[MEN573]
Advanced Control Systems I

Lecture 12 – Stability
Part Ⅲ The Lyapunov Equation

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Lyapunov stability theorems (LTI)

The origin 0 of the n-th order LTI system

$$\dot{x} = A\,x$$

is **<u>stable in the sense of Lyapunov</u>** if there exists a **Lyapunov function** *V(x)* for some for some $r > 0$, *i.e.*

$$V(x) \succ 0, \qquad \forall |x| < r$$

$$\dot{V}(x) \preceq 0, \qquad \forall |x| < r$$

Lyapunov stability theorems (LTI)

The origin 0 of the n-th order LTI system

$$\dot{x} = A\,x$$

is **asymptotically stable** if there exists a **Lyapunov function** *V(x)* such that

$$V(x) \succ 0 \qquad \textbf{PDF}$$

$$\dot{V}(x) \prec 0 \qquad \textbf{NDF}$$

Lyapunov stability theorems (LTI)

Lets consider a quadratic Lyapunov function candidate:

$$V(x) = x^T P x$$

where

$$P^T = P \qquad\qquad P \succ 0$$

and compute $\dot{V}(x)$ along $\dot{x} = A x$

Lyapunov stability theorem for LTI systems (CT)

$$V(x) = x^T P x \qquad P^T = P \qquad \dot{x} = \boxed{A\,x}$$

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^T P x + x^T P \underline{\dot{x}} \\
&= \underline{\dot{x}}^T P x + x^T P \overbrace{A\,x} \\
&= \overbrace{x^T A^T} P x + x^T P A x
\end{aligned}$$

$$\boxed{\dot{V}(x) = x^T \left[A^T P + P A\right] x}$$

Lyapunov stability theorem for LTI systems (CT)

Thus,

$$P^T = P$$
$$P \succ 0$$

$$V(x) = x^T P x$$

is a Lyapunov function for the system $\dot{x} = A x$ when

$$\left[A^T P + P A\right] \preceq 0$$ (negative semi-definite)

and the origin is **stable in the sense of Lyapunov**.

Lyapunov stability theorem for LTI systems (CT)

Therefore, the origin of the system

$$\dot{x} = A\,x$$

is **stable in the sense of Lyapunov**, if

there exists a symmetric matrix

$$P \succ 0$$ (positive definite)

such that

$$\left[A^T P + P\,A\right] \preceq 0$$ (negative semi-definite)

Lyapunov stability theorem for LTI systems (CT)

Moreover, the origin of the system

$$\dot{x} = A\,x$$

is **globally asymptotically stable**, if

there exists a symmetric matrix

$P \succ 0$ (positive definite)

such that

$\left[A^T P + P\,A\right] \prec 0$ (negative definite)

The Lyapunov Equation

It turns out that much stronger stability results can be obtained for CT LTI systems by analyzing the following Lyapunov equation,

$$A^T P + P A = -Q$$

Given

Find

Exponential Stability Theorem (CT)

The origin of the n-th order LTI system

$$\dot{x} = A\,x$$

is globally exponentially stable ***iff*** (if and only if),

for ***any*** symmetric matrix $Q \succ 0$

there exist a symmetric matrix $P \succ 0$

which is the ***unique*** solution of the Lyapunov equation

$$A^T P + P\,A = -Q$$

Exponential Stability Theorem (CT)

The matrix

$$A \in \mathcal{R}^{n \times n}$$

is Hurwitz ***iff*** (if and only if),

for ***every*** symmetric matrix $Q \succ 0$

there exist a symmetric matrix $P \succ 0$

which is the ***unique*** solution of the Lyapunov equation

$$A^T P + P A = -Q$$

Stability Analysis (CT)

How to use the Lyapunov equation:

- Given a matrix $\boldsymbol{A}$**,** select an arbitrary positive definite symmetric matrix $\boldsymbol{Q}$ (for example $\boldsymbol{I}$) .
- Attempt to find a solution to the Lyapunov equation

$$A^T P + P A = -Q$$

1. If a solution $\boldsymbol{P}$ cannot be found**,** $\boldsymbol{A}$ is not Hurwitz.
2. If a solution $\boldsymbol{P}$ is found, check for its sign definiteness:
   - If $\boldsymbol{P}$ is positive definite, then $\boldsymbol{A}$ is Hurwitz.
   - If $\boldsymbol{P}$ is not positive definite, then $\boldsymbol{A}$ has at least one eigenvalue with a positive real part (unstable).

Lyapunov equation

It is important to note that the Lyapunov equation:

$$A^T P + P A = -Q$$

is a linear algebraic equation in $\boldsymbol{P}$

Thus, it is easy to solve!

