Linear System Theory

Jun Moon

Optimal Control: Linear Quadratic Regulator (LQR)

May 9, 2018

Recap

- State space equation
- Linear Algebra
- Solutions of LTI and LTV systems
- Stability
- Controllability & observability
- State feedback & output feedback

We will study

- Optimal state feedback: Linear quadratic regulator (LQR)

Linear Quadratic Regulator (LQR)

- Finite-Horizon Problem

$$J(u) = \int_0^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt + x^T(t_f)Q_f x(t_f) \ \rightarrow \ \text{minimize}$$

subject to $\dot{x} = Ax + Bu, \ x(0) = x_0$

where $Q_f \geq 0$, $Q \geq 0$, and $R > 0$

- Infinite-Horizon Problem

$$J(u) = \int_0^{\infty} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \ \rightarrow \ \text{minimize}$$

subject to $\dot{x} = Ax + Bu, \ x(0) = x_0$

where $Q_f \geq 0$, $Q \geq 0$, and $R > 0$

- $J(u)$: cost function (energy minimization problem)
- Q, Q_f, R: weighting parameters (design parameters)
- State feedback problem

Linear Quadratic Regulator (LQR)

- Finite-Horizon Problem

$$J(u) = \int_0^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt + x^T(t_f)Q_f x(t_f) \ \rightarrow \ \text{minimize}$$

subject to $\dot{x} = Ax + Bu,\ x(0) = x_0\ (x \in \mathbb{R}^n)$

where $Q_f \geq 0,\ Q \geq 0$, and $R > 0$

Linear Quadratic Regulator (LQR)

Three methods to solve LQR

- The LQR problem is invented by R. Kalman
- Maximum Principle (Pontryagin, Russia, 1956)
- Dynamic Programming: Hamilton-Jacobi-Bellman equation (Bellman, USA, 1953)
- "Completion of squares" (mostly used when we know and want to verify the optimal solution)
- The problem can also be extended to the time-varying case ($A(t)$, $B(t)$, $Q(t) \geq 0$, $R(t) > 0$, $\forall t \geq 0$)

LQR: "Completion of Squares"

We introduce the following differential equation

$$-\frac{dP(t)}{dt} = A^T P(t) + P(t)A + Q - P(t)BR^{-1}B^T P(t), \; P(t_f) = Q_f$$

- The above equation is called "Riccati differential equation (RDE)"
- The RDE is an $n \times n$ matrix differential equation
- The RDE is a *nonlinear* matrix differential equation
- The RDE is a backward differential equation (the terminal condition is given, instead of the initial condition)

LQR: "Completion of Squares"

Fact 1:
If $Q, Q_f \geq 0$ and $R > 0$, there is a unique positive-semi definite matrix $P(\cdot)$ solving the RDE

LQR: "Completion of Squares"

Fact 2:
The unique positive-definite solution to the RDE can be expressed as

$$P(t) = Y(t)X^{-1}(t),\ 0 \le t \le t_f,\ P(t_f) = Q_f,$$

where X and Y satisfy the following *linear* matrix differential equation

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \mathcal{H} \begin{pmatrix} X \\ Y \end{pmatrix},\ X(t_f) = I,\ Y(t_f) = Q_f$$

where $\mathcal{H}$ is the Hamiltonian matrix, given by

$$\mathcal{H} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}$$

LQR: "Completion of Squares"

We claim that the optimal controller that minimizes the cost function is

$$u^*(t) = -R^{-1}B^T P(t)x(t)$$

We will prove this by "completion of squares"

Completion of squares
Example:
$x^2 + 2xy = x^2 + 2xy + y^2 - y^2 = (x+y)^2 - y^2$

LQR: "Completion of Squares"

Proof

$$
\begin{aligned}
&x^T(t_f)Q_f x(t_f) - x_0^T P(0)x_0 \\
&= \int_0^{t_f} \frac{d}{dt}[x^T(t)P(t)x(t)]dt \\
&= \int_0^{t_f} [(Ax(t) + Bu(t))^T P(t)x(t) + x^T P(t)(Ax(t) + Bu(t)) \\
&\qquad + x^T(t)\frac{dP(t)}{dt}x(t)]dt \\
&= \int_0^{t_f} [u^T(t)B^T P(t)x(t) + x^T(t)P(t)Bu(t) \\
&\qquad - x^T(t)(Q - P(t)BR^{-1}B^T P(t))x(t)]dt \\
&\qquad + \int_0^{t_f} [u^T(t)Ru(t) - u^T(t)Ru(t)]dt
\end{aligned}
$$

LQR: "Completion of Squares"

Proof

Rearranging the above equation and "completion of squares" lead to

$$
\begin{aligned}
J(u) &= \int_0^{t_f} x^T(t)Qx(t) + u^T(t)Ru(t)dt + x^T(t_f)Q_f x(t_f) \\
&= x_0^T P(0)x_0 \\
&\quad + \int_0^{t_f} (u(t) + R^{-1}B^T P(t)x(t))^T R(u(t) + R^{-1}B^T P(t)x(t))dt \\
&\geq x_0^T P(0)x_0 \quad \text{when } u(t) = -R^{-1}B^T P(t)x(t)
\end{aligned}
$$

