[MEN573] Advanced Control Systems I

Lecture 12 – Stability
Part II Lyapunov's Direct Method

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Outline

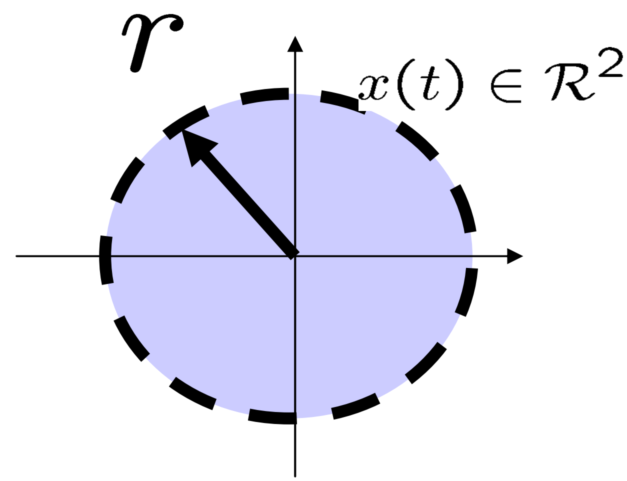
- Positive Definite and Quadratic Functions
- The Direct Method of Lyapunov
- Lyapunov Stability Theorems
  - Stability in the sense of Lyapunov
  - Asymptotic stability
- L'Salles Asymptotic Stability Theorem
- Instability Theorems

Locally Positive Definite Functions (LPDF)

A continuous LPDF $W : \mathcal{R}^n \rightarrow \mathcal{R}_+$ *satisfies:*

(i) $W(x) > 0$ $\quad \forall x \neq 0$ and $|x| < r$

(ii) $W(0) = 0$



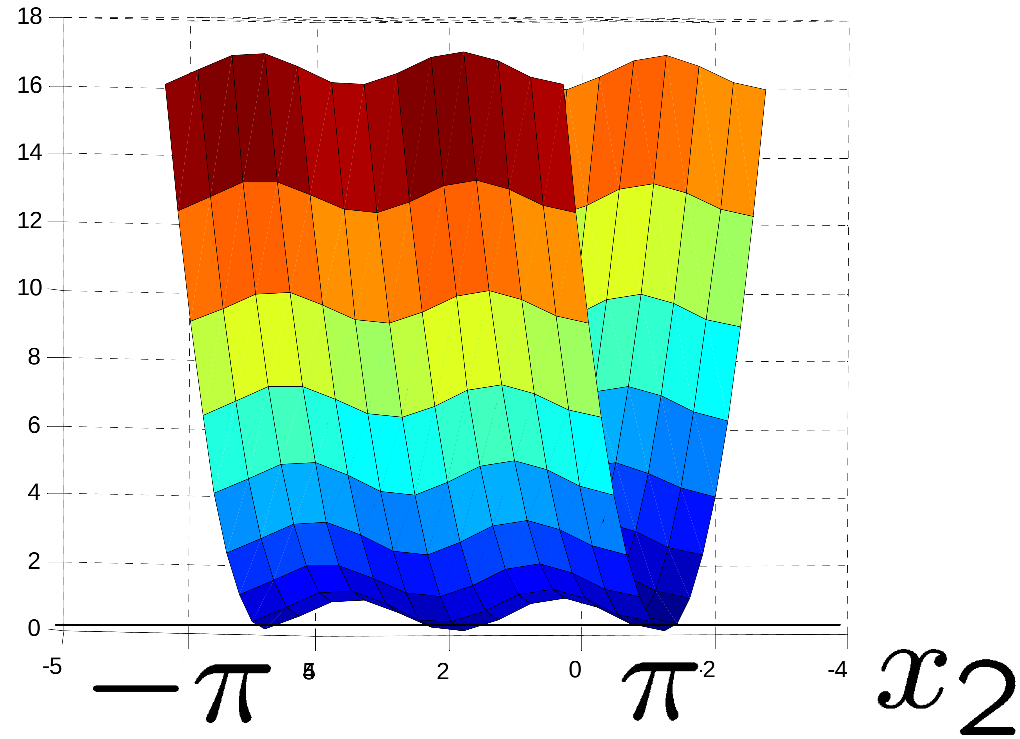
Locally Positive Definite Functions (LPDF)

Example:

$$W(x_1, x_2) = x_1^2 + \sin^2(x_2)$$

is PDF for

$$x_1 \in \mathcal{R} \text{ and } |x_2| < \pi$$



Positive Definite Functions (PDF)

