# [MEN573]
# Advanced Control Systems I

## Lecture 2 – Laplace Transform

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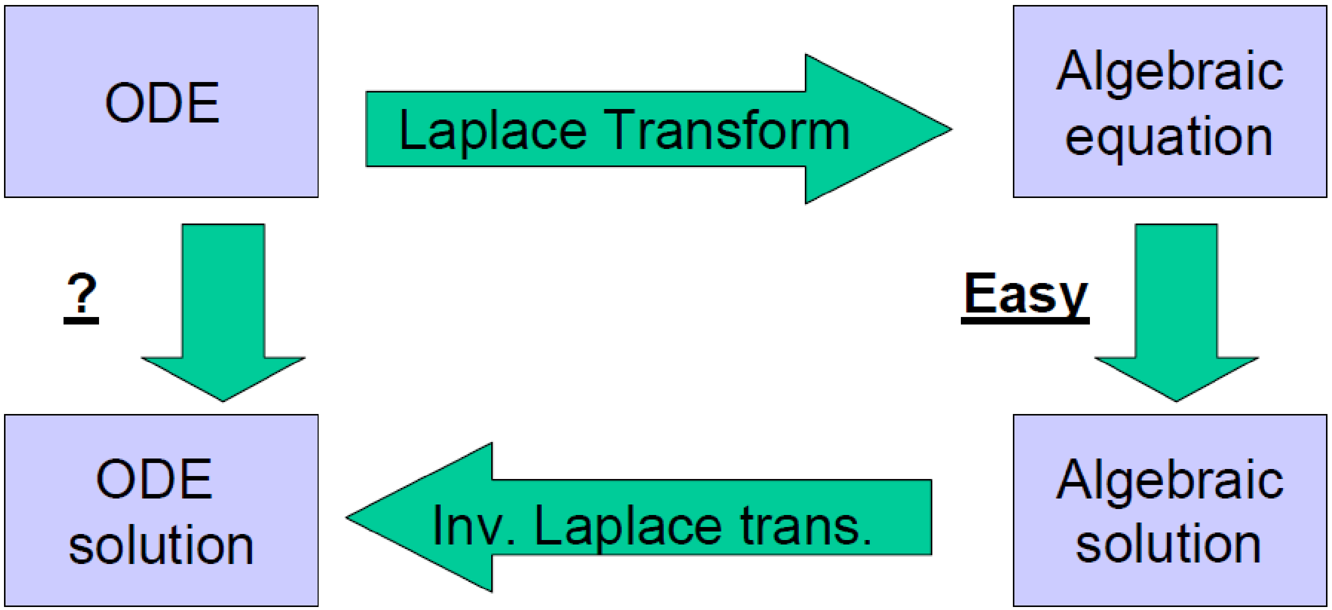
# Outline

- Introduction
- Continuous time function
- Laplace transform definition
- Examples
- Laplace transform properties
- Applications of Laplace transform

# Brief review of Laplace transformation

- The Laplace transform is a powerful tool formulated to solve a wide variety of Ordinary Differential Equations (ODEs).



# Brief review of Laplace transformation

- Definition

$$\boxed{F(s) = \mathcal{L}\{f(t)\}}$$

$$\boxed{\mathcal{L}\{f(t)\} \triangleq \int_{0-}^{\infty} f(t)e^{-st}dt}$$

$s \in \mathcal{C}$ Complex number

# Brief review of Laplace transformation

- Exponential

$$\boxed{\begin{array}{l} f(t) = e^{-at} \\ \\ F(s) = \dfrac{1}{s+a} \end{array}}$$

$a \in \mathcal{C}$

$$\begin{aligned} F(s) &= \int_{0-}^{\infty} e^{-at} e^{-st} dt = \int_{0-}^{\infty} e^{-(s+a)t} dt \\ &= \frac{-1}{s+a} \left\{ e^{-(s+a)t} \right\}_{t=0}^{t\to\infty} = \frac{-1}{s+a} \left\{ \underbrace{\lim_{t\to\infty} e^{-(s+a)t}}_{=0} - 1 \right\} \\ &= \frac{1}{s+a} \end{aligned}$$

