[MEN573]
Advanced Control Systems I

Lecture 3 – Z Transform

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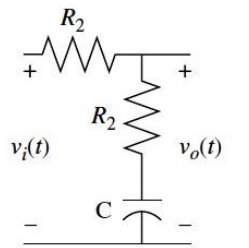
Outline

- Digital control
- Introduction
- Discrete time sequence
- Z transform definition
- Examples
- Properties of the Z transform
- Applications of Z transformation

Digital Control

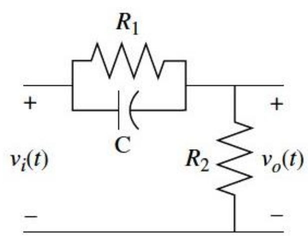
- Most of the controllers we have studied so far were described by the Laplace transform or differential equations which, strictly speaking, are assumed to be built using analog electronics.

Lag compensation



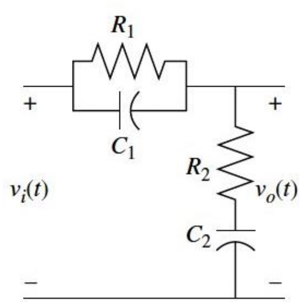
$$\frac{R_2}{R_1+R_2}\frac{s+\dfrac{1}{R_2C}}{s+\dfrac{1}{(R_1+R_2)C}}$$

