Final Exam: Linear System Theory (ECE532)

Instructor: Jun Moon

June 10, 2018

Time: June 10, 6:30am - June 11, 11:30am

Name / Student ID:

Note:

- Write your name and student ID on the answer sheet.
- You must submit your MATLAB code.
- You must use LaTeX
- Cheating is not allowed.
- The point value of each sub-problem is indicated in parentheses.

GOOD LUCK!!!

Problem	1	2	3	4	5	6	7			Total
Max Point	45	30	10	25	30	45	45			230
Point										

Problem 1 (45 point)

Consider the following system

$$\ddot{x} = x + u, \ x(0) = x_0$$

(i) 5 point Obtain the state-space representation of the above system. Namely, for the given system

$$\dot{x} = Ax + Bu, \ y = Cx$$

find A, B and C.

(ii) 5 point Is the system controllable? Is the realization of the system unique? Is it the minimal realization? Justify your answers.

Now, assume that you can measure the state, x and $\dot{x}$.

(iii) 5 point Find the state feedback gain K that guarantees the eigenvalues of the closed-loop system being -0.5 and -4. Does there exist a unique K? Justify your answer.

(iv) 10 point Consider the finite-horizon LQR control problem:

$$J(u) = \int_0^{10} [x_1^2 + 4x_1x_2 + 4x_2^2 + 4u^2]dt$$

Obtain the linear-quadratic regulator (LQR) controller. You should write the equation the Riccati differential equation. Does there exist a unique optimal controller? Why? Justify your answer. Also,

- Obtain the time response plot of the Riccati differential equation using MATLAB. That is you need to solve the Riccati differential equation using MATLAB, and obtain the time response plot of each component of the Riccati differential equation.
- Compute the minimum cost when the initial condition is $x(0) = (1\ 0)^T$.

(v) 15 point Consider the infinite-horizon problem of (iv):

- Is there a unique solution to the corresponding algebraic Riccati equation in the set of positive-definite matrices? Justify your answer. Compute the solution of the algebraic Riccati equation using hand computation and MATLAB ("care").

- With the infinite-horizon LQR controller, determine the eigenvalues of the closed-loop system.
- What is the minimum value of J with the infinite-horizon LQR controller when the initial condition is $x(0) = (1\ 0)^T$?

(vi) 5 point Use MATLAB to obtain the time-response plot of the closed-loop system with the following controllers:

- (iii): pole-placement controller
- (iv): finite-horizon LQR controller
- (v): infinite-horizon LQR controller

Note that the initial condition is $x(0) = (1\ 0)^T$. Compare the time-response plots.

Problem 2 (30 point)

Consider the undamped, control harmonic oscillator:

$$\ddot{x} = -\omega x + u$$

where x is the position, $\dot{x}$ is the velocity, and $\omega > 0$ is a constant. Suppose that we can measure only the velocity. We now consider the output feedback control problem.

(i) 5 point Obtain the state-space representation of the above system. Namely, for the given system

$$\dot{x} = Ax + Bu, \ y = Cx$$

find A, B and C. Is the system controllable and observable? Justify your answer.

(ii) 10 point Design an observer-based (output) feedback controller to control the position x. Specifically, place the state feedback controller poles at $s = -\omega \pm i\omega$, and both observer poles at $s = -\omega$. Justify that both the state feedback controller and the observer exist.

(iii) 5 point Compute the controller and the observer in (ii) when $\omega = 2$.

(iv) 5 point What are the eigenvalues of the closed-loop system, $\begin{pmatrix} x \\ \hat{x} \end{pmatrix}$, when $\omega = 2$?

What is $\lim_{t\to\infty} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}$? Justify your answer.

(v) 5 point Obtain the time-response plot of the output feedback controller in (iii) when the initial condition is $x(0) = [4\ 6]^T$ and the initial condition of the observer is $[1\ 4]^T$.

Problem 3 (10 point)

Let $\mathcal{X} = \mathbb{R}^{n\times n}$ be the vector space of $n \times n$ matrices with the field $\mathbb{R}$. Let $\mathcal{V} = \{V \in \mathcal{X} : \ V = V^T\}$ be the set of symmetric matrices, and $\mathcal{W} = \{W \in \mathcal{X} : \ W = -W^T\}$ be the set of skew-symmetric matrices.

(i) 3 point Show that $\mathcal{V}$ and $\mathcal{W}$ are subspaces of $\mathcal{X}$.

(ii) 3 point Show that $\mathcal{V} \cap \mathcal{W} = \{0\}$.

(iii) 4 point Show that $\mathcal{V} + \mathcal{W} = \mathcal{X}$.