Solving the Lyapunov equation with matlab

$$A^T P + P A = -Q$$

Matlab functions:

- Lyapunov equation: **`P = lyap(A',Q)`**

  `(if` $\boldsymbol{P}$ cannot be found, it returns an error message)

- The definiteness of $\boldsymbol{P}$ can be checked with the Cholesky factorization function: **`N = chol(P)`**

  it returns a upper triangular matrix $\boldsymbol{N}$, such that

  $P = N^T N$ (when $P \succ 0$ *)*

(otherwise it returns an error message)

Examples using matlab (CT)

1. ***A*** is Hurwitz

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

- **P = lyap(A’,Q)**
- **N = chol(P)**

$$P = \begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}, \; N = \begin{bmatrix} 0.7071 & 0.3536 \\ 0 & 0.7906 \end{bmatrix} \qquad P = N^T N$$

P is positive definite

2. ***A*** is not Hurwitz but limitedly stable

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

- **P = lyap(A’,Q)**

P could not be found

Examples using matlab (CT)

3. $\boldsymbol{A}$ is not Hurwitz and has an unstable eigenvalue

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

- `P = lyap(A',Q)`
- `N = chol(P)`

$$P = \begin{bmatrix} 0.50 & -0.50 \\ -0.50 & 0 \end{bmatrix},$$

N could not be found

$\boldsymbol{P}$ is not positive definite

4. $\boldsymbol{A}$ is not Hurwitz and has an unstable eigenvalue

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

- `P = lyap(A',Q)`

P could not be found

Solving the Lyapunov Equation

Lets consider the solution of the Lyapunov equation when $A \in \mathcal{R}^{2\times 2}$,

Let: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, and

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}, \qquad P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

first column of $\boldsymbol{Q}$ *second column of* $\boldsymbol{P}$

we will latter generalize the result for $\mathcal{R}^{n\times n}$

Solving the Lyapunov Equation

Expanding element by element the matrices in

$$A^T P + P A = -Q$$

we obtain

$$\underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} = - \underbrace{\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}}_{Q}$$

$$\Downarrow$$

$$A^T \begin{bmatrix} p_1 & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = -\begin{bmatrix} q_1 & q_2 \end{bmatrix}$$

Solving the Lyapunov Equation

$$A^T \begin{bmatrix} p_1 & p_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = -\begin{bmatrix} q_1 & q_2 \end{bmatrix}$$

Lining up one column on top of the other

$$\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = -\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\Downarrow$$

$$\left\{ \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix} \right\} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = -\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Solving the Lyapunov Equation

$$\left\{ \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix} \right\} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = -\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\Downarrow$$

$$\underbrace{\left[I \otimes A^T + A^T \otimes I \right]}_{\mathbf{L}_A} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = -\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

where we have used the Kronecker product $\otimes$

Solving the Lyapunov Equation

The Kronecker product $\otimes$ between two matrices is defined as follows:

Let $B \in \mathcal{R}^{m\times n}$ and C of arbitrary dimension,

is defined as

$$B \otimes C = \begin{bmatrix} b_{11}C & b_{12}C & \cdots & b_{1n}C \\ b_{21}C & b_{22}C & \cdots & b_{2n}C \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1}C & b_{m2}C & \cdots & b_{mn}C \end{bmatrix}$$

Solving the Lyapunov Equation

We can now consider the solution of the Lyapunov equation

$$A^T P + P A = -Q$$

where

$$A \in \mathcal{R}^{n \times n} \;\; P \in \mathcal{R}^{n \times n} \;\; Q \in \mathcal{R}^{n \times n}$$

Solving the Lyapunov Equation

$$A^T P + P A = -Q$$

First stack the columns of matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$

$$P = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \qquad Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

as follows,

$$\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \in \mathcal{R}^{n^2} \qquad \mathbf{Q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \in \mathcal{R}^{n^2}$$

Solving the Lyapunov Equation

$$A^T P + P A = -Q$$

$$\boxed{\mathbf{L}_A \mathbf{P} = -\mathbf{Q}}$$

$$\mathbf{L}_A \in \mathcal{R}^{n^2 \times n^2}$$

$$\mathbf{L}_A = \left\{ A^T \otimes I + I \otimes A^T \right\}$$

There is a unique solution for $\boldsymbol{P}$ iff $\mathbf{L}_A$ is nonsingular.

Theorem LTI S-2

- Let the i^{th} eigenvalue of the matrix A be λ_i

- Let the l^{th} eigenvalue of the matrix $\mathbf{L}_A$ be μ_l

where,

$$\mathbf{L}_A = \left\{ A^T \otimes I + I \otimes A^T \right\}$$

Then, the $\boldsymbol{n^2}$ eigenvalues $\mu_l{}'s$ are given by

$$\boxed{\mu_l = \lambda_i + \lambda_j \,,}$$

$$l = 1,\, 2,\, \cdots,\, n^2$$
$$i = 1,\, 2,\, \cdots,\, n$$
$$j = 1,\, 2,\, \cdots,\, n$$

Example:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$

$$\mathbf{L}_A \;=\; \left\{A^T \otimes I + I \otimes A^T\right\}$$

$$\mu_l = \lambda_i + \lambda_j \,,$$

$$\mathbf{L}_A \;=\; \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\mu_1 = -2$$
$$\mu_2 = -2$$
$$\mu_3 = -2$$
$$\mu_4 = -2$$

