Linear System Theory

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Chapter 8: State Feedback and State Estimators

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Recap

- State space equation
- Linear Algebra
- Solutions of LTI and LTV systems
- Stability
- Controllability & observability

We will study

- State feedback control (pole-placement)
- Output feedback control (state estimator or state observer)
- We will cover Sections 8.1, 8.2 (not 8.2.1), 8.3.2, 8.4 (not 8.4.1), 8.5, 8.6 (not 8.6.1-8.6.4), 8.7, 8.8
- Optimal control? (linear-quadratic regulator)

State Feedback: Overview

State x

- position, velocity, current, voltage, etc.
- If you have sensors, you can measure the state
- State feedback: you have all the sensors to measure the state
- Based on the measurement, you want to control the system $\Rightarrow$ Feedback!!!
- You want to control the system
  $\Leftrightarrow$ The system must be controllable!! (or stabilizable)

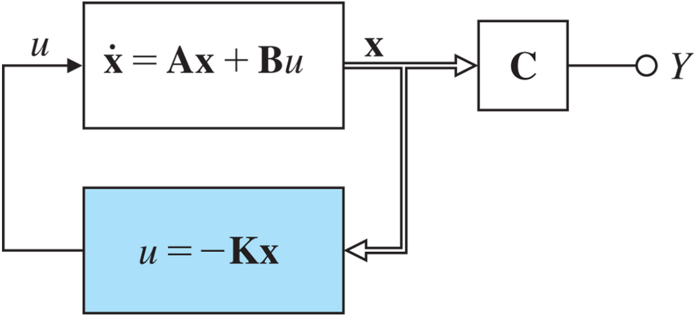
State Feedback: Overview

State Feedback

$$\dot{x} = Ax + Bu, \;\; u = -Kx + r, \;\; \text{linear controller}$$

- K: state feedback controller (linear controller)
- r: reference input ($r = 0$: regulation)
- We want to design K to stabilize the system
- Optimal K: Linear-quadratic regulator

State Feedback: Overview



Output Feedback: Overview

Output y

- Sensor measurement
- You have less number of sensors than the number of state
- You need to estimate all the state from the output
- The system needs to be observable!!! (or detectable!!!)

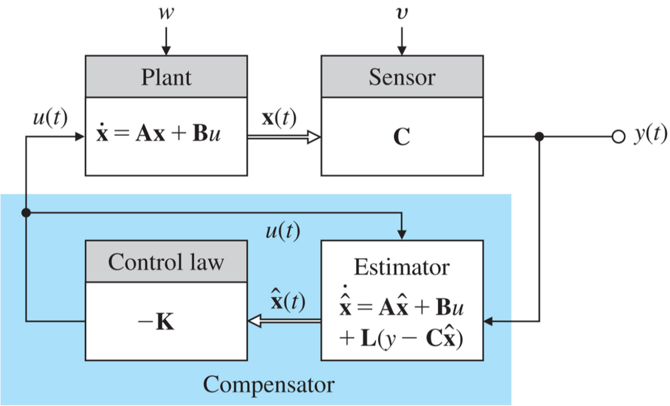
Output Feedback: Overview

Output Feedback

$$
\begin{aligned}
&\dot{x} = Ax + Bu, \;\; y = Cx, \;\; (C = I\text{: state feedback})\\
&\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \;\; \text{linear estimator (observer)}\\
&u = -K\hat{x} + r
\end{aligned}
$$

- $\hat{x}$: (linear) state estimator (state observer)
- K: state feedback controller
- L: observer gain (estimator gain)
- The controller uses the estimate $\hat{x}$ to control the system
- We want to design both K and L to stabilize the system
- Optimal L: Kalman filter

Output Feedback: Overview



Example

$$\dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u, \;\; y = (1\ 0\ 1)x$$

$$u = -Kx + r = -(k_1\ \ k_2\ \ k_3)x + r$$

$$\dot{x} = (A - BK)x + Br =: A_{cl}x + Br, \;\; y = Cx$$

$$A_{cl} = \begin{pmatrix} 1 - k_1 & -k_2 & -k_3 \\ -k_1 & 2 - k_2 & -k_3 \\ 0 & 0 & -1 \end{pmatrix}$$

