[MEN573] Advanced Control Systems I

Lecture 17
State Variable Feedback Control

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State Variable Feedback

Consider an nth order LTI continuous time:

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x \end{aligned}$$

$$\begin{aligned} y &\in \mathcal{R}^m \\ u &\in \mathcal{R}^m \end{aligned}$$

In the Laplace domain

$$\begin{aligned} X(s) &= [sI - A]^{-1} B\,u \\ Y(s) &= C\,X(s) \end{aligned}$$

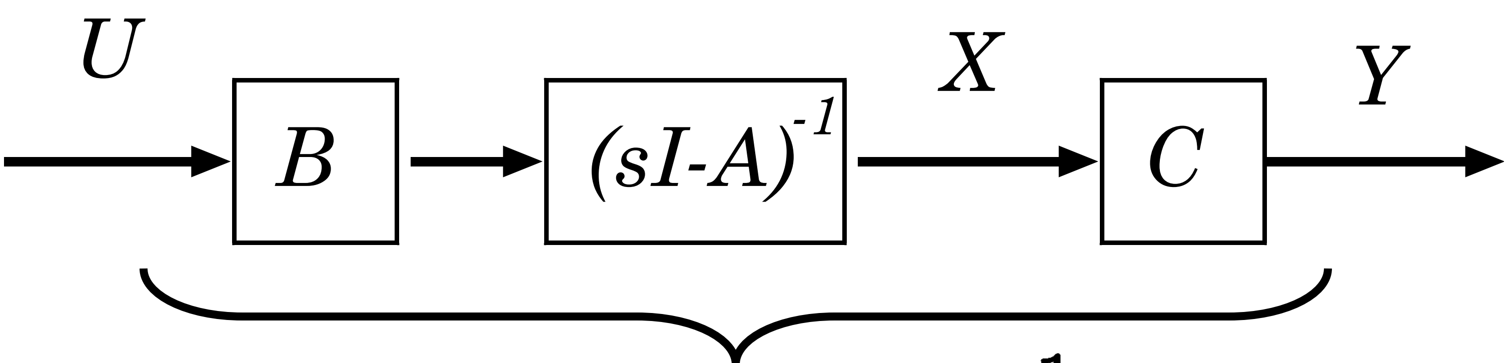
State Variable Feedback

Consider an nth order LTI continuous time:

$$X(s) = [sI - A]^{-1} B\, U(s) \qquad y \in \mathcal{R}^m$$

$$Y(s) = C\, X(s) \qquad u \in \mathcal{R}^m$$

In the Laplace domain



$$G(s) = \frac{1}{A(s)} B(s)$$

State Variable Feedback

Solution matrix in Laplace domain

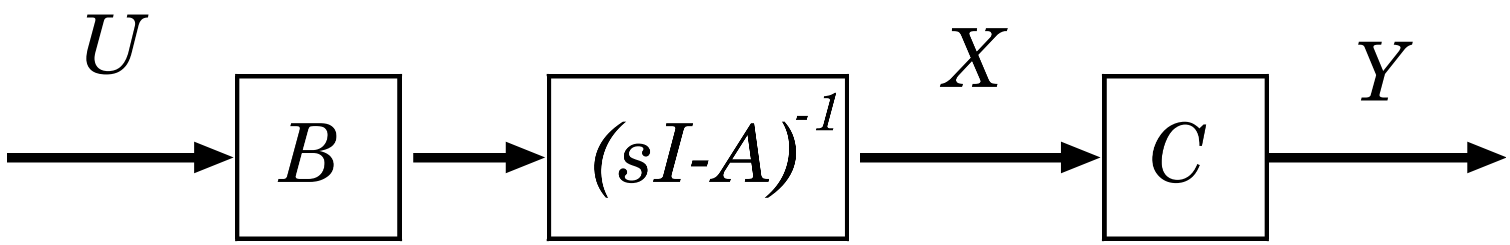
$$[sI - A]^{-1} = \mathcal{L}\{e^{At}\}$$

$$[sI - A]^{-1} = \frac{1}{\mathrm{Det}\{[sI - A]\}} \mathrm{Adj}\{[sI - A]\}$$

Common denominator:

$$\begin{aligned} A(s) &= \mathrm{Det}\{[sI - A]\} \\ &= s^n + a_{n-1}s^{n-1} + \cdots + a_0 \end{aligned}$$