Nonsingular

$$\mu_1 = \lambda_1 + \lambda_1 = -2$$

$$\mu_3 = \lambda_2 + \lambda_1 = -2$$

$$\mu_2 = \lambda_1 + \lambda_2 = -2$$

$$\mu_4 = \lambda_2 + \lambda_2 = -2$$

Example:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{L}_A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathbf{Q} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{P} = -\mathbf{L}_A^{-1}\mathbf{Q} \Rightarrow \mathbf{P} = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.25 \\ 0.75 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0.50 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

$$P \succ 0$$

Example:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = -1$$

$$\lambda_2 = 1$$

$$\mathbf{L}_A \;=\; \left\{ A^T \otimes I + I \otimes A^T \right\}$$

$$\mu_l = \lambda_i + \lambda_j \, ,$$

$$\mathbf{L}_A \;=\; \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\mu_1 = -2$$

$$\mu_2 = 0$$

$$\mu_3 = 0$$

$$\mu_4 = 2$$

Singular

$$\mu_1 = \lambda_1 + \lambda_1 = -2$$

$$\mu_2 = \lambda_1 + \lambda_2 = 0$$

$$\mu_3 = \lambda_2 + \lambda_1 = 0$$

$$\mu_4 = \lambda_2 + \lambda_2 = 2$$

Theorem LTI S-2

- Let the i^{th} eigenvalue of the matrix A be λ_i

- Let the l^{th} eigenvalue of the matrix $\mathbf{L}_A$ be μ_l

where,

$$\mathbf{L}_A = \left\{ A^T \otimes I + I \otimes A^T \right\}$$

Then, the $\boldsymbol{n^2}$ eigenvalues $\mu_l{}'s$ are given by

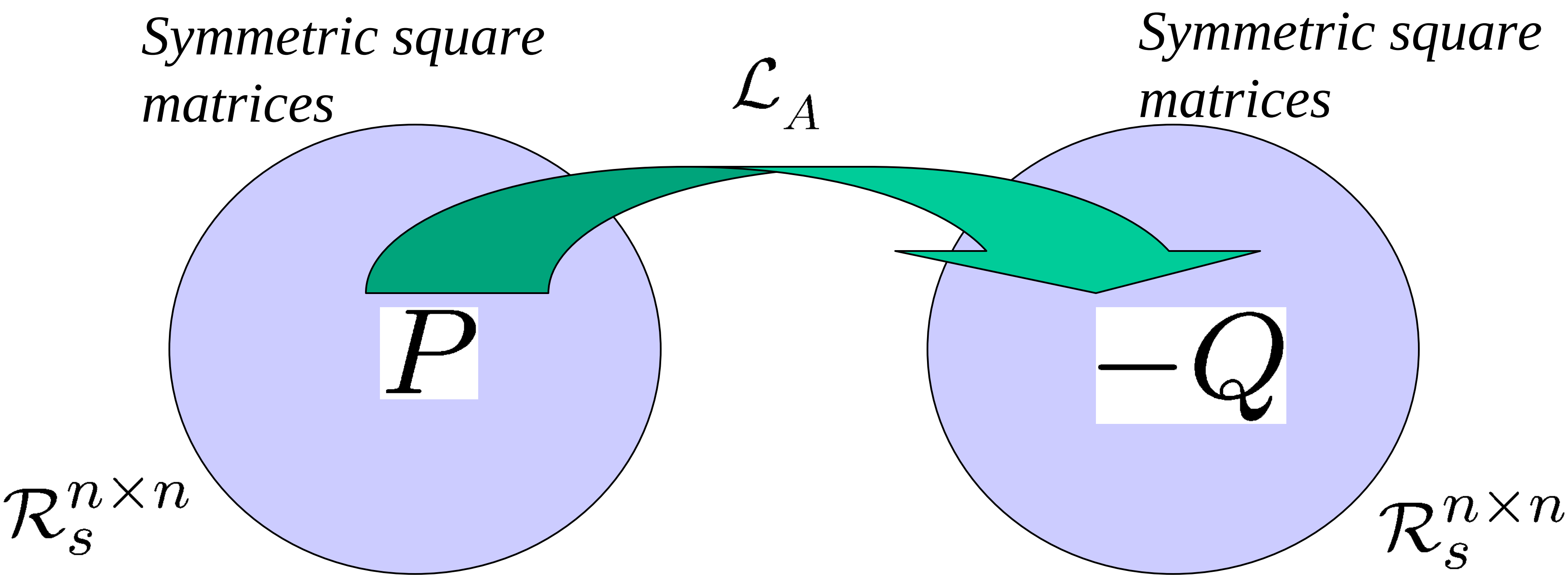
$$\boxed{\mu_l = \lambda_i + \lambda_j \,,} \qquad \begin{array}{l} l = 1,\, 2,\, \cdots\, ,n^2 \\ i = 1,\, 2,\, \cdots\, ,n \\ j = 1,\, 2,\, \cdots\, ,n \end{array}$$

Proof of Theorem LTI S-2:

Consider the Lyapunov equation in an abstract sense:

$$A^T P + P A = -Q$$

The left hand side of this equation is a linear map:



Proof of Theorem LTI S-2:

Consider the Lyapunov equation in an abstract sense:

$$A^T P + P A = -Q$$

The left hand side of this equation is a linear map:

$$\mathcal{L}_A(P) = A^T P + P A$$

$$\mathcal{L}_A : \mathcal{R}_s^{n \times n} \to \mathcal{R}_s^{n \times n}$$

where $\mathcal{R}_s^{n \times n}$ is the vector space of symmetric $\boldsymbol{nxn}$ matrices

Coordinate representations

$$\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \in \mathcal{R}^{n^2}$$

is the coordinate representation of the vector $P \in \mathcal{R}^{n \times n}$

$$\mathbf{L}_A = \left\{ A^T \otimes I + I \otimes A^T \right\}$$

is the matrix representation of the linear map $\mathcal{L}_A(\cdot)$

$$\mathbf{L}_A \mathbf{P} \iff \mathcal{L}_A(P) = A^T P + P A$$

Proof of Theorem LTI S-2:

Let λ_i and t_i be respectively an eigenvalue and corresponding eigenvector of A

$$At_i = \lambda_i t_i$$

Also, let λ_i and v_i be respectively an eigenvalue and corresponding eigenvector of A^T

$$A^T v_i = \lambda_i v_i$$

Remember that the eigenvalues of $\boldsymbol{A}$ are invariant under matrix transposition.

$$det(\lambda I - A) = det(\lambda I - A^T)$$

Proof of Theorem LTI S-2:

If,

$$A^T v_i = \lambda_i \, v_i \qquad \text{and} \qquad A^T v_j = \lambda_j \, v_j$$

Then,

$$\boxed{\begin{array}{l} \mu_l = \lambda_i + \lambda_j \qquad \text{and} \qquad V_l = \left[v_i \, v_j^T + v_j \, v_i^T \right] \\ \text{are respectively an eigenvalue and eigenvector of } \mathcal{L}_A(\cdot) \end{array}}$$

$$\mathcal{L}_A(V_l) = \mu_l \, V_l \qquad\qquad A^T \, V_l + V_l \, A = \mu_l \, V_l$$

Proof of Theorem LTI S-2:

Lets compute: $\mathcal{L}_A(V_l) = A^T V_l + V_l A$

$$\mathcal{L}_A(V_l) \;=\; A^T \left[v_i\, v_j^T + v_j\, v_i^T\right] + \left[v_i\, v_j^T + v_j\, v_i^T\right] A$$

$$\mathcal{L}_A(V_l) \;=\; \left[\underline{A^T v_i}\, v_j^T + \underline{A^T v_j}\, v_i^T\right] + \left[v_i\, \underline{v_j^T A} + v_j\, \underline{v_i^T A}\right]$$

$$\mathcal{L}_A(V_l) \;=\; \left[\lambda_i v_i\, v_j^T + \lambda_j v_j\, v_i^T\right] + \left[v_i\, \lambda_j v_j^T + v_j\, \lambda_i v_i^T\right]$$

$$\mathcal{L}_A(V_l) \;=\; \left[\lambda_i v_i\, v_j^T + \lambda_j v_j\, v_i^T\right] + \left[\lambda_j v_i\, v_j^T + \lambda_i v_j\, v_i^T\right]$$

$$\mathcal{L}_A(V_l) \;=\; \left[\lambda_i + \lambda_j\right] v_i\, v_j^T + \left[\lambda_j + \lambda_i\right] v_j\, v_i^T$$

Proof of Theorem LTI S-2:

Lets compute: $\mathcal{L}_A(V_l) = A^T V_l + V_l A$

$$\mathcal{L}_A(V_l) \;=\; \left[\lambda_i + \lambda_j\right] v_i\, v_j^T + \left[\lambda_j + \lambda_i\right] v_j\, v_i^T$$

$$\mathcal{L}_A(V_l) \;=\; \left[\lambda_i + \lambda_j\right] \left[v_i\, v_j^T + v_j\, v_i^T\right] = \mu_l\, V_l$$

$$\mathcal{L}_A(V_l) = \mu_l\, V_l \qquad\qquad A^T\, V_l + V_l\, A = \mu_l\, V_l$$

Q.E.D.

Solving the Lyapunov Equation, Examples

1. $\boldsymbol{A}$ is Hurwitz

$$\boxed{\mu_l = \lambda_i + \lambda_j\,,}$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},\ \lambda_1 = -1,\ \lambda_2 = -1$$

$$\boxed{\begin{array}{l} \mu_1 = -2,\ \mu_2 = -2, \\ \mu_3 = -2,\ \mu_4 = -2, \end{array}}$$

$\mathbf{L}_A$ is nonsingular and $\mathbf{P} = -\mathbf{L}_A^{-1}\mathbf{Q}$ is unique

2. $\boldsymbol{A}$ is not Hurwitz but limitedly stable

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},\ \lambda_1 = -1,\ \lambda_2 = 0$$

$$\boxed{\begin{array}{l} \mu_1 = 0,\ \mu_2 = -1, \\ \mu_3 = -1,\ \mu_4 = -2, \end{array}}$$

$\mathbf{L}_A$ is singular

Solving the Lyapunov Equation, Examples

3. $\boldsymbol{A}$ is not Hurwitz and has an unstable eigenvalue

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \ \lambda_1 = -1, \ \lambda_2 = 2$$

$$\boxed{\mu_l = \lambda_i + \lambda_j \, ,}$$

$$\boxed{\begin{array}{l} \mu_1 = -2, \ \mu_2 = 1, \\ \mu_3 = 1, \ \mu_4 = 4, \end{array}}$$

$\mathbf{L}_A$ is nonsingular and

$\mathbf{P} = -\mathbf{L}_A^{-1}\mathbf{Q}$ is unique

4. $\boldsymbol{A}$ is not Hurwitz and has an unstable eigenvalue

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \ \lambda_1 = -1, \ \lambda_2 = 1$$

$$\boxed{\begin{array}{l} \mu_1 = 0, \ \mu_2 = 0, \\ \mu_3 = -2, \ \mu_4 = 2, \end{array}}$$

