UNIST
Department of Mechanical Engineering

MEN 573: Advanced Control Systems I		**Spring, 2016**
Homework #6	Assigned: Wednesday, April 27, 2016	
Solution	Due: Monday, May 9, 2016 (in class)	

Problem 1.

$$s^4 + 2.9s^3 + 2.7s^2 + 0.7s + (K - 0.1) = 0$$

Routh array,

s^4	1	2.7	$K-0.1$
s^3	2.9	0.7	0
s^2	$a_1 = 2.46$	$a_2 = K - 0.1$	$a_3 = 0$
s^1	$b_1 = -1.18K + 0.82$	$b_2 = 0$	
s^0	$c_1 = K - 0.1$		

$$a_1 = -\frac{\begin{vmatrix} 1 & 2.7 \\ 2.9 & 0.7 \end{vmatrix}}{2.9} = -\frac{0.7 - 2.7 \times 2.9}{2.9} \cong 2.46$$

$$a_2 = -\frac{\begin{vmatrix} 1 & K-0.1 \\ 2.9 & 0 \end{vmatrix}}{2.9} = K - 0.1$$

$$a_3 = -\frac{\begin{vmatrix} 1 & 0 \\ 2.9 & 0 \end{vmatrix}}{2.9} = 0$$

$$b_1 = -\frac{\begin{vmatrix} 2.9 & 0.7 \\ 2.46 & K-0.1 \end{vmatrix}}{2.46} = -1.18K + 0.82$$

$$b_2 = -\frac{\begin{vmatrix} 2.9 & 0 \\ 2.46 & 0 \end{vmatrix}}{2.46} = 0$$

$$c_1 = -\frac{\begin{vmatrix} 2.46 & K-0.1 \\ -1.18K+0.82 & 0 \end{vmatrix}}{-1.18K+0.82} = K - 0.1$$

Not to possess any root in the RHP, there are no sign changes in the first column.

$$b_1 = -1.18K + 0.82 > 0 \qquad \Rightarrow K < 0.69$$

$$c_1 = K - 0.1 > 0 \qquad \Rightarrow K < 0.1$$

$$\therefore 0.1 < K < 0.69$$

Problem 2.

$$x(k+1)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.008 & 0.008 & -0.79 & -0.8 \end{bmatrix} x(k) = Ax(k)$$

$$A(z)=det(zI-A)=\begin{vmatrix} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 0 & 0 & z & -1 \\ -0.008 & -0.008 & 0.79 & z+0.8 \end{vmatrix} = z\begin{vmatrix} z & -1 & 0 \\ 0 & z & -1 \\ -0.008 & 0.79 & z+0.8 \end{vmatrix} +1\begin{vmatrix} 0 & -1 & 0 \\ 0 & z & -1 \\ -0.008 & 0.79 & z+0.8 \end{vmatrix}$$

$$\begin{aligned} &= z[z\{z(z+0.8)+0.79\}-0.008]-0.008 \\ &= z(z^3+0.8z^2+0.79z-0.008)-0.008 \\ &= z^4+0.8z^3+0.79z^2-0.008z-0.008 \end{aligned}$$

$$A^*(s)=A(z)\big|_{z=\frac{1+s}{1-s}}(1-s)^4$$

$$=(1+s)^4+0.8(1+s)^3(1-s)+0.79(1+s)^2(1-s)^2-0.008(1+s)(1-s)^3-0.008(1-s)$$

$$=0.99s^4+2.416s^3+4.372s^2+5.648s+2.574$$

Routh array

s^4	0.99	4.372	2.574
s^3	2.416	5.648	0
s^2	2.058	2.574	0
s^1	2.626	0	
s^0	2.574		

All elements in the first column are positive. Thus, this system is asymptotically stable.

Problem 3.

$z = \frac{r(1+s)}{1-s}$ **, Let** s **be** $j\omega$ **, the imaginary axis of the s-plane.**

$$z = \frac{r(1+j\omega)}{1-j\omega} = \frac{r(1+j\omega)^2}{(1-j\omega)(1+j\omega)} = \frac{r(1-\omega^2+2j\omega)}{1+\omega^2} = r\left(\frac{1-\omega^2}{1+\omega^2} + j\frac{2j\omega}{1+\omega^2}\right)$$

$$\Rightarrow \left\{\left(\frac{r(1-\omega^2)}{1+\omega^2}\right)^2 + \left(\frac{r\cdot 2\omega}{1+\omega^2}\right)^2\right\}^{1/2} = r$$

$\therefore$ **Circle (radius:** r **, center: origin)**

Problem 4.