State Variable Feedback



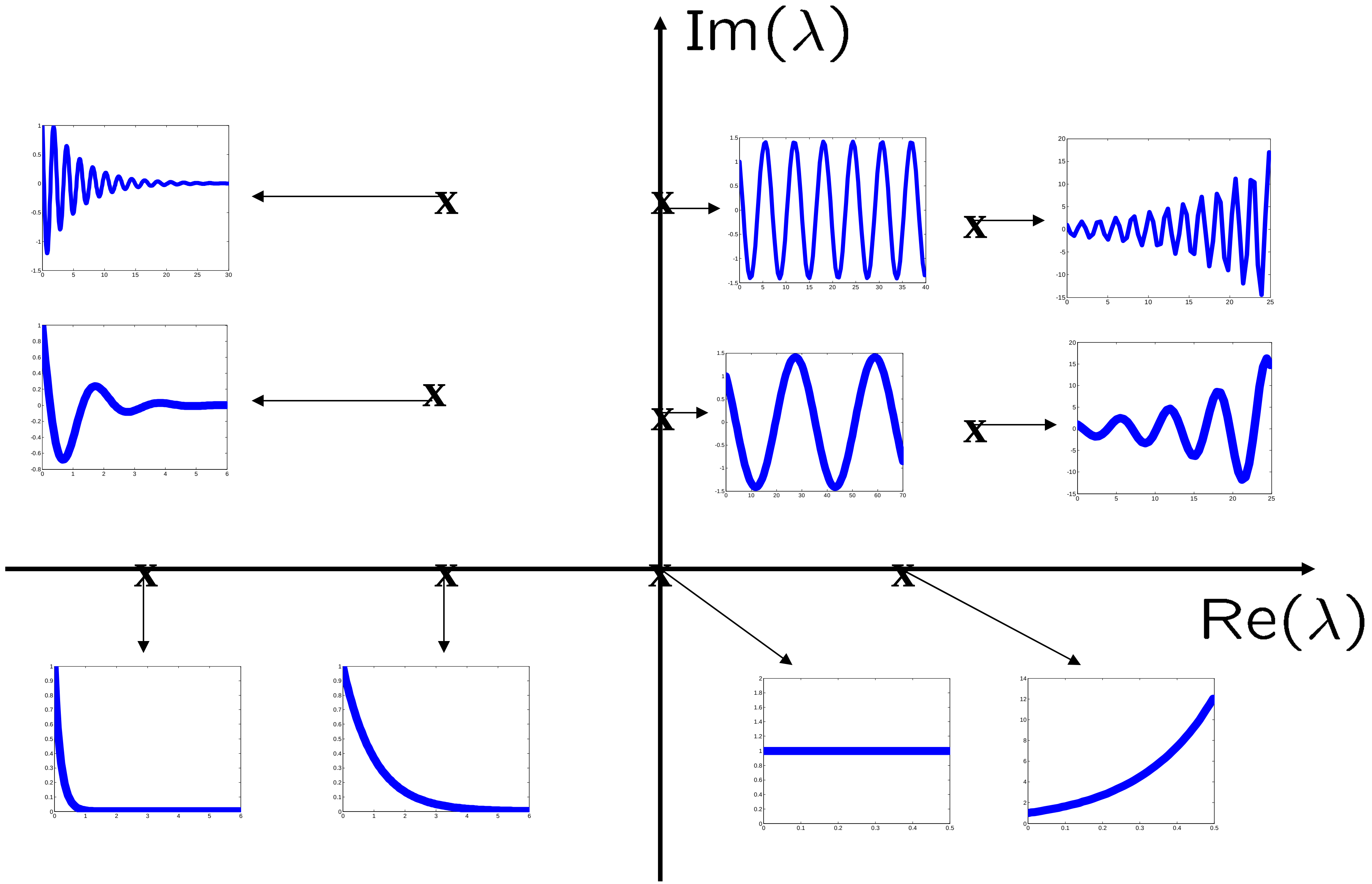
Open loop characteristic equation:

$$A(s) \;=\; \mathsf{Det}\{[sI - A]\} = 0$$

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0$$

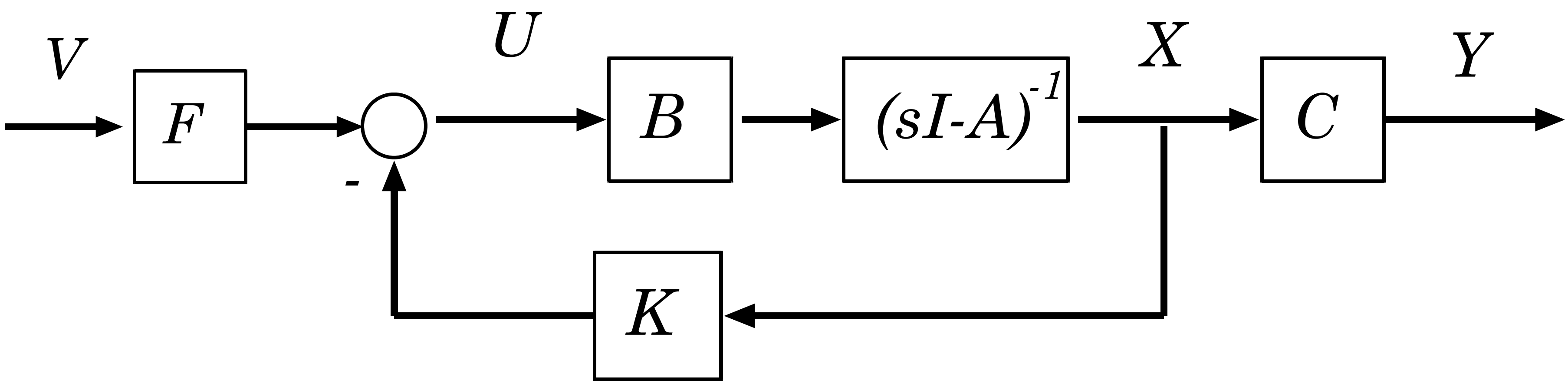
Its roots are the eigenvalues of ***A***

