UNIST
Department of Mechanical Engineering

MEN 573: Advanced Control Systems I — **Spring, 2016**

Homework #1	Assigned: Saturday, March 12, 2016	
Solution	Due: Monday, March 21, 2016 (in class)	

Problem 1.

1) (i) $f(t)=\int_0^t e^{-5\tau}\sin(3\tau)\,d\tau$

Let $g(\tau)=e^{-5\tau}\sin(3\tau)$, then $f(t)=\int_0^t g(\tau)\,d\tau$

From the property of Laplace Transform $\left(\mathcal{L}\{\int_0^t f(\tau)d\tau\}=\frac{F(s)}{s}\right)$,

$$\mathcal{L}\{f(t)\}=\mathcal{L}\{\int_0^t g(\tau)\,d\tau\}=\frac{G(s)}{s}$$

$$G(s)=\mathcal{L}\{g(t)\}=\mathcal{L}\{e^{-5t}\sin(3t)\}=\frac{3}{(s+5)^2+9}\ (\because \mathcal{L}\{e^{-at}f(t)\}=F(s+a))$$

$$\therefore \mathcal{L}\{\int_0^t e^{-5\tau}\sin(3\tau\,d\tau\}=\frac{3}{s((s+5)^2+9)} \vee$$

(ii) $f(t)=t^2e^{-2t}$.

From the property of L.T. $(\mathcal{L}\{e^{-at}f(t)=F(s+a)\})$,

$$\mathcal{L}\{t^2e^{-2t}\}=\frac{2}{(s+2)^3} \vee$$

(iii) $f(t)=\{\begin{array}{l} e^{at} \text{ for } 0\le t\le T \\ 0 \text{ for } t\ge T\end{array}$

$$\mathcal{L}\{f(t)\}=\int_{0^-}^{\infty} f(t)e^{-st}dt=\int_{0^-}^{T} e^{at}e^{-st}dt=\frac{1}{a\cdot s}e^{(a-s)t}\Big|_{0^-}^{T}=\frac{e^{(a-s)T}-1}{a-s}$$

(iv) $f(k)=\{\begin{array}{l} 1 \text{ for even } k\ (k=0,2,4,\ldots) \\ 0 \text{ for odd } k\ (k=1,3,5,\ldots)\end{array}$

$$\mathcal{Z}\{f(k)\}=\sum_{k=0}^{\infty} f(k)z^{-k}=1\cdot z^0+0\cdot z^{-1}+1\cdot z^{-2}+0\cdot z^{-3}+\ldots$$

$$=1+z^{-2}+z^{-4}+\ldots=\frac{1}{1-z^{-2}}=\frac{z^2}{z^2-1}$$

$$(v)\ f(k)=f(k-N);\text{where}\{f(j)\big|0\le j<N\}$$

$$\mathscr{Z}\{f(k)\}=\sum_{k=0}^{\infty}f(k)z^{-k}=f(0)z^{0}+f(1)z^{-1}+f(2)z^{-2}+\ldots+f(N)z^{-N}+f(N+1)z^{-(N+1)}+\ldots$$

$$=f(0)z^{0}+f(1)z^{-1}+\ldots+f(0)z^{-N}+f(1)z^{-(N+1)}+\ldots$$

$$=(f(0)z^{0}+f(1)z^{-1}+\ldots+f(N-1)z^{-(N+1)})+z^{-N}(f(0)z^{0}+f(1)z^{-1}+\ldots+$$

$$f(N-1)z^{-(N+1)})+z^{-2N}(f(0)z^{0}+f(1)z^{-1}+\ldots+f(N-1)z^{-(N+1)})+\ldots$$

$$=(\sum_{k=0}^{N-1}f(k)z^{-k})(\frac{1}{1-z^{-N}})\vee$$

$$(vi)\ \mathscr{Z}\{f(k)\}=F(z)$$

$$g(k)=\begin{cases}f(k)\ \text{for}\ k=4k',k'=0,1,2,\ldots\\ 0\quad \text{for}\ k\ne 4k',k'=0,1,2,\ldots\end{cases}$$

$$G(z)=\mathscr{Z}\{g(k)\}=\sum_{k=0}^{\infty}g(k)z^{-k}=g(0)z^{0}+g(1)z^{-1}+g(2)z^{-2}+\ldots=f(0)+f(4)z^{-4}+f(8)z^{-8}\ldots$$

$$F(z)=f(0)+f(1)z^{-1}+f(2)z^{-2}+\ldots$$

$$F(\alpha z)=f(0)+f(1)(\alpha z)^{-1}+f(2)(\alpha z)^{-2}+\ldots$$

$$F(\beta z)=f(0)+f(1)(\beta z)^{-1}+f(2)(\beta z)^{-2}+\ldots \qquad \to F(z)+F(\alpha z)+F(\beta z)+F(\gamma z)=4G(z)$$

