[MEN573]
Advanced Control Systems I

Lecture 7 – Solution of LTI State Equations

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Differential Equations

$$\dot{x} = f(x,t) \qquad x(t) \in R^n \quad x(t_0) = x_0$$

$$f(x,t) : R^n \times R_+ \to R^n$$

- Under what conditions,

(a) Does a solution exist, i.e., meaning that $x(t_0) = x_0$ guarantees that $x(t)$ is defined for all $t \geq t_0$?

(b) Is the solution unique?

Piecewise Continuous

- Definition

$f(x,t): R^n \times R_+ \to R^n$ is <u>**piecewise continuous in** ***t***</u> for all x if $f(x,\cdot): R^n \to R^n$ is continuous except at points of discontinuity, and these can only be finitely many points of discontinuity in any compact interval.

Lipschitz Condition

- Definition

$f(x,t): R^n \times R_+ \to R^n$ is <u>**Lipschitz continuous in *x***</u> for all t if there exists a piecewise continuous function $k(\cdot): R_+ \to R_+$ such that

$$\| f(x,t) - f(y,t) \| \leq k(t) \| x - y \| \qquad \begin{array}{l} \forall x, y \in R^n \\ \forall t \in R_+ \end{array}$$

This inequality is called the Lipschitz condition.

Fundamental Theorem of Differential Equations

- Consider $\dot{x} = f(x,t),\ x(t_0) = x_0$ with $f(x,t)$ piecewise continuous in t and Lipschitz continuous in x. Then there **<u>exists</u>** a **<u>unique</u>** function of time $\phi(\cdot): R_+ \to R^n$ which is almost everywhere satisfying:

$$\phi(t_0) = x_0$$

$$\dot{\phi}(t) = f(\phi(t),t) \qquad \forall t \in R_+ \setminus D$$

where D is the set of discontinuity points for f as a function of t.

Fundamental Theorem of Differential Equations

- Consider, for example,

$$\dot{x} = f(x,u,t) \qquad f : R^n \times R^{n_i} \times R_+ \to R^n$$

$$y = h(x,u,t) \qquad h : R^n \times R^{n_i} \times R_+ \to R^{n_o}$$

If f is Lipschitz continuous in x, continuous in u, and piecewise continuous in t, we are guaranteed that given $x(t_0) = x_0$, $\exists! \, x(t) \in R^n$ satisfying the differential equation. With this, $\exists! \, y(t) \in R^{n_o}$ called the output of the system.

Fundamental Theorem of Differential Equations

- If the Lipschitz condition does not hold, it may be that the solution cannot be continued beyond a certain time.
- Example: $\dot{\xi}(t) = \xi(t)^2 \qquad \xi(0) = \dfrac{1}{c} \qquad c \neq 0$

  where $\xi(t) : R_+ \to R$

  This differential equation had the solution

$$\xi(t) = \frac{1}{c-t} \qquad t \in (-\infty, c)$$

when $t \to c$, $\|\xi(t)\| \to \infty$

which means "finite escape time at *c*"

Bellman-Gronwall Lemma

- Let $u(\cdot)$, $k(\cdot)$ be real-valued, piecewise continuous function on R_+ ; and assume $u(\cdot) \geq 0$, $k(\cdot) > 0$ on R_+. Assume $c_1 \geq 0$, $t_0 \in R_+$. Then, if

$$u(t) \leq c_1 + \int_{t_0}^{t} k(\tau) u(\tau) d\tau$$

then,

$$u(t) \leq c_1 e^{\int_{t_0}^{t} k(\tau) d\tau}$$

- For more details, refer *Linear System Theory*, F. M. Callier and C. A. Desoer, Springer-Verlag. (Appendix B)

Continuous time first order system

$$\boxed{\begin{aligned} \frac{d}{dt}x(t) &= a\,x(t) + b\,u(t) \\ x(t_o) &= x_0 \end{aligned}}$$

- **Solution**:

$$\boxed{x(t) = \underbrace{e^{a(t-t_o)}\,x(t_o)}_{\text{free response}} + \underbrace{\int_{t_o}^{t} e^{a(t-\tau)}\,b\,u(\tau)d\tau}_{\text{forced response}}}$$

