# Linear System Theory

Jun Moon

Chapter 6: Controllability & Observability

Chapter 7: Minimal Realizations

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# Recap

- State space equation
- Linear Algebra
- Solutions of LTI and LTV system
- Stability

We will study

- Controllability & Observability
- Kalman Decomposition
- Minimal realizations

# Controllability & Observability

Controllability (informal): we want to know whether the state of the system is controllable or not from the input

- Analyze the system structure from the input
- With the input, we want to move the state to the desired point in a finite time.

Observability (informal): we want to observe the initial state of the system from the output and input to quantify the behavior of the system

- State: position, velocity, acceleration, etc
- Sensors are required to measure the state. We are not able to use many sensors in real applications.

Controllability & Observability

- Important concepts in control, estimation, and filtering problems
- Optimal control (LQG, Kalman filtering, etc.)

# Controllability & Observability

Example

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u, \; x = (x_1 \; x_2)^T$$

- $(b_1, b_2)^T = (-1, 1)^T$: can move both eigenvalues $\Leftrightarrow$ can control the state $x_1$ and $x_2$
- $(b_1, b_2)^T = (1, 0)^T$: cannot move the eigenvalue 3 $\Leftrightarrow$ cannot control state $x_2$
- $(b_1, b_2)^T = (1, 0)^T$: No matter input, $x_2$ diverges $\Leftrightarrow$ we cannot control $x_2$

# Controllability & Observability

Example

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} x, \; y = \begin{pmatrix} c_1 & c_2 \end{pmatrix} x$$

- $(c_1, c_2) = (1, 1)$: can observe the state $x_1$ and $x_2$
- $(c_1, c_2) = (1, 0)$: cannot observe the state $x_2$
- $(c_1, c_2) = (1, 0)$: Output is always stable, but the system is internally unstable

# Controllability

$$\dot{x}(t) = Ax(t) + Bu(t),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

Definition (Definition 6.1)

The state equation with the pair $(A, B)$ is said to be controllable if for any initial state $x(0) = x_0$, any final state $x_1$, there exists an input that transfers $x_0$ to $x_1$ in a finite time.

# Controllability

Equivalent Definition:
A system is controllable at time $t_0$ if there exists a finite time $t_f$ such that for any initial condition $x_0$, and any final state $x_f$, there is a control input $u$ defined on $[t_0, t_f]$ such that $x(t_f) = x_f$.

- We need an input $u$ to transfer the state from the initial to the final state
- Given initial and finial state conditions in $\mathbb{R}^n$, is it possible to steer $x(t)$ to the final state by choosing an appropriate input $u(t)$?

# Controllability: A Preview

Discrete-time LTI system

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k),\ x(0) = 0,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \\
x(1) &= Bu(0) \\
x(2) &= Ax(1) + Bu(1) = ABu(0) + Bu(1) \\
x(3) &= A^2Bu(0) + ABu(1) + Bu(2) \\
&\vdots \\
x(r) &= A^{r-1}Bu(0) + A^{r-1}Bu(1) + \cdots + Bu(r-1) \\
x(r) &= \begin{pmatrix} B & AB & \cdots A^{r-1}B \end{pmatrix} \begin{pmatrix} u(r-1) \\ u(r-2) \\ \vdots \\ u(0) \end{pmatrix}
\end{aligned}
$$

# Controllability: A Preview

$$x(r) = \begin{pmatrix} B & AB & \cdots A^{r-1}B \end{pmatrix} \begin{pmatrix} u(r-1) \\ u(r-2) \\ \vdots \\ u(0) \end{pmatrix}$$

$$R((B \ AB \ \cdots \ A^{r-1}B)) = \{z \in \mathbb{R}^n, \ z = (B \ AB \ \cdots \ A^{r-1}B)p, \ p \in \mathbb{R}^{nm}\}$$

If $x_f \in R((B \ AB \ \cdots \ A^{r-1}B))$, then $x_f$ is reachable

Namely, there exists a sequence of control $\{u(0), \dots, u(r-1)\}$ that transfers the state to $x_f$.