$$F(\alpha z)=f(0)+f(1)(\gamma z)^{-1}+f(2)(\gamma z)^{-2}+\ldots$$

$$\Rightarrow 1+\alpha^{-1}+\beta^{-1}+\gamma^{-1}=0$$

$$1+\alpha^{-2}+\beta^{-2}+\gamma^{-2}=0 \qquad \to \alpha=-1,\beta=j,\gamma=-j$$

$$1+\alpha^{-3}+\beta^{-3}+\gamma^{-3}=0$$

$$\therefore G(z)=\frac{1}{4}\{F(z)+F(-z)+F(jz)+F(-jz)\}\vee$$

Problem 2.

$$F(s)=\frac{(K_1-K_2\tau)s+(K_1-K_2)}{s(\tau s+1)(s+1)}$$

(a) Assume that $f(0^+)=\lim_{t\to0^+} f(t)$ and $\lim_{t\to\infty} f(t)$ exist, so we can use initial value theorem and final value theorem.

For a negative initial slope, $\left.\frac{df}{dt}\right|_{t=0}<0$

Using initial value theorem ($\lim_{t\to\infty} f(t)=f(0^+)=\lim_{s\to0} sF(s)$),

$$\left.\frac{df}{dt}\right|_{t=0}=\lim_{t\to0}\dot{f}(t)=\lim_{s\to\infty} s(sF(s)-f(0))=\lim_{s\to\infty}(s^2F(s)-sf(0))$$

$$=\lim_{s\to\infty}\frac{(K_1-K_2\tau)s^2+(K_1-K_2)s}{(\tau s+1)(s+1)}\ (\because f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\frac{(K_1-K_2\tau)s+(K_1-K_2)}{(\tau s+1)(s+1)}=0$$

from initial value theorem.)

$$=\frac{K_1-K_2\tau}{\tau}=\frac{K_1}{\tau}-K_2<0\to\frac{K_1}{\tau}<K_2\ ...(1)$$

For a positive final value, $\lim_{t\to\infty} f(t)>0$

Using final value theorem,

$$\lim_{t\to\infty} f(t)=\lim_{s\to0} sF(s)=\lim_{s\to0}\frac{(K_1-K_2\tau)s+(K_1-K_2)}{(\tau s+1)(s+1)}=K_1-K_2>0\to K_1>K_2\ ...(2)$$

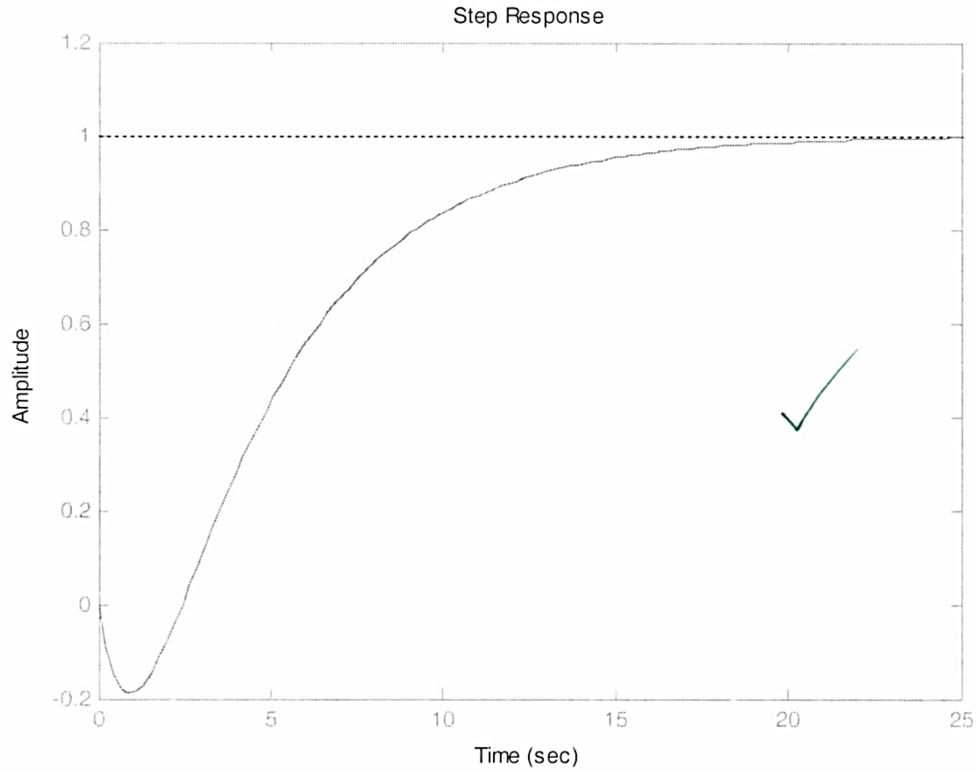
From (1), (2), $K_1>K_2>\frac{K_1}{\tau}\vee$

