[MEN573]
Advanced Control Systems I

Lecture 20 - Properties of
Optimal Linear Quadratic Regulators (LQR)

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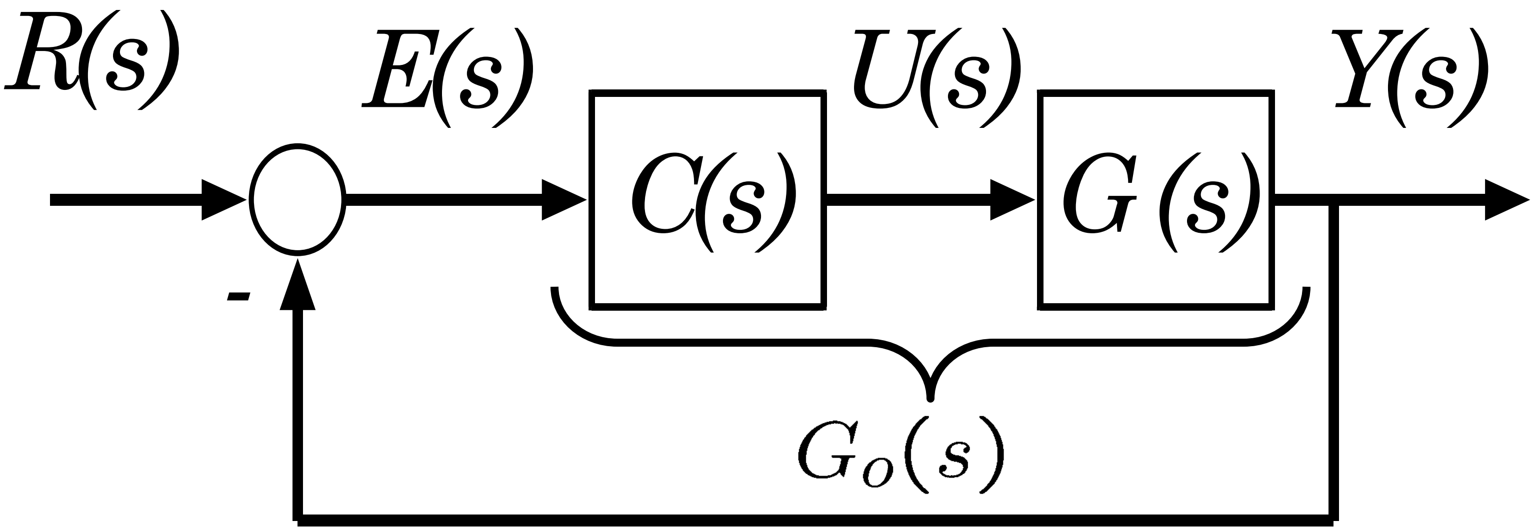
Outline

- Brief review of SISO control systems
  - gain and phase margins
  - root locus
- LQR problem
- Return difference equality for LQR
  - SISO systems
- Guaranteed gain and phase margins of LQR
- Symmetric root locus for a SISO LQR

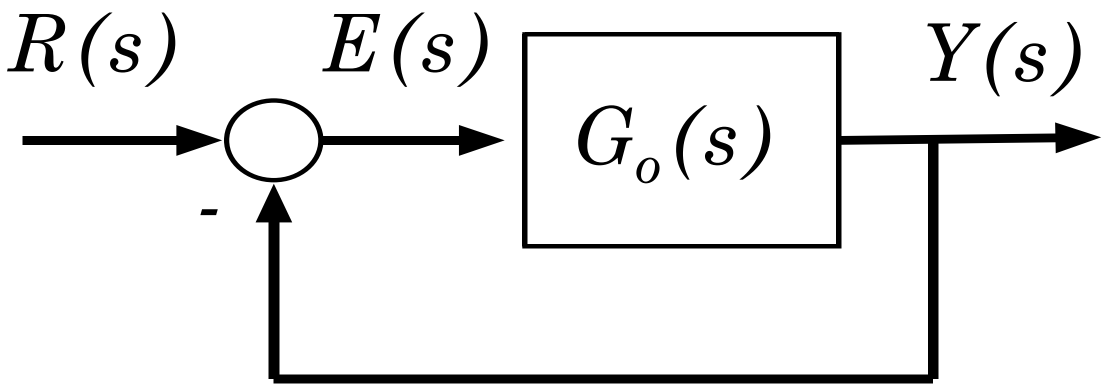
Basic Review: SISO Feedback System



Open loop Transfer Function:

$$G_o(s) = C(s)\,G(s) \quad = \frac{B(s)}{A(s)}$$

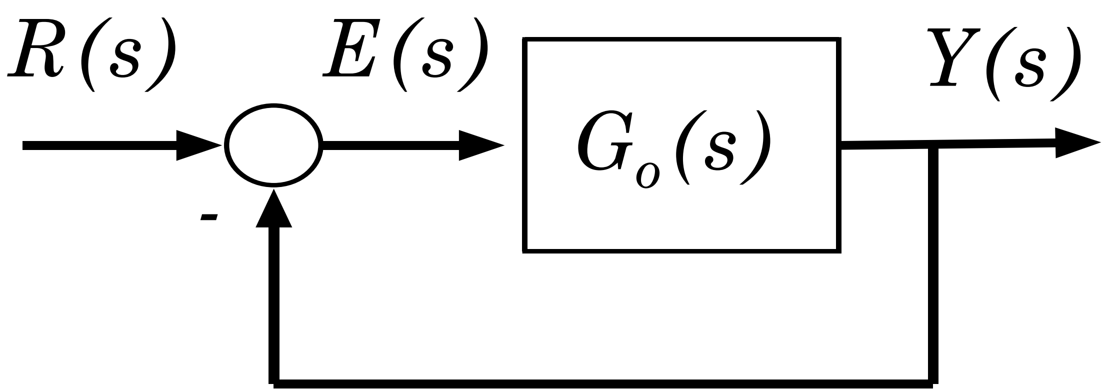
Basic Review: SISO Feedback System



Open loop Transfer Function:

$$G_o(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + \cdots + b_o}{s^n + \cdots + a_o}$$

Basic Review: SISO Feedback System



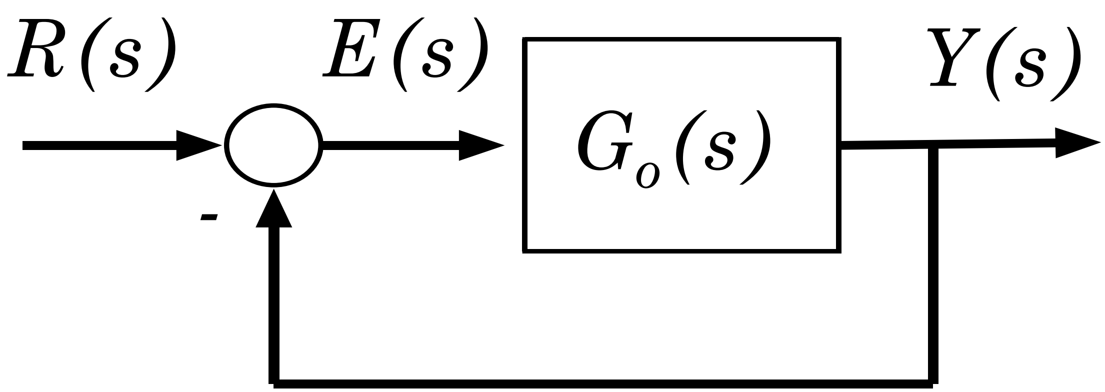
Open loop Transfer Function:

open loop zeros

$$G_o(s) = \frac{B(s)}{A(s)} = \frac{b_m(s - z_{o1}) \cdots (s - z_{om})}{(s - p_{o1}) \cdots (s - p_{on})}$$

open loop poles

Basic Review: SISO Feedback System



Closed Transfer Functions:

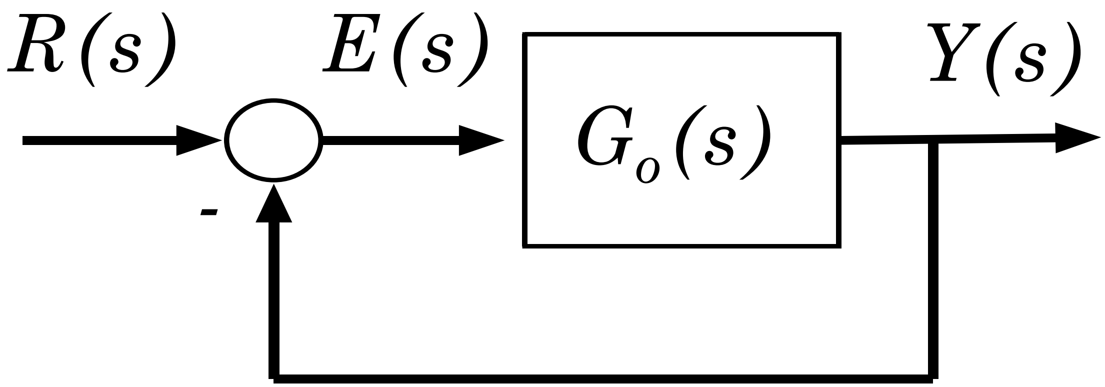
$$\frac{Y(s)}{R(s)} = \frac{G_o(s)}{1+G_o(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G_o(s)}$$

Return difference:

$$1+G_o(s) = D(s)$$

Basic Review: SISO Feedback System

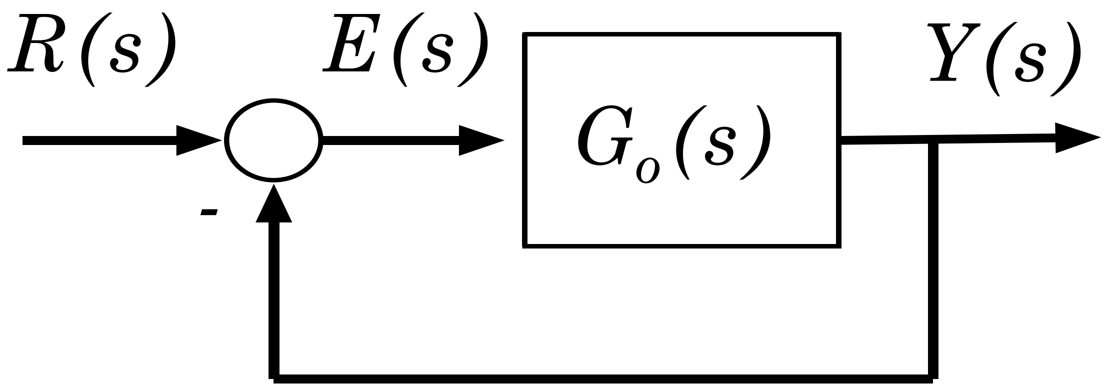


Closed Loop poles: roots of the return difference

$$D(s) = 1 + G_o(s) = 0$$

$$D(s) \;=\; 1 + \frac{B(s)}{A(s)} \;=\; \frac{\overbrace{A(s) + B(s)}^{A_c(s) = 0}}{A(s)} = 0$$

Basic Review: SISO Feedback System



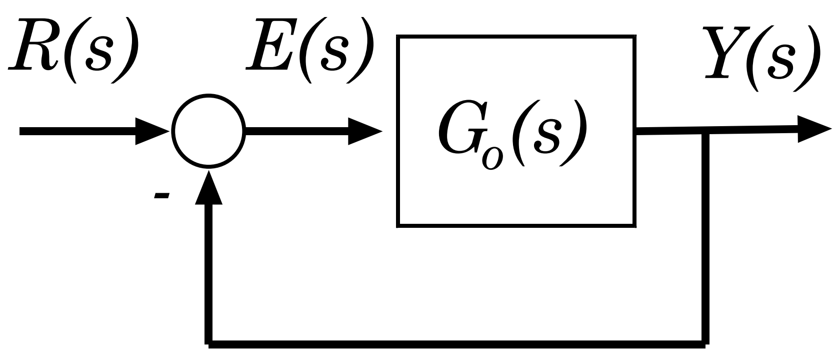
Return difference: $D(s) = 1 + G_o(s)$

close loop poles

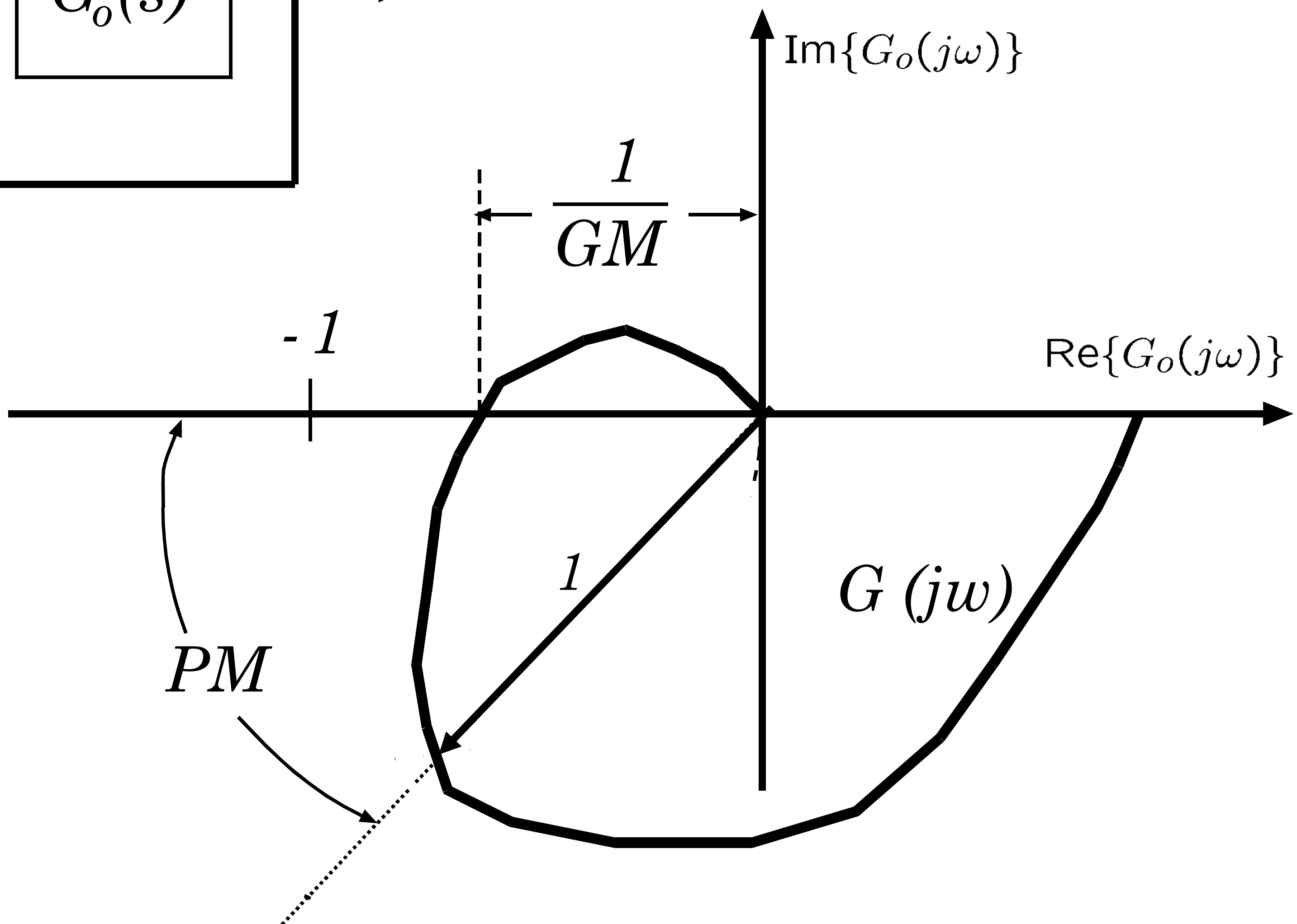
$$D(s) = \frac{A_c(s)}{A(s)} = \frac{(s - p_{c1}) \cdots (s - p_{cn})}{(s - p_{o1}) \cdots (s - p_{on})}$$

