CSE530: Algorithms & Complexity
Solutions to Exercise Set 3

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1. Let $A[1 \ldots n]$ be an array of n numbers. A *maximum sum subarray* of A is a subarray $A[p \ldots q]$ such that the sum $\sum_{i=p}^{q} A[i]$ of its elements is maximum. For instance, if $A = [-2, 1, -3, 4, -1, 2, 1, -5, 4]$, then a maximum sum subarray is $[4, -1, 2, 1]$, and its sum is 6.

Give an algorithm that computes the sum of a maximum sum subarray in $O(n^2)$ time. For instance, in the example above, your algorithm should return 6.

Answer. We use dynamic programming to compute all the sums $S_i^j = \sum_{k=i}^{j} A[k]$, and then we return the maximum value. (See Algorithm 1.) It can be done in quadratic time using the relation $S_i^j = S_i^{j-1} + A[j]$

Algorithm 1

procedure MaximumSum($A[1 \ldots n]$)
    result $\leftarrow A[1]$
    for $i \leftarrow 1, n$ **do**
        $S \leftarrow 0$
        for $j \leftarrow i, n$ **do**
            $S \leftarrow S + A[j]$ ▷ Now $S = \sum_{k=i}^{j} A[k]$
            result $\leftarrow \max(S, \text{result})$
    return result

Remark. This problem can even be solved in linear time. See the wikipedia page on the *Maximum subarray problem*.

2. You are given k types of coins with values $v_1, \ldots, v_k \in \mathbb{N}$ and a cost $C \in \mathbb{N}$. You may assume $v_1 = 1$ so that it is always possible to make any cost. You want to find the smallest number of coins required to sum to C exactly. For example, assume you have coins of values 1, 5, and 10. Then the smallest number of coins to make 26 is 4: take 2 coins of value 10, 1 coin of value 5, and 1 coin of value 1.

Give an algorithm for this problem. Its running time should be polynomial in $n = \max(k, C)$.

Answer. Let $S(C)$ denote the smallest number of coins needed to make cost C. Then it satisfies the relation

$$S(C) = \begin{cases} 0 & \text{if } C = 0 \\ 1 + \min\{S(C - v_j) \mid 1 \leqslant j \leqslant k \text{ and } v_j \leqslant C\} & \text{if } C > 0. \end{cases}$$

Based on this recurrence relation, we can compute $S(C)$ in $O(kC) = O(n^2)$ time with the following algorithm.

Algorithm 2

procedure CHANGE($C, V[1 \dots k]$) $\triangleright$ $V[i]$ is v_i
    $S[0 \dots C] \leftarrow$ new array
    $S[0] \leftarrow 0$
    for $i \leftarrow 1, C$ **do**
        $S[i] \leftarrow 1 + S[i-1]$
        for $j \leftarrow 2, k$ **do**
            if $v_j \leqslant i$ **then**
                $S[i] \leftarrow \min(S[i], 1 + S[i - v_j])$
    return $S[C]$

3. Suppose General Motors makes a profit of \$100 on each Chevrolet, \$200 on each Buick and \$400 on each Cadillac. These cars get 20, 17, and 14 miles a gallon respectively, and it takes respectively 1, 2, and 3 minutes to assemble one Chevrolet, one Buick, and one Cadillac. The government requires that the fuel efficiency of the cars produced by General Motors be on average at least 18 miles a gallon. General Motors needs to determine the optimal number of cars that can be assembled in an 8-hours day, so that the profit is maximized.

(a) Formulate this problem as a linear program.

(b) Solve this problem numerically using a linear program solver.

