[MEN573]
Advanced Control Systems I

Lecture 8 – Similarity Transformations using Eigenvalues and Eigenvectors

Associate Professor Joonbum Bae
Department of Mechanical Engineering
UNIST

Motivation

- In the previous lecture, we learned how to calculate the exponential matrix for specific classes of matrices:

- Diagonal:

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- Jordan:

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

- Complex:

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Motivation

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{n!}A^nt^n + \cdots$$

- Diagonal: $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ $\qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$
- Jordan: $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ $\qquad e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$
- Complex: $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$ $\qquad e^{At} = e^{\sigma t}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$

Motivation

- In the this lecture, we will learn how to calculate the matrix exponential for general matrices:
  - Determine a similarity transformation *T* and a matrix *B*

$$A = TBT^{-1} \qquad \text{and} \qquad e^{BT} \qquad \text{is known}$$

  - Then, calculate

$$e^{At} = Te^{Bt}T^{-1}$$

Similarity Transformation

- If two matrices $A, B \in C^{m \times n}$ are similar

$$A = TBT^{-1} \qquad T \in C^{m \times n}$$

- Then, their exponential matrices are also similar

$$e^{At} = Te^{Bt}T^{-1}$$

Similarity Transformation - Proof

- Notice that if $A = TBT^{-1}$
  then, $AA = (TBT^{-1})(TBT^{-1})$

$$= TBBT^{-1}$$

- Repeating this process for n=2, 3, 4,…

$$A^n = TB^nT^{-1} \qquad n = 0,1,2,\cdots$$

Similarity Transformation - Proof

- Therefore, since $TB^nT^{-1} = A^n$

$$
\begin{aligned}
Te^{Bt}T^{-1} &= T(I + Bt + \frac{1}{2}B^2t^2 + \cdots + \frac{1}{n!}B^nt^n + \cdots)T^{-1} \\
&= I + TBT^{-1}t + \cdots + \frac{1}{n!}T^{-1}B^nT^{-1}t^n + \cdots \\
&= I + At + \cdots + \frac{1}{n!}A^nt^n + \cdots \\
&= e^{At}
\end{aligned}
$$

Similarity Transformation – Example

- Consider $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

and $T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ $T^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$

Notice that $T^{-1}AT = \Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$

and $A = T\Lambda T^{-1}$

Similarity Transformation – Example

- Since $A = T\Lambda T^{-1} \;\Rightarrow\; e^{At} = Te^{\Lambda t}T^{-1}$

  and $e^{\Lambda t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$

  Then, $e^{At} = T\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}T^{-1}$

$$e^{At} = \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Eigenvalues and Eigenvectors

- Consider the vector space V, a field F, and a linear map $A : V \to V$

- The scalar $\lambda \in F$ is an eigenvalue of A iff there exists an associated eigenvector $v \neq 0 \in V$ such that

$$A(v) = \lambda v$$

i.e. $A(v)$ and λv are linearly dependent.

Eigenvalues and Eigenvectors

- Consider the matrix $A \in R^{n \times n}$

  The scalar $\lambda \in C$ is an eigenvalue of A iff there exists an associated eigenvector $t \neq 0 \in C^n$ such that

$$At = \lambda t$$

$$(\lambda I - A)t = 0 \quad \Rightarrow \quad \det(\lambda I - A) = 0$$

Eigenvalues and Eigenvectors

- Consider the matrix $A \in R^{n \times n}$

  Let $\lambda_i, \lambda_j \in C$ be two non-repeated eigenvalues of A with associated eigenvectors: $t_i, t_j \in C^n$

$$At_i = \lambda_i t_i \qquad\qquad At_j = \lambda_j t_j$$

$$\lambda_i \neq \lambda_j \quad \Rightarrow \quad t_i \neq t_j$$

- Distinct eigenvalues have distinct associated eigenvectors.

Eigenvalues and Eigenvectors

- Prove that $\lambda_i \neq \lambda_j \quad \Rightarrow \quad t_i \neq t_j$

- Assume that $\lambda_i \neq \lambda_j$ and $t_i = t_j \neq 0$

  Then, $At_i = \lambda_i t_i$ and $At_i = \lambda_j t_i$

  Therefore,

$$A(t_i - t_i) = (\lambda_i - \lambda_j)t_i$$

$$\Rightarrow 0 = (\lambda_i - \lambda_j)t_i \Rightarrow \lambda_i = \lambda_j$$

Contradiction!

Matrix Diagonalization

- Consider the matrix $A \in R^{n \times n}$
- Assume that all eigenvalues of A are distinct.

$$\det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

$$\lambda_i = \lambda_j \Leftrightarrow i = j$$

- For each eigenvalue $\lambda_i \in C$ there exists an associated eigenvector $t_i \neq 0 \in C^n$

$$(\lambda_i I - A)t_i = 0$$

Matrix Diagonalization

- All eigenvalues of A are distinct $\lambda_i = \lambda_j \Leftrightarrow i = j$
- For each eigenvalue $\lambda_i \in C$

$$(\lambda_i I - A)t_i = 0$$

- The matrix $T = \left[t_1 \ t_2 \ \cdots t_n\right]$ is nonsingular.