Eigenvalue location and associated response mode $e^{\lambda t}$



State Feedback Control

State feedback control law with exogenous input



$$\boxed{u = -Kx + Fv}$$

$$K \in \mathcal{R}^{m \times n}$$

$$F \in \mathcal{R}^{m \times m}$$

State Feedback Control

$$X(s) = [sI - A]^{-1} B\, U(s)$$

$$[sI - A] X(s) = B\, U(s)$$

$$U(s) = -K\, X(s) + F\, V(s)$$

$$[sI - A] X(s) = -BK\, X(s) + BF\, V(s)$$

$$[sI - A + BK] X(s) = BF\, V(s)$$

State Feedback Control

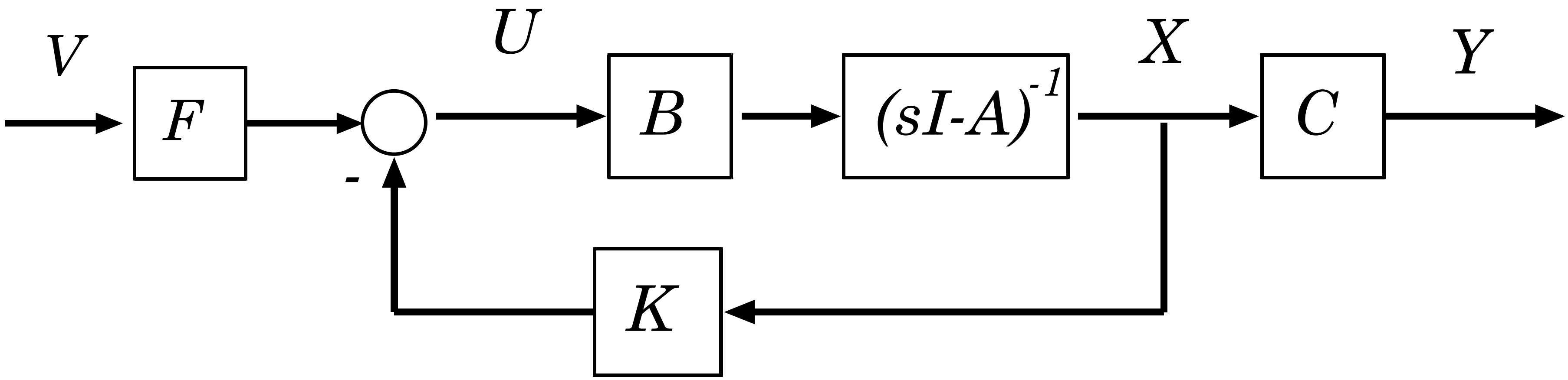
Closed loop system

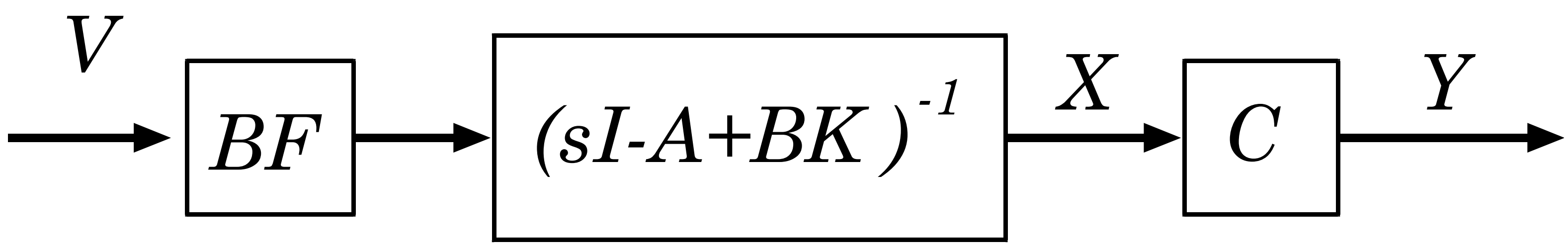
$$[sI - \underbrace{A + BK}_{A_c = A - BK}]X(s) \;=\; \underbrace{BF}_{B_c = BF}V(s)$$