$\mathbf{L}_A$ is singular

Solving the Lyapunov Equation

Corollary LTI S-1:

Let the matrix A be Hurwitz

Then the matrix

$$\mathbf{L}_A = \left\{A^T \otimes I + I \otimes A^T\right\}$$

is also Hurwitz and

$\mathbf{P} = -\mathbf{L}_A^{-1}\mathbf{Q}$ always exists and is unique

Solving the Lyapunov Equation

Corollary LTI S-1:

Let the matrix A be Hurwitz

Then the solution $\boldsymbol{P}$ of the Lyapunov equation

$$A^T P + P A = -Q$$

Always exists and is unique for any matrix $\boldsymbol{Q}$

Exponential Stability Theorem (CT)

The origin of the n-th order LTI system

$$\dot{x} = A\,x$$

is globally exponentially stable ***iff*** (if and only if),

for ***any*** symmetric matrix $Q \succ 0$

there exist a symmetric matrix $P \succ 0$

which is the ***unique*** solution of the Lyapunov equation

$$A^T P + P\,A = -Q$$

Proof of sufficiency $\Longleftarrow$

Proof of asymptotic stability:

Assume that, for a symmetric matrix $Q \succ 0$

there exists a symmetric matrix $P \succ 0$

which is the solution of the Lyapunov equation

$$A^T P + P A = -Q$$

We need to show that the origin of

$\dot{x} = A x$ is asymptotically stable

Proof of sufficiency $\Longleftarrow$

Proof of asymptotic stability:

Assume that, for a symmetric matrix $Q \succ 0$

there exists a symmetric matrix $P \succ 0$

which is the solution of the Lyapunov equation

$$A^T P + P A = -Q$$

Define the Lyapunov function candidate

$$V(x) = x^T P x \succ 0$$

Proof of sufficiency $\Longleftarrow$

Proof of asymptotic stability:

$$V(x) = x^T P\, x \succ 0$$

Taking the derivative along $\dot{x} = A\,x$

$$\begin{aligned} \dot{V}(x) &= x^T \left\{ A^T P + P\,A \right\} x \\ &= -x^T Q\, x \prec 0 \end{aligned}$$

Global asymptotic stability follows from Lyapunov's theorem.

Proof of sufficiency $\Longleftarrow$

Proof of exponential stability: We have shown that

$$V(x) = x^T P\, x \succ 0$$

and

$$\dot{V}(x) \;=\; -x^T Q\, x \;\prec\; 0$$

We will now show that

$$\|x(t)\|_2 \;\leq\; e^{-\beta t} M \,\|x(0)\|_2$$

for $\beta > 0$ $\quad 0 < M < \infty$

Positive definite matrices

Let $P^T = P \qquad P \succ 0$

We will use the following facts about symmetric matrices:

Fact 1: All eigenvalues of a symmetric matrix are real.

Fact 2: Distinct eigenvectors of a symmetric matrix are orthogonal

Fact 3: Symmetric matrices can always be diagonalized

Positive definite matrices

A consequence of these facts is:

$$P^T = P \qquad P \succ 0 \qquad\qquad \underline{\textbf{iff}}$$

there exists unitary and diagonal matrices

$$U^T = U^{-1} \qquad \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \succ 0$$

such that

$$U^T P U = \Lambda$$

Positive definite matrices

Let $P^T = P \qquad P \succ 0$

and define its minimum and maximum eigenvalues

$$0 < (\lambda_P)_{min} \le (\lambda_P)_i \le (\lambda_P)_{max} < \infty$$

Then, for any vector $x \in \mathcal{R}^n \qquad \|x\|_2^2 = x^T x$

$$(\lambda_P)_{min} \, \|x\|_2^2 \le x^T P x \le (\lambda_P)_{max} \, \|x\|_2^2$$

Proof of sufficiency $\Longleftarrow$

Proof of exponential stability: Define:

- $(\lambda_Q)_{min} > 0$ the minimum eigenvalue of $\boldsymbol{Q}$
- $(\lambda_P)_{max} > 0$ the maximum eigenvalue of $\boldsymbol{P}$
- $(\lambda_P)_{min} > 0$ the minimum eigenvalue of $\boldsymbol{P}$

and

$$\alpha = \frac{(\lambda_Q)_{min}}{(\lambda_P)_{max}} > 0$$

Also remember that $\|x\|_2^2 = x^T x$

Proof of sufficiency $\Longleftarrow$

Proof of exponential stability:

Notice that:

$$\dot{V}(x) \;=\; -x^T Q\, x$$

since

$$x^T Q\, x \;\geq\; (\lambda_Q)_{min} \|x\|_2^2$$

$$-x^T Q\, x \;\leq\; -(\lambda_Q)_{min} \|x\|_2^2$$

$$\boxed{\dot{V}(x) \;\leq\; -(\lambda_Q)_{min} \|x\|_2^2}$$

Proof of sufficiency $\Longleftarrow$

also

$$V(x) \;=\; x^T P\, x$$

and

$$(\lambda_P)_{min}\,\|x\|_2^2 \le x^T P\, x \le (\lambda_P)_{max}\,\|x\|_2^2$$

$$\Updownarrow$$

$$\underbrace{(\lambda_P)_{min}\,\|x\|_2^2 \le V(x)} \le (\lambda_P)_{max}\,\|x\|_2^2$$

