# [MEN573]
# Advanced Control Systems I

## Lecture 11 – State Space Models of Energetic Systems

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# References


- “Modern Control Theory”, 3rd Ed., by William L. Brogan, Prentice Hall. (Section 1.3.1 and Section 3.4.5)

- "Introduction to System Dynamics," by Shearer, Murphy and Richardson, Addison Wesley, 1967.

- "Understanding Dynamic Systems: Approaches to Modeling, Analysis, and Design," by Dorny, Prentice Hall, 1993

# Goal

## Minimum Number of State Variables

How do we know the minimum number of state variables to select? Typically, the minimum number required equals the order of the differential equation describing the system. For example, if a third-order differential equation describes the system, then three simultaneous, first-order differential equations are required along with three state variables. From the perspective of the transfer function, the order of the differential equation is the order of the denominator of the transfer function after canceling common factors in the numerator and denominator.

In most cases, another way to determine the number of state variables is to count the number of independent energy-storage elements in the system.[5] The number of these energy-storage elements equals the order of the differential equation and the number of state variables. In Figure 3.2 there are two energy-storage elements, the capacitor and the inductor. Hence, two state variables and two state equations are required for the system.

# Goal

- Given an "**engineering system**," composed of diverse elements, derive the state and output equations that govern the behavior of that system.

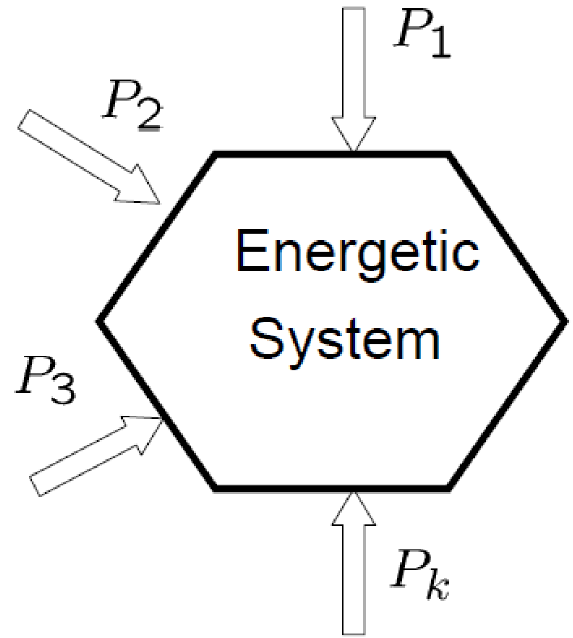
- We will analyze the diverse elements of an engineering system by the manner in which power
  – flows through, or
  – is generated by, or
  – is dissipated by, or
  – is stored by
  them.

- **Energetic systems**: mechanical (translational and rotational), electrical, fluid and thermal systems.

# Energetic Systems

The fundamental law for energetic systems is "**energy conservation**".

$$\sum_{k=1}^{n} P_k(t) = \frac{d}{dt}E(t) + P_D$$




- $P_k$ : power delivered to the system through the ***k-th*** energy port
- $E$ : energy stored by the system
- $P_D$: power dissipated by the system

# Power

- *Power* $P \in \mathcal{R}$ is the rate of flow of work or energy. It is represented by the product of two fundamental variables such as velocity and force for mechanical systems.

$$P_{\text{mech}} = \langle \underbrace{\text{velocity}}_{\text{across variable}}, \underbrace{\text{force}}_{\text{through variable}} \rangle$$

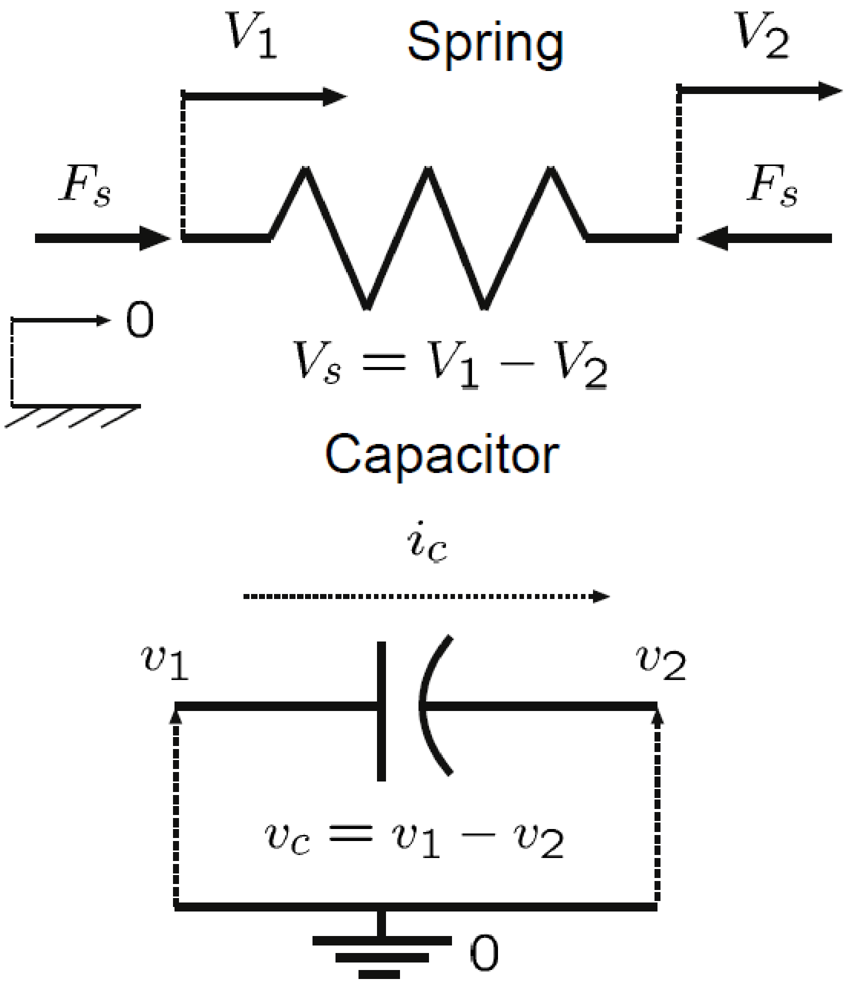
$$P_{\text{elec}} = \langle \underbrace{\text{voltage}}_{\text{across variable}}, \underbrace{\text{current}}_{\text{through variable}} \rangle$$

$$P_{\text{fluid}} = \langle \underbrace{\text{pressure}}_{\text{across variable}}, \underbrace{\text{vol. flowrate}}_{\text{through variable}} \rangle$$

# Across and through variables

- **Through variable** (generalized flow):
  - Flows though an element
- **Across variable** (generalized potential):
  - Has differential values across an element
  - Always measured relative to some reference value
  - Example:
    - Voltage: measured relative to ground (0 voltage)
    - Velocity: measured relative to inertial frame (0 velocity)
    - Pressure: measured relative to atmospheric pressure (0 pressure)

# Examples of across and through variable




$F_s$ : force through the spring (compression)

$V_s$ : velocity across the spring (compression)

$i_c$ : current through the capacitor

$v_c$ : voltage across the capacitor

# Power

- In these lectures we will only consider scalar "***across***" and "***through***" variables.

$$P_{\text{mech}} = \underbrace{V}_{\text{velocity}} \underbrace{\text{F}}_{\text{force}}$$

$$P_{\text{elec}} = \underbrace{v}_{\text{voltage}} \underbrace{\text{i}}_{\text{current}}$$

$$P_{\text{fluid}} = \underbrace{p}_{\text{pressure}} \underbrace{\text{Q}}_{\text{vol. flow}}$$

***<u>Through variable:</u>*** flows though an element

- Force
- Current
- Volumetric flow

***<u>Across variable:</u>*** has differential values across an element

- Velocity
- Voltage
- Pressure

# One-Port Energetic Elements

- Interact with other elements through one energy port and only require one pair of across and through variables.
- Passive Elements:
  - **Dissipative elements**: dissipate power
  - **Energy storage elements**: store energy
    - <u>A-type energy storage elements</u>: accumulate the across variable
    - <u>T-type energy storage elements</u>: accumulate the through variable
- Active elements:
  - **Sources**: generate power
    - <u>A-type sources</u>: sources of across variables
    - <u>T-type sources</u>: sources of through variables

# One-Port Mechanical Elements

$$P_{\text{mech}} = \underbrace{V}_{\text{velocity}} \underbrace{\mathsf{F}}_{\text{force}}$$