Lead compensation



$$\frac{s+\dfrac{1}{R_1C}}{s+\dfrac{1}{R_1C}+\dfrac{1}{R_2C}}$$

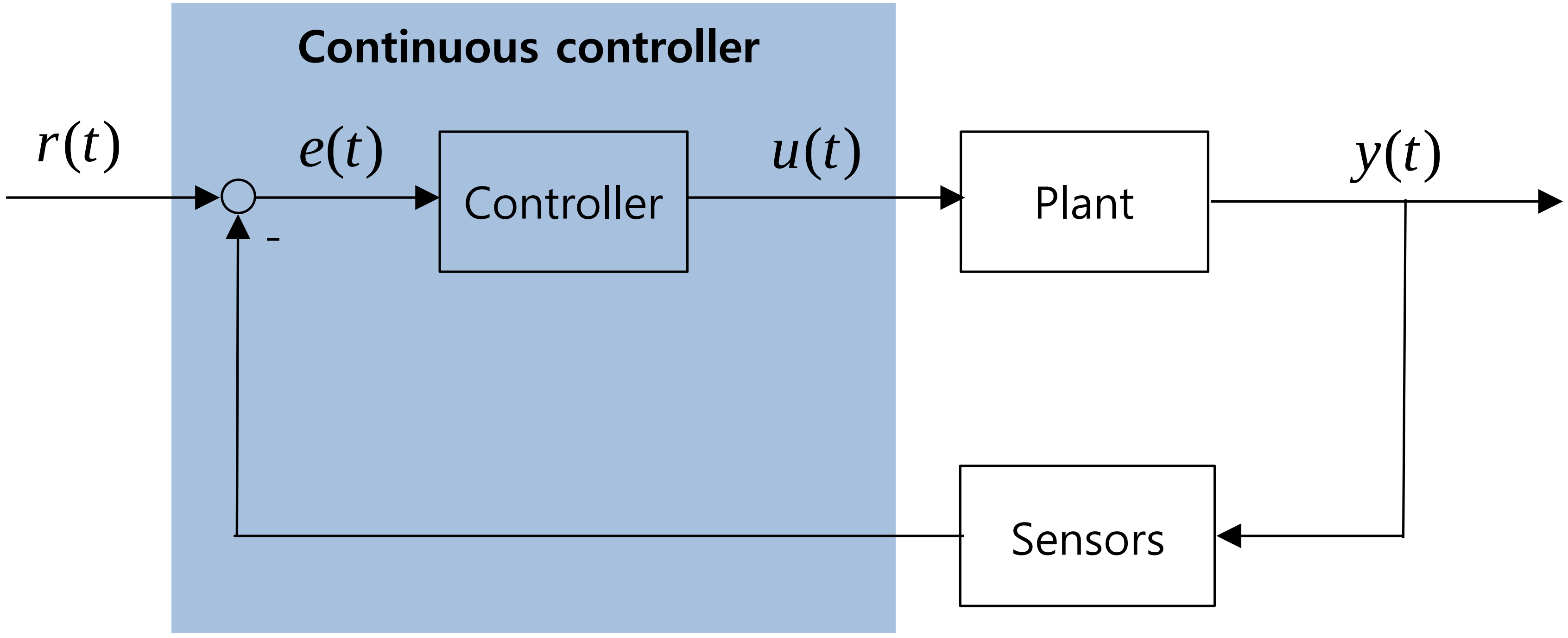
Lag-lead compensation



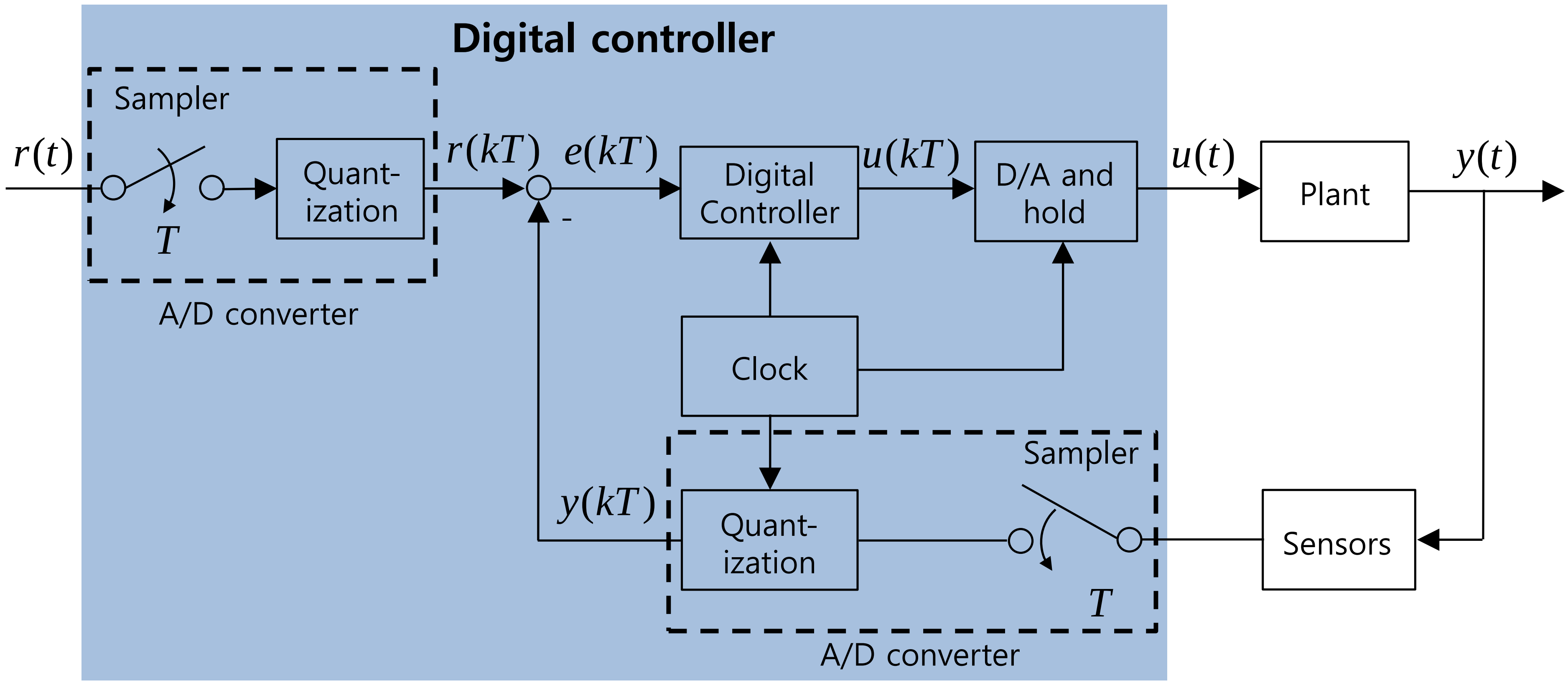
$$\frac{\left(s+\dfrac{1}{R_1C_1}\right)\left(s+\dfrac{1}{R_2C_2}\right)}{s^2+\left(\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_2}+\dfrac{1}{R_2C_1}\right)s+\dfrac{1}{R_1R_2C_1C_2}}$$

Digital Control

- However, most of control systems today use digital computers (usually microprocessors) to implement the controllers.

