{Y}}_1\}], \\ M_{Z,1}(P, Q) &:= \text{Var}_{P,Q}[\mathbb{E}\{\check{h}_{ts}(\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2; \tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2) \mid \tilde{\mathbf{Z}}_1\}], \\ M_{Y,Z,2}(P, Q) &:= \max\{\mathbb{E}[(\tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{Y}}_2)^2], \mathbb{E}[(\tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{Z}}_1)^2], \mathbb{E}[(\tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{Z}}_2)^2]\}. \end{aligned} \quad (36)$$

Using these moments, we rephrase the two moments method under the setting of LDP.

Lemma 21 (Two moments method) *Let U_{n_1, n_2} be a two-sample U -statistic based on the kernel given in (35). Assume that the samples are privatized through an α -LDP mechanism Q . Then there exists a sufficiently large constant $C > 0$ such that if*

$$\mathbb{E}[U_{n_1, n_2}] \geq C \sqrt{\max \left\{ \frac{M_{Y,1}(P, Q)}{\beta n_1}, \frac{M_{Z,1}(P, Q)}{\beta n_2}, \frac{M_{YZ,2}(P, Q)}{\gamma \beta} \left(\frac{1}{n_1} + \frac{1}{n_2} \right)^2 \right\}} \quad (37)$$

for all pairs of distributions $P = (P_Y, P_Z) \in \mathcal{P}_{1, \text{multi}}(\rho_{n_1, n_2})$, then the type II error of the permutation test over $\mathcal{P}_{1, \text{multi}}(\rho_{n_1, n_2})$ is uniformly bounded by β as in (1).

Having stated the two moments method, our goal is to verify that inequality (37) holds under the separation conditions described in Section 3.2. We provide separate proofs for the Laplace-noise based mechanisms (LapU and DiscLapU) and the RAPPOR mechanism. However, both proofs follow the same two steps:

1. Derive upper bounds for the moments $M_{Y,1}(P, Q)$, $M_{Z,1}(P, Q)$, and $M_{YZ,2}(P, Q)$ in (36).
2. Using the upper bounds established in Step 1, show that condition (37) in the two moments method is fulfilled as long as inequality (34) in Lemma 20 holds.

We now verify the previous preliminary steps in order, first assuming the Laplace-noise based mechanisms.

D.1.1 UPPER BOUND THROUGH LAPLACE OR DISCRETE LAPLACE MECHANISM

Since the analysis for Laplace and discrete Laplace mechanism is similar, we present the proof using the former (see Remark 22 for details).

Step 1: Upper bounds for the moments. We start by bounding the variance of conditional expectation terms. Recall the notation $M_{Y,1}(P, Q) = \text{Var}_{P, Q}[\mathbb{E}\{\tilde{h}_{ts}(\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2; \tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2) | \tilde{\mathbf{Y}}_1\}]$. To upper bound this quantity, we first calculate the conditional expectation of the kernel function, namely $A := \mathbb{E}\{\tilde{h}_{ts}(\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2; \tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2) | \tilde{\mathbf{Y}}_1\}$. Then we find an upper bound of the variance of A .

For the conditional expectation, recall the m th component of $\mathbf{p}_Y, \mathbf{p}_Z, \tilde{\mathbf{Y}}_i, \tilde{\mathbf{Z}}_j$ are denoted as $p_{Ym}, p_{Zm}, \tilde{Y}_{im}$ and \tilde{Z}_{jm} , respectively. Due to i.i.d. assumptions, regardless of privacy mechanism, we have the following equalities and inequalities. First we calculate the conditional expectation:

$$A := \mathbb{E}[\tilde{h}_{ts}(\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2; \tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2) | \tilde{\mathbf{Y}}_1] = (\tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Z}}_1])^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1]).$$

Then we calculate the unconditional expectation:

$$\mathbb{E}[A] = (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1])^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1]) = \|\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1]\|_2^2.$$

Then the squared and centered conditional expectation is calculated as follows:

$$A - \mathbb{E}[A] = (\tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1])^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1]).$$

Based on this, the variance is calculated as

$$\mathbb{E}[(A - \mathbb{E}[A])^2] = \mathbb{E}[\{(\tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1])^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1])\}^2] \quad (38)$$

$$\begin{aligned} &= \mathbb{E}[\{\tilde{\mathbf{Y}}_1^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1]) - \mathbb{E}[\tilde{\mathbf{Y}}_1]^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1])\}^2] \\ &\leq 2\mathbb{E}[\{\tilde{\mathbf{Y}}_1^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1])\}^2] + 2[\{\mathbb{E}[\tilde{\mathbf{Y}}_1]^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1])\}^2], \end{aligned} \quad (39)$$

where the last inequality uses $(x + y)^2 \leq 2x^2 + 2y^2$. Using $\|\mathbf{p}_Y\|_2 \leq 1$, we bound the second term in (39) by

$$2k^2\|\mathbf{p}_Y\|_2\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \quad (40)$$

For the first term, we leverage the structure of our privacy mechanism, which adds independent and centered noises. Applying the Cauchy–Schwarz inequality then provides an upper bound, reducing the order of k compared to a direct application of Cauchy–Schwarz. More formally,

$$\begin{aligned} & 2\mathbb{E}\left\{\tilde{\mathbf{Y}}_1^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1])\right\}^2 \\ &= 2 \sum_{m=1}^k \sum_{m'=1}^k \mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}](\mathbb{E}[\tilde{Y}_{1m}] - \mathbb{E}[\tilde{Z}_{1m}])(\mathbb{E}[\tilde{Y}_{1m'}] - \mathbb{E}[\tilde{Z}_{1m'}]) \\ &\stackrel{(a)}{=} 2 \sum_{m=1}^k \mathbb{E}[\tilde{Y}_{1m}^2](\mathbb{E}[\tilde{Y}_{1m}] - \mathbb{E}[\tilde{Z}_{1m}])^2 \\ &\stackrel{(b)}{=} 2 \sum_{m=1}^k \{kp_{Ym} + \sigma_\alpha^2\}(\mathbb{E}[\tilde{Y}_{1m}] - \mathbb{E}[\tilde{Z}_{1m}])^2 \\ &= 2 \sum_{m=1}^k kp_{Ym}(\mathbb{E}[\tilde{Y}_{1m}] - \mathbb{E}[\tilde{Z}_{1m}])^2 + 2k\sigma_\alpha^2\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \\ &\stackrel{(c)}{\leq} 2k^2\|\mathbf{p}_Y\|_2\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + 2k\sigma_\alpha^2\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2, \end{aligned} \quad (41)$$

where step (a) uses $\mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}] = 0$ for $m \neq m'$, step (b) uses $\mathbb{E}[\tilde{Y}_{1m}^2] = kp_{Ym} + \sigma_\alpha^2$, and step (c) uses the Cauchy–Schwarz inequality, $\|\mathbf{p}_Y\|_2^2 \leq \|\mathbf{p}_Y\|_2$, and monotonicity of ℓ_p norm, specifically $\ell_4 \leq \ell_2$. Combining the results in (40) and (41), we achieve an upper bound for $M_{Y,1}(P, Q)$ given as

$$M_{Y,1}(P, Q) \leq (4k^2\|\mathbf{p}_Y\|_2 + 2k\sigma_\alpha^2)\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \quad (42)$$

By symmetry, we also have

$$M_{Z,1}(P, Q) \leq (4k^2\|\mathbf{p}_Z\|_2 + 2k\sigma_\alpha^2)\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \quad (43)$$

Combining the upper bound results of (42) and (43), and keeping in mind that $b = \max\{\|\mathbf{p}_Y\|_2^2, \|\mathbf{p}_Z\|_2^2\}$, we obtain the following upper bound:

$$\max\{M_{Y,1}(P, Q), M_{Z,1}(P, Q)\} \leq (4k^2b^{1/2} + 2k\sigma_\alpha^2)\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \quad (44)$$

We now turn our attention to the expectation of square terms:

$$M_{YZ,2}(P, Q) := \max\{\mathbb{E}[(\tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{Y}}_2)^2], \mathbb{E}[(\tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{Z}}_2)^2], \mathbb{E}[(\tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{Z}}_1)^2]\}.$$

First, we calculate $\mathbb{E}[(\tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{Y}}_2)^2]$ and $\mathbb{E}[(\tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{Z}}_2)^2]$. Note that

$$\begin{aligned} \mathbb{E}[(\tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{Y}}_2)^2] &= \mathbb{E}\left[\left(\sum_{m=1}^k \tilde{Y}_{1m}\tilde{Y}_{2m}\right)^2\right] \stackrel{(a)}{=} \sum_{m=1}^k \sum_{m'=1}^k \mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}]\mathbb{E}[\tilde{Y}_{2m}\tilde{Y}_{2m'}] \\ &\stackrel{(b)}{=} \sum_{m=1}^k \{kp_{Ym} + \sigma_\alpha^2\}^2 \\ &= \sum_{m=1}^k \{k^2p_Y^2(m) + 2k\sigma_\alpha^2p_{Ym} + \sigma_\alpha^4\} \\ &= k^2\|\mathbf{p}_Y\|_2^2 + 2k\sigma_\alpha^2 + k\sigma_\alpha^4, \end{aligned} \quad (45)$$

where step (a) uses the independence between different samples, and step (b) uses $\mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}] = (kp_{Ym} + \sigma_\alpha^2)\mathbb{1}(m = m')$ (and similar equality for \tilde{Y}_2). By symmetry, $\mathbb{E}[(\tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{Z}}_2)^2]$ is also calculated as

$$\mathbb{E}[(\tilde{\mathbf{Z}}_1^\top \tilde{\mathbf{Z}}_2)^2] = k^2 \|\mathbf{p}_Z\|_2^2 + 2k\sigma_\alpha^2 + k\sigma_\alpha^4. \quad (46)$$

Moving on, we upper bound $\mathbb{E}[(\tilde{\mathbf{Y}}_1^\top \tilde{\mathbf{Z}}_1)^2]$ as follows:

$$\begin{aligned} \mathbb{E}[(\tilde{Y}_{1m}\tilde{Z}_{1m})^2] &= \mathbb{E}\left[\left(\sum_{m=1}^k \tilde{Y}_{1m}\tilde{Z}_{1m}\right)^2\right] \\ &\stackrel{(a)}{=} \sum_{m=1}^k \sum_{m'=1}^k \mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}]\mathbb{E}[\tilde{Z}_{1m}\tilde{Z}_{1m'}] \\ &\stackrel{(b)}{=} \sum_{m=1}^k \{kp_{Ym} + \sigma_\alpha^2\} \{kp_{Zm} + \sigma_\alpha^2\} \\ &\stackrel{(c)}{\leq} \frac{1}{2} \sum_{m=1}^k [\{kp_{Ym} + \sigma_\alpha^2\}^2 + \{kp_{Zm} + \sigma_\alpha^2\}^2] \\ &= \frac{k^2}{2} \|\mathbf{p}_Y\|_2^2 + \frac{k^2}{2} \|\mathbf{p}_Z\|_2^2 + 2k\sigma_\alpha^2 + k\sigma_\alpha^4, \end{aligned} \quad (47)$$

where step (a) uses the independence between $\tilde{\mathbf{Y}}_1$ and $\tilde{\mathbf{Z}}_1$, step (b) uses $\mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}] = (kp_{Ym} + \sigma_\alpha^2)\mathbb{1}(m = m')$ (and similar equality for $\tilde{\mathbf{Z}}_1$), and step (c) applies the inequality $xy \leq x^2/2 + y^2/2$.

Finally, by combining (45), (46) and (47), we obtain the following upper bound:

$$M_{YZ,2}(P, Q) \leq 2(k^2b + k\sigma_\alpha^2 + k\sigma_\alpha^4). \quad (48)$$

Remark 22 (Proving the upper bound using DiscLapU) *In our proof with continuous Laplace noise from LapU, we use the independence and the equality $\mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}] = (kp_{Ym} + \sigma_\alpha^2)\mathbb{1}(m = m')$, which holds due to the Laplace noise's moments: mean zero and variance $\sigma_\alpha^2 = 8k/\alpha^2$. The discrete Laplace noise of DiscLapU also satisfies these independence and moment conditions, with variance upper bounded by $8k/\alpha^2$ (Lemma 13). Due to these properties, the use of DiscLapU also leads to $\mathbb{E}[\tilde{Y}_{1m}\tilde{Y}_{1m'}] \leq (kp_{Ym} + \sigma_\alpha^2)\mathbb{1}(m = m')$. Given this inequality, the entire proof of this section remains valid for DiscLapU as well.*

Step 2: apply the two moments method. Using the bounds (44) and (48), we show that condition (37) in the two moments method holds if the separation condition (34) in Lemma 20 is met. Since $\mathbb{E}[U_{n_1, n_2}] = k\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2$, assuming $n_1 \leq n_2$, condition (37) of Lemma 21 is satisfied when

$$k\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq C_1 \sqrt{\frac{(k^2b^{1/2} + k\sigma_\alpha^2)\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2}{\beta n_1}} \quad \text{from (44), and} \quad (49)$$

$$k\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq C_2 \sqrt{\frac{k^2b + k\sigma_\alpha^2 + k\sigma_\alpha^4}{\gamma\beta n_1^2}} \quad \text{from (48),} \quad (50)$$

for all pairs $P = (P_Y, P_Z) \in \mathcal{P}_{1, \text{multi}}(\rho_{n_1, n_2})$. Since $\sigma_\alpha = 2\sqrt{2k}/\alpha$, condition (49) is satisfied when

$$k\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq C_3 \sqrt{\frac{(k^2b^{1/2} + k^2/\alpha^2)\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2}{\beta n_1}}.$$

Using $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \leq 2 \max\{\sqrt{x}, \sqrt{y}\}$ for $x, y \geq 0$, the condition above is implied by:

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq C_4(\beta) \frac{\max\{b^{1/4}, 1/\alpha\}}{n_1^{1/2}}. \quad (51)$$

On the other hand, condition (50) is satisfied when

$$k\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq C_5 \sqrt{\frac{k^2 b + k^2/\alpha^2 + k^3/\alpha^4}{\gamma\beta n_1^2}}.$$

Using $\sqrt{x+y+z} \leq 3 \max\{\sqrt{x}, \sqrt{y}, \sqrt{z}\}$ for $x, y, z \geq 0$, the condition above is implied by:

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq C_6(\gamma, \beta) \frac{\max\{b^{1/4}, 1/\sqrt{\alpha}, k^{1/4}/\alpha\}}{n_1^{1/2}}. \quad (52)$$

Then, by combining (51) and (52), the condition (37) of Lemma 21 is satisfied when

$$\begin{aligned} \rho_{n_1, n_2} &\geq C_6(\gamma, \beta) \frac{1}{n_1^{1/2}} \max\left\{\max\left(b^{1/4}, \frac{1}{\alpha}\right), \max\left(b^{1/4}, \frac{1}{\sqrt{\alpha}}, \frac{k^{1/4}}{\alpha}\right)\right\} \\ &= C_6(\gamma, \beta) \frac{1}{n_1^{1/2}} \max\left\{b^{1/4}, \max\left(\frac{1}{\alpha}, \frac{1}{\sqrt{\alpha}}, \frac{k^{1/4}}{\alpha}\right)\right\} \\ &= \begin{cases} C_6(\gamma, \beta) \frac{1}{n_1^{1/2}} \max\left(b^{1/4}, \frac{k^{1/4}}{\alpha}\right) & \text{if } \alpha \leq k^{1/2}, \\ C_6(\gamma, \beta) \frac{1}{n_1^{1/2}} \max\left(b^{1/4}, \frac{1}{\sqrt{\alpha}}\right) & \text{if } \alpha \geq k^{1/2} \end{cases} \\ &\stackrel{(a)}{=} C_6(\gamma, \beta) \frac{1}{n_1^{1/2}} \max\left(b^{1/4}, \frac{k^{1/4}}{\alpha}\right) \\ &= C_6(\gamma, \beta) \max\left(\frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4}}{(n_1 \alpha^2)^{1/2}}\right), \end{aligned}$$

where the step (a) holds because if $\alpha \geq k^{1/2}$, then we have $b^{1/4} \geq 1/k^{1/4} \geq 1/\sqrt{\alpha} \geq k^{1/4}/\alpha$. This completes the proof of the upper bound through the Laplace and discrete Laplace mechanisms.

D.1.2 UPPER BOUND THROUGH RAPPOR MECHANISM

While we follow the same two main steps as in Appendix D.1.1, proving a tight upper bound under the RAPPOR mechanism requires a more delicate analysis due to the dependence and bias inherent in private views. To elaborate, note that the entries of an α -LDP view under the RAPPOR mechanism are dependent Bernoulli random variables. Specifically, the m th entry of $\tilde{\mathbf{Y}}_i$, denoted as \tilde{Y}_{im} , and the m' th entry of $\tilde{\mathbf{Z}}_j$, denoted as $\tilde{Z}_{jm'}$, follow the following distributions:

$$\tilde{Y}_{im} \sim \text{Ber}(\alpha_{\text{bf}} \mathbb{1}(Y_i = m) + \delta_{\text{bf}}) \quad \text{and} \quad \tilde{Z}_{jm'} \sim \text{Ber}(\alpha_{\text{bf}} \mathbb{1}(Z_j = m') + \delta_{\text{bf}}),$$

where

$$\alpha_{\text{bf}} := \frac{e^{\alpha/2} - 1}{e^{\alpha/2} + 1} \quad \text{and} \quad \delta_{\text{bf}} := \frac{1}{e^{\alpha/2} + 1}.$$

As in Appendix D.1.1, we employ the two moments method (Lemma 21), which compares the expectation of the U-statistic with a variance proxy. The challenge is that, unlike LapU, for each $m \in [k]$, the m th entry of each α -LDP view in RAPPOR is not centered at the scaled multinomial probability:

$$\mathbb{E}[\tilde{Y}_{im}] = \mathbb{E}[\mathbb{E}[\tilde{Y}_{im} \mid Y_{im}]] = \alpha_{\text{bf}} p_{Ym} + \delta_{\text{bf}}, \quad (53)$$

and pairwise negatively correlated for $m, m' \in [k]$ such that $m \neq m'$:

$$\text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) = -\alpha_{\text{bf}}^2 p_{Ym} p_{Ym'}, \quad (54)$$

which is proved in Lemma 14. Keeping these facts in mind, we follow the two steps of Appendix D.1.1.