A continuous PDF $W : \mathcal{R}^n \to \mathcal{R}_+$ *satisfies:*

(i) $W(x) > 0 \quad \forall x \neq 0$

(ii) $W(0) = 0$

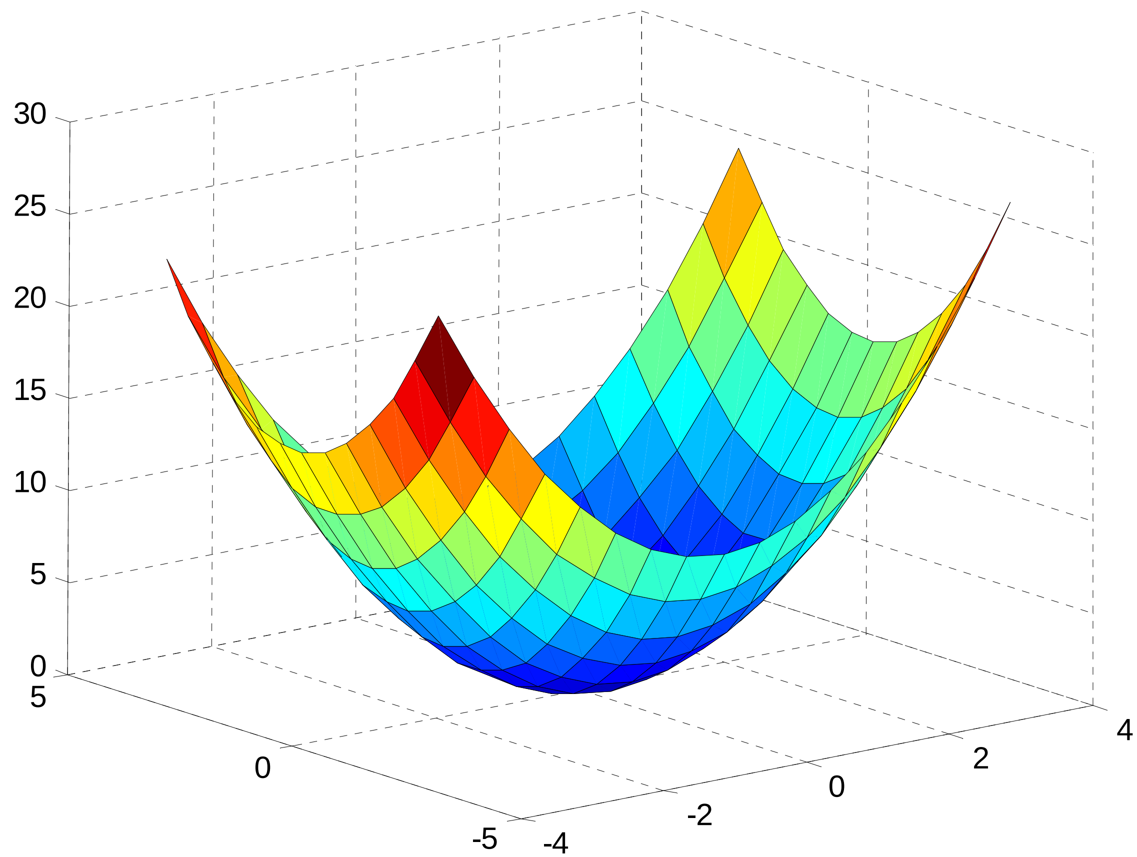
(iii) $W(x) \to \infty$ as $|x| \to \infty$, uniformly in x

Positive Definite Functions (PDF)

Example:

$$W(x_1, x_2) = x_1^2 + x_2^2$$

is a PDF



Positive Semi-Definite Functions (PSDF)