T.F. : $\dfrac{GC}{1+GC}=\dfrac{\dfrac{k}{z(z-0.8)}}{1+\dfrac{k}{z(z-0.8)}}=\dfrac{k}{z^2-0.8z+k}$

$$A(z)=z^2-0.8z+k$$

$$z=\frac{r(1+s)}{1-s}=0.5\frac{(1+s)}{(1-s)}$$

$$A(s)=0.25\frac{(1+s)^2}{(1-s)^2}-0.4\frac{1+s}{1-s}+k=c$$

$$\Rightarrow A(s)=0.25(1+s)^2-0.4(1+s)(1-s)+k(1-s)^2=(0.65+k)s^2+(0.5-2k)s+(k-0.15)=1$$

Routh array,

s^2	$0.65+k$	$k-0.15$
s^1	$0.5-2k$	0
s^0	$k-0.15$	

The closed loop poles are inside of a circle with radius 0.5.

$\Rightarrow$ **No sign changes in the first column**

$0.65+k>0 \Rightarrow k>-0.65$		$0.65+k<0 \Rightarrow k<-0.65$
$0.5-2k>0 \Rightarrow k<0.25$	**or**	$0.5-2k<0 \Rightarrow k>0.25$
$k-0.15>0 \Rightarrow k>0.15$		$k-0.15<0 \Rightarrow k<0.15$
$\Rightarrow 0.15<k<0.25$		$\Rightarrow$ **No range.**

$\therefore 015<k<0.25$

Problem 5.

(a) $X_1 = y \qquad \rightarrow \dot{X}_1 = \dot{y} = X_2$

$X_2 = \dot{y} \qquad \rightarrow \dot{X}_2 = \ddot{y} = -[a + b\cos(y)]\dot{y} - c\sin(y) = -[a + b\cos(X_1)]X_2 - c\sin(X_1)$

4…①

Note that the origin is an equilibrium point.

(b) $V(X) = c(1 - \cos(x_1)) + \frac{1}{2}x_2^2$

$$\left.\begin{array}{l} V(X) > 0 \ (\forall x \neq 0) \\ V(0) = 0 \end{array}\right\} \Rightarrow V(X) \succ 0$$

$\dot{V}(X) = c\sin(x_1) \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2 = c\sin(x_1)x_2 + x_2(-[a + b\cos(x_1)]x_2 - c\sin(x_1)])$

$= -[a + b\cos(x_1)]x_2^2 \leq 0$ **…②**

$$\left.\begin{array}{l} \dot{V}(0) = 0 \\ \dot{V}(X) \leq 0 \end{array}\right\} \Rightarrow$$ **Since** $-1 \leq \cos(x_1) \leq 1$**,** $a \geq 0$**,** $b \geq 0$**, thus if** $a \geq b \Rightarrow \dot{V}(x) \leq 0$

$\therefore$ **If** $a \geq b \geq 0$**, the origin is an asymptotically stable system.**

(c) If $a \geq b \geq 0$**, then** $\dot{V}(x) \leq 0$ **&** $\Rightarrow V(X) \succ 0 \Rightarrow$ **Stable in the sense of Lyapunov.**

$s = \{x : V(x) \leq m, \dot{V}(x) = 0\}$**,** $m = \sup_{|x| \leq r} V(x)$

Let $|x_1| < \pi$ **since** $\dot{V}(X) = 0$ **at** $x_1 = \pm\pi$

$\dot{V}(X) = 0 \;\Rightarrow\; x_2 = 0$ **(by ②)**

Using ①, $x_2 = 0 \;\Rightarrow\; \dot{x}_1 = 0$ **and** $0 = -c\sin(x_1)$

Thus $x_1 = 0$ **and it is in the range,** $|x_1| < \pi$**.**

$\therefore x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Rightarrow$ **satisfying Lasalle's theorem.**

$\therefore$ **The origin is asymptotically stable system if** $a > b \geq 0$**.**

Problem 6.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}, \; A^T = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$$