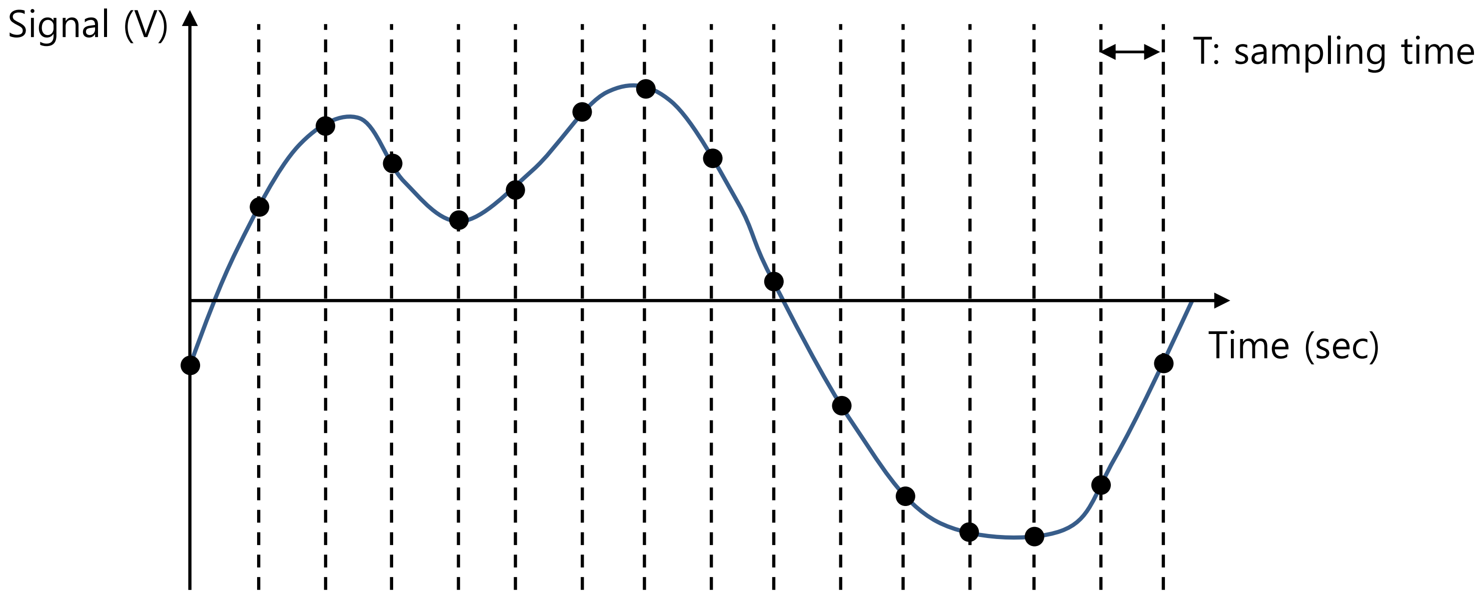


Digital Control



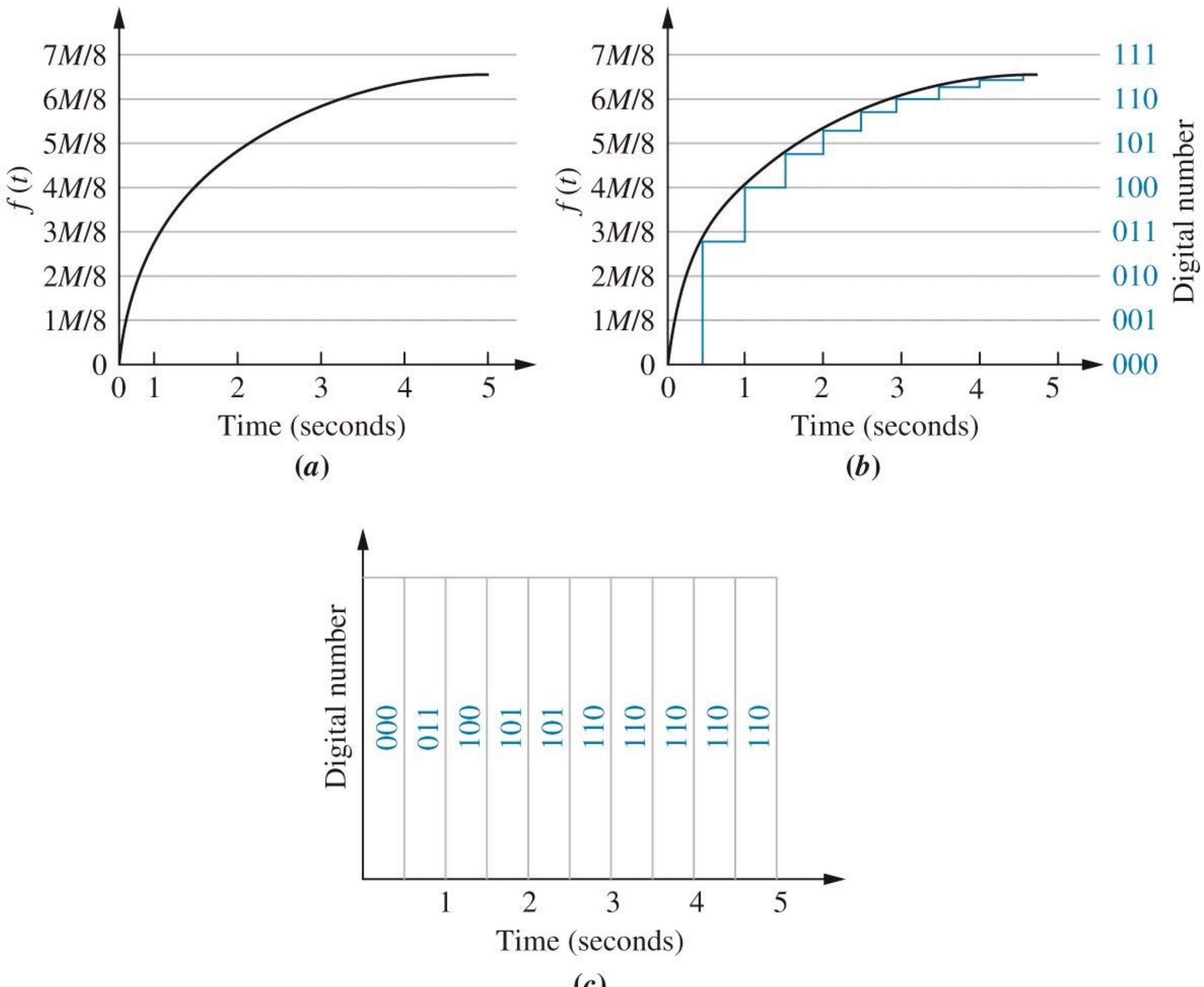
A/D Converter

- Analog-to-Digital (A/D) Converter
  - To convert a voltage level to the correct bit pattern usable by the computer. (sampler + quantization)
  - Bit resolution (ex. ±10 V to 16 bit, $20/2^{16}=0.0003$)



▲ Sampler

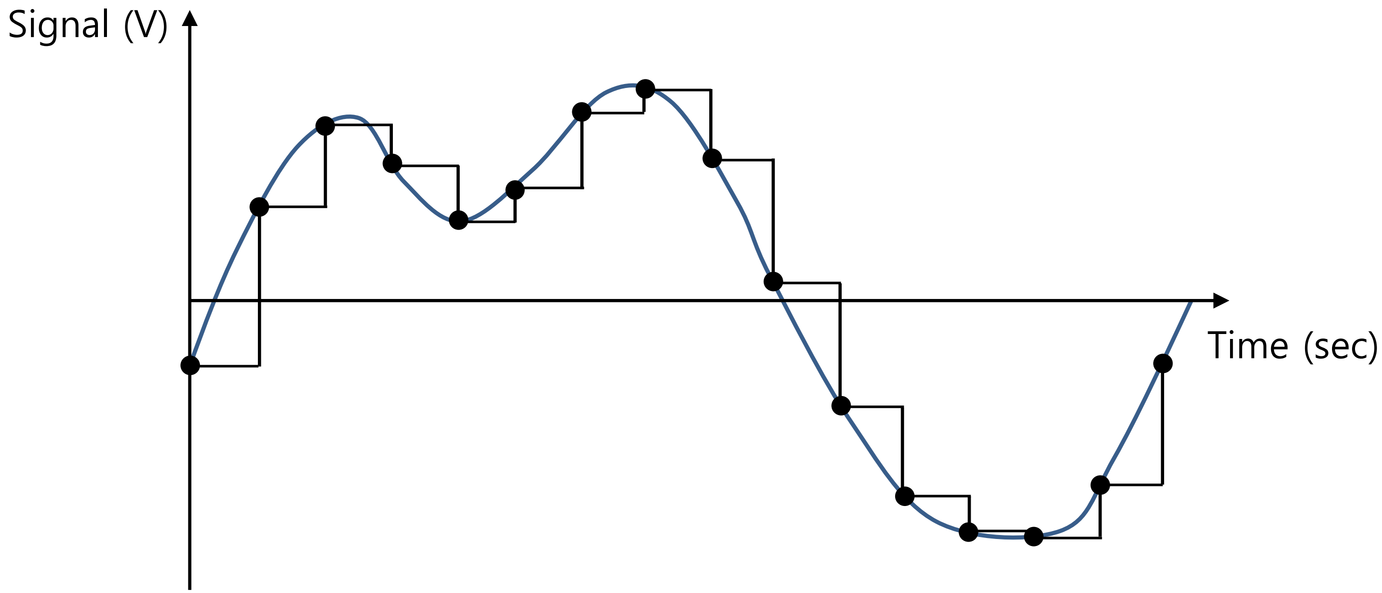
A/D Converter



▲ Quantization

D/A Converter

- Digital-to-Analog (D/A) Converter
  - To convert the digital words from the computer to a voltage level
  - Zero order holder (ZOH), First order holder (FOH), etc.