# Controllability: A Preview

This implies that we can reach arbitrary $x_f \in \mathbb{R}^n$ at time $t_f = r$ if and only if $R((B\ AB\ \cdots\ A^{r-1}B)) = \mathbb{R}^n$ that is equivalent to $rank((B\ AB\ \cdots\ A^{r-1}B)) = n$

Rank of $(B\ AB\ \cdots\ A^{r-1}B))$

- By C-H theorem, $A^k$ is a linear combination of $\{I, A, \ldots, A^{n-1}\}$
- For $r \geq n$, the rank of $(B\ AB\ \cdots\ A^{r-1}B))$ cannot increase

Hence, if $rank((B\ AB\ \cdots\ A^{n-1}B)) = n$, then we can find $u$ for an arbitrary $x_f \in \mathbb{R}^n$ for any finite time

# Controllability

$$\dot{x}(t) = Ax(t) + Bu(t), \ x \in \mathbb{R}^n$$
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

The system is controllable (page 213 of the textbook)

- $\Leftrightarrow$ for any $x_0$, there exists $u(t)$ on $[t_0, t_f]$ that transfers $x_0$ to the origin at $t_f$ (controllability to the origin)
- $\Leftrightarrow$ there exists $u(t)$ on $[t_0, t_f]$ that transfers state from the origin to any final state $x_f$ at $t_f$ (reachability)

Proof: Exercise!! (note that $e^{A(t-t_0)}$ is always invertible!)

# Controllability

$$\dot{x}(t) = Ax(t) + Bu(t),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ x_0 = 0$$

- Set of reachable state for a fixed time $t$:

$$\mathcal{R}_t = \{\xi \in \mathbb{R}^n,\ \text{there exists } u \text{ such that } x(t) = \xi\}$$

Note that $\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$

# Controllability

$$\dot{x}(t) = Ax(t) + Bu(t),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ x_0 = 0$$

- Controllability matrix and controllability subspace

$$\mathcal{C}_{AB} = \{\xi \in \mathbb{R}^n :\ \xi = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} z,\ z \in \mathbb{R}^{nm}\}$$

$\mathcal{C}_{AB}$: range space of $\mathcal{C}$, where $\mathcal{C} = (B\ AB\ \cdots\ A^{n-1}B) \in \mathbb{R}^{n \times nm}$

# Controllability

$$\dot{x}(t) = Ax(t) + Bu(t),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ x_0 = 0$$

- Controllability Gramian

$$W_t = \int_0^t e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} d\tau = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau \geq 0$$

$R(W_t)$: the range space of $W_t$. Note that $W_t$ is a symmetric positive semi-definite matrix

# Controllability

Theorem: Controllability (Theorem 6.1 of the textbook)
For each time $t > 0$, the following set equality holds:

$$\mathcal{R}_t = \mathcal{C}_{AB} = R(W_t).$$

- $\mathcal{C} = (B \ AB \ \cdots \ A^{n-1}B)$: controllability matrix
- Hence if $\dim \mathcal{C}_{AB} = rank((B \ AB \ \cdots \ A^{n-1}B)) = n$, the system is controllable
- Due to $\mathcal{C}_{AB}$, the controllability is independent of the time
- If the system is controllable, then $\mathcal{R}_t = \mathbb{R}^n$, all the states are reachable by an appropriate choice of the control $u$

# Controllability

We will show that

- $\mathcal{R}_t \subset \mathcal{C}_{AB}$, $\mathcal{C}_{AB} \subset R(W_t)$, $R(W_t) \subset \mathcal{R}_t$

Required tools

- C-H theorem (Chapter 3)
- $R(A^T) = (N(A))^\perp$

# Controllability

Theorem: $\mathcal{R}_t \subset \mathcal{C}_{AB}$

Proof:
Fix $t > 0$, and choose any reachable state $\xi \in \mathcal{R}_t$. We need to show that $\xi \in \mathcal{R}_t$ implies $\xi \in \mathcal{C}_{AB}$.

