UNIST
Department of Mechanical Engineering

MEN 573: Advanced Control Systems I **Spring, 2016**

Homework #7 Assigned: Wednesday, May 4, 2016
Due: Wednesday, May 18, 2016 (in class)

Problem 1.

Consider the asymptotically stable LTI system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Utilize the Lyapunov equation to find an expression for

$$J = \int_0^\infty \left\{2x_1^2(t) + q\, x_2^2(t)\right\}\, dt$$

in terms of q, where q is some nonnegative number.

Hint: Notice that J can be written as

$$\begin{aligned} J &= \int_0^\infty \left\{x^T(t)\begin{bmatrix} 2 & 0 \\ 0 & q \end{bmatrix} x(t)\right\} dt \\ &= x^T(0)\left\{\int_0^\infty e^{A^T t}\begin{bmatrix} 2 & 0 \\ 0 & q \end{bmatrix} e^{At} dt\right\} x(0) \end{aligned}$$

Problem 2.

Consider the continuous time system described by

$$\dot{x} = Ax$$

where $A \in \mathcal{R}^{n\times n}$. Define a new matrix $\overline{A}$ by

$$\overline{A} = A + \lambda_p I$$

where $\lambda_p \in \mathcal{R}$ and I is the identity matrix.

(a) Let λ_{Ai} be the ith eigenvalue of A and $\lambda_{\overline{A}i}$ be the ith eigenvalue of $\overline{A}$, show that

$$\lambda_{\overline{A}i} = \lambda_{Ai} + \lambda_p$$

(b) Now consider a Lyapunov like equation

$$A\,P + P\,A^T = -\left(Q + \sigma P\right)$$

and explain how you can use this equation to check whether or not all the eigenvalues of A have real parts smaller than $-\lambda_p$.

Problem 3.

Consider the discrete time Lyapunov function $\mathcal{D}_{_A} : \mathcal{R}^{n\times n} \to \mathcal{R}^{n\times n}$, which is defined as follows:

$$\mathcal{D}_{_A}(P) = A^T\, P\, A - P,$$

where $A \in \mathcal{R}^{n\times n}$.

(a) Show that $\mathcal{D}_{_A}(\cdot)$ is a linear function of P, i.e. show that, for any two matrices $P_1 \in \mathcal{R}^{n\times n}$ and $P_2 \in \mathcal{R}^{n\times n}$ and two scalars $\alpha_1 \in \mathcal{R}$ and $\alpha_2 \in \mathcal{R}$,

$$\mathcal{D}_{_A}(\alpha_1\, P_1 + \alpha_2 P_2) = \alpha_1\, \mathcal{D}_{_A}(P_1) + \alpha_2\, \mathcal{D}_{_A}(P_2)$$

(b) Let the λ_i be the ith eigenvalue of A^T and let v_i be its associated eigenvector, i.e.

$$A^T\, v_i = \lambda_i\, v_i\,.$$

Given two eigenvectors v_i and v_j, define the matrix

$$V_k = v_i\, v_j^T\,.$$

Notice that you can construct n^2 matrices $V_k's$. Show that

$$\begin{aligned} \mathcal{D}_{_A}(V_k) &= \mathcal{D}_{_A}(v_i\, v_j^T) \\ &= \mu_k\, V_k \end{aligned}$$

and determine an expression for the scalar μ_k in terms of the eigenvalues λ_i and λ_j. Notice that μ_k is an eigenvalue for the linear function $\mathcal{D}_{_A}(\cdot)$ and the matrix $V_k = v_i\, v_j^T \in \mathcal{R}^{n\times n}$ is its corresponding eigenvector.

(c) For the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix} \tag{1}$$

(i) Find the eigenvalues $\mu_k's$ of the linear function $\mathcal{D}_{_A}(\cdot)$ and their corresponding eigenvectors $V_k's$.