LQR: "Completion of Squares"

We claim that the optimal controller that minimizes the cost function is

$$u^*(t) = -R^{-1}B^T P(t)x(t)$$

- $P(t)$ is the solution of the RDE
- The optimal controller is linear in x!!!
- The optimal controller is time-varying (due to the finite-horizon t_f)
- The optimal controller always exists (since the solution of the RDE always exists)
- The optimal controller is unique!!!
- The optimal cost is

$$J(u^*) = x_0^T P(0)x_0$$

The optimal cost depends on the initial condition x_0, and the flow of the RDE from t_f to 0 (Backward!!!)

LQR: Infinite-Horizon, $t_f \to \infty$

The infinite-horizon problem

$$J(u) = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt$$

$$\text{subject to } \dot{x} = Ax + Bu, \; x(0) = x_0$$

- Steady-state problem
- Since $Q \geq 0$, there is C such that $Q = C^T C$
- We need (A, B) controllable and (C, A) observable
- The additional constraint is the asymptotic stability. Namely, the optimal controller should guarantee that the closed-loop system is asymptotically stable

LQR: Infinite-Horizon, $t_f \to \infty$

Fact 3: If (A, B) is controllable and (C, A) is observable, then

$$\lim_{t_f \to \infty} P(t, t_f) = P > 0, \text{ for all fixed } t \geq 0,$$

where $P > 0$ is a positive-definite matrix, which is a unique solution to the following algebraic Riccati equation (ARE)

$$0 = A^T P + PA + Q - PBR^{-1}B^T P$$

- As $t_f \to \infty$, the solution of the RDE converges to the positive definite matrix, which solves the ARE.
- The ARE can solved by MATLAB ("care")

LQR: Infinite-Horizon, $t_f \to \infty$

Proof (sketch)

- For $t \leq t_1 \leq t_2$, $P(t, t_1) \leq P(t, t_2)$, i.e., for a fixed $t \geq 0$, the RDE is monotonically nondecreasing
- There exists $\bar{P}$ such that $\bar{P} \geq P(t)$ for all $t \geq 0$, i.e., the RDE is uniformly bounded above by $\bar{P}$ (due to controllability)
- The monotonically nondecreasing and bounded sequence is convergent to $P > 0$ (the monotonic sequence convergence theorem)

LQR: Infinite-Horizon, $t_f \to \infty$

Theorem
Assume that (A, B) is controllable and (C, A) is observable, where $Q = C^T C$. Then

- P above is the unique positive definite solution of the ARE in the class of positive-semi definite matrices
- The optimal controller that minimizes $J(u)$ is

$$u^*(t) = -R^{-1}B^T Px(t)$$

- The minimum cost is $J(u^*) = x_0^T P x_0$
- The closed-loop system

$$\dot{x}(t) = (A - BR^{-1}B^T P)x(t)$$

  is asymptotically stable, that is, the real part of eigenvalues of $A - BR^{-1}B^T P$ is negative

LQR: Infinite-Horizon, $t_f \to \infty$

The Hamiltonian

$$\mathcal{H} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}, \ \Pi = \begin{pmatrix} I_n & O_n \\ P & I_n \end{pmatrix}, \ \Pi^{-1} = \begin{pmatrix} I_n & O_n \\ -P & I_n \end{pmatrix}$$

$$W = \Pi^{-1}\mathcal{H}\Pi = \begin{pmatrix} A - BR^{-1}B^T P & -BR^{-1}B^T \\ 0 & -(A - BR^{-1}B^T P)^T \end{pmatrix}$$

- Note that eigenvalues of $\mathcal{H}$ are the same as those of $F = A - BR^{-1}B^T P$ and $-F^T$
- If λ is an eigenvalue of $\mathcal{H}$, then $-\lambda$ is also an eigenvalue of $\mathcal{H}$
- If (A, B) is controllable and (C, A) is observable, then F is stable (due to the theorem), and $\mathcal{H}$ has n stable and n unstable eigenvalues
- It leads to the symmetric root locus

LQR: Robustness

Kalman's inequality
Let $H(jw) = K(jwI - A)^{-1}B$, where H is the loop gain, K is the LQR controller, and $R = \alpha I > 0$ $(\alpha > 0)$. Then

$$(I + H(jw))^*(I + H(jw)) \geq I$$

For the SISO case, this implies that

$$|1 + H(iw)| \geq 1, \ \forall w$$

This implies that the Nyquist plot of the loop gain $H(ij)$ must lie outside a unit circle centered at the -1 point.

This also implies that with the LQR, the gain margin is infinity, and the phase margin is more than 60 degree.