Step 1: Upper bounds for the moments. We start by bounding the variance of conditional expectation terms. We begin with the intermediate form of $M_{Y,1}(P, Q)$ (38) found in Appendix D.1.1:

$$\begin{aligned} M_{Y,1}(P, Q) &= \mathbb{E} \left[\left\{ (\tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1])^\top (\mathbb{E}[\tilde{\mathbf{Y}}_1] - \mathbb{E}[\tilde{\mathbf{Z}}_1]) \right\}^2 \right] \\ &\stackrel{(a)}{=} \mathbb{E} \left[\left\{ (\tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1])^\top \alpha_{\text{bf}} (\mathbf{p}_Y - \mathbf{p}_Z) \right\}^2 \right] \\ &= \alpha_{\text{bf}}^2 \mathbb{E} \left[\left\{ \sum_{m=1}^k (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}]) (p_{Ym} - p_{Zm}) \right\}^2 \right] \\ &= \alpha_{\text{bf}}^2 \mathbb{E} \left[\sum_{m=1}^k (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])^2 (p_{Ym} - p_{Zm})^2 \right] \\ &\quad + \alpha_{\text{bf}}^2 \mathbb{E} \left[\sum_{1 \leq m \neq m' \leq k} (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}]) (\tilde{Y}_{1m'} - \mathbb{E}[\tilde{Y}_{1m'}]) (p_{Ym} - p_{Zm}) (p_{Ym'} - p_{Zm'}) \right] \\ &= \alpha_{\text{bf}}^2 \left[\sum_{m=1}^k \text{Var}(\tilde{Y}_{1m}) (p_{Ym} - p_{Zm})^2 \right] \\ &\quad + \alpha_{\text{bf}}^2 \left[\sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) (p_{Ym} - p_{Zm}) (p_{Ym'} - p_{Zm'}) \right] \\ &\stackrel{(b)}{\leq} \alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2, \end{aligned} \quad (55)$$

where step (a) uses (53), and step (b) uses (22) in Lemma 15 for the first term and (54) for the second term. Therefore we have

$$M_{Y,1}(P, Q) \leq \alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2, \quad (56)$$

and by symmetry, we also have

$$M_{Z,1}(P, Q) \leq \alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Z - \mathbf{p}_Y\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \quad (57)$$

Next, we examine the expectation of square terms $M_{YZ,2}(P, Q)$. The proof's key technique involves rewriting the kernel $(\tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Z}}_1)^\top (\tilde{\mathbf{Y}}_2 - \tilde{\mathbf{Z}}_2)$ (35) of the U-statistic (9) using $\tilde{\mathbf{Y}}_1 := \tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1]$, $\tilde{\mathbf{Z}}_1 := \tilde{\mathbf{Z}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1]$, $\tilde{\mathbf{Y}}_2 := \tilde{\mathbf{Y}}_2 - \mathbb{E}[\tilde{\mathbf{Y}}_2]$, and $\tilde{\mathbf{Z}}_2 := \tilde{\mathbf{Z}}_2 - \mathbb{E}[\tilde{\mathbf{Y}}_2]$. Formally, we re-express the kernel (35) as the following:

$$\begin{aligned} (\tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Z}}_1)^\top (\tilde{\mathbf{Y}}_2 - \tilde{\mathbf{Z}}_2) &= (\tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1] + \mathbb{E}[\tilde{\mathbf{Y}}_1] - \tilde{\mathbf{Z}}_1)^\top (\tilde{\mathbf{Y}}_2 - \mathbb{E}[\tilde{\mathbf{Y}}_2] + \mathbb{E}[\tilde{\mathbf{Y}}_2] - \tilde{\mathbf{Z}}_2) \\ &= \{(\tilde{\mathbf{Y}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1]) - (\tilde{\mathbf{Z}}_1 - \mathbb{E}[\tilde{\mathbf{Y}}_1])\}^\top \{(\tilde{\mathbf{Y}}_2 - \mathbb{E}[\tilde{\mathbf{Y}}_2]) - (\tilde{\mathbf{Z}}_2 - \mathbb{E}[\tilde{\mathbf{Y}}_2])\} \\ &= (\tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Z}}_1)^\top (\tilde{\mathbf{Y}}_2 - \tilde{\mathbf{Z}}_2) \end{aligned} \quad (58)$$

Let \bar{U}_{n_1, n_2} denote the U-statistic defined by the kernel $(\bar{\mathbf{Y}}_1 - \bar{\mathbf{Z}}_1)^\top (\bar{\mathbf{Y}}_2 - \bar{\mathbf{Z}}_2)$ in (58). Then \bar{U}_{n_1, n_2} is equal to our original U-statistic (9). Thus, it suffices to prove Lemma 20 with respect to \bar{U}_{n_1, n_2} . Let $\bar{M}_{Y,1}(P, Q)$, $\bar{M}_{Z,1}(P, Q)$, and $\bar{M}_{Y,Z,2}(P, Q)$ be the moments defined as in (36) using the kernel $(\bar{\mathbf{Y}}_1 - \bar{\mathbf{Z}}_1)^\top (\bar{\mathbf{Y}}_2 - \bar{\mathbf{Z}}_2)$. Since $\bar{M}_{Y,1}(P, Q) = M_{Y,1}(P, Q)$ and $\bar{M}_{Z,1}(P, Q) = M_{Z,1}(P, Q)$, the bounds (56) and (57) are also valid for $\bar{M}_{Y,1}(P, Q)$ and $\bar{M}_{Z,1}(P, Q)$. Now, we move on to bounding $\bar{M}_{Y,Z,2}(P, Q)$ given as

$$\bar{M}_{Y,Z,2}(P, Q) = \max\{\mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Y}}_2)^2], \mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Z}}_1)^2], \mathbb{E}[(\bar{\mathbf{Z}}_1^\top \bar{\mathbf{Z}}_2)^2]\}.$$

Let us start by upper bounding $\mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Y}}_2)^2]$:

$$\begin{aligned} \mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Y}}_2)^2] &= \mathbb{E}\left[\left\{\sum_{m=1}^k (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])(\tilde{Y}_{2m} - \mathbb{E}[\tilde{Y}_{2m}])\right\}^2\right] \\ &= \mathbb{E}\left[\sum_{m=1}^k (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])^2 (\tilde{Y}_{2m} - \mathbb{E}[\tilde{Y}_{2m}])^2\right] \\ &\quad + \mathbb{E}\left[\sum_{1 \leq m \neq m' \leq k} (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])(\tilde{Y}_{1m'} - \mathbb{E}[\tilde{Y}_{1m'}])(\tilde{Y}_{2m} - \mathbb{E}[\tilde{Y}_{2m}])(\tilde{Y}_{2m'} - \mathbb{E}[\tilde{Y}_{2m'}])\right] \\ &= \sum_{m=1}^k \text{Var}(\tilde{Y}_{1m})^2 + \sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'})^2 \\ &\leq 3\alpha_{\text{bf}}^2 b + 2\delta_{\text{bf}}^2 k, \end{aligned} \tag{59}$$

where the last inequality uses inequalities (23) and (24) in Lemma 15. By symmetry, we also have

$$\mathbb{E}[(\bar{\mathbf{Z}}_1^\top \bar{\mathbf{Z}}_2)^2] \leq 3\alpha_{\text{bf}}^2 b + 2\delta_{\text{bf}}^2 k. \tag{60}$$

Now we move on to bounding $\mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Z}}_1)^2]$, which is expanded as:

$$\begin{aligned} \mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Z}}_1)^2] &= \mathbb{E}\left[\left\{\sum_{m=1}^k (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}])\right\}^2\right] \\ &= \mathbb{E}\left[\sum_{m=1}^k (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])^2 (\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}])^2\right] \\ &\quad + \mathbb{E}\left[\sum_{1 \leq m \neq m' \leq k} (\tilde{Y}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])(\tilde{Y}_{1m'} - \mathbb{E}[\tilde{Y}_{1m'}])(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}])(\tilde{Z}_{1m'} - \mathbb{E}[\tilde{Z}_{1m'}])\right] \\ &= \sum_{m=1}^k \text{Var}(\tilde{Y}_{1m})\mathbb{E}(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}])^2 \end{aligned} \tag{61}$$

$$+ \sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'})\mathbb{E}(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}])(\tilde{Z}_{1m'} - \mathbb{E}[\tilde{Z}_{1m'}]), \tag{62}$$

where the last equality uses Lemma 14. We bound the terms (61) and (62) separately. For the term (61), we utilize the following equality that holds for each of $m \in [k]$:

$$\begin{aligned} \mathbb{E}[\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}]]^2 &= \mathbb{E}[\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}] + \mathbb{E}[\tilde{Z}_{1m}] - \mathbb{E}[\tilde{Y}_{1m}]]^2 \\ &= \mathbb{E}[\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}]]^2 + [\mathbb{E}[\tilde{Z}_{1m}] - \mathbb{E}[\tilde{Y}_{1m}]]^2 + 2\mathbb{E}[\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}]](\mathbb{E}[\tilde{Z}_{1m}] - \mathbb{E}[\tilde{Y}_{1m}]) \\ &= \text{Var}(\tilde{Z}_{1m}) + [\mathbb{E}[\tilde{Z}_{1m}] - \mathbb{E}[\tilde{Y}_{1m}]]^2 \\ &= \text{Var}(\tilde{Z}_{1m}) + \alpha_{\text{bf}}^2 (p_{Ym} - p_{Zm})^2, \end{aligned} \tag{63}$$

where the last equality uses (53). Using this equality, the term (61) is bounded as:

$$\begin{aligned}
& \sum_{m=1}^k \text{Var}(\tilde{Y}_{1m}) \mathbb{E}(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])^2 \\
&= \sum_{m=1}^k \text{Var}(\tilde{Y}_{1m}) \text{Var}(\tilde{Z}_{1m}) + \sum_{m=1}^k \text{Var}(\tilde{Y}_{1m}) \alpha_{\text{bf}}^2 (p_{Ym} - p_{Zm})^2 \\
&\stackrel{(a)}{\leq} \frac{1}{2} \sum_{m=1}^k \text{Var}(\tilde{Y}_{1m})^2 + \frac{1}{2} \sum_{m=1}^k \text{Var}(\tilde{Z}_{1m})^2 + \alpha_{\text{bf}}^2 \sum_{m=1}^k \text{Var}(\tilde{Y}_{1m}) (p_{Ym} - p_{Zm})^2 \\
&\stackrel{(b)}{\leq} \alpha_{\text{bf}}^2 b + \alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2,
\end{aligned}$$

where step (a) uses $2ab \leq a^2 + b^2$, and step (b) uses Lemma 15. For the term (62), first note that:

$$\begin{aligned}
& \mathbb{E}[(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])(\tilde{Z}_{1m'} - \mathbb{E}[\tilde{Y}_{1m'}])] \\
&= \mathbb{E}[(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}] + \mathbb{E}[\tilde{Z}_{1m}] - \mathbb{E}[\tilde{Y}_{1m}])(\tilde{Z}_{1m'} - \mathbb{E}[\tilde{Z}_{1m'}] + \mathbb{E}[\tilde{Z}_{1m'}] - \mathbb{E}[\tilde{Y}_{1m'}])] \\
&= \mathbb{E}[(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Z}_{1m}])(\tilde{Z}_{1m'} - \mathbb{E}[\tilde{Z}_{1m'}])] + (\mathbb{E}[\tilde{Z}_{1m}] - \mathbb{E}[\tilde{Y}_{1m}])(\mathbb{E}[\tilde{Z}_{1m'}] - \mathbb{E}[\tilde{Y}_{1m'}]) \\
&= \text{Cov}(\tilde{Z}_{1m}, \tilde{Z}_{1m'}) + \alpha_{\text{bf}}^2 (p_{Ym} - p_{Zm})(p_{Ym'} - p_{Zm'}),
\end{aligned}$$

for each $m \in [k]$. Using this equality, the term (62) is bounded as:

$$\begin{aligned}
& \sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) \mathbb{E}(\tilde{Z}_{1m} - \mathbb{E}[\tilde{Y}_{1m}])(\tilde{Z}_{1m'} - \mathbb{E}[\tilde{Y}_{1m'}]) \\
&= \sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) \text{Cov}(\tilde{Z}_{1m}, \tilde{Z}_{1m'}) \\
&\quad + \sum_{1 \leq m \neq m' \leq k} \text{Cov}(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) \alpha_{\text{bf}}^2 (p_{Ym} - p_{Zm})(p_{Ym'} - p_{Zm'}) \\
&\stackrel{(a)}{\leq} \frac{1}{2} \sum_{1 \leq m \neq m' \leq k} \text{Cov}^2(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) + \frac{1}{2} \sum_{1 \leq m \neq m' \leq k} \text{Cov}^2(\tilde{Z}_{1m}, \tilde{Z}_{1m'}) + \frac{1}{2} \sum_{1 \leq m \neq m' \leq k} \text{Cov}^2(\tilde{Y}_{1m}, \tilde{Y}_{1m'}) \\
&\quad + \frac{1}{2} \sum_{1 \leq m \neq m' \leq k} \alpha_{\text{bf}}^4 (p_{Ym} - p_{Zm})^2 (p_{Ym'} - p_{Zm'})^2 \\
&\stackrel{(b)}{\leq} \frac{3}{2} \alpha_{\text{bf}}^2 b + \frac{\alpha_{\text{bf}}^4}{2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \\
&\stackrel{(c)}{\leq} \frac{5}{2} \alpha_{\text{bf}}^2 b,
\end{aligned}$$

where step (a) uses $ab \leq a^2/2 + b^2/2$, step (b) uses Lemma 15, and step (c) uses the fact that $0 < \alpha_{\text{bf}} < 1$ for any $\alpha > 0$ and $\mathbf{p}_Y^\top \mathbf{p}_Z > 0$. Collecting the bounds for (61) and (62), we finally bound $\mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Z}}_1)^2]$ as

$$\mathbb{E}[(\bar{\mathbf{Y}}_1^\top \bar{\mathbf{Z}}_1)^2] \leq \frac{7}{2} \alpha_{\text{bf}}^2 b + \alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2. \quad (64)$$

Collecting the bounds (59), (60), and (64), we finally bound $\bar{M}_{YZ,2}(P, Q)$ as

$$\bar{M}_{YZ,2}(P, Q) \leq \frac{7}{2} \alpha_{\text{bf}}^2 b + \alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + 2\delta_{\text{bf}}^2 k. \quad (65)$$

Step 2: apply the two moments method. Using the bounds in (56), (57) and (65), we show that condition (37) in the two moments method holds if the separation condition (34) in Lemma 20 is met. Since $\mathbb{E}[U_{n_1, n_2}] = \alpha_{\text{bf}}^2 \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2$, assuming $n_1 \leq n_2$ and $\gamma = \beta$ for simplicity, the condition (37) of Lemma 21 is satisfied if, for all pairs of distributions $P = (P_Y, P_Z) \in \mathcal{P}_{1, \text{multi}}(\rho_{n_1, n_2})$, the following conditions hold:

$$\begin{aligned} \alpha_{\text{bf}}^2 \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 &\geq \sqrt{\frac{\alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2}{\beta n_1}} \quad (\text{from (56), (57)}), \text{ and} \\ \alpha_{\text{bf}}^2 \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 &\geq \sqrt{\frac{\frac{7}{2} \alpha_{\text{bf}}^2 b + \alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \delta_{\text{bf}}^2 k}{\gamma \beta n_1^2}} \quad (\text{from (65)}). \end{aligned}$$

Since $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$ for any nonnegative a and b , the conditions above are satisfied when

$$\alpha_{\text{bf}}^2 \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq \sqrt{\frac{\alpha_{\text{bf}}^3 b^{1/2} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 + \alpha_{\text{bf}}^2 \delta_{\text{bf}} \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2}{\beta n_1}} \quad \text{and} \quad \alpha_{\text{bf}}^2 \|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq \sqrt{\frac{\frac{7}{2} \alpha_{\text{bf}}^2 b + \delta_{\text{bf}}^2 k}{\gamma \beta n_1^2}}.$$

Since $k \geq 2$, the above inequalities are further satisfied when

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2^2 \geq C_1(\beta) \max \left\{ \frac{b^{1/4}}{\alpha_{\text{bf}}^{1/2} n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\}.$$

Therefore, to obtain our desire result, it suffices to show that

$$\max \left\{ \frac{b^{1/4}}{\alpha_{\text{bf}}^{1/2} n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\} \leq 6 \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4}}{n_1^{1/2} \alpha} \right\}, \quad (66)$$

and set $C_u(\gamma, \beta) = 6C_1(\beta)$. Using an indicator function, we separately consider the cases of high privacy ($\alpha \leq 1$) and low privacy ($\alpha > 1$). Then we derive (66) for both cases. First, assuming $0 < \alpha \leq 1$, we have

$$\max \left\{ \frac{b^{1/4}}{\alpha_{\text{bf}}^{1/2} n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\} \leq \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\} \leq \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{6k^{1/4}}{\alpha n_1^{1/2}} \right\}, \quad (67)$$

where the last inequality uses the fact that for $0 \leq \alpha \leq 1$, we have

$$\frac{\delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}}} = \frac{\sqrt{e^{\alpha/2} + 1}}{(e^{\alpha/2} - 1)} < \frac{6}{\alpha}.$$

Next, assuming $\alpha > 1$, we have

$$\begin{aligned} \max \left\{ \frac{b^{1/4}}{\alpha_{\text{bf}}^{1/2} n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\} &= \max \left\{ \frac{b^{1/4}}{n_1^{1/2}} \sqrt{\frac{e^{\alpha/2} + 1}{e^{\alpha/2} - 1}} + \frac{k^{1/4}}{n_1^{1/2}} \frac{1}{\sqrt{e^{\alpha/2} + 1}} \frac{e^{\alpha/2} + 1}{e^{\alpha/2} - 1} \right\} \\ &= \sqrt{\frac{e^{\alpha/2} + 1}{e^{\alpha/2} - 1}} \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4}}{n_1^{1/2} \sqrt{e^{\alpha/2} - 1}} \right\} \\ &\stackrel{(a)}{\leq} 3 \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4}}{n_1^{1/2} \sqrt{e^{\alpha/2} - 1}} \right\} \\ &\stackrel{(b)}{\leq} 3 \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{2k^{1/4}}{n_1^{1/2} \alpha} \right\}, \end{aligned} \quad (68)$$

where step (a) and step (b) use the following inequalities:

$$\sqrt{\frac{e^{\alpha/2} + 1}{e^{\alpha/2} - 1}} \leq 3, \text{ and } \frac{1}{\sqrt{e^{\alpha/2} - 1}} < \frac{2}{\alpha},$$

respectively, both of which holds for $\alpha > 1$. Combining (67) and (68), we obtain the following inequality:

$$\begin{aligned} \max \left\{ \frac{b^{1/4}}{\alpha_{\text{bf}}^{1/2} n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\} &= \mathbb{1}(\alpha \leq 1) \cdot \max \left\{ \frac{b^{1/4}}{\alpha_{\text{bf}}^{1/2} n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\} + \mathbb{1}(\alpha > 1) \cdot \max \left\{ \frac{b^{1/4}}{\alpha_{\text{bf}}^{1/2} n_1^{1/2}}, \frac{k^{1/4} \delta_{\text{bf}}^{1/2}}{\alpha_{\text{bf}} n_1^{1/2}} \right\} \\ &\leq \mathbb{1}(\alpha \leq 1) \cdot 6 \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4}}{n_1^{1/2} \alpha} \right\} + \mathbb{1}(\alpha > 1) \cdot 6 \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4}}{n_1^{1/2} \alpha} \right\} \\ &= 6 \max \left\{ \frac{b^{1/4}}{n_1^{1/2}}, \frac{k^{1/4}}{n_1^{1/2} \alpha} \right\}. \end{aligned}$$

This completes the proof of the upper bound through RAPPOR mechanism.