open loop poles

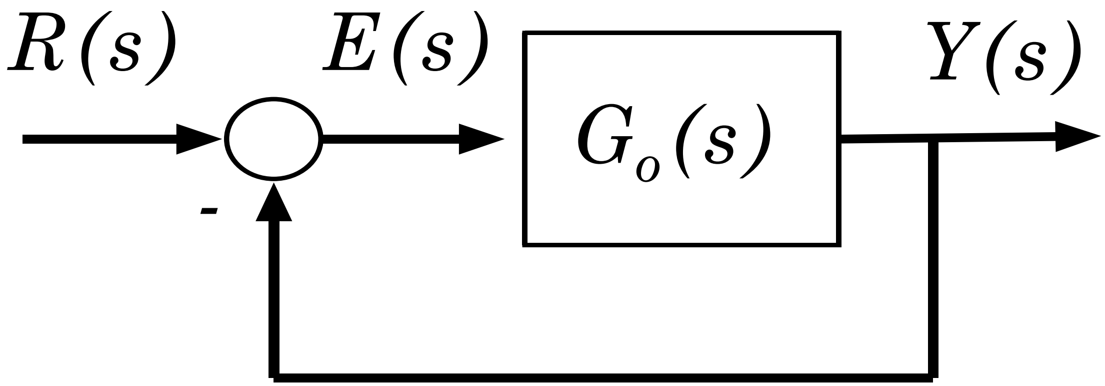
Basic Review: Gain and Phase Margins



Nyquist plot of $G_o(j\omega)$



Basic Review: Root Locus

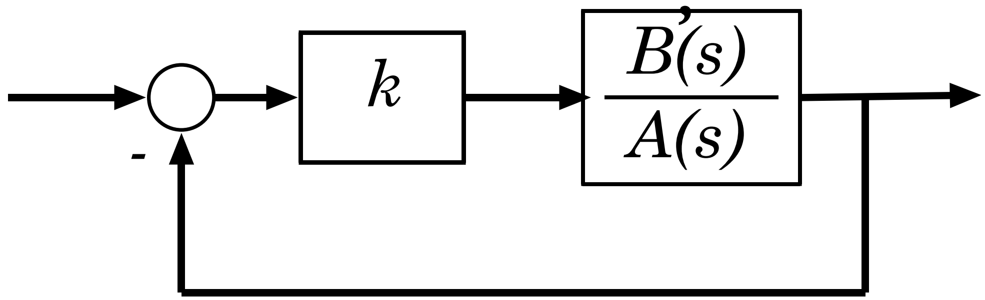


Closed Loop poles: roots of the return difference

$$D(s) = \frac{A_c(s)}{A(s)} = 1 + \frac{B(s)}{A(s)} = 1 + \frac{\overbrace{b_m}^{k} s^m + \cdots + b_o}{s^n + \cdots + a_o}$$

$$= 1 + k\,\frac{s^m + \cdots + b_o^{'}}{s^n + \cdots + a_o} = 1 + k\,\frac{B^{'}(s)}{A(s)}$$

Basic Review: Root Locus



- **Root Locus:** How close loop poles change with $\boldsymbol{k}$

closed loop poles

open loop zeros

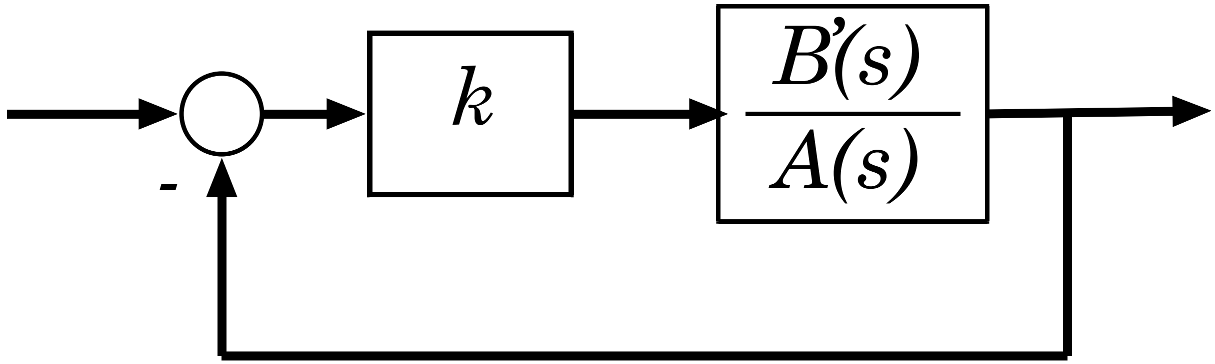
$$\frac{A_c(s)}{A(s)} = 1 + k\frac{B'(s)}{A(s)}$$

open loop poles

- **All polynomials must be monic,**

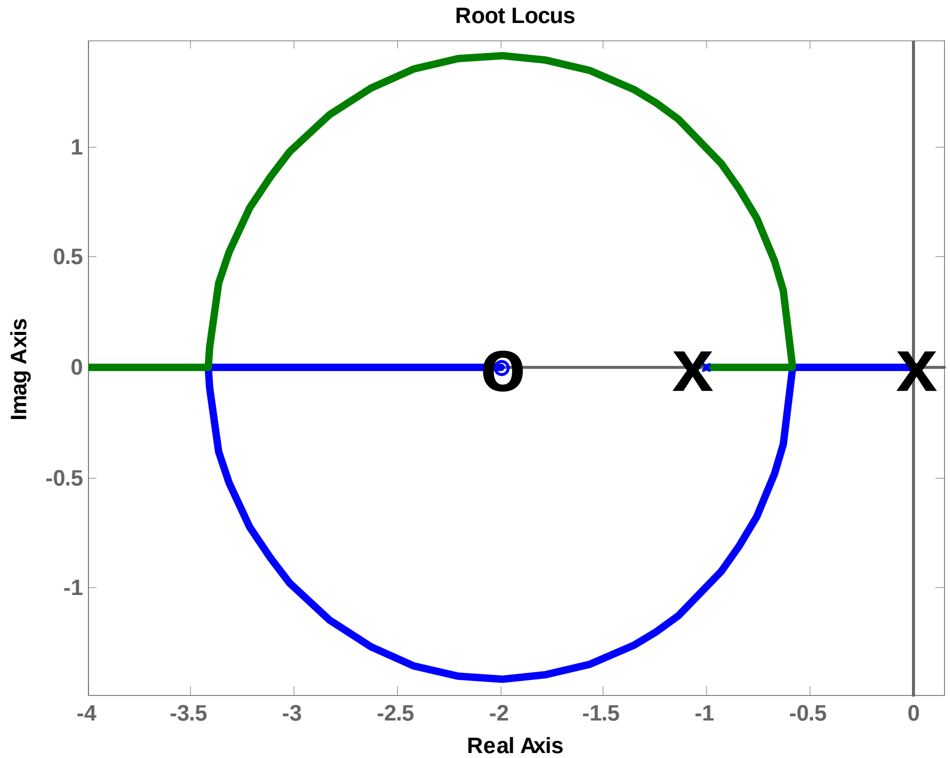
$$A(s) = s^n + \cdots + a_o$$

Basic Review: Root Locus



- **Example:**

$$\frac{B'(s)}{A(s)} = \frac{s+2}{s(s+1)}$$



Infinite Horizon LQ Regulator (LQR)

In the remainder we will analyze LQRs and show:

- LQR exhibit some nice robustness properties
  - guaranteed gain and phase margins
- Closed loop eigenvalues of LQR can be plotted as the function of the control input weight
  - Symmetric root locus techniques

Infinite Horizon LQ Regulator (LQR)

Consider a controllable and observable nth order LTI system:

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t) \qquad\qquad x(0) = x_o$$

Under the optimal control:

$$u(t) \;=\; -K\,x(t)$$

which minimizes the cost functional:

$$J = \frac{1}{2}\int_0^\infty \left\{x^T Q\,x + u^T R\,u\right\} dt$$

$$Q = Q^T \succeq 0 \qquad\qquad R = R^T \succ 0$$

Optimal LQR

The optimal control law is given by:

$$u(t) \;=\; -K\,x(t)$$

where:

$$K \;=\; R^{-1}\,B^T\,P$$

and $\boldsymbol{P}$ satisfies the following ***Algebraic Riccati Equation*** (ARE) :

$$0 \;=\; A^T P + P\,A + C^T C - P\,B\,R^{-1}\,B^T P$$

Cost function

Notice that the cost functional

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T \underbrace{Q}_{Q = C^T C} x + u^T R\, u \right\} dt$$

defining,

$$y \;=\; C\, x$$

we have,

$$J = \frac{1}{2}\int_0^\infty \left\{ y^T y + u^T R\, u \right\} dt$$

Cost function

$$J = \frac{1}{2}\int_0^\infty \left\{ y^T\, y + u^T\, R\, u \right\} dt$$

Where,

$$y \;=\; C\,x \qquad\qquad Q = C^T\, C$$

define the transfer function :

$$G(s) \;=\; C\,\underbrace{(sI - A)^{-1}}_{\phi(s) = (sI - A)^{-1} = \mathcal{L}\{\phi(t)\} = \mathcal{L}\{e^{At}\}}\,B = \; C\,\Phi(s)\,B$$

Optimal LQR

The closed loop system,

$$\begin{aligned}\dot{x}(t) &= A\,x(t) + B\,u(t) \\ u(t) &= -K\,x(t) + v(t)\end{aligned}$$

$$v = 0$$

exogenous reference input

Output (for cost function)

$$y = C\,x$$

$$J = \frac{1}{2}\int_0^{\infty} \left\{ y^T\,y + u^T\,R\,u \right\}\,dt$$

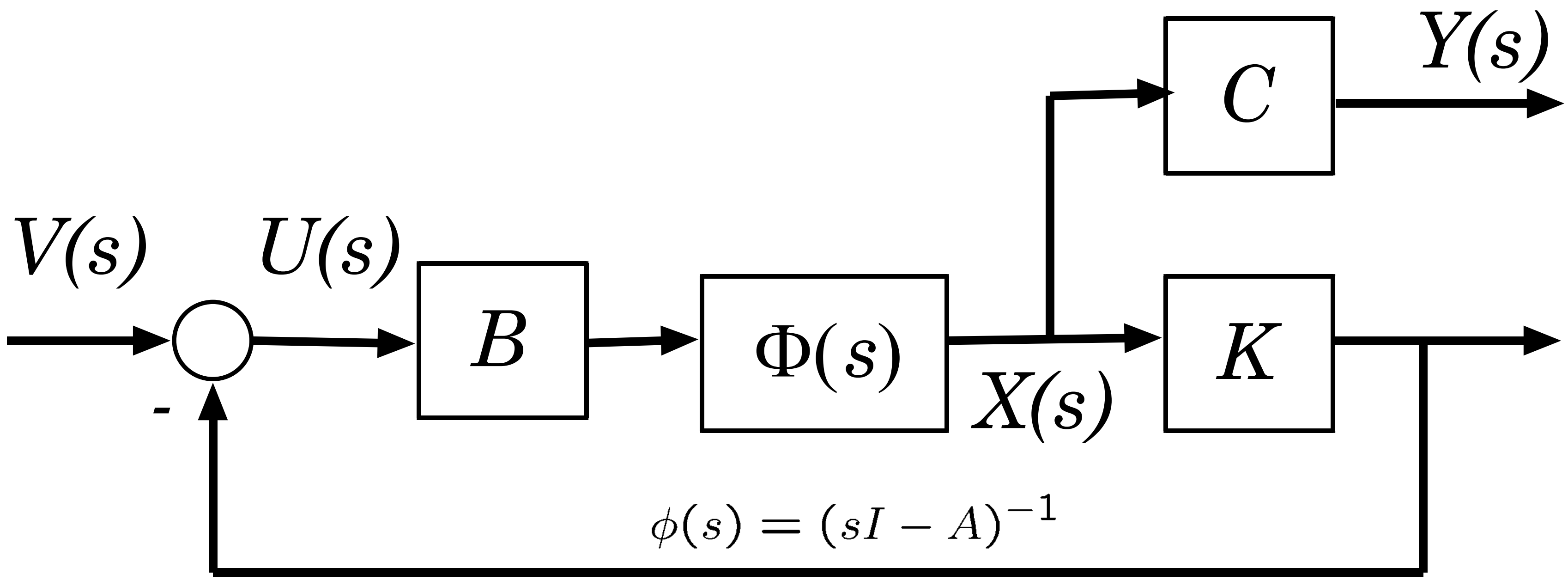
Optimal LQR

The closed loop system,

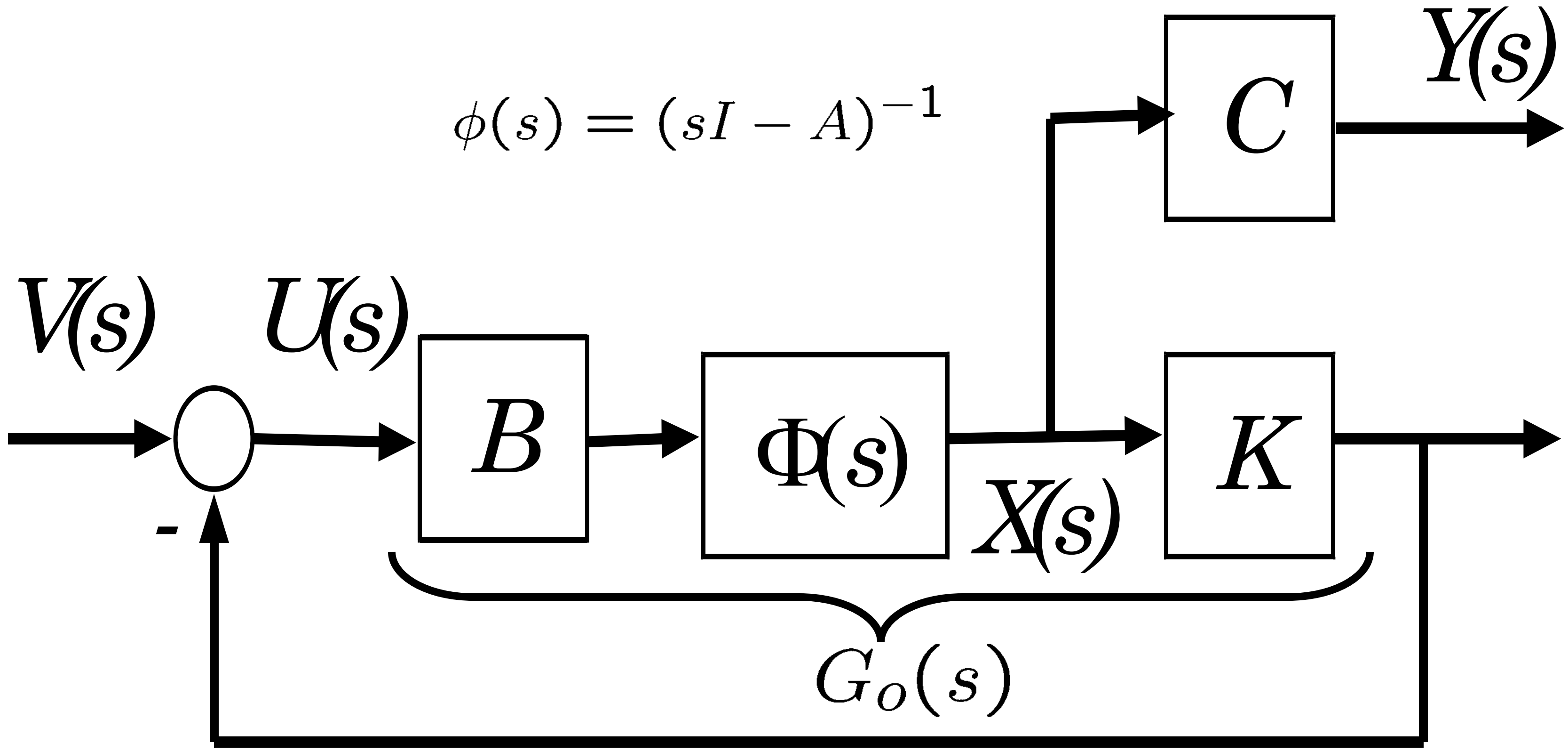
$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$u(t) = -K\,x(t) + v(t) \qquad y = C\,x$$