$$\boxed{\|x\|_2^2 \;\le\; \frac{1}{(\lambda_P)_{min}} V(x)}$$

Proof of sufficiency $\Longleftarrow$

also

$$V(x) \;=\; x^T P\, x$$

and

$$(\lambda_P)_{min}\,\|x\|_2^2 \le \underbrace{V(x) \le (\lambda_P)_{max}\,\|x\|_2^2}$$

$$\boxed{\|x\|_2^2 \;\ge\; \frac{1}{(\lambda_P)_{max}} V(x)}$$

Proof of sufficiency $\Longleftarrow$

also

$$V(x) \;=\; x^T P\, x$$

and

$$(\lambda_P)_{min}\, \|x\|_2^2 \le \underbrace{V(x) \le (\lambda_P)_{max}\, \|x\|_2^2}$$

$$\boxed{-\|x\|_2^2 \;\le\; -\frac{1}{(\lambda_P)_{max}} V(x)}$$

Proof of sufficiency $\Longleftarrow$

since

$$\dot{V}(x) \leq -(\lambda_Q)_{min} \|x\|_2^2$$

$$-\|x\|_2^2 \leq -\frac{1}{(\lambda_P)_{max}} V(x)$$

therefore

$$\dot{V}(x) \leq -\frac{(\lambda_Q)_{min}}{(\lambda_P)_{max}} V(x)$$

Proof of sufficiency $\Longleftarrow$

since

$$\dot{V}(x) \;\leq\; -(\lambda_Q)_{min} \, \|x\|_2^2$$

$$-\|x\|_2^2 \;\leq\; -\frac{1}{(\lambda_P)_{max}} V(x)$$

therefore

$$\boxed{\dot{V}(x) \;\leq\; -\alpha \, V(x)}$$

$$\alpha = \frac{(\lambda_Q)_{min}}{(\lambda_P)_{max}} > 0$$

Proof of sufficiency $\Longleftarrow$

Considering V as a function of time

$$\dot{V}(t) \le -\alpha V(t) \quad \text{and} \quad V(t) \ge 0$$

integrating the inequality we obtain,

$$V(t) \le e^{-\alpha t} V(0)$$

$$\boxed{V(x(t)) \le e^{-\alpha t} V(x(0))}$$

Proof of sufficiency $\Longleftarrow$

$$V(x(t)) \leq e^{-\alpha t} V(x(0))$$

$$\|x(t)\|_2^2 \;\leq\; \frac{1}{(\lambda_P)_{min}} V(x(t))$$

$$\|x(t)\|_2^2 \;\leq\; \frac{1}{(\lambda_P)_{min}} e^{-\alpha t} V(x(0))$$

Proof of sufficiency $\Longleftarrow$

$$\|x(t)\|_2^2 \;\le\; \frac{1}{(\lambda_P)_{min}} e^{-\alpha t} V(x(0))$$

$$V(x(0)) \le (\lambda_P)_{max} \|x(0)\|_2^2$$

$$\|x(t)\|_2^2 \;\le\; \frac{(\lambda_P)_{max}}{(\lambda_P)_{min}} e^{-\alpha t} \|x(0))\|_2^2$$

Proof of sufficiency $\Longleftarrow$

$$\|x(t)\|_2^2 \;\; \leq \;\; e^{-\alpha t} \left(\frac{(\lambda_P)_{max}}{(\lambda_P)_{min}} \right) \|x(0)\|_2^2$$

Taking square roots, we obtain

$$\boxed{\|x(t)\|_2 \;\; \leq \;\; e^{-\beta t} \, M \, \|x(0)\|_2}$$

$$\beta = \frac{(\lambda_Q)_{min}}{2\,(\lambda_P)_{max}} \qquad\qquad M = \left(\frac{(\lambda_P)_{max}}{(\lambda_P)_{min}} \right)^{\frac{1}{2}}$$

Q.E.D.

Exponential Stability Theorem (CT)

The origin of the n-th order LTI system

$$\dot{x} = A\,x$$

is globally exponentially stable ***iff*** (if and only if),

for ***any*** symmetric matrix $Q \succ 0$

there exist a symmetric matrix $P \succ 0$

which is the ***unique*** solution of the Lyapunov equation

$$A^T P + P\,A = -Q$$

Proof of necessity $\Longrightarrow$

Part 1):

We will first show that

A is Hurwitz $\Rightarrow$ There is a unique solution to

$$A^T P + P A = -Q$$

Corollary LTI S-1

Let the matrix A be Hurwitz

Then the matrix

$$\mathbf{L}_A = \left\{ A^T \otimes I + I \otimes A^T \right\}$$

is also Hurwitz and

$\mathbf{P} = -\mathbf{L}_A^{-1}\mathbf{Q}$ always exists and is unique

Proof of necessity $\Longrightarrow$

- According to the Corollary LTI-S1, if the matrix $\boldsymbol{A}$ is Hurwitz, then the matrix

$$\mathbf{L}_A = \left\{ A^T \otimes I + I \otimes A^T \right\}$$

is Hurwitz and nonsingular.

