[MEN573]
Advanced Control Systems I

Lecture 12 – Stability
Part I Definitions & Routh-Hurwitz Test

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Outline

- Review of finite dimensional vector norms
- Equilibrium state of CT unforced systems
- Lyapunov’s definitions of stability
- Stability of CT LTI Systems
  - Stability analysis using the Routh-Hurwitz criterion
- Stability of DT LTI Systems
  - Stability analysis using the Routh-Hurwitz criterion

Vector Norm, Review

Let $v \in \mathcal{R}^n$

1. 1- norm

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

2. (Euclidean)

$$\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2 \right)^{\frac{1}{2}} = \left(v^T v \right)^{\frac{1}{2}}$$

Vector Norm, Review

Let $v \in \mathcal{R}^n$

3. p – norm

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}}, \ 1 \leq p < \infty$$

4. sup - norm

$$\|v\|_\infty = \max_i \ |v_i|$$

Finite dimensional vector norms

All vector norms in $\mathcal{R}^n$ and $\mathcal{C}^n$ **are equivalent**:

- Let $\|\cdot\|_a$, $\|\cdot\|_b$ be ***any*** two vector norms,

then, there exists a pair of constants k_1, k_2 such that

$$\boxed{k_1\|v\|_a \leq \|v\|_b \leq k_2\|v\|_a \quad \text{for all } v \in \mathcal{R}^n}$$

Finite dimensional vector norms

- In this class, we uses the symbol $\|\cdot\|$ to denote the Euclidean vector norm $\|\cdot\|_2$.

$$\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2 \right)^{\frac{1}{2}} = \left(v^T v\right)^{\frac{1}{2}}$$

- However, most results can be applied to any vector norm.

Vector norms - convention

- For $v \in \mathcal{R}^n$, $|v|$ means a vector norm of v

- For $v \in \mathcal{R}$, $|v|$ means the absolute value of v

Continuous Time Unforced Systems

n-th order system:

$$\dot{x} = f(x,t)\ , \quad x(t_o) = x_o$$

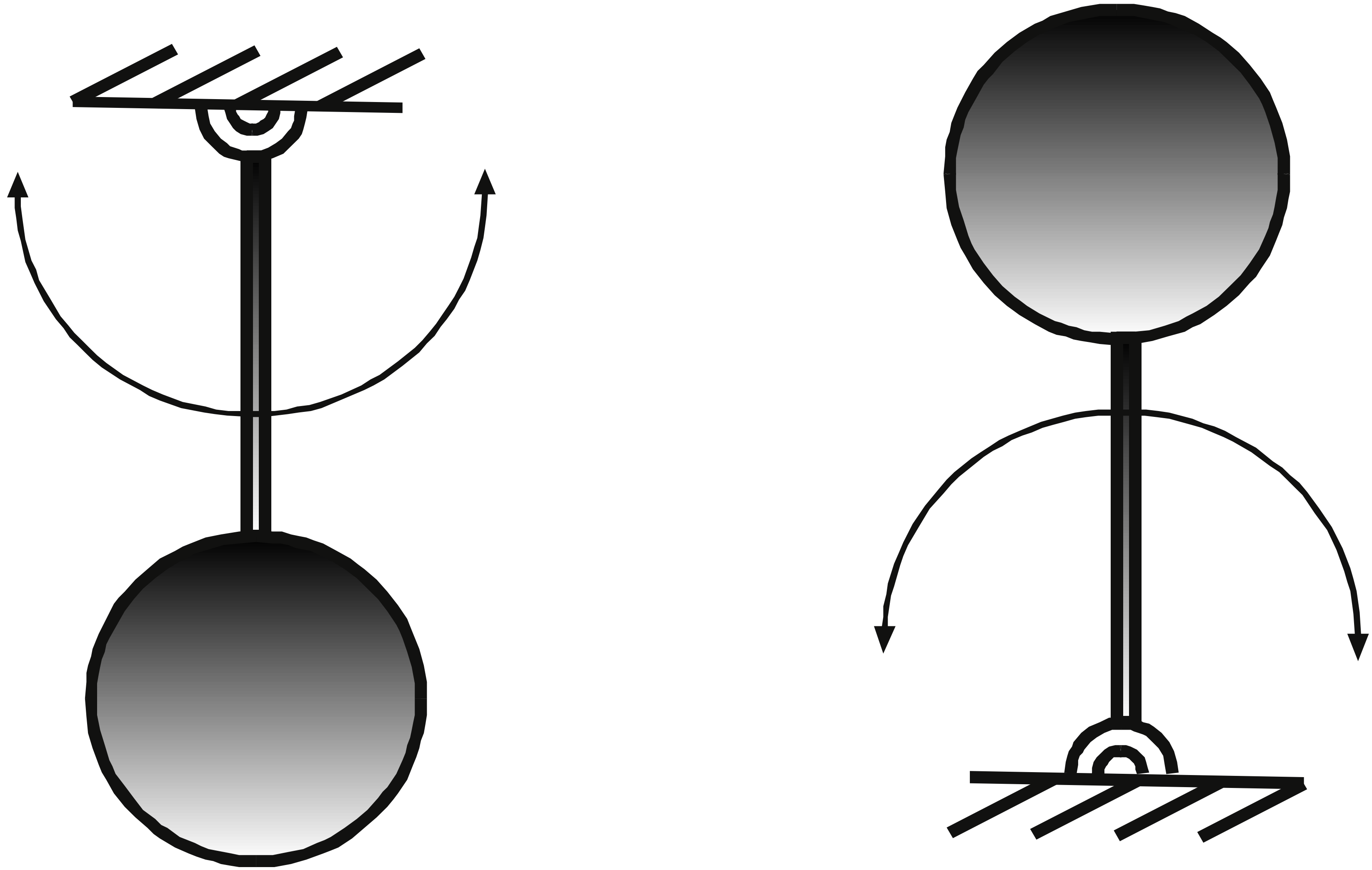
An equilibrium state x_e is such that:

$$f(x_e, t) = 0 \qquad \forall t$$

Without loss of generality, we will assume that 0 is an equilibrium state.