D.2 Lower bound

The lower bound result follows by combining the lower bound results of Theorem 3.2 in (Lam-Weil et al., 2022) for the one-sample problem under LDP with results from (Chan et al., 2014b; Kim et al., 2022a) for the two-sample problem without privacy constraints. Hence we omit the details.

Appendix E. Proof of Theorem 11

Here we prove the upper bound and lower bound results for density two-sample testing. We start with the upper bound (Appendix E.1) and move onto the lower bound (Appendix E.2).

E.1 Upper bound

The proof of the upper bound proceeds in two steps, where step 1 leverage the result from multinomial testing (Lemma 20), and step 2 utilizes the discretization error analysis (Appendix B.4).

Step 1: separation condition for the probability vectors. We derive a separation condition for the probability vectors \mathbf{p}_Y and \mathbf{p}_Z of κ^d categories, obtained by binning the densities f_Y and f_Z , that ensures our multinomial test distinguishes between them using the samples. By substituting $(\mathbf{p}_Y, \mathbf{p}_Z)$ with $(\mathbf{p}_Y, \mathbf{p}_Z)$ and k with κ^d in the separation condition (34) of Lemma 20, we obtain the following separation condition:

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2 \geq C_u(\gamma, \beta) \left(\frac{\kappa^{d/4}}{(n_1 \alpha^2)^{1/2}} \vee \frac{\max\{\|\mathbf{p}_Y\|_2^{1/2}, \|\mathbf{p}_Z\|_2^{1/2}\}}{n_1^{1/2}} \right). \quad (69)$$

Since we assume that $\|f_Y\|_{\mathbb{L}_\infty} < R$ and $\|f_Z\|_{\mathbb{L}_\infty} < R$ in Definition 10, $\|\mathbf{p}_Y\|_2^2$ is upper bounded as

$$\|\mathbf{p}_Y\|_2^2 = \sum_{m \in [\kappa^d]} \left(\int_{B_m} f_Y(\mathbf{t}) d\mathbf{t} \right)^2 \leq R \sum_{\mathbf{k} \in [\kappa]^d} \left(\int_{B_m} f_Y(\mathbf{t}) d\mathbf{t} \right)^2 = R \kappa^d (\kappa^{-d})^2 = R \kappa^{-d},$$

and in a similar manner, we also have $\|\mathbf{p}_Z\|_2^2 \leq R \kappa^{-d}$. Thus, the condition (69) is implied by:

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2 \geq C_1(R, \gamma, \beta) \left(\frac{\kappa^{d/4}}{(n_1 \alpha^2)^{1/2}} \vee \frac{\kappa^{-d/4}}{n_1^{1/2}} \right). \quad (70)$$

Step 2: separation condition for the densities. Now we find a density separation condition that ensures the probability vector separation condition (70). From the discretization error analysis (27) and (28) in Appendix B.4, both for the Hölder and Besov case, we have the following inequality:

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2 \geq C_2(s, R, d, \gamma, \beta) \kappa^{-d/2} (\|f_Y - f_Z\|_{\mathbb{L}_2} - \kappa^{-s}).$$

Using the above inequality, we derive the following sufficient condition for (70):

$$\|f_Y - f_Z\|_{\mathbb{L}_2} \geq C_3(s, R, d, \gamma, \beta) \max \left\{ \frac{\kappa^{3d/4}}{(n_1 \alpha^2)^{1/2}} + \kappa^{-s}, \frac{\kappa^{d/4}}{n_1^{1/2}} + \kappa^{-s} \right\}.$$

Recalling from (13) that $\kappa \leq (n^{2/(4s+d)} \wedge (n\alpha^2)^{2/(4s+3d)})$, and utilizing

$$\frac{2}{4s+3d} \frac{3d}{4} - \frac{1}{2} = \frac{-2s}{4s+3d} \text{ and } \frac{2}{4s+d} \frac{d}{4} - \frac{1}{2} = \frac{-2s}{4s+d},$$

the condition above is implied by:

$$\|f_Y - f_Z\|_{\mathbb{L}_2} \geq C_3(s, R, d, \gamma, \beta) \max \left\{ (n_1 \alpha^2)^{\frac{-2s}{4s+3d}} + \kappa^{-s}, n_1^{\frac{-2s}{4s+d}} + \kappa^{-s} \right\}.$$

Since $(n^{2/(4s+d)} \wedge (n\alpha^2)^{2/(4s+3d)}) = \kappa + \delta$, where $0 \leq \delta < 1$ and $\kappa \geq 1$, we have $2\kappa \geq (n^{2/(4s+d)} \wedge (n\alpha^2)^{2/(4s+3d)})$. Therefore we have $\kappa^{-s} \leq 2^s (n^{2/(4s+d)} \wedge (n\alpha^2)^{2/(4s+3d)})$. Thus the condition above is implied by:

$$\|f_Y - f_Z\|_{\mathbb{L}_2} \geq C_4(s, R, d, \gamma, \beta) \max \left\{ (n_1 \alpha^2)^{\frac{-2s}{4s+3d}} + n_1^{\frac{-2s}{4s+d}} \right\}.$$

This concludes the proof of the upper bound result of Theorem 11.

E.2 Lower bound

By the argument in Arias-Castro et al. (2018), the lower bound for the minimax testing rate in the one-sample problem, denoted as $\rho_{n_1, \alpha}^*$, also provides a lower bound for the minimax rate in two-sample case, denoted as $\rho_{n_1, n_2, \alpha}^*$. Thus, we focus on bounding $\rho_{n_1, \alpha}^*$. In the one-sample setting, let f_0 denote the known density f_Z , and assume that the data-generating distribution P_Y lies within the following class:

Definition 23 (Smooth distribution classes) Let $\mathcal{P}_{d,s}^{\text{H},1}(R)$ denote the set of distributions P_Y whose density function f_Y satisfies $(f_Y - f_0) \in \mathcal{B}_{d,s}^{\text{H}}(R)$ and $\|f_Y - f_0\|_{\mathbb{L}_2} \leq R$. Similarly, define $\mathcal{P}_{d,s,q}^{\text{B},1}(R)$ by replacing $\mathcal{B}_{d,s}^{\text{H}}(R)$ with $\mathcal{B}_{d,s,q}^{\text{B}}(R)$.

The one-sample density testing problem is defined as follows: Given $\{\tilde{Y}_i\}_{i \in [n_1]}$ generated from P_Y and privatized via α -LDP mechanism Q , decide whether P_Y came from

$$\mathcal{P}_0 = \{P_Y \in \mathcal{P} : f_Y = f_0\} \quad \text{or} \quad \mathcal{P}_1(\rho_{n_1}) = \{P_Y \in \mathcal{P} : \|f_Y - f_0\|_{\mathbb{L}_2} \geq \rho_{n_1}\}. \quad (71)$$

We consider two problems: (i) $\mathcal{P} = \mathcal{P}_{d,s}^{\text{H},1}(R)$, and (ii) $\mathcal{P} = \mathcal{P}_{d,s,q}^{\text{B},1}(R)$, with significant overlap in most of the proof steps.

We derive the lower bound by testing the uniform null hypothesis against a carefully constructed mixture alternative. This alternative balances two objectives regarding the distances between the distributions: achieving sufficient separation in \mathbb{L}_2 distance without the α -LDP constraint, and ensuring indistinguishability in the total variation distance under the α -LDP constraint. The mixture

construction follows the approach in Appendix A.2.2 of [Lam-Weil et al. \(2022\)](#), where uniform densities are perturbed by eigenfunctions of the integral operator created from the α -LDP mechanism. We generalize this construction to multivariate Besov and Hölder smoothness classes.

As defined in Definition 1, let $Q \in \mathcal{Q}_\alpha$ be a non-interactive α -LDP mechanism, and $\{Q_i\}_{i \in [n_1]}$ be its marginals. By Lemma B.1 of [Lam-Weil et al. \(2022\)](#), for each $i \in [n_1]$, there exists a probability measure μ_i such that $Q_i(\cdot | \mathbf{y}_i)$ is absolutely continuous with respect to μ_i for all $\mathbf{y}_i \in [0, 1]^d$. Denote its density with respect to μ_i by $q_i(\tilde{\mathbf{y}}_i | \mathbf{y}_i)$. Using this, we define the private counterpart of a density function on $[0, 1]^d$:

Definition 24 (Privatized density) *Let f be a probability density on $[0, 1]^d$ and Q be an α -LDP mechanism with marginal conditional densities $q_i(\tilde{\mathbf{y}}_i | \mathbf{y})$ with respect to μ_i . The private counterpart of f is defined as*

$$\tilde{f}_i(\tilde{\mathbf{y}}_i) := \int_{[0, 1]^d} q_i(\tilde{\mathbf{y}}_i | \mathbf{y}) f(\mathbf{y}) d\mathbf{y}.$$

Let f_0 denote the uniform density supported on $[0, 1]^d$. Using its private counterpart, we introduce the following integral operator whose kernel resembles the likelihood ratio of the privatized densities:

Definition 25 (Privacy mechanism integral operator) *For $i \in [n_1]$, define $L_i : [0, 1]^d \times \tilde{\mathcal{X}} \rightarrow \mathbb{R}$ as:*

$$L_i(\mathbf{s}, \mathbf{t}) := \frac{q_i(\mathbf{t} | \mathbf{s})}{\sqrt{\tilde{f}_{0,i}(\mathbf{t})}}.$$

Using L_i as a kernel function, we define an integral operator $K_i : \mathbb{L}_2([0, 1]^d) \rightarrow \mathbb{L}_2(\tilde{\mathcal{X}}, d\mu_i)$ as:

$$(K_i f)(\cdot) := \int_{[0, 1]^d} L_i(\mathbf{y}, \cdot) f(\mathbf{y}) d\mathbf{y}.$$

Let K_i^* denote the adjoint of K_i . Finally, by aggregating all the operators for $i \in [n_1]$, we define a symmetric and positive semidefinite integral operator:

$$K := \sum_{i=1}^{n_1} \frac{1}{n_1} K_i^* K_i.$$

For $f \in \mathbb{L}_2([0, 1]^d)$, the operator K yields the expected squared privatized likelihood ratio:

$$\begin{aligned} \langle Kf, f \rangle &= \frac{1}{n_1} \sum_{i=1}^n \int_{\tilde{\mathcal{X}}_i} \frac{(\int_{[0, 1]^d} q_i(\tilde{\mathbf{y}}_i | \mathbf{y}) f(\mathbf{y}) d\mathbf{y})^2}{\tilde{f}_{0,i}(\tilde{\mathbf{y}}_i)} d\mu_i(\tilde{\mathbf{y}}_i) \\ &= \frac{1}{n_1} \sum_{i=1}^n \int_{\tilde{\mathcal{X}}_i} \frac{\tilde{f}_i(\tilde{\mathbf{y}}_i)^2}{\tilde{f}_{0,i}(\tilde{\mathbf{y}}_i)} \frac{dQ_{f_0}}{\tilde{f}_{0,i}(\tilde{\mathbf{y}}_i)} = \frac{1}{n_1} \sum_{i=1}^n \mathbb{E}_{Q_{f_0}} \left[\frac{\tilde{f}_i^2(\tilde{\mathbf{y}}_i)}{\tilde{f}_{0,i}^2(\tilde{\mathbf{y}}_i)} \right]. \end{aligned} \quad (72)$$

We construct each mixture element by adding carefully chosen eigenfunctions of K as bumps to f_0 , ensuring the resulting densities belong to the Hölder or Besov classes. Specifically, we define two orthonormal sets of smoothness-inducing functions—one for Besov smoothness and one for Hölder smoothness. For a positive integer J and $\omega > 0$, we set $\kappa = 2^J$ and determine the number of functions as κ^d . The parameter ω controls the perturbation height. The value of J and ω will be optimized later to achieve a tight lower bound. Both orthonormal sets have cardinality κ^d , the supports of their elements partition $[0, 1]^d$ without overlap, each element has the \mathbb{L}_∞ norm bounded by $\kappa^{d/2}$, and is orthogonal to f_0 . We start by defining the orthonormal set for the Besov ball, consisting of bounded step functions from the multivariate Haar basis.

Definition 26 (Orthonormal set for the Besov ball) Consider the Haar wavelet basis on $[0, 1]^d$ with a prime resolution level $J \in \mathbb{N}$, as defined in Appendix B.3. Set $\kappa = 2^J$ and define $\Lambda(J) := \{0, 1, \dots, (\kappa - 1)\}^d$. Fix a d -dimensional on-off multi-index $\epsilon^* := \{1, 0, 0, \dots, 0\}$. Let \mathbb{B}_B represent the elements with this multi-index, drawn from the mixed tensor product set at the lowest resolution level J . Specifically, we define:

$$\mathbb{B}_B := \{\psi_{J, \ell}^{\epsilon^*}\}_{\ell \in \Lambda(J)}.$$

Since all results ahead hold for any non-zero ϵ^* , the specific choice in Definition 26 is arbitrary. Each element of \mathbb{B}_B is orthogonal to f_0 because its multi-index has at least one non-zero entry. In that dimension of non-zero entry, the scaled and shifted Haar wavelet function $\psi(x) = \mathbb{1}_{[0, 1/2)}(x) - \mathbb{1}_{[1/2, 1)}(x)$, which integrates to zero, contributes to the integral. By Fubini's theorem, this ensures that the entire integral equals zero. Next we define the orthonormal set for Hölder ball:

Definition 27 (Orthonormal set for the Hölder ball) For a smoothness parameter $s > 0$, let $\bar{\varphi} : \mathbb{R}^d \rightarrow \mathbb{R}$ be an infinitely differentiable function supported on $[0, 1]^d$ satisfying $\|\bar{\varphi}\|_{\mathbb{L}_2} = 1$, $\|\bar{\varphi}\|_{\mathbb{L}_\infty} < \infty$, $\int_{[0, 1]^d} \bar{\varphi} = 0$, and $\|\bar{\varphi}^{(s')}\|_\infty < \infty$ for $s' \in \{1, \dots, (\lfloor s \rfloor + 1)\}$. For $J \in \mathbb{N}$, set $\kappa = 2^J$ and define $\Lambda(J) := \{0, 1, \dots, (\kappa - 1)\}^d$, $\varphi_{J, \ell}(\mathbf{y}) := \kappa^{d/2} \bar{\varphi}(\kappa \mathbf{y} - \ell)$ for $\ell \in \Lambda(J)$, and $\mathbb{B}_H := \{\varphi_{J, \ell}\}_{\ell \in \Lambda(J)}$, i.e. scaled and shifted copies of $\bar{\varphi}$.

Now we are ready to define the mixtures, one for Besov and one for Hölder ball.

Definition 28 (Mixture alternatives) For the Hölder ball, let $V = \text{Span}(\{f_0\} \cup \mathbb{B}_H)$. Construct an orthonormal basis of V as $\{f_0\} \cup \{u_t\}_{t=1}^{\kappa^d}$, where each u_t is an eigenfunction of K with eigenvalue λ_t . For each $t \in [\kappa^d]$, since u_t is orthogonal to f_0 , it integrates to zero. Recall that $z_\alpha = e^{2\alpha} - e^{-2\alpha}$ for $\alpha > 0$, and set $\tilde{\lambda}_t = (\lambda_t / z_\alpha^2) \vee \kappa^{-d}$. Fix $\omega > 0$. For any $\boldsymbol{\eta} = (\eta_1, \dots, \eta_{\kappa^d}) \in \{-1, 1\}^{\kappa^d}$, define the perturbed density as:

$$f_{\omega, \kappa}^{\boldsymbol{\eta}} := f_0 + \omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} u_t. \quad (73)$$

Since $u_t \in V$ for each $t \in [\kappa^d]$, we can also write:

$$f_{\omega, \kappa}^{\boldsymbol{\eta}} = f_0 + \omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \sum_{\psi \in \mathbb{B}_H} \theta_\psi(u_t) \psi, \quad (74)$$

where $\theta_\psi(u_t)$ is the Fourier coefficient of u_t with respect to ψ . The mixture for Hölder ball, denoted as ν_ρ^H , is defined as the uniform probability measure over $\{f_{\omega, \kappa}^{\boldsymbol{\eta}} : \boldsymbol{\eta} \in \{-1, 1\}^{\kappa^d}\}$. The mixture for the Besov ball, denoted as ν_ρ^B , is defined through the same procedure as above, with \mathbb{B}_H replaced by \mathbb{B}_B .

We now outline the properties of these mixtures, beginning with their \mathbb{L}_2 separation from the null:

Lemma 29 (\mathbb{L}_2 separation) For any density $f_{\omega, \kappa}^{\boldsymbol{\eta}}$ drawn from ν_ρ , the \mathbb{L}_2 distance from f_0 is bounded as:

$$\|f_{\omega, \kappa}^{\boldsymbol{\eta}} - f_0\|_{\mathbb{L}_2} \geq \kappa^d \omega \sqrt{3/4}. \quad (75)$$

We next establish a series of conditions on κ and ω to ensure that the mixture elements belong to the smooth density class with high probability.

Lemma 30 (Nonnegativity) *For any $f_{\omega,\kappa}^\eta$ drawn from either the distribution ν_ρ^H or ν_ρ^B , we always have $\int_{[0,1]^d} f_{\omega,\kappa}^\eta(\mathbf{y}) d\mathbf{y} = 1$. Also, $f_{\omega,\kappa}^\eta \geq 0$ with a probability greater than $1 - \gamma$ if*

$$\omega \leq \frac{\kappa^{-d}}{\sqrt{2 \log(2\kappa^d/\gamma)}}. \quad (76)$$

Lemma 31 (Hölder class) *With probability at least $1 - \gamma$, a function drawn from ν_ρ^H lies in $\mathcal{B}_{d,s}^H(R)$ if*

$$\omega \leq \frac{R\kappa^{-(d+s)}}{C(s, \bar{\varphi}) \sqrt{2 \log(2\kappa^d/\gamma)}},$$

where $C(s, \bar{\varphi}) := \max_{s' \in \{1, \dots, \lfloor s \rfloor + 1\}} 4 \|\bar{\varphi}^{(s')}\|_{\mathbb{L}_\infty}$.