▲ ZOH D/A converter

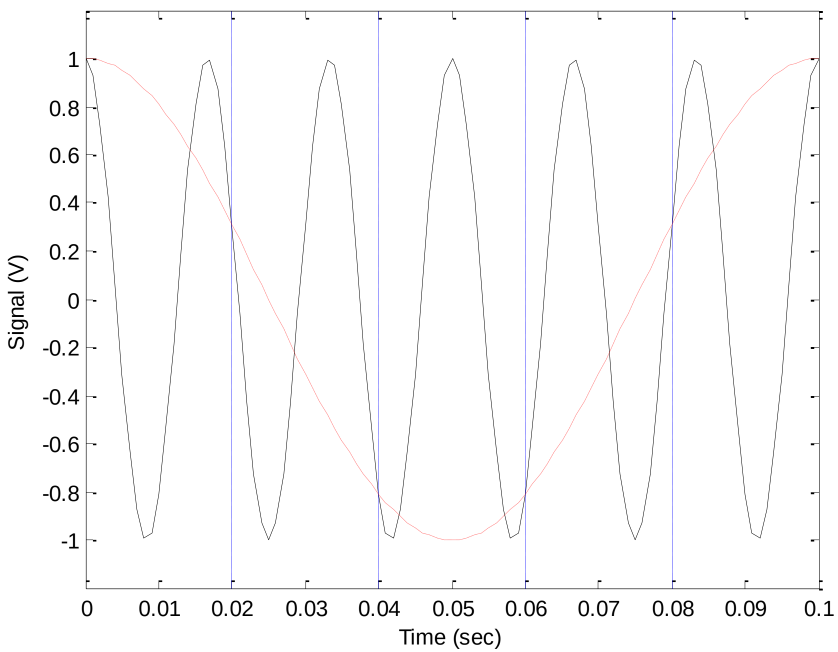
Anti-Alias Prefilters

- Nyquist-Shannon sampling theorem
  - For the signal to be accurately **reconstructed** from the samples, it must have **no frequency component greater than half the sample rate**.
  - Aliasing will occur any time the sample rate is not at least twice as fast as any of the frequencies in the signal being sampled. (at least 120 Hz sample rate is required for the 60 Hz signal)
  - To place an analog prefilter before the sampler. In many cases, a simple first-order low-pass filter will do.

$$H_p(s) = \frac{a}{s+a}$$

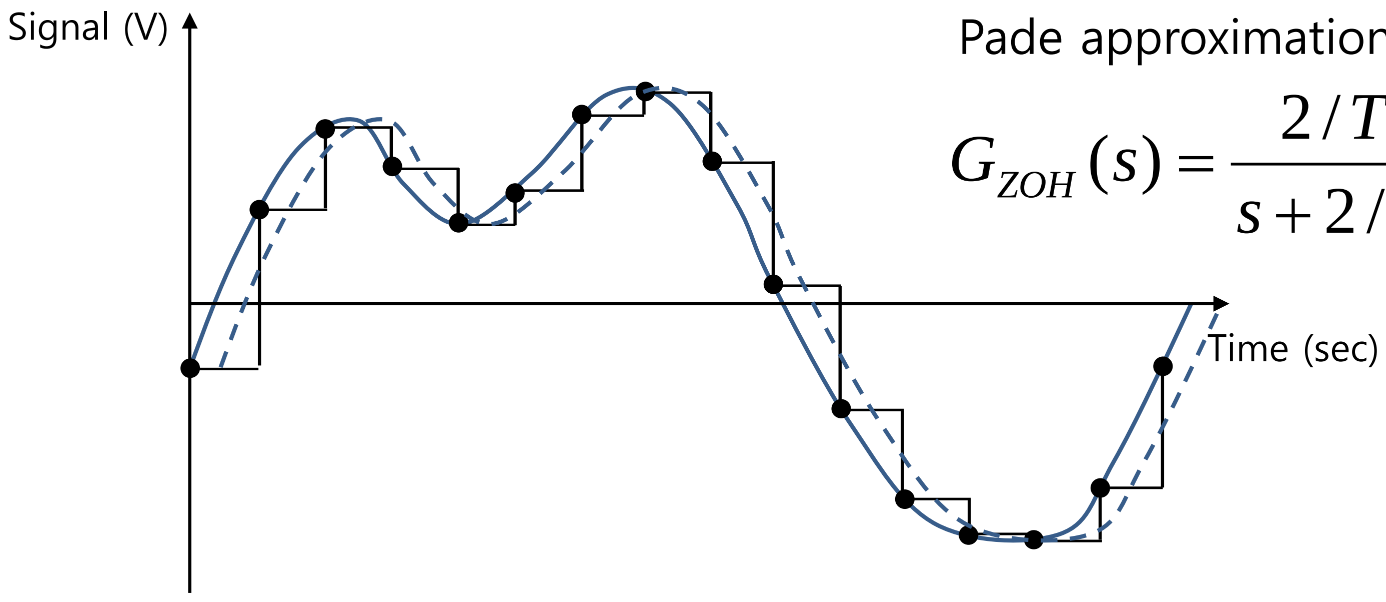
Anti-Alias Prefilters

- Anti-Alias Prefilters
  - To reduce the higher-frequency noise components in the analog signal in order to prevent aliasing, that is, having the noise be modulated to a lower frequency by the sampling process.
  - All spectral content above the half-sampling frequency will be removed by a low-pass anti-alias filter.



Effect of Sampling

- The most important impact of implementing a control system digitally is the delay associated with the hold. → degrades the stability and damping of the system.
- The error come about because the technique ignores the lagging effect of the ZOH which, on the average, is T/2.



Pade approximation

$$G_{ZOH}(s) = \frac{2/T}{s + 2/T}$$

Effect of Sampling

- Generally, sample rates should be faster than 30 times the bandwidth in order assure that the digital controller can be made to closely match the performance of the continuous controller.