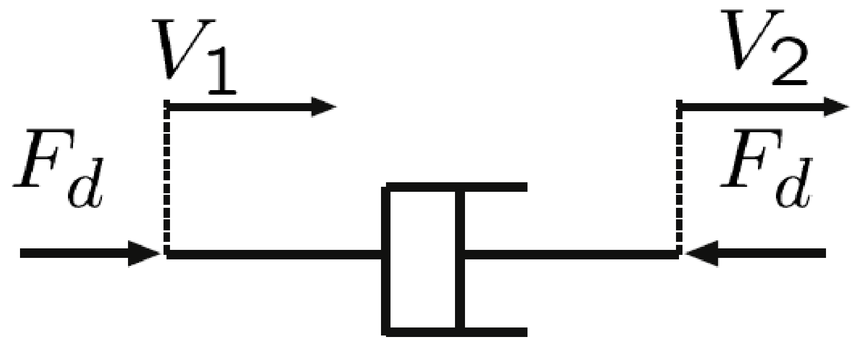
- **Passive Elements**:
  - Dissipative element:
    - D-type dissipative element: ***damper***
  - Storage elements:
    - A-type energy storage element: ***mass***
    - T-type energy storage element: ***spring***
- **Active elements**:
  - Generate power
    - A-type source: ***velocity source***
    - T-type source: ***force source***

# One-Port Mechanical D-type Dissipative Element

- ***Damper***

1) Nonlinear constitutive relation ***(static)***

$$F_d = \Phi_d(V_d)$$



$$V_d = V_1 - V_2$$

2) Linear constitutive relation

$$F_d = B\,V_d$$

Power dissipated
(linear CR)

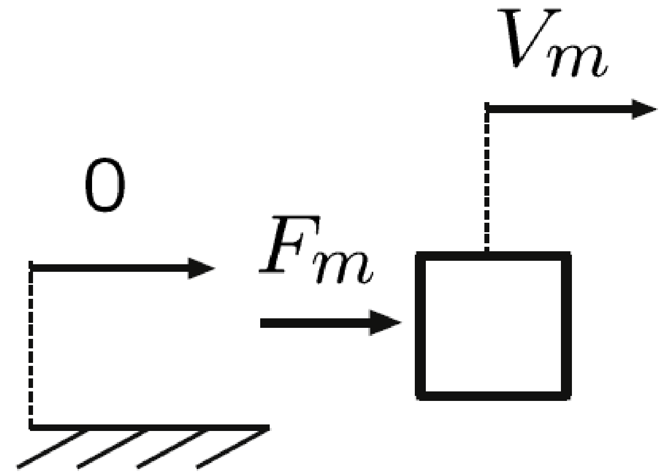
$$P_d = F_d V_d = B\,V_d^2$$

# One-Port Mechanical A-type Energy Storage Element

- ***Mass***

1) Nonlinear relation ***(dynamic)***

> Do not worry about it.




2) Linear relation ***(dynamic)***

$P_m$ : momentum

$$\frac{d}{dt}P_m = F_m$$

$$V_m = \frac{1}{m}P_m$$

$$\Rightarrow \boxed{\frac{d}{dt}V_m = \frac{1}{m}F_m}$$

# One-Port Mechanical A-type Energy Storage Element

- ***Mass***

$F_m$ : input

$V_m$ : output

2) Power storage (linear)

$$\frac{d}{dt}E_m = V_m F_m = \frac{P_m}{m}\frac{dP_m}{dt}$$

$$\frac{d}{dt}V_m = \frac{1}{m}F_m$$

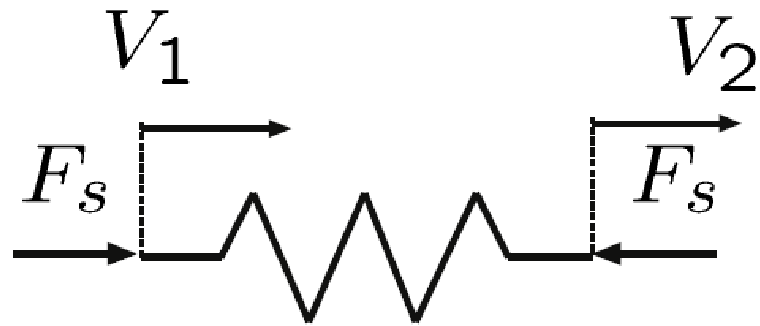
$$\boxed{E_m(t) = \int_0^{P_m(t)} \frac{P_m}{m} dP_m = \frac{P_m^2(t)}{2m} = \frac{1}{2} m V_m^2}$$

# One-Port Mechanical T-type Energy Storage Element

- ***Spring***

1) Nonlinear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt}x_s &= V_s \\ F_s &= \Phi_s(x_s) \end{aligned}}$$




$$V_s = V_1 - V_2$$

$x_s$ : comp. displacement

2) Linear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt}x_s &= V_s \\ F_s &= K\,x_s \end{aligned}} \Rightarrow \boxed{\frac{d}{dt}F_s = K\,V_s}$$

# One-Port Mechanical T-type Energy Storage Element

- ***Spring***

1) Power storage (non-linear)

$V_s$: input
$F_s$: output
$x_s$: displacement

$$\frac{d}{dt}E_s = F_s V_s = \Phi_s(x_s)\frac{dx_s}{dt}$$

$$\begin{aligned} \frac{d}{dt}x_s &= V_s \\ F_s &= \Phi_s(x_s) \end{aligned}$$

$$\boxed{E_s(t) = \int_0^{x_s(t)} \Phi_s(\eta)\, d\eta}$$

# One-Port Mechanical T-type Energy Storage Element

- ***Spring***

$V_s$: input
$F_s$: output
$x_s$: displacement

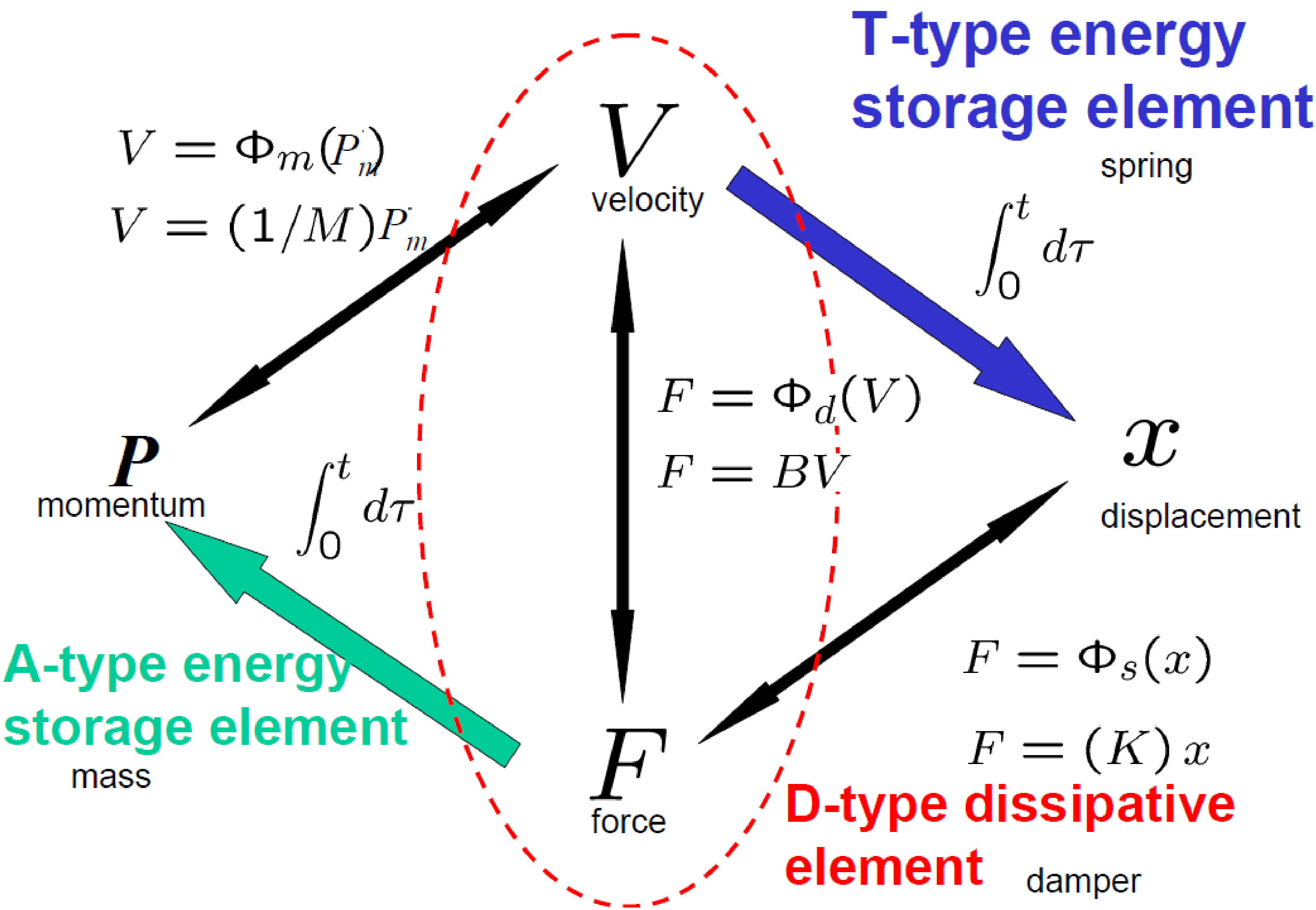
2) Power storage (linear)