(b) $G(s)=\frac{(K_1-K_2\tau)s+(K_1-K_2)}{(\tau s+1)(s+1)}\quad K_1=2, K_2=1, \tau=4.$

MATLAB code

```
% hw1, prob 2.(b)
num = [ 1/2 ] ; % (s-1/2)
den = [ -1/4 -1 ] ; % (s+1/4)(s+1)
f = zpk( num, den, -1/2 ) ; % transfer function with gain -1/2
step( f ) % put step input
```

MATLAB graph



$$(c)\ F(s)=\frac{(K_1-K_2\tau)s+(K_1-K_2)}{s(\tau s+1)(s+1)}\quad (K_1=2, K_2=1, \tau=4)$$

$$=\frac{-2s+1}{s(4s+1)(s+1)}=\frac{A}{s}+\frac{B}{4s+1}+\frac{C}{s+1}=\frac{1}{s}-\frac{2}{s+\frac{1}{4}}+\frac{1}{s+1}$$

$$\because \begin{array}{c} A(4s+1)(s+1)=A(4s^2+5s+1) \\ B(s+1)s=B(s^2+s) \\ C(4s+1)s=C(4s^2+s) \end{array} \Rightarrow \begin{array}{c}(4A+B+4C)s^2+(5A+B+C)s+A \\ =-2s+1\end{array}$$

$$\begin{array}{ll} s^2: 4A+B+4C=0 & A=1 \\ s: 5A+B+C=-2 \Rightarrow & B=-8 \\ s^0: A=1 & C=1 \end{array}$$

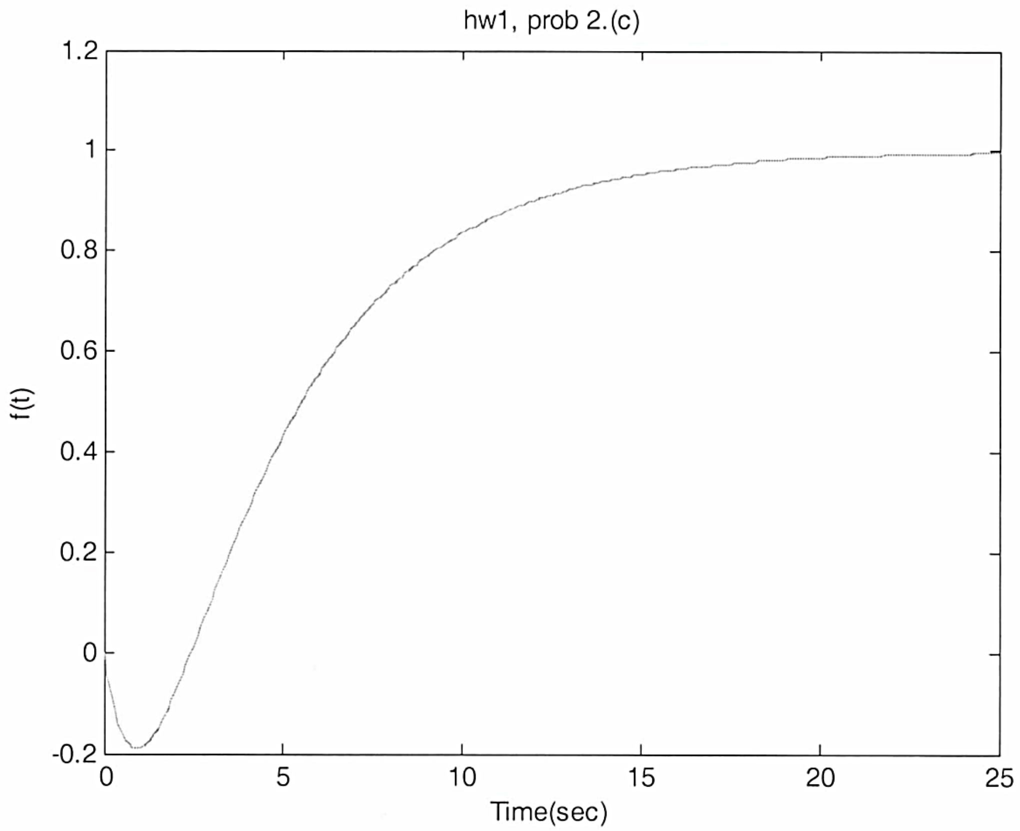
$$f(t)=L^{-1}\{F(s)\}=L^{-1}\{\frac{1}{s}-\frac{2}{s+\frac{1}{4}}+\frac{1}{s+1}\}=1(t)-2e^{-\frac{1}{4}t}+e^{-t}\ \vee$$

