Algorithms & Complexity
Lecture 8
The Simplex Algorithm

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Introduction

- Midterm: Wednesday 04.18 14:30–17:00, room TBA. Closed book.
- Assignment 2 due on Monday 04.09

- The *simplex algorithm* is a practical algorithm for linear programming.
- It does not run in polynomial time:
  - Examples are known where it runs in exponential time.
  - But in practice, it usually runs in polynomial time.
    - For a justification, see *smoothed analysis*. (Not covered in CSE530.)
- Polynomial time algorithms are known, but:
  - They are only *weakly polynomial*.
  - See last lecture of this half-semester.
  - Finding a strongly polynomial algorithm for LP is an important open problem.

- Reference: Chapter 29.3 of the textbook Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.

Example

We want to solve the following linear program, given in standard form:

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & 3x_3 & \leqslant & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leqslant & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leqslant & 36 \\
 & & & & & x_1, x_2, x_3 & \geqslant & 0
\end{array}
$$

We first convert it into slack form:

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

Example

- The slack form above has basic variables x_4, x_5, x_6 and nonbasic variables x_1, x_2, x_3.
- So $N = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$.
- The *basic solution* corresponding to this slack form is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 30, 24, 36)$$

  with value $z = 0$.
  - In the basic solution, all nonbasic variables are set to 0.
- Approach: Modify N and B, keeping an equivalent program, and increasing the value of the basic solution.

Example

- In the slack form from slide 4, increasing x_1 would increase z.
- Due to the 3rd constraint, we can increase x_1 to 9 at most.
- We move x_1 to LHS in 3rd constraint, and obtain:

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6.$$

- We now replace x_1 with $9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$ in the other equations and obtain:

Example

$$\begin{array}{rcrcrcrcr}
z & = & 27 & + & \frac{1}{4}x_2 & + & \frac{1}{2}x_3 & - & \frac{3}{4}x_6 \\
x_1 & = & 9 & - & \frac{1}{4}x_2 & - & \frac{1}{2}x_3 & - & \frac{1}{4}x_6 \\
x_4 & = & 21 & - & \frac{3}{4}x_2 & - & \frac{5}{2}x_3 & + & \frac{1}{4}x_6 \\
x_5 & = & 6 & - & \frac{3}{2}x_2 & - & 4x_3 & + & \frac{1}{2}x_6
\end{array}$$

- This operation is called a *pivot*.
- Now $N = \{2, 3, 6\}$ and $B = \{1, 4, 5\}$.
  - x_1 has become a basic variable, and x_6 has become nonbasic.
  - x_1 is called the *entering variable*, and x_6 is the *leaving variable*.
- The basic solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (9, 0, 0, 21, 6, 0)$.
- The value of the objective function at this solution is 27.
  - We can read it from the top line of this new slack form.
  - Or we can substitute $x_1 = 9$, $x_2 = 0$ and $x_3 = 0$ in Slide 4.
- It will always be the case: At each step of the simplex algorithm, the new LP is equivalent to the LP at the previous step.

Example

- We can now increase x_2 or x_3, but not x_6 as its coefficient is negative and thus it would decrease the value of the objective function.
- We choose x_3.
- The limiting constraint is the last one, with $x_3 = \frac{3}{2}$.
- So we have

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6.$$

and

$$\begin{array}{rcrcrcrcr} z & = & \frac{111}{4} & + & \frac{1}{16}x_2 & - & \frac{1}{8}x_5 & - & \frac{11}{16}x_6 \\ x_1 & = & \frac{33}{4} & - & \frac{1}{16}x_2 & + & \frac{1}{8}x_5 & - & \frac{5}{16}x_6 \\ x_3 & = & \frac{3}{2} & - & \frac{3}{8}x_2 & - & \frac{1}{4}x_5 & + & \frac{1}{8}x_6 \\ x_4 & = & \frac{69}{4} & + & \frac{3}{16}x_2 & + & \frac{5}{8}x_5 & - & \frac{1}{16}x_6 \end{array}$$

Example

- Last step: Increase x_2 by 4:

$$\begin{array}{rcrcrcrcr} z & = & 28 & - & \frac{1}{6}x_3 & - & \frac{1}{6}x_5 & - & \frac{2}{3}x_6 \\ x_1 & = & 8 & + & \frac{1}{6}x_3 & + & \frac{1}{6}x_5 & - & \frac{1}{3}x_6 \\ x_2 & = & 4 & - & \frac{8}{3}x_3 & - & \frac{2}{3}x_5 & + & \frac{1}{3}x_6 \\ x_4 & = & 18 & - & \frac{1}{2}x_3 & + & \frac{1}{2}x_5 & & \end{array}$$

- The optimal solution is $(x_1, x_2, x_3, x_4, x_5, x_6) = (8, 4, 0, 18, 0, 0)$ with value 28.
  - Proof?
- The solution to the original problem is $(x_1, x_2, x_3) = (8, 4, 0)$ and the optimal value is also 28.