Matrix Diagonalization

Since $\boxed{A\, t_i = \lambda_i\, t_i}$

Then, lining up each eigenvector column-wise,

$$A \underbrace{[t_1\, t_2 \cdots t_n]}_{T} = [(\lambda_1\, t_1)(\lambda_2\, t_2) \cdots (\lambda_n t_n)]$$

$$= \underbrace{[t_1\, t_2 \cdots t_n]}_{T} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$\boxed{A\, T = T\, \Lambda}$$

Matrix Diagonalization

Since, $$A\,T = T\Lambda \;\Rightarrow\; \boxed{e^{At}\,T = T\,e^{\Lambda t}}$$

Lining up each eigenvector column-wise,

$$e^{At}\,\underbrace{[t_1\, t_2\, \cdots\, t_n]}_{T} = \left[(e^{\lambda_1 t}\, t_1)\,(e^{\lambda_2 t}\, t_2)\,\cdots\,(e^{\lambda_n t} t_n)\right]$$

Thus, for each eigenvector t_i with associated $\lambda_i \in \mathcal{F}$

$$\boxed{\underbrace{\left[e^{At}\right]}_{\mathcal{R}^{n\times n}}\; t_i = \underbrace{\left(e^{\lambda_i t}\right)}_{\mathcal{F}}\; t_i}$$

field

$\mathcal{C}$

Matrix Diagonalization Example

Given $$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

1) Find eigenvalues:

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda & -1 \\ 2 & (\lambda + 3) \end{bmatrix} \\ &= (\lambda + 2)(\lambda + 1) \end{aligned}$$

2) Find associate eigenvectors:

- $\lambda_1 = -2$: $\quad (\lambda_1 I - A)\, t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

- $\lambda_2 = -1$: $\quad (\lambda_2 I - A)\, t_2 = 0 \Rightarrow t_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Matrix Diagonalization Example

3) Define T and Λ

$$T = \begin{array}{c} \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \end{array} \qquad \Lambda = \begin{array}{c} \begin{array}{cc} \lambda_1 & \lambda_2 \end{array} \\ \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda}$$

Matrix Diagonalization Example

4) Compute

$$\boxed{e^{At} = Te^{\Lambda t}T^{-1}}$$

$$e^{At} = T\underbrace{\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}}_{e^{\Lambda t}}T^{-1}$$

$$= \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Physical Interpretation of Eigenvalues and Eigenvectors

Consider now the LTI dynamic system:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

With free response

$$x(t) = e^{At}\, x(0)$$

Physical Interpretation of Eigenvalues and Eigenvectors

Express $x(0)$ as a linear combination of the eigenvectors

$$\boxed{x(0) = \alpha_1 t_1 + \alpha_2 t_2}$$

Then

$$\begin{aligned} x(t) &= \left[e^{At}\right](\alpha_1 t_1 + \alpha_2 t_2) \\ &= \alpha_1 \left[e^{At}\right] t_1 + \alpha_2 \left[e^{At}\right] t_2 \end{aligned}$$

$$\boxed{x(t) = \underbrace{(\alpha_1 e^{\lambda_1 t})}_{\mathcal{R}} t_1 + \underbrace{(\alpha_2 e^{\lambda_2 t})}_{\mathcal{R}} t_2}$$

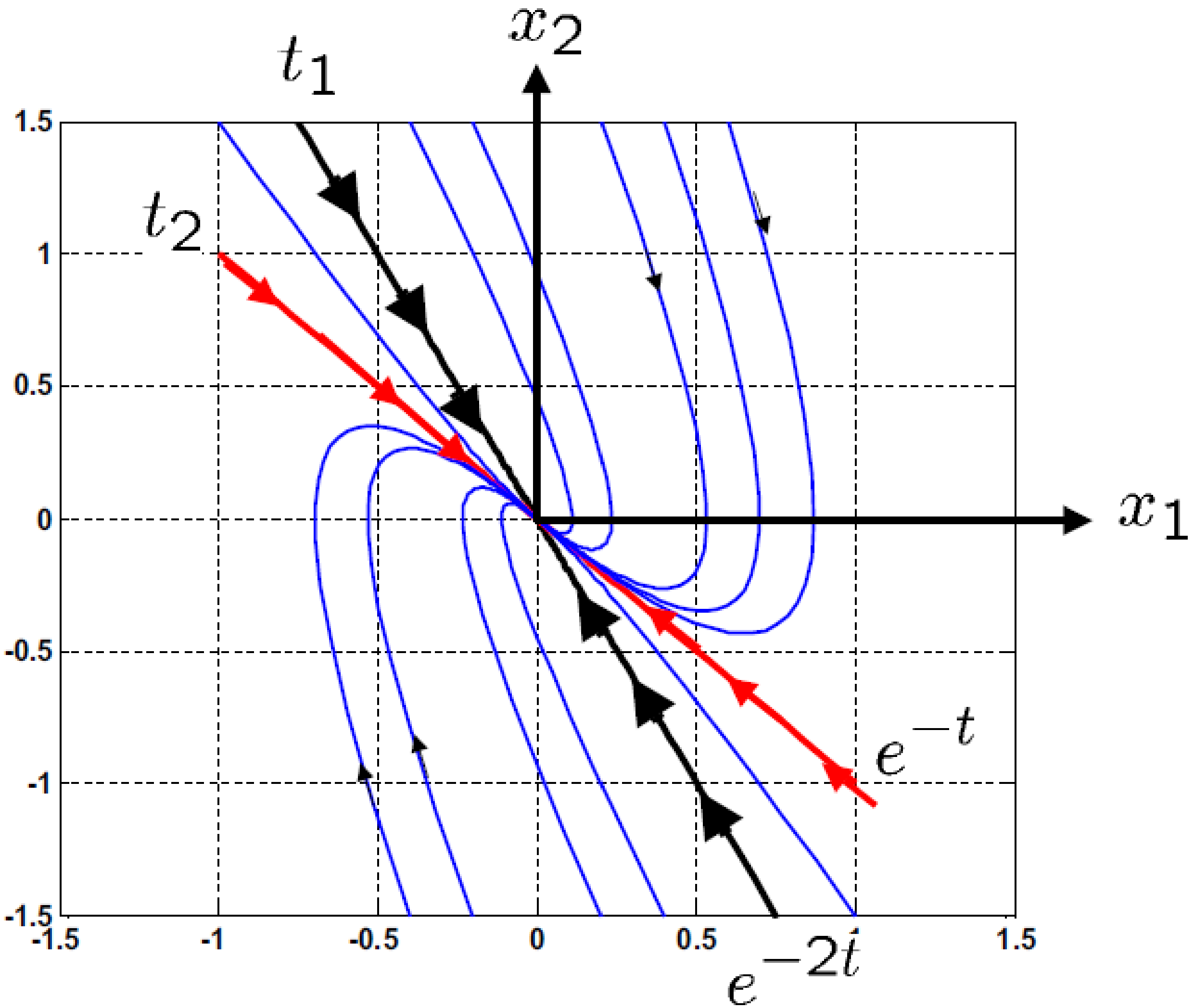
Physical Interpretation of Eigenvalues and Eigenvectors