Continuous time first order system

- Derivation:

1. Use $$\frac{d}{dt}e^{at} = a\,e^{at}$$

$$\begin{aligned}
\frac{d}{dt}x(t) &= a\,x(t) + b\,u(t) \\
\frac{d}{dt}x(t) - a\,x(t) &= b\,u(t) \\
e^{-at}\,\frac{d}{dt}x(t) - a\,e^{-at}\,x(t) &= e^{-at}\,b\,u(t) \\
\frac{d}{dt}\left\{e^{-at}\,x(t)\right\} &= e^{-at}\,b\,u(t)
\end{aligned}$$

Continuous time first order system

- Derivation:

2. Integrate

$$\begin{aligned}
\int_{t_o}^{t} \frac{d}{d\tau}\left\{e^{-a\tau}\,x(\tau)\right\} d\tau &= \int_{t_o}^{t} e^{-a\tau}\,b\,u(\tau)d\tau \\
e^{-at}\,x(t) - e^{-at_o}\,x(t_o) &= \int_{t_o}^{t} e^{-a\tau}\,b\,u(\tau)d\tau \\
e^{-at}\,x(t) &= e^{-at_o}\,x(t_o) + \int_{t_o}^{t} e^{-a\tau}\,b\,u(\tau)d\tau \\
x(t) &= e^{at}\,e^{-at_o}\,x(t_o) + \int_{t_o}^{t} e^{at}\,e^{-a\tau}\,b\,u(\tau)d\tau
\end{aligned}$$

$$\boxed{x(t) = e^{a(t-t_o)}\,x(t_o) + \int_{t_o}^{t} e^{a(t-\tau)}\,b\,u(\tau)d\tau}$$

Continuous time first order system

$$\begin{aligned} \frac{d}{dt}x(t) &= a\,x(t) + b\,u(t) \\ x(0) &= x_0 \end{aligned}$$

- **Free response**:

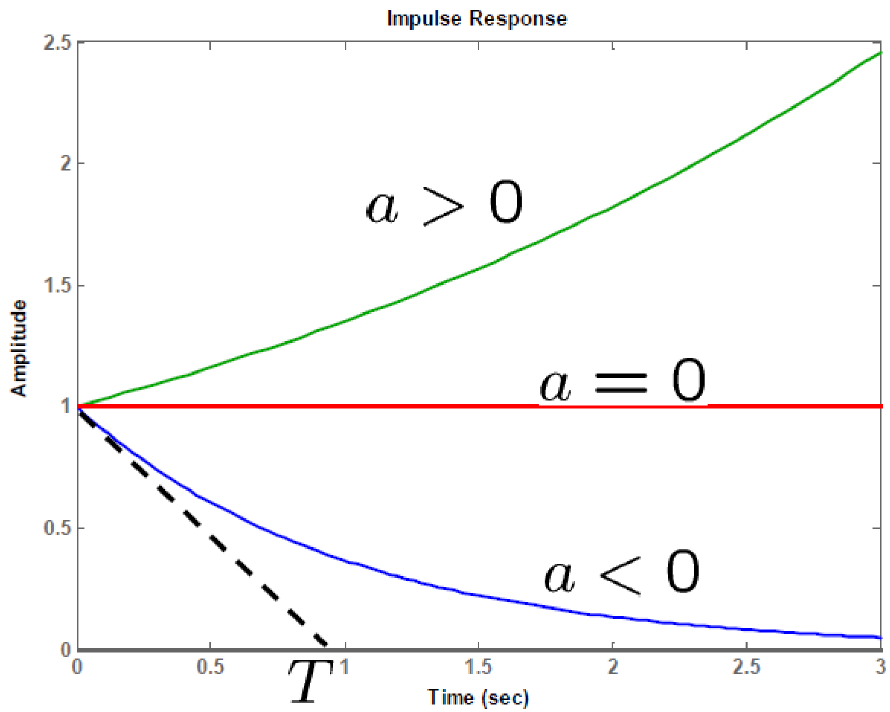
$$x(t) = e^{at}\,x(0)$$

- Time constant:

$$T = \frac{1}{|a|},\ a < 0$$

$$\left.\frac{dx}{dt}\right|_{t=0} = ax_0 = -\frac{1}{T}$$

$$\begin{aligned} x(T) &= e^{-1}\,x(0) \\ &\approx 0.37\,x(0) \end{aligned}$$

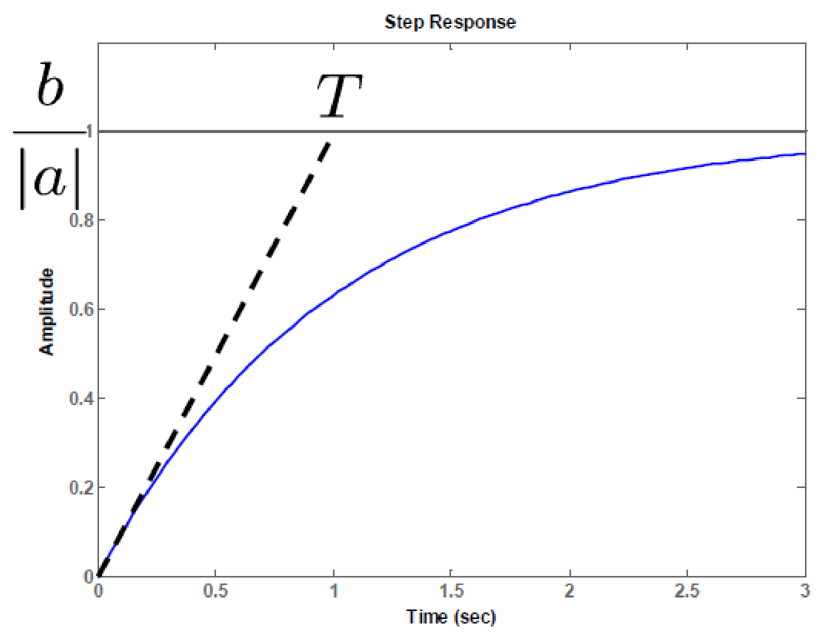


Continuous time first order system

$$\boxed{\frac{d}{dt}x(t) \;=\; a\,x(t) + b\,u(t)}$$

- **Unit step forced response (** $a < 0$ **):**

$$x(t) \;=\; \frac{b}{|a|}(1 - e^{at})$$



- Time constant:

$$T \;=\; \frac{1}{|a|},\; a < 0$$

State transition scalar

- **Scalar case :** $a \in \mathcal{R}$

$$\boxed{e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots \frac{1}{n!}(at)^n + \cdots}$$

- **Transition scalar:** $\boxed{\Phi(t, t_o) = e^{a(t-t_o)}}$

$$\Phi(t,t) = e^{a(t-t)} = 1$$

$$\Phi(t_3,t_2)\Phi(t_2,t_1) = e^{a(t_3-t_2)}\, e^{a(t_2-t_1)} = e^{a(t_3-t_1)} = \Phi(t_3,t_1)$$

$$\Phi(t_2,t_1) = e^{a(t_2-t_1)} = \left(e^{a(t_1-t_2)}\right)^{-1} = \Phi^{-1}(t_1,t_2)$$

Continuous time first order system

$$\boxed{\begin{aligned} \frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) \\ x(t_o) &= x_0 \end{aligned}}$$

$$\begin{aligned} x &\in \mathcal{R}^n \\ A &\in \mathcal{R}^{n\times n} \end{aligned}$$

- **Solution**:

$$\boxed{x(t) = \underbrace{e^{A(t-t_o)}\,x(t_o)}_{\text{free response}} + \underbrace{\int_{t_o}^{t} e^{A(t-\tau)}\,B\,u(\tau)d\tau}_{\text{forced response}}}$$