(ii) Obtain a matrix representation of the linear function $\mathcal{D}_{_A}(\cdot)$ when A is given by (1), using the following procedure: Let

$$P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \qquad Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

$$\mathcal{D}_{_A}(P) = A \begin{bmatrix} p_1 & p_2 \end{bmatrix} A^T - \begin{bmatrix} p_1 & p_2 \end{bmatrix} = - \begin{bmatrix} q_1 & q_2 \end{bmatrix}$$

Stacking up the columns of the matrices $\mathbf{P} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$, and $\mathbf{Q} = -\begin{bmatrix} q_1 & q_2 \end{bmatrix}$ to form column vectors

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} p_{11} & p_{21} & p_{12} & p_{22} \end{bmatrix}^T \\ -\mathbf{Q} &= -\begin{bmatrix} q_{11} & q_{21} & q_{12} & q_{22} \end{bmatrix}^T \end{aligned}$$

determine the matrix $\mathbf{D}_{_A} \in \mathcal{R}^{4\times 4}$ which is the coordinate representation of the linear function $\mathcal{D}_{_A}(\cdot)$

$$APA^T - P = -Q \;\Leftrightarrow\; \mathbf{D}_{_A}\, \mathbf{P} = -\mathbf{Q}\,.$$

(iii) Use matlab to verify that the eigenvalues $\mu_k's$ of the linear function $\mathcal{D}_A(\cdot)$ are also the eigenvalues of the matrix $\mathbf{D}_A$. Also determine the four eigenvectors $u_k's \in \mathcal{R}^4$ of the matrix $\mathbf{D}_A$, such that

$$\mathbf{D}_A u_k = \mu_k u_k \,.$$

(d) Consider now the solution of the discrete time Lyapunov-like equation: Given a matrix Q, find a matrix P such that

$$\mathcal{D}_A(P) = -Q \tag{2}$$

i.e. find the matrix P such that

$$A^T P A - P = -Q$$

The necessary and sufficient condition for the solution to be unique is that the null space of $\mathcal{D}_A(\cdot)$ must be empty, i.e., all eigenvalues $\mu_k's$ of $\mathcal{D}_A(\cdot)$ must be nonzero. Determine the condition that the eigenvalues $\lambda_i's$ of A must satisfy for Eq. (2) to have a unique solution.

(e) Using the results of the of part d), prove the following threorem:

Theorem The following two conditions are equivalent

1. The matrix $A \in \mathcal{R}^{n\times n}$ is Schur.
2. For any symmetric matrix $Q \in \mathcal{R}^{n\times n}$ with bounded elements, there exists a symmetric matrix P with bounded elements, which is the unique solution of the Lyapunov equation

$$A^T P A - P = -Q \,.$$

Problem 4.

Let the matrix $A \in \mathcal{R}^{n\times n}$ be Schur. We will now show that, for any symmetric matrix $Q \succ 0$, the unique solution of the Lyapunov equation

$$A^T P A - P = -Q \tag{3}$$

is

$$P = \sum_{k=0}^{\infty} (A^k)^T Q A^k \,. \tag{4}$$

To do so, consider the LTI system

$$x(k+1) = A\,x(k) \qquad x(0) = x_o$$

where A is Schur and the initial condition $x_o \in \mathcal{R}^n$ is arbitrary. Also, define the PDF $V(x) \succ 0$ by

$$V(x) = x^T P x$$

where $P = P^T$, $P \succ 0$ satisfies the Lyapunov equation (3).

(a) Show:

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) = -x^T(k)\, Q\, x(k)$$

and

$$\lim_{k\to\infty} V(x(k)) = 0$$

(b) Show: [1]

$$V(x(0)) = \sum_{k=0}^{\infty} x^T(k)\, Q\, x(k)$$

(c) Finally, utilizing

$$x(k) = A^k x_o \qquad\qquad V(x(0)) = x_o^T\, P\, x_o$$

where x_o is arbitrary, obtain Eq. (4).