- $\lambda_1 = 1$: controllable and observable
- $\lambda_2 = 2$: controllable but not observable
- $\lambda_3 = -1$: not controllable but observable

Example

Fact 1: The rank of $\mathcal{C} = (B \ A_{cl}B \ A_{cl}^2B)$ is 2 $\Rightarrow$ the controllability is invariant under the state feedback

Fact 2: Uncontrollable is invariant under the state feedback

Example

Fact 3: The observability matrix

$$\mathcal{O} = \begin{pmatrix} C \\ CA_{cl} \\ CA_{cl}^2 \end{pmatrix}$$

The rank of $\mathcal{O}$ can be 1, 2, or 3 depending on the values of k_i, $i = 1, 2, 3$ $\Leftrightarrow$ Observability is NOT invariant under the state feedback

Example

Fact 4: The characteristic equation of A_{cl} is

$$\det(\lambda I - A_{cl}) = (\lambda + 1)(\lambda^2 - (3 - k_1 - k_2)\lambda + 2 - 2k_1 - k_2)$$

You can control the controllable eigenvalues to arbitrary values $\Leftrightarrow$ Arbitrary pole-placement is possible by linear state feedback, in the controllable subspace

State Feedback

$$\dot{x} = Ax + Bu, \;\; u = -Kx, \quad \dot{x} = (A - BK)x$$

Theorem (Theorems 8.1, 8.2, 8.3 and 8.M1, 8.M3)

- The pair $(A - BK, B)$ is controllable if and only if the pair (A, B) is controllable
- The eigenvalues of $(A - BK)$ can be placed arbitrarily, respecting complex conjugate constraints, if and only if (A, B) is controllable.
- The controllable system can be transformed into the controllable canonical form (Theorem 8.2 for the SISO system, and the general MIMO case, please see the reference)

State Feedback

Note that with $u = -Kx + r$, we have the same result, since r does not affect the eigenvalues of $(A - BK)$

Why do we need the complex conjugate constraint? $\Rightarrow$ The eigenvalues of the matrix are of the form of the complex conjugate!!! (Think about the roots of the polynomial equation)

State Feedback

SISO system with the controllable canonical form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & \cdots & \alpha_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u, \;\; u = -Kx$$

State Feedback

$$A - BK = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n - k_1 & -\alpha_{n-1} - k_2 & \cdots & \cdots & \alpha_1 - k_n \end{pmatrix}$$

$$\det(sI - (A - BK)) = s^n + (\alpha_1 + k_n)s^{n-1} + \cdots + (\alpha_n + k_1)$$

State Feedback

Assume that we want the characteristic equation of $(A - BK)$ being

$$s^n + \beta_n s^{n-1} + \cdots + \beta_1$$

Then by comparing β_i and $(\alpha_j + k_i)$, we can design k_i

State Feedback

Proof of the theorems: SISO case: See the textbook

We prove Theorem 8.M1 (the MIMO state feedback)

Lemma (not in the textbook, no proof)
Suppose that (A, B) is controllable where $B \in \mathbb{R}^{n \times m}$. If $B_1 \in \mathbb{R}^n$ is any nonzero column of B; then there exists an $K_1 \in \mathbb{R}^{m \times n}$ such that the matrix pair $(A - BK_1, B_1)$ is controllable

State Feedback

Proof of Theorem 8.M1 (sufficiency, necessity: exercise)
If (A, B) is controllable, by the lemma, there exists K_1 such that $(A - BK_1, B_1)$ is controllable. Since $(A - BK_1, B_1)$ is the SISO system, from the SISO state feedback theorem, there exists $K_2 \in \mathbb{R}^{1\times n}$ such that

$$(A - BK_1) - B_1 K_2$$

have any desired eigenvalues. Hence, by the definition of B_1, we can choose a matrix K such that

$$(A - BK_1) - B_1 K_2 = A - BK.$$

State Feedback

What if (A, B) is NOT controllable? $\Rightarrow$ We can do the Kalman decomposition studied in Chapter 6:

$$\dot{x} = \begin{pmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{pmatrix} x + \begin{pmatrix} B_c \\ 0 \end{pmatrix} u, \; x = (x_c^T \; x_{\bar{c}}^T)^T$$

$$u = -Kx = -(K_1 \; K_2) \begin{pmatrix} x_c \\ x_{\bar{c}} \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} A_c - B_c K_1 & A_{12} - B_c K_2 \\ 0 & A_{\bar{c}} \end{pmatrix} x$$