Equilibrium state of unforced systems

Examples:



Equilibrium state of linear systems

For linear systems,

$$\dot{x} = A(t)\,x\ , \quad x(t_o) = x_o$$

- 0 is an equilibrium state, although not necessarily the only one.

- When $A(.)$ is singular, there are multiple equilibrium states.

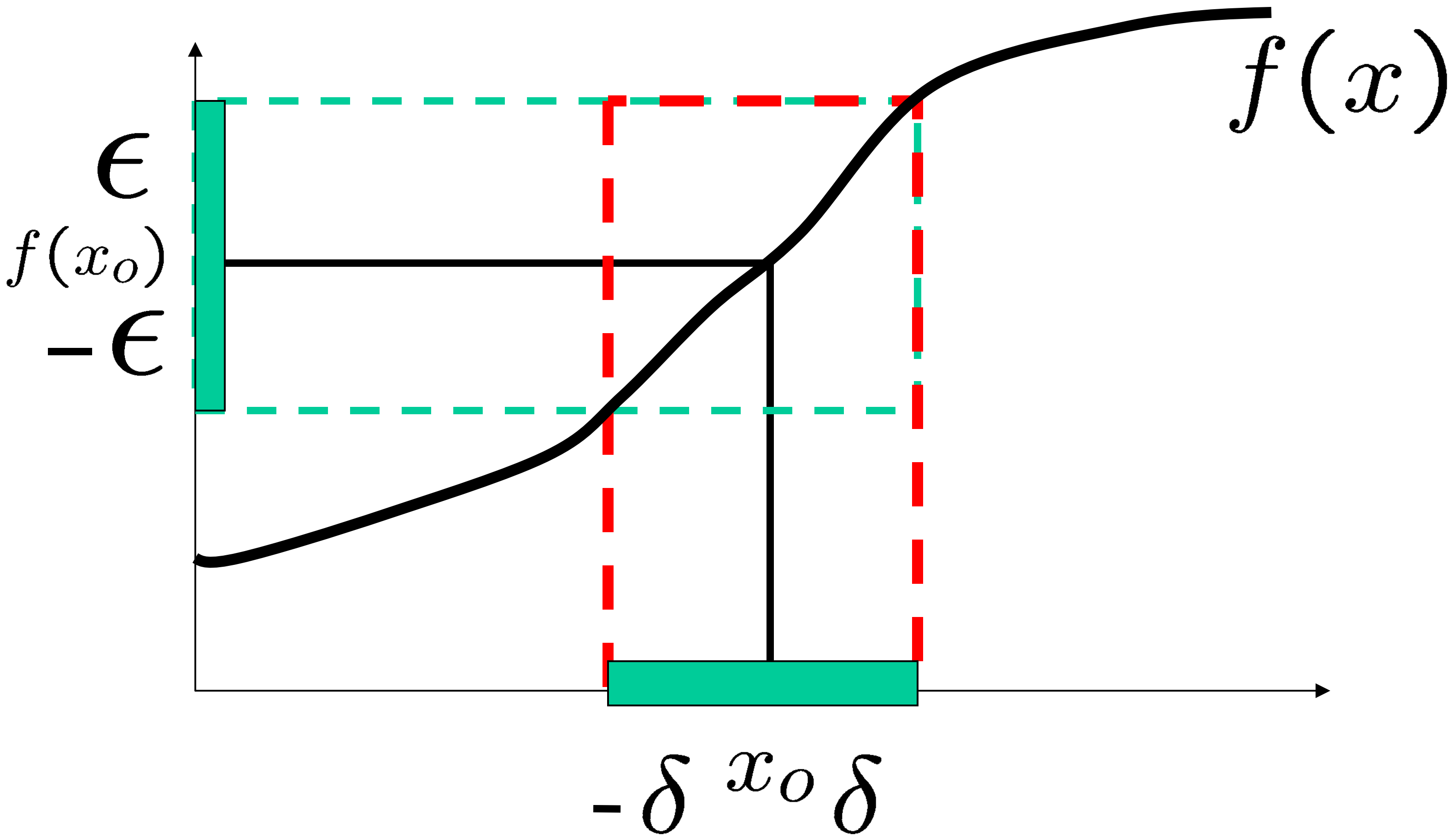
Continuous Function

The function $f : \mathcal{R} \rightarrow \mathcal{R}$ is continuous at x_o if

- for every $\epsilon > 0$, there exists a $\delta(x_o, \epsilon) > 0$ such that

$$|x - x_o| < \delta \quad \Longrightarrow \quad |f(x) - f(x_o)| < \epsilon$$

Continuous Function



$$|x - x_o| < \delta \quad \Rightarrow \quad |f(x) - f(x_o)| < \epsilon$$

Uniformly Continuous Function

The function $f : \mathcal{R} \rightarrow \mathcal{R}$ is ***uniformly continuous*** if

- for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$|x - x_o| < \delta \Longrightarrow |f(x) - f(x_o)| < \epsilon$$