# Controllability

We have $\xi \in \mathcal{R}_t$, which implies $\xi = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$. Then by C-H theorem, $e^{At} = \beta_0(t) I + \cdots + \beta_{n-1}(t) A^{n-1}$ ($\beta_i(t)$: scalar function); hence,

$$
\begin{aligned}
\xi =& B \int_0^t \beta_0(t-\tau) u(\tau) d\tau + \cdots + A^{n-1} B \int_0^t \beta_{n-1}(t-\tau) u(\tau) d\tau \\
=& \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} \underbrace{\begin{pmatrix} \int_0^t \beta_0(t-\tau) u(\tau) d\tau \\ \vdots \\ \int_0^t \beta_{n-1}(t-\tau) u(\tau) d\tau \end{pmatrix}}_{\in \mathbb{R}^{nm}}
\end{aligned}
$$

Hence, $\xi \in \mathcal{C}_{AB}$

# Controllability

Theorem: $\mathcal{C}_{AB} \subset R(W_t)$

Proof:
Since $\mathcal{C}_{AB} \subset R(W_t)$ is equivalent to $\mathcal{C}_{AB}^{\perp} \supset R(W_t)^{\perp}$ (proof: exercise!!), we will show that $\mathcal{C}_{AB}^{\perp} \supset R(W_t)^{\perp}$.

From Problem 1 in HW3, $R(W_t) = (N(W_t))^{\perp}$, which is equivalent to $(R(W_t))^{\perp} = N(W_t)$, and similarly, $\mathcal{C}_{AB}^{\perp} = N((B \ AB \ \cdots \ A^{n-1}B))$.

# Controllability

Hence we need to show that if $\xi \in N(W_t)$, then $\xi \in N((B\ AB\ \cdots\ A^{n-1}B))$.

Let $\xi \in N(W_t)$, then $W_t\xi = 0 \in \mathbb{R}^n$, which also implies $\xi^T W_t \xi = 0 \in \mathbb{R}$. Then

$$0 = \xi^T \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau \xi = \int_0^t \|B^T e^{A^T\tau}\xi\|^2 d\tau$$

$$\Leftrightarrow B^T e^{A^T\tau}\xi = 0,\ \forall \tau \in [0, t]$$

# Controllability

Since $y(\tau) = \xi^T e^{A\tau} B = 0$, $\forall \tau \in [0, t]$, we have

$$\xi^T \left( \frac{d^k}{d\tau^k} e^{A\tau} \right) \Big|_{\tau=0} B = \xi^T A^k B = 0, \ \forall k \geq 0$$
$$\Rightarrow \xi^T \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = 0 \Rightarrow \xi \in N((B \ AB \ \cdots \ A^{n-1}B)) = \mathcal{C}_{AB}^{\perp}$$

Therefore, we have the desired result, i.e., $\mathcal{C}_{AB} \subset R(W_t)$.

# Controllability

Theorem: $R(W_t) \subset \mathcal{R}_t$

Proof:
Let $\xi \in R(W_t)$. Then there exists $v \in \mathbb{R}^n$ such that

$$\xi = W_t v = \int_0^t e^{A\tau} B B^T e^{A^T \tau} v d\tau$$

Define $u(\tau) = B^T e^{A^T (t-\tau)} v, \; \tau \in [0, t]$

# Controllability

Then, since $\dot{x} = Ax + Bu$ with $x(0) = 0$, we have

$$\begin{aligned} x(t) &= \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \\ &= \int_0^t e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} v d\tau = W_t v = \xi \end{aligned}$$

This means that $\xi \in \mathcal{R}_t$, since we have found the control $u$ that steers the state to $\xi$ from the origin. Hence, $R(W_t) \subset \mathcal{R}_t$.