A continuous PSDF $W : \mathcal{R}^n \to \mathcal{R}_+$ *satisfies:*

(i) $W(x) \geq 0 \qquad \forall x$

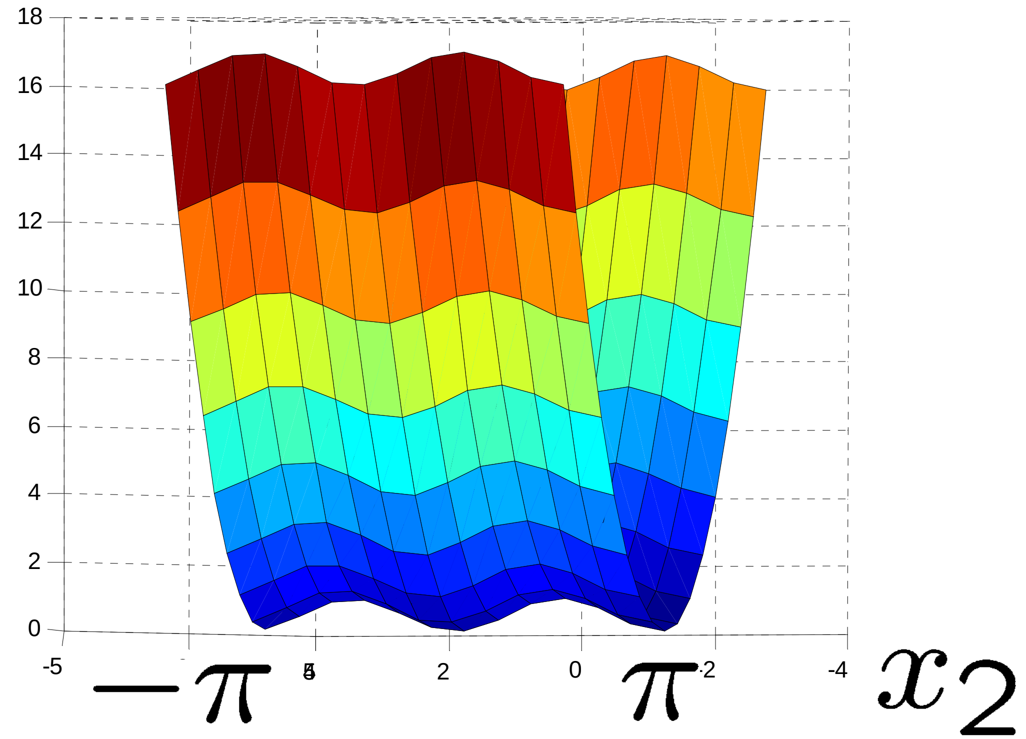
(ii) $W(0) = 0$

Positive Semi-Definite Functions (PSDF)

Example:

$$W(x_1, x_2) = x_1^2 + \sin^2(x_2)$$

is a PSDF



Quadratic functions

A quadratic function $Q : \mathcal{R}^n \to \mathcal{R}$ *is a function of the form:*

$$Q(x) = \sum_{i}^{n} \sum_{j}^{n} p_{ij} x_i x_j \qquad p_{ij} = p_{ji}$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{Q(x) = x^T P x \,, \qquad P^T = P}$$

Quadratic functions

We will generally express quadratic forms as

$$Q(x) = x^T P x\,, \qquad\qquad P^T = P$$

since all quadratic forms can be expressed in this manner.

Quadratic functions

All square matrices can be expressed as a sum of a symmetric matrix and a skew symmetric matrix:

$$M = P + S$$

$$P = P^T \qquad S = -S^T$$

$$P = \frac{1}{2}M + \frac{1}{2}M^T \qquad S = \frac{1}{2}M - \frac{1}{2}M^T$$

$$x^T M x = x^T P x \qquad x^T S x = 0$$

Symmetric Matrices

Fact 1: All eigenvalues of a symmetric matrix are real.

Fact 2: Distinct eigenvectors of a symmetric matrix are orthogonal

Fact 3: Symmetric matrices can always be diagonalized

Symmetric Matrices

Fact 1: All eigenvalues of a symmetric matrix are real.

Proof: Let λ_i be the ith eigenvalue of P and let

$$Pv_i = \lambda_i v_i$$

Assume that λ_i and v_i are both complex and

Let $v_i^* = \overline{v_i}^T$ (complex conjugate transpose)

Then it must also be true that $(\bar{\lambda}_i \lambda_i = |\lambda_i|)$

$$P\bar{v}_i = \bar{\lambda}_i \bar{v}_i$$

Symmetric Matrices

Lets now compute the quadratic form $v_i^* P v_i$,

$$v_i^*(Pv_i) \;=\; v_i^*(\lambda_i v_i) = \lambda_i v_i^* v_i = \lambda_i |v_i|$$

and, by the fact that P is symmetric,

$$(v_i^* P)\, v_i \;=\; (P\overline{v}_i)^T v_i = \overline{\lambda}_i v_i^* v_i = \overline{\lambda}_i |v_i|$$

Therefore,

$$\boxed{\overline{\lambda}_i = \lambda_i \Leftrightarrow \lambda_i \in \mathcal{R} \text{ and } v_i \in \mathcal{R}^n}$$