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x(0) = \alpha_1 \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{t_1} + \alpha_2 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{t_2}$$

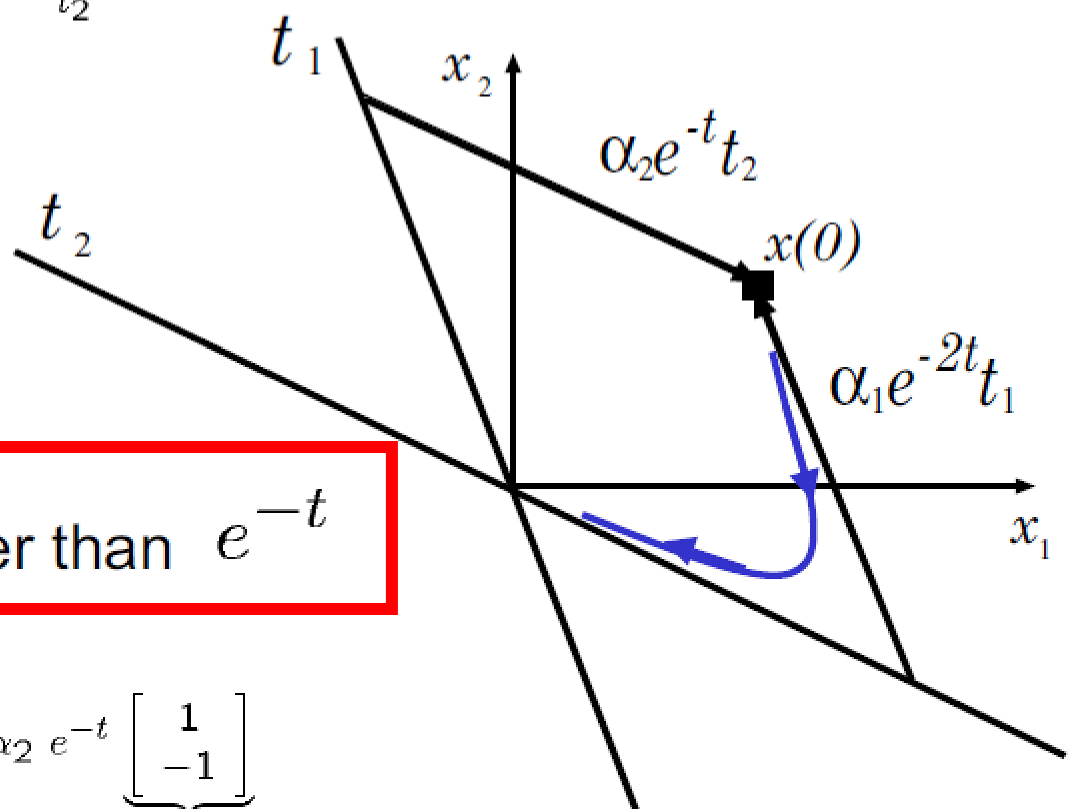
$$\boxed{x(t) = \alpha_1 e^{-2t} \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{t_1} + \alpha_2 e^{-t} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{t_2}}$$

Physical Interpretation of Eigenvalues and Eigenvectors



Physical Interpretation of Eigenvalues and Eigenvectors

$$x(0) = \alpha_1 \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{t_1} + \alpha_2 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{t_2}$$

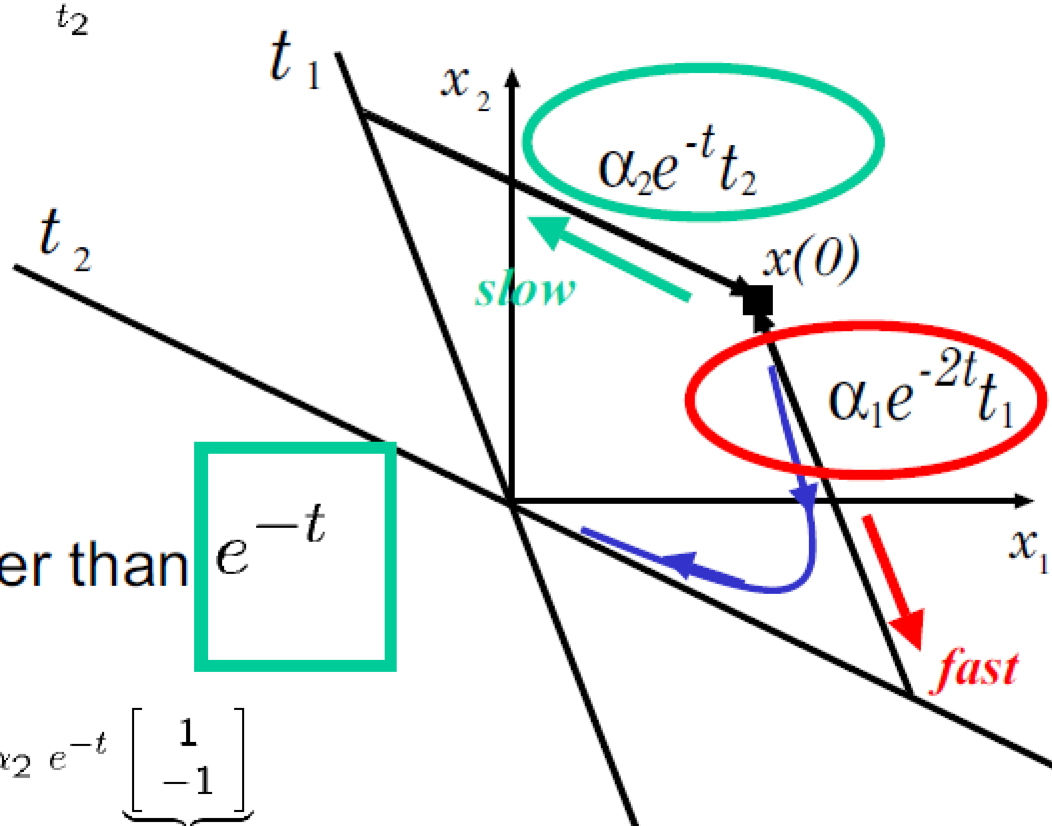


$\boxed{e^{-2t} \text{ decays faster than } e^{-t}}$

$$x(t) = \alpha_1 e^{-2t} \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{t_1} + \alpha_2 e^{-t} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{t_2}$$

