

[MEN573]

Advanced Control Systems I

Lecture 13

Input-Output Stability of LTI Systems

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Bounded-input bounded-output (BIBO) stability (CT)

The LTI system

$$\dot{x} = A x + B u$$

$$y = C x$$

is **BIBO** if,

- for **every** bounded input $u(t)$ which satisfies

$$|u(t)| < u_{max}, \quad \forall t \geq 0$$

the output $y(t)$ is also bounded **and**

- there exist finite constants k and b such that

$$|y(t)| \leq k u_{max} + b, \quad \forall t \geq 0$$

Bounded-input bounded-output (BIBO) stability (CT)

Theorem

The controllable and observable LTI system

$$\dot{x} = A x + B u$$

$$y = C x$$

is BIBO **iff** (if and only if) the matrix A is Hurwitz
(all eigenvalues have negative real parts).

Note: controllability and observability is needed only for necessity, not sufficiency.

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Proof of sufficiency \Leftarrow

Assume that the matrix A of the LTI system

$$\dot{x} = A x + B u$$

$$y = C x$$

is Hurwitz. We need to show that the system is BIBO.

Define the transfer function

$$G(s) = C (sI - A)^{-1} B$$

and the impulse response

$$g(t) = \mathcal{L}^{-1} \{G(s)\}$$

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Proof of sufficiency \Leftarrow

Lemma1:

Let

$$G(s) = C (sI - A)^{-1} B$$

$$g(t) = \mathcal{L}^{-1} \{G(s)\}$$

where the matrix A is Hurwitz.

Then

$$\int_0^{\infty} |g(t)| dt = k < \infty$$

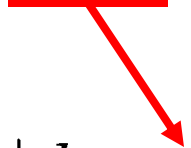
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Proof of sufficiency \Leftarrow

Consider now the forced response

$$y_f(t) = \int_0^t g(t - \tau) u(\tau) d\tau$$

Since $|u(t)| < u_{max}$ and $\int_0^\infty |g(t)| dt = k < \infty$

$$\begin{aligned} |y_f(t)| &\leq \int_0^t |g(t - \tau) \underline{u(\tau)}| d\tau \\ &\leq \int_0^t |g(t - \tau)| d\tau u_{max} \\ &\leq k u_{max} \end{aligned}$$


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Proof of necessity \Rightarrow

Assume that the matrix A of the LTI system

$$\dot{x} = A x + B u$$

$$y = C x$$

is not Hurwitz and the system is controllable and observable (i.e. no pole-zero cancellation of unstable or limitedly stable poles can take place).

Then, the impulse response,

$$g(t) = \mathcal{L}^{-1} \{G(s)\}$$

will have either an unstable and/or marginally stable mode.

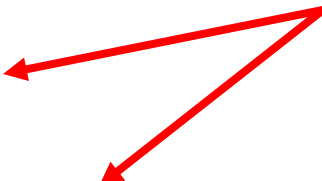
Bounded-input bounded-output (BIBO) stability (CT)

Proof of necessity \Rightarrow

By partial fraction expansions, $g(t)$ will contain at least one unstable and/or marginally stable mode

$$\begin{aligned}
 g(t) = & g_1(t) \\
 & + e^{\lambda_u t} b_u \\
 & + e^{\sigma_u t} \{c_u \cos(\omega_u t) + d_u \sin(\omega_u t)\}
 \end{aligned}$$

unstable or marginally stable modes



$(\lambda_u \geq 0) \quad (\sigma_u \geq 0) \quad (\omega_u > 0)$

with either \mathbf{b}_u , \mathbf{c}_u and/or \mathbf{d}_u not zero.

Bounded-input bounded-output (BIBO) stability (CT)

Proof of necessity \Rightarrow

$$\begin{aligned}
 g(t) &= g_1(t) & (\lambda_u \geq 0) \\
 &+ e^{\lambda_u t} b_u & (\sigma_u \geq 0) \\
 &+ e^{\sigma_u t} \{c_u \cos(\omega_u t) + d_u \sin(\omega_u t)\} & (\omega_u > 0)
 \end{aligned}$$

- If either $\lambda_u \geq 0$ or $\sigma_u > 0$, then $u(t) = 1$ will result in an unbounded output, e.g.

$$(\lambda_u = 0) (b_u = 1)$$

$$\begin{aligned}
 g(t) &= 1 \\
 u(t) &= 1
 \end{aligned}
 \quad \Rightarrow \quad
 y_f(t) = \int_0^t d\tau = t$$

Bounded-input bounded-output (BIBO) stability (CT)

Proof of necessity \Rightarrow

$$g(t) = g_1(t) + e^{\sigma_u t} \{c_u \cos(\omega_u t) + d_u \sin(\omega_u t)\}$$

- If $\sigma_u = 0$ then $u(t) = \cos(\omega_u t)$ will result in an unbounded output (resonance). E.g.

$$g(t) = \cos(\omega_u t) \qquad u(t) = \cos(\omega_u t)$$

$$\begin{aligned} y_f(t) &= \int_0^t \cos(\omega_u(t - \tau)) \cos(\omega_u \tau) d\tau \\ &= \underbrace{\cos(\omega_u t) \int_0^t \cos^2(\omega_u \tau) d\tau}_{= \cos(\omega_u t) [t + \frac{1}{2\omega_u} \sin(2\omega_u t)]} + \sin(\omega_u t) \int_0^t \sin(\omega_u \tau) \cos(\omega_u \tau) d\tau \end{aligned}$$

Q.E.D.

Bounded-input bounded-output (BIBO) stability (DT)

The LTI system

$$x(k+1) = A x(k) + B u(k)$$

$$y(k) = C x(k)$$

is **BIBO** stable if,

- for every bounded input $u(k)$ which satisfies

$$|u(k)| < u_{max}, \quad \forall k \geq 0$$

the output $y(k)$ is also bounded **and**

- there exist finite constants k and b such that

$$|y(k)| \leq k u_{max} + b, \quad \forall k \geq 0$$

Bounded-input bounded-output (BIBO) stability (DT)

Theorem

The controllable and observable LTI system

$$\begin{aligned}x(k+1) &= A x(k) + B u(k) \\ y(k) &= C x(k)\end{aligned}$$

is BIBO **iff** (if and only if) the matrix A is Schur
(all eigenvalues are inside the unit circle).

Note: controllability and observability is needed only for the necessity, not sufficiency.