•for **every** symmetric $Q \succ 0$

•there exists a symmetric P

•which is the **unique** solution of the Lyapunov equation

$$A^T P + P A = -Q$$

Proof of necessity $\Longrightarrow$

- According to the Corollary LTI-S1, if the matrix $\boldsymbol{A}$ is Hurwitz, then the matrix

$$\mathbf{L}_A = \left\{A^T \otimes I + I \otimes A^T\right\}$$

is Hurwitz and nonsingular.

We still need to prove two things:

- All elements of P are bounded

- $P \succ 0$

Proof of necessity $\Longrightarrow$

Proof that all elements of P are bounded:

- $Q \succ 0$ has all bounded elements $\quad \mathbf{Q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$

- $\mathbf{L}_A = \left\{ A^T \otimes I + I \otimes A^T \right\}$ is Hurwitz

- $\mathbf{P} = -\mathbf{L}_A^{-1}\mathbf{Q}$ is unique $\quad \mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$

Thus,
$$\mathbf{P}^T\mathbf{P} = \mathbf{Q}^T \underbrace{\left[\mathbf{L}_A^{-T}\mathbf{L}_A^{-1}\right]}_{\succ 0} \mathbf{Q}$$

and
$$\boxed{\|\mathbf{Q}\|_2 < \infty \;\Rightarrow\; \|\mathbf{P}\|_2 < \infty}$$

Proof of necessity $\Longrightarrow$

Part 2): We will now show that

A is Hurwitz ⇒ The unique solution to

$$A^T P + P A = -Q$$

is

$$\boxed{P = \int_0^\infty e^{A^T t} Q \, e^{At} \, dt}$$

This result proves that $P \succ 0$

Aside

Notice that, if

$$P = \int_0^\infty e^{A^T t} Q \, e^{At} \, dt$$

Since:

- $Q \succ 0$
- $\Phi(t) = e^{At}$ is **nonsingular**

Then,

$$M(t) = \Phi(t)^T Q \, \Phi(t) \succ 0$$

$$\Rightarrow \quad P = \int_0^\infty M(t) \, dt \succ 0$$

Proof of necessity $\Longrightarrow$ $P = \int_0^\infty e^{A^T t}\, Q\, e^{At}\, dt$

Part 2):

$\dot{x} = A\,x$ is exponentially stable.

Therefore,

$$x(t) = e^{At}\,x(0)$$

$$\lim_{t \to \infty} x(t) = 0$$

Proof of necessity $\Longrightarrow$ $P = \int_0^\infty e^{A^T t}\, Q\, e^{At}\, dt$

Part 2): since,

$$x(t) = e^{At}\, x(0) \qquad \lim_{t \to \infty} x(t) = 0$$

Then,

$$V(t) \;=\; x^T(t)\, P x(t)$$

Satisfies:

$$V(0) \;=\; x^T(0)\, P\, x(0) \qquad \lim_{t \to \infty} V(t) = 0$$

Proof of necessity $\Longrightarrow$ $P = \int_0^{\infty} e^{A^T t}\, Q\, e^{At}\, dt$

Part 2): also since,

$$V(t) \;=\; x^T(t)\, P x(t) \qquad\qquad A^T P + P\, A = -Q$$

Than,

$$\dot{V}(t) \;=\; \frac{d}{dt}\left\{x^T(t) P x(t)\right\}$$

$$= \; x^T(t)\{\underbrace{A^T P + P A}_{-Q}\}x(t)$$

$$\dot{V}(t) \;=\; -x^T(t)\, Q\, x(t)$$

Proof of necessity $\Longrightarrow$ $P = \int_0^\infty e^{A^T t}\, Q\, e^{At}\, dt$

Part 2): Integrate with respect to time

$$\dot{V}(t) \;=\; -x^T(t)\, Q\, x(t)$$

$$\int_0^\infty \dot{V}(t)dt \;=\; -\int_0^\infty x^T(t)\, Q\, x(t)dt$$

$$\cancelto{0}{\lim_{t\to\infty} V(t)} - V(0) \;=\; -\int_0^\infty x^T(t)\, Q\, x(t)dt$$

$$V(0) \;=\; \int_0^\infty x^T(t)\, Q\, x(t)dt$$

Proof of necessity $\Longrightarrow$ $P = \int_0^\infty e^{A^T t}\, Q\, e^{At}\, dt$

Part 2): Evaluate both sides

$$V(0) \;=\; \int_0^\infty x^T(t)\, Q\, x(t) dt$$

using $$x(t) = e^{At}\, x(0)$$

$$x^T(0)\, P\, x(0) \;=\; \int_0^\infty \overbrace{x^T(0)\, e^{A^T t}}^{x^T(t)}\, Q\, \overbrace{e^{At}\, x(0)}^{x(t)} dt$$

$$x^T(0)\, P\, x(0) \;=\; x^T(0) \left[\int_0^\infty e^{A^T t}\, Q\, e^{At} dt\right] x(0)$$

Proof of necessity $\Longrightarrow$ $P = \int_0^\infty e^{A^T t}\, Q\, e^{At}\, dt$

Part 2): Examine both sides

$$x^T(0)\, P\, x(0) \;=\; x^T(0)\left[\int_0^\infty e^{A^T t}\, Q\, e^{At} dt\right] x(0)$$

Since $x(0)$ is completely arbitrary

$$\Downarrow$$

$$P = \int_0^\infty e^{A^T t}\, Q\, e^{At}\, dt$$

Q.E.D.

