[MEN573] Advanced Control Systems I

Lecture 4 – State Space Models of Dynamic Systems

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Static System

- Its present output depends only on its **present** input

$$y(k) = f(u(k))$$

- *memory-less* systems
- described by *algebraic* equations

Dynamic System

- Its present output depends **past as well as present** inputs

$$y(k) = f(u(k), u(k-1), \cdots, u(k-n), \cdots)$$

- Systems that have *memory*
- described by *differential* or *difference* equations

State of a Dynamic Systems

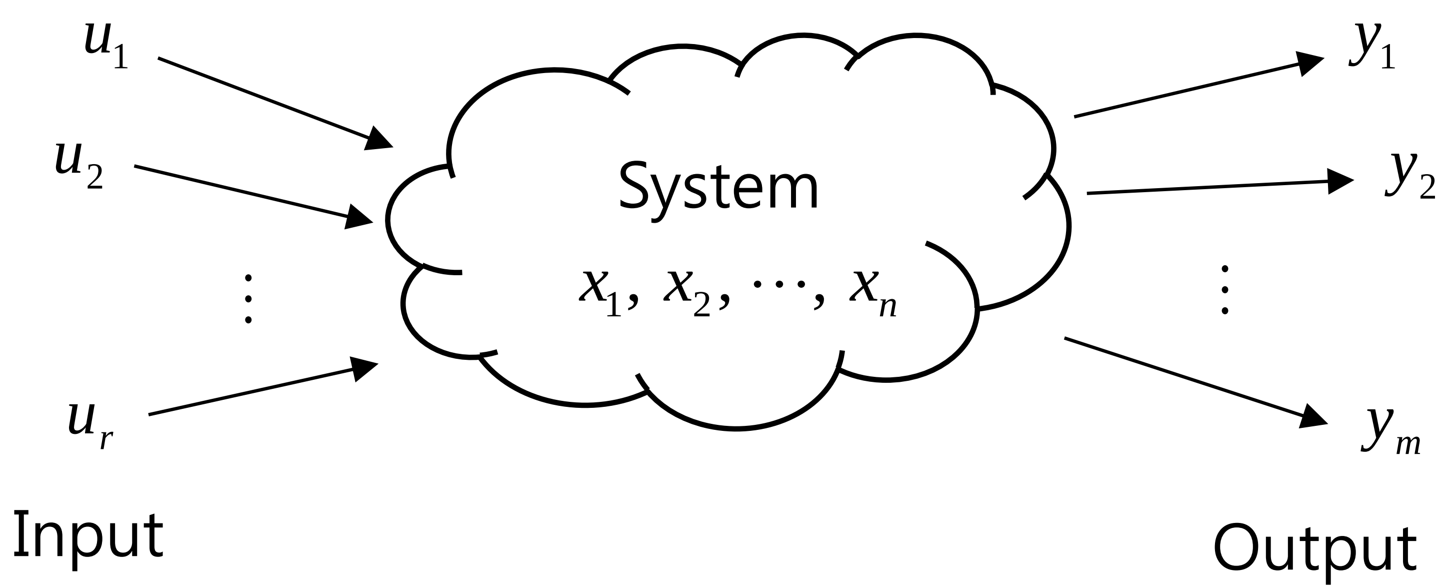
- Loose definitions:
  - The "aggregated effect of past inputs"
  - the necessary ***memory*** that the dynamic system keeps at each time.

- A state determined model of a system is a system model such
  - (1) the **description** of the system
  - (2) the **specification** of a limited set of parameters at time t_0, i.e. x_1, x_2, … , x_n, and
  - (3) the specification of system **inputs** for all $t \geq t_0$

  is necessary and sufficient to uniquely determine the system behavior for all instances $t \geq t_0$.

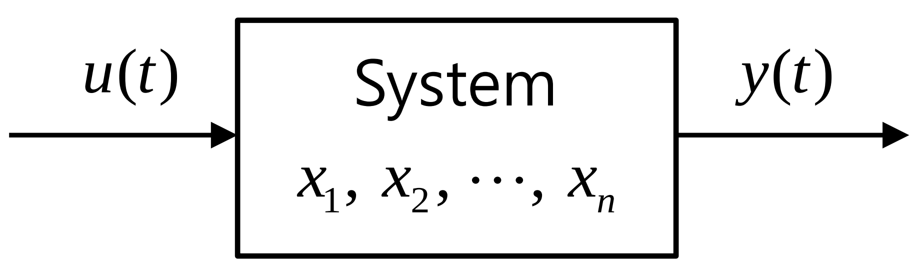
State of a Dynamic Systems

- The state of a system at any time t_0 is the **minimum** set of numbers, i.e. values of system variables required to **uniquely describe** the system.
- The minimum set of variables are the system **state variables**.