Lemma 32 (Besov class) *With probability at least $1 - \gamma$, a function drawn from ν_ρ^B lies in $\mathcal{B}_{d,s,q}^B(R)$ if*

$$\omega \leq \frac{R\kappa^{-(s+d)}}{\sqrt{2 \log(\kappa^d/\gamma)}}. \quad (77)$$

Finally, we characterize the condition for κ and ω that ensures any valid private test fails.

Lemma 33 (Indistinguishability) *Under the conditions of Lemmas 30 and 31, if*

$$\omega \leq (nz_\alpha^2)^{-1/2} \left(\frac{\log [1 + 4(1 - 2\gamma - \beta)^2]}{\kappa^d} \right)^{1/4}, \quad (78)$$

then no valid α -LDP test of level γ can distinguish between the uniform density f_0 and the alternatives in (71) (with $\mathcal{P} = \mathcal{P}_{d,s}^{H,1}(R)$) with type II error less than β . The same conclusion holds when $\mathcal{P}_{d,s}^{H,1}(R)$ and the condition in 31 is replaced by $\mathcal{P}_{d,s,q}^{B,1}(R)$ and the condition in 32, respectively.

Conclusion. Recall from Lemma 29 that $\|f_{\omega,\kappa}^\eta - f_0\|_{\mathbb{L}_2} \geq \kappa^d \omega \sqrt{3/4}$. To maximize this separation while satisfying the conditions of Lemmas 30, 31, 32, and 33, we choose ω as the largest possible value:

$$\omega = (nz_\alpha^2)^{-1/2} \left(\frac{\log [1 + 4(1 - 2\gamma - \beta)^2]}{\kappa^d} \right)^{1/4} \wedge \frac{(R \wedge 1) \kappa^{-(d+s)}}{C(s, \bar{\varphi}) \sqrt{2 \log(2\kappa^d/\gamma)}}.$$

Substituting this ω into the \mathbb{L}_2 separation expression, and collecting the constant terms, we can show that any α -LDP test fails if the \mathbb{L}_2 separation between the hypotheses is larger than:

$$C_1(\gamma, \beta, R, s) \left(\frac{\kappa^{3d/4}}{(nz_\alpha^2)^{1/2}} \wedge \frac{\kappa^{-s}}{\sqrt{\log(\kappa^d)}} \right).$$

Taking J as the largest integer such that $2^J \leq C_1(\gamma, \beta, R, s)(n_1 z_\alpha^2)^{2/(4s+3d)}$ and taking $\kappa = 2^J$ leads to the following lower bound for the minimax testing rate:

$$\rho_{n_1, \alpha}^* \geq C_2(\gamma, \beta, R, s) \left[\frac{(nz_\alpha^2)^{-2s/(4s+3d)}}{\sqrt{\log(nz_\alpha^2)}} \right].$$

As justified in [Lam-Weil et al. \(2022\)](#), the α -LDP minimax testing rate is lower bounded by its non-private counterpart. Therefore, combining the result above with the non-private minimax testing rate $n_1^{-2s/(4s+d)}$ of [Arias-Castro et al. \(2018\)](#), we obtain the final result:

$$\rho_{n_1, \alpha}^* \geq C_3(\gamma, \beta, R, s) \left[\frac{(n_1 z_\alpha^2)^{-2s/(4s+3d)}}{\sqrt{\log(n_1 z_\alpha^2)}} \vee n^{-2s/(4s+d)} \right].$$

This completes the proof for the lower bound result in [Theorem 11](#).

E.3 Proofs of the Lemmas in [Appendix E.2](#)

This section provides proofs for the lemmas used in the lower bound proof given in [Appendix E.2](#).

E.3.1 PROOF OF [LEMMA 29](#)

The proof follows the same reasoning as inequality (31), [Lemma B.4](#), and inequality (32) in the [Appendix of Lam-Weil et al. \(2022\)](#), with the only difference being the number of orthonormal basis functions. The key steps are as follows: Since $f_{\omega, \kappa}^\eta - f_0$ is a weighted sum of orthonormal singular vectors of Q , its \mathbb{L}_2 norm $\|f_\eta - f_0\|_{\mathbb{L}_2}$ is bounded below by a quantity involving the sum of the corresponding singular values of Q . Bounding this sum by $z_\alpha^2 = e^{2\alpha} - e^{-2\alpha}$, using the definition of the α -LDP constraint, completes the proof.

E.3.2 PROOF OF [LEMMA 33](#)

As the proof follows [Appendix B.4](#) and [Lemma 3.1](#) in [Lam-Weil et al. \(2022\)](#), we outline only the key steps. The argument applies to both $(\mathcal{P}_{d,s}^{H,1}(R), \nu_\rho^H)$ and $(\mathcal{P}_{d,s,q}^{B,1}(R), \nu_\rho^B)$. For simplicity, denote the distribution class and mixture as \mathcal{P} and ν_ρ , respectively. Let $L_{Q_{\nu_\rho}^n}(\tilde{\mathbf{Y}}_1, \dots, \tilde{\mathbf{Y}}_{n_1})$ be the likelihood ratio between $Q_{\nu_\rho}^n$ and $Q_{f_0}^n$. Define the total variation distance as $\|\mathbb{P}_{Q_{\nu_\rho}^n} - \mathbb{P}_{Q_{f_0}^n}\|_{\text{TV}} := \frac{1}{2} \int |L_{Q_{\nu_\rho}^n} - 1| d\mathbb{P}_{Q_{f_0}^n}$, and chi-square divergence as $\chi^2(Q_{\nu_\rho}^n \| Q_{f_0}^n) := \frac{1}{2} (\mathbb{E}_{Q_{f_0}^n} [L_{Q_{\nu_\rho}^n}(\tilde{\mathbf{Y}}_1, \dots, \tilde{\mathbf{Y}}_{n_1})^2] - 1)^{1/2}$. The minimax type II error is lower bounded as:

$$\begin{aligned} \inf_{\Delta_{\gamma, Q}} \sup_{f \in \mathcal{P}} \mathbb{P}_{Q_f^n} \left(\Delta_{\gamma, Q}(\tilde{\mathbf{Y}}_1, \dots, \tilde{\mathbf{Y}}_{n_1}) = 0 \right) &\geq \inf_{\Delta_{\gamma, Q}} \mathbb{P}_{Q_{\nu_\rho}^n} \left(\Delta_{\gamma, Q}(\tilde{\mathbf{Y}}_1, \dots, \tilde{\mathbf{Y}}_{n_1}) = 0 \right) - \gamma \\ &\geq 1 - 2\gamma - \|Q_{\nu_\rho}^n - Q_{f_0}^n\|_{\text{TV}} \\ &\geq 1 - 2\gamma - \frac{1}{2} \sqrt{\chi^2(Q_{\nu_\rho}^n \| Q_{f_0}^n)}, \end{aligned} \quad (79)$$

where the last inequality utilizes the inequality between chi-square divergence and total variation distance $\chi^2(Q_{\nu_\rho}^n \| Q_{f_0}^n) \geq 4\|Q_{\nu_\rho}^n - Q_{f_0}^n\|_{\text{TV}}^2$ (see, for example, [Lemma B.2](#) of [Lam-Weil et al. \(2022\)](#)). Thus it suffices to verify that the condition in (78) implies that the last right-hand side term in (79) is lower bounded by β . Thus the proof is completed by bounding the chi-square divergence as $\chi^2(Q_{\nu_\rho}^n \| Q_{f_0}^n) \leq 1 + \exp(n_1^2 \omega^4 z_\alpha^4 \kappa^d)$, by utilizing the equation (72) and the definition in (73) involving the orthonormal singular vectors of Q .

E.3.3 PROOF OF [LEMMA 30](#)

We begin by verifying that $f_{\omega, \kappa}^\eta$ integrates to one. Using the expansion of $f_{\omega, \kappa}^\eta$ in terms of the singular vectors from (73), which are orthogonal to f_0 for any η , the integral is computed as follows:

$$\int_{[0,1]^d} f_{\omega, \kappa}^\eta(\mathbf{y}) d\mathbf{y} = \langle f_{\omega, \kappa}^\eta, f_0 \rangle_{\mathbb{L}_2} = \langle f_0, f_0 \rangle_{\mathbb{L}_2} + \sum_{t=1}^{\kappa^d} \omega \eta_t \tilde{\lambda}_t^{-1/2} \langle f_0, u_t \rangle = 1,$$

where the final equality follows from the orthogonality between f_0 and the u_t 's. Note that this equation holds for both ν_ρ^H and ν_ρ^B .

Next, we move on to the nonnegativeness property. This proof follows a similar approach to that of Lemma B.5 in [Lam-Weil et al. \(2022\)](#), with an adjustment for the change of basis. Since the proof is the same for ν_ρ^H and ν_ρ^B , we state it only for ν_ρ^B . For each $\psi \in \mathbb{B}_B$, define an event A_ψ as:

$$A_\psi := \left[\left| \omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_\psi(u_t) \right| \geq \kappa^{-d/2} \right],$$

which belongs to σ -algebra generated by $\boldsymbol{\eta}$. Since the sum of independent scaled Rademacher variables inside A_ψ is sub-Gaussian with variance proxy $\omega^2 \tilde{\lambda}_t^{-1} \sum_{t=1}^{\kappa^d} \theta_\psi^2(u_t)$, by the union bound, we have:

$$\begin{aligned} \mathbb{P}_{\nu_\rho^B} \left(\bigcup_{\psi \in \mathbb{B}_B} A_\psi \right) &\leq \sum_{\psi \in \mathbb{B}_B} 2 \exp \left(\frac{-\kappa^{-d}}{2\omega^2 \tilde{\lambda}_t^{-1} \sum_{t=1}^{\kappa^d} \theta_\psi^2(u_t)} \right) \\ &\stackrel{(a)}{\leq} 2 \sum_{\psi \in \mathbb{B}_B} \exp \left(\frac{-\kappa^{-2d}}{2\omega^2 \sum_{t=1}^{\kappa^d} \theta_\psi^2(u_t)} \right) \\ &\stackrel{(b)}{=} 2\kappa^d \exp \left(\frac{-\kappa^{-2d}}{2\omega^2} \right), \end{aligned} \tag{80}$$

where step (a) uses $\tilde{\lambda}_t \leq \kappa^{-d}$ and step (b) uses the following equality that utilizes the fact that u_t 's are orthonormal eigenbasis of V and the orthonormality of \mathbb{B}_B :

$$\sum_{t=1}^{\kappa^d} \theta_\psi^2(u_t) = \langle \psi, \sum_{t=1}^{\kappa^d} \langle \psi, u_t \rangle_{\mathbb{L}_2} u_t \rangle_{\mathbb{L}_2} = \langle \psi, \psi \rangle_{\mathbb{L}_2} = 1.$$

Utilizing the expansion in (74), the expression $f_{\omega, \kappa}^\eta - f_0$ can be represented as:

$$f_{\omega, \kappa}^\eta - f_0 = \sum_{\psi \in \mathbb{B}_B} \left(\omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_\psi(u_t) \right) \psi.$$

Each $\psi \in \mathbb{B}_B$ has mutually disjoint support, and the maximum magnitude of ψ across its domain is $\kappa^{d/2}$. Therefore, $f_{\omega, \kappa}^\eta$ is nonnegative if the event $\bigcap_{\psi \in \mathbb{B}_B} A_\psi^c$ holds, which occurs with a probability exceeding $1 - 2\kappa^d \exp(-\kappa^{-2d}/2\omega^2)$. Finally, solving $\gamma \leq 2\kappa^d \exp(-\kappa^{-2d}/2\omega^2)$ for ω confirms the condition (76).

E.3.4 PROOF OF LEMMA 31

For a $f_{\omega, \kappa}^\eta$ drawn from ν_ρ^H , we utilize the basis expansion form in (74) using \mathbb{B}_H (defined in (27)):

$$f_{\omega, \kappa}^\eta = f_0 + \omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \sum_{\varphi \in \mathbb{B}_H} \theta_\varphi(u_t) \varphi.$$

We begin by identifying a sufficient condition for the Lipschitz property and then proceed to establish a sufficient condition for bounded derivatives.

Lipschitz property. Fix arbitrary $\mathbf{y}_1, \mathbf{y}_2 \in [0, 1]^d$. Since each $\varphi \in \mathbb{B}_H$ has disjoint support, we can specify the element of \mathbb{B}_H whose $[s]$ -derivative has support containing \mathbf{y}_1 and \mathbf{y}_2 , say $\varphi_{\mathbf{y}_1}$ and $\varphi_{\mathbf{y}_2}$. Then using the fact that $\varphi_{\mathbf{y}_1}^{([s])}(\mathbf{y}_2) - \varphi_{\mathbf{y}_2}^{([s])}(\mathbf{y}_1) = 0$, we can write the difference of the $[s]$ -derivative of $f_{\omega, \kappa}^{\boldsymbol{\eta}}$ as

$$\begin{aligned} |(f_{\omega, \kappa}^{\boldsymbol{\eta}})^{([s])}(\mathbf{y}_1) - (f_{\omega, \kappa}^{\boldsymbol{\eta}})^{([s])}(\mathbf{y}_2)| &= \omega \left| \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \{ \theta_{\varphi_{\mathbf{y}_1}}(u_t) \varphi_{\mathbf{y}_1}^{([s])}(\mathbf{y}_1) - \theta_{\varphi_{\mathbf{y}_2}}(u_t) \varphi_{\mathbf{y}_2}^{([s])}(\mathbf{y}_2) \} \right| \\ &\leq \omega \left| \sum_{t=1}^{\kappa^d} \theta_{\varphi_{\mathbf{y}_1}}(u_t) \eta_t \tilde{\lambda}_t^{-1/2} \left(\varphi_{\mathbf{y}_1}^{([s])}(\mathbf{y}_1) - \varphi_{\mathbf{y}_1}^{([s])}(\mathbf{y}_2) \right) \right| \end{aligned} \quad (81)$$

$$+ \omega \left| \sum_{t=1}^{\kappa^d} \theta_{\varphi_{\mathbf{y}_2}}(u_t) \eta_t \tilde{\lambda}_t^{-1/2} \left(\varphi_{\mathbf{y}_2}^{([s])}(\mathbf{y}_1) - \varphi_{\mathbf{y}_2}^{([s])}(\mathbf{y}_2) \right) \right|, \quad (82)$$

where the last inequality uses the triangle inequality. We separately bound the terms (81) and (82), starting with (81). For $\mathbf{y} \in [0, 1]^d$, recall from (27) that $\varphi_{\mathbf{y}_1}$ has the form $\kappa^{d/2} \varphi(\kappa \mathbf{y} - \boldsymbol{\ell})$. By the chain rule, its $[s]$ -mixed partial derivatives are written as

$$\varphi_{\mathbf{y}_1}^{([s])}(\mathbf{y}) = \kappa^{(d/2)+[s]} \bar{\varphi}^{([s])}(\kappa \mathbf{y} - \boldsymbol{\ell}). \quad (83)$$

Using the above equality, the mean value theorem and the boundedness of all partial mixed derivatives specified in Definition 27, the term (81) is upper bounded by

$$\omega \kappa^{(d/2)+[s]} \left| \sum_{t=1}^{\kappa^d} \theta_{\varphi_{\mathbf{y}_1}}(u_t) \eta_t \tilde{\lambda}_t^{-1/2} \right| C_1(s, \bar{\varphi}) \kappa \|\mathbf{y}_1 - \mathbf{y}_2\|,$$

where $C_1(s, \bar{\varphi}) := (2\|\bar{\varphi}^{([s])}\|_{L_\infty} \wedge \|\bar{\varphi}^{([s+1])}\|_{L_\infty})$. Bounding the term (82) using the same argument, we obtain the following bound:

$$\begin{aligned} |(f_{\omega, \kappa}^{\boldsymbol{\eta}})^{([s])}(\mathbf{y}_1) - (f_{\omega, \kappa}^{\boldsymbol{\eta}})^{([s])}(\mathbf{y}_2)| &\leq \omega \kappa^{(d/2)+[s]} \left(\left| \sum_{t=1}^{\kappa^d} \theta_{\varphi_{\mathbf{y}_1}}(u_t) \eta_t \tilde{\lambda}_t^{-1/2} \right| + \left| \sum_{t=1}^{\kappa^d} \theta_{\varphi_{\mathbf{y}_2}}(u_t) \eta_t \tilde{\lambda}_t^{-1/2} \right| \right) C_1(s, \bar{\varphi}) \kappa \|\mathbf{y}_1 - \mathbf{y}_2\|. \end{aligned} \quad (84)$$

For each $\varphi \in \mathbb{B}_H$, let A_φ denote an event within the σ -algebra generated by $\boldsymbol{\eta}$, defined as follows:

$$A_\varphi := \left[\left| \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_\varphi(u_t) \right| \geq \sqrt{2\kappa^d \log(2\kappa^d/\gamma)} \right]. \quad (85)$$

Since the sum of independent scaled Rademacher variables inside A_φ is sub-Gaussian with variance proxy $\tilde{\lambda}_t^{-1} \sum_{t=1}^{\kappa^d} \theta_\varphi^2(u_t)$, by the same technique as (80), we have $\mathbb{P}_{\nu_\rho^H}(\bigcap_{\varphi \in \mathbb{B}_H} A_\varphi^c) \geq 1 - \gamma/2$. Under the event $\bigcap_{\varphi \in \mathbb{B}_H} A_\varphi^c$, the bound (84) continues to

$$\begin{aligned} |(f_{\omega, \kappa}^{\boldsymbol{\eta}})^{([s])}(\mathbf{y}_1) - (f_{\omega, \kappa}^{\boldsymbol{\eta}})^{([s])}(\mathbf{y}_2)| &\stackrel{(a)}{\leq} \omega \kappa^{(d/2)+[s]} 2\sqrt{2\kappa^d \log(2\kappa^d/\gamma)} C_1(s, \bar{\varphi}) \kappa \|\mathbf{y}_1 - \mathbf{y}_2\| \\ &\stackrel{(b)}{\leq} \omega \kappa^{(d/2)+[s]} \sqrt{2\kappa^d \log(2\kappa^d/\gamma)} C_2(s, \bar{\varphi}) (1 \wedge \kappa \|\mathbf{y}_1 - \mathbf{y}_2\|) \\ &\stackrel{(c)}{\leq} \omega \kappa^{(d/2)+[s]} \sqrt{2\kappa^d \log(2\kappa^d/\gamma)} C_2(s, \bar{\varphi}) \kappa^{(s-[s])} \|\mathbf{y}_1 - \mathbf{y}_2\|^{s-[s]} \\ &= \omega \kappa^{d+s} \sqrt{2 \log(2\kappa^d/\gamma)} C_2(s, \bar{\varphi}) \|\mathbf{y}_1 - \mathbf{y}_2\|^{s-[s]}, \end{aligned} \quad (86)$$

where $C_2(s, \bar{\varphi}) := (4\|\bar{\varphi}^{(\lfloor s \rfloor)}\|_{L_\infty} \vee 2\|\bar{\varphi}^{(\lfloor s+1 \rfloor)}\|_{L_\infty})$, step (a) uses the event $\bigcap_{\varphi \in \mathbb{B}_H} A_\varphi^c$, step (b) uses the basic inequality that

$$(a \times 1) \wedge (b \times c) \leq (a \vee b) \times (1 \wedge c) \quad \text{for } a, b, c \geq 0,$$

and step (c) holds by the following fact: When $(1 \wedge c) \leq 1$ and $0 \leq s - \lfloor s \rfloor < 1$, it holds that

$$(1 \wedge c) \leq (1 \wedge c)^{s - \lfloor s \rfloor} \leq c^{s - \lfloor s \rfloor}.$$

Starting from inequality (86), we solve for ω in $\omega \kappa^{d+s} \sqrt{2 \log(2\kappa^d/\gamma)} C_2(s, \bar{\varphi}) \leq R$. Solving for ω , we obtain

$$\omega \leq \frac{R \kappa^{-d+s}}{C_2(s, \bar{\varphi}) \sqrt{2 \log(2\kappa^d/\gamma)}}. \quad (87)$$

Therefore, if ω satisfies this inequality, then with probability at least $1 - \gamma/2$, the Lipschitz property from Definition 8 holds for a function $f_{\omega, \kappa}^\eta$ drawn from ν_ρ^H .

