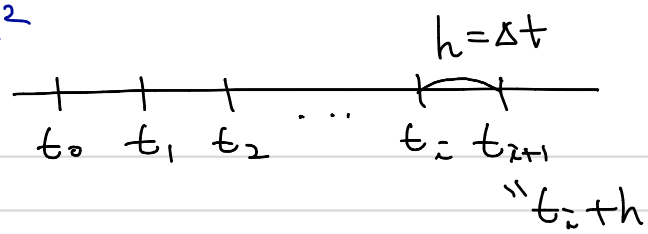
$y_1(t) = y(t)$ & $y_2(t) = y_1'(t) = v(t)$ 이므로

$$\begin{cases} y_{i+1} = y_i + v_i \cdot h + \frac{1}{2}(-g)h^2 \\ v_{i+1} = v_i - g \cdot h \end{cases}$$



A Bouncing Ball

① $x'(t) = f(t, x) = v_0^H, \quad x(t_0) = x_0$

$x_i \equiv x(t_i) = x(t_0 + i \cdot h)$ ⇒ $t_i \rightarrow t_{i+1}$ 일때 $x_i \rightarrow x_{i+1}$?

(i) (t_i, x_i)에 기반을 둔 미분 정보

$$k_1 = v_0^H$$

(ii) $(t_i + h, x_i + v_0^H \cdot h)$에 기반을 둔 미분 정보

※ 이 경우 1st-order Euler와 동일

$$k_2 = v_0^H$$

$$\Rightarrow x_{i+1} = x_i + \frac{h}{2}(k_1 + k_2) = x_i + v_0^H \cdot h$$

↑ 2nd-order modified Euler

② $y''(t) = -g, \quad y(t_0) = y_0, \quad y'(t_0) = v_0^V$

$$\begin{cases} y_1(t) = y(t) \\ y_2(t) = y_1'(t) \end{cases} \Rightarrow \begin{cases} y_1'(t) = f_1(t, y_1, y_2) = y_2, \quad y_1(t_0) = y_0 \\ y_2'(t) = f_2(t, y_1, y_2) = -g, \quad y_2(t_0) = v_0^V \end{cases}$$

$y_{1i} \equiv y_1(t_i)$ & $y_{2i} \equiv y_2(t_i)$ ⇒ $t_i \rightarrow t_{i+1}$ 일때 $\begin{cases} y_{1i} \rightarrow y_{1,i+1} \\ y_{2i} \rightarrow y_{2,i+1} \end{cases}$?

(i) (t_i, y_{1i}, y_{2i})에 기반을 둔 미분 정보

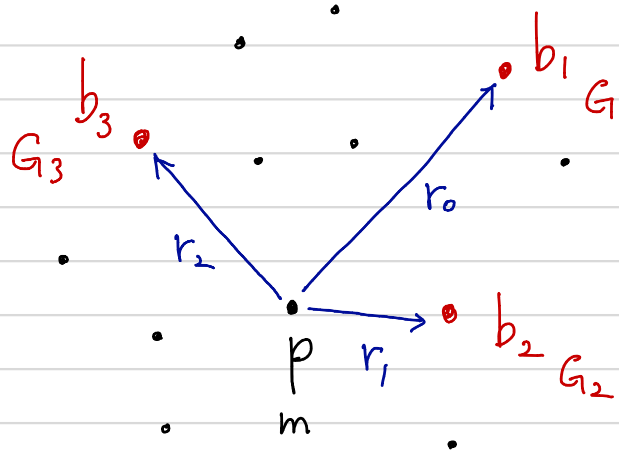
$$k_{11} = y_{2i}, \quad k_{12} = -g$$

(ii) $(t_i, y_{1i} + y_{2i} \cdot h, y_{2i} + (-g) \cdot h)$에 기반을 둔 미분정보

$$k_{21} = y_{2i} + (-g) \cdot h, \quad k_{22} = -g$$

$$\Rightarrow \begin{cases} y_{1,i+1} = y_{1i} + \frac{h}{2}(k_{11} + k_{21}) = y_{1i} + \frac{h}{2}(y_{2i} + y_{2i} + (-g) \cdot h) \\ \qquad = y_{1i} + y_{2i} \cdot h + \frac{1}{2}(-g) \cdot h^2 \\ y_{2,i+1} = y_{2i} + \frac{h}{2}(k_{12} + k_{22}) = y_{2i} + \frac{h}{2}((-g) + (-g)) = y_{2i} - g \cdot h \end{cases}$$