Matrix exponential

- **Matrix case** : $A \in \mathcal{R}^{n\times n}$

$$\boxed{e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^n t^n + \cdots \in \mathcal{R}^{n\times n}}$$

$$A^n = \underbrace{A\cdots A}_{n \text{ times}}$$

$$\left(e^{At}\right)^{-1} = e^{A(-t)}$$

$$e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$$

$$e^{At}A = Ae^{At}$$

$$\boxed{e^{At}e^{Bt} = e^{(A+B)t} \Leftrightarrow AB = BA}$$

State transition matrix

- **Matrix case** : $A \in \mathcal{R}^{n\times n}$

$$\boxed{e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^nt^n + \cdots}$$

- **Transition Matrix** : $\boxed{\Phi(t,t_o) = e^{A(t-t_o)} \in \mathcal{R}^{n\times n}}$

$$\Phi(t,t) = e^{A(t-t)} = I_n$$

$$\Phi(t_3,t_2)\Phi(t_2,t_1) = e^{A(t_3-t_2)}\,e^{A(t_2-t_1)} = e^{A(t_3-t_1)} = \Phi(t_3,t_1)$$

$$\Phi(t_2,t_1) = e^{A(t_2-t_1)} = \left(e^{A(t_1-t_2)}\right)^{-1} = \Phi^{-1}(t_1,t_2)$$

$x(t) = e^{At}x_0$ → *i*-th column of the solution matrix is the response of the system for $x_0 = [0, 0, \ldots, 1, 0, \ldots, 0]$ (i-th element is 1)

Computation of the Solution Matrix

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^n t^n + \cdots$$

- **Case 1** : Diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$
- **Case 2** : Jordan canonical form $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$
- **Case 3** : Complex eigenvalues $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

Computation of the Solution Matrix

- **Case 1** : Diagonal matrix

Notice that

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \Longrightarrow \quad A^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

Thus, all matrices in the series expansion

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^nt^n + \cdots$$

are diagonal!

Computation of the Solution Matrix

- **Case 1** : Diagonal matrix

Notice that

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \Rightarrow \quad A^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

$$e^{At} \;=\; I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^n t^n + \cdots$$

$$\Rightarrow \qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

Computation of the Solution Matrix

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^nt^n + \cdots$$

- **Case 1** : $\boxed{A = \text{Diag}\{\lambda_1, \cdots, \lambda_n\}}$ $(\lambda_i \in \mathcal{C})$

$$\boxed{e^{At} = \text{Diag}\{e^{\lambda_1 t}, \cdots, e^{\lambda_n t}\}}$$

Since: $A^n = \text{Diag}\{\lambda_1^n, \cdots, \lambda_n^n\}$

$$\text{Diag}\{\lambda_1, \lambda_2\} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Computation of the Solution Matrix

$$e^{At} \;=\; I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^nt^n + \cdots$$

- **Case 2** : Jordan canonical form $A \in \mathcal{R}^{n\times n}$

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & \cdots & 0 & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$e^{At} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{3!}t^3 & \cdots & \frac{1}{n!}t^n \\ 0 & 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(n-1)!}t^{(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & t \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution matrix for the Jordan canonical form

Jordan canonical form, $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, notice that

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N}$$

N is nilpotent, i.e $\det(N) = 0$ and

$$N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = N^4 = N^5 \cdots$$

Solution matrix for the Jordan canonical form

I_3 is the identity matrix , and $\boxed{A = \lambda I_3 + N}$

Notice that, because $I_3 N = N$ and $\lambda \in \mathcal{R}$

$$\lambda I_3 N = N \lambda I_3$$

$$\Rightarrow e^{(\lambda I_3 + N)t} = e^{\lambda I_3 t} e^{N t}$$

$$\boxed{e^{A t} = e^{\lambda t} e^{N t}}$$

Solution matrix for the Jordan canonical form

Also, since N is nilpotent, with

$$N^3 = N^4 = N^5 = \cdots = 0\, I_3$$

$$\boxed{e^{Nt} = I_3 + Nt + \frac{1}{2}N^2t^2}$$

$$e^{Nt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{t^2}{2}$$

$$= \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Solution matrix for the Jordan canonical form

Therefore, $e^{At} = e^{\lambda t} e^{Nt}$

$$e^{At} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}} \Rightarrow \boxed{e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}}$$

Computation of the Solution Matrix

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^n t^n + \cdots$$

- **Case 3** : Complex eigenvalues, $(\lambda = \sigma \pm \omega j)$

$$\boxed{A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}$$

$$\boxed{e^{At} = e^{\sigma t} \underbrace{\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}}_{\text{rotation matrix}}}$$