(Hint: Any matrix A can be decomposed as sum of $\frac{1}{2}(A + A^T)$ and $\frac{1}{2}(A - A^T)$.)

Problem 4 (25 point)

Consider the LTI system

$$\dot{x} = Ax + Bu, \ y = Cx$$

where $A \in \mathbb{R}^{3\times 3}$ matrix with eigenvalues of 0.5, -0.5 and 3. Let $A - BK$ be a closed-loop matrix. Verify whether each of the following statements is true or false and provide a brief justification of the following statements is true or false and provide a brief justification or give a counter-example. You CANNOT say that the answer is true or false due to the computation result of MATLAB.

(i) 5 point If for a particular K, the closed-loop eigenvalues are at 1.3, -1.3 and 3, then the following is necessarily true

$$\operatorname{rank}\begin{pmatrix} A - 3I & B \end{pmatrix} < 3$$

(ii) 5 point If for a particular K, the closed-loop eigenvalues are still 0.5, -0.5 and 3, then the following is necessarily true

$$\operatorname{rank}\begin{pmatrix} A - 3I & B \end{pmatrix} < 3$$

(iii) 5 point If for all K, the closed-loop eigenvalues always include 0.5, then (A, B) is necessarily not stabilizable.

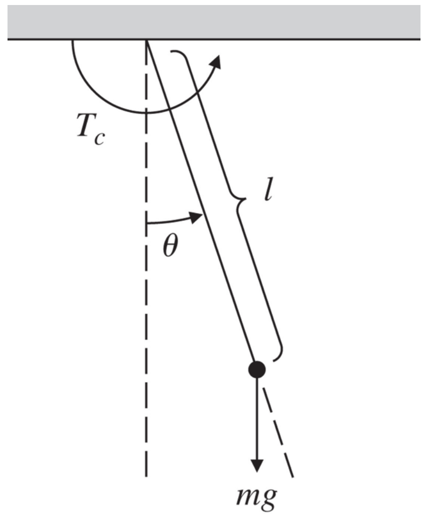
(iv) 5 point If for a particular K, the closed-loop eigenvalues are at 0.5, -0.5 and 1/3, then the following is necessarily true

$$\operatorname{rank}\begin{pmatrix} A - 3I & B \end{pmatrix} = 3$$

(v) 5 point If for a particular nonzero $x(0) = x_0$, we obtain $\lim_{t\to\infty} y(t) = 0$ when $u(t) = 0$ (the control is identically zero), then (C, A) is not detectable.

Problem 5 (30 point)

Consider the following pendulum system



The motion of the pendulum system can be written as

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}, \tag{1}$$

where θ is an angle, T_c is the torque (input), l is the length of the pendulum, m is the mass, and g is the gravity. Note that θ is the output of the pendulum system.

(i) 5 point Obtain the *nonlinear* state space equation of the pendulum system (1) when the output is θ and the input is T_c.

(ii) 5 point Linearize the system in (i) around $\theta \approx 0$. Write down the linearized state space equation in the controller canonical form. Is the system controllable and observable?

(iii) 5 point Assume that $l = m = g = 1$. Obtain the discrete-time system of (ii) using MATLAB "c2d" with the sampling time $T_s = 0.01$ and the "zoh" method. Namely, consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \; y = Cx_k. \tag{2}$$

Find A, B and C. Is the discretized system controllable and observable? Justify your answer.

(iv) 5 point Consider the discrete-time output feedback control problem

$$\begin{aligned} u_k &= -K\hat{x}_k \\ \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L(C\hat{x}_k - y_k) \end{aligned}$$

where $\hat{x}_k$ is the state of the observer. Show that the closed-loop system (2) with the above discrete-time output feedback controller holds the separation principle.

(v) 5 point Design K and L such that the closed-loop poles are

$$0.7 \pm 0.5j \text{ controller eigenvalues}$$
$$0.01 \pm 0.1j \text{ observer eigenvalues.}$$

(vi) 5 point By using MATLAB, obtain the time-response plot when $x_0 = [2\ 1]^T$ and $\hat{x}_0 = [4\ 1]^T$.

Problem 6 (45 point)

Consider the discrete-time LQR control problem

$$x_{k+1} = Ax_k + Bu_k$$
$$J(u) = \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T M x_N$$

where $R > 0$ and $Q, M \geq 0$.

(i) 10 point Solve the discrete-time LQR control problem using dynamic programming. Namely, obtain the optimal controller. Does there exist a unique optimal controller? Why? justify your answer.

Now, suppose that $Q = R = M = I$ where I is an identity matrix. Also, we use the system parameters in Problem 5, and assume that we can measure all the state, i.e. θ and $\dot{\theta}$.