Symmetric Matrices

Fact 2: Distinct eigenvectors of a symmetric matrix are orthogonal

Proof: Let λ_i and λ_j be two distinct eigenvalues of a symmetric matrix $\boldsymbol{P}$ with associated eigenvectors $\boldsymbol{v_i}$ and $\boldsymbol{v_j}$. $(Pv_i = \lambda_i v_i \text{ and } Pv_j = \lambda_j v_j)$

$$v_j^T P v_i = \lambda_i v_j^T v_i$$

$$(Pv_j)^T v_i = \lambda_i v_j^T v_i$$

$$\boxed{\lambda_j v_j^T v_i = \lambda_i v_j^T v_i}$$

Thus,

$$\boxed{\lambda_i \neq \lambda_j \Leftrightarrow v_j^T v_i = 0}$$

Symmetric Matrices

Fact 3: Symmetric matrices can always be diagonalized

Proof: Let λ_m be *a repeated eigenvalue of* P of multiplicity 2 and assume that v_1 is one of its associated eigenvectors. We will show that

$$\text{nullity}\{[\lambda_m I - P]\} = 2$$

Attempt to compute a ***generalized eigenvector*** v_2

$$Pv_2 = \lambda_m v_2 + v_1$$

Symmetric Matrices

Then,

$$v_1^T P v_2 \;=\; \lambda_m v_1^T v_2 + v_1^T v_1$$

and, by the fact that P is symmetric and v_1 is an eigenvector,

$$\begin{aligned} (Pv_1)^T v_2 &= \lambda_m v_1^T v_2 + |v_1|^2 \\ \cancel{\lambda_m v_1^T v_2} &= \cancel{\lambda_m v_1^T v_2} + |v_1|^2 \end{aligned}$$

Therefore, a generalized eigenvector ***cannot*** be found, which in turn means that

$$\mathrm{nullity}\{[\lambda_m I - P]\} = 2$$

and two (orthogonal) eigenvectors can be found for the repeated eigenvalue λ_m

Symmetric Matrices

Fact 3: Symmetric matrices can always be diagonalized

Thus,

- The normalized eigenvectors of every symmetric matrix $P \in \mathcal{R}^{n \times n}$ form an ***orthonormal basis*** for $\mathcal{R}^n$.
- There exist an orthogonal coordinate transformation matrix $\boldsymbol{T}$ and a diagonal matrix Λ such that

$$\boxed{T^T P T = \Lambda \qquad T^T T = I}$$

Positive Definite Matrix

A symmetric matrix $P = P^T$

is positive definite iff the quadratic function

$$Q(x) = x^T P x$$

is a Positive Definite Function (PDF).

Positive definite symbol

We will use the symbol: $P \succ 0$

to denote that $\boldsymbol{P}$ is a **symmetric and positive definite matrix**.

We will use the symbol: $W(x) \succ 0$

to denote that $\boldsymbol{W(x)}$ is positive definite function (PDF).

Positive Semi-Definite Matrix

A symmetric matrix $P = P^T$

is positive semi-definite iff the quadratic function

$$Q(x) = x^T P x$$

is a Positive Semi-definite Function (PSDF).

Positive semi-definite symbol

We will use the symbol: $$P \succeq 0$$

to denote that $\boldsymbol{P}$ is a **symmetric and positive semi-definite matrix.**

We will use the symbol: $$W(x) \succeq 0$$

to denote that $\boldsymbol{W(x)}$ is positive semi-definite function (PDF).

Determining if a matrix is positive definite

For a symmetric matrix $\boldsymbol{P}$, the following statements are equivalent:

1) $P \succ 0$ (positive definite)

2) **<u>ALL</u>** eigenvalues of $\boldsymbol{P}$ are positive

3) **<u>ALL</u>** leading principal minors of $\boldsymbol{P}$ are positive

4) There exist a **<u>non-singular</u>** matrix $\boldsymbol{N}$ such that

$$P = N^* N$$

Determining if a matrix is positive semi-definite

For a symmetric matrix $\boldsymbol{P}$, the following statements are equivalent:

1) $P \succeq 0$ (positive semi-definite)

2) **ALL** eigenvalues of $\boldsymbol{P}$ are greater than or equal to zero, and at least one is zero.

3) **ALL** leading principal minors of $\boldsymbol{P}$ greater than or equal to zero, and at least one is zero.