Effect of Sampling - Example

Controller

$$D(s) = 70\frac{s+2}{s+10}$$

Plant

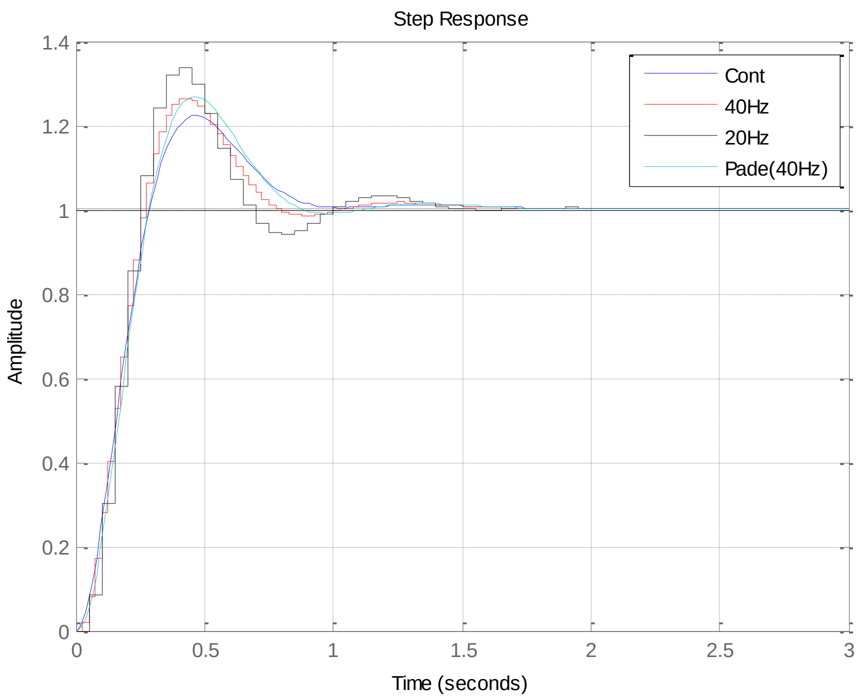
$$G(s) = \frac{1}{s(s+1)}$$

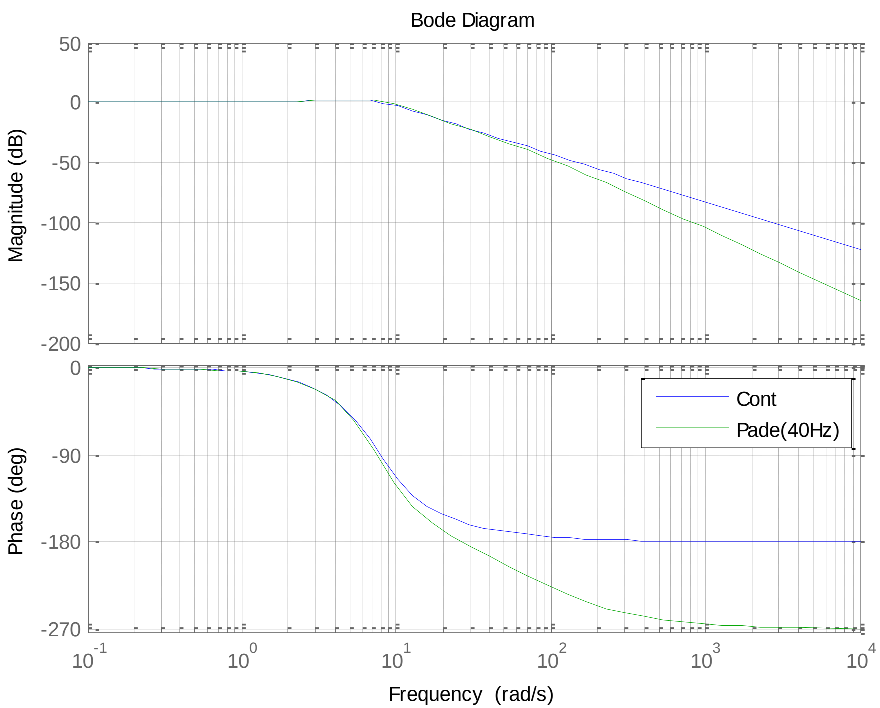
Feedback system.
Sampling 40Hz and 20Hz.

```
% continuous-time system
sysD_c = 70*tf([1 2], [1 10]);
sysG_c = tf(1, [1 1 0]);
sys_c = sysD_c*sysG_c;
sysCL_c = feedback (sys_c, 1);
figure (1);
step(sysCL_c); grid on; hold on;

% discrete-time system, 40 Hz
Ts1 = 1/40;
sysD_d1 = c2d(sysD_c, Ts1, 'zoh');
sysG_d1 = c2d(sysG_c, Ts1, 'zoh');
sys_d1 = sysD_d1*sysG_d1;
sysCL_d1 = feedback (sys_d1, 1);
step(sysCL_d1, 'r-'); grid on; hold on;

% discrete-time system, 20 Hz
Ts2 = 1/20;
sysD_d2 = c2d(sysD_c, Ts2, 'zoh');
sysG_d2 = c2d(sysG_c, Ts2, 'zoh');
sys_d2 = sysD_d2*sysG_d2;
sysCL_d2 = feedback (sys_d2, 1);
step(sysCL_d2, 'k-'); grid on; hold on;
```

```
% Pade approximation
T = Ts1;
numDL = 2/T; denDL = [1 2/T];
sys2 = tf(numDL, denDL)*sys_c;
sysCL_c_delay = feedback(sys2, 1);
step(sysCL_c_delay, 'c-'); grid on; hold on;
legend('Cont', '40Hz', '20Hz', 'Pade(40Hz)');

% bode plot
figure(2);
bode(sysCL_c); hold on;
bode(sysCL_c_delay); grid on;
legend('Cont', 'Pade(40Hz)');

[Gm1,Pm1,Wcg1,Wcp1] = margin(sysCL_c);
[Gm2,Pm2,Wcg2,Wcp2] = margin(sysCL_c_delay);

fprintf('Gain margin (cont/delay): %f/%f, Phase margin
(cont/delay): %f/%f₩n', Gm1, Gm2, Pm1, Pm2);
```