Answer a. Let x, y, z represent the number of Chevrolet, Buick and Cadillac assembled in one day, respectively. So we need $x + 2y + 3z$ minutes to assemble all the cars, which gives the constraint

$$x + 2y + 3z \leqslant 480. \tag{1}$$

The average mileage per gallon is

$$\frac{20x + 17y + 14z}{x + y + z}$$

This quantity should be at least 18, so we obtain the second constraint:

$$2x - y - 4z \geqslant 0. \tag{2}$$

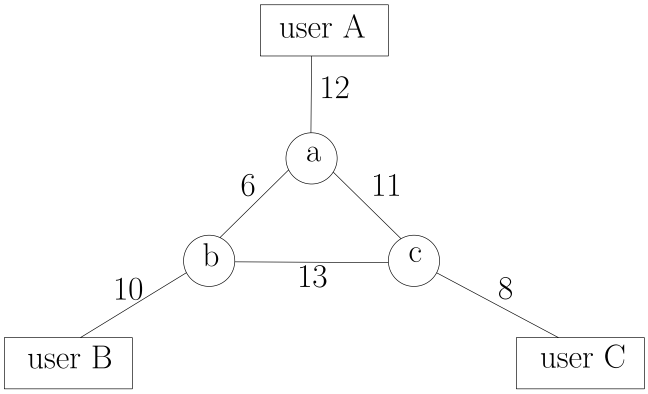
We also require that $x, y, z \geqslant 0$ as these quantities are numbers of cars produced.

As our goal is to maximize the profit, which is $100x + 200y + 400z$ dollars, we obtain the following linear program:

$$\begin{array}{lrcrcrcr}
\text{Maximize} & 100x & + & 200y & + & 400z & & \\
\text{Subject to} & x & + & 2y & + & 3z & \leqslant & 480 \\
 & 2x & - & y & - & 4z & \geqslant & 0 \\
 & & & & & x, y, z & \geqslant & 0.
\end{array}$$

Answer b. Using the solver at `http://www.zweigmedia.com/RealWorld/simplex.html`, the optimal profit is $\$57,600$, with $x = 192$, $y = 0$ and $z = 96$.

4. Suppose that you are managing a network whose lines have the bandwidths shown in the figure below, and you need to establish three connections: between users A and B, between B and C, and between A and C. Each connection requires at least two units of bandwidth, but can be assigned more. Connection AB pays \$3 per unit of bandwidth, and connections BC and AC pay \$2 and \$4, respectively. Each connection can be routed in two ways, a long path and a short path, or by a combination: for instance, between A and B, two units of bandwidth via the short route AabB, and one via the long route AacbB. Your goal is to route these connections so as to maximize the network's revenue. Formulate this problem as a linear program.



Answer. We use 6 variables: the variable x_{AB} is the bandwidth allocated to the short path between A and B, x'_{AB} is the bandwidth allocated to the long path between A and B ... So the revenue, that we want to maximize, is

$$3x_{AB} + 3x'_{AB} + 2x_{BC} + 2x'_{BC} + 4x_{CA} + 4x'_{CA}.$$

We have two types of constraints. First, we want each user to be allocated at least 2 units of bandwidth, so we must have $x_{AB} + x'_{AB} \geqslant 2$ ... Second we have the bandwidth constraint on each link. So if we consider the link bc, for instance, we must have $x_{BC} + x'_{AB} + x'_{AC} \leqslant 13$. So we obtain the following linear program:

$$\begin{aligned}
\text{Maximize} \quad & 3x_{AB} + 3x'_{AB} + 2x_{BC} + 2x'_{BC} + 4x_{CA} + 4x'_{CA} \\
\text{subject to} \quad & x_{AB} + x_{AC} + x'_{AB} + x'_{AC} \leqslant 10 \\
& x_{AB} + x_{BC} + x'_{AB} + x'_{BC} \leqslant 12 \\
& x_{AC} + x_{BC} + x'_{AC} + x'_{BC} \leqslant 8 \\
& x_{AB} + x'_{AC} + x'_{BC} \leqslant 6 \\
& x_{AC} + x'_{AB} + x'_{BC} \leqslant 11 \\
& x_{BC} + x'_{AB} + x'_{AC} \leqslant 13 \\
& x_{AB} + x'_{AB} \geqslant 2 \\
& x_{AC} + x'_{AC} \geqslant 2 \\
& x_{BC} + x'_{BC} \geqslant 2 \\
& x_{AB}, x'_{AB}, x_{AC}, x'_{AC}, x_{BC}, x'_{BC} \geqslant 0
\end{aligned}$$