(i) 5 point Obtain the finite-horizon LQR controller.

(ii) 5 point Use MATLAB to obtain the time response plot of the Riccati difference equation. You must obtain the time response plot of each component of the Riccati difference equation. Also, compute the minimum cost when the initial condition is $x_0 = [2\ 1]^T$.

(iii) 20 point Consider the infinite-horizon discrete-time LQR problem:

- Does the Riccati difference equation in (ii) converge to a unique value? Why? Justify your answer.
- Is there a unique solution to the corresponding algebraic Riccati equation in the set of positive-definite matrices? Justify your answer. Compute the solution of the algebraic Riccati equation using hand computation and MATLAB (hint: "dare").
- With the infinite-horizon LQR controller, determine the eigenvalues of the closed-loop system.
- What is the minimum cost with the infinite-horizon controller when the initial condition is $x_0 = [2\ 1]^T$?

(iv) 5 point Use MATLAB to obtain the time-response plot of the closed-loop system with the finite-horizon and infinite-horizon LQR controllers when the initial condition is $x_0 = [2\ 1]^T$. Compare the time-response plots obtained by

- output feedback controller (Problem 5, (vi))
- finite-horizon LQR
- infinite-horizon LQR

Problem 7 (45 point): Minimum Energy Control

Consider

$$\dot{x} = Ax + Bu, \; x(t_0) = x_0.$$

Let

$$W_c(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_0-\tau)} B B^T e^{A^T(t_0-\tau)} d\tau$$

represent the associated controllability gramian. We also note that for any vector-valued signal $f(t)$, $\int_{t_0}^{t_f} f^T(\tau) f(\tau) d\tau$ is the energy of the signal $f(t)$.

(i) 5 point Show that $W_c(t_0, t_f) = W_c(0, t_f - t_0)$ for any $t_0, t_f \in \mathbb{R}$.

(ii) 5 point Show that

$$W_1(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_f-\tau)} B B^T e^{A^T(t_f-\tau)} d\tau$$

is positive definite if and only if $W_c(t_0, t_f)$ is positive definite (Note that a $n \times n$ matrix X is positive definite if $X = X^T$ and for any $z \in \mathbb{R}^n$, $z^T X z > 0$). (Hint: first show that $W_1(t_0, t_f) = e^{A(t_f-t_0)} W_c(t_0, t_f) e^{A^T(t_f-t_0)}$.)

(iii) 5 point Suppose that the system is controllable at time t_0. Show that the control design

$$u(t) = B^T e^{A^T(t_f-t)} W_1^{-1}(t_0, t_f)(x_f - e^{A(t_f-t_0)} x_0)$$

drives the system from the initial condition x_0 at time t_0 to x_f at time t_f.

(iv) 25 point Show that the controller in (iii) is the minimum energy control (and is unique) that achieves the desired transfer from t_0 to t_f. You can deduce this by proving the following:

5 point Show that for any control u_1 that achieves the same transfer (that is u_1 transfers the state from x_0 at t_0 to x_f at t_f),

$$\int_{t_0}^{t_f} e^{A(t_f-\tau)} B(u(\tau) - u_1(\tau)) d\tau = 0$$

5 point Use the above result to prove that

$$\int_{t_0}^{t_f} u^T(\tau)(u(\tau) - u_1(\tau)) d\tau = 0$$

5 point Use the above result to prove that

$$\int_{t_0}^{t_f} u_1^T(\tau)u_1(\tau)d\tau = \int_{t_0}^{t_f} u^T(\tau)u(\tau)d\tau + \int_{t_0}^{t_f} (u_1(\tau) - u(\tau))^T(u_1(\tau) - u(\tau))d\tau$$

5 point Deduce that the design $u(t)$ in (iii) achieves the state transfer with minimum energy.

5 point Prove that the design $u(t)$ in (iii) is the unique minimum energy control.

(v) 5 point Assume that we have two LTI systems, where the the first LTI system has $W_1(t_0, t_f) = 1$, $x_0 = 0$, and $x_f = 1$. Moreover, the second LTI system has $W_1(t_0, t_f) = 2$, $x_0 = 0$, and $x_f = 1$. For these two LTI systems, if u_1 is the corresponding minimum energy control for the first LTI system, and u_2 is the corresponding minimum energy control for the second LTI system, compute

$$\frac{\int_{t_0}^{t_f} u_2^2(\tau)d\tau}{\int_{t_0}^{t_f} u_1^2(\tau)d\tau}.$$

(Hint: for the scalar case, note that $\int_{t_0}^{t_f} f^T(\tau)f(\tau)d\tau = \int_{t_0}^{t_f} f^2(\tau)d\tau$).