δ is not a function of x_o , ONLY of ϵ

Stability in the sense of **Lyapunov**

The equilibrium state $\boldsymbol{0}$ of $$\dot{x} = f(x,t)$$

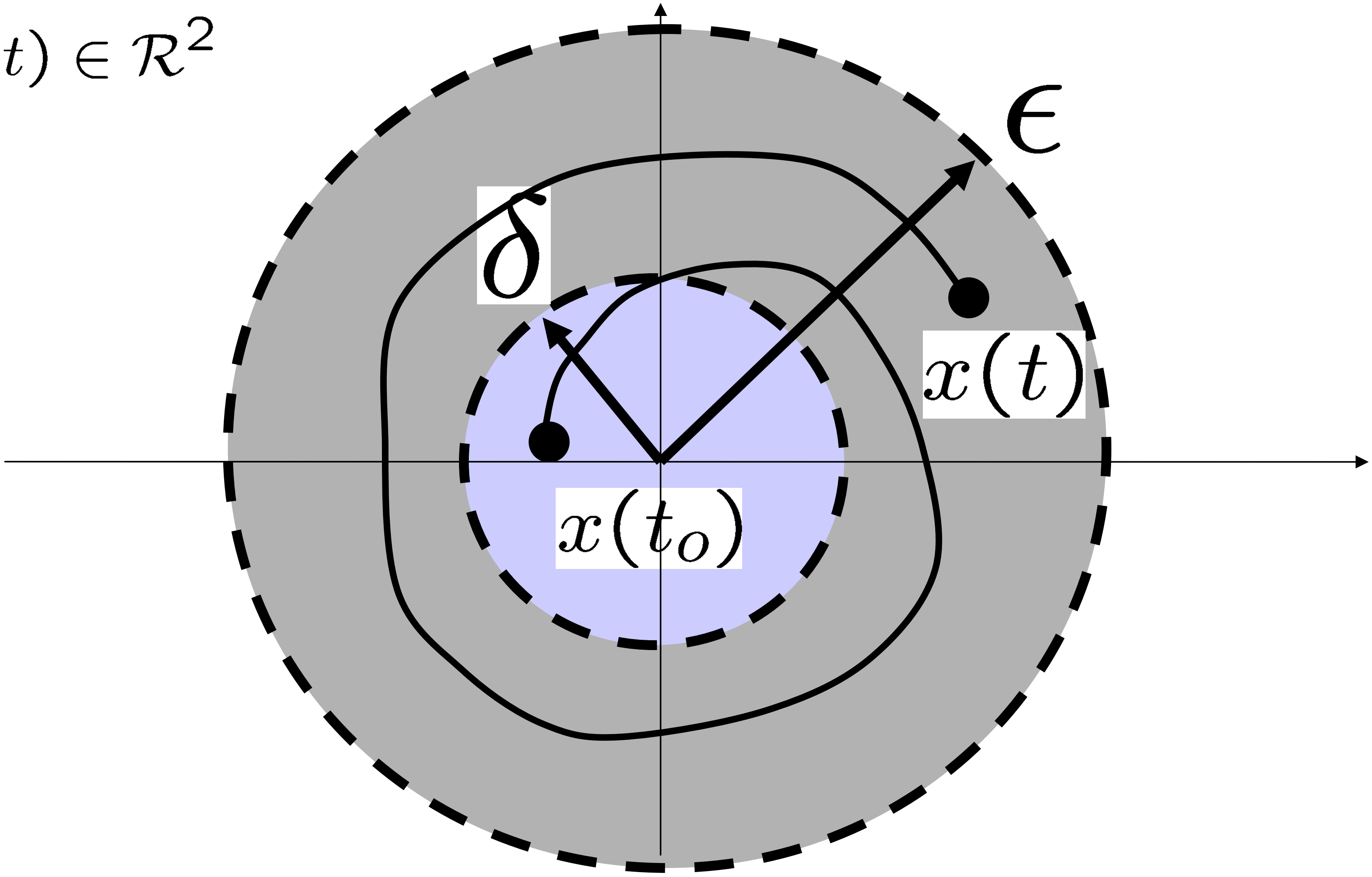
is stable in the sense of Lyapunov if

- for every $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_o) > 0$ such that

$$|x(t_o)| < \delta \quad \Longrightarrow \quad |x(t)| < \epsilon \qquad \forall\, t \geq t_o$$

Stability in the sense of **Lyapunov**

$x(t) \in \mathcal{R}^2$



$$|x(t_o)| < \delta \quad \Rightarrow \quad |x(t)| < \epsilon \quad \forall\, t \geq t_o$$

Stability in the sense of Lyapunov

The equilibrium state *0* of $\dot{x} = f(x,t)$

is ***uniformly stable*** in the sense of Lyapunov if

for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$|x(t_o)| < \delta \quad \Rightarrow \quad |x(t)| < \epsilon \qquad \forall\, t \geq t_o$$

$\boxed{\delta \text{ is not a function of } t_o}$

Asymptotic Stability

The equilibrium state ***0*** of $\dot{x} = f(x,t)$

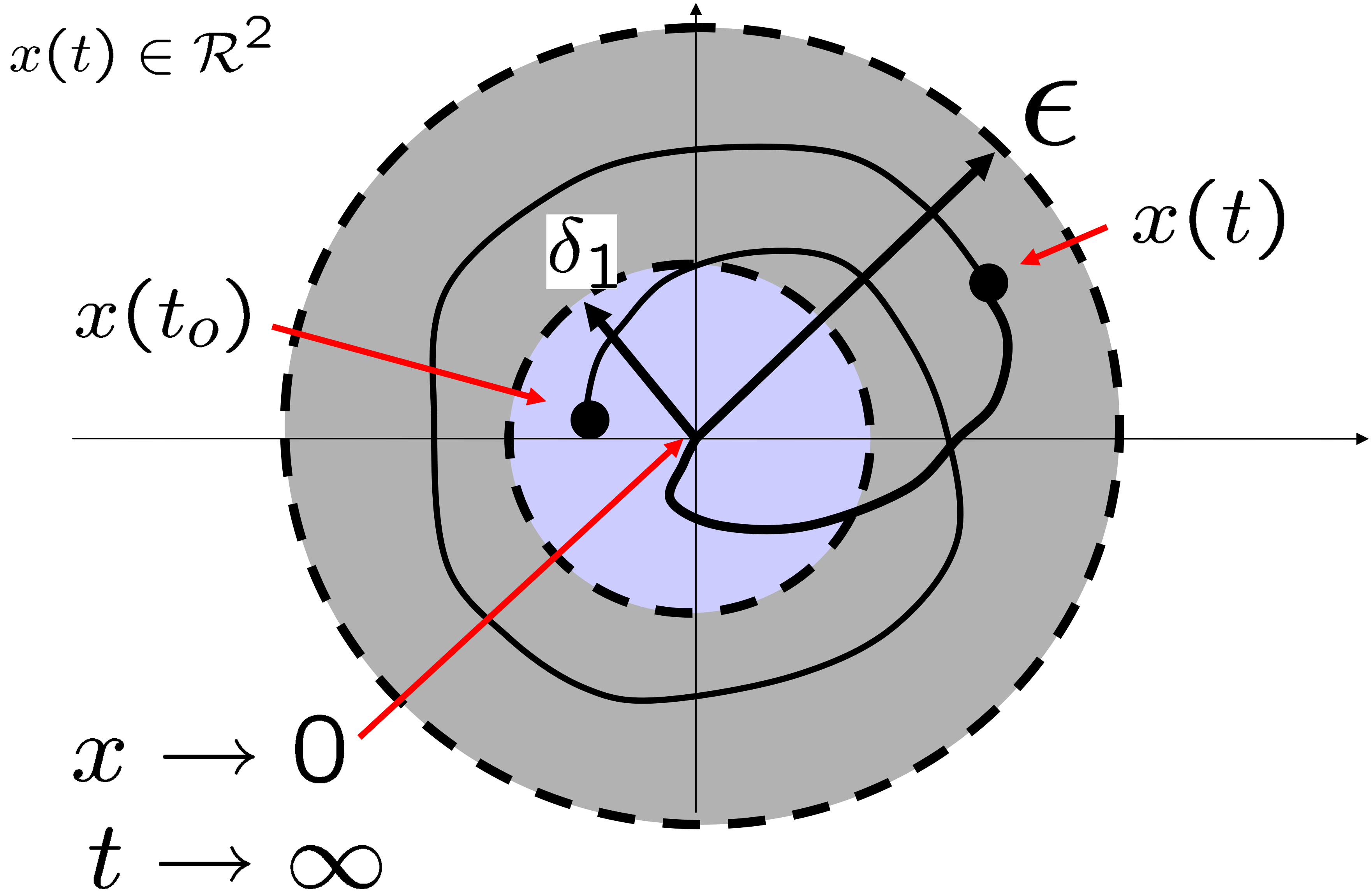
is ***asymptotically stable*** if:

1. it is stable in the sense of Lyapunov and

2. there exists a $\delta_1(t_o) > 0$ such that

$$|x(t_o)| < \delta_1 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

Asymptotic Stability



Uniform Asymptotic Stability

The equilibrium state 0 of $\dot{x} = f(x,t)$

is ***uniformly asymptotically*** stable if:

1. it is uniformly stable in the sense of Lyapunov and

2. there exists a $\delta_1 > 0$ such that

$$\boxed{|x(t_o)| < \delta_1 \Rightarrow \lim_{t \to \infty} x(t) = 0}$$

$\delta, \ \delta_1$ are not functions of t_o

Global Asymptotic Stability

The equilibrium state 0 of $\dot{x} = f(x,t)$

is ***globally asymptotically*** stable if:

- it is asymptotically stable for **any** $\delta_1 > 0$

$$|x(t_o)| < \delta_1$$

$\boxed{\delta_1 \text{ is arbitrarily large}}$

It does not matter how far is the initial condition from the origin

Exponential Stability

The equilibrium state 0 of $\dot{x} = f(x,t)$

is uniformly ***exponentially*** stable if:

- it is stable in the sense of Lyapunov and

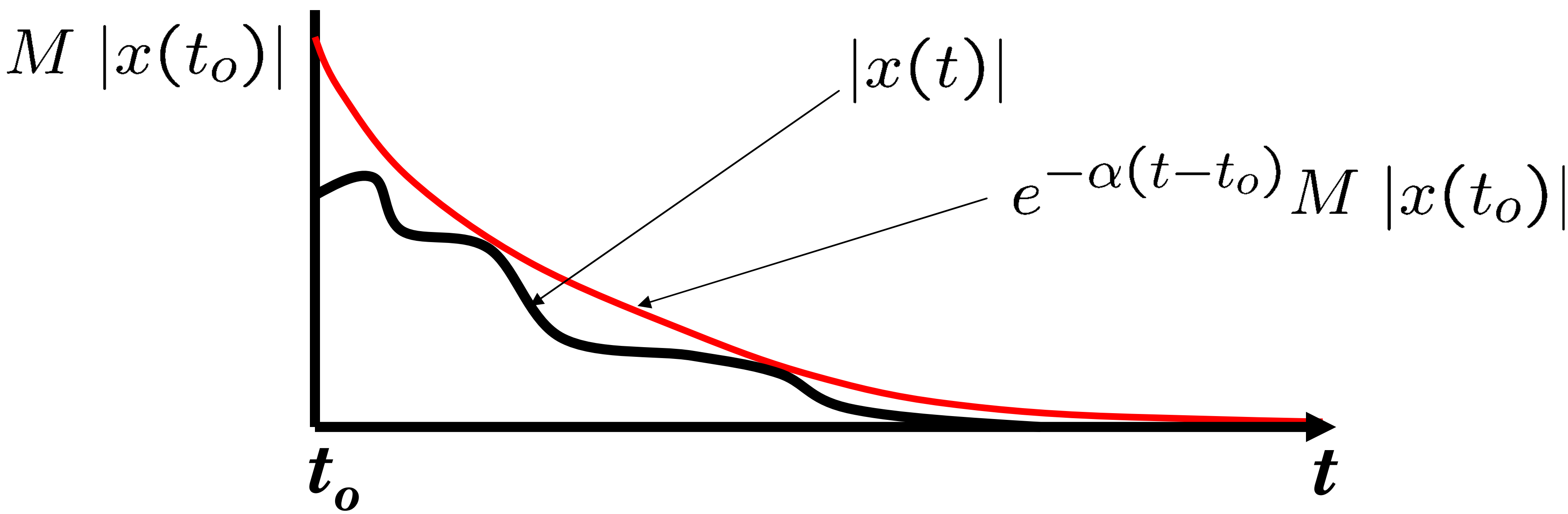
- there exists a $\underline{\delta > 0}$ and constants $\underline{M < \infty}$ and $\underline{\alpha > 0}$ such that

$$|x(t)| \leq e^{-\alpha(t-t_o)} M \; |x(t_o)|$$

for all $| \; x(t_o) \; | < \delta$

Exponential Stability

$$|x(t)| \le e^{-\alpha(t-t_o)} M \; |x(t_o)|$$



α is called the rate of exponential convergence.

exponential stab. is stronger than asymptotic stab.