# Controllability

Theorem (Theorem 6.1 of the textbook)
If $(A, B)$ is controllable, and $A$ is stable (eigenvalues of $A$ have negative real parts), then there exists a unique solution of

$$AP + PA^T = -BB^T,$$

where $P = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau > 0$

- Note that $BB^T \geq 0$
- In Chapter 5, $AP + PA^T = -Q$ where $Q > 0$

# Controllability

Theorem (Theorem 6.1 of the textbook)
$(A, B)$ is controllable if and only if $rank((A - \lambda I \;\; B)) = n$ for all eigenvalues, $\lambda$, of $A$.

- Hautus-Rosenbrock test

# Controllability

Theorem (Theorem 6.1 of the textbook)
$(A, B)$ is controllable if and only if $W_t > 0$, that is, the controllability Gramian is non-singular

Theorem (Theorem 6.2 of the textbook)
Let $\bar{A} = PAP^{-1}$ and $\bar{B} = PB$. Then $(A, B)$ is controllable if and only if $(\bar{A}, \bar{B})$ is controllable

- Controllability is invariant under the similarity transformation

Fact: The state space equation with the controllable canonical form is always controllable.

# Controllability

$$\dot{x} = Ax + Bu, \; G(s) = \frac{X(s)}{U(s)} = (sI - A)^{-1}B$$

Kalman Decomposition Theorem (Theorem 6.6 of the textbook): Suppose that $\mathcal{C}_{AB} = r < n$. Let

$$P = (v_1, \ldots, v_r, v_{r+1}, \ldots, v_n)$$

where $v_i$, $i = 1, 2, \ldots, r$ is eigenvectors of $\mathcal{C}$, and $v_{r+1}, \ldots, v_n$ are arbitrary vectors that guarantees $P$ being nonsingular.

# Controllability

Let $z = Px$. Then

$$\dot{z} = PAP^{-1}z + PBu$$

$$\bar{A} = PAP^{-1} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix}, \ \bar{B} = PB = \begin{pmatrix} \bar{B}_1 \\ 0 \end{pmatrix}$$

$$\bar{A}_{11} \in \mathbb{R}^{r \times r}, \ \bar{B}_1 \in \mathbb{R}^{r \times m}$$

Also, $(\bar{A}_{11}, \bar{B}_1)$ is controllable, and $G(s) = (sI - \bar{A}_{11})^{-1}\bar{B}_1$.

# Observability

$$\dot{x} = Ax + Bu,\ \ y = Cx,\ \ x \in \mathbb{R}^n,\ \ y \in \mathbb{R}^p$$

Definition (Definition 6.O1)
The state-space equation is said to be observable if for any unknown initial condition, there exists a finite $t_1$ such that the knowledge of the input and the output over $[0, t_1]$ is suffices to determine uniquely the initial condition $x(0)$.

W.L.G., $u = 0$, (since $u$ is completely known)

# Observability

Note that

$$y(t) = Ce^{At}x(0)$$

Hence, if $N(Ce^{At}) = \emptyset$, i.e., $\dim(N(Ce^{At})) = nullity(Ce^{At}) = 0$, then the system is observable.

- $N(Ce^{At})$: unobservable subspace

# Observability

Let

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

$\mathcal{O}$: Observability matrix, $\mathcal{O} \in \mathbb{R}^{pn \times n}$

# Observability

Theorem: $N(Ce^{At}) = N(\mathcal{O})$
Proof: We will show that $N(Ce^{At}) \subset N(\mathcal{O})$ and $N(Ce^{At}) \supset N(\mathcal{O})$.

If $x_0 \in N(Ce^{At})$, then

$$0 = Ce^{At}x_0 \Rightarrow 0 = C\left(\frac{d}{dt}e^{At}\right)\Big|_{t=0} x_0 \Rightarrow 0 = CA^k x_0, \;\; \forall k \geq 0$$

Hence, $x_0 \in N(\mathcal{O})$.
If $x_0 \in N(\mathcal{O})$. then $x_0 \in N(Ce^{At})$, since by C-H Theorem, we have

$$Ce^{At} = C\beta_0(t)I + \cdots + CA^{n-1}\beta_{n-1}(t)$$

# Observability

If $N(Ce^{At}) = \emptyset$, i.e., $\dim(N(Ce^{At})) = \mathit{nullity}(Ce^{At}) = 0$, then the system is observable.