General Case

- LP in slack form:

$$z = \nu + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j, \qquad i \in B.$$

- Next slide: Pseudocode for the Pivot operation, the input is $(N, B, A, b, c, \nu, \ell, e)$ where e is the index of the entering variable, and ℓ the leaving variable.
- The output is $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{\nu})$.

Pivot$(N, B, A, b, c, \nu, \ell, e)$

$\hat{b}_e \leftarrow b_\ell / a_{\ell e}$ $\triangleright$ Coefs. of entering variable x_e

for each $j \in N \setminus \{e\}$ **do**

  $\hat{a}_{ej} \leftarrow a_{\ell j} / a_{\ell e}$

$\hat{a}_{e\ell} \leftarrow 1/a_{\ell e}$

for each $i \in B \setminus \{\ell\}$ **do** $\triangleright$ Coefs. of other constraints

  $\hat{b}_i \leftarrow b_i - a_{ie}\hat{b}_e$

  for each $j \in N \setminus \{e\}$ **do**

    $\hat{a}_{ij} \leftarrow a_{ij} - a_{ie}\hat{a}_{ej}$

  $\hat{a}_{i\ell} \leftarrow -a_{ie}\hat{a}_{e\ell}$

$\hat{\nu} \leftarrow \nu + c_e\hat{b}_e$ $\triangleright$ Coefs. of objective function

for each $j \in N \setminus \{e\}$ **do**

  $\hat{c}_j \leftarrow c_j - c_e\hat{a}_{ej}$

$\hat{c}_\ell \leftarrow -c_e\hat{a}_{e\ell}$

$\hat{N} \leftarrow N \setminus \{e\} \cup \{\ell\}$, $\hat{B} \leftarrow B \setminus \{\ell\} \cup \{e\}$

return $(\hat{N}, \hat{B}, \hat{A}, \hat{b}, \hat{c}, \hat{\nu})$

General Case

- We assume that we start with a slack form (N, B, A, b, c, ν) whose basic solution is feasible.
  - We explain at the end of this lecture how to find this solution.
- If all coefficients are negative in the objective function, we just return the basic solution restricted to $(x_1, \ldots, x_n)$.
- Otherwise we increase as much as possible one of the variables x_e with nonnegative coefficient.
  - If we can increase to ∞, then return *unbounded*.
  - Otherwise perform a pivot using x_e as entering variable, and using the leaving variable x_ℓ corresponding to the saturated constraint.
- *Next slide: pseudocode.*

General Case

Simplex(A, b, c)

$(N, B, A, b, c, \nu) \leftarrow$ Initialize-Simplex(A, b, c)
while $\exists j : c_j > 0$ **do**
    Choose e such that $c_e > 0$
    for each $i \in B$ **do**
        if $a_{ie} > 0$ **then** $\Delta_i \leftarrow b_i / a_{ie}$
        else $\Delta_i \leftarrow \infty$
    Choose ℓ that minimizes Δ_ℓ
    if $\Delta_\ell = \infty$ **then** **return** unbounded
    else $(N, B, A, b, c, \nu) \leftarrow$ Pivot($N, B, A, b, c, \nu, \ell, e$)
for $i \leftarrow 1, n$ **do**
    if $i \in B$ **then** $\bar{x}_i \leftarrow b_i$
    else $\bar{x}_i \leftarrow 0$
return $(\bar{x}_1, \ldots, \bar{x}_n)$

Degeneracy

- By construction, the value of the objective function never decreases during the course of the simplex algorithm.
- But in some cases, it may remain the same after one step:

$$\begin{array}{rcrcrcrcr} z & = & & & x_1 & + & x_2 & + & x_3 \\ x_4 & = & 8 & - & x_1 & - & x_2 & & \\ x_5 & = & & & & & x_2 & - & x_3 \end{array}$$

- Assume we choose $e = 1$, and thus $\ell = 4$. We get:

$$\begin{array}{rcrcrcrcr} z & = & 8 & & & + & x_3 & - & x_4 \\ x_1 & = & 8 & - & x_2 & & & - & x_4 \\ x_5 & = & & & x_2 & - & x_3 & & \end{array}$$

- The value of the basic solution is 8.