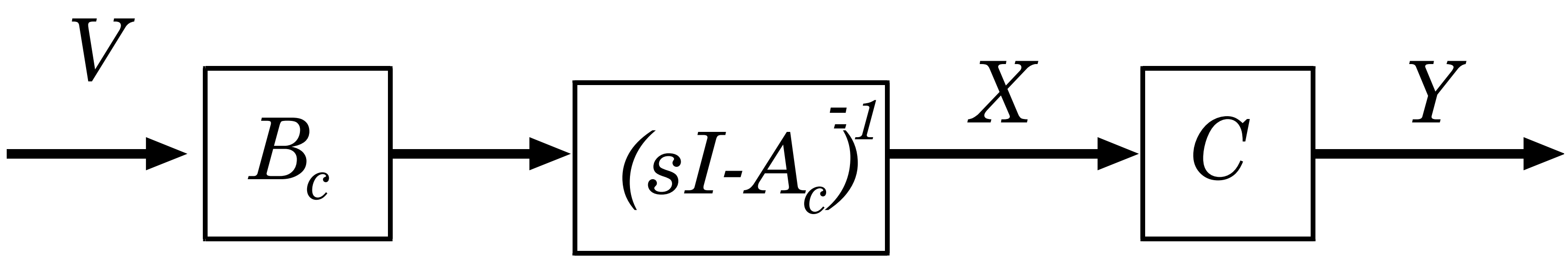
$$[sI - A_c]X(s) \;=\; B_c V(s)$$

$$X(s) \;=\; [sI - A_c]^{-1} B_c V(s)$$

State Feedback Control







Closed Loop System

Resulting closed loop system:

$$\dot{x} = A_c\, x + B_c\, v \qquad\qquad y \in \mathcal{R}^m$$

$$y = C\, x \qquad\qquad v \in \mathcal{R}^m$$

where

$$\boxed{A_c = A - B\, K} \qquad \boxed{B_c = B\, F}$$

Closed Loop System

Resulting closed loop system:

$$\begin{aligned} \dot{x} &= A_c\, x + B_c\, v \\ y &= C\, x \end{aligned} \qquad \begin{aligned} y &\in \mathcal{R}^m \\ v &\in \mathcal{R}^m \end{aligned}$$

Closed loop characteristic polynomial:

$$\begin{aligned} A_c(s) &= \text{Det}\{[sI - A_c]\} = 0 \\ &= \text{Det}\{[sI - A + B\, K]\} = 0 \end{aligned}$$

$$A_c(s) = s^n + a_{c(n-1)} s^{n-1} + \cdots + a_{c0} = 0$$

Closed Loop System

Resulting closed loop system:

$$\begin{aligned} \dot{x} &= A_c\, x + B_c\, v \\ y &= C\, x \end{aligned} \qquad \begin{aligned} y &\in \mathcal{R}^m \\ v &\in \mathcal{R}^m \end{aligned}$$