Is represented by the following block diagram:



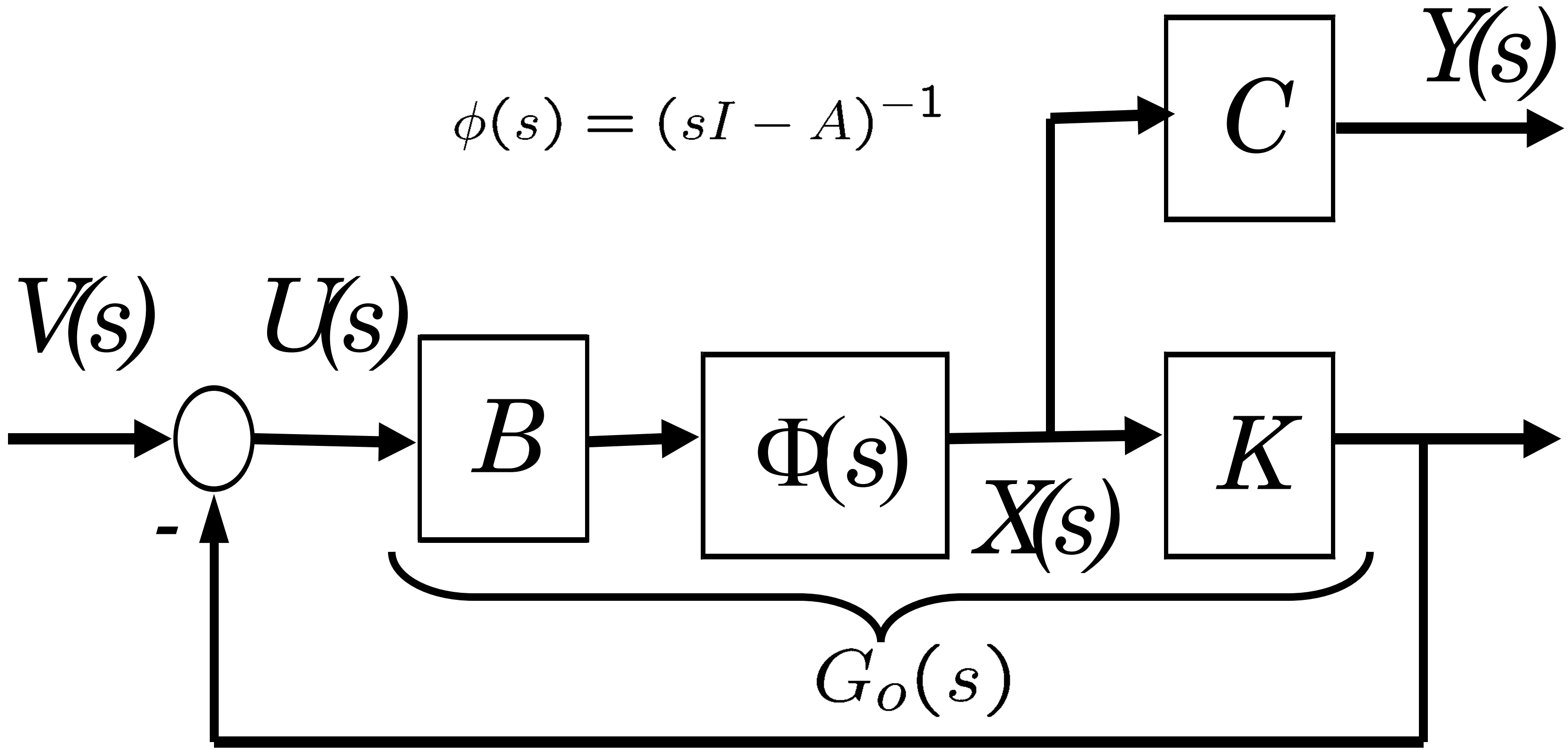
Open Loop & Closed Loop Transfer Functions of the Optimal LQR



The open loop transfer function:

$$G_O(s) = K\Phi(s)B$$

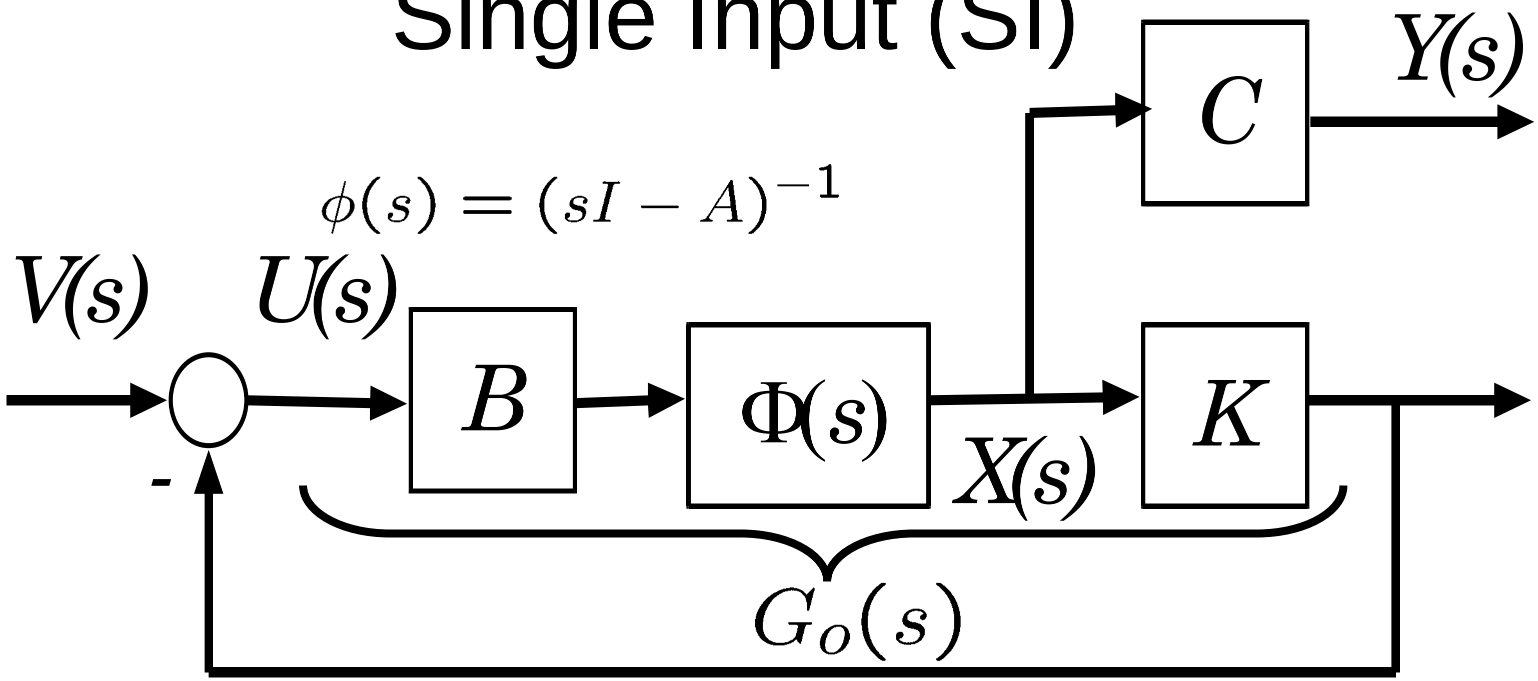
Open Loop & Closed Loop Transfer Functions of the Optimal LQR



The closed loop transfer function from *V(s)* to *U(s)*:

$$U(s) = (I + K\Phi(s)B)^{-1} V(s)$$

Single Input (SI)



The open loop transfer function:

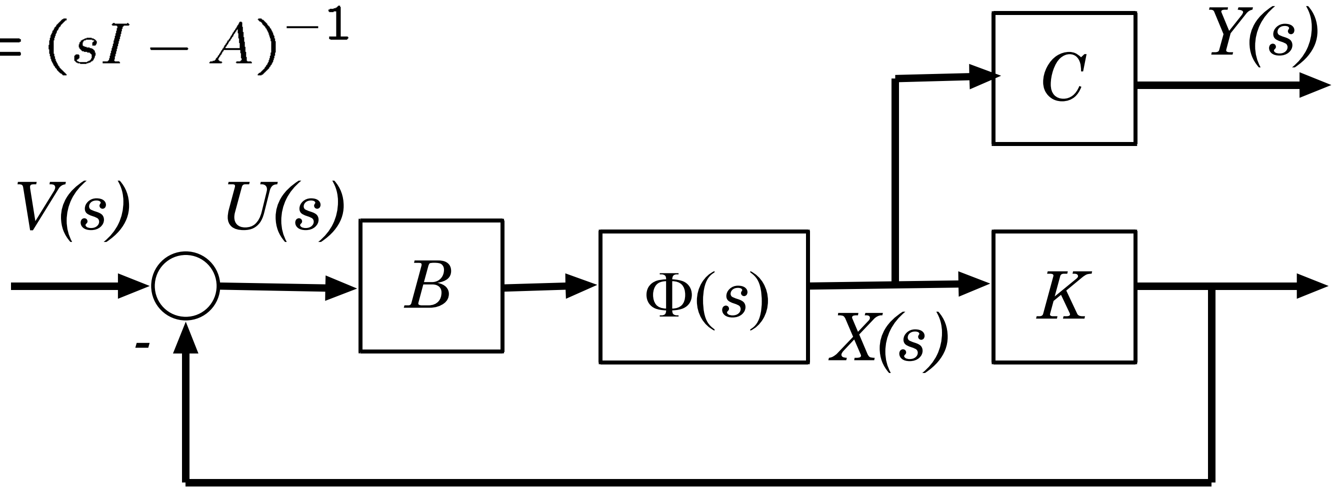
$$G_o(s) = K\Phi(s)B$$

The closed loop transfer function from *V(s)* to *U(s)*:

$$\frac{U(s)}{V(s)} = \frac{1}{(1 + G_o(s))} = \frac{1}{(1 + K\Phi(s)B)}$$

Return difference

$$\phi(s) = (sI - A)^{-1}$$



Return difference:

$$D(s) = (I + G_o(s))$$

Open loop transfer function:

$$G_o(s) = K\Phi(s)B$$

TF from $U(s)$ to $Y(s)$:

$$G(s) = C\Phi(s)B$$

Optimal LQR

The cost functional can be rewritten as:

$$J = \frac{1}{2}\int_0^\infty \left\{ y^T\, y + u^T\, R\, u \right\}\, dt$$

Where,

$$\begin{aligned} y &= C\, x \\ u &= -K\, x \end{aligned}$$

$$\phi(s) = (sI - A)^{-1}$$

Return difference:

$$D(s) = (I + K\Phi(s)B)$$

TF from *U(s)* to *Y(s)*:

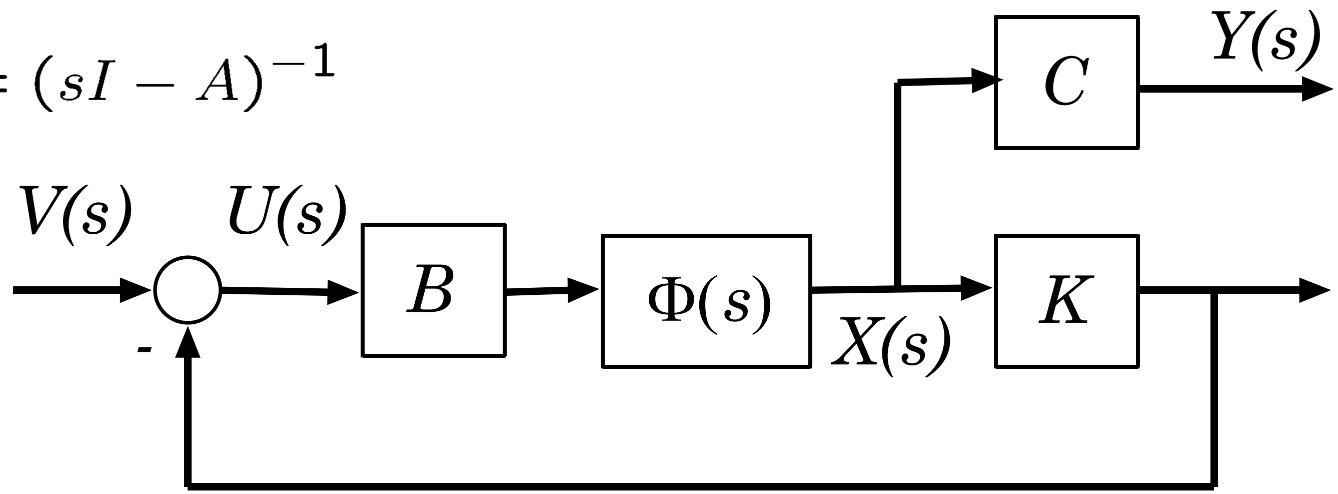
$$G(s) = C\Phi(s)B$$

Open loop transfer function:

$$G_o(s) = K\Phi(s)B$$

Return difference equality

$$\phi(s) = (sI - A)^{-1}$$



Return difference equality:

$$\boxed{\left(I + G_o^T(-s)\right) R \left(I + G_o(s)\right) = R + G^T(-s)\, G(s)}$$