Degeneracy

- At this point, we can only choose $e = 3$ and $\ell = 5$:

$$\begin{array}{rcrcrcrcr} z & = & 8 & + & x_2 & - & x_4 & - & x_5 \\ x_1 & = & 8 & - & x_2 & - & x_4 & & \\ x_3 & = & & & x_2 & & & - & x_5 \end{array}$$

- The value of the basic solution is still 8.
- So we may not make progress at each step of the simplex algorithm.
  - This is called *degeneracy*.
- Here, fortunately, if we pivot again, we have $e = 2$ and $\ell = 1$, and then the value increases to 16.

Cycling

- In some degenerate cases however, the simplex algorithm may go back to the same slack form repeatedly.
- Then the value does not increase, and the algorithm does not terminate.
- This is called *cycling*.
- An example with 6 variables and 3 equations is known.
- Cycling can be avoided by a careful choice of the pivot.
- For instance, using *Bland's rule*: Choose the entering variable with smallest index, and then the leaving variable with smallest index.

Theorem

Using Bland's rule, the simplex algorithm never cycles.

- We will not prove it in this course.

Proof of Correctness

Definition

We say that two slack forms are *equivalent* if they have the same set of feasible solutions.

Lemma

All the slack forms produced by the simplex algorithm are equivalent.

Proof.

At each pivot, we first move x_e to the LHS, obtaining an equivalent equation. Then this equation multiplied by a constant is added to each other equality constraint. As in Gaussian elimination, it produces an equivalent system of equations. $\square$

Proof of Correctness

Lemma

For a given LP, and for a given choice of basic variables, the simplex algorithm cannot produce two different slack forms.

Proof.

Done in class. Lemma 29.3 and 29.4 page 876 in the textbook. □

It follows that:

Corollary

If cycling does not occur, then the simplex algorithm terminates in at most $\binom{n+m}{n}$ *steps.*

Proof of Correctness

In this slide, we assume that the `Initialize-Simplex` procedure in Slide 13 returns a slack form whose basic solution is feasible. (This procedure is described in the textbook, Section 29.5.) Proofs for the lemmas and theorem below are given in class.

Lemma

If the simplex algorithm returns `unbounded`*, then the linear program is unbounded.*

Lemma

If the simplex algorithm returns $(\bar{x}_1, \dots, \bar{x}_n)$, then it is an optimal solution.

Theorem

If cycling does not occur, then the simplex algorithm returns a correct answer after at most $\binom{n+m}{n}$ iterations.

The Initial Basic Feasible Solution

- Consider the following LP:

$$\begin{array}{lrcrl} \text{maximize} & 2x_1 & - & x_2 & \\ \text{subject to} & 2x_1 & - & x_2 & \leqslant 2 \\ & x_1 & - & 5x_2 & \leqslant -4 \\ & & & x_1, x_2 & \geqslant 0 \end{array}$$

- Suppose we want to solve it with the simplex algorithm.
- After converting into slack form:

$$\begin{array}{rcrcrcr} z & = & & & 2x_1 & - & x_2 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 \end{array}$$

- What is the problem?
  - The basic solution is not feasible.

Auxiliary Linear Program

- Let L be a LP in standard form:

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, && i = 1, \dots, m, \\ & x_j \geqslant 0, && j = 1, \dots, n. \end{aligned}$$

- The *auxiliary linear program* L_{aux} is:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leqslant b_i, && i = 1, \dots, m, \\ & x_j \geqslant 0, && j = 0, \dots, n. \end{aligned}$$

Auxiliary Linear Program

Proposition

The linear program L is feasible if and only if the optimal objective value of L_{aux} is 0.

Proof: Done in class.

Example

- The auxiliary LP for the LP in Slide 20 is:

$$\begin{array}{llllllll} \text{maximize} & & & & - & x_0 & & \\ \text{subject to} & 2x_1 & - & x_2 & - & x_0 & \leqslant & 2 \\ & x_1 & - & 5x_2 & - & x_0 & \leqslant & -4 \\ & & & & x_1, x_2, & x_0 & \geqslant & 0 \end{array}$$

- We solve this LP using the simplex algorithm.
- The first slack form is:

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

- The basic solution is not feasible.
- We choose x_0 and x_4 as the entering and leaving variables, respectively.