Lyapunov stability theorems (DT)

The origin 0 of the n-th order LTI system

$$x(k+1) = A\,x(k)$$

is **stable in the sense of Lyapunov** if there exists a **Lyapunov function** $V(x)$ for some for some $r > 0$, *i.e.*

$$V(x) \succ 0, \qquad \forall |x| < r$$

$$\Delta V(x) \preceq 0, \qquad \forall |x| < r$$

Lyapunov stability theorems (DT)

The origin 0 of the n-th order LTI system

$$x(k+1) = A\,x(k)$$

is **asymptotically stable** if there exists a **Lyapunov function** $V(x)$ such that

$$V(x) \succ 0 \qquad \textbf{PDF}$$

$$\Delta V(x) \prec 0 \qquad \textbf{NDF}$$

Lyapunov stability theorems (DT)

Lets consider a quadratic Lyapunov function candidate:

$$V(x) = x^T P x$$

where

$$P^T = P \qquad\qquad P \succ 0$$

and compute $\Delta V(x)$ along $x(k+1) = A\,x(k)$

Lyapunov stability theorem for LTI systems (DT)

$$V(x) = x^T P x \qquad P^T = P \qquad x(k+1) = \boxed{A\,x(k)}$$

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k))$$

$$\Delta V(x(k)) = V(Ax(k)) - V(x(k))$$

$$\begin{aligned}\Delta V(x) &= V(Ax) - V(x)\\ &= x^T A^T P A x - x^T P x\end{aligned}$$

$$\boxed{\Delta V(x) = x^T \left[A^T P A - P\right] x}$$

Lyapunov stability theorem for LTI systems (DT)

Thus,

$$V(x) = x^T P\, x \qquad\qquad P^T = P, \quad P \succ 0$$

is a Lyapunov function for the system when

$$x(k+1) = A\, x(k)$$

$$\left[A^T P A - P\right] \preceq 0 \qquad \text{(negative semi-definite)}$$

and the origin is <u>**stable in the sense of Lyapunov**</u>.

Lyapunov stability theorem for LTI systems (DT)

Therefore, the origin of the system

$$x(k+1) = A\,x(k)$$

is **stable in the sense of Lyapunov**, if

there exists a symmetric matrix

$$P \succ 0$$ (positive definite)

such that

$$\left[A^T P A - P\right] \preceq 0$$ (negative semi-definite)

Lyapunov stability theorem for LTI systems (DT)

Moreover, the origin of the system

$$x(k+1) = A\,x(k)$$

is **globally asymptotically stable**, if

there exists a symmetric matrix

$$P \succ 0 \qquad \text{(positive definite)}$$

such that

$$\left[A^T P A - P\right] \prec 0 \qquad \text{(negative definite)}$$

Lyapunov stability theorem for LTI systems (DT)

The matrix $A \in \mathcal{R}^{n \times n}$

is **Schur** (i.e. all its eigenvalues are inside the unit circle)

if there exists a symmetric matrix

$$P \succ 0$$ (positive definite)

such that

$$\left[A^T P A - P\right] \prec 0$$ (negative definite)

The Lyapunov Equation

It turns out that much stronger stability results can be obtained for DT LTI systems by analyzing the following discrete time Lyapunov equation,

$$A^T P A - P = -Q$$

Given

Find

Exponential Stability Theorem (DT)

The origin of the n-th order LTI system

$$x(k+1) = A\,x(k)$$

is globally exponentially stable ***iff*** (if and only if),

for ***any*** symmetric matrix $Q \succ 0$

there exist a symmetric matrix $P \succ 0$

which is the ***unique*** solution of the Lyapunov equation

$$A^T P\,A - P = -Q$$

Exponential Stability Theorem (DT)

The matrix $A \in \mathcal{R}^{n \times n}$

is **<u>Schur</u>** (i.e. all its eigenvalues are inside the unit circle)

for ***any*** symmetric matrix $Q \succ 0$

there exist a symmetric matrix $P \succ 0$

which is the ***<u>unique</u>*** solution of the Lyapunov equation

$$A^T P A - P = -Q$$

Stability Analysis (DT)

How to use the discrete time Lyapunov equation:

- Given a matrix $\boldsymbol{A}$**,** select an arbitrary positive definite symmetric matrix $\boldsymbol{Q}$ (for example $\boldsymbol{I}$) .
- Attempt to find a solution to the Lyapunov equation

$$A^T P A - P = -Q$$

1. If a solution $\boldsymbol{P}$ cannot be found**,** $\boldsymbol{A}$ is not Hurwitz.
2. If a solution $\boldsymbol{P}$ is found, check for its sign definiteness:
   - If $\boldsymbol{P}$ is positive definite, then $\boldsymbol{A}$ is Hurwitz.
   - If $\boldsymbol{P}$ is not positive definite, then $\boldsymbol{A}$ has at least one eigenvalue outside the unit circle (unstable).

Stability Analysis (DT)

It is important to note that the Lyapunov equation is a linear algebraic equation. Thus, it is easy to solve!

$$A^T P A - P = -Q$$

Matlab functions

- Discrete time Lyapunov equation: **`P = dlyap(A',Q)`**

  `(if` P cannot be found, it returns an error message)
- The definiteness of P can be check with the Cholesky factorization function: **`N = chol(P)`**
- which returns a upper triangular matrix N, such that $P = N^T N$ when P is positive definite (otherwise it returns an error message)