HW9: Linear System Theory (ECE532)

Instructor: Jun Moon

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Due Date: June 4 (Monday) at the beginning of the class.

Problem 1

Consider the LQR problem

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ J(u) &= \int_0^{t_1} x_2^2 + u^2 dt\end{aligned}$$

Write down the Riccati differential equation (with its boundary condition), and the expressions for the optimal cost and the optimal state feedback controller u^* (these expressions will depend on the solution P to the Riccati differential equation, but you don't need to compute this solution).

Problem 2 Consider the infinite-horizon LQR problem

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ J(u) &= \int_0^{\infty} x_2^2 + u^2 dt\end{aligned}$$

Show that the closed-loop system is stable (in the sense of Lyapunov), but not asymptotically stable. Which condition of the theorem that guarantees closed-loop system asymptotically is violated?

Problem 3 Consider the algebraic Riccati equation (ARE):

$$A^T P + PA + Q - PBR^{-1}B^T P = 0,\ A = \begin{pmatrix} 0 & 0 \\ -1 & -\sqrt{2} \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ R = 1, Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

First justify that the ARE admits a unique positive definite solution, and then find solution. Verify your solution by MATLAB "care".

Problem 4 Consider the linearized magnetically suspended ball:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_1 + u$$

with the cost function

$$J(u) = \int_0^\infty x_1^2 + 2x_1x_2 + x_2^2 + 4u^2 dt$$

(i) Obtain the LQR controller

(ii) Determine the eigenvalues of the closed-loop system using the solution in (i).

(iii) Determine the eigenvalues of the closed-loop system using directly the Hamiltonian matrix associated with this system

(iv) What is the minimum value of J as a function of $x(0)$?

Problem 5 Consider the scalar LQR problem

$$\dot{x} = ax + bu, \ J(u) = \int_0^\infty qx^2 + ru^2 dt$$

where a, q, r are positive, and b is arbitrary. (Note that with $u = 0$, the system is unstable). Suppose that a, b, q are fixed, but r can vary.

- Show that in the limit as $r \to \infty$ (the expensive control), the optimal control yields the closed-loop dynamics $\dot{x} = -ax$. In other words, the eigenvalue of the optimal closed-loop system tends to $-a$, the opposite of the open-loop eigenvalue.
- Show that in the limit as $r \to 0$ (the cheap control), the eigenvalue of the optimal closed-loop system moves off to $-\infty$.

Problem 6 Consider the discrete-time LQR problem

$$J(u) = \sum_{k=0}^{N-1} x_k^T Q x + u_k^T R u_k + x_N^T M x_N$$
$$x_{k+1} = Ax_k + Bu_k, \ x(0) = x_0$$

where $R > 0$ and $Q, M \geq 0$.

- By using completion of squares, show that the following control is the optimal control

  $$u_k^* = -(R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A x_k$$

  where P is the Riccati difference equation

  $$P_k = A^T P_{k+1} A + Q - A^T P_{k+1} B (R + B^T P_{k+1} B)^{-1} B^T P_{k+1} A, \ P_N = M.$$

- Suppose that (A, B) is controllable and (A, C) is observable where $Q = C^T C$. Show that

  $$\lim_{N \to \infty} P_{k,N} = Z$$

  where Z is the solution to the following algebraic Riccati equation

  $$0 = A^T Z A + Q - A^T Z B (R + B^T Z B)^{-1} B^T Z A$$

Also, show that the close-loop matrix

$$A_{cl} = A - (R + B^T Z B)^{-1} B^T Z$$

has eigenvalues inside a unit circle. Hence,

$$x_{k+1} = A_{cl} x_k$$

is asymptotically stable.

- Suppose

$$\dot{x} = Ax + Bu, \qquad A = \begin{bmatrix} 0 & 1 \\ 2 & 8 \end{bmatrix}, \; B = [0 \; 1]^T.$$

Obtain the discrete-time system by using the sampling rate $20Hz$. Design the discrete-time LQR controller by choosing appropriate R and Q. Use MATLAB "dare" to compute the solution of the algebraic Riccati equation. Use MATLAB, show that the closed-loop system is asymptotically stable.