Example

- The new slack form is:

$$
\begin{array}{rcrcrcrcr}
z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4
\end{array}
$$

- The basic solution is now feasible. (It will always be the case.)
- We now run the simplex algorithm until we find an optimal solution.
- We pick $x_e = x_2$ and $x_\ell = x_0$, and thus:

$$
\begin{array}{rcrcrcrcr}
z & = & & - & x_0 & & & & \\
x_2 & = & \frac{4}{5} & - & \frac{1}{5}x_0 & + & \frac{1}{5}x_1 & + & \frac{1}{5}x_4 \\
x_3 & = & \frac{14}{5} & + & \frac{4}{5}x_0 & - & \frac{9}{5}x_1 & + & \frac{1}{5}x_4
\end{array}
$$

- The optimal value for L_{aux} is 0, so the original LP is feasible.

Example

- As $x_0 = 0$, we remove it from the slack form:

$$\begin{array}{rcrcrcr} z & = & ? & & & & \\ x_2 & = & \frac{4}{5} & + & \frac{1}{5}x_1 & + & \frac{1}{5}x_4 \\ x_3 & = & \frac{14}{5} & - & \frac{9}{5}x_1 & + & \frac{1}{5}x_4 \end{array}$$

- We restore the original objective function

$$\begin{aligned} z &= 2x_1 - x_2 \\ &= 2x_1 - \left(\frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4\right) \end{aligned}$$

Example

- We obtain the following slack form, equivalent to the original LP:

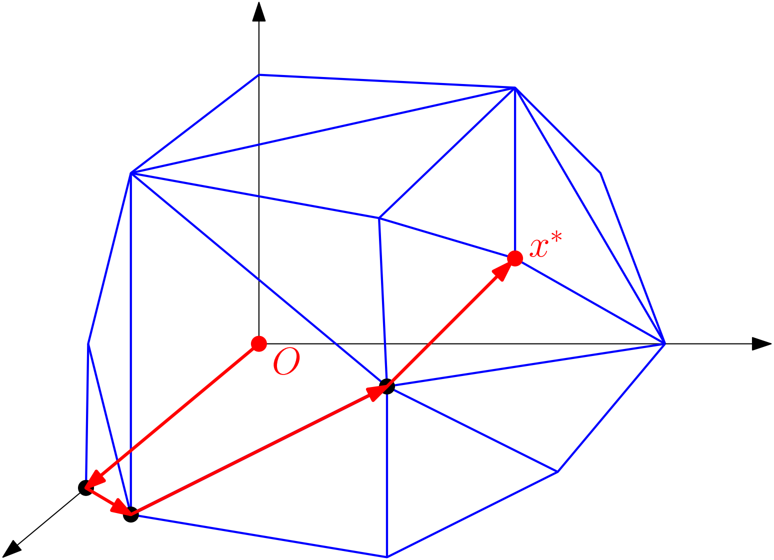
$$
\begin{array}{rcrcrcr}
z & = & -\frac{4}{5} & + & \frac{9}{5}x_1 & - & \frac{1}{5}x_4 \\
x_2 & = & \frac{4}{5} & + & \frac{1}{5}x_1 & + & \frac{1}{5}x_4 \\
x_3 & = & \frac{14}{5} & - & \frac{9}{5}x_1 & + & \frac{1}{5}x_4
\end{array}
$$

- This slack form has a feasible basic solution, so this completes the execution of Initialize-Simplex.

General Case

- Construct L_{aux}
- Make a first pivot with $\ell = 0$ and $e = k$ such that b_k is minimum
- The basic solution of L_{aux} is now feasible
- Solve L_{aux} with the simplex algorithm
- If $x_0 = 0$, then use the solution as an initial basic feasible solution
- Otherwise, the original LP is not feasible.
- More details can be found in textbook Section 29.5

Geometry



Geometry

- The simplex algorithm moves from one vertex of the feasible region to a neighboring vertex.
- At each move, the objective function does not decrease.
- For instance, it starts from the vertex $(x_1, \dots, x_n) = (0, \dots, 0)$, which is the initial basic solution restricted to $(x_1, \dots, x_n)$.
- At each step, the n nonbasic variables N give a set of n variables that are set to 0 in the basic solution.
- It corresponds to n of the constraints of the original LP being satisfied.
- In other words, the current basic solution is at the intersection of n hyperplanes bounding the feasible region.
- So it is a vertex of the feasible region.