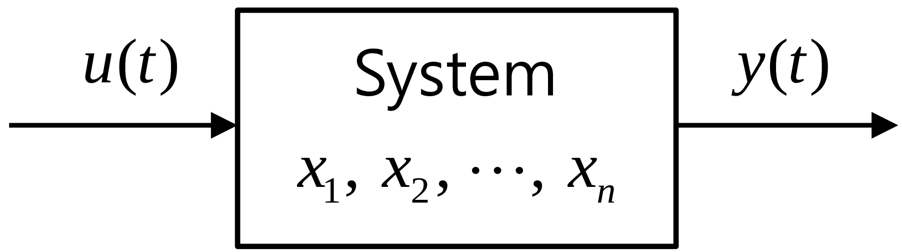


State Variables

- The **order** of a dynamic system is the number, n of state variables that is **necessary and sufficient to uniquely describe** the system.
- For a given dynamic system, **the choice of state variables is not unique**. However, its **order *n* is fixed**; i.e. you need not more than n but not less than n state variables.



Continuous-time State Space Description

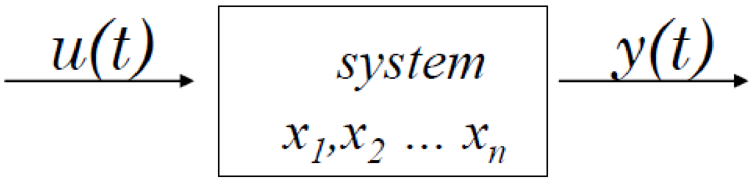


$$\frac{d}{dt}x(t) = f\,(x(t), u(t), t)$$

$$y(t) = h(x(t), u(t), t)$$

$u(t)$ Input $\quad$ $y(t)$ Output $\quad$ $x(t)$ State

Continuous-time State Space Description

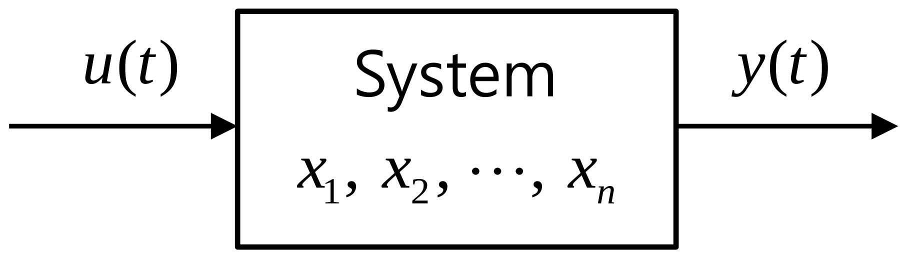


$$\boxed{\begin{aligned} \frac{d}{dt}x(t) &= f(x(t), u(t), t) \\ y(t) &= h(x(t), u(t), t) \end{aligned}}$$

- Input vector: $u(t) = \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_r(t) \end{bmatrix}^T \in \mathcal{U} \subset \mathcal{R}^r$
- Output vector: $y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_m(t) \end{bmatrix}^T \in \mathcal{Y} \subset \mathcal{R}^m$
- State vector: $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^T \in \mathcal{X} \subset \mathcal{R}^n$

- State function: $f : \mathcal{X} \times \mathcal{U} \times \mathcal{R} \to \mathcal{X}$
- Output function $h : \mathcal{X} \times \mathcal{U} \times \mathcal{R} \to \mathcal{Y}$

Continuous-time LTI State Space Description

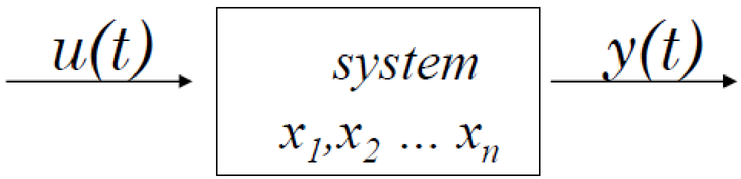


$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$u(t)$ Input $\quad$ $y(t)$ Output $\quad$ $x(t)$ State

Continuous-time LTI
State Space Description



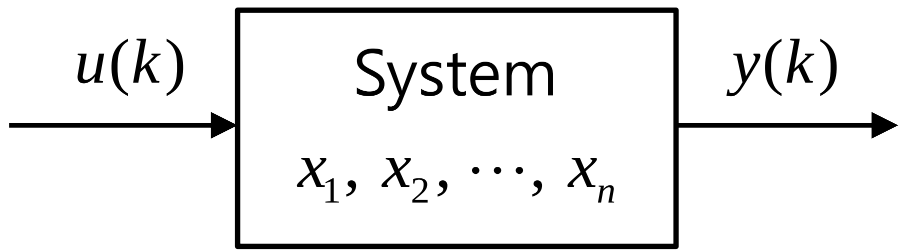
$$
\begin{aligned}
\frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) \\
y(t) &= C\,x(t) + D\,u(t)
\end{aligned}
$$