Physical Interpretation of Eigenvalues and Eigenvectors

$$x(0) = \alpha_1 \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{t_1} + \alpha_2 \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{t_2}$$



$\boxed{e^{-2t}}$ decays faster than $\boxed{e^{-t}}$

$$x(t) = \alpha_1 e^{-2t} \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{t_1} + \alpha_2 e^{-t} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{t_2}$$

Phase Plane Analysis

- **Phase plane analysis (phase portrait)** is a **graphical method for studying second-order systems**, which was introduced by mathematicians such as Henri Poincare.

- The basic idea of the method is to generate, in the **state space of a second-order dynamic system** (a two-dimensional plane called the **phase plane**), **motion trajectories corresponding to various initial conditions**, and then to examine the qualitative features of the trajectories.

Phase Plane Analysis

- As a graphical method, it allows us to visualize what goes on in a linear/nonlinear system starting from various initial conditions, **without having to solve the governing equation analytically**.

- It is not restricted to small or smooth nonlinearities, but applies equally well to **strong and hard nonlinearities**.

- Some practical control systems can be adequately **approximated as second –order systems**.

- It is **restricted to second-order systems**, because the graphical study of higher-order systems is computationally and geometrically complex.

Phase Plane Analysis

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

- The state space of this system is a plane having x_1 and x_2 as coordinates. → phase plane (getting rid of time).
- Let $x(t) = (x_1(t), x_2(t))$ be the solution that starts at a certain initial state $x_0 = (x_{10}, x_{20})$; that is, $x(0) = x_0$.
- The locus in the x_1-x_2 plane of the solution $x(t)$ for all $t>=0$ is a curve that passes through the point x_0, called a trajectory or orbit.

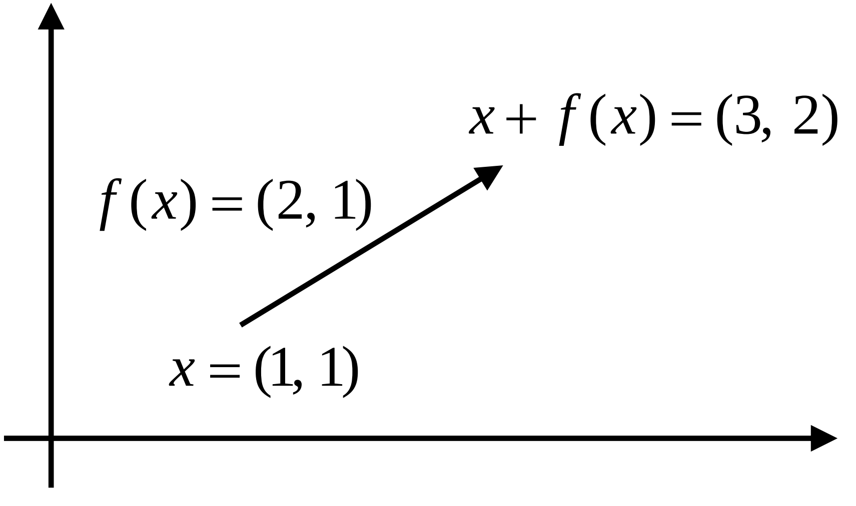
Phase Plane Analysis

- Using the vector notation

$$\dot{x} = f(x)$$

where $f(x)$ is the vector $(f_1(\mathrm{x}), f_2(\mathrm{x}))$, we consider $f(x)$ as a vector filed on the state plane, which means that to each point x in the plane, we assign a vector $f(x)$.

- For example, $f(x) = (2x_1^{\ 2},\ x_2)$ at $x = (1, 1)$.



Constructing Phase Portraits

- There are a number of methods for constructing phase plane trajectories for linear or nonlinear systems such as the so-called analytical method, the method of isoclines, the delta method, Lienard’s method, and Pell’s method.

Analytic Method

- Analytical Method: leading a functional relation between the two variables x_1 and x_2 in the form

$$g(x_1,\ x_2,\ c) = 0$$

where c represents the effects of initial conditions.

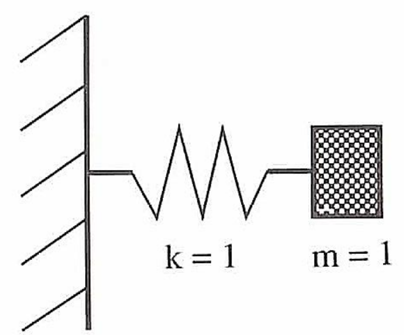
- Solving the ODE for x_1 and x_2 as functions of time t,

$$x_1(t) = g_1(t) \qquad\qquad x_2(t) = g_2(t)$$

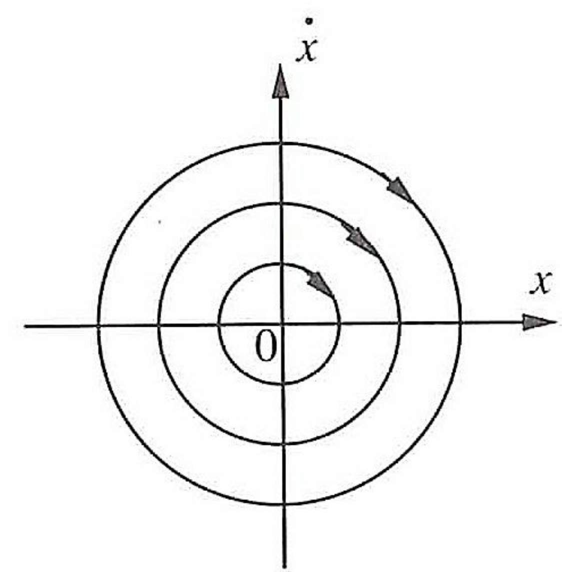
and then eliminating time t from these equations.