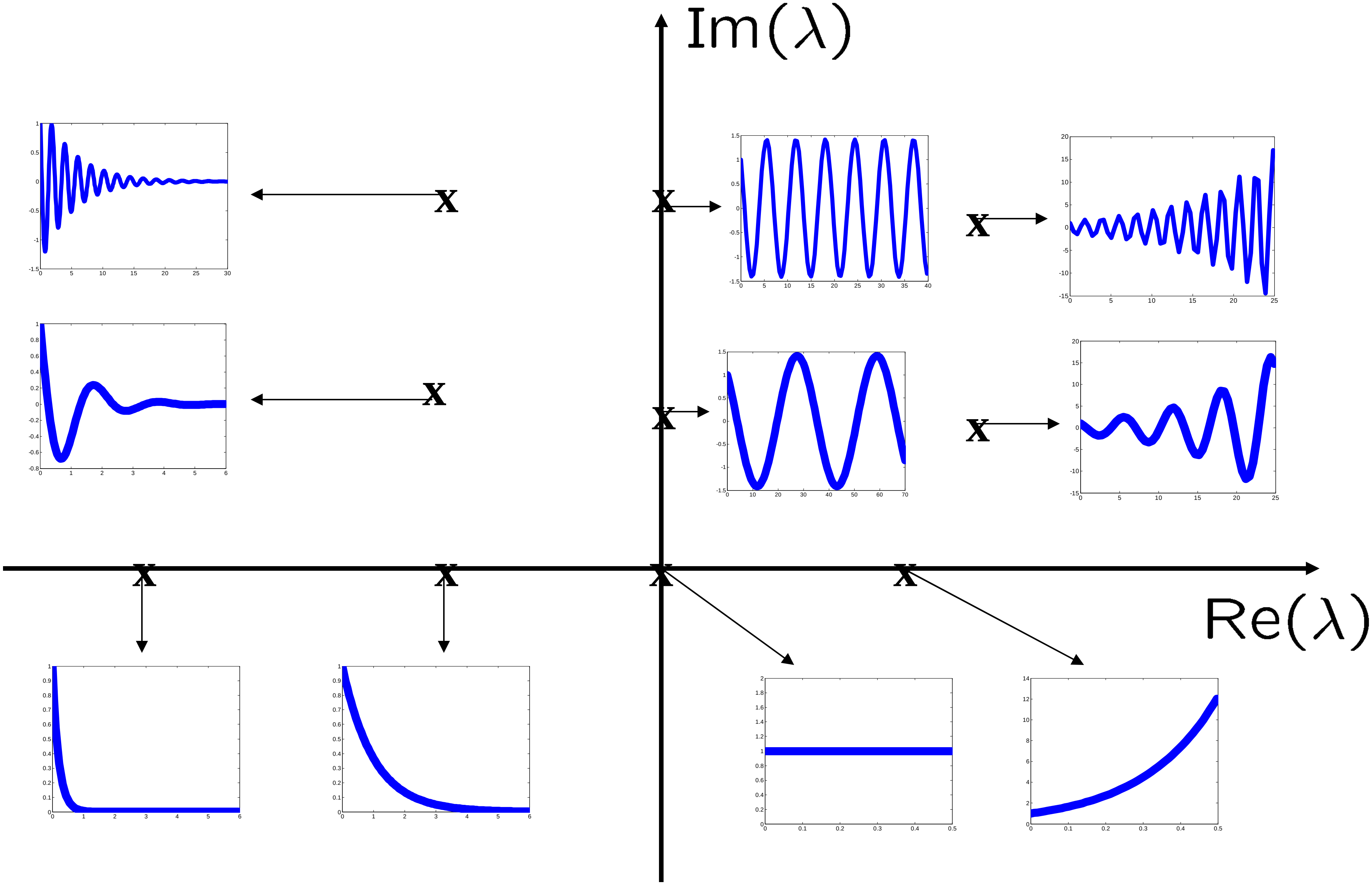
Closed loop characteristic polynomial:

$$A_c(s) \;=\; s^n + a_{c(n-1)} s^{n-1} + \cdots + a_{c0} = 0$$

The roots of $\boldsymbol{A_c(s) = 0}$ are the ***closed loop eigenvalues*** of $\boldsymbol{A_c}$

The roots of $\boldsymbol{A_c(s) = 0}$ are also the ***closed loop poles***

Eigenvalue location and associated response mode $e^{\lambda t}$



State Variable Feedback

Theorem:

If the pair $\{A,\ B\}$ is controllable, then the roots of the closed loop characteristic equation (closed loop poles)

$$A_c(s) \;=\; s^n + a_{c(n-1)}s^{n-1} + \cdots + a_{c0} = 0$$

can be ***arbitrarily**** assigned in the complex plane using state variable feedback.

***** (complex roots must be accompanied by their complex conjugates – symmetry about real axis)

Eigenvalue (pole) placement problem

Given a set of ***desired*** closed loop eigenvalues

$$\{\lambda_{c1},\, \lambda_{c2},\, \cdots,\, \lambda_{cn}\}$$

Find the state feedback gain K such that

$$A_c = A - B\,K$$

and the closed loop characteristic polynomial satisfies:

$$A_c(s) \;=\; (s-\lambda_{c1})(s-\lambda_{c2})\cdots(s-\lambda_{cn})$$

Eigenvalue (pole) placement problem

1. Convert the original realization to the controllable canonical realization using a ***similarity transformation.***

2. Find the state feedback gain matrix that will place the poles of a controllable canonical realization to the desired location.

3. After the feedback gain matrix is found, convert the system back to the original realization.

Controllable Canonical Realization

$$\frac{d}{dt}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_0 & -a_1 & -a_2\end{bmatrix}}_{\bar{A}}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}_{\bar{B}} u$$

$$y = \underbrace{\begin{bmatrix}b_o & b_1 & b_2\end{bmatrix}}_{\bar{C}}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}$$

and

$$G(s) = \bar{C}\,(s\,I - \bar{A})^{-1}\,\bar{B} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Single input controllable canonical realization

Lemma:

If the pair $\{A,\ B\}$ is controllable, then there exists a similarity transformation matrix Q such that

$$\bar{A} = Q^{-1} A\, Q \qquad \bar{B} = Q^{-1}\, B$$

is the controllable canonical pair

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Single input controllable canonical realization

Note: The transformation matrix Q such that

$$\bar{A} = Q^{-1} A Q \qquad \bar{B} = Q^{-1} B$$

is the controllable canonical pair, **is not** the observability matrix.