$$\frac{d}{dt}E_s \;=\; F_s\,V_s = F_s\,\frac{1}{K}\frac{dF_s}{dt}$$

$$\frac{d}{dt}F_s \;=\; K\,V_s$$

$$\boxed{E_s(t) \;=\; \int_0^{F_s(t)} \frac{1}{K}\eta\,d\eta = \frac{1}{2K}F_s^2(t) = \frac{K}{2}x_s^2(t)}$$

# Overview of 1-port passive mechanical elements



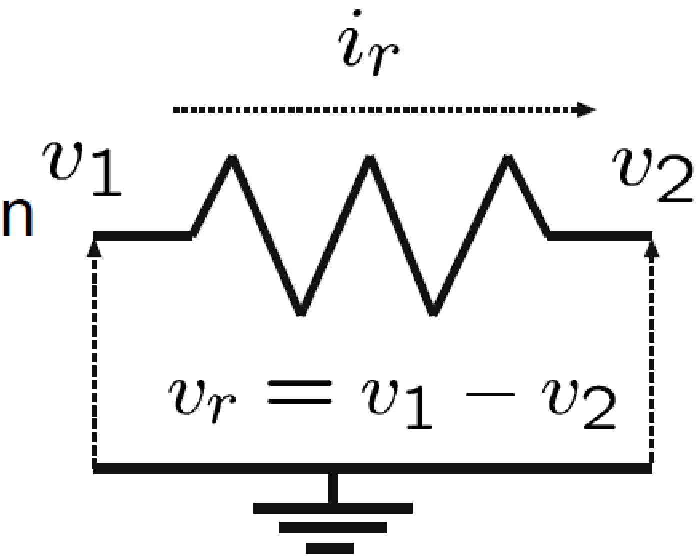
# One-Port Electrical Elements

$$P_{\text{elec}} = \underbrace{v}_{\text{voltage}} \underbrace{i}_{\text{current}}$$

- **Passive Elements**:
  - Dissipative element:
    - D-type dissipative element: ***resistor***
  - Storage elements:
    - A-type energy storage element: ***capacitor***
    - T-type energy storage element: ***inductor***
- **Active elements**:
  - Generate power
    - A-type source: ***voltage source*** (battery)
    - T-type source: ***current source***

# One-Port Electrical R-Element
# (D-type dissipative element)

- ***Resistor***




1) Nonlinear constitutive relation ***(static)***

$$\boxed{v_r = \Phi_r(i_r)}$$

2) Linear constitutive relation

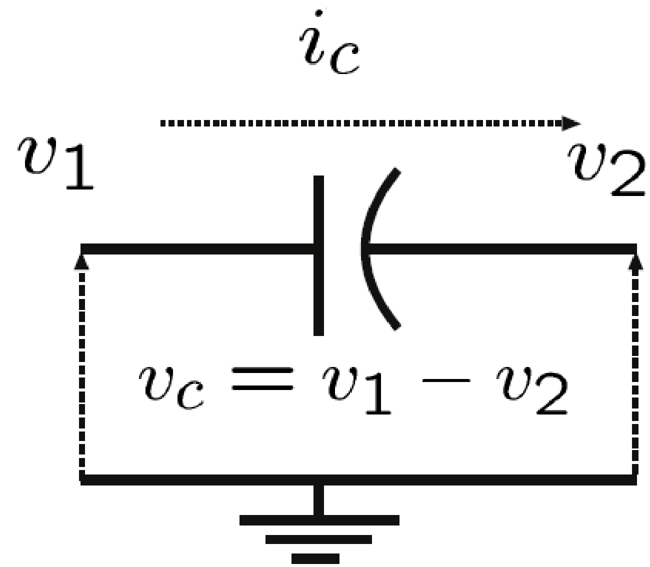
$$\boxed{v_r = R\, i_r}$$

Power dissipated
(linear CR)

$$\boxed{P_r = v_r\, i_r = R\, i_r^2}$$

# One-Port Electrical C-Element (A-type energy storage element)

- ***Capacitor***




1) Nonlinear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt} q_c &= i_c \\ v_c &= \Phi_c(q_c) \end{aligned}}$$

2) Linear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt} q_c &= i_c \\ v_c &= \frac{1}{C} q_c \end{aligned}} \Rightarrow \boxed{\frac{d}{dt} v_c = \frac{1}{C} i_c}$$

$q_c$ : charge

# One-Port Electrical C-Element (A-type energy storage element)

- ***Capacitor***

$i_C$ : input
$v_C$ : output

1) Power storage (linear)

$$\frac{d}{dt}E_C = v_C i_C = v_C C\frac{dv_C}{dt}$$

$$\boxed{\frac{d}{dt}v_C = \frac{1}{C}i_C}$$

$$dE_C = C\, v_C\, dv_C$$

$$\boxed{\begin{aligned} E_C(t) &= \int_{v_C(0)}^{v_C(t)} C\eta\, d\eta + E_C(0) \\ &= \frac{C}{2}\left[v_C^2(t) - \cancel{v_C^2(0)}\right] + \cancel{E_C(0)} = \frac{C}{2}v_C^2(t) = \frac{q_c^2(t)}{2C} \end{aligned}}$$

# One-Port Electrical C-Element
# (A-type energy storage element)

- ***Capacitor***

$i_c$ : input
$v_c$ : output

2) Power storage (non-linear)

$$\frac{d}{dt}E_c = v_c i_c = \Phi_c(q_c)\frac{dq_c}{dt}$$

$$\begin{aligned}\frac{d}{dt}q_c &= i_c \\ v_c &= \Phi_c(q_c)\end{aligned}$$

$$dE_c = \Phi_c(q_c)dq_c$$

$$\boxed{E_c(t) = \int_0^{q_c(t)} \Phi_c(\eta)\, d\eta}$$

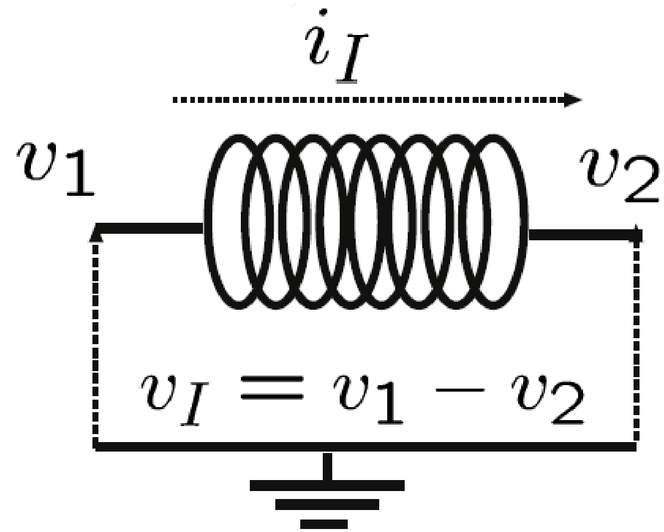
Note: Stored energy of a pure capacitor is expressed by voltage (across variable) or charge ($q_c$), which is algebraically related to the across variable.