Analytical Method

- Example) $\ddot{x} + x = 0$ with initial condition x_0



Then, $x(t) = x_0 \cos t \qquad \dot{x}(t) = -x_0 \sin t$



$$x^2 + \dot{x}^2 = x_0{}^2$$

Analytical Method

- Directly eliminating the time variable,

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1,\ x_2)}{f_1(x_1,\ x_2)}$$

- Example) $\ddot{x} + x = 0$ with initial condition x_0

  By noting that $\ddot{x} = (d\dot{x}/dx)(dx/dt)$, then

$$\dot{x}\frac{d\dot{x}}{dx} + x = 0$$

  Integration of the equation yields

$$\dot{x}^2 + x^2 = {x_0}^2$$

Method of Isoclines

- An isocline is defined to be the locus of the points with a given tangent slope.
- An isocline with slope α is defined to be

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1,\ x_2)}{f_1(x_1,\ x_2)} = \alpha$$

- Example) $\ddot{x} + x = 0$ with initial condition x_0

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 = f_1(x_1,\ x_2) \\ \dot{x}_2 &= -x_1 = f_2(x_1,\ x_2) \end{aligned} \quad \Rightarrow \quad \frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

Qualitative Behavior of Linear Systems

- Consider the linear time-invariant system

$$\dot{x} = Ax$$

where A is a 2x2 real matrix.

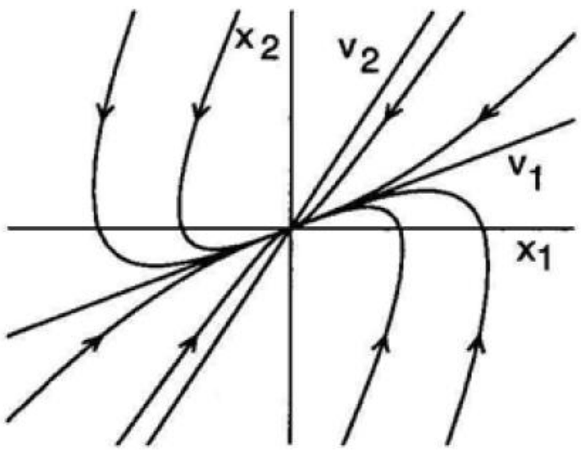
- Depending on the eigenvalues of A

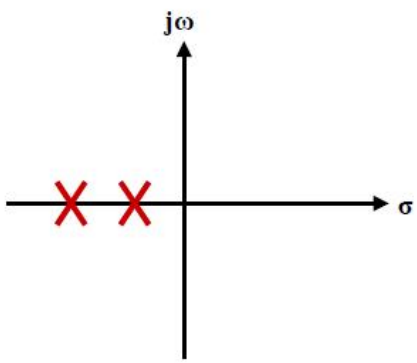
λ_1, λ_2 real and negative	Stable node
λ_1, λ_2 real and positive	Unstable node
λ_1, λ_2 real and opposite signs	Saddle point
λ_1, λ_2 complex and negative real parts	Stable focus
λ_1, λ_2 complex and positive real parts	Unstable focus
λ_1, λ_2 complex and zero real parts	Center