Bounded derivatives property. Fix an integer $s' \in [\lfloor s \rfloor]$ and an arbitrary $\mathbf{y} \in [0, 1]^d$. Since each $\varphi \in \mathbb{B}_H$ has disjoint support, we can specify the element of \mathbb{B}_H whose s' -derivative has support containing \mathbf{y} , say $\varphi_{\mathbf{y}}$. Utilizing (83) to the basis expansion of $f_{\omega, \kappa}^\eta$ with respect to \mathbb{B}_H , we compute the s' -derivative of $f_{\omega, \kappa}^\eta$ as:

$$(f_{\omega, \kappa}^\eta)^{(s')}(\mathbf{y}) = \omega \kappa^{(d/2)+s'} \left(\sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_{\varphi_{\mathbf{y}}}(u_t) \right) \bar{\varphi}^{(s')}(\kappa \mathbf{y} - \ell).$$

Under the event $\bigcap_{\varphi \in \mathbb{B}_H} A_\varphi^c$ as defined in (85), the derivative above is bounded by $\kappa^{d+s'} \|\varphi^{(s')}\|_{\mathbb{L}_\infty} \omega \sqrt{2 \log(2\kappa^d/\gamma)}$. We solve for ω in $\kappa^{d+s'} \|\varphi^{(s')}\|_{\mathbb{L}_\infty} \omega \sqrt{2 \log(2\kappa^d/\gamma)} \leq R$. Solving for ω , we obtain

$$\omega \leq \frac{R \kappa^{-(d+s')}}{\|\varphi^{(s')}\|_{\mathbb{L}_\infty} \sqrt{2 \log(2\kappa^d/\gamma)}}. \quad (88)$$

Therefore, if ω satisfies this inequality, then with probability at least $1 - \gamma/2$, the s' -derivative of $f_{\omega, \kappa}^\eta$ drawn from ν_ρ^H is bounded by R . Combining the above condition for $s' \in [\lfloor s \rfloor]$, we conclude that if

$$\omega \leq \frac{R \kappa^{-(d+s)}}{C_3(s, \bar{\varphi}) \sqrt{2 \log(2\kappa^d/\gamma)}}, \quad (89)$$

where $C_3(s, \bar{\varphi}) := \max_{s' \in \{1, \dots, \lfloor s \rfloor\}} \|\bar{\varphi}^{(s')}\|_{\mathbb{L}_\infty}$, then with probability at least $1 - \gamma/2$, the bounded derivatives property of Definition 8 holds for a function $f_{\omega, \kappa}^\eta$ drawn from ν_ρ^H .

Conclusion. Combining the conditions (87) and (89), we conclude that if

$$\omega \leq \frac{R \kappa^{-(d+s)}}{C_4(s, \bar{\varphi}) \sqrt{2 \log(2\kappa^d/\gamma)}}, \quad (90)$$

where $C_4(s, \bar{\varphi}) := \max_{s' \in \{1, \dots, \lfloor s \rfloor + 1\}} 4\|\bar{\varphi}^{(s')}\|_{\mathbb{L}_\infty}$, then with probability at least $1 - \gamma$, for a function $f_{\omega, \kappa}^\eta$ drawn from ν_ρ^H , we have $f_{\omega, \kappa}^\eta \in \mathcal{B}_{d, s}^H(R)$.

E.3.5 PROOF OF LEMMA 32

Recall from Definition 26 that \mathbb{B}_B is a subset of the multivariate Haar wavelet basis defined in Appendix B.3, specifically the mixed tensor products at resolution level J with on-off multi-index $\epsilon^* = \{1, 0, 0, \dots, 0\}$. Using the orthonormality of the Haar wavelet basis and the basis expansion of $f_{\omega, \kappa}^\eta$ in (74), the Fourier coefficient of $f_{\omega, \kappa}^\eta - f_0$ with respect to a Haar wavelet ψ is calculated as

$$\theta_\psi(f_{\omega, \kappa}^\eta - f_0) = \begin{cases} \omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_\psi(u_t), & \text{if } \psi \in \mathbb{B}_B, \\ 0, & \text{otherwise.} \end{cases} \quad (91)$$

due to the orthonormality of Haar wavelets (see Appendix B.3 for details). Recall that we set $\kappa = 2^J$. We proceed the analysis for the case of $1 \leq q < \infty$ and $q = \infty$.

The case of $1 \leq q < \infty$. The q th power of the Besov seminorm (Definition 9) of $f_{\omega, \kappa}^\eta - f_0$ is written as:

$$\begin{aligned} \|f_{\omega, \kappa}^\eta - f_0\|_{s, 2, q}^q &= \kappa^{sq} \left(\sum_{\psi \in \mathbb{B}_B} \theta_\psi^2(f_{\omega, \kappa}^\eta - f_0) \right)^{q/2} \\ &= \kappa^{sq} \left\{ \sum_{\psi \in \mathbb{B}_B} \left(\omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_\psi(u_t) \right)^2 \right\}^{q/2}. \end{aligned} \quad (92)$$

For each $\psi \in \mathbb{B}_B$, let E_ψ denote an event within the σ -algebra generated by η , defined as follows:

$$E_\psi := \left[\left| \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_\psi(u_t) \right| \geq \sqrt{2\kappa^d \log(\kappa^d/\gamma)} \right].$$

Since the sum of independent scaled Rademacher variables is sub-Gaussian with variance proxy $\tilde{\lambda}_t^{-1} \sum_{t=1}^{\kappa^d} \theta_\psi^2(u_t)$, by the same technique as (80), we have $\mathbb{P}_{\nu_\rho^B}(\bigcap_{\psi \in \mathbb{B}_B} E_\psi^c) \geq 1 - \gamma$. Under the event $\bigcap_{\psi \in \mathbb{B}_B} E_\psi^c$, and noting that the cardinality of \mathbb{B}_B is κ^d , we bound the quantity in (92) as:

$$\|f_{\omega, \kappa}^\eta - f_0\|_{s, 2, q}^q \leq \kappa^{sq} \left(\sum_{\psi \in \mathbb{B}_B} \omega^2 2\kappa^d \log(\kappa^d/\gamma) \right)^{q/2} = \kappa^{(s+d)q} \omega^q (2 \log(\kappa^d/\gamma))^{q/2},$$

which is equivalent to

$$\|f_{\omega, \kappa}^\eta - f_0\|_{s, 2, q} \leq \kappa^{(s+d)} \omega \sqrt{2 \log(\kappa^d/\gamma)}.$$

The case of $q = \infty$. Under the event $\bigcap_{\psi \in \mathbb{B}_B} E_\psi^c$, and noting that the cardinality of \mathbb{B}_B is κ^d , the Besov seminorm (Definition 9) of $(f_{\omega, \kappa}^\eta - f_0)$ is bounded as:

$$\begin{aligned} \|f_{\omega, \kappa}^\eta - f_0\|_{s, 2, q} &= \kappa^s \left(\sum_{\psi \in \mathbb{B}_B} \theta_\psi^2(f_{\omega, \kappa}^\eta - f_0) \right)^{1/2} \\ &= \kappa^s \left\{ \sum_{\psi \in \mathbb{B}_B} \left(\omega \sum_{t=1}^{\kappa^d} \eta_t \tilde{\lambda}_t^{-1/2} \theta_\psi(u_t) \right)^2 \right\}^{1/2} \\ &\leq \kappa^{(s+d)} \omega \sqrt{2 \log(\kappa^d/\gamma)}. \end{aligned} \quad (93)$$

Conclusion. From (92) and (93), we have $\|f_{\omega,\kappa}^\eta - f_0\|_{s,2,q} \leq \kappa^{(s+d)}\omega\sqrt{2\log(\kappa^d/\gamma)}$ for all possible values of q . We solve for ω in $\kappa^{(s+d)}\omega\sqrt{2\log(\kappa^d/\gamma)} \leq R$. Solving for ω , we obtain

$$\omega \leq \frac{R\kappa^{-(s+d)}}{\sqrt{2\log(\kappa^d/\gamma)}}. \quad (94)$$

Therefore, if ω satisfies this inequality, then with probability at least $1 - \gamma$, a function $f_{\omega,\kappa}^\eta$ drawn from ν_ρ^B lies in the Besov ball $\mathcal{B}_{d,s,q}^B(R)$.

Appendix F. Proof of Theorem 12

This section proves the upper bound for the adaptive density test presented in Theorem 12. For any $(P_Y, P_Z) \in \mathcal{P}_0$, the type I error is controlled through the union bound:

$$\mathbb{E}[\Delta_\gamma^{\text{adapt}}] \leq \sum_{t \in [\mathcal{N}]} \mathbb{E}[\Delta_{\gamma/\mathcal{N}}^t] \leq \mathcal{N} \cdot \frac{\gamma}{\mathcal{N}} = \gamma.$$

Now we move onto the type II error guarantee, following the proof strategy of Proposition 7.1 of Kim et al. (2022a) for non-private adaptive two-sample test. Recall from (14) that the number of tests is defined as

$$\mathcal{N} = \left\lceil \frac{2}{d} \log_2 \left(\frac{n_1}{\log \log n_1} \right) \wedge \frac{2}{3d} \log_2 \left(\frac{n_1 \alpha^2}{(\log n_1)^2 \log \log n_1} \right) \right\rceil.$$

Since $2/d > 2/(d+4s)$ and $2/3d > 2/(3d+4s)$ for $s > 0$, there exists an integer $t^* \in [\mathcal{N}]$ such that

$$\kappa^* := 2^{t^*} \leq 2 \left[\left(\frac{n_1}{\log \log n_1} \right)^{2/(4s+d)} \wedge \left(\frac{n_1 \alpha^2}{(\log n_1)^2 \log \log n_1} \right)^{2/(4s+3d)} \right] \leq 2^{t^*+1}. \quad (95)$$

Since the type II error of $\Delta_\gamma^{\text{adapt}}$ is upper bounded by that of a single inner test $\Delta_{\gamma/\mathcal{N}}^t$, it suffices to control the type II error of $\Delta_{\gamma/\mathcal{N}}^{t^*}$. The dependence of the two-moments method on the significance level as $1/(\gamma/\mathcal{N})$ significantly affects the resulting rate. Therefore, we use the following improved two-moments method (Lemma 34), which has a logarithmic dependence on $1/(\gamma/\mathcal{N})$, at the cost of assuming $n_1 \asymp n_2$:

Lemma 34 (Lemma C.1 of Kim et al. (2022a)) *For $0 < \gamma < 1/e$, suppose that there is a sufficiently large constant $C > 0$ such that*

$$\mathbb{E}[U_{n_1, n_2}] \geq C \max \left\{ \sqrt{\frac{M_{Y,1}(P, Q)}{\beta n_1}}, \sqrt{\frac{M_{Z,1}(P, Q)}{\beta n_2}}, \sqrt{\frac{M_{YZ,2}(P, Q)}{\beta}} \log \left(\frac{1}{\gamma} \right) \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \right\}$$

for all pairs of distributions $P = (P_Y, P_Z) \in \mathcal{P}_1(\rho_{n_1, n_2})$. Then under the assumptions that $n_1 \asymp n_2$, the type II error of the permutation test over $\mathcal{P}_1(\rho_{n_1, n_2})$ is uniformly bounded by β .

Applying Lemma 34 on the upper bounds of moments in (44) and (48) (for LapU or DiscLapU) or the bounds in (56), (57) and (65) (for RAPPOR), one can verify that the type II error of the two-sample multinomial test with k categories is at most β if

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2 \geq C_1(\beta) \sqrt{\log(\mathcal{N}/\gamma)} \left(\frac{k^{1/4}}{(n_1 \alpha^2)^{1/2}} \vee \frac{\max\{\|\mathbf{p}_Y\|_2^{1/2}, \|\mathbf{p}_Z\|_2^{1/2}\}}{n_1^{1/2}} \right).$$

Note that since $\Delta_{\gamma/\mathcal{N}}^{t^*}$ is (α/\mathcal{N}) -LDP, the scaling factor for Laplace noise is multiplied by \mathcal{N} . As in Appendix E.1, we substitute $(\mathbf{p}_Y, \mathbf{p}_Z)$ with $(\mathbf{p}_Y, \mathbf{p}_Z)$ and k with $(\kappa^*)^d$ and utilize the discretization error analysis result:

$$\|\mathbf{p}_Y - \mathbf{p}_Z\|_2 \geq C_2(s, R, d, \gamma, \beta) (\kappa^*)^{-d/2} (\|f_Y - f_Z\|_{\mathbb{L}_2} - (\kappa^*)^{-s}).$$

As a result, type II error of $\Delta_{\gamma/\mathcal{N}}^{t^*}$ is at most β if:

$$\|f_Y - f_Z\|_{\mathbb{L}_2} \geq C_3(s, R, d, \gamma, \beta) \max \left\{ \mathcal{N} \sqrt{\log \mathcal{N}} \frac{(\kappa^*)^{3d/4}}{(n_1 \alpha^2)^{1/2}} + (\kappa^*)^{-s}, \sqrt{\log \mathcal{N}} \frac{(\kappa^*)^{d/4}}{n_1^{1/2}} + (\kappa^*)^{-s} \right\}.$$

Since we have $\mathcal{N} \leq C(d) \log n_1$ for any $d \geq 1$ and $n_1 \geq e^e$, the condition above is satisfied if $\|f_Y - f_Z\|_{\mathbb{L}_2}$ is larger than:

$$C_4(s, R, d, \gamma, \beta) \max \left\{ (\kappa^*)^{3d/4} \left(\frac{(\log n_1)^2 \log \log n_1}{n_1 \alpha^2} \right)^{1/2} + (\kappa^*)^{-s}, (\kappa^*)^{d/4} \left(\frac{\log \log n_1}{n_1} \right)^{1/2} + (\kappa^*)^{-s} \right\}.$$

Applying the left part of inequality (95), the condition above is implied by:

$$\|f_Y - f_Z\|_{\mathbb{L}_2} \geq C_5(s, R, d, \gamma, \beta) \left\{ (\kappa^*)^{-s} + \left(\frac{n_1 \alpha^2}{(\log n_1)^2 \log \log n_1} \right)^{\frac{-2s}{4s+3d}} + \left(\frac{n_1}{\log \log n_1} \right)^{\frac{-2s}{4s+d}} \right\}.$$

Applying the right part of inequality (95), the condition above is implied by:

$$\|f_Y - f_Z\|_{\mathbb{L}_2} \geq C_6(s, R, d, \gamma, \beta) \left\{ \left(\frac{n_1 \alpha^2}{(\log n_1)^2 \log \log n_1} \right)^{\frac{-2s}{4s+3d}} \vee \left(\frac{n_1}{\log \log n_1} \right)^{\frac{-2s}{4s+d}} \right\}.$$

This completes the proof of Theorem 12.

Appendix G. Supplementary information for the numerical results

This section describes the baseline methods for evaluating our proposed methods in Section 5 and presents additional numerical results.

G.1 Baseline methods

To the best of our knowledge, no methods exist for nonparametric two-sample testing under LDP. For a reliable evaluation, we adapt two one-sample testing methods from Gaboardi and Rogers (2018) to the two-sample setting for our simulations. The first combines the generalized randomized response (GenRR) with a two-sample chi-square statistic (Chi), while the second integrates the RAPPOR mechanism with a projected two-sample chi-square statistic (ProjChi), both calibrated using asymptotic chi-square null distributions. Lastly, we adopt the combination of the generalized randomized response mechanism and ℓ_2 -type U-statistic (9) as the third baseline method.

Baseline method 1: GenRR+Chi. We first introduce the generalized randomized response mechanism.

Definition 35 (Generalized randomized response mechanism for multinomial data: GenRR)

Consider a pooled raw multinomial sample $\{X_i\}_{i \in [n]}$ with k categories. Fix the privacy level $\alpha > 0$. Each data owner perturbs their data point, resulting in the privatized sample $\{\tilde{X}_i\}_{i \in [n]} = \{\tilde{Y}_i\}_{i \in [n_1]} \cup \{\tilde{Z}_i\}_{i \in [n_2]}$; a multinomial random sample where the i th sample \tilde{X}_i represents a modified version of

the original i th sample X_i , with category change from $m \in [k]$ to $\tilde{m} \in [k]$ occurring according to the following conditional probabilities:

$$\mathbb{P}(\tilde{X}_i = \tilde{m} \mid X_i = m) = \frac{\exp(\alpha \mathbb{1}(\tilde{m} = m))}{\exp(\alpha) + k - 1}. \quad (96)$$

Gaboardi and Rogers (2018) invoke the exponential mechanism argument to prove the privacy guarantee. We here present an alternative proof which directly uses the definition of LDP (Definition 1).

