[MEN573]
Advanced Control Systems I

Lecture 13
Input-Output Stability of LTI Systems

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Bounded-input bounded-output (BIBO) stability (CT)

The LTI system

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned}$$

is ***BIBO*** if,

- for ***every*** bounded input $u(t)$ which satisfies

$$|u(t)| < u_{max}\,, \quad \forall\ t \geq 0$$

the output $y(t)$ is also bounded ***and***

- there exist finite constants k and b such that

$$|y(t)| \leq k\,u_{max} + b\,, \quad \forall\ t \geq 0$$

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Theorem

The controllable and observable LTI system

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned}$$

is BIBO **iff** (if and only if) the matrix $\boldsymbol{A}$ is Hurwitz

(all eigenvalues have negative real parts).

Note: controllability and observability is needed only for necessity, not sufficiency.

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Proof of sufficiency $\Longleftarrow$

Assume that the matrix $\boldsymbol{A}$ of the LTI system

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned}$$

is Hurwitz. We need to show that the system is BIBO.

Define the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B$$

and the impulse response

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\}$$

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Proof of sufficiency $\Leftarrow$

Lemma1:

Let

$$G(s) = C\,(sI - A)^{-1}\,B$$

$$g(t) = \mathcal{L}^{-1}\,\{G(s)\}$$

where the matrix $\boldsymbol{A}$ is Hurwitz.

Then

$$\boxed{\int_0^{\infty} |g(t)|dt = k < \infty}$$

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Proof of sufficiency $\Longleftarrow$

Consider now the forced response

$$y_f(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Since $|u(t)| < u_{max}$ and $\int_0^\infty |g(t)|dt = k < \infty$

$$\begin{aligned}
|y_f(t)| &\leq \int_0^t |g(t-\tau)\underline{u(\tau)}|d\tau \\
&\leq \int_0^t |g(t-\tau)|d\tau\, u_{max} \\
&\leq k\, u_{max}
\end{aligned}$$

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Proof of necessity $\Longrightarrow$

Assume that the matrix $\boldsymbol{A}$ of the LTI system

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned}$$

is not Hurwitz and the system is controllable and observable (i.e. no pole-zero cancellation of unstable or limitedly stable poles can take place).

Then, the impulse response,

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\}$$

will have either an unstable and/or marginally stable mode.

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Proof of necessity $\Longrightarrow$

By partial fraction expansions, $g(t)$ will contain at least one unstable and/or marginally stable mode

$$\begin{aligned} g(t) \;=\; & g_1(t) \\ & + e^{\lambda_u t}\, b_u \\ & + e^{\sigma_u t}\,\{c_u \cos(\omega_u t) + d_u \sin(\omega_u t)\} \end{aligned}$$

unstable or marginally stable modes

$$(\lambda_u \geq 0) \qquad (\sigma_u \geq 0) \qquad (\omega_u > 0)$$

with either $\boldsymbol{b_u}$, $\boldsymbol{c_u}$ and/or $\boldsymbol{d_u}$ not zero.

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Proof of necessity $\Rightarrow$

$$\begin{aligned} g(t) &= g_1(t) & (\lambda_u \geq 0) \\ &\quad + e^{\lambda_u t} b_u & (\sigma_u \geq 0) \\ &\quad + e^{\sigma_u t}\{c_u \cos(\omega_u t) + d_u \sin(\omega_u t)\} & (\omega_u > 0) \end{aligned}$$

- If either $\lambda_u \geq 0$ or $\sigma_u > 0$, then *u(t)* = *1* will result in an unbounded output, e.g. $(\lambda_u = 0)\ (b_u = 1)$

$$\begin{aligned} g(t) &= 1 \\ u(t) &= 1 \end{aligned} \quad \Rightarrow \quad y_f(t) = \int_0^t d\tau = t$$

Bounded-input bounded-output (BIBO) stability (CT)

Proof of necessity $\Longrightarrow$

$$g(t) \;=\; g_1(t) + e^{\sigma_u t}\{c_u \cos(\omega_u t) + d_u \sin(\omega_u t)\}$$

- If $\sigma_u = 0$ then $u(t) = \cos(\omega_u t)$ will result in an unbounded output (resonance). E.g.

$$g(t) \;=\; \cos(\omega_u\, t) \qquad\qquad u(t) = \cos(\omega_u t)$$

$$\begin{aligned} y_f(t) &= \int_0^t \cos(\omega_u(t-\tau))\cos(\omega_u\tau)d\tau \\ &= \underbrace{\cos(\omega_u t)\int_0^t \cos^2(\omega_u\tau)d\tau}_{=\cos(\omega_u t)[t + \frac{1}{2\omega_u}\sin(2\omega_u t)]} + \sin(\omega_u t)\int_0^t \sin(\omega_u\tau)\cos(\omega_u\tau)d\tau \end{aligned}$$

Q.E.D.

Bounded-input bounded-output (BIBO) stability (DT)

The LTI system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

is ***BIBO*** stable if,

- for every bounded input $u(k)$ which satisfies

$$|u(k)| < u_{max}\,, \quad \forall\ k \geq 0$$

the output $y(k)$ is also bounded ***and***

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Bounded-input bounded-output (BIBO) stability (DT)

Theorem

The controllable and observable LTI system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

is BIBO **iff** (if and only if) the matrix $\boldsymbol{A}$ is Schur

(all eigenvalues are inside the unit circle).

Note: controllability and observability is needed only
for the necessity, not sufficiency.