4) There exist a **singular** matrix $\boldsymbol{N}$ such that

$$P = N^* N$$

Leading principal minors

The leading principal minors of

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$$

are

$$p_{11} \qquad \det\left(\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\right) \qquad \det(P)$$

Negative (semi-) definite symbol

$$P \prec 0 \quad \Longleftrightarrow \quad -P \succ 0$$

$$P \preceq 0 \quad \Longleftrightarrow \quad -P \succeq 0$$

Derivative along a state trajectory

- Consider an n-th order nonlinear time invariant continuous time system of the form:

$$\dot{x} = f(x) \qquad \text{with} \qquad f(0) = 0$$

- Let the function $W : \mathcal{R}^n \rightarrow \mathcal{R}_+$ have continuous partial derivatives.
- Denote the gradient of W with respect to x by

$$\nabla W(x) = \begin{bmatrix} \frac{\partial W}{\partial x_1} & \cdots & \frac{\partial W}{\partial x_n} \end{bmatrix}^T$$

Derivative along a state trajectory

- The derivative of ***W(x)*** along the state trajectory

$$\dot{x} = f(x)$$

is

$$\frac{dW(x)}{dt} = \nabla W^T(x)\, f(x)$$

$$= \begin{bmatrix} \frac{\partial W}{\partial x_1} & \cdots & \frac{\partial W}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

Derivative along a state trajectory

The expression

$$\nabla W^T(x)\, f(x) = \begin{bmatrix} \frac{\partial W}{\partial x_1} & \cdots & \frac{\partial W}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

Is also called the Lie derivative of ***W(x)*** with respect to the vector field ***f(x)***.

$$L_f W(x) = \nabla W^T(x)\, f(x)$$

The Direct Method of Lyapunov

- The direct method of Lyapunov lets us study stability of dynamic systems:

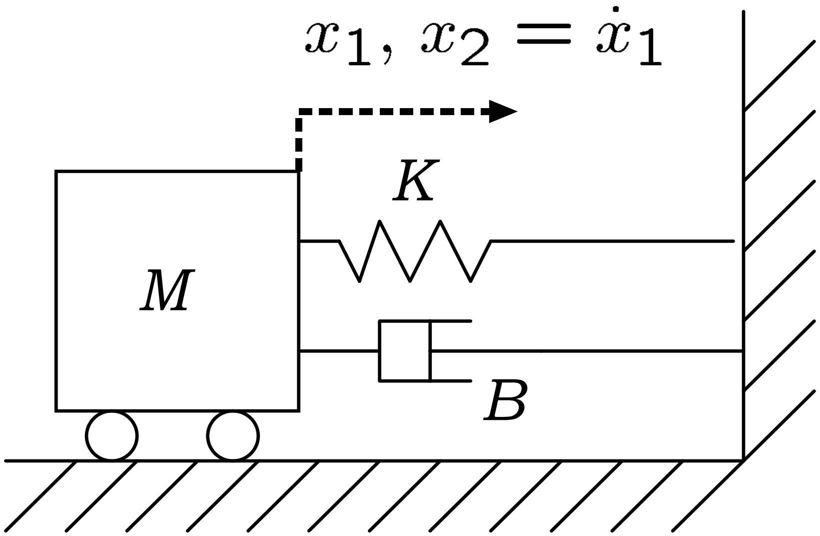
  – linear/nonlinear,
  – time-invariant/time varying

  without solving the state equations explicitly.

- It is based on computing a scalar function called the ***Lyapunov function candidate***, which can be viewed as a generalization of the *energy function*.

Stability from the Energy Viewpoint

- Consider the mechanical system:



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

position

velocity

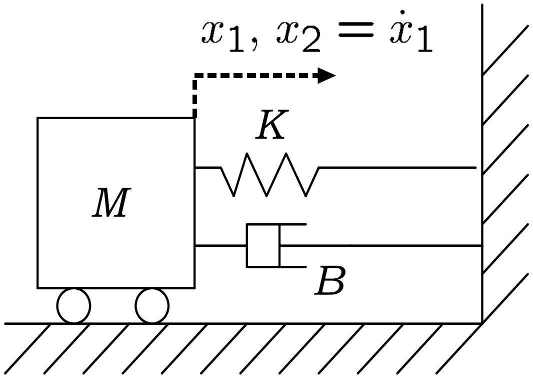
- Energy:

$$E(t) = \underbrace{\frac{1}{2} K\, x_1^2(t)}_{\text{potential}} + \underbrace{\frac{1}{2} M\, x_2^2(t)}_{\text{kinetic}}$$