Lyapunov equation: $A^T P + PA = -Q$

Let $Q = I$ **for an arbitrary positive definite symmetric matrix.**

Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$

$$A^T P + PA = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$$

$$= \begin{bmatrix} aP_{11} + bP_{12} & aP_{12} + bP_{22} \\ P_{11} & P_{22} \end{bmatrix} + \begin{bmatrix} aP_{11} + bP_{12} & P_{11} \\ aP_{12} + bP_{22} & P_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 2aP_{11} + 2bP_{12} & P_{11} + aP_{12} + bP_{22} \\ P_{11} + aP_{12} + bP_{22} & 2P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\left.\begin{array}{l} aP_{11} + bP_{12} = -\frac{1}{2} \\ \\ P_{11} + aP_{12} + bP_{22} = 0 \\ \\ P_{12} = -\frac{1}{2} \end{array}\right\} \Rightarrow \begin{array}{l} P_{11} = \frac{1}{a}\left(-bP_{12} - \frac{1}{2}\right) = \frac{b-1}{2a} \\ \\ P_{22} = \frac{1}{b}\left(-P_{11} - aP_{12}\right) = \frac{1}{b}\left(-\frac{b-1}{2a} + \frac{a}{2}\right) = \frac{1}{b}\left(\frac{-b+1+a^2}{2a}\right) = \frac{a^2 + (1-b)}{2ab} \\ \\ P_{12} = -\frac{1}{2} \end{array}$$

$$\therefore P = \begin{bmatrix} \frac{b-1}{2a} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{a^2 + (1-b)}{2ab} \end{bmatrix} \Rightarrow \textbf{Positive definite.}$$

$\Rightarrow$ **All the leading principle minors should be positive.**

$$P_{11} = \frac{b-1}{2a} > 0 \;\Rightarrow\; a(b-1) > 0$$

$$det(P) = \frac{b-1}{2a} \cdot \frac{a^2 + (1-b)}{2ab} - \frac{1}{4} > 0 \;\Rightarrow\; \frac{(b-1)\{a^2 + (1-b)\} - a^2 b}{4a^2 b} > 0$$

$$\Rightarrow b(b-1)\{a^2 + (1-b) - a^2 b^2\} \qquad = b\{a^2 b - a^2 - (b-1)^2 - a^2 b\} > 0$$

$$\Rightarrow b\{a^2 + (b-1)^2\} < 0 \qquad \Rightarrow\; b < 0$$

$$P_{11} = (b-1)/2a > 0 \Rightarrow a(b-1) > 0 \qquad \Rightarrow\; a < 0$$

$\therefore$ **The system is asymptotically stable iff** $a < 0$ **and** $b < 0$**.**

Problem 7.

(a) $\left|u^T Pv\right| \le \lambda_{max}(P)\|u\|_2\|v\|_2$, $\lambda_{max}(P) > 0$: the largest eigenvalue of P .

Prove: If $P = P^T$ **,** $P > 0$ **,** $P = T\Lambda T^T$ **…①**

$$\begin{cases} \because det(\lambda I - P) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda) \qquad \lambda_i \in \Re,\ \lambda_i > 0 \\ \Rightarrow \left.\begin{matrix} Pv_1 = \lambda_1 v_1 \\ \vdots \\ Pv_n = \lambda_n v_n \end{matrix}\right\} \lambda_i : \textbf{eigenvalue, } v_i : \textbf{eigenvector} \\ \Rightarrow P\underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{T} = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \\ \Rightarrow PT = T\Lambda \\ v_i^T v_j = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases} \quad \Rightarrow T^T T = I \quad \Rightarrow T^T = T^{-1} \\ \Rightarrow P = T\Lambda T^{-1} = T\Lambda T^T \end{cases}$$

$$\Rightarrow \left|u^T Pv\right| \le \|u\|_2 \left\{(\lambda_{max}(P))^2 \left\|T^T v\right\|_2^{\ 2}\right\}^{\frac{1}{2}}$$

$$= \|u\|_2 (\lambda_{max}(P)) \left\|T^T v\right\|_2$$

$$= \|u\|_2 (\lambda_{max}(P)) \|v\|_2$$ **(** $\because \left\|T^T v\right\|_2 = \|v\|_2$ **, since T is unitary matrix)**

$$\therefore \left|u^T Pv\right| \le (\lambda_{max}(P))\|u\|_2\|v\|_2$$

(b) Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **,** $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **,** $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$det(\lambda I - P) = (\lambda - 1)^2 = 0 \ \Rightarrow\ \lambda_1 = \lambda_2 = 1 \ \Rightarrow\ \lambda_{min} = 1$

$\left|u^T Pv\right| = \left|\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right| = 0$

$\lambda_{min}(P)\|u\|_2\|v\|_2 = 1 \times 1 \times 1 = 1$

$\therefore \left|u^T Pv\right| \ge \lambda_{min}(P)\|u\|_2\|v\|_2$ **is not true.**