Open loop transfer function:

$$G_o(s) = K\Phi(s)B$$

TF from *U(s)* to *Y(s)*:

$$G(s) = C\Phi(s)B$$

Return difference equality (SISO)

$$(1+G_o(s))\,R\,(1+G_o(s)) = R + G(-s)\,G(s)$$

Multiply both sides by $\dfrac{1}{R}$

$$\boxed{(1+G_o(-s)\,)(1+G_o(s)\,) = 1+\frac{1}{R}G(-s)\,G(s)}$$

Open loop transfer function:

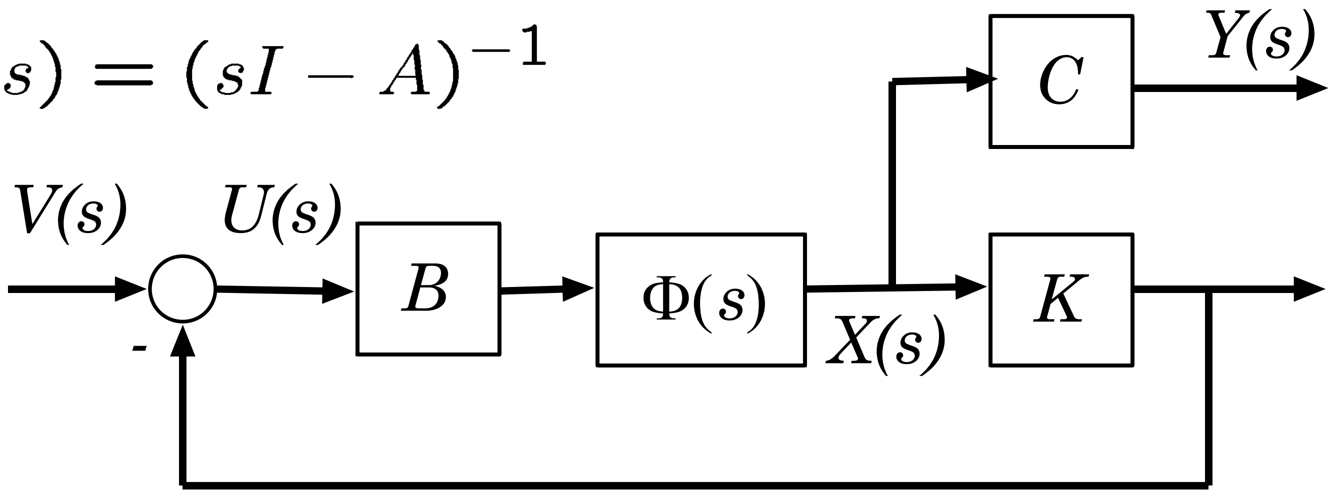
$$G_o(s) = K\Phi(s)B$$

TF from *U(s)* to *Y(s)*:

$$G(s) = C\Phi(s)B$$

Return difference equality (SISO)

$$\Phi(s) = (sI - A)^{-1}$$



Return difference equality:

$$\boxed{(1 + G_o(-s)\,)(1 + G_o(s)\,) = 1 + \frac{1}{R} G(-s)\, G(s)}$$

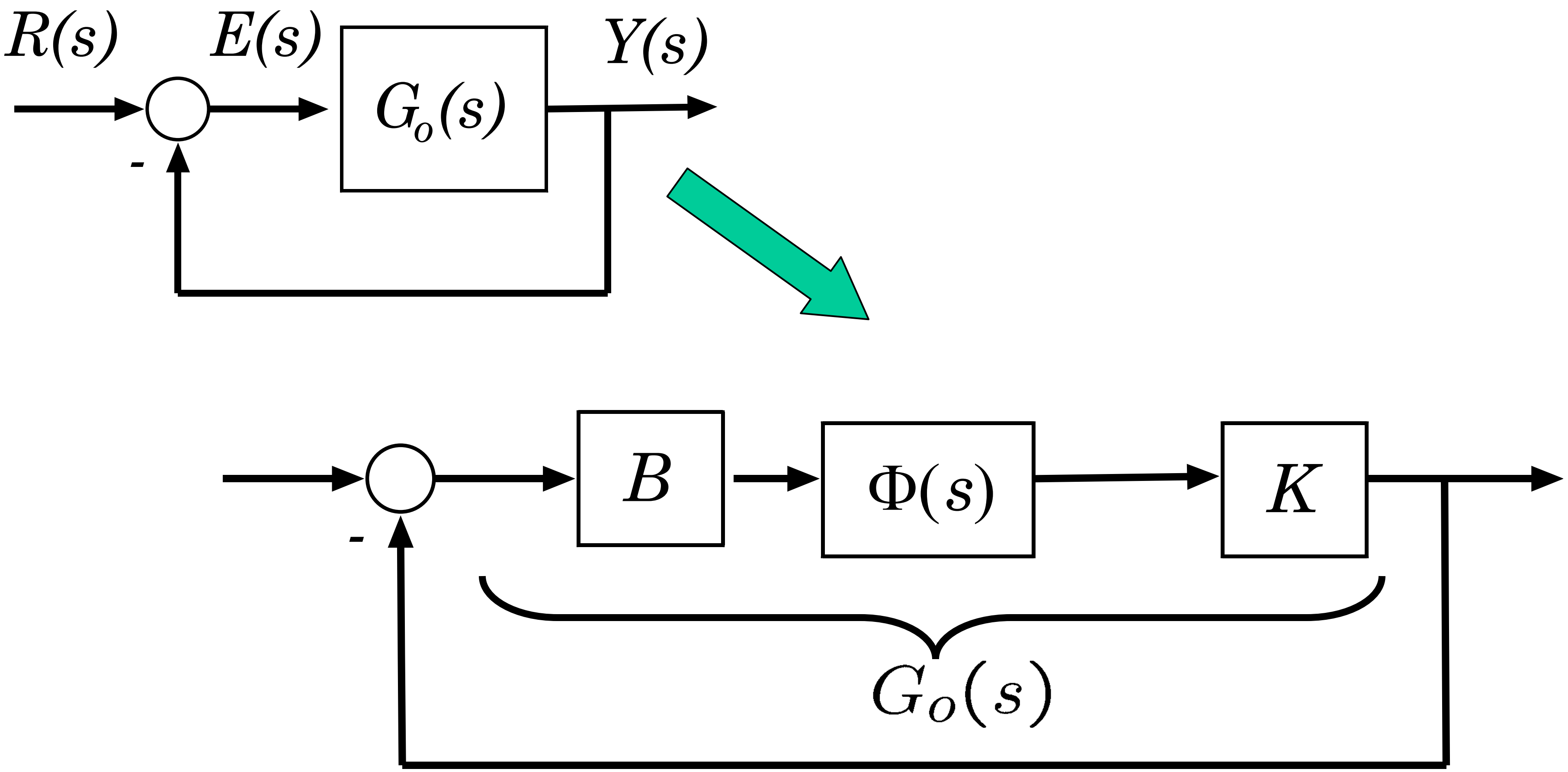
Open loop transfer function:

$$G_o(s) = K\Phi(s)B$$

TF from $U(s)$ to $Y(s)$:

$$G(s) = C\Phi(s)B$$

Gain and Phase Margins for LQR



$$(1 + G_o(-s))(1 + G_o(s)) = 1 + \frac{1}{R} G(-s)\, G(s)$$

Stability Margins of SISO LQR

$$(1+G_o(-s))(1+G_o(s)) = 1 + \frac{1}{R} G(-s)\, G(s)$$

Set $s = j\, w$:

$$\underbrace{\underbrace{\underbrace{(1+G_o(-j\omega))}_{D^*(j\omega)}\;\underbrace{(1+G_o(j\omega))}_{D(j\omega)}}_{|D(j\omega)|^2}}_{|1+G_o(j\omega)|^2} = \underbrace{1+\frac{1}{R}\underbrace{G(-j\omega)}_{G^*(j\omega)}\;\underbrace{G(j\omega)}_{G(j\omega)}}_{1+\frac{1}{R}|G(j\omega)|^2}$$

Stability Margins of SISO LQR

$$(1+G_o(-s)\,)(1+G_o(s)\,)=1+\frac{1}{R}G(-s)\,G(s)$$

Set $s = j\,w$:

$$\underbrace{(1+G_o(-j\omega))}_{D^*(j\omega)}\,\underbrace{(1+G_o(j\omega))}_{D(j\omega)}=1+\frac{1}{R}\underbrace{G(-j\omega)}_{G^*(j\omega)}\,\underbrace{G(j\omega)}_{G(j\omega)}$$

$$\boxed{|(1+G_o(j\omega))|^2=1+\frac{1}{R}|G(j\omega)|^2}$$

Stability Margins of SISO LQR

$$\underbrace{|(1+G_o(j\omega))|^2 = 1 + \frac{1}{R}|G(j\omega)|^2}_{\geq 0}$$

$$\Downarrow$$

$$\boxed{|(1+G_o(j\omega))|^2 \geq 1}$$

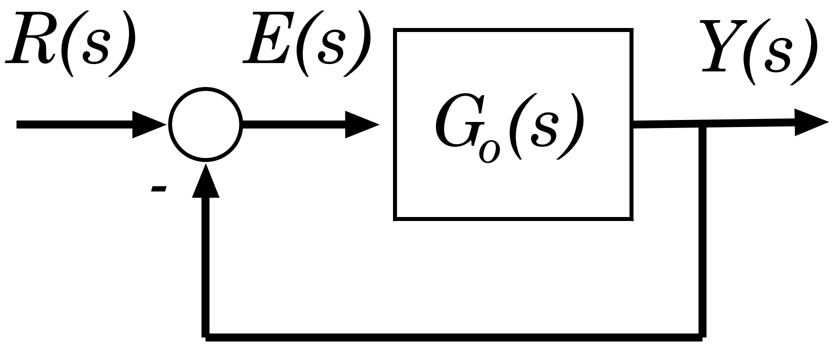
Open loop transfer function:

$$G_o(s) = K\Phi(s)B$$

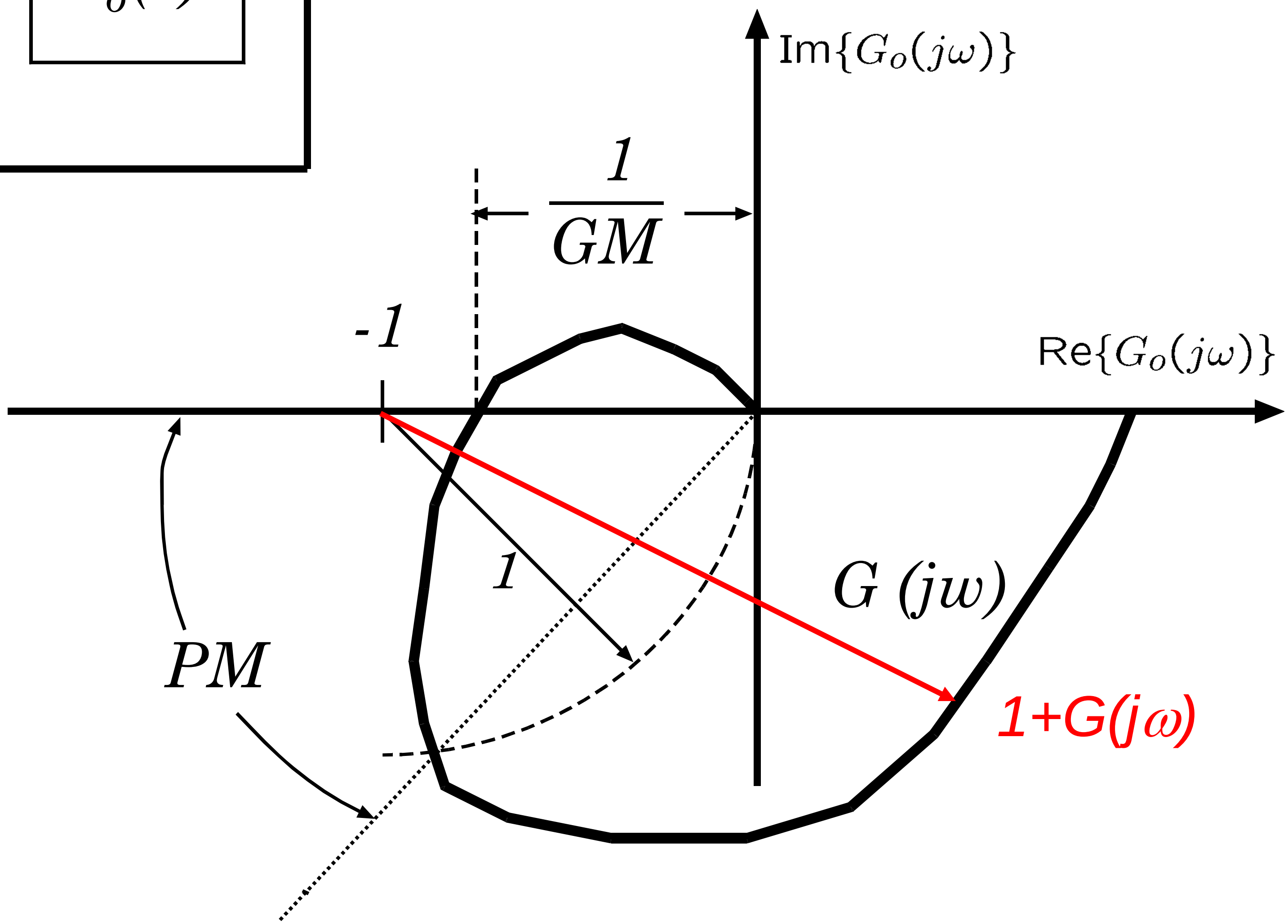
TF from *U(s)* to *Y(s)*:

$$G(s) = C\Phi(s)B$$

Gain and Phase Margins



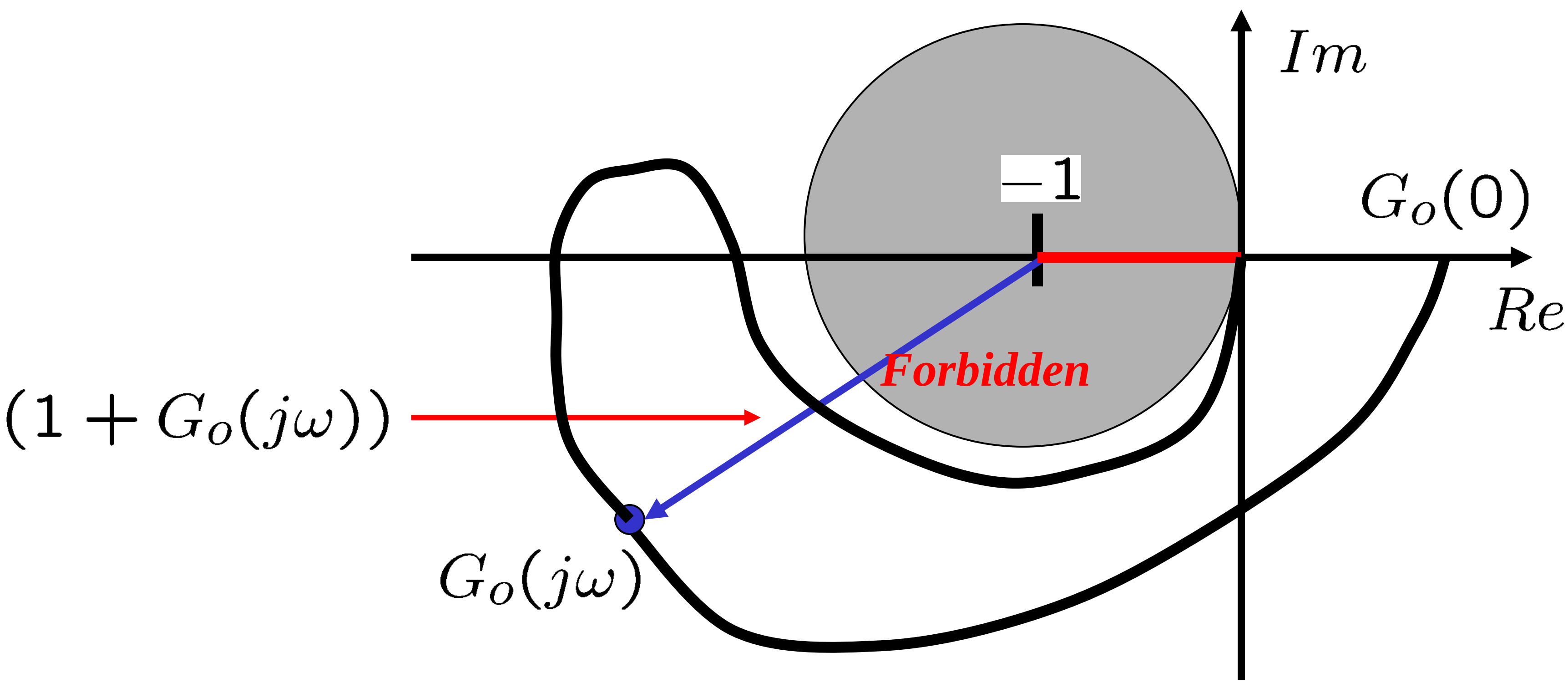
- Nyquist plot of $G_o(j\omega)$



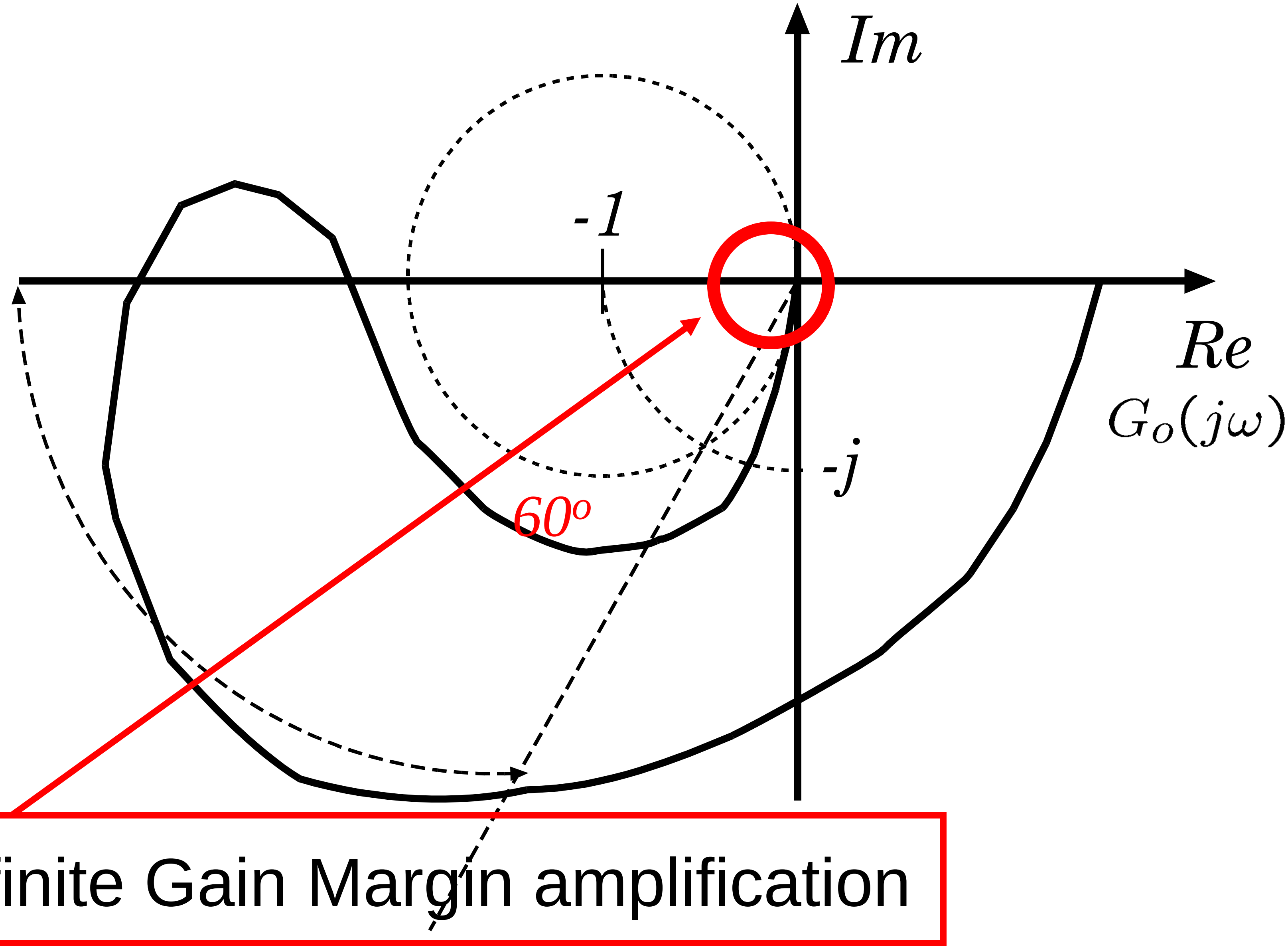
Gain and Phase Margins for SI LQR

$$\boxed{|(1 + G_o(j\omega))|^2 \geq 1}$$

Nyquist plot of $G_o(j\omega)$

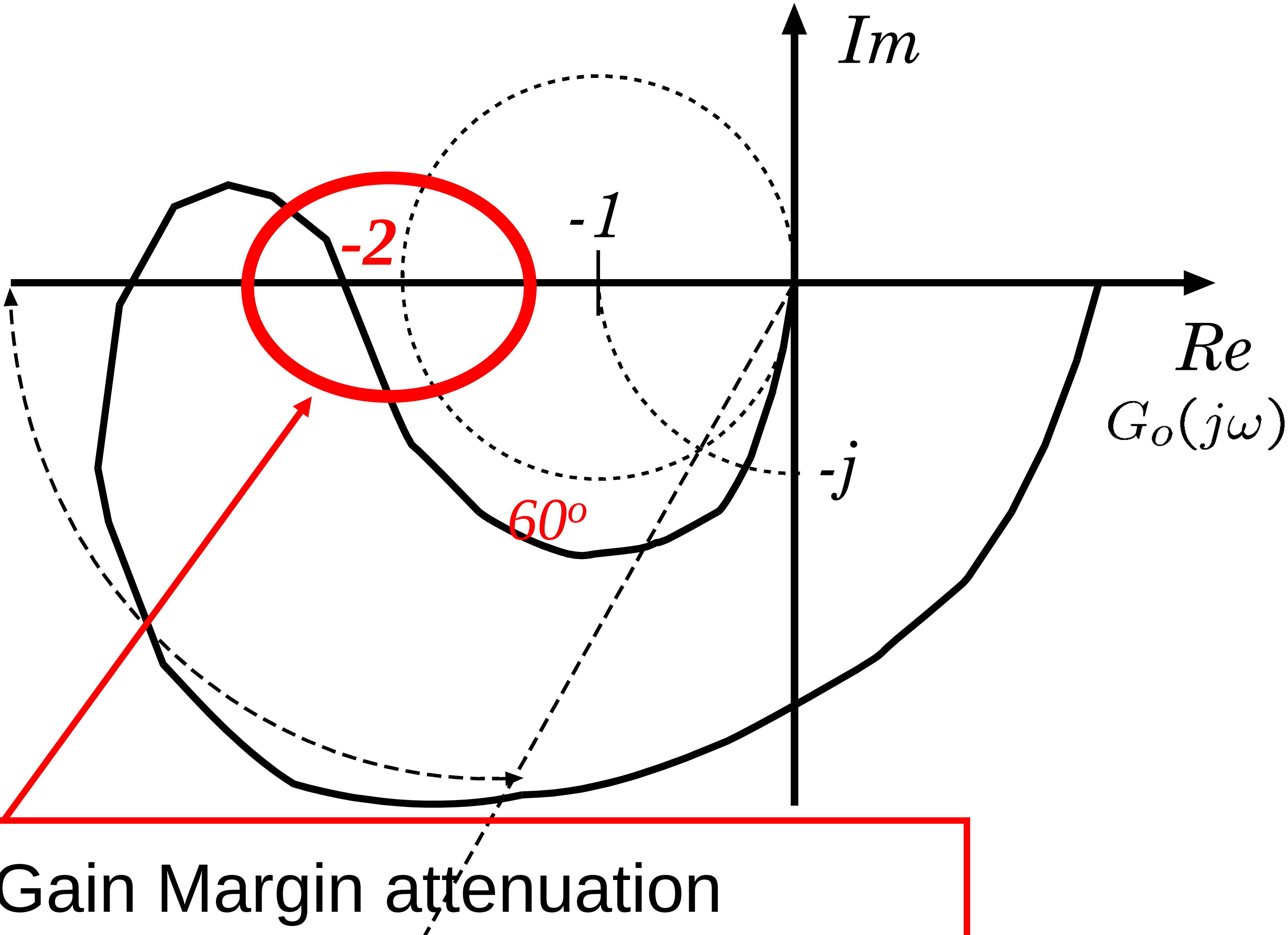


Gain margin amplification



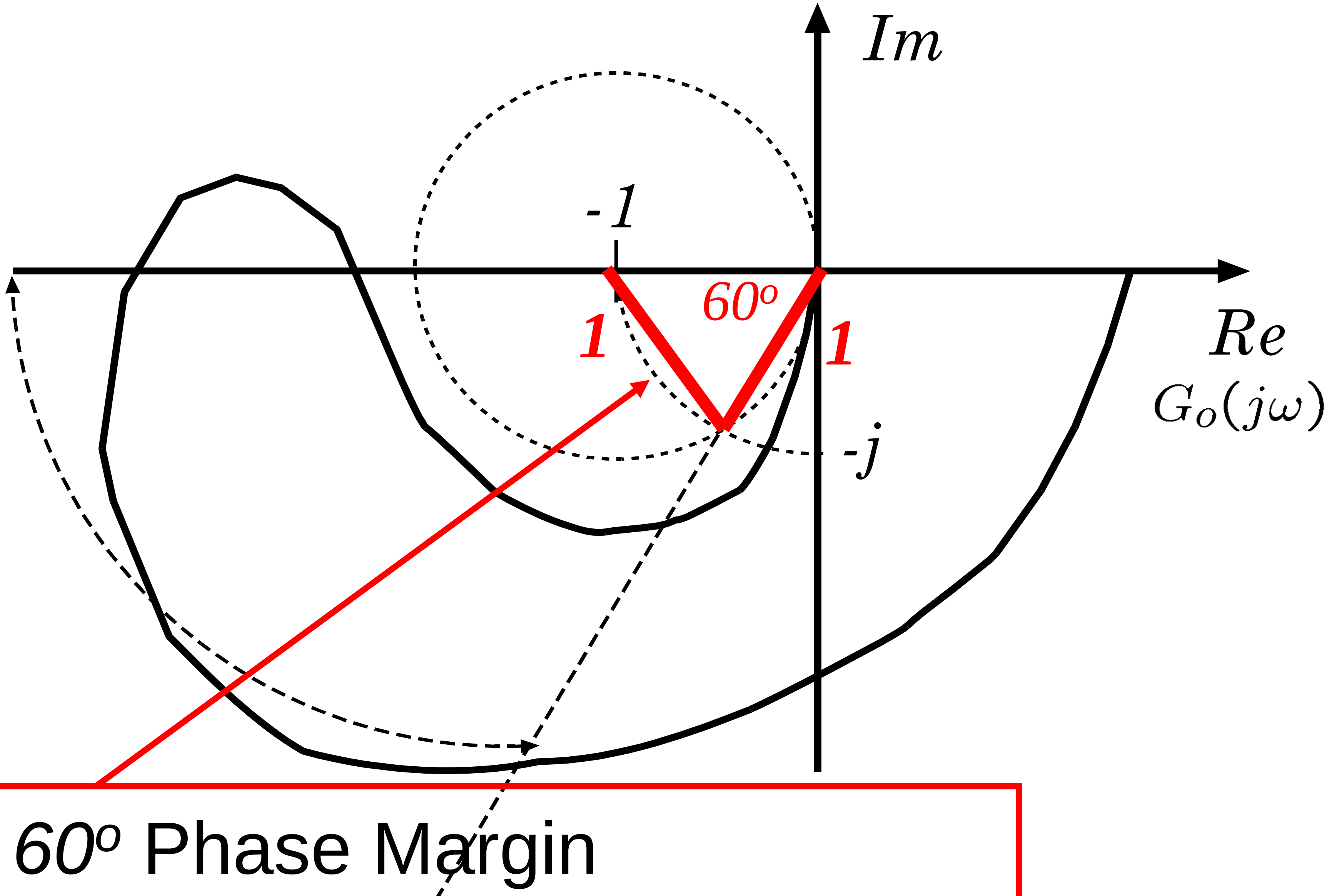
Infinite Gain Margin amplification

Gain margin attenuation



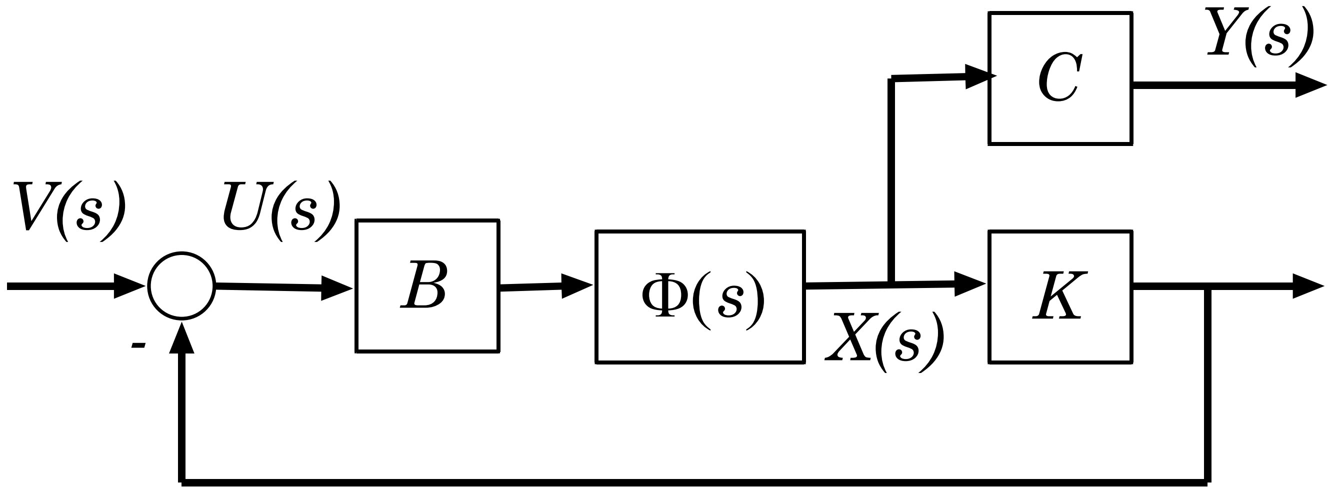
½ Gain Margin attenuation

Phase margin



60^o Phase Margin

LQR gain and phase margins

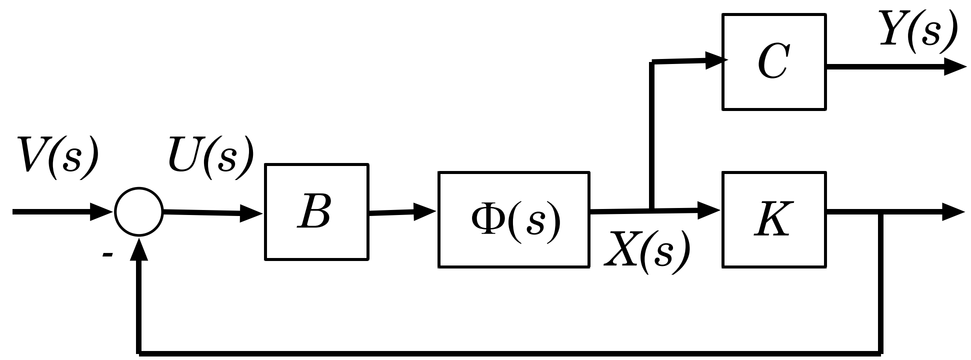


- Gain margins:
  - infinite gain margin amplification
  - ½ gain margin attenuation
- 60^o phase margin