# Brief review of Laplace transformation

- Step function

$$\begin{aligned} f(t) &= 1(t) \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$

$$F(s) = \frac{1}{s}$$

Calculation: Use

$$1 = e^{0t} \qquad t \geq 0$$

$$e^{-at} \leftrightarrow \frac{1}{s+a}, \quad a = 0$$

# Brief review of Laplace transformation

- Sine

$$f(t) = \sin(\omega t)$$

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

- Calculation: Use

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \qquad \mathcal{L}\left\{e^{j\omega t}\right\} = \frac{1}{s - j\omega}$$

# Brief review of Laplace transformation

- Cosine

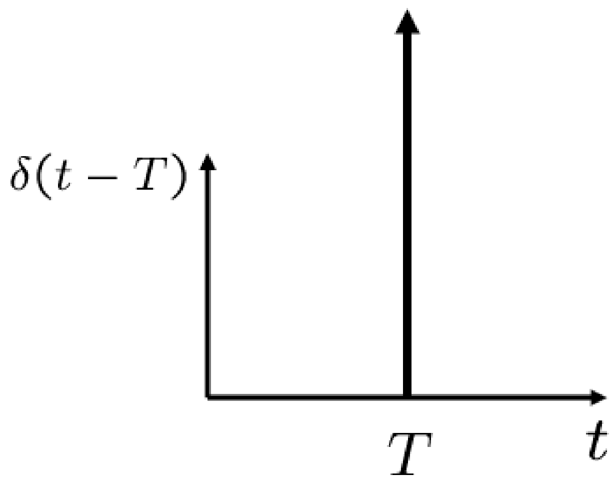
$$\boxed{\begin{aligned} f(t) &= \cos(\omega t) \\ F(s) &= \frac{s}{s^2+\omega^2} \end{aligned}}$$

- Calculation: Use

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \mathcal{L}\left\{e^{j\omega t}\right\} = \frac{1}{s - j\omega}$$

# Brief review of Laplace transformation

- Dirac impulse




$$\int_0^\infty \delta(t-T)dt = 1$$

$$\int_0^\infty \delta(t-T)f(t)dt = f(T)$$

$$\boxed{\begin{array}{c} f(t) = \delta(t) \\ \\ F(s) = 1 \end{array}}$$

- Calculation: $$\int_0^\infty \delta(t)f(t)dt = f(0)$$

$$\mathcal{L}\{\delta(t)\} = \int_{0-}^\infty e^{-st}\delta(t)dt = e^{-s0} = 1$$

# Laplace transformation properties

- Linearity

For any $\alpha,\ \beta \in \mathcal{C}$ and functions $f(t),\ g(t)$

$$F(s) = \mathcal{L}\left\{f(t)\right\} \qquad G(s) = \mathcal{L}\left\{g(t)\right\}$$

$$\boxed{\begin{aligned} \mathcal{L}\left\{\alpha\, f(t) + \beta\, g(t)\right\} &= \alpha\,\mathcal{L}\left\{f(t)\right\} + \beta\,\mathcal{L}\left\{g(t)\right\} \\ &= \alpha\, F(s) + \beta\, G(s) \end{aligned}}$$

# Laplace transformation properties

- Differentiation

Defining $$\dot{f}(t) = \frac{df(t)}{dt}$$

$$F(s) = \mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\left\{\dot{f}(t)\right\} = sF(s) - f(0-)}$$

# Laplace transformation properties

- Differentiation

Integration by parts

$$
\begin{aligned}
\mathcal{L}\left\{(\dot{f}(t)\right\} &= \int_{0-}^{\infty} e^{-st}\dot{f}(t)dt \\
&= -\int_{0-}^{\infty} \frac{de^{-st}}{dt} f(t)dt + \left\{e^{-st}f(t)\right\}_{t=0-}^{t\to\infty} \\
&= s\int_{0-}^{\infty} e^{-st}f(t)dt - f(0-) \\
&= sF(s) - f(0-)
\end{aligned}
$$

# Laplace transformation properties

- Integration

Defining $F(s) = \mathcal{L}\{f(t)\}$

$$\boxed{\mathcal{L}\left\{\int_{0-}^{t} f(\tau)d\tau\right\} = \frac{1}{s}F(s)}$$