Particle Simulation



$$p = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \; : \text{ particle position}$$

m : particle mass

$$b_i = \begin{pmatrix} b_{xi} \\ b_{yi} \\ b_{zi} \end{pmatrix} \; : \text{ the position of the } i\text{-th attractor (black hole)}$$

$(i = 1, 2, \cdots, N)$

G_i : the strength of the i-th attractor

$$r_i = b_i - p$$

$(i = 1, 2, \cdots, N)$

* Problem : Given the force $F(p)$ such that

$$F(p) = \sum_{i=1}^{N} \frac{G_i}{|r_i|} \cdot \frac{r_i}{|r_i|},$$

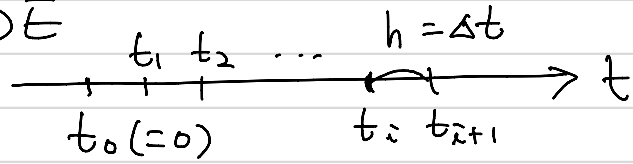
find the trajectory of the particle : $p(t) \equiv \begin{pmatrix} p_x(t) \\ p_y(t) \\ p_z(t) \end{pmatrix}$, $t \geq t_0$.

* Solution :

$$v(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = p'(t), \quad a(t) = \begin{pmatrix} a_x(t) \\ a_y(t) \\ a_z(t) \end{pmatrix} = v'(t) = p''(t)$$

$$\Rightarrow \boxed{p''(t) = \frac{F(p(t))}{m}, \quad p(t_0) = p_0, \quad p'(t_0) = v_0}$$

the 2nd-order ODE



Let $\begin{cases} y_1(t) \equiv p(t) \\ y_2(t) \equiv y_1'(t) = p'(t) = v(t), \end{cases}$ then

$$\begin{cases} y_1'(t) = f_1(t, y_1, y_2) = y_2, & y_1(t_0) = p_0 \\ y_2'(t) = f_2(t, y_1, y_2) = \dfrac{F(y_1)}{m}, & y_2(t_0) = v_0 \end{cases}$$

$$\Rightarrow \begin{cases} p'(t) = f_1(t, p, v) = v, & p(t_0) = p_0 \\ v'(t) = f_2(t, p, v) = \dfrac{F(p)}{m}, & v(t_0) = v_0 \end{cases}$$

Let $p_i = p(t_i)$ & $v_i = v(t_i)$. Then whar are

$p_{i+1} = p(t_{i+1})$ & $v_{i+1} = v(t_{i+1})$?