Stability of LTI Systems

The stability of the equilibrium state 0 for a linear systems

$$\dot{x} = A\,x$$

can be concluded immediately based on the eigenvalues of A.

Let λ_i be the ith eigenvalue of A.

Stability of Continuous Time LTI Systems

Unstable	$Re\{\lambda_i\} > 0$ *for at least one* λ_i, or $Re\{\lambda_i\} \le 0 \ \forall \lambda_i's$, but for a repeated λ_j on the imaginary axis with multiplicity m_j, ***nullity*** $[\lambda_j I - A] < m_j$ (Jordan form)
Stable in the sense of Lyapunov	$Re\{\lambda_i\} \le 0 \ \forall \lambda_i's$, but for any repeated λ_j on the imaginary axis with multiplicity m_j, ***nullity*** $[\lambda_j I - A] = m_j$ (diagonal form)
Exponentially stable	$Re\{\lambda_i\} < 0 \ \forall \lambda_i$,

λ_i is the ith eigenvalue of A.

Hurwitz Matrix

A matrix $A \in \mathcal{R}^{n \times n}$ is ***<u>Hurwitz</u>*** if

$$\boldsymbol{Re\{\lambda_i\} < 0} \qquad \forall \boldsymbol{\lambda_i},$$

λ_i is the ith eigenvalue of A.

$\dot{x} = Ax$ is exponentially stable

Instability

Unstable	$Re\{\lambda_i\} > 0$ *for at least one* λ_i, or $Re\{\lambda_i\} \leq 0 \ \forall \lambda_i$'s, but for a repeated λ_j on the imaginary axis with multiplicity m_j, ***nullity*** $[\lambda_j I - A] < m_j$ (Jordan form)

λ_i is the ith eigenvalue of A.

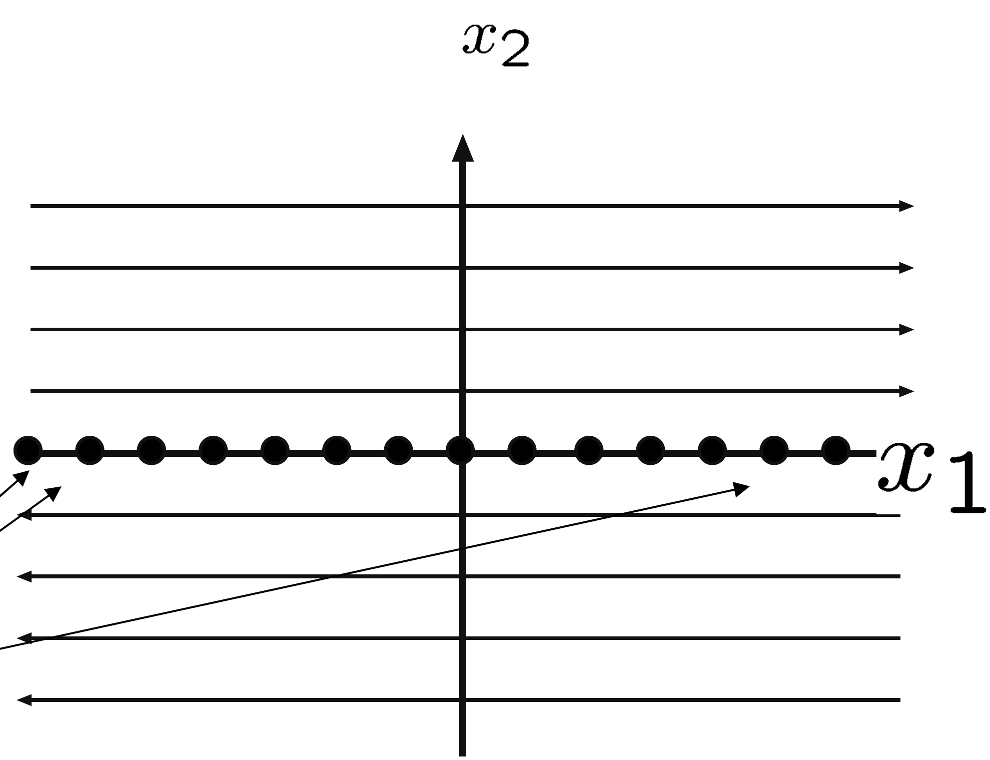
Unstable system

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 0$$

$$\text{nullity}\,\{\lambda_i I - A\} = 1$$

Jordan canonical form



equilibrium states

Limited stability

Stable in the sense of Lyapunov	$\boldsymbol{Re\{\lambda_i\} \le 0\ \forall\, \lambda_i's}$, but for any repeated $\boldsymbol{\lambda_j}$ on the imaginary axis with multiplicity $\boldsymbol{m_j}$, ***nullity*** $\boldsymbol{[\lambda_j I - A] = m_j}$ (diagonal form)

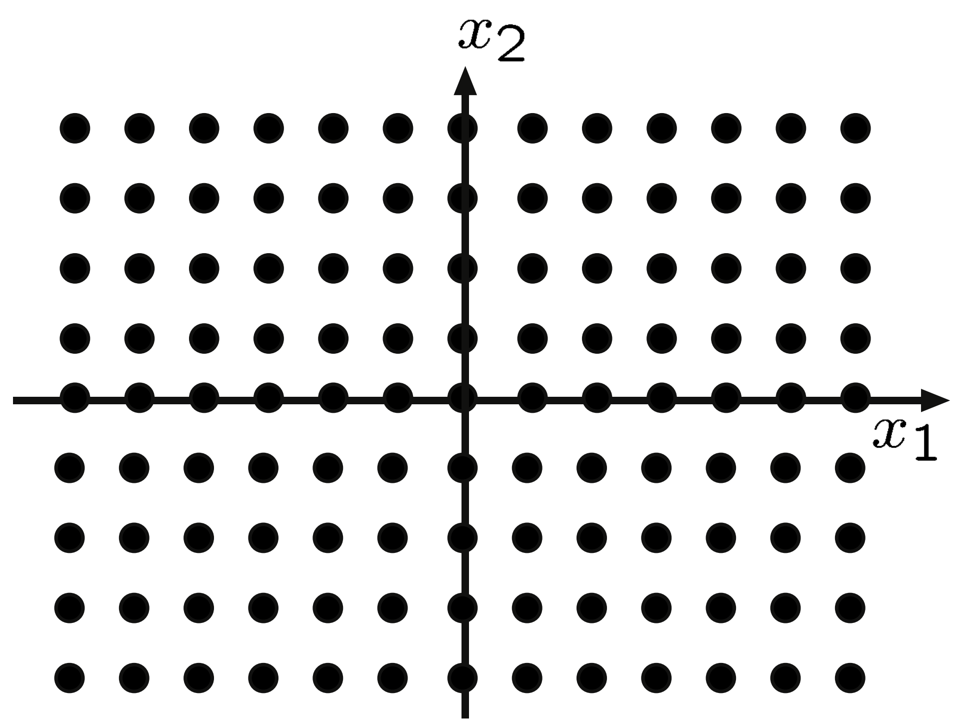
λ_i is the ith eigenvalue of A.