Energy is a PDF

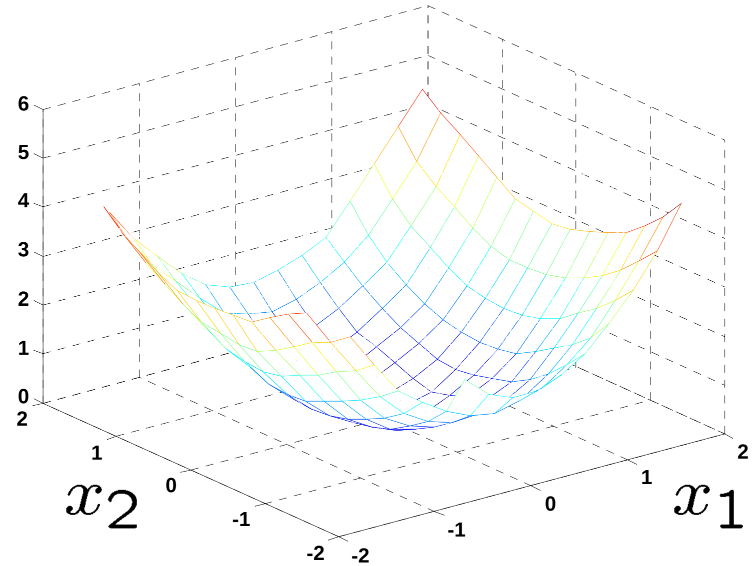
- Energy:

$$E(x_1, x_2) = \frac{1}{2}\left(K\,x_1^2 + M\,x_2^2\right)$$



$$E(x_1, x_2) = \frac{1}{2}\left[\begin{array}{cc} x_1 & x_2 \end{array}\right]\left[\begin{array}{cc} K & 0 \\ 0 & M \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \succ 0$$

PDF



Stability from the Energy Viewpoint

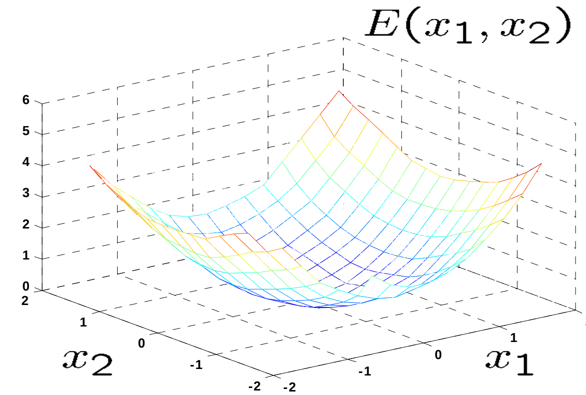
$$E(x_1, x_2) = \frac{1}{2}\left(K\,x_1^2 + M\,x_2^2\right)$$

- Time derivative of the energy

$$\begin{aligned}
\dot{E} &= (K\,x_1\,\dot{x}_1 + M\,x_2\,\dot{x}_2) \\
&= \cancel{K\,x_1\,x_2} + M\,x_2\left(\cancel{-\frac{K}{M}x_1} - \frac{B}{M}x_2\right) \\
&= -B\,x_2^2 \preceq 0
\end{aligned}$$

is negative, as long as $x_2 \neq 0$

Stability from the Energy Viewpoint



- Thus, since

$$\dot{E}(t) < 0 \Leftrightarrow x_2 \neq 0$$

- Energy will continue to decrease unless $x_2 = 0$

- Since $(x_1 \neq 0, x_2 = 0)$ is not an equilibrium state, the motion cannot stop at $(x_1 \neq 0, x_2 = 0)$

Thus,

$$E \to 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Lyapunov Function

V(x) is a Lyapunov function for the system

$$\dot{x} = f(x)$$

if there exists a constant ***r > 0*** such that:

1) $$V(x) \succ 0 \qquad \forall\, |x| < r$$

2) $$\dot{V}(x) = \nabla V^T(x)\, f(x) \preceq 0$$

for all $|x| < r$

Lyapunov Function

Lyapunov function ***$V(x)$*** must be:

1) <u>**Locally Positive Definite (LPDF)**</u>

$$V(x) \succ 0 \qquad \forall\, |x| < r$$

2) its derivative along the state trajectory must be
<u>**Locally Negative Semi-definite**</u>:

$$\dot{V}(x) \preceq 0 \qquad \forall |x| < r$$

Lyapunov Stability Theorem (CT)

The equilibrium state 0 of an n-th order nonlinear time invariant continuous time system of the form:

$$\dot{x} = f(x)$$

is stable in the sense of Lyapunov if there exists a Lyapunov function $\boldsymbol{V(x)}$ for this system.