# One-Port Electrical I-Element
# (T-type energy storage element)

- ***Inductor***

1) Nonlinear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt}p_I &= v_I \\ i_I &= \Phi_I(p_I) \end{aligned}}$$




2) Linear relation ***(dynamic)***

$p_I$ : Flux linkage

$$\boxed{\begin{aligned} \frac{d}{dt}p_I &= v_I \\ i_I &= \frac{1}{I}p_I \end{aligned}} \Rightarrow \boxed{\frac{d}{dt}i_I = \frac{1}{I}v_I}$$

# One-Port Electrical I-Element (T-type energy storage element)

- ***Inductor***

$v_I$ : input
$i_I$ : output

1) Power storage (linear)

$$\frac{d}{dt}E_I = i_I v_I = i_I I \frac{di_I}{dt}$$

$$\frac{d}{dt}i_I = \frac{1}{I}v_I$$

$$dE_I = i_I I \, di_I$$

$$\boxed{E_I(t) = \int_0^{i_I(t)} I\eta \, d\eta = \frac{I}{2}i_I^2(t) = \frac{p_I^2(t)}{2I}}$$

# One-Port Electrical I-Element (T-type energy storage element)

- ***Inductor***

$v_I$ : input
$i_I$ : output

2) Power storage (non-linear)

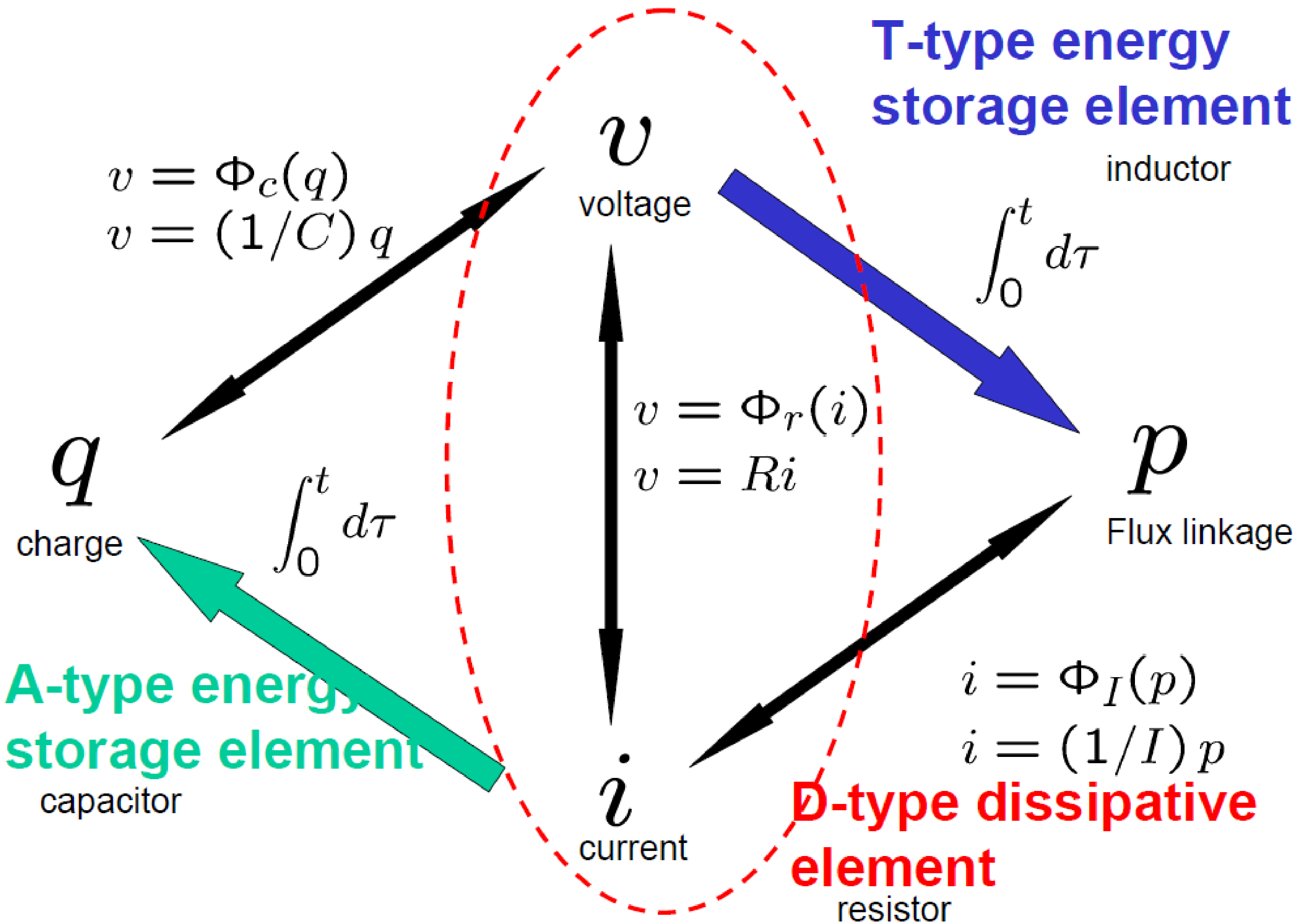
$$
\begin{aligned}
\frac{d}{dt}E_I &= i_I v_I = \Phi_I(P_I)\,\frac{dp_I}{dt} \\
dE_I &= \Phi_I(P_I)\,dp_I
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}p_I &= v_I \\
i_I &= \Phi_I(p_i)
\end{aligned}
$$

$$
\boxed{E_I(t) = \int_0^{p_I(t)} \Phi_I(\eta)\,d\eta}
$$

Note: Stored energy of a pure inductor is expressed by current (through variable) or flux linkage ($p_I$), which is algebraically related to the through variable.

# Overview of 1-port passive electrical elements

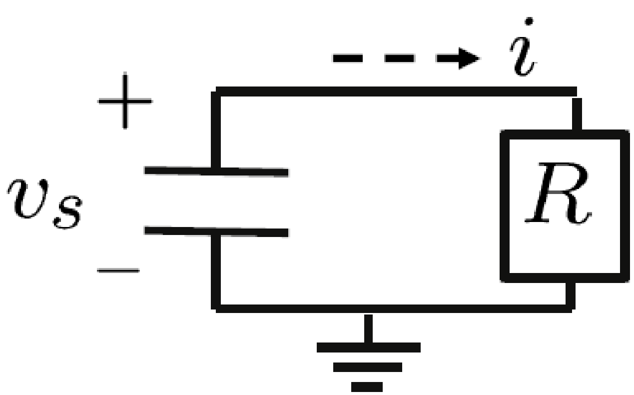


# One-Port Electrical Sources

## *1. A-type Source: Voltage source (battery)*

- Voltage source produces a specified voltage $v_s$ across it, regardless of the elements that it is connected to.

- The current that flows through the source $i$ will depend on what the source is connected to.



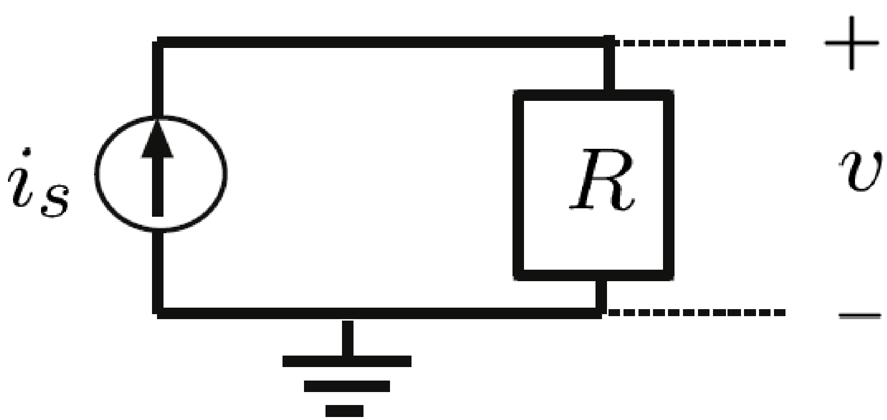

$$i = \frac{v_s}{R}$$

# One-Port Electrical Sources

## *2. T-type Source: Current source*

- Current source produces a specified current $i_s$ through it, regardless of the elements that it is connected to.