MATLAB code

```
t = [ 0 : 0.1 : 25 ] ; % time range
y = 1 - 2*exp(-t/4) + exp( -t ) ; % f(t)
plot( t, y )
```

```
xlabel( 'Time(sec)' ) ; ylabel( 'f(t)' ) ; title( 'hw1, prob 2.(c)' ) ;
```

MATLAB graph



Problem 3.

$$X(z)=\frac{z^{-1}}{(1-z^{-1})(1-1.4z^{-1}+0.48z^{-2})}$$

$$\text{By initial value theorem, } X(0)=\lim_{z\to\infty}X(z)=\lim_{z\to\infty}\frac{z^{-1}}{(1-z^{-1})(1-1.4z^{-1}+0.48z^{-2})}=0.$$

Assume that $\lim_{k\to\infty}X(k)$ exists, by final value theorem

$$\lim_{k\to\infty}X(k)=\lim_{z\to1}(z-1)X(z)=\lim_{z\to1}\frac{(z-1)z^{-1}}{(1-z^{-1})(1-1.4z^{-1}+0.48z^{-2})}\left(\frac{z}{z}\right)$$

$$=\lim_{z\to1}\frac{(z-1)\cdot 1}{(z-1)(1-1.4z^{-1}+0.48z^{-2})}=\frac{1}{1-1.4-0.48}=12.5$$

$$\text{(a) } X(z)=\frac{z^{-1}}{(1-z^{-1})(1-0.6z^{-1})(1-0.8z^{-1})}=\frac{A}{(1-z^{-1})}+\frac{B}{(1-0.6z^{-1})}+\frac{C}{(1-0.8z^{-1})}$$

$$\begin{aligned}
A(1-0.6z^{-1})(1-0.8z^{-1})&=A(1-1.4z^{-1}+0.48z^{-2}) && (A+B+C)-(1.4A+1.8B+1.6C)z^{-1}\\
B(1-z^{-1})(1-0.8z^{-1})&=B(1-1.8z^{-1}0.8z^{-2}) \quad \to && +(0.48A+0.8B+0.6C)z^{-2}\\
C(1-z^{-1})(1-0.6z^{-1})&=C(1-1.6z^{-1}+0.6z^{-2}) && =z^{-1}
\end{aligned}$$

$$\begin{aligned}
&A+B+C=0 && A=12.5\\
\to\ &1.4A+1.8B+1.6C=-1 \ \to && B=7.5\\
&0.48A+0.8B+0.6C=0 && C=-20
\end{aligned}$$

$$=\frac{12.5}{(1-z^{-1})}+\frac{7.5}{(1-0.6z^{-1})}-\frac{20}{(1-0.8z^{-1})}$$

$$\therefore X(k)=Z^{-1}\{X(z)\}=12.5\cdot 1(k)+7.5\cdot(0.6)^k-20\cdot(0.8)^k.\vee$$

3. (b)

MATLAB code

```
num = [ 0 1 ] ;
den1 = [ 1 -1 ] ; den2 = [ 1 -1.4 0.48 ] ; den = conv( den1, den2 ) ;
[ R, P, K ] = residuez( num, den )
```