Solution matrix for $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

Notice that $A = \underbrace{\begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}}_{\sigma I_2} + \underbrace{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}}_{S}$

$$\sigma\, I_2\, S = S\, \sigma\, I_2 \Rightarrow e^{(\sigma I_2 + S)\, t} = e^{\sigma I_2\, t}\, e^{S\, t}$$

$$\boxed{e^{A\, t} = e^{\sigma\, t}\, e^{S\, t}}$$

$S = \underbrace{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}}_{\text{skew-symmetric}}$ $\qquad S^T = -S$

Solution matrix for $S = \underbrace{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}}_{\text{skew-symmetric}} = -S^T$

Evaluating the series expansion,

$$e^{St} = \begin{bmatrix} \left(\sum^{\infty}_{\substack{n \geq 0 \\ n\ even}} \frac{(-1)^{\frac{n}{2}}}{n!} (\omega t)^n \right) & \left(\sum^{\infty}_{\substack{n \geq 0 \\ n\ odd}} \frac{(-1)^{\frac{(n-1)}{2}}}{n!} (\omega t)^n \right) \\ -\left(\sum^{\infty}_{\substack{n \geq 0 \\ n\ odd}} \frac{(-1)^{\frac{(n-1)}{2}}}{n!} (\omega t)^n \right) & \left(\sum^{\infty}_{\substack{n \geq 0 \\ n\ even}} \frac{(-1)^{\frac{n}{2}}}{n!} (\omega t)^n \right) \end{bmatrix}$$

$$e^{St} = \underbrace{\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}}_{\text{rotation matrix}}$$

Computation of the Solution Matrix

$$e^{At}$$

- **Case 1** : Diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

$$\boxed{\lim_{t \to \infty} e^{\lambda_i t} = 0 \Leftrightarrow \text{Real}(\lambda_i) < 0}$$

Computation of the Solution Matrix

$$e^{At}$$

- **Case 2** : Jordan canonical form

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

$$\boxed{\lim_{t \to \infty} t^m e^{\lambda t} = 0 \Leftrightarrow \mathrm{Real}(\lambda) < 0}$$

Computation of the Solution Matrix

$$e^{At}$$

- **Case 3** : complex eigenvalues

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \quad e^{At} = e^{\sigma\, t} \underbrace{\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}}_{\text{rotation matrix}}$$

$$\boxed{\lim_{t \to \infty} e^{\sigma t} \cos(\omega t) = 0 \Leftrightarrow \text{Real}(\sigma) < 0}$$

Discrete time first order system

$$\boxed{\begin{aligned} x(k+1) &= a\,x(k) + b\,u(k) \\ x(k_0) &= x_{k_0} \end{aligned}}$$

- **Solution**:

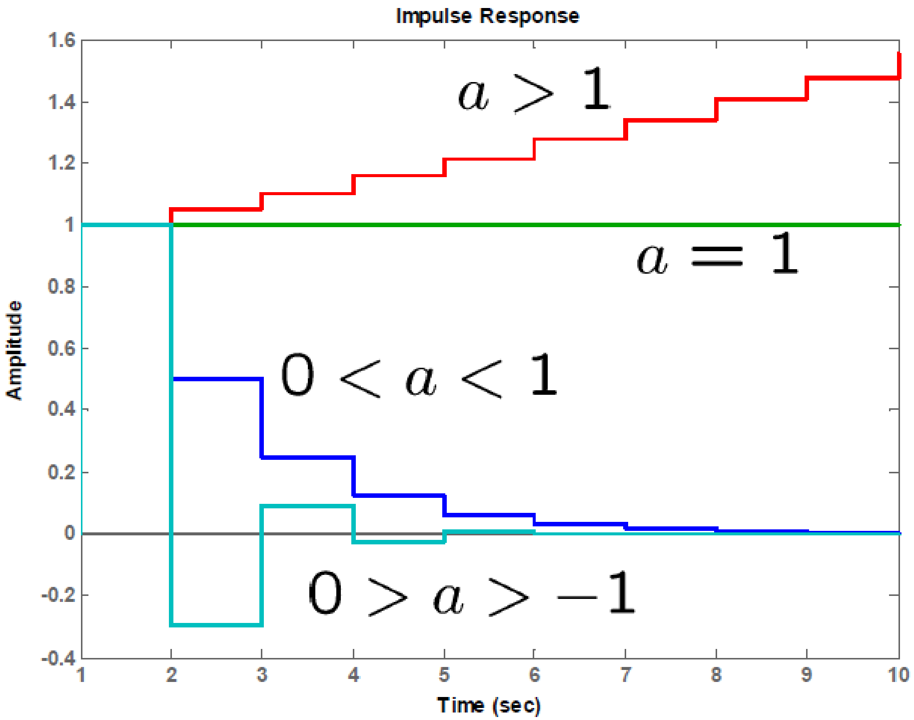
$$\boxed{x(k) \;=\; \underbrace{a^{(k-k_o)}\,x(k_o)}_{\text{free response}} + \underbrace{\sum_{j=k_o}^{k-1} a^{(k-1-j)}\,b\,u(j)}_{\text{forced response}}}$$