- The voltage across the source $v$ will depend on what the source is connected to.




$$v = R i_s$$

# One-Port Fluid Elements

$$\boxed{P_{\text{fluid}} = \underbrace{p}_{\text{pressure}} \; \underbrace{Q}_{\text{vol. flow}}}$$

- **Passive Elements**:
  - Dissipative element:
    - D-type dissipative element: ***orifice, porous plug***
  - Storage elements:
    - A-type energy storage element: ***tank***
    - T-type energy storage element: ***fluid inertance (pipe)***

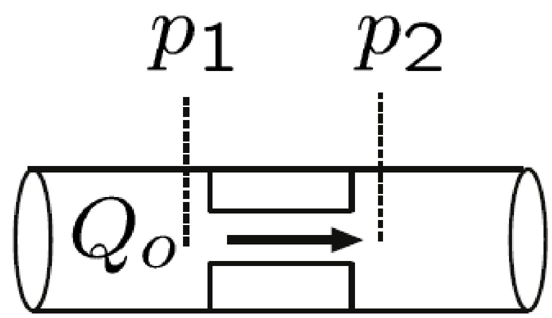
- **Active elements**:
  - Generate power
    - A-type source: ***pressure source***
    - T-type source: ***flow source***

# One-Port Fluid D-type Dissipative Element

- ***Orifice, porous plug***

1) Nonlinear constitutive relation ***(static)***

$$\boxed{p_o = \Phi_o(Q_o)}$$




$$p_o = p_1 - p_2$$

2) Linear constitutive relation

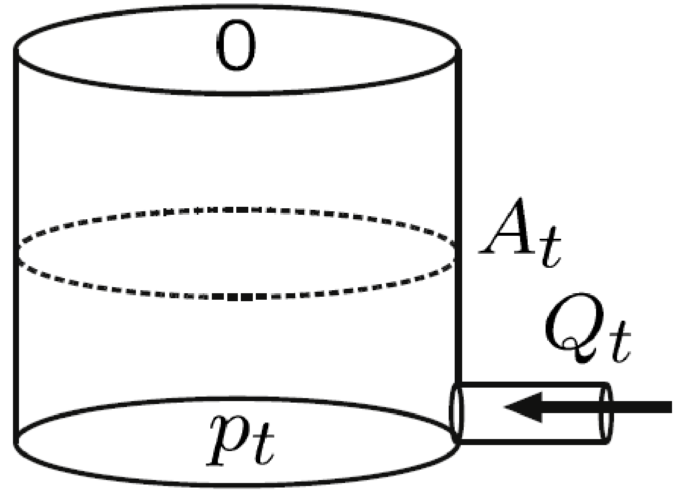
$$\boxed{p_o = R\,Q_o}$$

Power dissipated
(linear CR)

$$\boxed{P_o = p_o\,Q_o = R\,Q_o^2}$$

# One-Port Fluid A-type Energy Storage Element

- ***Tank:***




1) Nonlinear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt}V_t &= Q_t \\ p_t &= \Phi_t(V_t) \end{aligned}}$$

$V_t$ : volume

2) Linear relation ***(dynamic)***

$$C_t = \frac{A_t}{g\rho}$$

$g$ : grav. acce.
$\rho$ : fluid den.

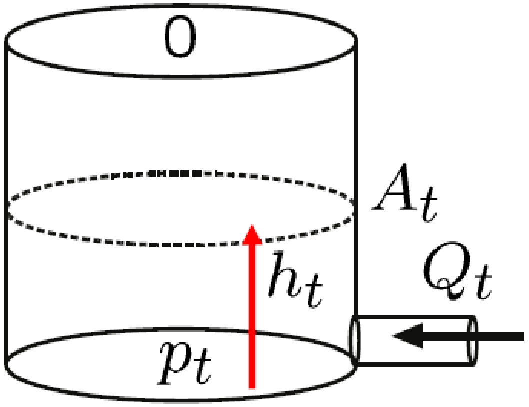
$$\boxed{\begin{aligned} \frac{d}{dt}V_t &= Q_t \\ p_t &= \frac{1}{C_t}V_t \end{aligned}} \Rightarrow \boxed{\frac{d}{dt}p_t = \frac{1}{C_t}Q_t}$$

# One-Port Fluid A-type Energy Storage Element

- ***Tank linear constitutive relation***

$$\frac{d}{dt}V_t = Q_t$$

$$p_t = \rho\, g\, h_t \qquad h_t = \frac{V_t}{A_t}$$




$$\frac{d}{dt}V_t = Q_t$$

$$p_t = \frac{\rho g}{A_t}V_t = \frac{1}{C_t}V_t \quad \Rightarrow \quad \boxed{\frac{d}{dt}p_t = \frac{1}{C_t}Q_t}$$

# One-Port Fluid A-type Energy Storage Element

- ***Tank:***

$Q_t$ : input

$p_t$ : output

1) Power storage (non-linear)

$$\frac{d}{dt}E_t = p_t Q_t = \Phi_t(V_t)\frac{dV_t}{dt}$$

$$\boxed{E_t(t) = \int_0^{V_t(t)} \Phi_t(\eta)\, d\eta}$$

2) Power storage (linear)

$$\frac{d}{dt}E_t = p_t Q_t = p_t C_t \frac{dp_t}{dt}$$

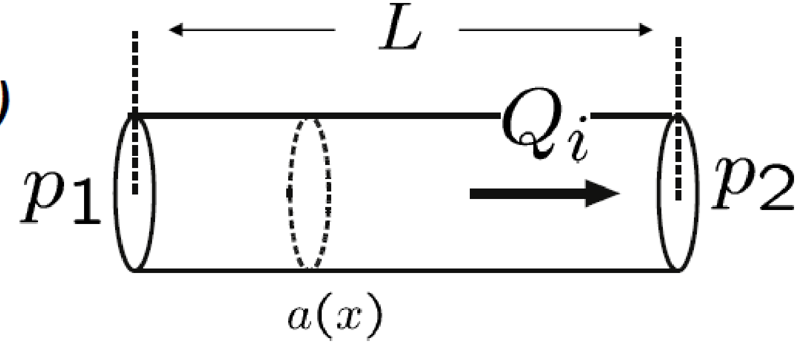
$$\boxed{E_t(t) = \int_0^{p_t(t)} C_t \eta\, d\eta = \frac{C_t}{2}p_t^2(t) = \frac{g\, A_t}{2\, \rho g}p_t^2(t)}$$

# One-Port Fluid T-type Energy Storage Element

- ***Fluid inertance***

1) Nonlinear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt}\mathcal{P}_i &= p_i \\ Q_i &= \Phi_i(\mathcal{P}) \end{aligned}}$$




$$p_i = p_1 - p_2$$

$\mathcal{P}_i$ : Pressure momentum

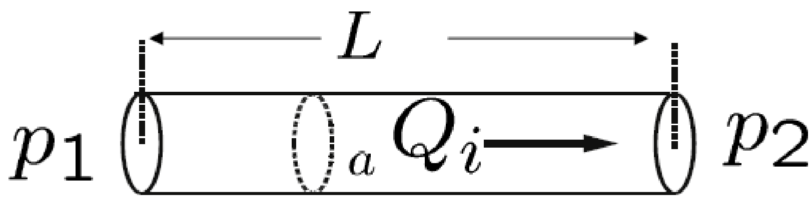
2) Linear relation ***(dynamic)***

$$\boxed{\begin{aligned} \frac{d}{dt}\mathcal{P}_i &= p_i \\ Q_i &= \frac{1}{I_f}\mathcal{P}_i \end{aligned}} \Rightarrow \boxed{\frac{d}{dt}Q_i = \frac{1}{I_f}p_i}$$

$$I_f = \int_0^L \frac{\rho}{a(x)}dx = \frac{\rho L}{a}$$

# One-Port Fluid T-type Energy Storage Element

- ***Fluid inertance linear constitutive relation***




$$p_i = p_1 - p_2$$

$$F_1 = a\,p_1 \qquad \boxed{m_i = \rho\, a\, L} \qquad F_2 = a\,p_2$$

$$\begin{aligned} \frac{d}{dt}Q_i &= a\frac{d}{dt}V_i = \frac{a}{m_i}\left[F_1 - F_2\right] \\ &= \frac{a}{\rho L}\left[p_1 - p_2\right] = \frac{1}{I_f}p_i \end{aligned} \qquad I_f = \frac{\rho L}{a}$$

# One-Port Fluid T-type Energy Storage Element

- ***Fluid inertance***

$p_i$ : input

$Q_i$ : output

1) Power storage (non-linear)

$$\frac{d}{dt}E_i = Q_i\,p_i = \Phi_i(\mathcal{P}_i)\,\frac{d\mathcal{P}_i}{dt}$$