Qualitative Behavior of Linear Systems

λ_1 and λ_2 are real and negative

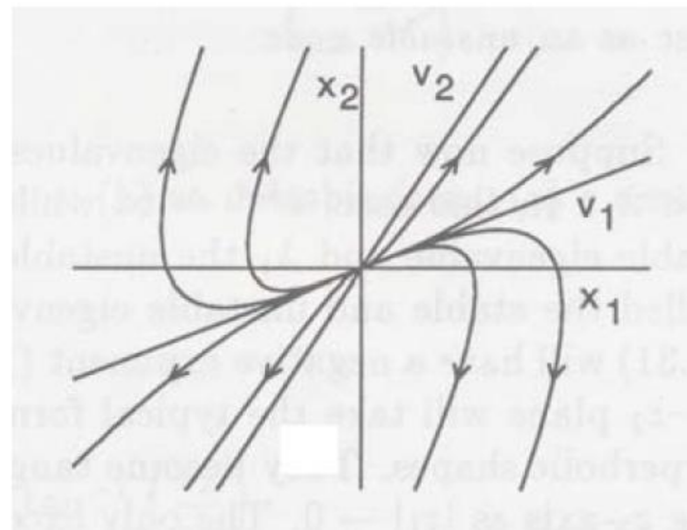
STABLE NODE

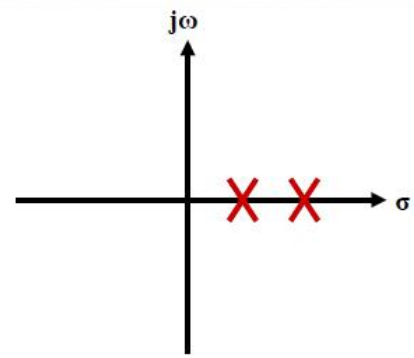




λ_1 and λ_2 are real and positive

UNSTABLE NODE

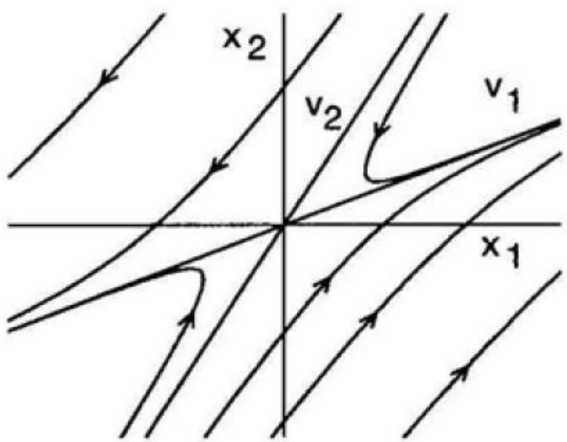


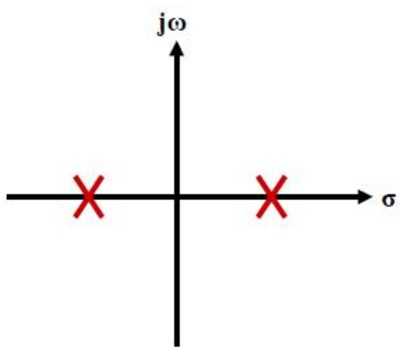


Qualitative Behavior of Linear Systems

λ_1 and λ_2 are real and of opposite sign

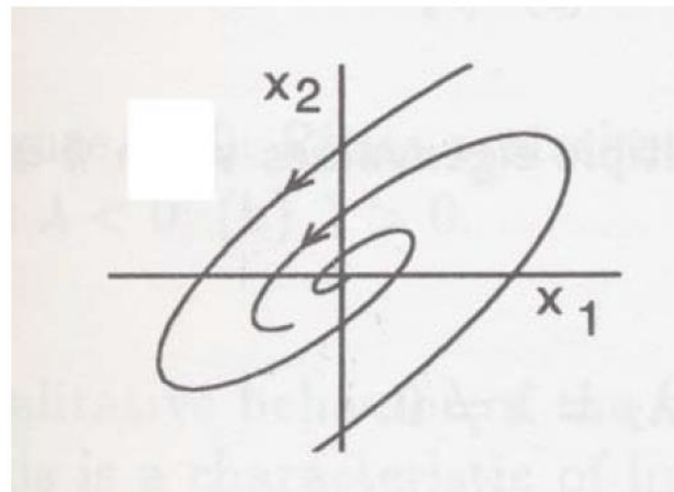
SADDLE POINT (UNSTABLE)

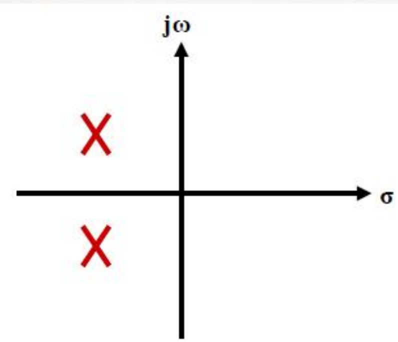




λ_1 and λ_2 are complex with negative real parts

STABLE FOCUS

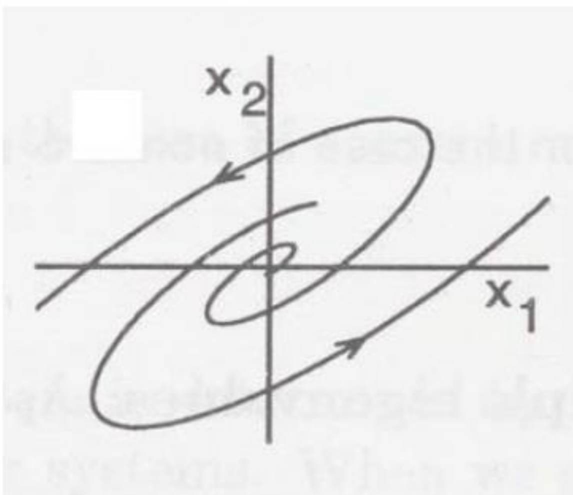


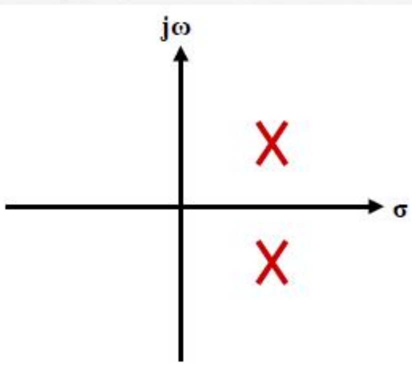


Qualitative Behavior of Linear Systems

λ_1 and λ_2 are complex with positive real parts

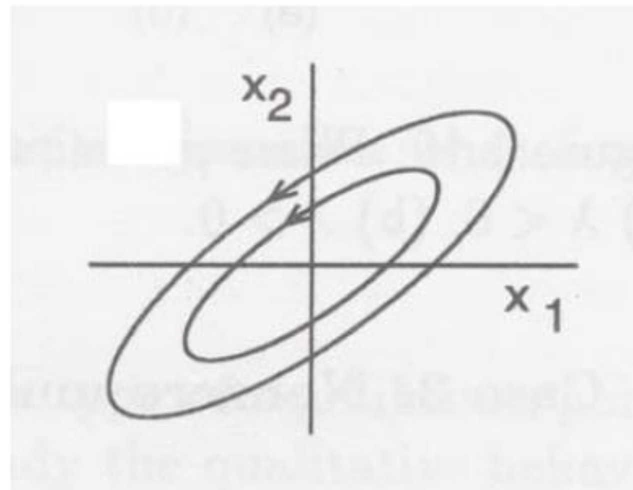
UNSTABLE FOCUS

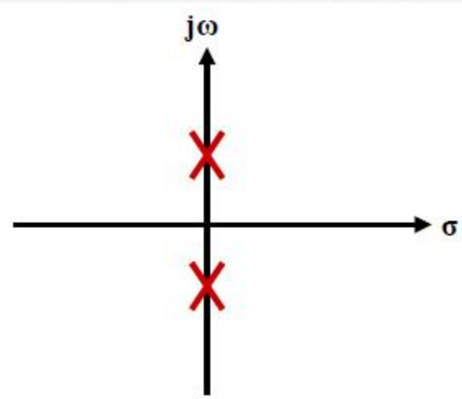




λ_1 and λ_2 are complex with zero real parts

CENTER





Matrix with complex eigenvalues

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \cdots \frac{1}{n!}A^n t^n + \cdots$$

$$\boxed{A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}$$

$$\boxed{e^{At} = e^{\sigma t} \underbrace{\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}}_{\text{rotation matrix}}}$$