Discrete time first order system

$$\boxed{\begin{aligned} x(k+1) &= a\,x(k) + b\,u(k) \\ x(0) &= x_0 \end{aligned}}$$

- **Free response:**

$$x(k) \;=\; a^k\,x(0)$$



Discrete time nth order system

$$\boxed{\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ x(k_0) &= x_0 \end{aligned}}$$

$$\begin{aligned} x &\in \mathcal{R}^n \\ A &\in \mathcal{R}^{n\times n} \end{aligned}$$

- **Solution**:

$$\boxed{x(k) = \underbrace{A^{(k-k_o)}\,x(k_o)}_{\text{free response}} + \underbrace{\sum_{j=k_o}^{(k-1)} A^{(k-1-j)}\,B\,u(j)}_{\text{forced response}}}$$

$$A^k = \underbrace{A\cdots A}_{k\,\text{times}}$$

Discrete time state transition matrix

- **Matrix case** : $A \in \mathcal{R}^{n \times n}$ $\quad A^k = \underbrace{A \cdots A}_{k\ \text{times}}$

- **Transition Matrix** :

$$\boxed{\Phi(k, k_o) = A^{(k-k_o)} \in \mathcal{R}^{n \times n}}$$

$$\Phi(k, k) = A^{(k-k)} = I_n$$

$$\left.\begin{array}{l} \Phi(k_3, k_2)\Phi(k_2, k_1) = \Phi(k_3, k_1) \\ \Phi(k_2, k_1) = \Phi^{-1}(k_1, k_2) \end{array}\right\} \Leftrightarrow \det(A) \neq 0$$

Computation of the Solution Matrix

$$A^k$$

- **Case 1** : Diagonal matrix $$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- **Case 2** : Jordan canonical form $$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

- **Case 3** : Complex eigenvalues $$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Computation of the Solution Matrix

$$A^k$$

- **Case 1** : Diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$$

$$\boxed{\lim_{k \to \infty} \lambda_i^k = 0 \Leftrightarrow |\lambda_i| < 1}$$

Computation of the Solution Matrix

$$A^k$$

- **Case 2** : Jordan canonical form $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

$$A^k = \begin{bmatrix} \lambda^k & k\lambda^{(k-1)} & \frac{k(k-1)}{2}\lambda^{(k-2)} \\ 0 & \lambda^k & k\lambda^{(k-1)} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

$$\boxed{\lim_{k\to\infty} k^m \lambda^k = 0 \Leftrightarrow |\lambda| < 1}$$

Computation of the Solution Matrix

$$A^k$$

- **Case 3** : Complex eigenvalues $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

$$A^k = r^k \begin{bmatrix} \cos(\theta k) & \sin(\theta k) \\ -\sin(\theta k) & \cos(\theta k) \end{bmatrix}$$

$$r = \sqrt{\sigma^2 + \omega^2}$$

$$\theta = \tan^{-1}\left(\frac{\omega}{\sigma}\right)$$

$$(\lambda = \sigma + \omega j)$$