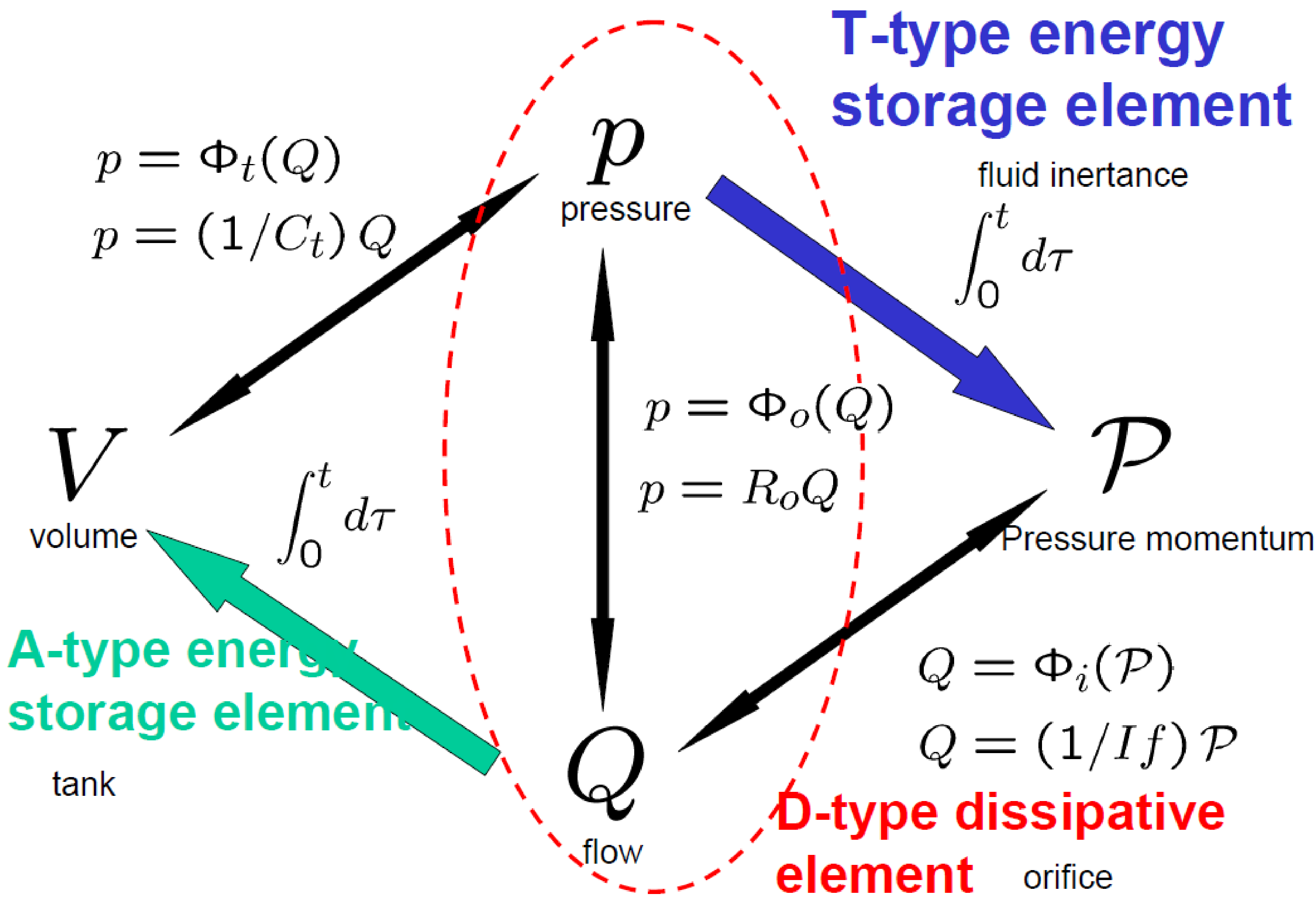
$$\boxed{E_i(t) = \int_0^{\mathcal{P}_i(t)} \Phi_i(\eta)\,d\eta}$$

2) Power storage (linear)

$$\frac{d}{dt}E_i = Q_i\,p_i = Q_i\,I_f\frac{dQ_i}{dt}$$

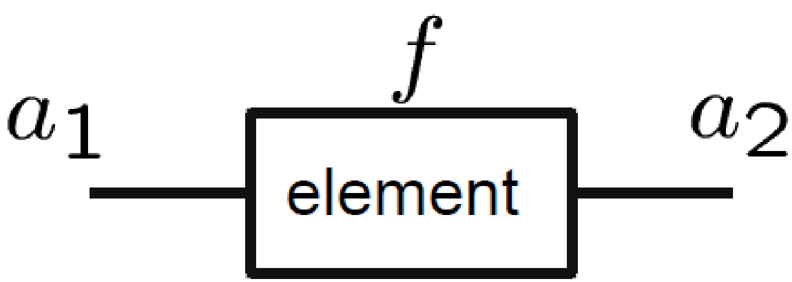
$$\boxed{E_i(t) = \int_0^{Q_i(t)} I_f\eta\,d\eta = \frac{I_f}{2}Q_i^2(t) = \frac{\rho L}{2\,a}Q_i^2(t)}$$

# Overview of 1-port passive fluid elements



# Linear Graph

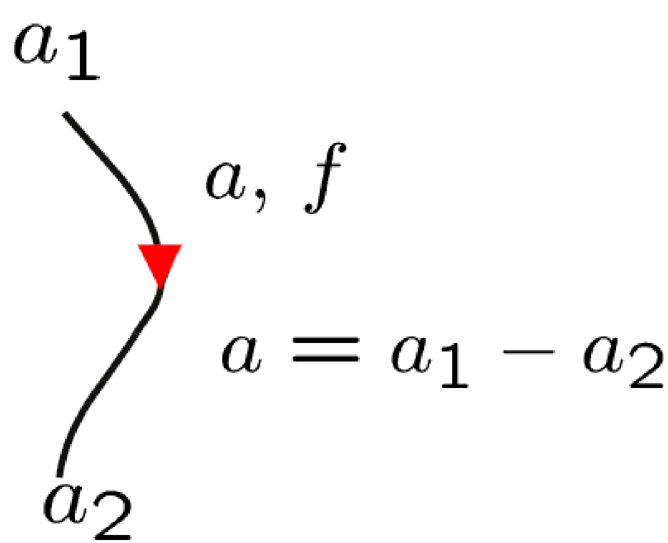
- Each one-port element has physically two terminals (two ends) and is characterized by functional relationships between its through and across variables $(a,\ f)$




$$a = a_1 - a_2$$

# Linear Graph

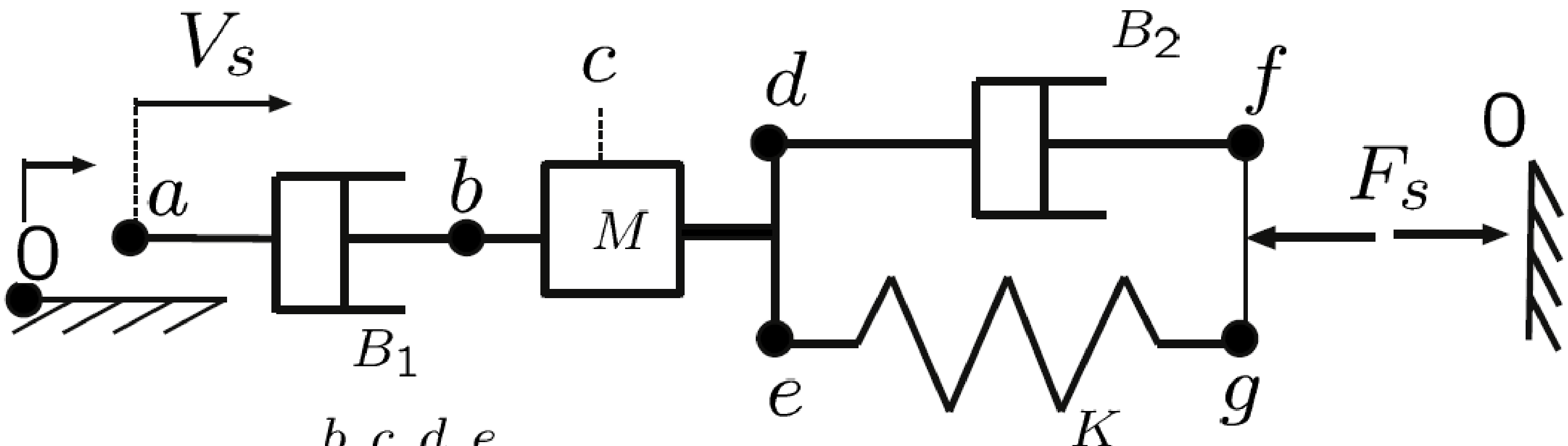
- A convenient symbol to express this relationship is the *linear graph*.