- We need $N(Ce^{At}) = N(\mathcal{O}) = \emptyset$
- Hence, by the rank-nullity theorem, the system is observable if $\mathit{rank}(\mathcal{O}) = n$
- We say that the system is observable if and only if the pair $(C, A)$ is observable
- Observability also does not depend on the time (by C-H Theorem)

# Observability

Duality Theorem (Theorem 6.5 of the textbook)
The following are equivalent:

- $(C, A)$ is observable
- $(A^T, C^T)$ is controllable

Proof: $(A^T, C^T)$ is controllable if and only if

$$\mathcal{O}^T = (C^T \; A^T C^T \; \cdots \; (A^T)^{n-1} C^T)$$

$$rank(\mathcal{O}^T) = n = rank(\mathcal{O})$$

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

# Observability

Theorem (Theorem 6.O1)
If $(A, C)$ is observable, and $A$ is stable, then there exists a unique solution of

$$A^T P + PA = -C^T C,$$

where $P = \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau > 0$.

Theorem (Theorem 6.O1)
$(C, A)$ is observable if and only if

$$rank \begin{pmatrix} C \\ A - \lambda I \end{pmatrix} = n$$

# Observability

Theorem (Theorem 6.O1)
$(C, A)$ is observable if and only if the observability Gramian

$$Q_t = \int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau > 0$$

Theorem (Theorem 6.O3) Let $\bar{A} = PAP^{-1}$ and $\bar{C} = CP^{-1}$. Then $(C, A)$ is observable if and only if $(\bar{C}, \bar{A})$ is observable.

# Observability

$$\dot{x} = Ax + Bu, \ \ y = Cx, \ \ G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Kalman Decomposition Theorem (Theorem 6.O6 of the textbook):
Suppose that $rank(\mathcal{O}) = q < n$. Let

$$P = \begin{pmatrix} v_1 \\ \vdots \\ v_q \\ v_{q+1} \\ \vdots \\ v_n \end{pmatrix}, \ \ v_1, \ldots, v_q\text{: eigenvectors.}$$

# Observability

Let $z = Px$. Then $\dot{z} = PAP^{-1}z + PBu$, $y(t) = CP^{-1}z$, and

$$\bar{A} = PAP^{-1} = \begin{pmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}, \ \bar{B} = PB = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}, \ \bar{C} = \begin{pmatrix} \bar{C}_1 & 0 \end{pmatrix}$$

$$\bar{A}_{11} \in \mathbb{R}^{q \times q}, \ \bar{C}_1 \in \mathbb{R}^{p \times q}$$

Also, $(\bar{C}_1, \bar{A}_{11})$ is observable, and $G(s) = \bar{C}_1(sI - A_{11})^{-1}\bar{B}_1$.

# Kalman Decomposition Theorem

Theorem (Theorem 6.7 of the textbook)
We can extract the state that is controllable and observable.

Fact: The state space equation with the observable canonical form is always observable

Discrete-time LTI system

$$x(k+1) = Ax(k) + Bu(k), \ y(k) = Cx(k)$$

- $(A, B)$ is controllable if and only if $rank(\mathcal{C}) = n$
- $(C, A)$ is observable if and only if $rank(\mathcal{O}) = n$

# Minimum Energy Control (page 189)

$$\dot{x} = Ax + Bu,\ x(0) = x_0,\ x(t_1) = x_f,\ (A, B) \text{ controllable}$$

$$W_{t_1} = \int_0^{t_1} e^{A\tau} BB^T e^{A^T\tau} d\tau > 0,\ \text{invertible}$$