Limited Stability

$$\dot{x} = A\,x \qquad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 0$$

$$\text{nullity}\,\{\lambda_i I - A\} = 2$$



Routh-Hurwitz criterion

The asymptotic stability of the equilibrium state *0*

$$\dot{x} = A\,x$$

can be concluded immediately based on the Routh-Hurwitz criterion.

The characteristic polynomial of the matrix $A \in \mathcal{R}^{n\times n}$ is

$$\begin{aligned} A(s) &= \mathrm{det}\{sI - A\} \\ &= s^n + a_1\,s^{n-1} + \cdots + a_n \end{aligned}$$

Hurwitz Characteristic Polynomial

Characteristic polynomial (CP) of the matrix $A \in \mathcal{R}^{n\times n}$

$$\begin{aligned} A(s) &= \mathrm{det}\{sI - A\} \\ &= s^n + a_1 s^{n-1} + \cdots + a_n \end{aligned}$$

is ***Hurwitz*** if

$$\boxed{A(s_o) = 0 \Leftrightarrow \mathrm{Re}\{s_o\} < 0}$$

A matrix $A \in \mathcal{R}^{n\times n}$ is Hurwitz if its CP is Hurwitz.

Routh Array

Let the characteristic polynomial (CP) of the matrix $A \in \mathcal{R}^{5\times 5}$

$$A(s) = a_0\, s^5 + a_1\, s^4 + a_2\, s^3 + a_3\, s^2 + a_4\, s + a_5$$

Routh Array:

a_0	a_2	a_4	0	s^5
a_1	a_3	a_5	0	s^4
$b_1 = a_2 - \frac{a_0 a_3}{a_1}$	$b_2 = a_4 - \frac{a_0 a_5}{a_1}$	$b_3 = 0 - \frac{a_0 0}{a_1} = 0$		s^3
$c_1 = a_3 - \frac{a_1 b_2}{b_1}$	$c_2 = a_5 - \frac{a_1 b_3}{b_1}$	0		s^2
$d_1 = b_2 - \frac{b_1 c_2}{c_1}$	0	0		s
$e_1 = c_2$	0			s^0

Routh-Hurwitz criterion (CT)

The characteristic polynomial (CP) of the matrix $A \in \mathcal{R}^{n\times n}$

$$A(s) \;=\; a_0\, s^n + a_1\, s^{n-1} + \cdots + a_n,\; a_o > 0$$

- is Hurwitz ***iff*** all elements of the first column of the Routh array are positive (necessary and sufficient condition)

- If Hurwitz, all coefficients of the polynomial are positive (necessary condition).
- The number of sign changes that occur on the first column of the Routh array equals the number of roots on the right half of the complex plane.

Routh-Hurwitz criterion (CT)

- See Nise chapter 6, or any undergraduate textbook, for a discussion of special cases such as:
  - "0" in the first column
  - all elements in one row are zero.

Routh-Hurwitz criterion (CT)

- Example:

$$A(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$$

s^4	2	3	10
s^3	1	5	0
s^2	3-2x5/1=-7	10	0
s^1	5-1x10/(-7)=6.43	0	
s^0	10		

- There are two sign changes in the first column.
- The system is unstable and there are two characteristic roots in the right half side of s-plane.

Discrete Time Unforced Systems

Consider an n-th order nonlinear time varying discrete time (DT) systems of the form:

$$x(k+1) = f(x(k), k)\ , \quad x(k_o) = x_o$$

Equilibrium state of unforced systems

An equilibrium state x_e is such that:

- $f(x_e, k) = x_e$, for all k, for DT systems.

Without loss of generality, we will assume that 0 is an equilibrium state.

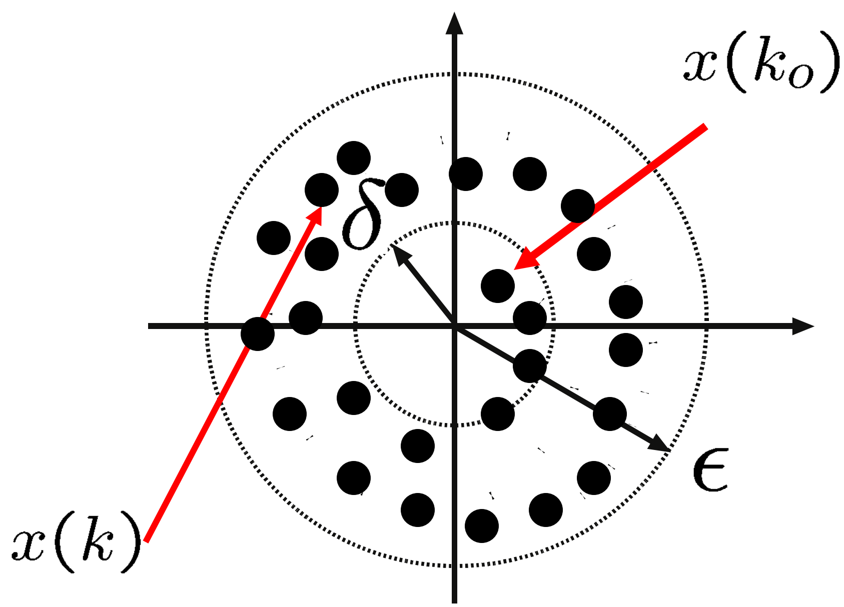
Stability in the sense of Lyapunov

The equilibrium state 0 of $\quad x(k+1) = f(x(k), k)$

is stable in the sense of Lyapunov if

- for every $\varepsilon > 0$, there exists a $\delta(\varepsilon, k_o) > 0$ such that

$$\boxed{|x(k_o)| < \delta \Rightarrow |x(k)| < \epsilon \qquad \forall\, k \geq k_o}$$