5. We consider a line-fitting problem similar with the one studied in class. The difference is that, instead of minimizing the sum of the vertical distances to the points, we want to minimize the maximum vertical distance. So the input is still a set of n points (x_i, y_i), and the output should be the coefficients a and b of a line $y = ax + b$ such that

$$\max_{i \in \{1,\dots,n\}} |ax_i + b - y_i|$$

is minimized.

Formulate this problem as a linear program.

Answer. We use three variables: The coefficients a and b of the line, and the "error" ε made by the line. So for each i, we must have $|ax_i + b - y_i| \leqslant \varepsilon$, which is equivalent to $ax_i + b - y_i \leqslant \varepsilon$ and $-ax_i - b + y_i \leqslant \varepsilon$. We want to minimize the error ε, so the linear program has three variables a, b and ε, and can be written:

$$\begin{array}{lrl} \text{minimize} & \varepsilon & \\ \text{subject to} & ax_i + b - \varepsilon \leqslant y_i, & i = 1, \dots, n \\ & -ax_i - b - \varepsilon \leqslant -y_i, & i = 1, \dots, n \end{array}$$

6. Show that the set of optimal solutions to a linear program is a convex set.

Answer. We may assume that the LP is feasible and bounded as otherwise, the set of optimal solutions is empty and hence convex. Let C_{opt} be the value of the optimal solutions. Let H denote the hyperplane with equation $f(x) = C_{opt}$, where f is the objective function. Then the set of optimal solutions is the intersection of the feasible region with H. The feasible region is convex, as we saw in class, and H is also convex, because any hyperplane is convex. So the set of optimal solutions is the intersection of two convex sets, and hence it is convex.

7. Consider the linear program below.

$$\begin{array}{lrcrcrl} \text{maximize} & 2x_1 & + & 3x_2 & + & 2x_3 & \\ \text{subject to} & x_1 & & & + & x_3 & \leqslant 1 \\ & & & x_2 & + & x_3 & \leqslant 6 \\ & x_1 & + & x_2 & - & x_3 & \leqslant 5 \\ & & & & & x_1, x_2, x_3 & \geqslant 0 \end{array}$$

Prove that $(x_1^*, x_2^*, x_3^*) = \left(\frac{1}{3}, \frac{16}{3}, \frac{2}{3}\right)$ is an optimal solution to this linear program.

Answer. The solution x^* is feasible and has value 18. Let C_1, C_2, C_3 denote the first three constraints of the linear program, in the same order as above. Then the linear combination $C_1 + 2C_3 + C_3$ gives $2x_1 + 3x_2 + 2x_3 \leqslant 18$, thus x^* is optimal.

8. Solve the linear program below using the simplex algorithm.

$$\begin{array}{lrcrcrl} \text{maximize} & 42x_1 & + & 39x_2 & + & 52x_3 & \\ \text{subject to} & 9x_1 & + & 5x_2 & + & 6x_3 & \leqslant 600 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leqslant 150 \\ & & & & & x_3 & \leqslant 60 \\ & x_1 & + & x_2 & + & x_3 & \leqslant 90 \\ & & & & & x_1, x_2, x_3 & \geqslant 0 \end{array}$$

Answer. The initial slack form is:

$$\begin{array}{rcrcrcrcr} z & = & & & 42x_1 & + & 39x_2 & + & 52x_3 \\ x_4 & = & 600 & - & 9x_1 & - & 5x_2 & - & 6x_3 \\ x_5 & = & 150 & - & 2x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 60 & & & & & - & x_3 \\ x_7 & = & 90 & - & x_1 & - & x_2 & - & x_3 \end{array}$$