We will later obtain a formula for Q

Similarity transformation

Use the similarity transformation Q as follows

$$\bar{x} = Q^{-1} x$$

on

$$\dot{x} = A x + B u$$

Similarity transformation

$$\dot{x} \;=\; A\,x + B\,u$$

Multiply on the left by Q^{-1}

$$Q^{-1}\dot{x} \;=\; Q^{-1}A\,\underbrace{\quad}_{QQ^{-1}=I}\,x + Q^{-1}\,B\,u$$

$$Q^{-1}\dot{x} \;=\; Q^{-1}AQ\,Q^{-1}x + Q^{-1}\,B\,u$$

Similarity transformation

$$\dot{x} = A\,x + B\,u \qquad \bar{x} = Q^{-1}\,x$$

$$\underbrace{Q^{-1}\dot{x}}_{\dot{\bar{x}}} = \underbrace{Q^{-1}AQ}_{\bar{A}}\,\underbrace{Q^{-1}x}_{\bar{x}} + \underbrace{Q^{-1}\,B}_{\bar{B}}\,u$$

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}\,u$$

Single input controllable canonical realization

$$\frac{d}{dt}\bar{x} \;=\; \bar{A}\,\bar{x} + \bar{B}\,u$$

$$\frac{d}{dt}\bar{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}}_{\bar{A}} \bar{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\bar{B}} u$$

$$\begin{aligned} \mathrm{Det}\{[sI - \bar{A}]\} \;&=\; \mathrm{Det}\{[sI - A]\} = A(s) \\ &=\; s^n + a_{n-1}s^{n-1} + \cdots + a_0 \end{aligned}$$

State variable feedback on the controllable canonical realization

$$\frac{d}{dt}\bar{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Lets use the state feedback:

$$u = -\bar{K}\,\bar{x} = -\begin{bmatrix} \bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_n \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

State variable feedback on the controllable canonical realization

$$\frac{d}{dt}\bar{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \underbrace{\begin{bmatrix} -\bar{k}_1 & -\bar{k}_2 & \cdots & -\bar{k}_n \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}}_{u = -\bar{K}\,\bar{x}}$$

State variable feedback on the controllable canonical realization

$$\frac{d}{dt}\bar{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -\bar{k}_1 & -\bar{k}_2 & -\bar{k}_3 & \cdots & -\bar{k}_n \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

State variable feedback on the controllable canonical realization

Closed loop controllable canonical realization:

$$\underbrace{\frac{d}{dt}\bar{x} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -(a_o + \bar{k}_1) & -(a_1 + \bar{k}_2) & \cdots & -(a_{n-1} + \bar{k}_n) \end{bmatrix} \bar{x}}_{\bar{A}_c = \bar{A} - \bar{B}\,\bar{K}}$$

State variable feedback on the controllable canonical realization

Closed loop controllable canonical $\boldsymbol{A_c}$ matrix:

$$\bar{A}_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\underbrace{(a_o + \bar{k}_1)}_{a_{c0}} & -\underbrace{(a_1 + \bar{k}_2)}_{a_{c1}} & \cdots & -\underbrace{(a_{n-1} + \bar{k}_n)}_{a_{c(n-1)}} \end{bmatrix}$$

Close loop coefficients

$$a_{c0} = a_o + \bar{k}_1$$

$$a_{c1} = a_1 + \bar{k}_2$$

$$a_{ci} = a_i + \bar{k}_{i+1}$$

$$\vdots$$

$$a_{c(n-1)} = a_{n-1} + \bar{k}_n$$

State variable feedback on the controllable canonical realization

$$\bar{A}_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\underbrace{(a_o + \bar{k}_1)}_{a_{c0}} & -\underbrace{(a_1 + \bar{k}_2)}_{a_{c1}} & \cdots & -\underbrace{(a_{n-1} + \bar{k}_n)}_{a_{c(n-1)}} \end{bmatrix}$$

Closed loop characteristic polynomial:

$$\begin{aligned} A_c(s) &= \mathrm{Det}\{[sI - \bar{A}_c]\} \\ &= s^n + \underbrace{(a_{n-1} + \bar{k}_n)}_{a_{c(n-1)}} s^{n-1} + \cdots + \underbrace{(a_0 + \bar{k}_1)}_{a_{c0}} \end{aligned}$$

Pole placement on controllable realization

Given a ***set of desired close loop eigenvalues***:

$$\{\lambda_{c1},\, \lambda_{c2},\, \cdots\, ,\, \lambda_{cn}\}$$

We can compute the coefficients of the closed loop characteristic polynomial from

$$\begin{aligned} A_c(s) \;&=\; (s-\lambda_{c1})(s-\lambda_{c2})\cdots(s-\lambda_{cn}) \\ &=\; s^n + a_{c(n-1)}s^{n-1} + \cdots + a_{c0} \end{aligned}$$