Exponential Stability

The equilibrium state 0 of $x(k+1) = f(x(k), k)$

is uniformly ***exponentially*** stable if:

- it is stable in the sense of Lyapunov and
- there exists a $\delta > 0$ and constants $M < \infty$ and

$0 < \rho < 1$ such that

$$\boxed{|x(k)| \leq \rho^{(k-k_o)} M \; |x(k_o)|}$$

for all $|\, x(k_o) \,| < \delta$

exponential stab. is stronger than asymptotic stab.

Equilibrium state of linear systems

For linear DT systems,

$$x(k+1) = A\,x(k)\ , \quad x(k_o) = x_o$$

- 0 is an equilibrium state, although not necessarily the only one.

- When $\boldsymbol{det(I - A) = 0}$, there are multiple equilibrium states.

Stability of Discrete Time (DT) Systems

$$x(k+1) = A\,x(k)$$

Unstable	$\mid \lambda_i \mid > 1$ *for at least one* λ_i, or $\mid \lambda_i \mid \leq 1 \; \forall \lambda_i's$, but for a repeated λ_j on the unit circle with multiplicity m_j ***nullity*** $[\lambda_j I - A] < m_j$ (Jordan form)
Stable in the sense of Lyapunov	$\mid \lambda_i \mid \leq 1 \; \forall \lambda_i's$, but for any repeated λ_j on the imaginary axis with multiplicity m_j ***nullity*** $[\lambda_j I - A] = m_j$
Exponentially stable	$\mid \lambda_i \mid < 1 \; \forall \lambda_i$,

λ_i is the ith eigenvalue of A.

Schur Matrix

A matrix $A \in \mathcal{R}^{n \times n}$ is ***Schur*** if

$$| \lambda_i | < 1 \qquad \forall \lambda_i,$$

λ_i is the ith eigenvalue of A.

$x(k+1) = A\,x(k)$ is exponentially stable

Schur Characteristic Polynomial (DT)

$$x(k+1) = A\,x(k)$$

Characteristic polynomial (CP) of the matrix $A \in \mathcal{R}^{n\times n}$

$$\begin{aligned} A(z) &= \det\{zI - A\} \\ &= z^n + a_1\, z^{n-1} + \cdots + a_n \end{aligned}$$

is ***Schur*** if $\boxed{A(z_o) = 0 \Leftrightarrow |z_o| < 1}$

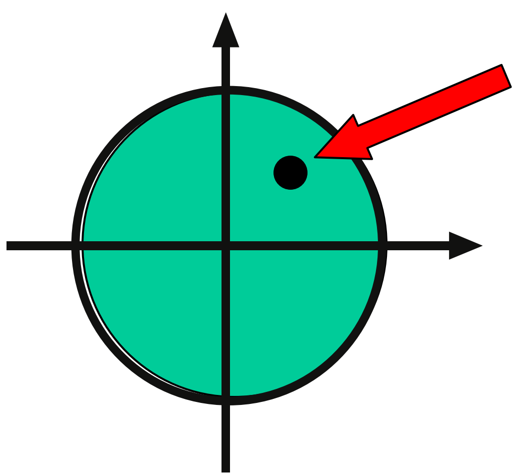
A matrix $A \in \mathcal{R}^{n\times n}$ is Schur if its CP is Schur.

Routh-Hurwitz criterion (DT)

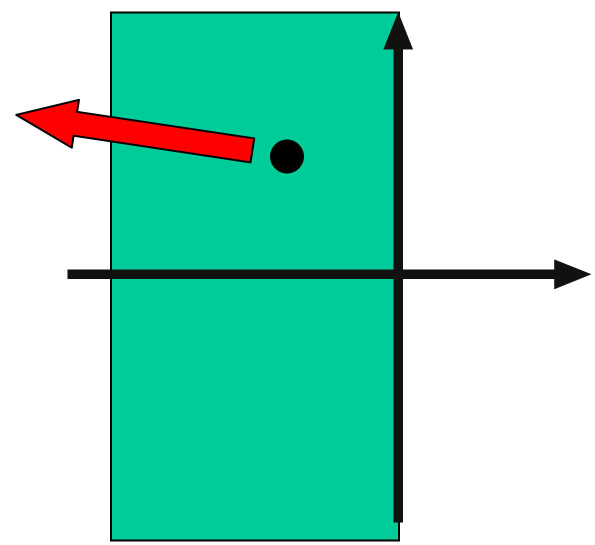
To apply the Routh-Hurwitz test to the discrete time characteristic polynomial (CP)

$$A(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

it is necessary to first apply a bilinear transformation:



$$z = \frac{1+s}{1-s}$$



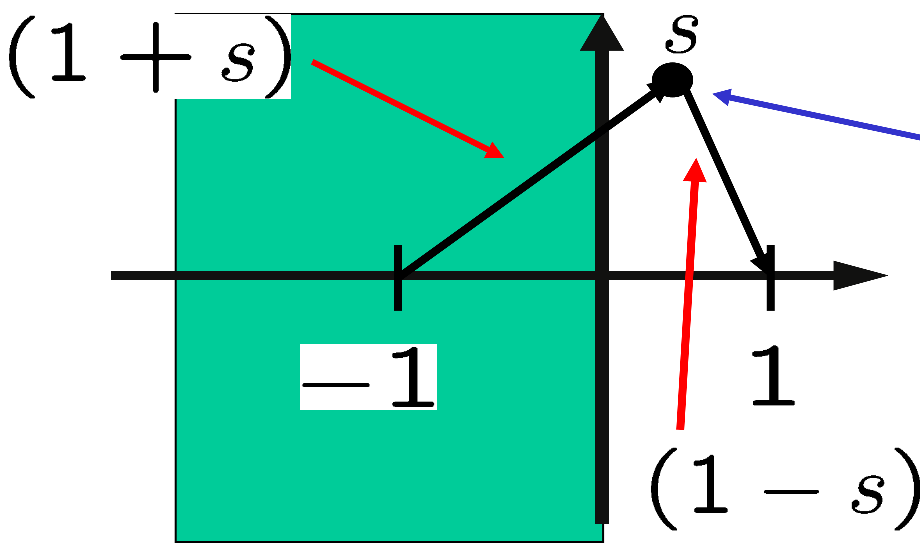
$$\{z \in \mathcal{C}\,,\ |z| \leq 1\} \leftarrow \{s \in \mathcal{C}\,,\ \mathrm{Re}\{s\} \leq 0\}$$

Bilinear Transformation

$$z = \frac{1+s}{1-s}$$

Unstable case,

$$\mathrm{Re}\{s\} > 0 \Rightarrow |z| > 1$$



$$\mathrm{Re}\{s\} > 0$$

$$|z| = \frac{|1+s|}{|1-s|} > 1$$

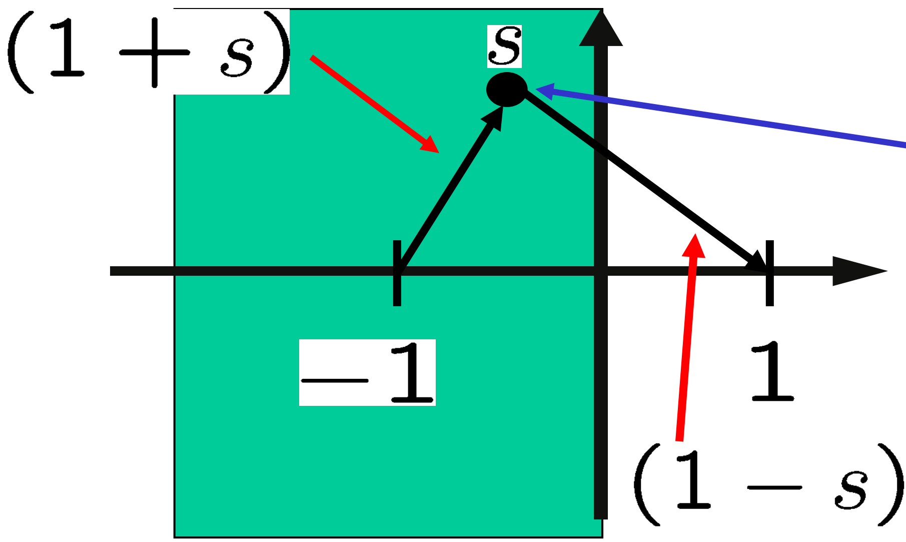
$$(1+s) = (s - (-1))$$

Bilinear Transformation

$$z = \frac{1+s}{1-s}$$

Stable case,

$$\mathrm{Re}\{s\} < 0 \Rightarrow |z| < 1$$



$$\mathrm{Re}\{s\} < 0$$

$$|z| = \frac{|1+s|}{|1-s|} < 1$$

$$(1+s) = (s-(-1))$$

Bilinear Transformation

Let $$A(z) = z^n + a_1\, z^{n-1} + \cdots + a_n$$

and apply the bilinear transformation,

$$\begin{aligned} A(z)|_{z=\frac{1+s}{1-s}} &= \left(\frac{1+s}{1-s}\right)^n + a_1 \left(\frac{1+s}{1-s}\right)^{n-1} + \cdots a_o \\ &= \frac{A^*(s)}{(1-s)^n} \end{aligned}$$

Then,

$$\begin{aligned} A^*(s) &= a_0^*\, s^n + a_1^*\, s^{n-1} + \cdots + a_n^* \\ &= A(z)|_{z=\frac{1+s}{1-s}}\,(1-s)^n \end{aligned}$$

Bilinear Transformation

Defining

$$A^*(s) \;=\; A(z)\big|_{z=\frac{1+s}{1-s}} (1-s)^n$$

then

$$\boxed{A^*(s_o) = 0 \Leftrightarrow A(z_o) = 0}$$

where,

$$z_o = \frac{1+s_o}{1-s_o}$$

and,

$$\boxed{\mathrm{Re}\{s_o\} < 0 \Leftrightarrow |z_o| < 1}$$

Routh-Hurwitz criterion (DT)

The discrete time characteristic polynomial

$$A(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

is Schur, i.e. $A(z_o) = 0 \Leftrightarrow |z_o| < 1$

- ***iff*** the polynomial

$$A^*(s) \;=\; A(z)|_{z=\frac{1+s}{1-s}} (1-s)^n$$

is Hurwitz

Example

Consider the discrete time characteristic polynomial

$$A(z) = z^3 + 0.8\,z^2 + 0.6z + 0.5$$

$$\begin{aligned} A^*(s) &= A(z)|_{z=\frac{1+s}{1-s}}(1-s)^3 \\ &= (1+s)^3 + 0.8(1+s)^2(1-s) \\ &\quad +0.6(1+s)(1-s)^2 + 0.5(1-s)^3 \end{aligned}$$

$$A^*(s) = 0.3s^3 + 3.1s^2 + 1.7s + 2.9$$

Discrete Time Example

Routh array of

$$A^*(s) = 0.3s^3 + 3.1s^2 + 1.7s + 2.9$$

All elements in first column are positive

0.3	1.7	s^3
3.1	2.9	s^2
$b_1 = 1.7 - \frac{0.3\,2.9}{3.1} = 1.42$	0	s
2.9		s^0

Thus,

$$A(z) = z^3 + 0.8\,z^2 + 0.6z + 0.5$$

is Schur.