Effect of Sampling - Example

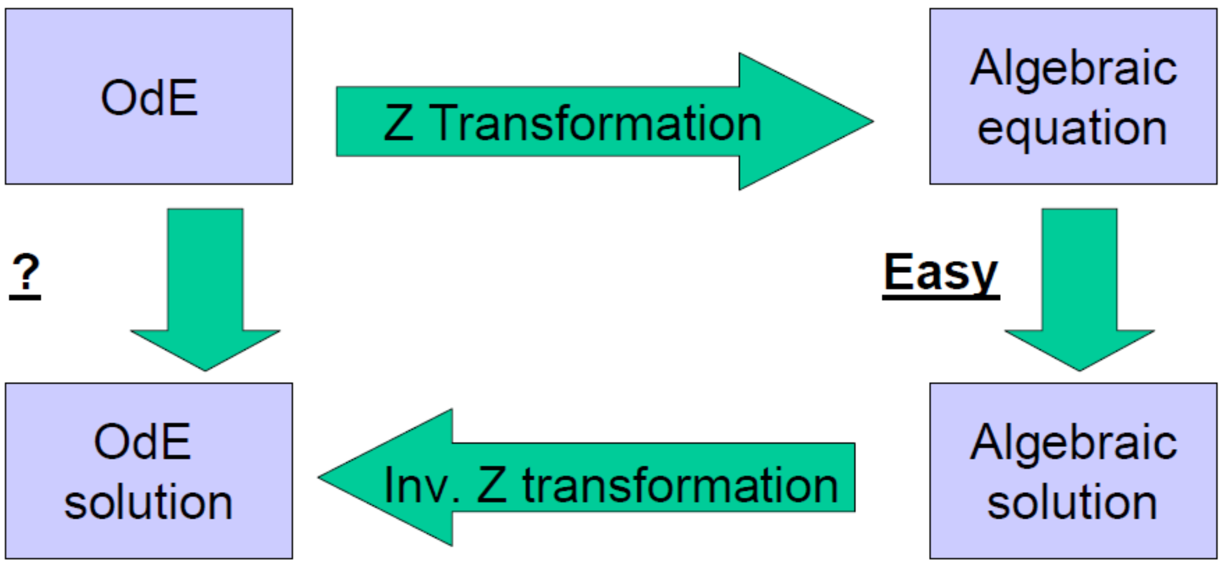




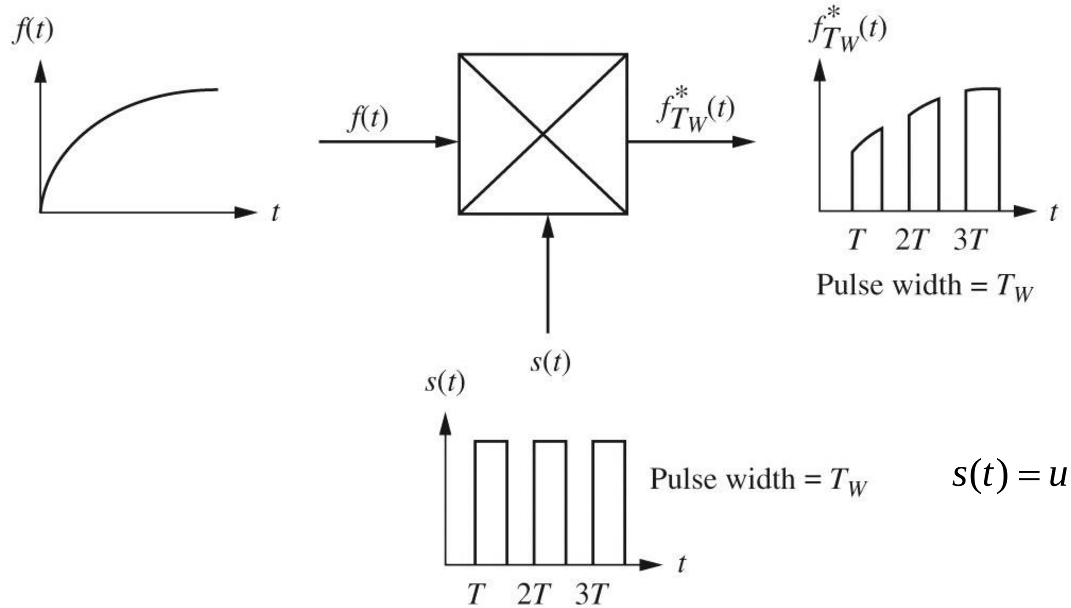
Gain margin (cont/delay): Inf/10.398993
Phase margin (cont/delay): 88.938724/73.255838

Introduction

- The Z transform is a powerful tool formulated to solve a wide variety of Ordinary difference Equations (OdEs).



Derivation of Z Transform



$$s(t) = u(t-kT) - u(t-kT-T_w)$$

$$f^*_{T_w} = f(t)s(t) = f(t)\sum_{k=-\infty}^{\infty} u(t-kT) - u(t-kT-T_w)$$

$$= \sum_{k=-\infty}^{\infty} f(kT)[u(t-kT) - u(t-kT-T_w)]$$

Laplace transform

$$F^*_{T_w}(s) = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-kTs-T_w s}}{s}\right] = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{1-e^{-T_w s}}{s}\right]e^{-kTs}$$

Derivation of Z Transform

Taylor series expansion

$$F_{T_w}^*(s) = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{1-\left\{1-T_w s+\frac{(T_w s)^2}{2!}-\cdots\right\}}{s}\right]e^{-kTs}$$

For small T_w ,

$$F_{T_w}^*(s) = \sum_{k=-\infty}^{\infty} f(kT)\left[\frac{T_w s}{s}\right]e^{-kTs} = \sum_{k=-\infty}^{\infty} f(kT)T_w e^{-kTs}$$

$$f_{T_w}^*(t) = \underbrace{T_w}_{\text{Pulse width}} \underbrace{\sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)}_{\text{Ideal sampler}}$$