Lemma 36 *The random variables $\{\tilde{X}_i\}_{i \in [n]}$ generated by **GenRR** are α -LDP views of $\{X_i\}_{i \in [n]}$.*

Proof

$$\begin{aligned} \sup_{\tilde{m} \in [k], (m, m') \in [k]^2} \frac{\mathbb{P}(\tilde{X}_i = \tilde{m} \mid X_i = m)}{\mathbb{P}(\tilde{X}_i = \tilde{m} \mid X_i = m')} &= \sup_{\tilde{m} \in [k], (m, m') \in [k]^2} \frac{\exp(\alpha \mathbb{1}(\tilde{m} = m)) / (\exp(\alpha) + k - 1)}{\exp(\alpha \mathbb{1}(\tilde{m} = m')) / (\exp(\alpha) + k - 1)} \\ &= \sup_{\tilde{m} \in [k], (m, m') \in [k]^2} \exp(\alpha (\mathbb{1}(\tilde{m} = m) - \mathbb{1}(\tilde{m} = m'))) \\ &= \exp(\alpha). \end{aligned}$$

■

Now we define a chi-square statistic, referred to as **Chi**, which takes the outputs of **GenRR** as inputs:

$$T_{n_1, n_2} := \left(\frac{1}{n_1} + \frac{1}{n_2} \right)^{-1} (\hat{\boldsymbol{\mu}}_{\tilde{Y}} - \hat{\boldsymbol{\mu}}_{\tilde{Z}})^\top \text{diag}(\hat{\mathbf{p}})^{-1} (\hat{\boldsymbol{\mu}}_{\tilde{Y}} - \hat{\boldsymbol{\mu}}_{\tilde{Z}}), \quad (97)$$

where for $m \in [k]$, the m th elements of $\hat{\boldsymbol{\mu}}_{\tilde{Y}}, \hat{\boldsymbol{\mu}}_{\tilde{Z}}$ and $\hat{\mathbf{p}}$ are defined as:

$$\begin{aligned} \hat{\mu}_{\tilde{Y}m} &:= \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{1}(\tilde{Y}_i = m), \quad \hat{\mu}_{\tilde{Z}m} := \frac{1}{n_2} \sum_{j=1}^{n_2} \mathbb{1}(\tilde{Z}_j = m), \text{ and} \\ \hat{p}_m &:= \frac{\sum_{i=1}^{n_1} \mathbb{1}(\tilde{Y}_i = m) + \sum_{j=1}^{n_2} \mathbb{1}(\tilde{Z}_j = m)}{n_1 + n_2}, \end{aligned}$$

respectively. The test statistic T_{n_1, n_2} (97) can be calibrated using their asymptotic null distributions:

Lemma 37 (Asymptotic null distribution of GenRR+Chi) *Fix the number of categories k and the multinomial probability vectors \mathbf{p}_Y and \mathbf{p}_Z , each contained in the interior of the set $\Omega := \{\mathbf{p} \in \mathbb{R}^k : p_m > 0, \sum_{m=1}^{k-1} p_m \leq 1, p_k = 1 - \sum_{m=1}^{k-1} p_m\}$. Fix the privacy level $\alpha > 0$. Under the null hypothesis $\mathbf{p}_Y = \mathbf{p}_Z$, for each pair of sample sizes (n_1, n_2) , compute the statistic T_{n_1, n_2} based on the **GenRR** α -LDP views $\{\tilde{Y}_i\}_{i \in [n_1]}$ and $\{\tilde{Z}_j\}_{j \in [n_2]}$, generated from \mathbf{p}_Y and \mathbf{p}_Z , respectively. Then, as $n_1, n_2 \rightarrow \infty$, we have $T_{n_1, n_2} \xrightarrow{d} \chi_{k-1}^2$.*

The proof given in Appendix G.4.1 follows directly from Lindberg's central limit theorem (CLT), as outlined in Proposition 2.27 of van der Vaart (1998), and the fact that a chi-square random variable arises from an asymptotic normal distribution with a projection matrix as its covariance.

Baseline method 2: RAPPOR+ProjChi. We propose a projected Hotelling's-type statistic, referred to as ProjChi, which takes the outputs of RAPPOR as input, defined as follows:

$$T_{n_1, n_2}^{\text{proj}} := \left(\frac{1}{n_1} + \frac{1}{n_2} \right)^{-1} (\check{\mathbf{Y}} - \check{\mathbf{Z}})^\top \Pi \hat{\Sigma}^{-1} \Pi (\check{\mathbf{Y}} - \check{\mathbf{Z}}), \quad (98)$$

where $\check{\mathbf{Y}} := \sum_{i=1}^{n_1} \tilde{\mathbf{Y}}_i / n_1$ and $\check{\mathbf{Z}} := \sum_{j=1}^{n_2} \tilde{\mathbf{Z}}_j / n_2$ are the sample means of the RAPPOR α -LDP views, $\hat{\Sigma}$ is the pooled empirical covariance matrix:

$$\hat{\Sigma} := \frac{(n_1 - 1)\hat{\Sigma}_1 + (n_2 - 1)\hat{\Sigma}_2}{n_1 + n_2 - 2}, \quad (99)$$

where

$$\begin{aligned} \hat{\Sigma}_1 &:= \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (\tilde{\mathbf{Y}}_i - \check{\mathbf{Y}})(\tilde{\mathbf{Y}}_i - \check{\mathbf{Y}})^\top, \text{ and} \\ \hat{\Sigma}_2 &:= \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (\tilde{\mathbf{Z}}_i - \check{\mathbf{Z}})(\tilde{\mathbf{Z}}_i - \check{\mathbf{Z}})^\top, \end{aligned}$$

and $\Pi = I_k - \mathbf{1}\mathbf{1}^\top$ is the projection matrix to the subspace spanned by the one-vector $\mathbf{1}$. The test statistic $T_{n_1, n_2}^{\text{proj}}$ (98) can be calibrated using its asymptotic null distribution:

Lemma 38 (Asymptotic null distribution of rappor+ProjChi) *Assume the same multinomial setting as in Lemma 37. Fix the privacy level $\alpha > 0$. Under the null hypothesis $\mathbf{p}_Y = \mathbf{p}_Z$, for each pair of sample sizes (n_1, n_2) , compute the statistic T_{n_1, n_2} based on the RAPPOR α -LDP views $\{\tilde{\mathbf{Y}}_i\}_{i \in [n_1]}$ and $\{\tilde{\mathbf{Z}}_i\}_{i \in [n_2]}$, generated from \mathbf{p}_Y and \mathbf{p}_Z , respectively. Then, as $n_1, n_2 \rightarrow \infty$, we have $T_{n_1, n_2}^{\text{proj}} \xrightarrow{d} \chi_{k-1}^2$.*

The proof, provided in Appendix G.4.2, follows a similar approach to the proof of Lemma 37. The key difference is that the RAPPOR α -LDP views are no longer multinomial samples.

Baseline method 3: GenRR+ ℓ_2 . This method applies the ℓ_2 -type U-statistic in (9) to the GenRR α -LDP views, with calibration via permutation procedures.

G.2 Power comparison of RAPPOR-based methods under additional scenarios

Within the tests based on the RAPPOR mechanism, as shown in Figures 2, 3, and 5, we observe that RAPPOR+ProjChi and RAPPOR+ ℓ_2 exhibit comparable performance. In certain cases, RAPPOR+ProjChi demonstrates slightly greater power. The dominance of one test over the other depends on the specific scenario, particularly on the privacy level α and the relative signal strength of the chi-square divergence versus the ℓ_2 distance.

In the perturbed uniform distribution scenario considered for Figure 2, the chi-square divergence is relatively strong, while the ℓ_2 distance is relatively weak compared to other potential scenarios. To contrast this, we now present a multinomial testing scenario where the chi-square divergence is relatively weak. In this scenario, the probability vectors follow a power law as in (17), but with different powers. Specifically, we consider the following setup: for $m = 1, \dots, k = 40$, the m th elements of \mathbf{p}_Y and \mathbf{p}_Z are defined as:

$$p_{Ym} \propto m^{2.45}, \text{ and } p_{Zm} \propto m^{2.3}. \quad (100)$$

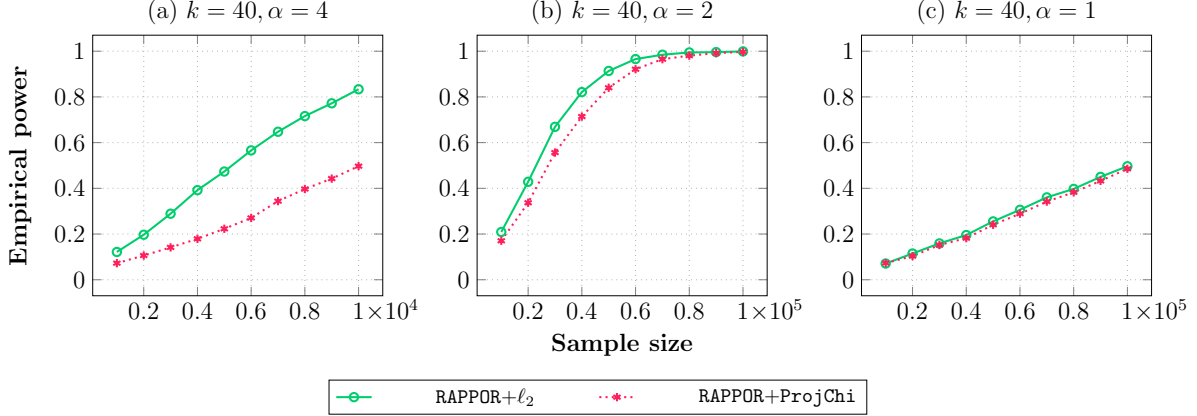


Figure 4: Comparison of the testing power between $\text{RAPPOR}+\ell_2$ and $\text{RAPPOR}+\text{ProjChi}$ under the power law alternatives in (100). To ensure a fair comparison, both tests are calibrated using permutation procedures at level $\gamma = 0.05$.

We also examine the effect of varying the privacy parameter $\alpha \in \{4, 2, 1\}$ to assess how the privacy level influences the power difference between the two methods. The results presented in Figure 4 demonstrate that in this power law scenario, where the signal is strong in ℓ_2 distance and weak in chi-square divergence, $\text{RAPPOR}+\ell_2$ outperforms $\text{RAPPOR}+\text{ProjChi}$. This difference is more pronounced in the low-privacy setting (i.e., larger α values) and becomes more subtle as the privacy level increases (i.e., smaller α values).

An intuition behind this phenomenon is as follows. For each $m \in [k]$, the m th entry of the RAPPOR output follows $\text{Ber}(\alpha_{\text{bf}}\mathbf{1}(Y_i = m) + \delta_{\text{bf}})$, where $\alpha_{\text{bf}} = \alpha/4 + o(\alpha)$ and $\delta_{\text{bf}} = 1/2 + o(1)$, with $o(1)$ representing a term that vanishes as $\alpha \rightarrow 0$. Therefore, as α decreases, the distributions approach uniform multinomial distributions, diminishing the difference between chi-square divergence and ℓ_2 distance in the signal.

G.3 Numerical result for density testing for scale alternatives

Similar to the location alternative scenario (19), we analyze scenarios involving covariance differences between two d -dimensional Gaussian distributions $P_{\mathbf{Y}} = \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Y}}, \Sigma_{\mathbf{Y}})$ and $P_{\mathbf{Z}} = \mathcal{N}(\boldsymbol{\mu}_{\mathbf{Z}}, \Sigma_{\mathbf{Z}})$. We set the mean vectors and covariance matrices of the Gaussian distributions as

$$\boldsymbol{\mu}_{\mathbf{Y}} = \boldsymbol{\mu}_{\mathbf{Z}} = \mathbf{0}_d, \Sigma_{\mathbf{Y}} = 0.5 \times \mathbf{J}_d + 0.5 \times \mathbf{I}_d, \quad \text{and} \quad \Sigma_{\mathbf{Z}} = 5 \times \Sigma_{\mathbf{Y}}. \quad (101)$$

The power results against the scale alternatives, provided in Figure 5, shows similar patterns to Figure 3.

G.4 Proof of asymptotic null distributions

This section presents the proof of chi-square asymptotic null distributions of $\text{GenRR}+\text{Chi}$ and $\text{RAPPOR}+\text{Chi}$ α -LDP tests introduced in Appendix G.1, starting with $\text{GenRR}+\text{Chi}$.

G.4.1 PROOF OF LEMMA 37

Throughout the proof, assume that the null hypothesis $\mathbf{p}_{\mathbf{Y}} = \mathbf{p}_{\mathbf{Z}}$ holds. Thus, for each $m \in [k]$, we have $p_{\tilde{Y}_m} = p_{\tilde{Z}_m}$. Without loss of generality, we denote all $p_{\tilde{Z}_m}$ as $p_{\tilde{Y}_m}$. For each $i \in [n_1]$

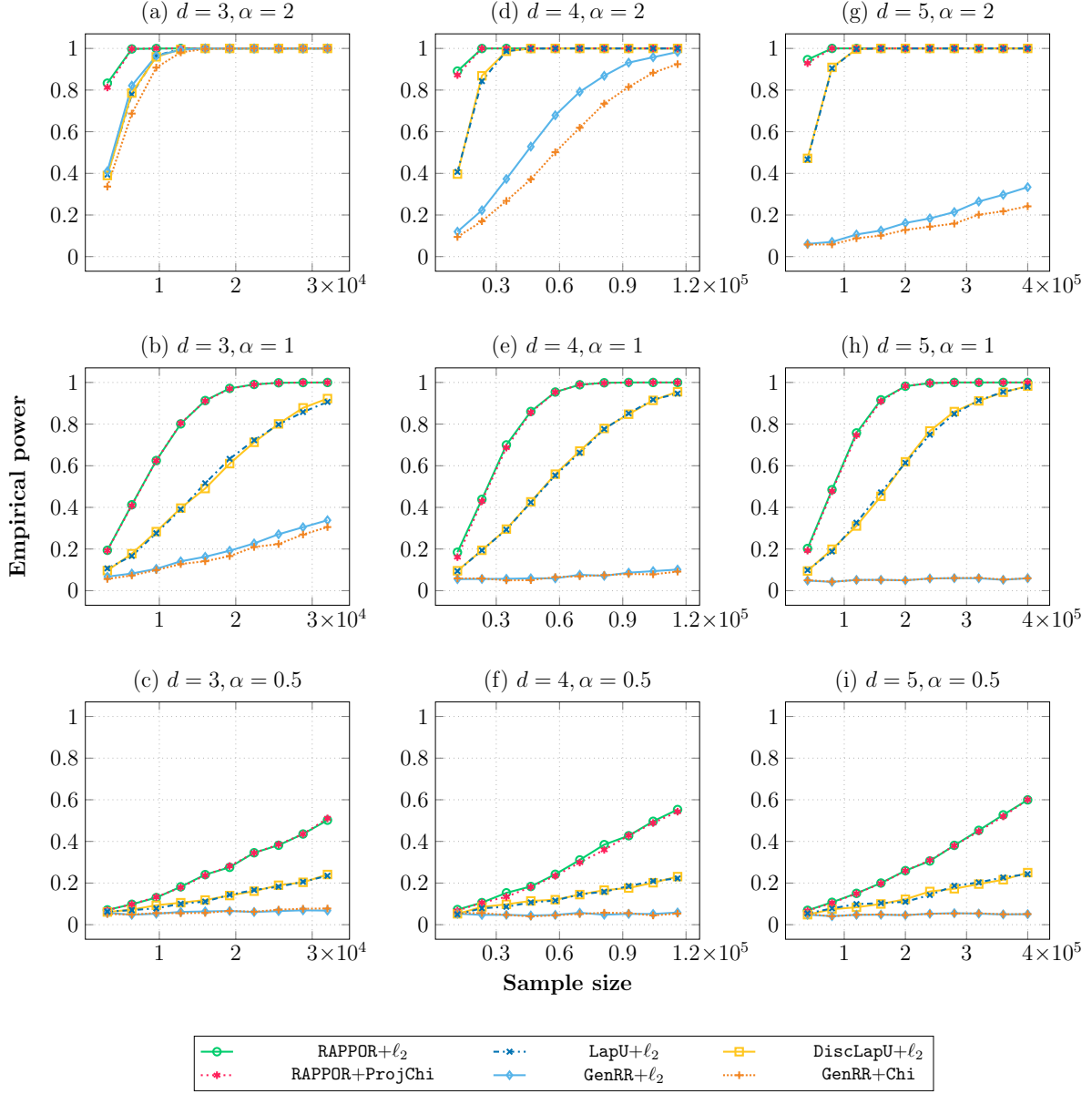


Figure 5: Comparison of the density testing power between our proposed methods (first row in the legend) and baseline methods (second row in the legend) under the scale alternatives in (101). To ensure a fair comparison, all methods are calibrated using permutation procedures at level $\gamma = 0.05$.

and $j \in [n_2]$, we can view \tilde{Y}_i and \tilde{Z}_j as drawn independently from a multinomial distribution with probability vector defined as:

$$\mathbf{p}_{\tilde{Y}} := \frac{\exp(\alpha)}{\exp(\alpha) + k - 1} \mathbf{p}_Y + \frac{1}{\exp(\alpha) + k - 1} (\mathbf{1}_k - \mathbf{p}_Y). \quad (102)$$

For each $m \in [k]$, the m th term of $\hat{\boldsymbol{\mu}}_{\tilde{Y}} - \hat{\boldsymbol{\mu}}_{\tilde{Z}}$ can be written as

$$\hat{\mu}_{\tilde{Y}m} - \hat{\mu}_{\tilde{Z}m} = \frac{1}{n_1} \sum_{i=1}^{n_1} (\mathbb{1}(\tilde{Y}_i = m) - p_{\tilde{Y}m}) - \frac{1}{n_2} \sum_{j=1}^{n_2} (\mathbb{1}(\tilde{Z}_j = m) - p_{\tilde{Y}m}).$$

To apply Lindeberg's CLT, we construct a triangular array sequence. Each row of the array corresponds to the pooled sample size $N := n_1 + n_2$. Within each row, the columns are indexed by $\ell \in [N]$. For each N and ℓ , we define a k -dimensional random vector $\tilde{\mathbf{X}}_{N[:,\ell]}$, whose m th element, denoted as $\tilde{\mathbf{X}}_{N[m,\ell]}$, is given by:

$$\tilde{\mathbf{X}}_{N[m,\ell]} := \begin{cases} \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{1}{n_1} (\mathbb{1}(\tilde{Y}_\ell = m) - p_{\tilde{Y}m}) & \text{if } 1 \leq \ell \leq n_1, \\ \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{1}{n_2} (p_{\tilde{Y}m} - \mathbb{1}(\tilde{Z}_{\ell-n_1} = m)) & \text{if } (n_1 + 1) \leq \ell \leq (n_1 + n_2), \end{cases}$$

for $m \in [k]$. Then for each $\ell \in [N]$, we have $\mathbb{E}[\tilde{\mathbf{X}}_{N[:,\ell]}] = 0$ and

$$\sum_{\ell=1}^N \tilde{\mathbf{X}}_{N[:,\ell]} = \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} (\hat{\boldsymbol{\mu}}_{\tilde{Y}} - \hat{\boldsymbol{\mu}}_{\tilde{Z}}).$$