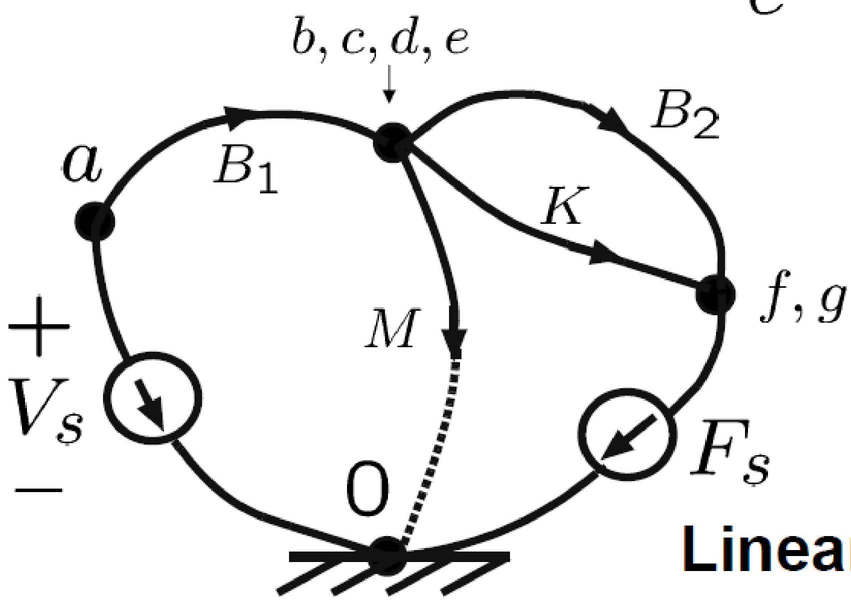



The two ends or terminals of this graph indicate the across variables for the element, and the line between these terminals represents the continuity of the through variable in the element.

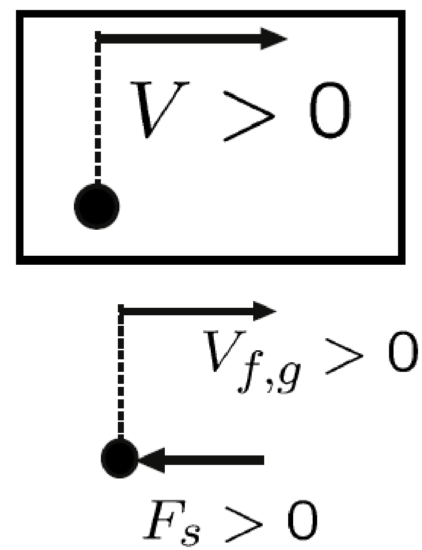
# Linear Graph: Example

**Mechanical:**



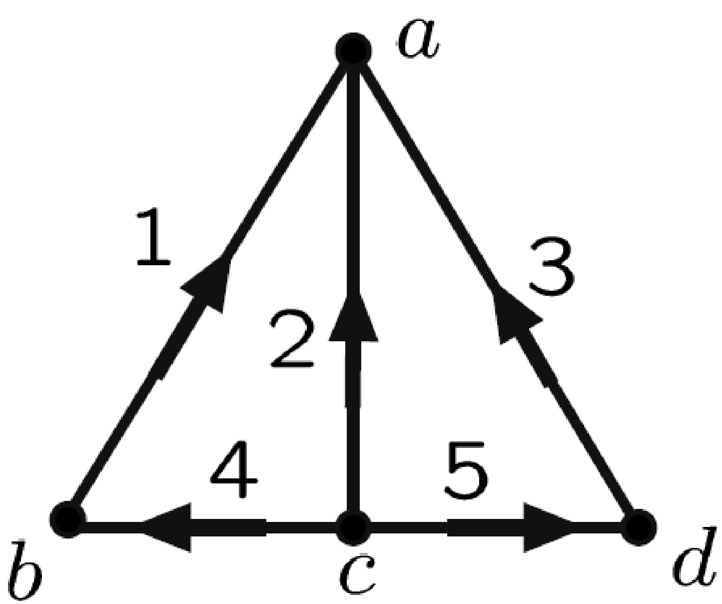




**Linear graph:**



# Graph Theory

- A ***graph*** consists of ***nodes*** and ***branches***.




$$N = 4$$

$$B = 5$$

**Graph**:

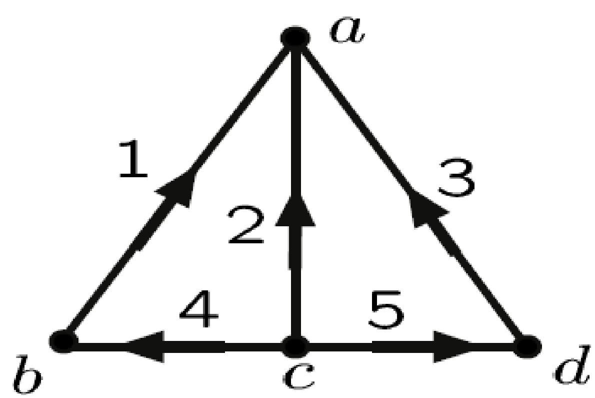
- 4 nodes: $\boldsymbol{a, b, c, d}$
- 5 branches: $\boldsymbol{1, 2, 3, 4, 5}$

$\boldsymbol{N}$: number of nodes

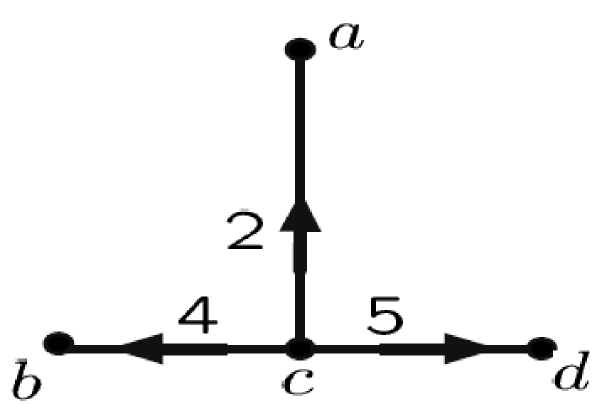
$\boldsymbol{B}$: number of branches

# Trees

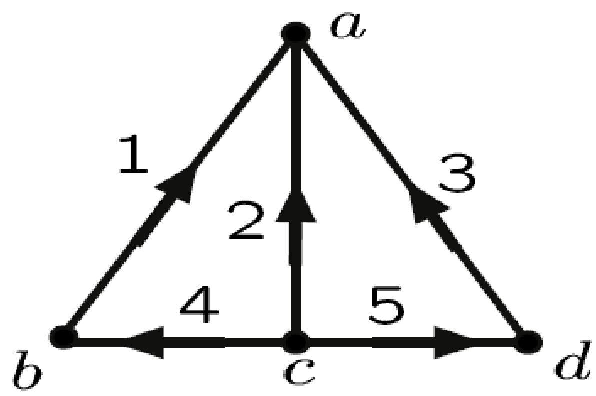
- A ***tree*** of a graph consists of a set of branches that:
  1. Contain every node of the graph
  2. Are connected
  3. Contain no loops




(Trees are obtained by pruning graphs)



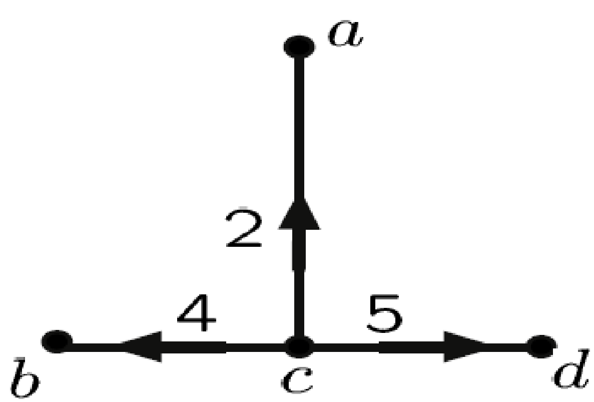
# Trees




***__Tree branch__*: **

graph branch ***included*** in the tree

(2,4,5)




***__Link__***:

graph branch ***not included*** in the tree

(1,3)

# Continuity and Compatibility

- Graphs must satisfy the following two conditions:

1. **Continuity**: The sum of through variables entering a node must be zero.

2. **Compatibility**: The sum of across variables around any closed loop must be zero.

- Analogous to Kirchhoff laws in electrical circuits

# Normal Tree

- Goals:

1. Identify all independent energy storage elements
   - **A-type energy storage** elements
   - **T-type energy storage** elements.