$\delta(t-kT)$: Dirac delta function

Derivation of Z Transform

- Our goal is to develop a transform that contains the information of sampling from which sampled-data systems can be modeled with transfer functions, analyzed, and designed with the ease and insight we enjoyed with the Laplace transform.

$$F'(s) = \sum_{k=0}^{\infty} f(kT)e^{-kTs}$$

- Now, letting $z = e^{Ts}$

$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k}$$

$$cf.\ F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Discrete time sequence

- We will use the notation $f(k)$ to denote a discrete time sequence.

$$f : Z_{+} \to \Re$$

$$\{f(0),\ f(1),\ f(2),\ f(3),\ \cdots\}$$

- The symbol Z_{+} denote the set of non-negative integers k. Also,

$$f(k) = 0 \quad for\ all\ k < 0$$

Z transform definition

- For a sequence

$$f : Z_+ \rightarrow \Re$$

- Z transform: $\boxed{F(z) = \mathcal{Z}\{f(k)\}}$ $\quad z \in \mathcal{C}$ Complex number

$$\boxed{\begin{aligned} \mathcal{Z}\{f(k)\} &\triangleq \sum_{k=0}^{\infty} f(k) z^{-k} \\ &= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots \end{aligned}}$$

Examples

- Geometric sequence:

$$f(k) = p^k = \{1, p, p^2, p^3, \cdots\}$$

$$\boxed{\begin{aligned} f(k) &= p^k \\ F(z) &= \frac{z}{z-p} \end{aligned}}$$

Examples

- Geometric sequence:

$$
\begin{aligned}
F(z) &= \sum_{k=0}^{\infty} p^k z^{-k} = \sum_{k=0}^{\infty} (pz^{-1})^k \\
&= 1 + \underbrace{pz^{-1} + (pz^{-1})^2 + (pz^{-1})^3 + \cdots}_{pz^{-1}\left[\underbrace{1 + pz^{-1} + (pz^{-1})^2 + (pz^{-1})^3 + \cdots}_{F(z)}\right]} \\
&= 1 + pz^{-1} F(z) \\
&= \frac{1}{1 - pz^{-1}} = \frac{z}{z - p}
\end{aligned}
$$

Examples

- Step sequence:

$$\begin{aligned} f(k) &= 1(k) \\ &= \{1,\,1,\,1,\,\cdots\} \end{aligned}$$

$$\boxed{F(z) = \frac{z}{z-1}}$$

- Calculation: set p=1

Examples

- Sine:

$$\boxed{\begin{array}{c} f(k) = \sin(\omega k) \\[2em] F(z) = \dfrac{\sin(\omega)\, z}{z^2 - 2\cos(\omega)\, z + 1} \end{array}}$$

- Notice that, if $\omega_1 = \omega + 2\pi$

$$\sin(\omega_1 k) = \sin([\omega + 2\pi]k) = \sin(\omega k)$$

Examples

- Calculation: Use

$$e^{j\omega} = \cos(\omega) + j\sin(\omega)$$

$$\cos(\omega k) = \frac{e^{j\omega k} + e^{-j\omega k}}{2}, \quad \sin(\omega k) = \frac{e^{j\omega k} - e^{-j\omega k}}{2j}$$

$$\begin{aligned} F(z) &= \sum_{k=0}^{\infty} \sin(\omega k) z^{-k} = \frac{1}{2j}\left[\frac{z}{z - e^{j\omega}} - \frac{z}{z - e^{-j\omega}}\right] \\ &= \frac{1}{2j}\left[\frac{z[(z - e^{-j\omega}) - (z - e^{j\omega})]}{(z - e^{j\omega})(z - e^{-j\omega})}\right] \\ &= \frac{\sin(\omega)\, z}{z^2 - 2\cos(\omega)\, z + 1} \end{aligned}$$

Examples

- Cosine:

$$\boxed{\begin{aligned} f(k) &= \cos(\omega k) \\ F(z) &= \frac{z\,(z-\cos(\omega))}{z^2 - 2\cos(\omega)\,z + 1} \end{aligned}}$$

- Calculation: Use

$$f(k) = \cos(\omega k) = \frac{1}{2}\left[e^{j\omega k} + e^{-j\omega k}\right]$$

Examples

- Pulse:

$$\boxed{\begin{array}{l} f(k) = \delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \\ \\ F(z) = 1 \end{array}}$$

- Calculation: Use

$$F(z) \;=\; \sum_{k=0}^{\infty} \delta(k) z^{-k} = 1 + \underbrace{0z^{-1} + 0z^{-2} + 0z^{-3} + \cdots}_{=0} = 1$$

Properties of the Z transform

- Linearity:

For any $\alpha,\ \beta \in \mathcal{C}$ and functions $f(k),\ g(k)$

$$F(z) = \mathcal{Z}\left\{f(k)\right\} \qquad\qquad G(z) = \mathcal{Z}\left\{g(k)\right\}$$

$$\boxed{\begin{aligned} \mathcal{Z}\left\{\alpha\, f(k) + \beta\, g(k)\right\} &= \alpha\, \mathcal{Z}\left\{f(k)\right\} + \beta\, \mathcal{Z}\left\{g(k)\right\} \\ &= \alpha\, F(z) + \beta\, G(z) \end{aligned}}$$

Properties of the Z transform

- One step advanced sequence:

$$f(k+1) \triangleq \{f(1),\, f(2),\, f(3), \cdots\}$$

- Let, $F(z) = \mathcal{Z}\{f(k)\}$

$$\boxed{\mathcal{Z}\{f(k+1)\} \quad = \quad z\,F(z) - z\,f(0)}$$

Properties of the Z transform

- Calculation: $f(k+1) = \{f(1),\, f(2),\, f(3), \cdots\}$

$$
\begin{aligned}
\mathcal{Z}\{f(k+1)\} &= \sum_{k=0}^{\infty} z^{-k} f(k+1) \\
&= \sum_{k=0}^{\infty} z^{-k} f(k+1) + \overbrace{z f(0) - z\, f(0)}^{=0} \\
&= \underbrace{z\, f(0) + z \sum_{k=1}^{\infty} z^{-k} f(k)}_{= z \sum_{k=0}^{\infty} z^{-k} f(k) = zF(z)} - z\, f(0) \\
&= zF(z) - z f(0)
\end{aligned}
$$