- Input vector: $u(t) = \left[\begin{array}{cccc} u_1(t) & u_2(t) & \cdots & u_r(t) \end{array}\right]^T \in \mathcal{U} \subset \mathcal{R}^r$
- Output vector: $y(t) = \left[\begin{array}{cccc} y_1(t) & y_2(t) & \cdots & y_m(t) \end{array}\right]^T \in \mathcal{Y} \subset \mathcal{R}^m$
- State vector: $x(t) = \left[\begin{array}{cccc} x_1(t) & x_2(t) & \cdots & x_n(t) \end{array}\right]^T \in \mathcal{X} \subset \mathcal{R}^n$

- State equation: $A \in \mathcal{R}^{n \times n}, \quad B \in \mathcal{R}^{n \times r}$
- Output equation $C \in \mathcal{R}^{m \times n}, \quad D \in \mathcal{R}^{m \times r}$

When {*A, B, C, D*} are constant, the linear system is time invariant.
When {*A, B, C, D*} depend on time, the linear system is time-varying.

Discrete-time State Space Description

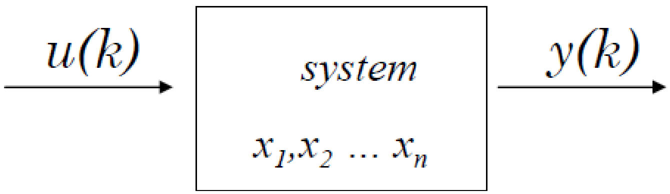


$$x(k+1) = f\,(x(k), u(k), k)$$

$$y(k) = h(x(k), u(k), k)$$

$u(k)$ Input $y(k)$ Output $x(k)$ State

Discrete-time State Space Description



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- Input vector: $u(k) = \begin{bmatrix} u_1(k) & u_2(k) & \cdots & u_r(k) \end{bmatrix}^T \in \mathcal{U} \subset \mathcal{R}^r$
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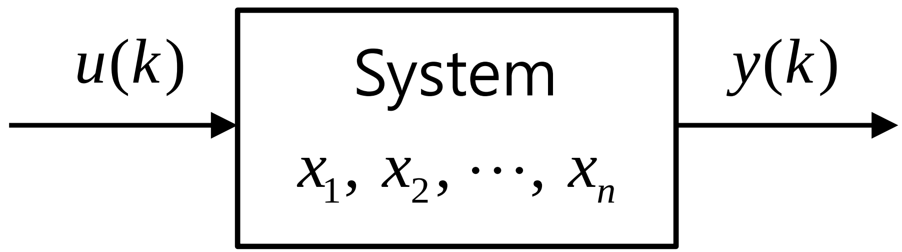
- State function: $f : \mathcal{X} \times \mathcal{U} \times \mathcal{Z}_+ \rightarrow \mathcal{X}$
- Output function $h : \mathcal{X} \times \mathcal{U} \times \mathcal{Z}_+ \rightarrow \mathcal{Y}$

Discrete-time LTI
State Space Description

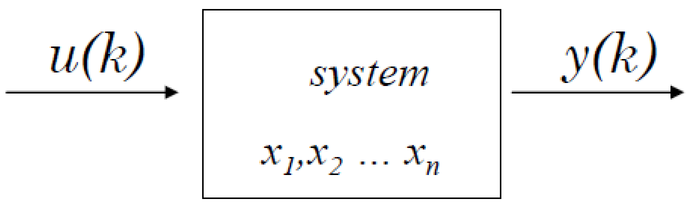


$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

$u(k)$ Input $y(k)$ Output $x(k)$ State

Discrete-time LTI
State Space Description



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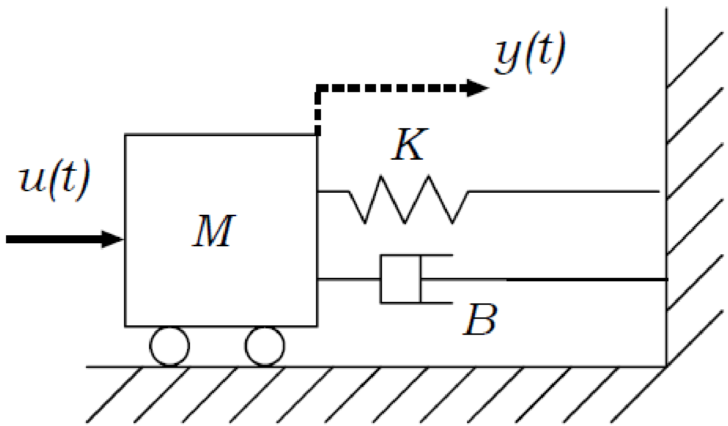
Example: Mass-spring-dashpot system



mass position

$$x(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} \in \mathcal{R}^2$$

mass velocity

$$\frac{d}{dt}\underbrace{\begin{bmatrix} p(t) \\ v(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} p(t) \\ v(t) \end{bmatrix}}_{x(t)}$$