We also have:

$$\text{Cov}(\tilde{\mathbf{X}}_{N[:,\ell]}) = \begin{cases} \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1} \frac{1}{n_1^2} (\text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top) & \text{if } 1 \leq \ell \leq n_1, \\ \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1} \frac{1}{n_2^2} (\text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top) & \text{if } (n_1 + 1) \leq \ell \leq (n_1 + n_2). \end{cases}$$

The first condition of Lindberg's CLT is

$$\sum_{\ell=1}^N \text{Cov}(\tilde{\mathbf{X}}_{N[:,\ell]}) \rightarrow \Sigma \text{ as } N \rightarrow \infty, \quad (103)$$

for some fixed covariance matrix Σ . The condition (103) is verified as follows. For each N ,

$$\begin{aligned} \sum_{\ell=1}^N \text{Cov}(\tilde{\mathbf{X}}_{N[:,\ell]}) &= \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1} \left(\sum_{i=1}^{n_1} \frac{1}{n_1^2} (\text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top) + \sum_{j=1}^{n_2} \frac{1}{n_2^2} (\text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top) \right) \\ &= \text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top. \end{aligned}$$

Since the covariance computed above depends on neither N nor ℓ , the condition (103) holds for $\Sigma = \text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top$. Now we check the second condition of Lindberg's CLT:

$$\sum_{\ell=1}^N \mathbb{E}[\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2^2 \mathbb{1}(\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2 > \epsilon)] \rightarrow 0 \text{ as } N \rightarrow \infty, \text{ for any } \epsilon > 0. \quad (104)$$

The condition (104) is verified as follows. For any fixed ℓ , we have:

$$\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2 = \begin{cases} \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{1}{n_1} \left\{ \sum_{m=1}^k (\mathbf{1}(\tilde{Y}_\ell = m) - p_{\tilde{Y}_m})^2 \right\}^{1/2} & \text{if } \ell = 1, \dots, n_1, \\ \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{1}{n_2} \left\{ \sum_{m=1}^k (\mathbf{1}(\tilde{Z}_{\ell-n_1} = m) - p_{\tilde{Y}_m})^2 \right\}^{1/2} & \text{if } (n_1 + 1) \leq \ell \leq (n_1 + n_2). \end{cases} \quad (105)$$

Note that $|\mathbf{1}(\tilde{Y}_\ell = m) - p_{\tilde{Y}_m}| \leq 1$ and $|\mathbf{1}(\tilde{Z}_{\ell-n_1} = m) - p_{\tilde{Y}_m}| \leq 1$ for $m \in [k]$. Therefore, for any $\ell \in [N]$, we have the following upper bound:

$$\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2 \leq \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{\sqrt{k}}{\min(n_1, n_2)},$$

where the right-hand side is not random and does not depend on ℓ . Therefore, when verifying the condition (104), we can pull the indicator out of the expectation and the sum:

$$\begin{aligned} \sum_{\ell=1}^N \mathbb{E}[\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2^2 \mathbf{1}(\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2 > \epsilon)] &\leq \sum_{\ell=1}^N \mathbb{E}\left[\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2^2 \mathbf{1}\left(\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{\sqrt{k}}{\min(n_1, n_2)} > \epsilon\right)\right] \\ &= \mathbf{1}\left(\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{\sqrt{k}}{\min(n_1, n_2)} > \epsilon\right) \sum_{\ell=1}^N \mathbb{E}[\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2^2]. \end{aligned}$$

From (105), it can be shown that $\sum_{\ell=1}^N \mathbb{E}[\|\tilde{\mathbf{X}}_{N[:,\ell]}\|_2^2] = 2\text{tr}(\text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top)$, where the right-hand side is finite and do not depend on N . For any n_1 and n_2 large enough, we have

$$\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{\sqrt{k}}{\min(n_1, n_2)} = \sqrt{\frac{n_1 n_2}{n_1 + n_2}} \frac{\sqrt{k}}{\min(n_1, n_2)} < \epsilon.$$

Therefore, the condition (104) is satisfied when $n_1, n_2 \rightarrow \infty$. Therefore by Lindberg's CLT, we have

$$\sum_{\ell=1}^N \tilde{\mathbf{X}}_{N[:,\ell]} = \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} (\hat{\mu}_{\tilde{Y}} - \hat{\mu}_{\tilde{Z}}) \xrightarrow{d} \mathcal{N}(0, \text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top).$$

By the weak law of large numbers and Slutsky's theorem, it can be shown that

$$\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \text{diag}(\hat{\mathbf{p}})^{-1/2} (\hat{\mu}_{\tilde{Y}} - \hat{\mu}_{\tilde{Z}}) \xrightarrow{d} \mathcal{N}(0, I_k - \text{diag}(\mathbf{p}_{\tilde{Y}})^{-1/2} \mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top \text{diag}(\mathbf{p}_{\tilde{Y}})^{-1/2}). \quad (106)$$

Since the covariance matrix in (106) is an identity matrix minus a rank-one matrix formed by an orthonormal vector, its eigenvalues are 0 (with multiplicity 1) and 1 (with multiplicity $k-1$). Thus, the covariance matrix is a projection matrix of rank $k-1$. By a standard result (e.g., Lemma 17.1 of van der Vaart (1998)), the test statistic in (97), equivalent to the squared ℓ_2 norm of the left-hand side of (106), converges to a chi-square distribution with $k-1$ degrees of freedom. This concludes the proof of Lemma 37.

G.4.2 PROOF OF LEMMA 38

Throughout the proof, assume that the null hypothesis $\mathbf{p}_Y = \mathbf{p}_Z$ holds. Thus, for each $m \in [k]$, we have $\mathbb{E}(\tilde{Y}_{1m}) = \mathbb{E}(\tilde{Z}_{1m})$. Without loss of generality, we denote all $\mathbb{E}(\tilde{Z}_{1m})$ as $\mathbb{E}(\tilde{Y}_{1m})$. The proof leverages the content and structure of the argument used in the proof for T_{n_1, n_2} in Appendix G.4.1.

To apply Lindeberg's CLT, we construct a triangular array sequence. Each row of the array corresponds to the pooled sample size $N := n_1 + n_2$. Within each row, the columns are indexed by $\ell \in [N]$. For each N and ℓ , we define a k -dimensional random vector $\tilde{\mathbf{X}}_{N[\cdot, \ell]}$, whose m th element, denoted as $\tilde{\mathbf{X}}_{N[m, \ell]}$, is given by:

$$\tilde{\mathbf{X}}_{N[m, \ell]} := \begin{cases} \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{1}{n_1} (\tilde{Y}_{\ell m} - \mathbb{E}(\tilde{Y}_{1m})) & \text{if } 1 \leq \ell \leq n_1, \\ \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} \frac{1}{n_2} (\mathbb{E}(\tilde{Y}_{1m}) - \tilde{Z}_{(\ell - n_1)m}) & \text{if } (n_1 + 1) \leq \ell \leq (n_1 + n_2), \end{cases}$$

for $m \in [k]$, where we recall from (53) that

$$\mathbb{E}(\tilde{Y}_{1m}) = \frac{(e^{\alpha/2} - 1)p_{Ym} + 1}{e^{\alpha/2} + 1}. \quad (107)$$

To verify the first condition (103) of Lindberg's CLT, first note that:

$$\begin{aligned} \sum_{\ell=1}^N \text{Cov}(\tilde{\mathbf{X}}_{N[\cdot, \ell]}) &= \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1} \left(\sum_{i=1}^{n_1} \frac{1}{n_1^2} \text{Cov}(\tilde{\mathbf{Y}}_i) + \sum_{j=1}^{n_2} \frac{1}{n_2^2} \text{Cov}(\tilde{\mathbf{Z}}_j) \right) \\ &= \text{Cov}(\tilde{\mathbf{Y}}_1) \\ &= \left(\frac{e^{\alpha/2} - 1}{e^{\alpha/2} + 1}\right)^2 (\text{diag}(\mathbf{p}_Y) - \mathbf{p}_Y \mathbf{p}_Y^\top) + \frac{e^{\alpha/2}}{(e^{\alpha/2} + 1)^2} I_d, \end{aligned}$$

where the last calculation is from Lemma 14. Since the covariance computed above depends on neither N nor ℓ , the condition (103) is satisfied with $\Sigma = \text{Cov}(\tilde{\mathbf{Y}}_1)$. To verify the second condition (104) of Lindberg's CLT, first note that $0 \leq \tilde{Y}_{\ell m} - \mathbb{E}(\tilde{Y}_{1m}) \leq 1$ and $0 \leq \mathbb{E}(\tilde{Y}_{1m}) - \tilde{Z}_{(\ell - n_1)m} \leq 1$. Therefore by the same analysis as the proof of T_{n_1, n_2} , the condition (104) is also satisfied. Therefore, by Linbderg's CLT, we have:

$$\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} (\tilde{\mathbf{Y}} - \tilde{\mathbf{Z}}) \xrightarrow{d} \mathcal{N}(0, \text{Cov}(\tilde{\mathbf{Y}}_1)). \quad (108)$$

Note that $\text{Cov}(\tilde{\mathbf{Y}}_1)$ has an eigenvector $\mathbf{1}$, which corresponds to an eigenvalue which is neither 0 nor 1. To turn the covariance matrix into a projector matrix, we delete the eigenvector $\mathbf{1}$ by pre-multiplying the random vector in the left-hand side of (108) by $\text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2} \Pi$. Then we have:

$$\text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2} \Pi \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} (\tilde{\mathbf{Y}} - \tilde{\mathbf{Z}}) \xrightarrow{d} \mathcal{N}(0, \text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2} \Pi \text{Cov}(\tilde{\mathbf{Y}}_1) \Pi \text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2}).$$

For a map $h : A \rightarrow A^{-1/2}$, define $\text{Disc}(h) := \{A \in \mathbb{R}^{p \times p} : h \text{ is not continuous at } A\}$. By Lemma 5.6 of Gaboardi and Rogers (2018), $\text{Cov}(\tilde{\mathbf{Y}}_1)$ is invertible for any $\alpha > 0$ and any $\mathbf{p}_Y \in \text{int}(\Omega)$.

Therefore we have $\text{Cov}(\tilde{\mathbf{Y}}_1) \notin \text{Disc}(h)$ almost surely. Thus by the weak law of large numbers and the continuous mapping theorem, we have $\hat{\Sigma}^{-1/2} \xrightarrow{P} \text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2}$. Then by Slutsky's theorem, we have

$$\hat{\Sigma}^{-1/2} \Pi \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-1}} (\tilde{\mathbf{Y}} - \tilde{\mathbf{Z}}) \xrightarrow{d} \mathcal{N}(0, \text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2} \Pi \text{Cov}(\tilde{\mathbf{Y}}_1) \Pi \text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2}). \quad (109)$$

By the analysis of [Gaboardi and Rogers \(2018\)](#), $\text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2} \Pi \text{Cov}(\tilde{\mathbf{Y}}_1) \Pi \text{Cov}(\tilde{\mathbf{Y}}_1)^{-1/2}$ is an identity matrix except one of the entries on the diagonal being zero, thus a projector matrix with rank $k - 1$. Therefore, by a standard result (e.g., Lemma 17.1 of [van der Vaart \(1998\)](#)), the test statistic (98), which is equivalent to squared ℓ_2 norm of the random vector on the left-hand side of (109), converges to a chi-square distribution with $k - 1$ degrees of freedom.

Appendix H. Suboptimality of the generalized randomized response

This section illustrates that the generalized randomized response mechanism in (96) can lead to suboptimal power. Specifically, we show that the **GenRR**+ ℓ_2 test performs suboptimally in certain privacy regimes where the number of categories k increases with the sample size n_1 , but the privacy level α is fixed. This negative result is further supported by our numerical studies in Section 5.

Theorem 39 (Asymptotic powerlessness of **GenRR+ ℓ_2 test)** *Let $(\mathbf{p}_Y, \mathbf{p}_Z)$ be a pair of data-generating multinomial distributions with k categories, where \mathbf{p}_Z is a uniform distribution and \mathbf{p}_Y is a perturbed uniform distribution parametrized by $\epsilon > 0$. Formally, for each $m \in [k]$, the m th entry of \mathbf{p}_Y and \mathbf{p}_Z are defined as:*

$$p_{Ym} := \frac{1}{k} + \frac{(-1)^m \epsilon}{\sqrt{k}}, \quad p_{Zm} := \frac{1}{k}, \quad (110)$$

ensuring that $\|\mathbf{p}_Y - \mathbf{p}_Z\|_2 = \epsilon$. Assume the privacy parameter $\alpha > 0$ is sufficiently small so that the minimax testing rate, given in equation (6), satisfies

$$\rho_{n_1, n_2, \alpha}^* \asymp \frac{k^{1/4}}{(n_1 \alpha^2)^{1/2}}. \quad (111)$$

For a sufficiently large, but fixed constant $C > 0$, set

$$\epsilon = \frac{C \cdot k^{1/4}}{(n_1 \alpha^2)^{1/2}}, \quad (112)$$

and consider the regime where

$$\sqrt{k} \epsilon = \frac{C \cdot k^{3/4}}{(n_1 \alpha^2)^{1/2}} \rightarrow 0, \quad (113)$$

*which guarantees that $0 < p_{Ym} < 1$ in (110) for each $m \in [k]$ and for sufficiently large n_1 . Under this regime, the **GenRR**+ ℓ_2 test is asymptotically powerless (i.e., its asymptotic power becomes at most the size).*

Given the setting in (112) from Theorem 39, an optimal test would achieve non-trivial power greater than the significance level by selecting a sufficiently large C . However, Theorem 39 proves that the **GenRR**+ ℓ_2 test becomes asymptotically powerless for any fixed $C > 0$, thereby underscoring its suboptimality.

Proof For the proof, we analyze the power function of the test statistic. To achieve this, we utilize high-dimensional asymptotics for U-statistics, specifically Corollary 3.3 and Theorem 3.3 from Kim (2020), which are restated as Theorem 18 and Theorem 19, respectively.

Let $w := (e^\alpha - 1)/(e^\alpha + k - 1)$. Recall from (102) that the α -LDP view $\{\tilde{Y}_i\}_{i \in [n_1]}$ via GenRR mechanism is equivalent to a random sample drawn from a mixture of \mathbf{p}_Y and \mathbf{p}_Z defined as follows:

$$\begin{aligned} \mathbf{p}_{\tilde{Y}} &:= \frac{e^\alpha}{e^\alpha + k - 1} \mathbf{p}_Y + \frac{1}{e^\alpha + k - 1} (\mathbf{1}_k - \mathbf{p}_Y) \\ &= \frac{e^\alpha - 1}{e^\alpha + k - 1} \mathbf{p}_Y + \frac{k}{e^\alpha + k - 1} \frac{1}{k} \mathbf{1}_k \\ &= w \mathbf{p}_Y + (1 - w) \mathbf{p}_Z, \end{aligned} \quad (114)$$

Similarly, $\{\tilde{Z}_i\}_{i \in [n_2]}$ can be viewed as a random sample drawn from $\mathbf{p}_{\tilde{Z}} := w \mathbf{p}_Z + (1 - w) \mathbf{p}_Z = \mathbf{p}_Z$. Note that the ℓ_2 distance between the probability vectors shrinks to $\|\mathbf{p}_{\tilde{Y}} - \mathbf{p}_{\tilde{Z}}\|_2 = w\epsilon$.

For convenience, let $\{\tilde{\mathbf{Y}}_i\}_{i \in [n_1]}$ be the one-hot vectorized version of $\{\tilde{Y}_i\}_{i \in [n_1]}$. We define a one-sample U-statistic for uniformity testing as follows:

$$U_{n_1} := \frac{2}{n_1(n_1 - 1)} \sum_{1 \leq i < j \leq n_1} h(\tilde{\mathbf{Y}}_i, \tilde{\mathbf{Y}}_j), \quad (115)$$

where $h(\tilde{\mathbf{Y}}_i, \tilde{\mathbf{Y}}_j) := (\tilde{\mathbf{Y}}_i - \mathbf{p}_Z)^\top (\tilde{\mathbf{Y}}_j - \mathbf{p}_Z)$, which is a special case of the two-sample statistic U_{n_1, n_2} in (9) with $n_2 = \infty$. Now we analyze the power function of the test based on U_{n_1} by specifying the critical value and the distribution of the test statistic under the alternative in order.

Critical value. We specify the critical value by establishing the asymptotic normality of the test statistic under the null by utilizing Theorem 18. Since the statistic (115) is equivalent to the statistic U_I in (31) of Theorem 18, it suffices to show that $n_1/\sqrt{k} \rightarrow \infty$ to establish the asymptotic normality. Note that the assumption $\sqrt{k}\epsilon = Ck^{3/4}/(n_1\alpha^2)^{1/2} \rightarrow 0$ in (113) implies $n_1/\sqrt{k} \rightarrow \infty$ for fixed α . Therefore, under the null of $\mathbf{p}_Y = \mathbf{p}_Z$, which implies $\mathbf{p}_{\tilde{Y}} = w\mathbf{p}_Z + (1 - w)\mathbf{p}_Z = \mathbf{p}_Z$, we have

$$\sqrt{\binom{n_1}{2}} \frac{U_{n_1}}{\sqrt{k^{-1}(1 - k^{-1})}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Based on this asymptotic normality, an asymptotically exact critical value is specified as follows:

$$\text{Reject } H_0 \text{ if } U_{n_1} \geq z_\gamma \sqrt{\frac{2(1 - 1/k)}{kn_1(n_1 - 1)}}. \quad (116)$$

Here z_γ denotes the upper γ quantile of $W \sim \mathcal{N}(0, 1)$ satisfying $\mathbb{P}(W > z_\gamma) = \gamma$.

Alternative distribution. Next we derive that the asymptotic distribution of the test statistic under the alternative specified in (110). By substituting $\|\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z\|_2^2 = w^2\epsilon^2$ into (32), we can establish the following asymptotic normality:

$$\sqrt{\frac{n_1(n_1 - 1)}{2}} \frac{U_{n_1} - w^2\epsilon^2}{\sqrt{\text{tr}(\Sigma_{\tilde{Y}}^2) + 2(n_1 - 1)(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)^\top \Sigma_{\tilde{Y}}(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)}} \xrightarrow{d} \mathcal{N}(0, 1), \quad (117)$$

provided that the conditions C1, C2, and C3 of Theorem 19 hold. We verify these conditions in order.