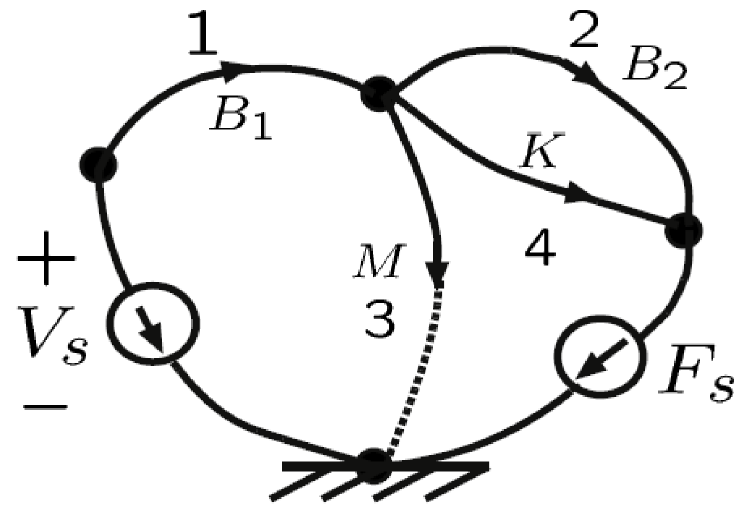
2. **Assign a state variable for each independent energy storage element**.

- Start from a linear graph.

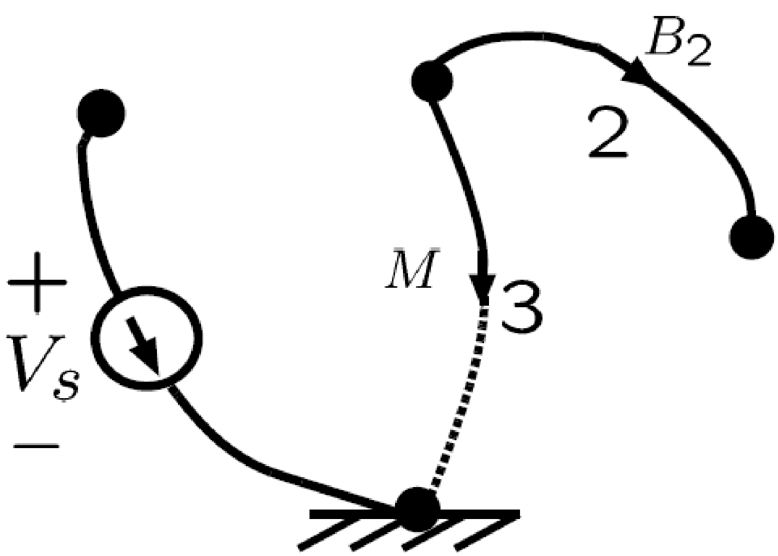
# Linear Graph: Example

**Change of notation**:

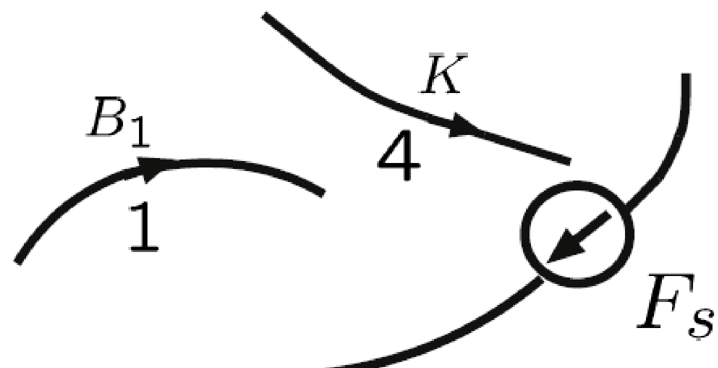
Use a number to denote the branch of each passive element




**Normal tree**:




**3 Links**:



# Selection of State Variables

- Linear Elements:

1. **A-type energy storage element in a normal tree**: the across variable

2. **T-type energy storage elements in a link**: the through variable

# Selection of State Variables

- <u>Nonlinear Elements</u>:

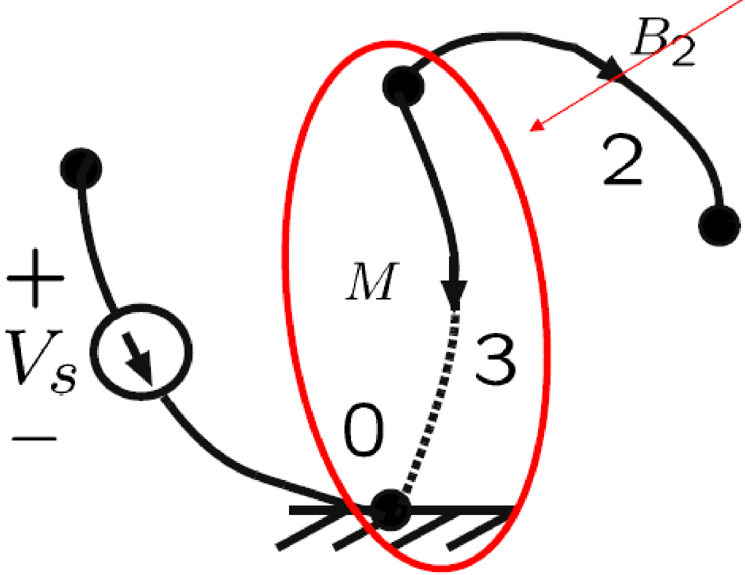
1. A-type energy storage element in a normal tree: the time integral of the through variable

2. T-type energy storage elements in a link: the time integral of the across variable

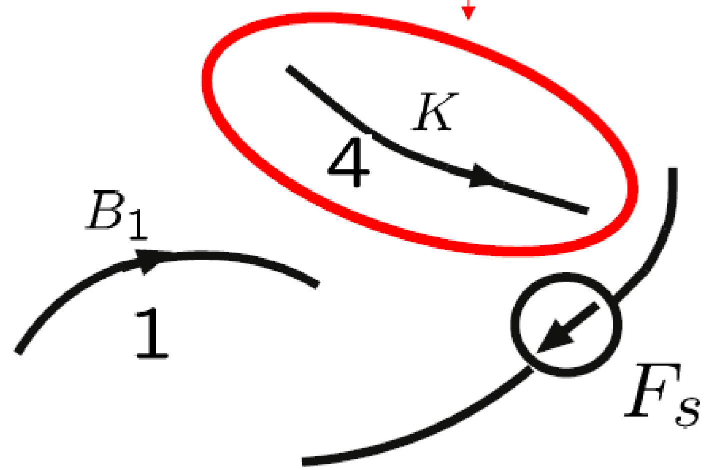
# Linear Graph: Example

**Normal tree**:

2 Independent energy storage elements:




**3 Links**:

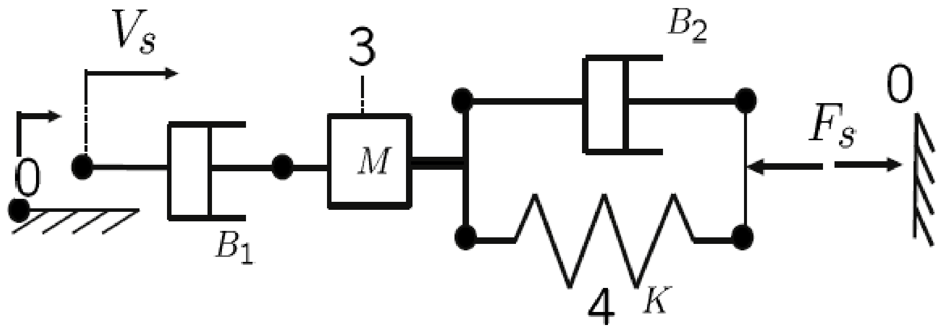


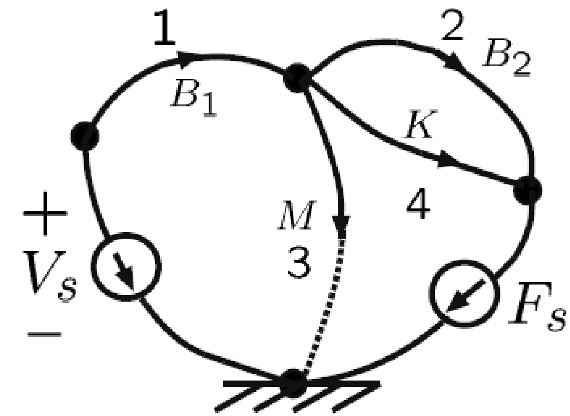

**2 State variables**:

$V_3$ : mass velocity

$F_4$ : spring force

# Linear Graph: Example




$$\frac{d}{dt}\underbrace{\begin{bmatrix} V_3 \\ F_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -\frac{B_1}{M} & 0 \\ 0 & -\frac{K}{B_2} \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} V_3 \\ F_4 \end{bmatrix}}_{x}$$

$$+\underbrace{\begin{bmatrix} \frac{B_1}{M} & -\frac{1}{M} \\ 0 & \frac{K}{B_2} \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} V_s \\ F_s \end{bmatrix}}_{u}$$