# Laplace transformation properties

- Multiplication by $e^{-at}$

Defining $F(s) = \mathcal{L}\left\{f(t)\right\}$

$$\boxed{\mathcal{L}\left\{e^{-at} f(t)\right\} \quad = \quad F(s+a)}$$

Example:

$$\mathcal{L}\left\{\sin(\omega t)\right\} = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\left\{e^{-at}\sin(\omega t)\right\} = \frac{\omega}{(s+a)^2+\omega^2}$$

# Laplace transformation properties

- Multiplication by $t$

Defining $F(s) = \mathcal{L}\{f(t)\}$

$$\boxed{\mathcal{L}\{t\,f(t)\} = -\frac{dF(s)}{ds}}$$

Example:

$$\mathcal{L}\{1(t)\} = \frac{1}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2}$$

# Laplace transformation properties

- Time delay $\tau$

Defining $F(s) = \mathcal{L}\{f(t)\}$

$$\boxed{\mathcal{L}\{f(t-\tau)\} \quad = \quad e^{-s\tau}F(s)}$$

Calculation:

$$\begin{aligned}
\mathcal{L}\{f(t-\tau)\} \quad &= \quad \int_{\tau}^{\infty} e^{-st} f(t-\tau)dt = e^{-s\tau}\int_{\tau}^{\infty} e^{-s(t-\tau)} f(t-\tau)dt \\
&= \quad e^{-s\tau}\int_{0}^{\infty} e^{-s(\eta)} f(\eta)d\eta = e^{-s\tau}F(s)
\end{aligned}$$

# Laplace transformation properties

- Convolution

Given f(t), g(t)

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

$$= (g \star f)(t)$$

$$\boxed{\mathcal{L}\{(f \star g)(t)\} = F(s)\,G(s)}$$

# Laplace transformation properties

$$\begin{aligned}
\mathcal{L}\{(f \star g)(t)\} &= \int_0^\infty e^{-st} \int_0^t f(t-\tau) g(\tau) d\tau dt \\
&= \int_0^\infty \int_0^t \left(e^{-s(t-\tau)} f(t-\tau)\right) \left(e^{-s\tau} g(\tau)\right) d\tau dt \\
&= \int_0^\infty \int_0^\infty \left(e^{-s(t-\tau)} f(t-\tau)\right) \left(e^{-s\tau} g(\tau)\right) d\tau dt
\end{aligned}$$

$(f(t-\tau) = 0$ for $\tau > t)$

$$\begin{aligned}
&= \int_0^\infty \left\{ \int_0^\infty e^{-s(t-\tau)} f(t-\tau) dt \right\} e^{-s\tau} g(\tau) d\tau \\
&= \int_0^\infty \left\{ \int_{-\tau}^\infty e^{-s\gamma} f(\gamma) d\gamma \right\} e^{-s\tau} g(\tau) d\tau
\end{aligned}$$

$(f(\gamma) = 0$ for $\gamma < 0)$

$$= \left\{ \int_0^\infty e^{-s\gamma} f(\gamma) d\gamma \right\} \left\{ \int_0^\infty e^{-s\tau} g(\tau) d\tau \right\} = F(s)G(s)$$

# Laplace transformation properties

- Initial Value Theorem

$$f(0+) = \lim_{t \to 0+} f(t) \qquad \text{exists}$$

$$\boxed{f(0+) = \lim_{s \to \infty} s\,F(s)}$$

# Laplace transformation properties

- Final Value Theorem

$$\lim_{t \to \infty} f(t) \quad \text{exists}$$

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)}$$

# Applications of Laplace transformation

- Solution of a first order ODE

Let: $a > 0,\ b > 0,\ y(0) = y_o \in \mathcal{R}$

Obtain the solution to the ODE:

$$\dot{y}(t) = -a\, y(t) + b\, 1(t)$$

$$1(t) \;=\; \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

# Applications of Laplace transformation

- Apply Laplace transform

$$\mathcal{L}\{\dot{y}(t)\} = -a\,\mathcal{L}\{y(t)\} + b\,\mathcal{L}\{1(t)\}$$

$$s\,Y(s) - y(0) = -a\,Y(s) + b\,\frac{1}{s}$$

- Algebraic manipulation:

$$\boxed{Y(s) = \frac{1}{s+a}\,y(0) + \frac{b}{s(s+a)}}$$

# Applications of Laplace transformation

- Apply inverse Laplace transform

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} y(0) + \mathcal{L}^{-1}\left\{\frac{b}{s(s+a)}\right\}$$