Pole placement on controllable realization

Since

$$A_c(s) = s^n + \underbrace{a_{c(n-1)}}_{(a_{n-1}+\bar{k}_n)} s^{n-1} + \cdots + \underbrace{a_{c0}}_{(a_0+\bar{k}_1)}$$

We can easily compute the controllable state feedback gain

$$\bar{K}^T = \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \vdots \\ \bar{k}_n \end{bmatrix} = \begin{bmatrix} a_{c0} - a_0 \\ a_{c1} - a_1 \\ \vdots \\ a_{c(n-1)} - a_{n-1} \end{bmatrix}$$

State variable feedback gain for the original realization

Closed loop control:

$$u = -\bar{K}\,\bar{x}$$

Since: $\bar{x} = Q^{-1}\,x$

$$u = -\underbrace{\bar{K}\,Q^{-1}}\,x$$

$$K = \bar{K}\,Q^{-1}$$

The controllable canonical transformation Q for Single Input Systems

Given:

1) a controllable pair $[\,A\,,\ B\,]$ $\qquad B \in \mathcal{R}^n$

2) the characteristic polynomial of the matrix $\boldsymbol{A}$

$$
\begin{aligned}
A(s) &= \mathrm{Det}\{[sI - A]\} \\
&= s^n + a_{n-1}s^{n-1} + \cdots + a_0
\end{aligned}
$$

Let a_j be the j-th coefficient of $A(s)$

The controllable canonical transformation $\boldsymbol{Q}$

$$Q = \left[\begin{array}{cccc} q_1 & q_2 \cdots q_{n-1} & q_n \end{array}\right]$$

Recursive formula for columns $\boldsymbol{q_j}$:

$$\boxed{\begin{aligned} q_n &= B \\ q_{j-1} &= A\,q_j + a_{j-1}\,B \qquad j \in [2, n] \end{aligned}}$$

and a_j is the j-th coefficient of $A(s)$

Eigenvalue placement algorithm

1) Select desired close loop eigenvalues:

$$\{\lambda_{c1},\, \lambda_{c2},\, \cdots,\, \lambda_{cn}\}$$

2) Compute $\left\{a_{c0},\, a_{c1},\, \cdots,\, a_{c(n-1)}\right\}$ such that

$$\begin{aligned} A_c(s) &= s^n + a_{c(n-1)}s^{n-1} + \cdots + a_{c0} \\ &= (s-\lambda_{c1})(s-\lambda_{c2})\cdots(s-\lambda_{cn}) \end{aligned}$$

Eigenvalue placement algorithm

3) Compute feedback control gains in controllable canonical form:

$$\bar{K}^T = \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \\ \vdots \\ \bar{k}_n \end{bmatrix} = \begin{bmatrix} a_{c0} - a_0 \\ a_{c1} - a_1 \\ \vdots \\ a_{c(n-1)} - a_{n-1} \end{bmatrix}$$

Eigenvalue placement algorithm

4) Compute similarity transformation $\boldsymbol{Q}$

$$Q = \left[\begin{array}{cccc} q_1 & q_2 \cdots q_{n-1} & q_n \end{array} \right]$$

$$\begin{aligned} q_n &= B \\ q_{j-1} &= A\,q_j + a_{j-1}\,B \qquad j \in [2, n] \end{aligned}$$

5) Compute feedback control gains:

$$K = \bar{K} Q^{-1}$$

Eigenvalue placement algorithm

6) The characteristic polynomial of the closed loop matrix

$$A_c = A - B\,K$$

satisfies:

$$A_c(s) \;=\; \mathrm{Det}\{[sI - A_c]\}$$

$$\boxed{A_c(s) \;=\; (s-\lambda_{c1})(s-\lambda_{c2})\cdots(s-\lambda_{cn})}$$

The controllable canonical transformation $\boldsymbol{Q}$

We now show that the controllable canonical transformation matrix is given by:

$$Q = \left[\begin{array}{cccc} q_1 & q_2 \cdots q_{n-1} & q_n \end{array} \right]$$

where, the columns $\boldsymbol{q_j}$ are computed recursively as follows:

$$\boxed{\begin{aligned} q_n &= B \\ q_{j-1} &= A\,q_j + a_{j-1}\,B \qquad j \in [2, n] \end{aligned}}$$

Review: Controllability matrix

Because the pair $\{A,\ B\}$ is controllable,

the controllability matrix

$$P = \left[\begin{array}{cccc} B & AB & \cdots A^{n-1}B \end{array}\right] \in \mathcal{R}^{n\times n}$$

is rank $\boldsymbol{n}$.