Matrix with complex eigenvalues

Consider a matrix $A \in \mathcal{R}^{2\times 2}$

- Assume that the eigenvalues of A are complex conjugates of each other

$$\det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

$$\boxed{\lambda_1 = \sigma + \omega j} \qquad \boxed{\lambda_2 = \sigma - \omega j}$$

- The eigenvectors are also complex conjugates

$$(\lambda_1 I - A)\, t_1 = 0 \ \Rightarrow \qquad t_1 = t_R + j\, t_I$$

$$(\lambda_2 I - A)\, t_2 = 0 \ \Rightarrow \qquad t_2 = t_R - j\, t_I$$

Matrix with complex eigenvalues

First approach: **Diagonalization**

- Define $T \in \mathcal{C}^{2\times 2}$ and $\Lambda \in \mathcal{C}^{2\times 2}$

$$T = \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_R & t_R \end{bmatrix} + j \begin{bmatrix} t_I & -t_I \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} + j \begin{bmatrix} \omega & 0 \\ 0 & -\omega \end{bmatrix}$$

$$\boxed{e^{At} = T e^{\Lambda t} T^{-1} \in \mathcal{R}^{2\times 2}}$$

Matrix with complex eigenvalues

Second approach: **Oscillatory canonical form**

- Define $T_o \in \mathcal{R}^{2\times 2}$ and $A_o \in \mathcal{R}^{2\times 2}$

$$\boxed{T_o = \begin{bmatrix} t_R & t_I \end{bmatrix}}$$

$$\boxed{A_o = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}$$

$$\boxed{A \underbrace{\begin{bmatrix} t_R & t_I \end{bmatrix}}_{T_o} = \underbrace{\begin{bmatrix} t_R & t_I \end{bmatrix}}_{T_o} \underbrace{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}_{A_o}}$$

Matrix with complex eigenvalues

$$\begin{aligned} e^{At} &= T_o\, e^{A_o t}\, T_o^{-1} \\ &= e^{\sigma\, t}\, T_o \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} T_o^{-1} \end{aligned}$$

$$\lambda_1 = \sigma + \omega j \qquad \lambda_2 = \sigma - \omega j$$

$$T_o = \begin{bmatrix} t_R & t_I \end{bmatrix}$$

$$t_1 = t_R + j\, t_I$$

$$t_2 = t_R - j\, t_I$$

Matrix with repeated eigenvalues

Consider a matrix $A \in \mathcal{R}^{3\times 3}$

- Assume that $\boldsymbol{A}$ has 3 repeated eigenvalues

$$\det(\lambda I - A) = (\lambda - \lambda_m)^3$$

The number of linearly independent eigenvectors of $\boldsymbol{A}$ is equal to the $\text{nullity}\{(\lambda_m I - A)\}$

Nullity

$A \in \mathcal{R}^{3\times 3}$ $\qquad$ $\det(\lambda I - A) = (\lambda - \lambda_m)^3$

$$\text{nullity}\{(\lambda_m I - A)\} = 3 - \underbrace{\text{rank}\{(\lambda_m I - A)\}}_{\text{Number of linearly independent columns of } (\lambda_m I - A)}$$

Example of a matrix with repeated eigenvalues

$$A = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -1.5 \end{bmatrix}$$

1) Find eigenvalues:

$$\det(\lambda I - A) = (\lambda + 1)^2 \qquad \boxed{\lambda_m = -1}$$

2) Determine nullity of A:

$$(-1\,I - A) = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad \boxed{\text{nullity}\{(-1I - A)\} = 1}$$

Case 1: $\text{nullity}\{(\lambda_m I - A)\} = 1$

1. Find the only eigenvector of $\boldsymbol{A}$

$$(\lambda_m I - A)\, t_1 = 0$$

2. Find two ***generalized*** eigenvectors for $\boldsymbol{A}$

$$(\lambda_m I - A)\, t_2 \;=\; -t_1$$

$$(\lambda_m I - A)\, t_3 \;=\; -t_2$$

Case 1: nullity$\{(\lambda_m I - A)\} = 1$

$$(\lambda_m I - A)\, t_1 = 0 \qquad A\, t_1 = \lambda_m\, t_1$$

$$(\lambda_m I - A)\, t_2 = -t_1 \qquad A\, t_2 = \lambda_m\, t_2 + t_1$$

$$(\lambda_m I - A)\, t_3 = -t_2 \qquad A t_3 = \lambda_m\, t_3 + t_2$$

3. Lineup previous 3 equations column wise

$$A \left[\; t_1 \,|\, t_2 \,|\, t_3 \;\right] = \left[\; \lambda_m t_1 \,|\, \lambda_m t_2 + t_1 \,|\, \lambda_m t_3 + t_2 \;\right]$$

Case 1: $\text{nullity}\{(\lambda_m I - A)\} = 1$

$$A \left[\; t_1 \,|\, t_2 \,|\, t_3 \;\right] = \left[\; \lambda_m t_1 \,|\, \lambda_m t_2 + t_1 \,|\, \lambda_m t_3 + t_2 \;\right]$$

$$\Downarrow$$

$$A \underbrace{\left[\; t_1 \,|\, t_2 \,|\, t_3 \;\right]}_{T} = \underbrace{\left[\; t_1 \,|\, t_2 \,|\, t_3 \;\right]}_{T} \underbrace{\begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}}_{J}$$

Case 1: nullity$\{(\lambda_m I - A)\} = 1$

4. Compute the solution matrix

$$
\boxed{
\begin{aligned}
e^{At} &= T\, e^{Jt}\, T^{-1} \\
&= e^{\lambda_m t}\, T \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} T^{-1}
\end{aligned}
}
$$