Let

$$u^*(t) = -B^T e^{A^T(t_1-t)} W_{t_1}^{-1}(e^{At_1}x_0 - x_f)$$

$$x(t_1) = e^{At_1}x_0 - \underbrace{\left(\int_0^{t_1} e^{A(t_1-\tau)} BB^T e^{A^T(t_1-\tau)} d\tau\right)}_{W_{t_1}} W_{t_1}^{-1}(e^{At_1}x_0 - x_f) = x_f$$

We can show that the controller $u^*$ is the minimum energy controller in the sense that for any controller $u$ that transfers the state from $x_0$ to $x_f$, we have

$$\int_0^{t_1} \|u(t)\|^2 dt \geq \int_0^{t_1} \|u^*(t)\|^2 dt,\ \forall u$$

# Stabilizability & Detectability

Weaker notions of controllability and observability

A system is stabilizable if and only if $\bar{A}_{22}$ is stable and $(\bar{A}_{11}, \bar{B}_1)$ is controllable

A system is detectable if and only if $\bar{A}_{22}$ is stable and $(\bar{C}_1, \bar{A}_{11})$ is observable

How about the example on pages 4-5. Is it stabilizable? Is it detectable?

# Controllability & Observability: LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \; y(t) = C(t)x(t)$$

$$W_t = \int_0^t \Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)d\tau \geq 0, \; \forall t \geq 0$$

$$Q_t = \int_0^t \Phi^T(t,\tau)C^T(\tau)C(\tau)\Phi(t,\tau)d\tau \geq 0, \; \forall t \geq 0$$

The LTV system is

- is controllable if and only if there exists $t_f > 0$ such that $W_{t_f} > 0$
- is observable if and only if there exists $t_f > 0$ such that $Q_{t_f} > 0$
- $W_t$: controllability Graminan
- $Q_t$: observability Graminan

# Minimal Realizations

We have seen that the realization of the state-space equation is not unique.

$$\dot{x} = Ax + Bu,\ \ y = Cx + Du,\ \ x(0) = 0$$
$$\dot{x} = A_1 x + B_1 u,\ \ y = C_1 x + D_1 u,\ \ x(0) = 0$$
$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = C_1\int_0^t e^{A_1(t-\tau)}B_1 u(\tau)d\tau$$

Lemmas (not in the textbook)

- Two system realizations $(A, B, C, D)$ and $(A_1, B_1, C_1, D_1)$ are equivalent if and only if $D = D_1$ and

$$Ce^{At}B = C_1 e^{A_1 t}B_1,\ \ \forall t \geq 0$$

- Two system realizations $(A, B, C, D)$ and $(A_1, B_1, C_1, D_1)$ are equivalent if and only if $D = D_1$ and

$$CA^k B = C_1 A_1^k B,\ \ \forall k \geq 0$$

# Minimal Realizations

In view of the Kalman decomposition, we have the following result:
$\Rightarrow$ Suppose $(A, B, C, D)$ is a system realization. If either $(C, A)$ is not observable or $(A, B)$ is not controllable, then there exists a lower-order realization $(A_1, B_1, C_1, D_1)$ for the system

Definition (page 233 of the textbook)
Realizations with the smallest possible dimension are called minimal realizations

Theorem (Theorem 7.M2 (page 254))
$(A, B, C, D)$ is a minimial realization of the transfer function $G(s)$ if and only if $(A, B)$ is controllable and $(C, A)$ is observable

If the system is not controllable or not observable (or not controllable and observable), then there are pole-zero cancellations in a transfer function.

# MATLAB Commands

- controllability matrix: $ctrb(A, B)$
- observability matrix: $ctrb(A^T, C^T)$
- minimal realization: $minreal(A, B, C, D) \Rightarrow$ reduce the system order that has only controllable and observable state
- Mostly, we use the balanced realization (Chapter 7.4) $\Rightarrow$ related to controllability and observability Gramians (robust control, advanced control topic)