- Verification of the condition C1. The condition C1 is restated as follows:

$$\frac{\text{tr}(\Sigma_{\tilde{Y}}^4)}{\{\text{tr}(\Sigma_{\tilde{Y}}^2)\}^2} \rightarrow 0.$$

We first analyze the denominator term $\{\text{tr}(\Sigma_{\tilde{Y}}^2)\}^2$. Since $\Sigma_{\tilde{Y}} = \text{diag}(\mathbf{p}_{\tilde{Y}}) - \mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top$, we have

$$\Sigma_{\tilde{Y}}^2 = \{\text{diag}(\mathbf{p}_{\tilde{Y}})\}^2 + (\mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top)^2 - \text{diag}(\mathbf{p}_{\tilde{Y}})\mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top - \mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top\text{diag}(\mathbf{p}_{\tilde{Y}}).$$

Utilizing the equalities $(\mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top)^2 = \|\mathbf{p}_{\tilde{Y}}\|_2^2\mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top$, $\text{tr}(\mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top) = \|\mathbf{p}_{\tilde{Y}}\|_2^2$, and $\text{tr}(\text{diag}(\mathbf{p}_{\tilde{Y}})\mathbf{p}_{\tilde{Y}}\mathbf{p}_{\tilde{Y}}^\top) = \mathbf{p}_{\tilde{Y}}^\top\text{diag}(\mathbf{p}_{\tilde{Y}})\mathbf{p}_{\tilde{Y}}$, we can calculate $\text{tr}(\Sigma_{\tilde{Y}}^2)$ as

$$\text{tr}(\Sigma_{\tilde{Y}}^2) = \|\mathbf{p}_{\tilde{Y}}\|_2^2 + \|\mathbf{p}_{\tilde{Y}}\|_2^4 - 2\|\mathbf{p}_{\tilde{Y}}\|_3^3.$$

In a similar manner, we can expand the numerator term as follows:

$$\text{tr}(\Sigma_{\tilde{Y}}^4) = \|\mathbf{p}_{\tilde{Y}}\|_4^4 + \|\mathbf{p}_{\tilde{Y}}\|_2^8 + 2\|\mathbf{p}_{\tilde{Y}}\|_3^6 + 4\|\mathbf{p}_{\tilde{Y}}\|_2^2\|\mathbf{p}_{\tilde{Y}}\|_4^4 - 4\|\mathbf{p}_{\tilde{Y}}\|_2^4\|\mathbf{p}_{\tilde{Y}}\|_3^3 - 4\|\mathbf{p}_{\tilde{Y}}\|_5^5.$$

Therefore, it suffices to study the asymptotic behavior of

$$\frac{\text{tr}(\Sigma_{\tilde{Y}}^4)}{\{\text{tr}(\Sigma_{\tilde{Y}}^2)\}^2} = \frac{(\|\mathbf{p}_{\tilde{Y}}\|_2^8 - 4\|\mathbf{p}_{\tilde{Y}}\|_2^4\|\mathbf{p}_{\tilde{Y}}\|_3^3 + 2\|\mathbf{p}_{\tilde{Y}}\|_3^6) + \|\mathbf{p}_{\tilde{Y}}\|_4^4 + 4\|\mathbf{p}_{\tilde{Y}}\|_2^2\|\mathbf{p}_{\tilde{Y}}\|_4^4 - 4\|\mathbf{p}_{\tilde{Y}}\|_5^5}{(\|\mathbf{p}_{\tilde{Y}}\|_2^8 - 4\|\mathbf{p}_{\tilde{Y}}\|_2^4\|\mathbf{p}_{\tilde{Y}}\|_3^3 + 4\|\mathbf{p}_{\tilde{Y}}\|_3^6) + \|\mathbf{p}_{\tilde{Y}}\|_2^4 + 2\|\mathbf{p}_{\tilde{Y}}\|_2^2\|\mathbf{p}_{\tilde{Y}}\|_4^4 - 4\|\mathbf{p}_{\tilde{Y}}\|_2^2\|\mathbf{p}_{\tilde{Y}}\|_3^3}, \quad (118)$$

where the parentheses group together terms of the same order. Since $p_{Ym} = 1/k + (-1)^m\epsilon/\sqrt{k}$, we have

$$p_{\tilde{Y}m} = \frac{w}{k} + \frac{w(-1)^m\sqrt{k}\epsilon}{k} + \frac{1}{k} - \frac{w}{k} = \frac{1}{k} + \frac{(-1)^mw\epsilon}{\sqrt{k}}, \quad (119)$$

for $m \in [k]$. Utilizing (119) and the fact that k is an even integer, the powers of norms of $\mathbf{p}_{\tilde{Y}}$ are calculated as follows:

$$\begin{aligned} \|\mathbf{p}_{\tilde{Y}}\|_2^2 &= \frac{1}{k} + w^2\epsilon^2, \quad \|\mathbf{p}_{\tilde{Y}}\|_3^3 = \frac{1}{k^2} + \frac{3w^2\epsilon^2}{k}, \\ \|\mathbf{p}_{\tilde{Y}}\|_4^4 &= \frac{1}{k^3} + 6\frac{w^2\epsilon^2}{k^2} + \frac{w^4\epsilon^4}{k}, \quad \|\mathbf{p}_{\tilde{Y}}\|_5^5 = \frac{1}{k^4} + 10\frac{w^2\epsilon^2}{k^3} + 5\frac{w^4\epsilon^4}{k^2}. \end{aligned} \quad (120)$$

By substituting (120) into (118), and recalling that $w\epsilon$ is of order $k^{-3/4}n_1^{-1/2}$, we find that among the terms in (118), the term that converges to 0 the most slowly is $-4/k^3$, which comes from $-4\|\mathbf{p}_{\tilde{Y}}\|_2^2\|\mathbf{p}_{\tilde{Y}}\|_3^3$ in the denominator. Consequently, the quantity in (120) converges to 0, confirming condition C1 of Theorem 19.

- Verification of the condition C2. The condition C2 is recalled below as

$$\frac{\mathbb{E}[W_1] + n_1\mathbb{E}[W_2]}{n_1^2\{\text{tr}(\Sigma_{\tilde{Y}}^2)\}^2} \rightarrow 0, \quad (121)$$

where

$$\begin{aligned} W_1 &:= \{(\tilde{\mathbf{Y}}_1 - \mathbf{p}_Z)^\top(\tilde{\mathbf{Y}}_2 - \mathbf{p}_Z)\}^4, \\ W_2 &:= \{(\tilde{\mathbf{Y}}_1 - \mathbf{p}_Z)^\top(\tilde{\mathbf{Y}}_2 - \mathbf{p}_Z)\}^2\{(\tilde{\mathbf{Y}}_1 - \mathbf{p}_Z)^\top(\tilde{\mathbf{Y}}_3 - \mathbf{p}_Z)\}^2. \end{aligned}$$

We separately analyze the terms in (121). First for $\mathbb{E}[W_1]$, the random variable W_1 takes two distinct values:

$$W_1 = \begin{cases} \left(1 - \frac{1}{k}\right)^4, & \tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_2, \\ \frac{1}{k^4}, & \text{otherwise.} \end{cases}$$

Observe that, according to (119), we have

$$\mathbb{P}(\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_2) = \sum_{m=1}^k p_{\tilde{\mathbf{Y}}_m}^2 = \sum_{m=1}^k \left(\frac{1}{k} + \frac{(-1)^m w \epsilon}{\sqrt{k}} \right)^2 = \frac{1}{k} + w^2 \epsilon^2. \quad (122)$$

Using the established probability (122), we calculate the target expected value as

$$\mathbb{E}[W_1] = -\frac{1}{k^5} + \frac{k}{k^5} - \frac{k\epsilon^2 w^2}{k^5} + \frac{(-1+k)^4}{k^5} + \frac{(-1+k)^4(1+k\epsilon^2 w^2)}{k^5}.$$

Among the terms in the expectation computed above, the term that converges to 0 most slowly is of order $1/k$, which arises from the fourth term of the numerator. Similarly, for $\mathbb{E}[W_2]$, the random variable W_2 takes on the values as:

$$W_2 = \begin{cases} \left(1 - \frac{1}{k}\right)^4, & \tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_2 = \tilde{\mathbf{Y}}_3, \\ \frac{1}{k^2} \left(1 - \frac{1}{k}\right)^2, & [\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_2, \tilde{\mathbf{Y}}_1 \neq \tilde{\mathbf{Y}}_3] \cup [\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_3, \tilde{\mathbf{Y}}_1 \neq \tilde{\mathbf{Y}}_2] \\ \frac{1}{k^4}, & \text{otherwise.} \end{cases} \quad (123)$$

The corresponding probabilities are calculated as

$$\begin{aligned} \mathbb{P}(\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_2 = \tilde{\mathbf{Y}}_3) &= \sum_{m=1}^k p_{\tilde{\mathbf{Y}}_m}^3 = \|\mathbf{p}_{\tilde{\mathbf{Y}}}\|_3^3 \stackrel{(a)}{=} \frac{1}{k^2} + \frac{3w^2\epsilon^2}{k}, \text{ and} \\ \mathbb{P}([\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_2, \tilde{\mathbf{Y}}_1 \neq \tilde{\mathbf{Y}}_3] \cup [\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_3, \tilde{\mathbf{Y}}_1 \neq \tilde{\mathbf{Y}}_2]) &\stackrel{(b)}{=} 2\mathbb{P}([\tilde{\mathbf{Y}}_1 = \tilde{\mathbf{Y}}_2, \tilde{\mathbf{Y}}_1 \neq \tilde{\mathbf{Y}}_3]) \\ &= 2 \sum_{m=1}^k p_{\tilde{\mathbf{Y}}_m}^2 (1 - p_{\tilde{\mathbf{Y}}_m}) \\ &= 2\|\mathbf{p}_{\tilde{\mathbf{Y}}}\|_2^2 - 2\|\mathbf{p}_{\tilde{\mathbf{Y}}}\|_3^3 \\ &\stackrel{(c)}{=} \frac{2}{k} + 2w^2\epsilon^2 - \frac{2}{k^2} - \frac{6w^2\epsilon^2}{k}, \end{aligned}$$

where (a) and (c) utilize the equations in (120) and (b) uses the independence between samples, as well as the exclusiveness and symmetry of the events. Now $\mathbb{E}[W_2]$ is calculated as

$$\begin{aligned} \mathbb{E}[W_2] &= \left(1 - \frac{1}{k}\right)^4 \left(\frac{1}{k^2} + \frac{3w^2\epsilon^2}{k} \right) + \left\{ \frac{1}{k^2} \left(1 - \frac{1}{k}\right)^2 \right\} \left(\frac{2}{k} + 2w^2\epsilon^2 - \frac{2}{k^2} - \frac{6w^2\epsilon^2}{k} \right) \\ &\quad + \frac{1}{k^4} \left(1 - \frac{1}{k^2} - \frac{3w^2\epsilon^2}{k} - \frac{2}{k} - 2w^2\epsilon^2 + \frac{2}{k^2} + \frac{6w^2\epsilon^2}{k} \right), \end{aligned} \quad (124)$$

where the term that converges to 0 the slowest is of order $1/k^2$, originating from the first term on the right-hand side of (124). Finally, the trace term in the denominator is calculated as

$$\begin{aligned} \text{tr}(\Sigma_{\tilde{Y}}^2) &= \|\mathbf{p}_{\tilde{Y}}\|_2^2 + \|\mathbf{p}_{\tilde{Y}}\|_2^4 - 2\|\mathbf{p}_{\tilde{Y}}\|_3^3 \\ &= \frac{1}{k} + w^2\epsilon^2 + \frac{1}{k^2} + w^4\epsilon^4 + \frac{2w^2\epsilon^2}{k} - \frac{2}{k^2} - \frac{6w^2\epsilon^2}{k} \\ &= \frac{1}{k} - \frac{1}{k^2} + w^2\epsilon^2 + w^4\epsilon^4 - \frac{4w^2\epsilon^2}{k}, \end{aligned} \quad (125)$$

where the term that converges to 0 the slowest is $1/k$. Now to verify the condition C2, we separately analyze the asymptotic behaviors of $\mathbb{E}[W_1]/n_1^2\{\text{tr}(\Sigma_{\tilde{Y}}^2)\}^2$ and $n_1\mathbb{E}[W_2]/n_1^2\{\text{tr}(\Sigma_{\tilde{Y}}^2)\}^2$, focusing on the terms that converges the slowest to 0 in the numerator and denominator, respectively. For the first term, it suffices to investigate $(1/k)/(n_1^2/k^2)$, which converges to 0 as long as $k/n_1^2 \rightarrow 0$. Note that $k/n_1^2 \rightarrow 0$ is implied by our assumption (113). For the second term, it suffices to investigate $(n_1/k^2)/(n_1^2/k^2)$, which converges to 0. Therefore the convergence in (121) is verified, confirming the condition C2 of Theorem 19.

- Verification of the condition C3. The condition C3 is restated as

$$(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)^\top \Sigma_{\tilde{Y}} (\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z) = o\left(n_1^{-1} \text{tr}(\Sigma_{\tilde{Y}}^2)\right). \quad (126)$$

To calculate the quadratic form therein, first note that since $\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z = w(\mathbf{p}_Y - \mathbf{p}_Z)$, we have

$$(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)^\top \Sigma_{\tilde{Y}} (\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z) = w^2(\mathbf{p}_Y - \mathbf{p}_Z)^\top \text{diag}(\mathbf{p}_{\tilde{Y}})(\mathbf{p}_Y - \mathbf{p}_Z) - w^2(\mathbf{p}_Y - \mathbf{p}_Z)^\top \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top (\mathbf{p}_Y - \mathbf{p}_Z).$$

We separately analyze the two terms on the right-hand side. For the first term, we have

$$\begin{aligned} w^2(\mathbf{p}_Y - \mathbf{p}_Z)^\top \text{diag}(\mathbf{p}_{\tilde{Y}})(\mathbf{p}_Y - \mathbf{p}_Z) &= w^2 \sum_{m=1}^k \frac{(-1)^m \epsilon}{\sqrt{k}} \left(\frac{1}{k} + \frac{(-1)^m w \epsilon}{\sqrt{k}} \right) \frac{(-1)^m \epsilon}{\sqrt{k}} \\ &= \frac{w^2 \epsilon^2}{k} \sum_{m=1}^k \left(\frac{1}{k} + \frac{(-1)^m w \epsilon}{\sqrt{k}} \right) \\ &= \frac{w^2 \epsilon^2}{k}, \end{aligned}$$

where the last equality uses the fact that k is an even integer. For the second term, we have

$$\begin{aligned} w^2(\mathbf{p}_Y - \mathbf{p}_Z)^\top \mathbf{p}_{\tilde{Y}} \mathbf{p}_{\tilde{Y}}^\top (\mathbf{p}_Y - \mathbf{p}_Z) &= w^2 \{(\mathbf{p}_Y - \mathbf{p}_Z)^\top \mathbf{p}_{\tilde{Y}}\}^2 \\ &= w^2 \left\{ \sum_{m=1}^k \frac{(-1)^m \epsilon}{\sqrt{k}} \left(\frac{1}{k} + \frac{(-1)^m w \epsilon}{\sqrt{k}} \right) \right\}^2 \\ &= w^4 \epsilon^4, \end{aligned}$$

where the last equality again uses the fact that k is an even integer. Therefore we have

$$(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)^\top \Sigma_{\tilde{Y}} (\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z) = \frac{w^2 \epsilon^2}{k} - w^4 \epsilon^4, \quad (127)$$

where $w^2 \epsilon^2/k$ is the term that converges to 0 most slowly. Utilizing the trace calculation (125), we can verify the condition C3 (126) by studying the asymptotic behavior of $(w^2 \epsilon^2/k)/(1/k n_1)$. Since this quantity converges to 0, the condition C3 is verified.

Since all three conditions of Theorem 19 are verified, the asymptotic distribution in (117) is confirmed.

Power function. Using the critical value in (116) and the asymptotic alternative distribution established in (117), under the asymptotic regime in (113), the power function can be written as

$$\begin{aligned}
\gamma_{n_1, k, \alpha}(\mathbf{p}_Y, \mathbf{p}_Z) &= \mathbb{P}_{H_1} \left(U_{n_1} \geq z_\gamma \sqrt{\frac{2(1 - 1/k)}{kn_1(n_1 - 1)}} \right) \\
&= \mathbb{P}_{H_1} \left(W \geq \sqrt{\frac{n_1(n_1 - 1)}{2}} \frac{z_\gamma \sqrt{2\{1 - (1/k)\}/\{kn_1(n_1 - 1)\}} - w^2\epsilon^2}{\sqrt{\text{tr}(\Sigma_Y^2) + 2(n_1 - 1)(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)^\top \Sigma_{\tilde{Y}}(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)}} \right) + o(1) \\
&= \Phi \left(\sqrt{\frac{n_1(n_1 - 1)}{2}} \frac{w^2\epsilon^2 - z_\gamma \sqrt{2\{1 - (1/k)\}/\{kn_1(n_1 - 1)\}}}{\sqrt{\text{tr}(\Sigma_Y^2) + 2(n_1 - 1)(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)^\top \Sigma_{\tilde{Y}}(\mathbf{p}_{\tilde{Y}} - \mathbf{p}_Z)}} \right) \\
&= \Phi \left(\frac{\sqrt{n_1(n_1 - 1)/2} w^2\epsilon^2 - z_\gamma \sqrt{(1/k) - (1/k^2)}}{\sqrt{\{(1/k) - (1/k^2) + w^2\epsilon^2 + w^4\epsilon^4 - (4w^2\epsilon^2/k)\} + 2(n_1 - 1)\{(w^2\epsilon^2/k) - w^4\epsilon^4\}}} \right) \\
&\rightarrow \Phi(-z_\gamma) = \gamma,
\end{aligned} \tag{128}$$

where $W \sim \mathcal{N}(0, 1)$, Φ denotes the cumulative distribution function of W , and $o(1)$ refers to a quantity that goes to 0 as n_1 and k go to ∞ , in the regime of (113). The last line in (128) uses the fact that given that $w\epsilon$ is of the order $k^{-3/4}n_1^{-1/2}$, the term in (128) that converges most slowly to 0 is $\sqrt{1/k}$, which appears both on the second term in the numerator and the first term in the denominator. It is important to note that this result holds for any sufficiently small α , indicating that the **GenRR**+ ℓ_2 test becomes asymptotically powerless in the regime where the minimax testing rate is $k^{1/4}/(n_1\alpha^2)^{1/2}$. ■

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