We choose $e = 3$, $\ell = 6$, and thus $x_3 = 60$. So the second slack form is:

$$\begin{array}{rcrcrcrcr} z & = & 3120 & + & 42x_1 & + & 39x_2 & - & 52x_6 \\ x_3 & = & 60 & & & & & - & x_6 \\ x_4 & = & 240 & - & 9x_1 & - & 5x_2 & + & 6x_6 \\ x_5 & = & 30 & - & 2x_1 & - & x_2 & + & 2x_6 \\ x_7 & = & 30 & - & x_1 & - & x_2 & + & x_6 \end{array}$$

We choose $e = 2$, $\ell = 5$, and $x_2 = 30$. We obtain:

$$\begin{array}{rcrcrcrcr} z & = & 4290 & - & 36x_1 & - & 39x_5 & + & 26x_6 \\ x_2 & = & 30 & - & 2x_1 & - & x_5 & + & 2x_6 \\ x_3 & = & 60 & & & & & - & x_6 \\ x_4 & = & 90 & + & x_1 & - & 5x_5 & - & 4x_6 \\ x_7 & = & & & x_1 & + & x_5 & - & x_6 \end{array}$$

We choose $e = 6$, $\ell = 7$, and $x_6 = 0$. We get:

$$\begin{array}{rcrcrcrcr} z & = & 4290 & - & 10x_1 & - & 13x_5 & - & 26x_7 \\ x_2 & = & 30 & & & + & x_5 & - & 2x_7 \\ x_3 & = & 60 & - & x_1 & - & x_5 & + & x_7 \\ x_4 & = & 90 & - & 3x_1 & - & 2x_5 & + & 4x_7 \\ x_6 & = & & & x_1 & + & x_5 & - & x_7 \end{array}$$

So the optimal value is 4290, and an optimal solution is $x_1^* = 0$, $x_2^* = 30$, $x_3^* = 60$.

9. Solve the linear program below using the simplex algorithm.

$$\begin{array}{ll} \text{Maximize} & x_1 - 2x_3 \\ \text{subject to} & \begin{array}{rl} x_1 - x_2 & \leqslant 1 \\ 2x_2 - x_3 & \leqslant 1 \\ x_1, x_2, x_3 & \geqslant 0 \end{array} \end{array}$$

Answer. We first write the slack form for this linear program:

$$\begin{array}{rcrcrcrcr} z & = & & & x_1 & & & - & 2x_3 \\ x_4 & = & 1 & - & x_1 & + & x_2 & & \\ x_5 & = & 1 & & & - & 2x_2 & + & x_3 \end{array}$$

So at the first step, the only choice for the entering and leaving variables are x_1 and x_4, respectively, with x_1 increasing by 1. We have $x_1 = 1 + x_2 - x_4$, and thus the new slack form is:

$$\begin{array}{rcrcrcrcr} z & = & 1 & + & x_2 & - & 2x_3 & - & x_4 \\ x_1 & = & 1 & + & x_2 & & & - & x_4 \\ x_5 & = & 1 & - & 2x_2 & + & x_3 & & \end{array}$$

We now have to take x_2 as the entering variable and x_5 the leaving variable, so x_2 increases by $\frac{1}{2}$. We have $x_2 = \frac{1}{2} + \frac{1}{2}x_3 - \frac{1}{2}x_5$, so we obtain the slack form:

$$\begin{array}{rcrcrcrcr} z & = & \frac{3}{2} & - & \frac{3}{2}x_3 & - & x_4 & - & \frac{1}{2}x_5 \\ x_1 & = & \frac{3}{2} & + & \frac{1}{2}x_3 & - & x_4 & - & \frac{1}{2}x_5 \\ x_2 & = & \frac{1}{2} & + & \frac{1}{2}x_3 & - & & & \frac{1}{2}x_5 \end{array}$$

The coefficients of the non-basic variables in the objective function are all negative, so the basic solution of this slack form is optimal. So the optimal value is $\frac{3}{2}$ for $(x_1^*, x_2^*, x_3^*) = (\frac{3}{2}, \frac{1}{2}, 0)$.