HW3-1: Linear System Theory (ECE532)

Instructor: Jun Moon

Due Date: April 2 at the beginning of the class (in addition to HW3)

Problem 1: You will prove the Rank-Nullity theorem discussed in class. Suppose that $A : \mathbb{V} \rightarrow \mathbb{W}$ is a linear operator, where both $\mathbb{V}$ and $\mathbb{W}$ are, respectively, n- and m-dimensional vector spaces (finite-dimensional vector spaces). Show that

$$\dim(R(A)) + \dim(N(A)) = \dim(\mathbb{V}) = n.$$

Problem 2: Suppose $x_1, \ldots, x_k$ is a list of numbers in $\mathbb{R}$ (the set of real numbers), and let $\mathbb{Q}$ denote the set of rational numbers. Define the set of real numbers

$$\mathbb{V}(x_1, x_2, \ldots, x_k) := \{v \in \mathbb{R} : \text{ there exists } \alpha_1, \ldots, \alpha_k \in \mathbb{Q} \text{ such that } v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k\}.$$

That is, $\mathbb{V}(x_1, x_2, \ldots, x_k)$ is the set of numbers that can be written in the form $v = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$, where $v \in \mathbb{V}(x_1, x_2, \ldots, x_k)$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{Q}$.

1. Show that $\mathbb{V}(x_1, x_2, \ldots, x_k)$ is the vector space over the field $\mathbb{Q}$.
2. What is the largest and smallest the dimension of $\mathbb{V}(x_1, x_2, \ldots, x_k)$ can be?
3. Determine the dimension of $\mathbb{V}(\sqrt{2}, \pi)$
4. Determine the dimension of $\mathbb{V}(\sqrt{2}, \pi, 2\sqrt{2} - \frac{2}{3}\pi)$