Lyapunov Asymptotic Stability Theorem (CT)

The equilibrium state 0 of an n-th order nonlinear system of the form:

$$\dot{x} = f(x)$$

is <u>locally asymptotically stable</u> if there exists a **Lyapunov function** $V(x)$ such that, for some $r > 0$

$$\dot{V}(x) \prec 0 \qquad \forall |x| < r$$

(local negative definite function)

Lyapunov Global Asymptotic Stability Theorem (CT)

The equilibrium state 0 of an n-th order nonlinear system of the form:

$$\dot{x} = f(x)$$

is <u>globally asymptotically stable</u> if there exists a **Lyapunov function, such that**

$$V(x) \succ 0$$ **PDF**

$$\dot{V}(x) \prec 0$$ **NDF**

LaSalle's Asymptotic Stability Theorem (CT)

Provides relaxed conditions for asymptotic stability

LaSalle's theorem only applies to:

- Time invariant systems $\dot{x} = f(x)$
- Periodic systems $\dot{x} = f(x,t)$

$T > 0$ $\boxed{f(x,t) = f(x,t+T)}$

LaSalle's Asymptotic Stability Theorem (CT)

The equilibrium state 0 of:

$$\dot{x} = f(x)$$

is locally asymptotically stable if:

1) There exist a Lyapunov function ***V(x).***

2) The set $S = \left\{ x : V(x) \leq m, \underline{\dot{V}(x) = 0} \right\}$

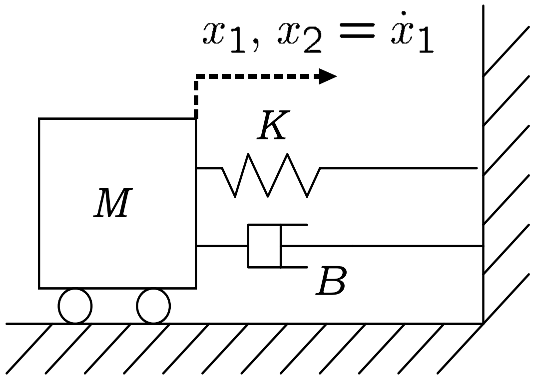
contains no trajectories other than ***x = 0***, where

$$m = \sup_{|x| \leq r} V(x)$$

LaSalle's Theorem Example

- Example:

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= -\frac{K}{M}x_1 - \frac{B}{M}x_2\end{aligned}$$



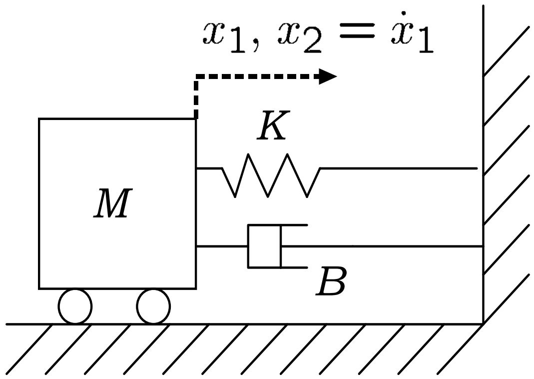
Consider the PDF

$$V(x) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succ 0$$

LaSalle's Theorem Example

- Example:

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = -\frac{K}{M}x_1 - \frac{B}{M}x_2$$



Take the time derivative of ***V(x)***:

$$\dot{V}(x) = -B\,x_2^2$$

$$\dot{V}(x) = -\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \preceq 0$$

LaSalle's Theorem Example

Thus:

$$V(x) \succ 0 \qquad\qquad \dot{V}(x) \preceq 0$$

$$\Downarrow$$

$V(x)$ is a Lyapunov function

$$\Downarrow$$

$$\boxed{\text{Stability in the sense of Lyapunov follows}}$$

LaSalle's Theorem Example

- Since $\dot{V}(x) = -Bx_2^2$

$$S = \left\{x : V(x) \le m, \dot{V}(x) = 0\right\} \qquad m = \sup_{|x| \le r} V(x)$$

$$\Downarrow$$

$$S = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \ |x_1| < m' \right\} \qquad m' = \sqrt{2m / K}$$

LaSalle's Theorem Example

To prove asymptotic stability, we must show that the only possible solution of the state equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -\frac{K}{M}x_1 - \frac{B}{M}x_2 \end{aligned}$$

that satisfies:

$$x_s(t) = \begin{bmatrix} x_{s1}(t) \\ x_{s2}(t) \end{bmatrix} = \begin{bmatrix} x_{s1}(t) \\ 0 \end{bmatrix}$$

