CSE530 Algorithms & Complexity
Lecture 1: Insertion Sort

Antoine Vigneron
antoine@unist.ac.kr

Ulsan National Institute of Science and Technology

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Introduction

Problem (Sorting)

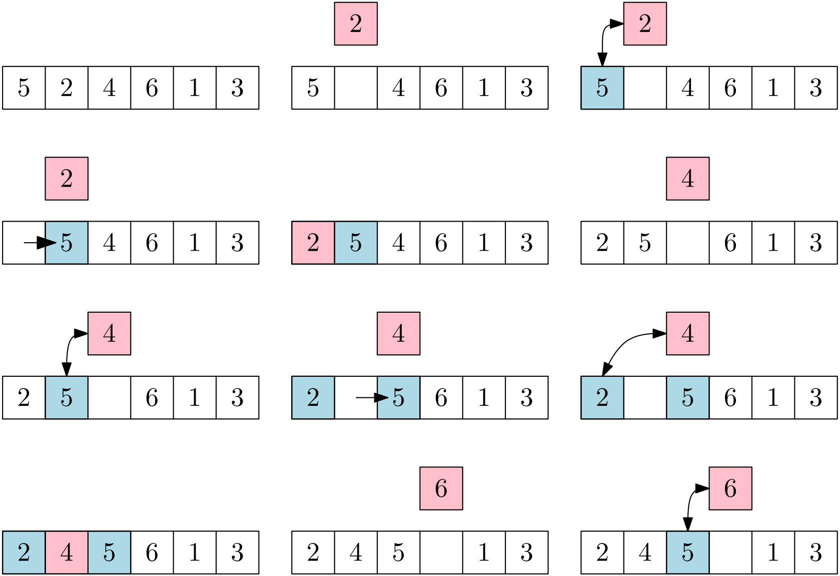
Given an input sequence of n numbers, the sorting problem is to find a permutation of the input sequence sorted in nondecreasing order.

- The sorting problem can also be stated as follows:
  - **Input:** a sequence of n numbers $(a_1, a_2, \ldots, a_n)$
  - **Output:** a permutation of the input sequence $(a'_1, a'_2, \ldots, a'_n)$ such that $a'_1 \leqslant a'_2 \leqslant \ldots \leqslant a'_n$
- Example:
  - **Input:** $(6, 1, 7, 6, 4)$
  - **Output:** $(1, 4, 6, 6, 7)$
- The numbers a_i that we wish to sort are also called the *keys*.

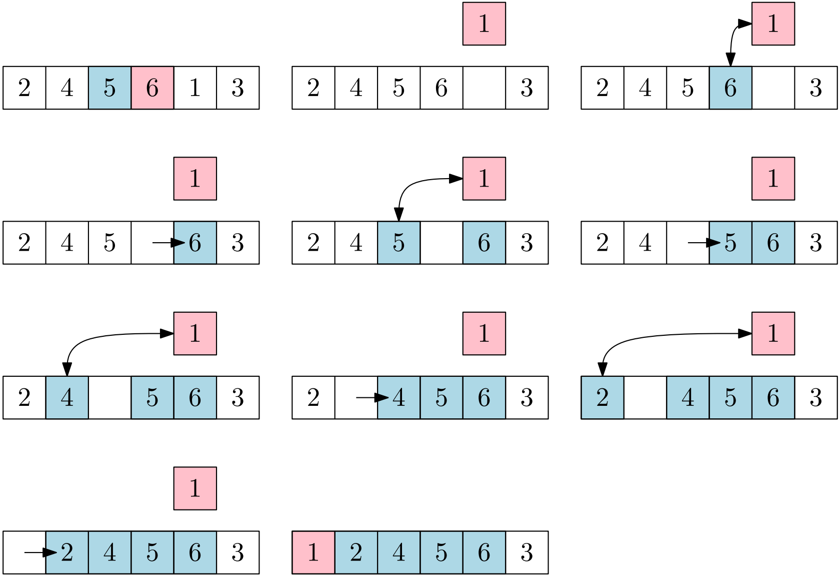
Introduction

- In this lecture, we present a first sorting algorithm.
  - **Reference**: Chapter 2 of the textbook Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.
  - Available online from the UNIST library website.
- The main goal is to introduce the framework of this course.
- Sorting is an important problem.
  - In the 60's, 25% of computing time was spent on sorting.
  - It allows to illustrate several algorithmic techniques.
- There will be more lectures on sorting later this semester.