  - Use table look up and partial fraction expansion

$$\boxed{y(t) = e^{-at} y(0) + \frac{b}{a}\left(1(t) - e^{-at}\right)}$$

# Applications of Laplace transformation

- Use table look up and partial fraction expansion

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1(t)$$

$$\frac{b}{s(s+a)} = \frac{b}{a}\left\{\frac{1}{s} - \frac{1}{s+a}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{s(s+a)}\right\} = \frac{b}{a}\left\{1(t) - e^{-at}\right\}$$

# Applications of Laplace transformation

- Solution in Laplace domain

$$Y(s) = \frac{1}{s+a} y(0) + \frac{b}{s(s+a)}$$

- Solution in time domain

$$y(t) = e^{-at} y(0) + \frac{b}{a} \left\{ 1(t) - e^{-at} \right\}$$

- Use initial value theorem:

$$\begin{aligned} y(0) &= \lim_{s \to \infty} \left\{ \frac{s}{s+a} y(0) + \frac{bs}{s(s+a)} \right\} \\ &= y(0) \end{aligned}$$

- Use final value theorem:

$$\begin{aligned} \lim_{t \to \infty} y(t) &= \lim_{s \to 0} \left\{ \frac{s}{s+a} y(0) + \frac{bs}{s(s+a)} \right\} \\ &= \frac{b}{a} \end{aligned}$$

# Applications of Laplace transformation

- Solution of a first order ODE

Let: $a > 0,\ b > 0,\ y(0-) = y_o \in \mathcal{R}$

Obtain the solution to the ODE:

$$\dot{y}(t) = -a\,y(t) + b\,\delta(t)$$

$\delta(t)$ : Dirac impulse

# Applications of Laplace transformation

- Apply Laplace transform

$$\mathcal{L}\{\dot{y}(t)\} \quad = \quad -a\,\mathcal{L}\{y(t)\} \; + b\,\mathcal{L}\{\delta(t)\}$$

$$s\,Y(s) - y(0-) \quad = \quad -a\,Y(s) + b$$

  - Algebraic manipulation:

$$\boxed{Y(s) \quad = \quad \frac{1}{s+a}\,\{y(0-) + b\}}$$

# Applications of Laplace transformation

- Apply inverse Laplace transform

$$\mathcal{L}^{-1}\{Y(s)\} \;=\; \mathcal{L}^{-1}\left\{\frac{1}{s+a}\,[y(0-)+b]\right\}$$

  - Use table look up

$$\boxed{y(t) \;=\; e^{-at}\,\{y(0-)+b\}}$$

**Notice that $y(t)$ is discontinuous at $0$**

$$y(0-) \;=\; y_o \qquad y(0+) \;=\; y_o + b$$

# Applications of Laplace transformation

- Solution in Laplace domain

$$Y(s) \;=\; \frac{1}{s+a}\,\{y(0-)+b\}$$

- Use initial value theorem:

$$\begin{aligned} y(0+) \;&=\; \lim_{s\to\infty}\left\{\frac{s}{s+a}\,[y(0-)+b]\right\} \\ &=\; y(0-)+b = y_o + b \end{aligned}$$

# Transfer Function

N-th order differential equation:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

$$y(0) = 0, \left.\frac{dy}{dt}\right|_{t=0} = 0, \cdots, \left.\frac{d^{n-1}y}{dt^{n-1}}\right|_{t=0} = 0$$

Apply Laplace transformation

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_0)Y(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)U(s)$$

$$\Rightarrow Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} U(s)$$

# Transfer Function

$$\boxed{G(s)=\frac{B(s)}{A(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_0}}$$

- ***A(s)=0***: Characteristic equation
- Roots of C.E. = Poles of ***G(s)***
- Roots of ***B(s)=0*** = Zeros of ***G(s)***
- $m \le n$ : realizability condition

- Note that pure differentiation (s) is an unrealizable operation: to find du(t)/dt you need to know u(t+ε) (ε>0) which is a future value.