Case 1: $\text{nullity}\{(\lambda_m I - A)\} = 2$

1. Find **2** linearly independent eigenvectors of $\boldsymbol{A}$

$$(\lambda_m I - A)\, t_1 = 0$$

$$(\lambda_m I - A)\, t_2 = 0$$

2. Find 1 ***generalized*** eigenvector for $\boldsymbol{A}$

$$(\lambda_m I - A)\, t_3 = -t_2$$

Case 2: nullity$\{(\lambda_m I - A)\} = 2$

$$(\lambda_m I - A)\,t_1 = 0 \qquad\qquad A\,t_1 = \lambda_m\,t_1$$

$$(\lambda_m I - A)\,t_2 = 0 \quad \Rightarrow \quad A\,t_2 = \lambda_m\,t_2$$

$$(\lambda_m I - A)\,t_3 = -t_2 \qquad\qquad At_3 = \lambda_m\,t_3 + t_2$$

3. Lineup previous 3 equations column wise

$$A\left[\ t_1 \,|\, t_2 \,|\, t_3\ \right] = \left[\ \lambda_m t_1 \,|\, \lambda_m t_2 \,|\, \lambda_m t_3 + t_2\ \right]$$

Case 2: nullity$\{(\lambda_m I - A)\} = 2$

$$A \left[\; t_1 \,|\, t_2 \,|\, t_3 \;\right] = \left[\; \lambda_m t_1 \,|\, \lambda_m t_2 \,|\, \lambda_m t_3 + t_2 \;\right]$$

$$\Downarrow$$

$$A \underbrace{\left[\; t_1 \,|\, t_2 \,|\, t_3 \;\right]}_{T} = \underbrace{\left[\; t_1 \,|\, t_2 \,|\, t_3 \;\right]}_{T} \underbrace{\begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}}_{J_d}$$

Case 2 : $\text{nullity}\{(\lambda_m I - A)\} = 2$

4. Compute the solution matrix

$$\boxed{\begin{aligned} e^{At} &= T\, e^{J_d t}\, T^{-1} \\ &= e^{\lambda_m t}\, T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} T^{-1} \end{aligned}}$$

Case 3 : $\text{nullity}\{(\lambda_m I - A)\} = 3$

Since $A \in \mathcal{R}^{3\times 3}$

$$\text{nullity}\{(\lambda_m I - A)\} = 3 \Leftrightarrow A = \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

$$\boxed{e^{At} \; = \; e^{\lambda_m t}\, I}$$

Example of a matrix with repeated eigenvalues

$$A = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -1.5 \end{bmatrix}$$

1) Find eigenvalues:

$$\det(\lambda I - A) = (\lambda + 1)^2 \qquad \boxed{\lambda_m = -1}$$

2) Determine nullity of A:

$$(-1\, I - A) = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad \boxed{\text{nullity}\{(-1I - A)\} = 1}$$

Example of a matrix with repeated eigenvalues

3) Find eigenvector t_1

$$\underbrace{\begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}}_{(-1I-A)} \underbrace{\begin{bmatrix} t_{11} \\ t_{12} \end{bmatrix}}_{t_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

4) Find generalized eigenvector t_2

$$\underbrace{\begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}}_{(-1I-A)} \underbrace{\begin{bmatrix} t_{21} \\ t_{22} \end{bmatrix}}_{t_2} = -\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{t_1} \Rightarrow t_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example of a matrix with repeated eigenvalues

5) Define T and J

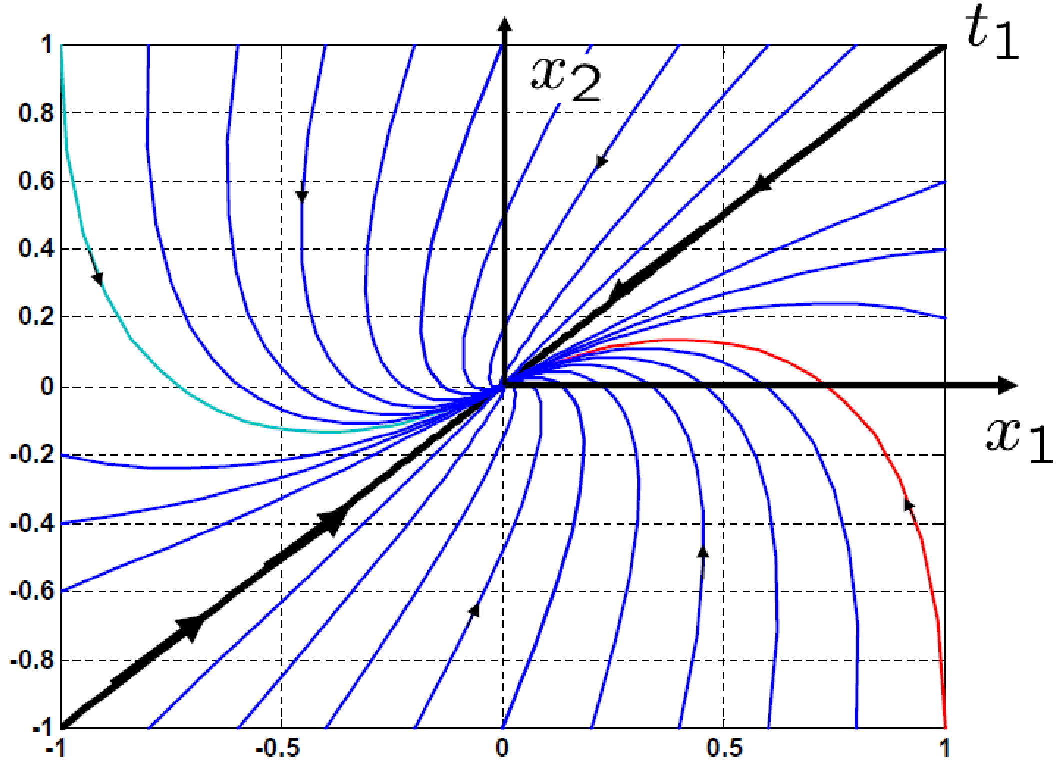
$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

6) Calculate e^{At}

$$\boxed{e^{At} = T \underbrace{e^{-t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}_{e^{Jt}} T^{-1}}$$

Example of a matrix with repeated eigenvalues

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -1.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Eigenvalue location and associated response mode $e^{\lambda t}$