State Feedback

Note that (A_c, B_c) is controllable!!! The characteristic equation is

$$\det(sI - (A - BK)) = \det(sI - (A_c - B_c K_1))\det(sI - A_{\bar{c}})$$

One can place the controllable eigenvalues to the arbitrary position, but the uncontrollable eigenvalues remain unchanged (see page 299 of the textbook)

Theorem (not in the textbook)
(A, B) is stabilizable if and only if the eigenvalues of the uncontrollable modes are in the strict left half plane

State Feedback

Some remarks on the state feedback

- For the SISO case, K is unique. But for the MIMO case, K may not be unique (think about the proof stated in the previous slide)
- You have to assign appropriate poles of the closed-loop system $(A - BK) \Rightarrow$ HOW???
- Selection of the good pole depends on applications
- To move the poles a long way requires large control gains

State Feedback

- Cannot consider the model uncertainty and noise (robustness)
- Computation: For the MIMO case, it is hard to compute K directly from the characteristic equation $\Rightarrow$ Numerically unstable
- MATLAB: "acker" (based on the Ackerman's Formula, see the reference) or "place" is also numerically unstable
  - place: you cannot place the repeated poles that are more than the number of inputs
  - acker: numerically unstable if the dimension is greater than 10

State Feedback

We want to select K optimally!! $\Rightarrow$ Optimal control

$$\min_u \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt, \;\; Q \geq 0, \;\; R > 0$$
$$\text{subject to } \dot{x} = Ax + Bu$$

We will study this problem at the end of this semester

- We assign the pole of the system by using Q and R
- Linear-quadratic regulator
- Minimizing the energy!!

Output Feedback

Output feedback control system: The most general form of the control system

$$\dot{x} = Ax + Bu, \ \ y = Cx \ (C = I\text{: state feedback})$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \text{ linear estimator (observer)}$$
$$u = -K\hat{x}$$

Output Feedback

Objective

Estimate x from u and y with the knowledge of A, B, and C

- The initial condition of x must be estimated $\Leftrightarrow$ (A, C) must be observable!!! (detectable)
- We must have $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$ for all initial conditions of x and $\hat{x}$: the estimation error must converge to zero

There are many ways of designing the state observer. But the one study in this course is linear estimator (observer)

We will see that the observer design is dual of the pole-placement (think about the duality between controllability and observability)

Output Feedback

Theorem (not in the textbook)
An observer exists if and only if (C, A) is observable (or detectable). Furthermore, in that case, one such observer is given by

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= (A - LC)\hat{x} + Bu + Ly,\end{aligned}$$

where the matrix L is chosen such that $(A - LC)$ is stable

Output Feedback

Proof (sufficiency)

We note that (C, A) observable is equivalent to (A^T, C^T) controllable (thinks about the duality theorem)

(A^T, C^T) controllable $\Leftrightarrow$ There exists a gain L^T such that we can assign the eigenvalues of $(A^T - C^T L^T)$ arbitrarily. Also, we have $(A^T - C^T L^T, C^T)$ controllable (via the pole-placement theorem), which is equivalent to $(C, A - LC)$ observable (think about the duality theorem)

Output Feedback

Theorem (not in the textbook)
An observer exists if and only if (C, A) is observable (or detectable).
Furthermore, in that case, one such observer is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - LC)\hat{x} + Bu + Ly,$$

where the matrix L is chosen such that $(A - LC)$ is stable

Output Feedback

Proof (sufficiency)

We need to show that with the above observer, $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$ for all initial conditions of x and $\hat{x}$

Note that

$$\begin{aligned}\dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly \\ &= (A - LC)\hat{x} + Bu + LCx\end{aligned}$$

Let $e = x - \hat{x}$ with $e(0) = x(0) - \hat{x}(0)$. Then

$$\dot{e} = (Ax + Bu) - \Big((A - LC)\hat{x} + Bu + LCx\Big) = (A - LC)e$$

Since $(A - LC)$ is stable, $e(t) \to 0$ as $t \to \infty$.