Closed Loop Eigenvalues (SISO)

$$\dot{x} = A\,x + B\,u$$

$$u = -K\,x + v$$



Open loop characteristic polynomial:

$$A(s) = \mathrm{det}(sI - A)$$

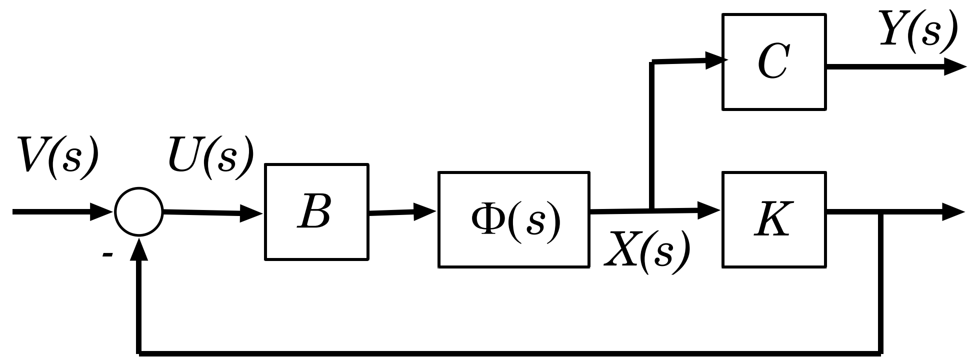
Closed loop characteristic polynomial:

$$A_c(s) = \mathrm{det}(sI - A + BK)$$

Closed Loop Eigenvalues (SISO)

$$\Phi(s) = (sI - A)^{-1}$$

$$G_o(s) = K\Phi(s)B$$



Notice that:

$$D(s) = 1 + G_o(s) = \frac{A_c(s)}{A(s)} = \frac{\det(sI - A + BK)}{\det(sI - A)}$$

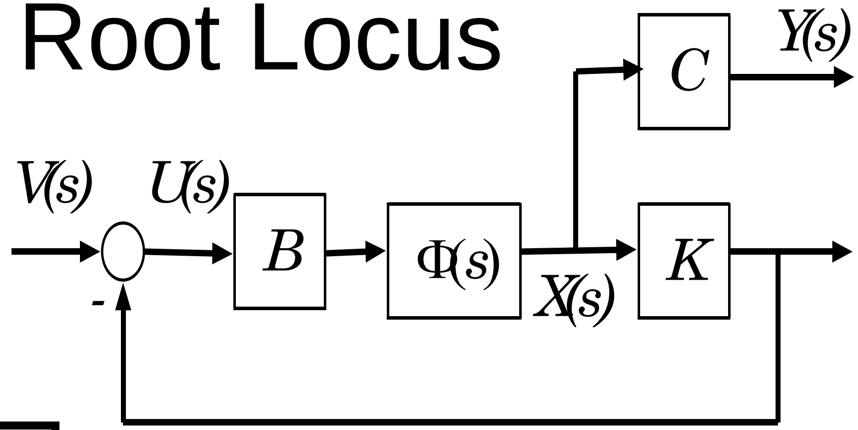
close loop poles

$$= \frac{(s - p_{c1})\cdots(s - p_{cn})}{(s - p_{o1})\cdots(s - p_{on})}$$

open loop poles

LQ Symmetric Root Locus

$$\boxed{1 + G_o(s) = \frac{A_c(s)}{A(s)}}$$



$$\boxed{G(s) = C\Phi(s)B = \frac{B(s)}{A(s)}}$$

Return difference equality:

$$(1 + G_o(-s))(1 + G_o(s)) = 1 + \frac{1}{R}G(-s)G(s)$$

$$\boxed{\frac{A_c(-s)}{A(-s)}\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\frac{B(-s)}{A(-s)}\frac{B(s)}{A(s)}}$$

LQ Symmetric Root Locus

Determine how closed loop eigenvalues change when the control weight, $\boldsymbol{R}$, is varied from infinity to zero.

$$\boxed{\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\,\frac{B(-s)}{A(-s)}\,\frac{B(s)}{A(s)}}$$

$$\boxed{A_c(-s)\,A_c(s) = A(s)\,A(-s) + \frac{1}{R}B(-s)\,B(s)}$$

LQ Symmetric Root Locus

$$A_c(-s)\,A_c(s) = A(s)\,A(-s) + \frac{1}{R}B(-s)\;B(s)$$

Open loop poles and zeros

$$A(s) = \prod_{i=1}^{n}(s - p_{oi}) \;= (s - p_{o1})\cdots(s - p_{on})$$

$$B(s) = b_m\;\prod_{i=1}^{m}(s - z_{oi}) \;= b_m(s - z_{o1})\cdots(s - z_{om})$$

Closed loop poles:

$$A_c(s) = \prod_{i=1}^{n}(s - p_{ci}) \;= (s - p_{c1})\cdots(s - p_{cn})$$

LQ Symmetric Root Locus

$$A_c(-s)\,A_c(s) = A(s)\,A(-s) + \frac{1}{R}B(-s)\;B(s)$$

e.g. What about?

$$A(s) = \prod_{i=1}^{n}(s - p_{oi}) \quad \Rightarrow \quad A(-s) = \prod_{i=1}^{n}(-s - p_{oi})$$

$$A(-s) = \prod_{i=1}^{n} -1(s + p_{oi}) = (-1)(s + p_{o1})\cdots(-1)(s + p_{on})$$

$$\boxed{A(-s) = (-1)^n \prod_{i=1}^{n}(s + p_{oi})}$$

LQ Symmetric Root Locus

$$\underline{A_c(-s)}\, A_c(s) = A(s)\, \underline{A(-s)} + \frac{1}{R}\, \underline{B(-s)}\, B(s)$$

$$A(-s) = (-1)^n \prod_{i=1}^{n} (s + p_{oi})$$

$$B(-s) = b_m\, (-1)^m \prod_{i=1}^{m} (s + z_{oi})$$

$$A_c(-s) = (-1)^n \prod_{i=1}^{n} (s + p_{ci})$$

LQ Symmetric Root Locus

Thus, from

$$A_c(-s)\,A_c(s) = A(-s)\,A(s) + \frac{1}{R}B(-s)\,B(s)$$

$$(-1)^n \prod_{i=1}^{n}(s+p_{ci}) \prod_{i=1}^{n}(s-p_{ci}) \;=\; (-1)^n \prod_{i=1}^{n}(s+p_{oi}) \prod_{i=1}^{n}(s-p_{0i})$$

$$+\frac{1}{R}(-1)^m\, b_m \prod_{i=1}^{m}(s+z_{oi})\, b_m \prod_{i=1}^{m}(s-z_{oi})$$

LQ Symmetric Root Locus

Thus, from

$$A_c(-s)\,A_c(s) = A(-s)\,A(s) + \frac{1}{R}\,B(-s)\;B(s)$$

$$(-1)^n \prod_{i=1}^{n}(s+p_{ci})\;\prod_{i=1}^{n}(s-p_{ci}) \;=\; (-1)^n \prod_{i=1}^{n}(s+p_{oi})\;\prod_{i=1}^{n}(s-p_{0i})$$

$$+\frac{1}{R}(-1)^m\, b_m \prod_{i=1}^{m}(s+z_{oi})\, b_m \prod_{i=1}^{m}(s-z_{oi})$$

LQ Symmetric Root Locus

$$(-1)^n \prod_{i=1}^{n} (s + p_{ci}) \prod_{i=1}^{n} (s - p_{ci}) \;=\; (-1)^n \prod_{i=1}^{n} (s + p_{oi}) \prod_{i=1}^{n} (s - p_{0i})$$

$$+ \frac{1}{R} (-1)^m \, b_m \prod_{i=1}^{m} (s + z_{oi}) \, b_m \prod_{i=1}^{m} (s - z_{oi})$$

Multiplying by *(–1)*^{n} and grouping b_m terms:

$$\prod_{i=1}^{n} (s + p_{ci}) \prod_{i=1}^{n} (s - p_{ci}) \;=\; \prod_{i=1}^{n} (s + p_{oi}) \prod_{i=1}^{n} (s - p_{0i})$$

$$+ \underbrace{\frac{b_m^2}{R}}_{} (-1)^{(n-m)} \prod_{i=1}^{m} (s + z_{oi}) \prod_{i=1}^{m} (s - z_{oi})$$

non negative number, function of $R \in (0, \infty)$

Return difference equality - Left

$$\prod_{i=1}^{n}(s+p_{ci})\ \prod_{i=1}^{n}(s-p_{ci}) \;=\; \prod_{i=1}^{n}(s+p_{oi})\ \prod_{i=1}^{n}(s-p_{oi})$$

$$+(-1)^{n-m}\frac{b_m^2}{R}\ \prod_{i=1}^{m}(s+z_{oi})\ \prod_{i=1}^{m}(s-z_{oi})$$

closed loop eigenvalues
(always asymptotically stable)

$R \in (0, \infty)$

negative of closed loop eigenvalues
(**always unstable**)

Return difference equality - Right

$$\prod_{i=1}^{n}(s+p_{ci})\prod_{i=1}^{n}(s-p_{ci}) = \prod_{i=1}^{n}(s+p_{oi})\prod_{i=1}^{n}(s-p_{oi})$$

negatives of open loop poles

open loop poles

$$+(-1)^{n-m}\frac{b_m^2}{R}\prod_{i=1}^{m}(s+z_{oi})\prod_{i=1}^{m}(s-z_{oi})$$

negatives of open loop zeros

open loop zeros

Return difference equality - Right

$$\prod_{i=1}^{n}(s+p_{ci})\prod_{i=1}^{n}(s-p_{ci}) = \prod_{i=1}^{n}(s+p_{oi})\prod_{i=1}^{n}(s-p_{oi})$$

$$+(-1)^{n-m}\frac{b_m^2}{R}\prod_{i=1}^{m}(s+z_{oi})\prod_{i=1}^{m}(s-z_{oi})$$

$n-m$ even $\Rightarrow$ negative feedback

$n-m$ odd $\Rightarrow$ positive feedback

LQ Symmetric Root Locus

$$\prod_{i=1}^{n}(s+p_{ci})\prod_{i=1}^{n}(s-p_{ci}) = \prod_{i=1}^{n}(s+p_{oi})\prod_{i=1}^{n}(s-p_{oi})$$

$$+(-1)^{n-m}\boxed{\frac{b_m^2}{R}}\prod_{i=1}^{m}(s+z_{oi})\prod_{i=1}^{m}(s-z_{oi})$$

$$R \to \infty \quad \Rightarrow \quad p_{ci} \to \text{Stable}\begin{cases} p_{oi} \\ \text{or} \\ -p_{oi} \end{cases}$$

$$R \to \infty \quad \Rightarrow \quad -p_{ci} \to \text{Unstable}\begin{cases} p_{oi} \\ \text{or} \\ -p_{oi} \end{cases}$$

LQ Symmetric Root Locus

$$\prod_{i=1}^{n}(s+p_{ci})\prod_{i=1}^{n}(s-p_{ci}) = \prod_{i=1}^{n}(s+p_{oi})\prod_{i=1}^{n}(s-p_{oi})$$

$$+(-1)^{n-m}\boxed{\frac{b_m^2}{R}}\prod_{i=1}^{m}(s+z_{oi})\prod_{i=1}^{m}(s-z_{oi})$$

$$R \rightarrow 0 \quad \Rightarrow \quad p_{ci} \rightarrow \text{Stable}\begin{cases} z_{oi} \\ \text{or} \\ -z_{oi} \end{cases}$$

$$R \rightarrow 0 \quad \Rightarrow \quad -p_{ci} \rightarrow \text{Unstable}\begin{cases} z_{oi} \\ \text{or} \\ -z_{oi} \end{cases}$$

$$\boxed{i = 1, \cdots, m}$$

LQ Symmetric Root Locus

$$\prod_{i=1}^{n}(s+p_{ci})\prod_{i=1}^{n}(s-p_{ci}) = \prod_{i=1}^{n}(s+p_{oi})\prod_{i=1}^{n}(s-p_{oi})$$

$$+(-1)^{n-m}\boxed{\frac{b_m^2}{R}}\prod_{i=1}^{m}(s+z_{oi})\prod_{i=1}^{m}(s-z_{oi})$$

$$\boxed{i = n-m, \cdots, n}$$

$$R \to 0 \;\Rightarrow\; |p_{ci}|, \; |-p_{ci}| \to \infty$$

$$\boxed{n-m : \text{ODD}}$$

Angle of asymptotes $\dfrac{l\,\pi}{n-m}$ $\quad l = 0, 1, \cdots, 2(n-m)-1$

LQ Symmetric Root Locus

$$\prod_{i=1}^{n}(s+p_{ci})\prod_{i=1}^{n}(s-p_{ci}) = \prod_{i=1}^{n}(s+p_{oi})\prod_{i=1}^{n}(s-p_{oi})$$

$$+(-1)^{n-m}\boxed{\frac{b_m^2}{R}}\prod_{i=1}^{m}(s+z_{oi})\prod_{i=1}^{m}(s-z_{oi})$$

$$\boxed{i = n-m, \cdots, n}$$

$$R \to 0 \Rightarrow |p_{ci}|, |-p_{ci}| \to \infty$$

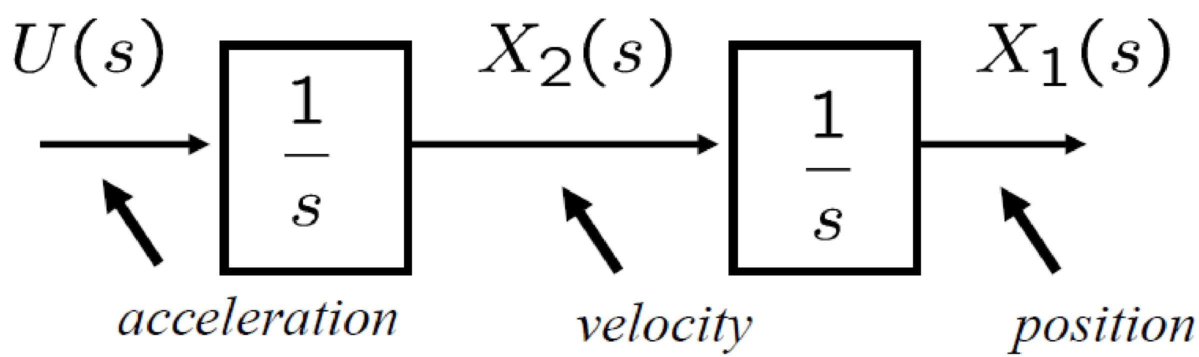
$$\boxed{n-m : \text{ EVEN}}$$

Angle of asymptotes $\dfrac{(l+\frac{1}{2})\pi}{n-m}$ $\quad l = 0, 1, \cdots, 2(n-m)-1$