MATLAB Result

```
R =
   12.5000
  -20.0000
```

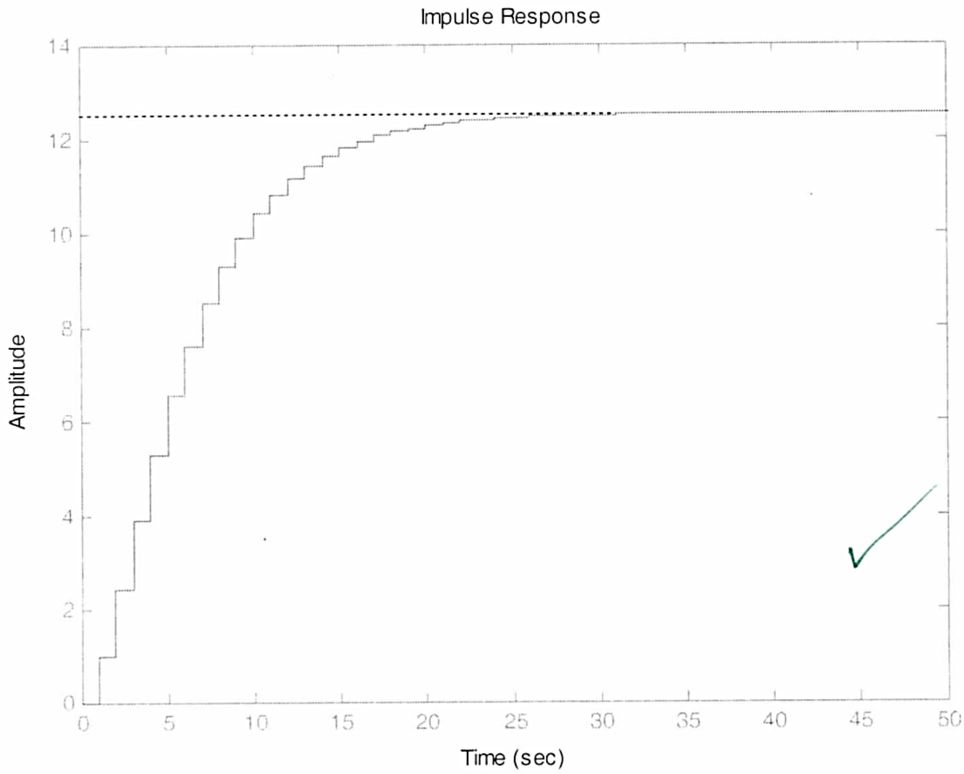
```
   7.5000
P =
   1.0000
   0.8000
   0.6000
K =
    []
```

3. (c)

MATLAB code

```
num = [ 1 0 0 ] ;
den1 = [ 1 -1 ] ; den2 = [ 1 -1.4 0.48 ] ; den = conv( den1, den2 ) ;
f = tf( num, den, -1 ) ;
impulse( f )
```

MATLAB graph



Problem 4.

$$y(k)=g(k)*u(k)=\sum_{j=0}^{k} g(k-j)u(j)$$

$$Y(z)=G(z)U(z)$$

(a) $$G(z)=\frac{0.8(z-1)}{z^2+0.2z-0.15}=\frac{0.8(z-1)}{(z+0.5)(z-0.3)}=\frac{A}{z+0.5}+\frac{B}{z-0.3}$$

$$A(z-0.3)+B(z+0.5)=(A+B)z+(-0.3A+0.5B)=0.8z-0.8$$

$$\rightarrow \begin{array}{c} A+B=0.8 \\ -0.3A+0.5B=-0.8 \end{array} \rightarrow \begin{array}{c} A=1.5 \\ B=-0.7 \end{array}$$

$$=\frac{1.5}{z+0.5}-\frac{0.7}{z-0.3}=z^{-1}\frac{1.5z}{z+0.5}-z^{-1}\frac{0.7z}{z-0.3}$$

From the property of z – transform ($\mathcal{Z}\{f(k-1)\}=z^{-1}F(z)$),

$$g(k)=1.5(-0.5)^{k-1}-0.7(0.3)^{k-1} \quad (\text{for } K\ge 1)$$

To find g(0), use initial value theorem.

$$g(0)=\lim_{z\to\infty} G(z)=0$$

$$\therefore g(k)=\begin{cases} 0 & \text{for } k=0 \\ 1.5(-0.5)^{k-1}-0.7(0.3)^{k-1} & \text{for } k\ge 1 \end{cases}$$

(b) $$p(k)=\sum_{j=0}^{k} g(j)$$

$$P(z)=\sum_{k=0}^{\infty} p(k)z^{-k}=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} g(j)\right)z^{-k}$$

$$=(g(0))z^{0}+(g(0)+g(1))z^{-1}+(g(0)+g(1)+g(2))z^{-2}+\ldots$$

$$=(g(0)+g(1)z^{-1}+g(2)z^{-2}+\ldots)+z^{-1}(g(0)+g(1)z^{-1}+g(2)z^{-2}+\ldots)$$

$$+z^{-2}(g(0)+g(1)z^{-1}+g(2)z^{-2}+\ldots)+\ldots$$

$$=(1+z^{-1}+z^{-2}+\ldots)G(z)=\frac{1}{1-z^{-1}}G(z)$$

From final value theorem,

$$p_{ss}=\lim_{k\to\infty} p(k)=\lim_{z\to 1}(z-1)P(z)=\lim_{z\to 1}(z-1)\frac{zG(z)}{(z-1)}=\lim_{z\to 1}\frac{0.8(z^2-z)}{z^2+0.2z-0.15}$$

$$=0$$