Output Feedback

Suppose that (C, A) is not observable. Then from the Kalman decomposition,

$$\dot{\hat{x}} = \begin{pmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{pmatrix} \hat{x} + \bar{B}u + L(y - (C_o \ 0)\hat{x}),$$

Then with $L = (L_1^T \ L_2^T)^T$, $(A - LC)$ becomes

$$A - LC = \begin{pmatrix} A_0 - L_1 C_o & 0 \\ A_{21} - L_2 C_o & A_{\bar{o}} \end{pmatrix}$$
$$\det(sI - (A - LC)) = \det(sI - A_{\bar{o}}) \det(sI - (A_o - L_1 C_o))$$

Output Feedback

Similar to the pole-placement, we can assign the observable eigenvalues to the arbitrary position, but the unobservable eigenvalues remain unchanged.

Hence, we need the detectability of (C, A). Namely, the real part of eigenvalues of $A_{\bar{o}}$ must be negative

Output Feedback

Assume that (A, B) controllable and (C, A) observable. The closed-loop system of the output feedback control system

$$\begin{aligned} \dot{x} &= Ax + Bu, \ \ y = Cx \\ u &= -K\hat{x}, \ \ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \end{aligned}$$

Let $z = (x^T \ \hat{x}^T)^T \in \mathbb{R}^{2n}$. Then the closed-loop system

$$\dot{z} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} z =: A_{cl} z$$

Output Feedback

We need stability of $z \Leftrightarrow z$ converges to zero as $t \to \infty$. That is, the real part of eigenvalues of A_{cl} must be negative!!!

It seems that we cannot design K and L independently to assign the eigenvalues of A_{cl}. However, we will show that K and L can be designed independently to assign the eigenvalues of A_{cl}.

Output Feedback

We have

$$\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} =: Pz \text{ Note that } P = P^{-1}$$

Then, we can show that (via the similarity transformation)

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BK & -BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

$$\det(sI - A_{cl}) = \det(sI - (A - BK)) \det(sI - (A - LC))$$

Output Feedback

Note that the similarity transformation does not affect the eigenvalues of the matrix.

The above result implies that we can choose K and L independently to assign the eigenvalues of A_{cl}

- This is known as the separation principle (Sections 8.5 and 8.8 of the textbook)

Output Feedback

Some remarks on the observer

- The observer is also known as the Luenberger observer
- Similar to the pole-placement, we have choose the arbitrary pole of the estimator (observer) to get the convergence of the estimation error e
- The assigning pole of the estimator depends on applications
- Similar to the pole-placement, there are numerical issues of selecting the eigenvalues of the estimator
- We need to select L optimally $\Rightarrow$ Kalman filter
- The dimension of the observer is the same as the dimension of the state $\rightarrow$ Full-dimensional (estimator) observer
- There are versions of the reduced-order estimator (will not be covered in this class, see page 303 and page 316 of the textbook)

Output Feedback

Some remarks on the output feedback controller

- The output feedback control system is widely used in many control applications
- We usually assign the pole of the estimator three times faster than the pole of the state feedback controller
- To design the output feedback system, we need (A, B) controllable (stabilizable) and (C, A) observable (detectable)
- Assign the eigenvalues of $(A - BK)$ and $(A - LC)$ determine the eigenvalues of the closed-loop system. Hence, we can design K and L independently. (Separation principle)
- Separation principle: One of the most important properties of the output feedback controller

The Next Class

We will study several control systems structures

- Tracking of the reference input ($r \neq 0$)
- Disturbance rejection
- Integral control

At the end of this semester, we will study optimal control