LQ optimal control example

Double integrator

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



LQR example 1

Double integrator

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{x^T C^T C x + R u^2\right\} dt$$

with

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad\qquad R > 0$$

Only position is penalized

LQR example 1

Double integrator

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\boxed{G(s) = C(sI - A)^{-1}B = \frac{1}{s^2}}$$

2 open loop poles at the origin

no open loop zeros

Return Difference Equality

$$\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\,\frac{B(-s)}{A(-s)}\,\frac{B(s)}{A(s)}$$

$$\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\left(\frac{1}{-s}\right)^2\left(\frac{1}{s}\right)^2$$

$$\frac{A_c(-s)}{(-s)^2}\,\frac{A_c(s)}{s^2} = 1 + \frac{1}{R}\left(\frac{1}{-s}\right)^2\left(\frac{1}{s}\right)^2$$

$$A_c(-s)\,A_c(s) = (s)^2\,(-s)^2 + \frac{1}{R}$$

LQR example 1

Thus, from

$$A_c(-s)\,A_c(s) = (s)^2\,(-s)^2 + \frac{1}{R}$$

We obtain

$$\prod_{i=1}^{2}(s+p_{ci})\;\prod_{i=1}^{2}(s-p_{ci}) \;=\; (s)^4 + (-1)^2\,\frac{1}{R}$$

$$\boxed{n - m = 2 \qquad \text{EVEN}}$$

LQR example 1

Symmetric Root Locus:

$$\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci}) \;=\; (s)^4+\frac{1}{R}$$

or

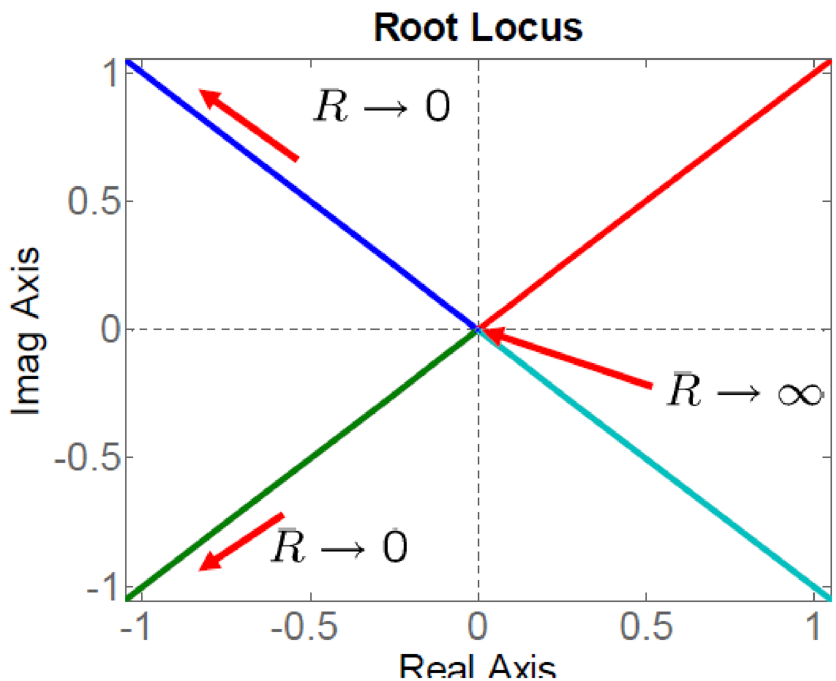
$$\boxed{\frac{\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci})}{s^4} \;=\; 1+\frac{1}{R}\,\frac{1}{s^4}}$$

LQR example 1

Double integrator

$$\boxed{1 + \frac{1}{R}\frac{1}{s^4} = 0}$$

$$R \rightarrow 0 \;\Rightarrow\; |p_{ci}|, |-p_{ci}| \rightarrow \infty$$



Asymptotes:

$$\frac{(l + \frac{1}{2})\,\pi}{2} \qquad l = 0, 1, 2, 3$$

$+45^o$

$+135^o$

-45^o

-135^o

LQR example 2

Double integrator (change state weight matrix $Q = C^T C$)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{x^T C^T C\, x + R\, u^2\right\} dt$$

with

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad R > 0$$

position and velocity are penalized

LQR example 2

Double integrator (change state weight matrix $Q = C^T C$)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

$$\boxed{G(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2}}$$

2 open loop poles at the origin

1 open loop zero at -1

LQR example 2

$$\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\,\frac{B(-s)}{A(-s)}\,\frac{B(s)}{A(s)}$$

$$\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\left(\frac{-s+1}{-s^2}\right)\left(\frac{s+1}{s^2}\right)$$

$$\frac{A_c(-s)}{(-s)^2}\,\frac{A_c(s)}{s^2} = 1 + \frac{1}{R}\left(\frac{-s+1}{-s^2}\right)\left(\frac{s+1}{s^2}\right)$$

$$A_c(-s)\,A_c(s) = (s)^2\,(-s)^2 + \frac{1}{R}(-s+1)(s+1)$$

LQR example 2

Thus, from

$$A_c(-s)\,A_c(s) = (s)^2\,(-s)^2 + \frac{1}{R}(-s+1)(s+1)$$

We obtain

$$\prod_{i=1}^{2}(s+p_{ci})\;\prod_{i=1}^{2}(s-p_{ci}) \;=\; (s)^4 + \frac{1}{R}(-1)^1(s+1)(s-1)$$

$$\boxed{n - m = 2 - 1 = 1 \qquad \text{EVEN}}$$

LQR example 2

Symmetric Root Locus:

$$\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci}) \;=\; (s)^4 - \frac{1}{R}(s+1)(s-1)$$

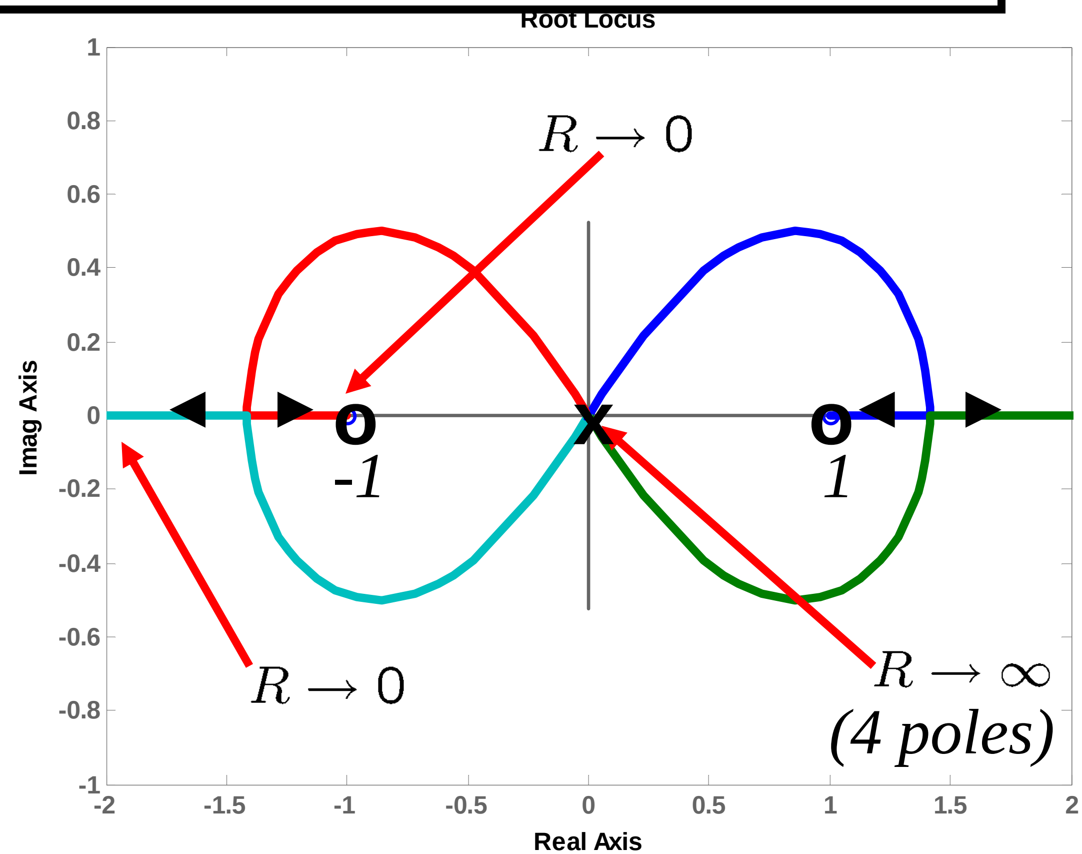
or

$$\boxed{\frac{\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci})}{s^4} \;=\; 1 - \frac{1}{R}\,\frac{(s+1)(s-1)}{s^4}}$$

LQR example 2

Double integrator (change state weight matrix $Q = C^T C$)

$$\boxed{1 - \frac{1}{R}\frac{(s+1)(s-1)}{s^4} = 0}$$



$$R \longrightarrow 0$$

$$p_{c1} \longrightarrow -1$$

$$p_{c2} \longrightarrow -\infty$$

$$-p_{c1} \longrightarrow 1$$

$$-p_{c2} \longrightarrow \infty$$

Asymptotes:

$$\frac{l}{\pi} \qquad l = 0,\ 1$$

$$+0^o$$

$$+180^o$$

LQR example 3

Open-loop unstable system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T C^T C\, x + R\, u^2 \right\} dt$$

with

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad\qquad R > 0$$

Only x_1 is penalized

LQR example 3

Open-loop unstable system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$\boxed{G(s) = C(sI - A)^{-1}B = \frac{1}{(s+1)(s-2)}}$$

LQR example 3

$$\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\,\frac{B(-s)}{A(-s)}\,\frac{B(s)}{A(s)}$$

$$\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\left(\frac{1}{(s+1)(s-2)}\right)\left(\frac{1}{(-s+1)(-s-2)}\right)$$

$$A_c(-s)\,A_c(s) = (s+1)(-s+1)\,(s-2)(-s-2) + \frac{1}{R}$$

LQR example 3

Thus, from

$$A_c(-s)\,A_c(s) = (s+1)(-s+1)\,(s-2)(-s-2) + \frac{1}{R}$$

We obtain

$$\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci}) \ = \ (s+1)(s-1)\,(s-2)(s+2) + \ (-1)^2\,\frac{1}{R}$$

$$\boxed{n-m=2 \qquad \text{EVEN}}$$

LQR example 3

Symmetric Root Locus:

$$\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci}) \;=\; (s+1)(s-1)\,(s-2)(s+2)+\frac{1}{R}$$

or

$$\frac{\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci})}{(s+1)(s-1)(s-2)(s+2)} \;=$$

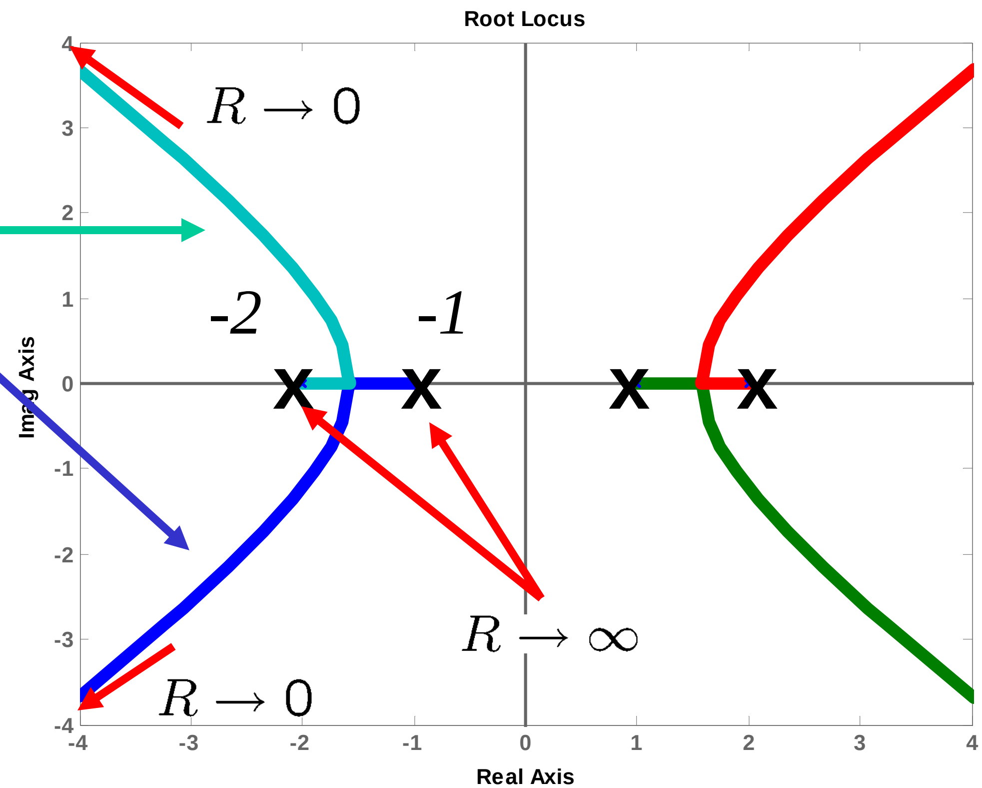
$$1+\frac{1}{R}\,\frac{1}{(s+1)(s-1)(s-2)(s+2)}$$