Algorithm

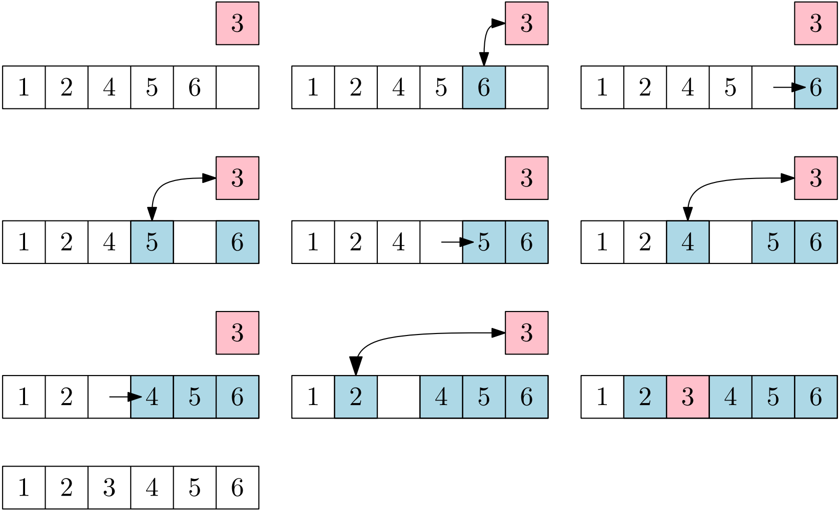

2
2
5 2 4 6 1 3
5 4 6 1 3
5 4 6 1 3
2
4
5 4 6 1 3
2 5 4 6 1 3
2 5 6 1 3
4
4
4
2 5 6 1 3
2 5 6 1 3
2 5 6 1 3
6
6
2 4 5 6 1 3
2 4 5 1 3
2 4 5 1 3

Algorithm



1
2 4 5 6 1 3
2 4 5 6 3
2 4 5 6 3
2 4 5 6 3
2 4 5 6 3
2 4 5 6 3
2 4 5 6 3
2 4 5 6 3
2 4 5 6 3
2 4 5 6 3
1 2 4 5 6 3

Algorithm



3
1 2 4 5 6
3
1 2 4 5 6
3
1 2 4 5 6
3
1 2 4 5 6
3
1 2 4 5 6
3
1 2 4 5 6
3
1 2 4 5 6
3
1 2 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6

Algorithm

- Insertion Sort proceeds from left to right. The current element $A[j]$ (red) is inserted into $A[1 \dots j-1]$.
- $A[j]$ is compared with all the blue keys.
- Insertion Sort is a very natural algorithm.
  - People use it to sort a deck of cards.

Algorithm

- *Pseudocode* of Insertion Sort:

Insertion Sort

```
procedure INSERTION-SORT(A[1...n])
    for j ← 2, n do
        key ← A[j]
        i ← j − 1
        while i > 0 and A[i] > key do
            A[i + 1] ← A[i]
            i ← i − 1
        A[i + 1] ← key
```

- We will present algorithms in pseudocode in this course.
  - Sometimes resembles C, Java, Python...
  - Sometimes uses plain English.
  - No strict rule.
  - Should be clear and concise.

Proof of Correctness

- We now want to prove that INSERTION SORT outputs a correct result.
  - i.e. at the end of the execution, A is sorted.
- Strategy: We use a *loop invariant*.

Loop invariant for INSERTION SORT

At the start of each iteration of the for loop, the subarray $A[1 \ldots j-1]$ consists of the elements originally in $A[1 \ldots j-1]$ in sorted order.

- A proof by loop invariant requires to prove 3 properties:
  - **Initialization**. It is true prior to the first iteration of the loop.
  - **Maintenance**. If it is true before an iteration of the loop, it remains true before the next iteration.
  - **Termination**. When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.
- Proofs done in class. See textbook page 19.