is $$x_s(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

LaSalle's Theorem Example

Assume: $$x_s(t_o) = \begin{bmatrix} x_{s1}(t_o) \\ 0 \end{bmatrix}$$ where

$$0 < |x_{s1}(t_0)| \leq \sqrt{2m/K}$$

Then:
$$\begin{aligned} \frac{dx_{s1}}{dt} &= x_{s2} \\ \frac{dx_{s2}}{dt} &= -\frac{K}{M}x_{s1} - \frac{B}{M}x_{s2} \end{aligned}$$

$$\Downarrow$$

$$\dot{x}_{s2}(t_o) = -\frac{K}{M}x_{s1}(t_o) \neq 0$$

LaSalle's Theorem Example

- Therefore, the only trajectory that remains in

$$S = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix},\ |x_1| < m' \right\} \qquad m' = \sqrt{2m/K}$$

is

$$x_s(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\text{Asymptotic stability follows from LaSalle's Theorem}}$$

Remarks

- The previous theorems only provide sufficient but not necessary conditions

  – i.e. failure to find a Lyapunov function does not imply instability.

- If a PDF function $W(x)$ does not satisfy the conditions of the theorems, nothing can be concluded.

  – It simply means that $W(x)$ *is not a Lyapunov function.*

Instability Theorem (CT)

The equilibrium state 0 of:

$$\dot{x} = f(x)$$

is <u>unstable</u> if there exists function ***W(x)*** such that:

1) $\dot{W}(x) \succ 0, \qquad \forall |x| < r$

(i.e $\dot{W}(x)$ is a **LPDF**).

2) There exists states ***x*** arbitrarily close to the origin such that

W(x) > 0.

Discrete Time Systems

All of the previous theorems have counterparts for discrete time systems of the form

$$x(k+1) = f(x(k))$$

$$f(0) = 0$$

Difference along a state trajectory

- Consider an n-th order discrete time system

$$x(k+1) = f(x(k))$$

- For a function $W : \mathcal{R}^n \to \mathcal{R}_+$

We denote

$$\Delta W(x(k)) = W(x(k+1)) - W(x(k))$$

$$= W(f(x(k))) - W(x(k))$$

Lyapunov Function

V(x) is a Lyapunov function for the system

$$x(k+1) = f(x(k))$$

if there exists a constant ***r > 0*** such that:

1) $$V(x) \succ 0 \qquad \forall\, |x| < r$$

2) $$\Delta V(x) \preceq 0$$

for all $|x| < r$

Lyapunov Function

Lyapunov function V(x) must be:

1) **<u>Locally Positive Definite (LPDF)</u>**

$$V(x) \succ 0 \qquad \forall\, |x| < r$$

2) its difference along the state trajectory must be **<u>Locally Negative Semi-definite</u>**:

$$\Delta V(x) \preceq 0 \qquad \forall |x| < r$$

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The equilibrium state 0 of an n-th order nonlinear time invariant continuous time system of the form:

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is stable in the sense of Lyapunov if there exists a Lyapunov function ***V(x)*** for this system.

Lyapunov Asymptotic Stability Theorem (DT)

The equilibrium state 0 of an n-th order nonlinear system of the form:

$$x(k+1) = f(x(k))$$

is <u>locally asymptotically stable</u> if there exists a **Lyapunov function** $V(x)$ such that, for some $r > 0$

$$\Delta V(x) \prec 0 \qquad \forall |x| < r$$

(local negative definite function)

LaSalle's Asymptotic Stability Theorem (DT)

Provides relaxed conditions for asymptotic stability

LaSalle's theorem only applies to:

- Time invariant systems $x(k+1) = f(x(k))$

- Periodic systems $x(k+1) = f(x(k), k)$

$L \geq 1$

$$\boxed{f(x, k) = f(x, k+L)}$$

LaSalle's Asymptotic Stability Theorem (DT)

The equilibrium state 0 of:

$$x(k+1) = f(x(k))$$

is <u>locally asymptotically stable</u> if:

1) There exist a Lyapunov function ***V(x)*.**

2) The set $S = \{x : V(x) \leq m, \underline{\Delta V(x) = 0}\}$

contains no trajectories other than ***x* = *0***, where

$$m = \sup_{|x| \leq r} V(x)$$

Remarks

- The previous theorems only provide sufficient but not necessary conditions
  - i.e. failure to find a Lyapunov function does not imply instability.
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Instability Theorem (DT)

The equilibrium state 0 of:

$$x(k+1) = f(x(k))$$

is <u>unstable</u> if there exists function ***W(x)*** such that:

1) $\Delta W(x) \succ 0, \quad \forall |x| < r$

(i.e $\Delta W(x)$ is a **LPDF**).

2) There exists states ***x*** arbitrarily close to the origin such that

$$\boldsymbol{W(x) > 0}.$$