LQR example 3

Open-loop unstable system:

$$\boxed{1 + \frac{1}{R}\frac{1}{(s+1)(s-1)(s-2)(s+2)} = 0}$$

Close-loop poles (always asymptotically stable) $R \in (0, \infty)$



Open-loop poles: -1,+2

LQR example 4

Open-loop unstable system(change state weight matrix $Q = C^T C$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T C^T C\, x + R\, u^2 \right\} dt$$

with

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad\qquad R > 0$$

x_1 and x_2 are penalized

LQR example 2

Open-loop unstable system(change state weight matrix $Q = C^T C$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

$$\boxed{G(s) = C(sI - A)^{-1}B = \frac{(s+2)}{(s+1)(s-2)}}$$

LQR example 4

$$\frac{A_c(-s)}{A(-s)}\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\frac{B(-s)}{A(-s)}\frac{B(s)}{A(s)}$$

$$\frac{A_c(-s)}{A(-s)}\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\left(\frac{s+2}{(s+1)(s-2)}\right)\left(\frac{-s+2}{(-s+1)(-s-2)}\right)$$

$$\prod_{i=1}^{2}(s+p_{ci})\prod_{i=1}^{2}(s-p_{ci}) \;=\; (s+1)(s-1)(s-2)(s+2) - \frac{1}{R}(s+2)(s-2)$$

LQR example 4

Symmetric Root Locus:

$$\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci}) \;=\; (s+1)(s-1)\,(s-2)(s+2)$$

$$-\frac{1}{R}(s+2)(s-2)$$

or

$$\frac{\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci})}{(s+1)(s-1)(s-2)(s+2)} \;=$$

$$1-\frac{1}{R}\,\frac{(s-2)(s+2)}{(s+1)(s-1)(s-2)(s+2)}$$

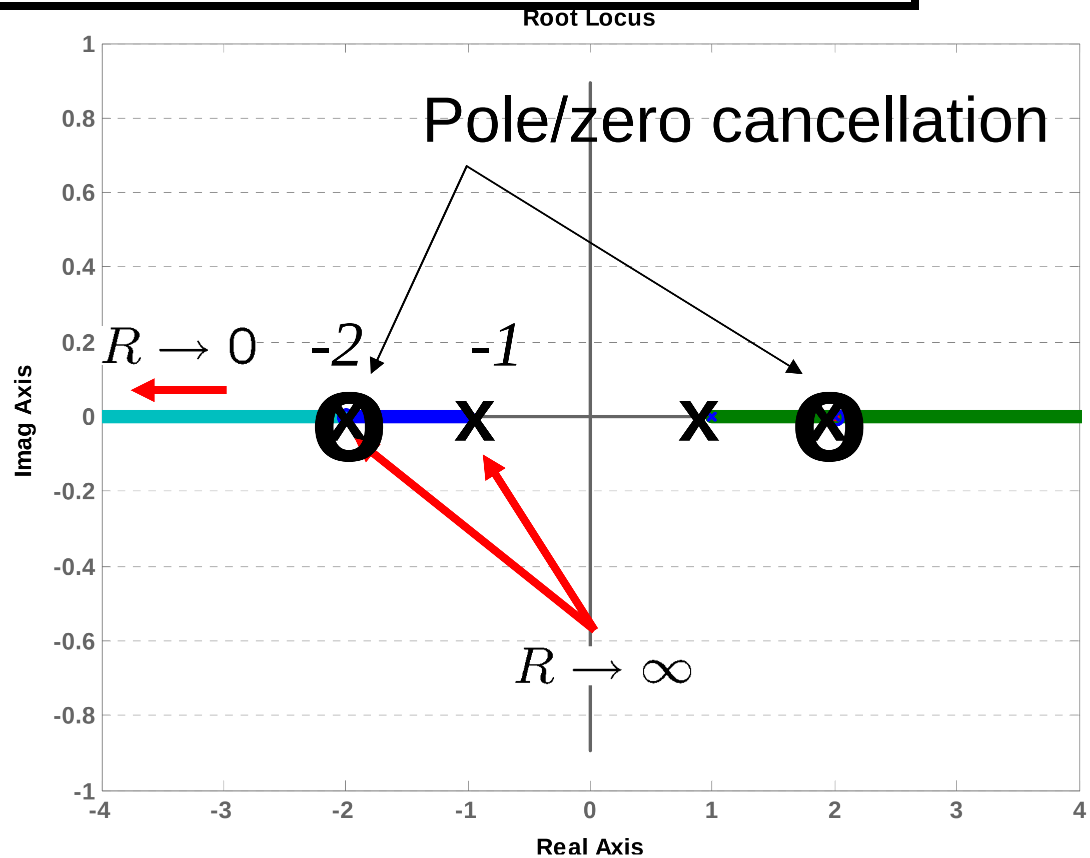
LQR example 4

Open-loop unstable system(change state weight matrix $Q = C^T C$)

$$\boxed{1 - \frac{1}{R}\frac{(s+2)(s-2)}{(s+1)(s-1)(s-2)(s+2)} = 0}$$

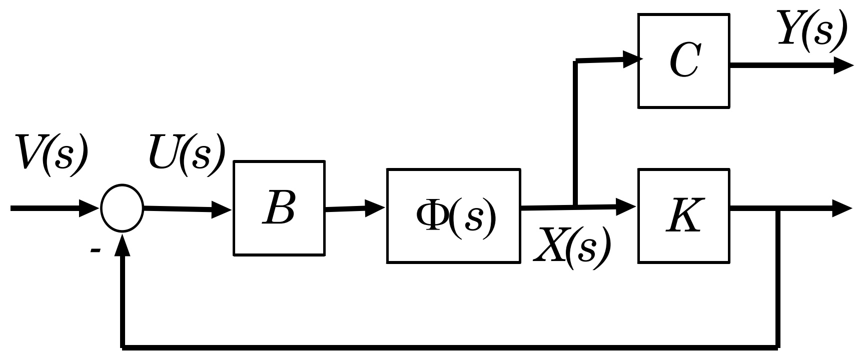
Close-loop poles (always asymptotically stable) $R \in (0, \infty)$

Open-loop poles: -1,+2



Symmetric root locus for SIMO Systems

$$\Phi(s) = (sI - A)^{-1}$$



$$\boxed{(1 + G_o(-s)\,)(1 + G_o(s)\,) = 1 + \frac{1}{R} G(-s)^T\, G(s)}$$

Let: $y(t) \in \mathcal{R}^p$ $\quad \Rightarrow \quad$ $G_o(s) = K\Phi(s)B \in H(s)$

$u(t) \in \mathcal{R}$ $\quad$ $G(s) = C\Phi(s)B \in H^p(s)$

where $H(s)$ is the class of rational functions of s

Symmetric root locus for SIMO Systems

$$G(s) = \frac{1}{A(s)} B(s) = \frac{1}{A(s)} \begin{bmatrix} B_1(s) \\ \vdots \\ B_p(s) \end{bmatrix}$$

$$G^T(-s)\, G(s) = \frac{B^T(-s) B(s)}{A(-s) A(s)} = \frac{\sum_{i=1}^{p} B_i(-s) B_i(s)}{A(-s) A(s)}$$

We can always find a polynomial

$$\bar{B}(s) = \bar{b}_m\, s^m + \bar{b}_{m-1}\, s^{m-1} \cdots + \bar{b}_o$$

such that:

$$\bar{B}(-s)\bar{B}(s) = \sum_{i=1}^{p} B_i(-s)\; B_i(s)$$

Symmetric root locus for SIMO Systems

$$(1+G_o(-s)\,)(1+G_o(s)\,) = 1+\frac{1}{R}G(-s)^T\,G(s)$$

for $\quad R \in (0,\infty)$

$$(1+G_o(s)) \;=\; \frac{A_c(s)}{A(s)} \qquad\qquad G(-s)^T G(s) \;=\; \frac{\bar{B}(-s)\bar{B}(s)}{A(-s)A(s)}$$

$$\boxed{\frac{A_c(-s)A_c(s)}{A(-s)A(s)} = 1+\frac{1}{R}\,\frac{\bar{B}(-s)\bar{B}(s)}{A(-s)A(s)}}$$

LQR Example 5

Double integrator (change state weight matrix $Q = C^T C$)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T C^T C\, x + R\, u^2 \right\} dt$$

with

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R > 0$$

y_1 =position and y_2 = velocity

LQR example 5

Double integrator (change state weight matrix $Q = C^T C$)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

$$J = \frac{1}{2}\int_0^\infty \left\{ y^T y + R\,u^2 \right\} dt$$

$$\boxed{G(s) = C(sI - A)^{-1} B = \frac{1}{s^2}\begin{bmatrix} 1 \\ s \end{bmatrix}}$$

LQR Example 5

$$G^T(-s)\,G(s) = \frac{B^T(-s)B(s)}{A(-s)A(s)} = \frac{1\times 1 + (-s)(s)}{((-s)^2)(s^2)} = \frac{(-s+1)(s+1)}{((-s)^2)(s^2)}$$

$$\frac{A_c(-s)}{A(-s)}\,\frac{A_c(s)}{A(s)} = 1 + \frac{1}{R}\left(\frac{-s+1}{(-s)^2}\right)\left(\frac{s+1}{s^2}\right)$$

$$A_c(-s)\,A_c(s) = A(-s)\,A(s) + \frac{1}{R}\,\bar{B}(-s)\;\bar{B}(s)$$

Open loop poles:

$$A(s) = s^2$$

Closed loop poles:

$$A_c(s) = \prod_{i=1}^{2}(s - p_{ci})$$

Open loop zero of $G(s)$:

$$\bar{B}(s) = (s+1)$$

The symmetric root locus equation is the same as the one in Example 2.

LQR example 5

Symmetric Root Locus:

$$\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci}) \ = \ (s)^4 - \frac{1}{R}(s+1)(s-1)$$

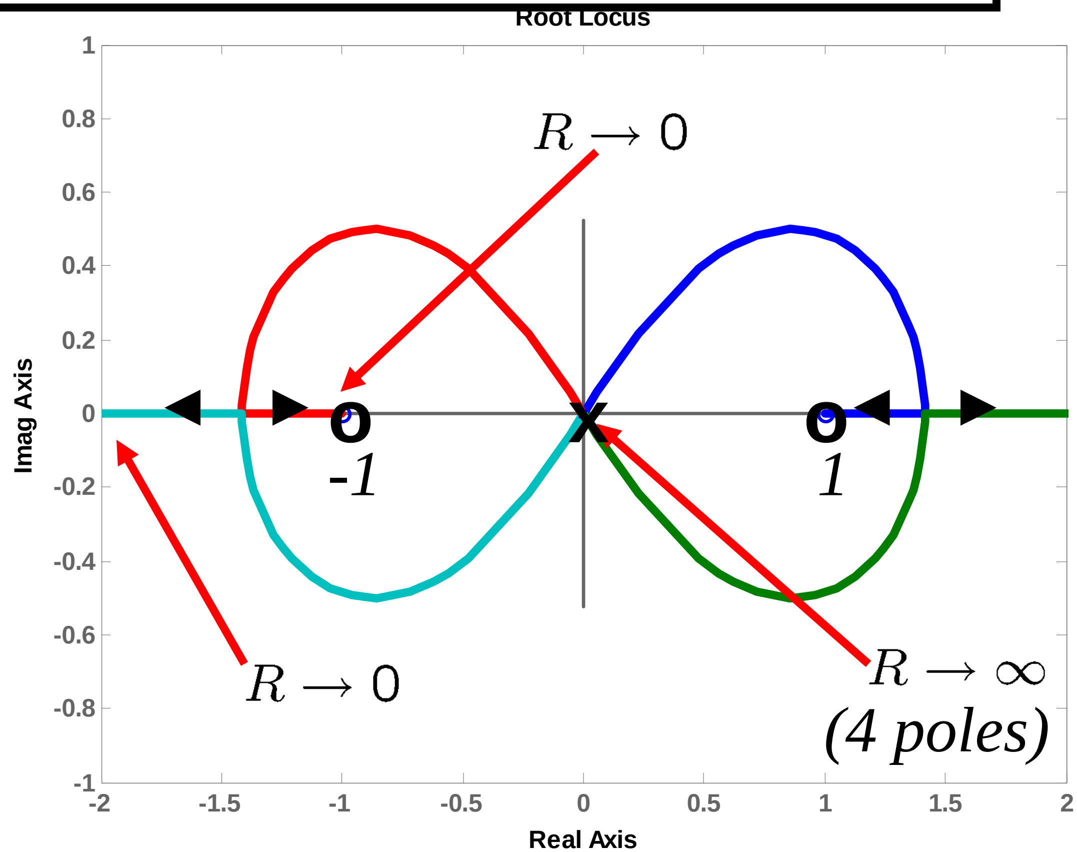
or

$$\frac{\prod_{i=1}^{2}(s+p_{ci})\ \prod_{i=1}^{2}(s-p_{ci})}{s^4} \ = \ 1 - \frac{1}{R}\,\frac{(s+1)(s-1)}{s^4}$$

LQR Example 5

Double integrator (change state weight matrix $Q = C^T C$)

$$\boxed{1 - \frac{1}{R}\frac{(s+1)(s-1)}{s^4} = 0}$$



$$R \rightarrow 0$$

$$p_{c1} \rightarrow -1$$

$$p_{c2} \rightarrow -\infty$$

$$-p_{c1} \rightarrow 1$$

$$-p_{c2} \rightarrow \infty$$

Asymptotes:

$$\frac{l}{\pi} \qquad l = 0,\ 1$$

$$+0^o$$

$$+180^o$$

Cost Functions in Examples 2 & 5

- Example 2:

$$J = \frac{1}{2} \int_0^{\infty} \left\{ x^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x + R\, u^2 \right\} dt$$

- Example 5:

$$J = \frac{1}{2} \int_0^{\infty} \left\{ x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + R\, u^2 \right\} dt$$

- The difference is in the off-diagonal elements of Q, but the two examples yielded the same symmetric root locus.

Cost Functions in Examples 2 & 5

- The off-diagonal element of Q gives:

$$J_{12} = \int_0^\infty \{x_1(t)\, x_2(t)\}\; dt$$

- Noting $\dfrac{d}{dt}x_1(t) \;=\; x_2(t)$

$$J_{12} = \int_0^\infty \left\{x_1(t)\, \frac{d}{dt}x_1(t)\right\}\; dt$$

$$= \frac{1}{2}\int_0^\infty \left\{\frac{d}{dt}x_1^2(t)\right\}\; dt = -\frac{1}{2}\, x_1^2(0)$$

where the asymptotic stability has been assumed.

- The above equation means that J_{12} does not depend on u(t).

Selection of Q and R in LQR

- Optimal control based on a wrong cost function (performance index) may performed poorly.

- In the quadratic cost function, the relative magnitudes between Q and R are important and not their absolute values.

Selection of Q and R in LQR

- In the absence of ideas about the structure of Q and R, start with diagonal Q and R.

- Diagonal elements of Q and R may be adjusted based on the expected (desired) magnitudes of state variables and inputs:

$$q_{ii} = 1/x_{i,max}^2 \qquad\qquad r_{ii} = 1/u_{i,max}^2$$

Selection of Q and R in LQR

- Q and R can be made frequency dependent: Frequency Shaped Linear Quadratic (FSLQ) control.

- In some design approaches, the choice of Q and R is not based on physical considerations such as the one on the previous page: e.g. Loop Transfer Recovery (LTR) method.