Remarks on Loop Invariants

- A loop invariant is a property that is true at the beginning (or at the end) of each iteration of a loop.
- You should always state the loop invariant precisely.
- A proof by loop invariant is essentially a proof by induction.
- It is useful when a detailed and rigorous proof is needed.

Analysis

- *Analyzing* an algorithm means predicting the amount of resources it uses.
  - Usually: estimate the *running time*, i.e. the time needed for the algorithm to complete.
  - It requires a model of computation.
- Our model of computation: The *Random Access Machine (RAM)*.
- RAM can perform in constant time simple instructions such as:
  - Arithmetic operations $+, -, \times, /$, remainder, floor, ceiling
  - Branching instructions (IF THEN ELSE,)
  - Copying a single variable (not a whole array)
  - Accessing an element of an array
- The *input size* n is the number of bits, or the number of words needed to encode the problem. We will specify it for each problem.
  - Here n is the size of the input array $A[1 \dots n]$.
- Data types:
  - Word size $c \log n$ for an input of size n, where c is a constant.
  - For instance, $c \log n$-bit integers.

Analysis

Insertion Sort

1: **procedure** Insertion-Sort($A[1 \dots n]$)
2:     **for** $j \leftarrow 2, n$ **do**
3:         $\text{key} \leftarrow A[j]$
4:         $i \leftarrow j - 1$
5:         **while** $i > 0$ and $A[i] > \text{key}$ **do**
6:             $A[i+1] \leftarrow A[i]$
7:             $i \leftarrow i - 1$
8:         $A[i+1] \leftarrow \text{key}$

line	cost	times
2	c_2	n
3	c_3	$n-1$
4	c_4	$n-1$
5	c_5	$\sum_{j=2}^{n} t_j$
6	c_6	$\sum_{j=2}^{n}(t_j - 1)$
7	c_7	$\sum_{j=2}^{n}(t_j - 1)$
8	c_8	$n-1$

- t_j: # of times the while loop test is performed
- c_k, $k = 2 \dots 8$ is the time taken to execute line k once
  - Unknown constant, depends on your hardware/OS/compiler ...
- Line 2 takes time $c_2 n$ because we count the nth iteration where $j = n+1$ and we check whether $j > n$, before exiting the loop.

Analysis

- So the running time is

$$T(n) = c_2 n + c_3(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j$$
$$+ c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1).$$

Analysis

- If the input is already sorted, then $t_j = 1$ for all j.
- So the running time on sorted input is

$$T(n) = (c_2 + c_3 + c_4 + c_5 + c_8)n - (c_3 + c_4 + c_5 + c_8)$$

- $T(n)$ cannot be smaller for any input of size n, as we have $t_j = 1$ for all j.
- It is the *best-case running time*.
- As $T(n) = an + b$ for two constants a, b, we say that it is a *linear function*.

Analysis

- Suppose that the input A is sorted in decreasing order: $A[1] > A[2] > \cdots > A[n]$.
- Then $t_j = j$ for all j.
- As $\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$ and $\sum_{j=2}^{n} j - 1 = \frac{n(n-1)}{2}$, we get:

$$T(n) = \left(\frac{c_5 + c_6 + c_7}{2}\right) n^2 + \left(c_2 + c_3 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right) n - (c_2 + c_4 + c_5 + c_8).$$

- As t_j cannot be larger, this is the *worst-case running time*.
- Since $T(n)$ can be written $an^2 + bn + c$ for some constants a, b, c, we say that it is a *quadratic function*.

Analysis

- We usually perform a worst-case analysis rather than best case.
- Reasons:
  - It gives a *guarantee* on the running time.
  - It often happens in practice.
  - The average case is often roughly as bad.
    - Example: Apply INSERTION SORT to a set of random numbers.
    - Then t_j is about $j/2$ on average.
    - So the average running time is still quadratic.
- When the running time is linear, we will write $T(n) = \Theta(n)$, and when it is quadratic, we will write $T(n) = \Theta(n^2)$.
  - We will study this in details in two weeks.
  - Intuition: Keep the dominant term, remove constant factors.