For the 2nd-order Modified Euler Method,

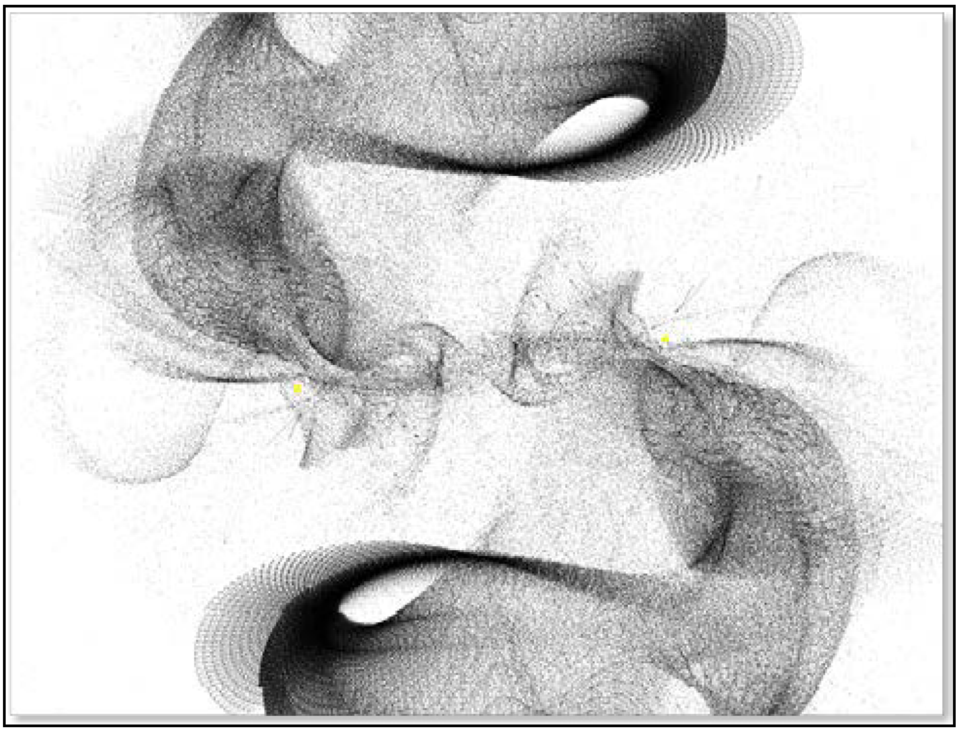
① (t_i, p_i, v_i)에 기반을 둔 미분정보

$$k_{1p} = v_i, \quad k_{1v} = \frac{F(p_i)}{m}$$

② $\left(t_{i+1}, p_i + v_i \cdot h, v_i + \frac{F(p_i)}{m} \cdot h\right)$에 기반을 둔 미분정보

$$k_{2p} = v_i + \frac{F(p_i)}{m} \cdot h, \quad k_{2v} = \frac{F(p_i + v_i \cdot h)}{m}$$

$$\begin{cases} p_{i+1} = p_i + \dfrac{h}{2}(k_{1p} + k_{2p}) = p_i + \dfrac{h}{2}\left(v_i + v_i + \dfrac{F(p_i)}{m} \cdot h\right) \\ \quad = p_i + v_i \cdot h + \dfrac{1}{2}\dfrac{F(p_i)}{m} \cdot h^2 \\ v_{i+1} = v_i + \dfrac{h}{2}(k_{1v} + k_{2v}) = v_i + \dfrac{h}{2}\left(\dfrac{F(p_i)}{m} + \dfrac{F(p_i + v_i \cdot h)}{m}\right) \\ \quad = v_i + \dfrac{h}{2m}\left(F(p_i) + F(p_i + v_i \cdot h)\right) \end{cases}$$



To implement the simulation, we compute the force on each particle and then update the position by integrating the Newtonian equations of motion. There are a number of well studied numerical techniques for integrating the equations of motion. For this simulation, the simple Euler method is sufficient. With the Euler method, the position of the particle at time **t + Δt** is given by the following equation:

$$\mathbf{P}(t+\Delta t)=\mathbf{P}(t)+\mathbf{v}(t)\Delta t+\frac{1}{2}\mathbf{a}(t)\Delta t^2$$

Where **P** is the position of the particle, **v** is the velocity, and **a** is the acceleration. Similarly, the updated velocity is determined by the following equation:

$$\mathbf{v}(t+\Delta t)=\mathbf{v}(t)+\mathbf{a}(t)\Delta t$$

These equations are derived from a Taylor expansion of the position function about time **t**. The result is dependent upon the size of the time step (**Δt**), and is more accurate when the time step is very small.

The acceleration is directly proportional to the force on the particle, so by calculating the force on the particle (using the preceding equation), we essentially have a value for the acceleration. To simulate the particle's motion, we track its position and velocity, determine the force on the particle due to the black holes, and then update the position and velocity using the equations.