Properties of the Z transform

- N steps advanced sequence:

$$f(k+N) \triangleq (f(N),\, f(N+1),\, f(N+2), \cdots)$$

$$\boxed{\mathcal{Z}\{f(k+N)\} \quad = \quad z^N F(z) - \sum_{j=1}^{N} z^j f(N-j)}$$

Properties of the Z transform

- One step delay:

$$f(k-1) = \begin{cases} 0 & k < 1 \\ f(k-1) & k \geq 1 \end{cases}$$

$$f(k-1) = \{0, f(0), f(1), f(2), \cdots\}$$

$$\boxed{\mathcal{Z}\{f(k-1)\} \quad = \quad z^{-1} F(z)}$$

Properties of the Z transform

- N step delay:

$$f(k-N)=\begin{cases}0 & k<N\\ f(k-N) & k\ge N\end{cases}$$

$$f(k-N)=\{0,\cdots,0,\quad f(0),\quad f(1),\quad f(2),\quad \cdots\qquad\}$$

$$\boxed{Z(f(k-N))=z^{-N}F(z)}$$

Properties of the Z transform

- Convolution: Given $f(k),\ g(k)$

$$(f \star g)(k) = \sum_{j=0}^{k} f(k-j)g(j)$$

$$= (g \star f)(k)$$

$$\boxed{\mathcal{Z}\{(f \star g)(k)\} = F(z)\,G(z)}$$

Properties of the Z transform

$$
\begin{aligned}
\mathcal{Z}\left\{(f \star g)(k)\right\} &= \sum_{k=0}^{\infty} z^{-k} \left\{ \sum_{j=0}^{k} f(k-j) g(j) \right\} \\
&= \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{\infty} z^{-(k-j)} f(k-j) z^{-j} g(j) \right\} \\
&\qquad (f(k-j) = 0 \text{ for } j > k) \\
&= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\infty} z^{-(k-j)} f(k-j) \right\} z^{-j} g(j) \\
&= \sum_{j=0}^{\infty} \left\{ \sum_{m=-j}^{\infty} z^{-m} f(m) \right\} z^{-j} g(j) \\
&\qquad (f(m) = 0 \text{ for } m < 0) \\
&= \left\{ \sum_{m=0}^{\infty} z^{-m} f(m) \right\} \left\{ \sum_{j=0}^{\infty} z^{-j} g(j) \right\} = F(z) G(z)
\end{aligned}
$$

Properties of the Z transform

- Initial Value Theorem:

$$\boxed{f(0) = \lim_{z\to\infty} F(z)}$$

- Final Value Theorem: If $\lim_{k\to\infty} f(k)$ exists,

$$\boxed{\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)F(z)}$$

Properties of the Z transform

- Final Value Theorem:

$$Z[f(k+1)-f(k)] = \sum_{j=0}^{\infty}\left[f(j+1)-f(j)\right]z^{-j}$$

$$zF(z)-zf(0)-F(z) = (z-1)F(z)-zF(0) = \lim_{k\to\infty}\sum_{j=0}^{k}\left[f(j+1)-f(j)\right]z^{-j}$$

$$\lim_{z\to 1}(z-1)F(z)-f(0) = \lim_{k\to\infty} f(k+1)-f(0) = \lim_{k\to\infty} f(k)-f(0)$$

$$\lim_{z\to 1}(z-1)F(z) = \lim_{k\to\infty} f(k)$$

Z transform application

- Solution of a first order OdE

Let: $a \neq 1,\, b \neq 0,\; y(0) = y_o \in \mathcal{R}$

Obtain the solution to the OdE:

$$y(k+1) = a\,y(k) + b\,1(k)$$

$$1(k) = \{1,\; 1,\; 1,\; \cdots\}$$

Z transform application

- Apply Z transform

$$\mathcal{Z}\{y(k+1)\} = a\,\mathcal{Z}\{y(k)\} + b\,\mathcal{Z}\{1(k)\}$$

$$z\,Y(z) - z\,y(0) = a\,Y(z) + b\,\frac{z}{z-1}$$

- Algebraic manipulation

$$\boxed{Y(z) = \frac{z}{z-a}y(0) + \frac{b\,z}{(z-1)(z-a)}}$$

Z transform application

- Apply inverse Z transform

$$\mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{z}{z-a}\right\} y(0) + \mathcal{Z}^{-1}\left\{\frac{b\,z}{(z-1)(z-a)}\right\}$$

- Use table loop up and partial fraction expansion

$$\boxed{y(k) = a^k\,y(0) + \frac{b}{1-a}\left\{1(k) - a^k\right\}}$$

Z transform application

- Use table loop up and partial fraction expansion

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-a}\right\} = a^k \qquad \mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = 1(k)$$

$$\frac{b\,z}{(z-1)(z-a)} = \frac{b}{1-a}\left\{\frac{z}{z-1} - \frac{z}{z-a}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{b\,z}{(z-1)(z-a)}\right\} = \frac{b}{1-a}\left\{1(k) - a^k\right\}$$

Transfer function

- N-th order difference equation:

$$y(k) + a_{n-1}y(k-1) + \cdots + y(k-n) = b_m u(k+m-n) + \cdots + b_0 u(k-n)$$

z transformation

$$\left(1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}\right)Y(z) = \left(b_m z^{-n+m} + b_{m-1}z^{-n+m-1} + \cdots + b_0 z^{-n}\right)U(z)$$

$$\Rightarrow Y(z) = \frac{b_m z^{-n+m} + b_{m-1}z^{-n+m-1} + \cdots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}}U(z)$$

$$\Rightarrow Y(z) = \frac{b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_0 z}U(z)$$

Transfer function

$$\boxed{G(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_0}}$$

- $\boldsymbol{A(z)=0}$: Characteristic equation
- Roots of C.E. = Poles of $\boldsymbol{G(z)}$
- Roots of $\boldsymbol{B(z)=0}$ = Zeros of $\boldsymbol{G(z)}$
- $\boldsymbol{m \leq n}$: realizability condition