Thus, $\left\{B\ AB\ \cdots\ A^{n-1}B\right\}$ are a basis of $\mathcal{R}^n$

Review: Cayley-Hamilton theorem

$$\left\{ B \; AB \; \cdots \; A^{n-1}B \right\} \quad \text{are a basis of} \quad \mathcal{R}^n$$

Also, by the Cayley-Hamilton theorem,

$$A^n B + a_{n-1} A^{n-1} B + \cdots + a_0 B = 0$$

Finding the similarity transformation $\boldsymbol{Q}$

We need to find the matrix

$$Q = \left[\begin{array}{cccc} q_1 & q_2 \cdots q_{n-1} & q_n \end{array}\right]$$

Such that

$$\bar{A} = Q^{-1} A Q \qquad \bar{B} = Q^{-1} B$$

$$\bar{A} = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{array}\right] \quad \bar{B} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array}\right]$$

Finding the controllable transformation $\boldsymbol{Q}$

Lets start with $\bar{B} = Q^{-1} B$ where $\bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

$$B = Q\,\bar{B}$$

$$B = \begin{bmatrix} q_1 & q_2 \cdots q_{n-1} & q_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = q_n$$

$$\boxed{q_n = B}$$

Finding the transformation $\boldsymbol{Q}$

Lets continue with $\bar{A} = Q^{-1} A Q$

$$A\,Q = Q\,\bar{A}$$

$$\left[\begin{array}{c} A\,q_1 \cdots A\,q_n \end{array}\right] = \left[\begin{array}{c} q_1 \cdots q_n \end{array}\right] \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{array}\right]$$

We now expand this equation column by column, from the last to the second

Finding the transformation $\boldsymbol{Q}$

Expanding column by column, from $\boldsymbol{n}$-th to second

$$\begin{bmatrix} Aq_1 & Aq_2 & \cdots & A\,q_n \end{bmatrix} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$\boxed{A\,q_n = q_{n-1} - a_{n-1}\,q_n}$$

$$A\,q_{n-1} = q_{n-2} - a_{n-2}\,q_n$$

$$A\,q_{n-2} = q_{n-3} - a_{n-3}\,q_n$$

$$\vdots$$

$$\boxed{A\,q_2 = q_1 - a_1\,q_n}$$

Finding the transformation $\boldsymbol{Q}$

Expanding column by column, from $\boldsymbol{n}$-th to second

$$\left[\, Aq_1\, Aq_2 \cdots A\, q_n \,\right] = \left[\, q_1 \cdots q_n \,\right] \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$\left.\begin{aligned} A\,q_n &= q_{n-1} - a_{n-1}\,q_n \\ A\,q_{n-1} &= q_{n-2} - a_{n-2}\,q_n \\ A\,q_{n-2} &= q_{n-3} - a_{n-3}\,q_n \\ &\vdots \\ A\,q_2 &= q_1 - a_1\,q_n \end{aligned}\right\}$$

Recursive formulation:

$$A\,q_j = q_{j-1} - a_{j-1}\,q_n$$

$$q_n = B$$

Finding the transformation $\boldsymbol{Q}$

Thus, from the equations in the previous slide we obtain a recursive formula for the columns of $\boldsymbol{Q}$:

$$Q = \left[\begin{array}{cccc} q_1 & q_2 \cdots q_{n-1} & q_n \end{array}\right]$$

Where,

$$\boxed{\begin{array}{rcl} q_n & = & B \\ q_{j-1} & = & A\,q_j + a_{j-1}\,B \quad\quad j \in [2, n] \end{array}}$$

Finding the transformation Q

What about the **first** column?

$$\begin{bmatrix} Aq_1 & Aq_2 & \cdots & A\,q_n \end{bmatrix} = \begin{bmatrix} q_1 \cdots q_n \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$\boxed{A\,q_1 \;=\; -a_0\,q_n}$$

$$q_n = B$$

Moreover, the second column states:

$$\boxed{A\,q_2 \;=\; q_1 - a_1\,q_n}$$

q_1 is already defined!

Finding the transformation $\boldsymbol{Q}$

Notice that the first column equation is different:

$$A\,q_1 \;=\; -a_0\,B \qquad\qquad (*)$$

Moreover, $\boldsymbol{q_1}$ is already defined:

$$q_1 \;=\; A\,q_2 + a_1\,B \qquad\qquad (**)$$

Equation $(*)$ is satisfied because of the Cayley-Hamilton theorem!

Finding the transformation $\boldsymbol{Q}$

Expanding $(**)$ utilizing results from previous columns

$$q_1 = A\underbrace{q_2}_{(A\,q_3 + a_2\,B)} + a_1\,B$$

$$= A(A\underbrace{q_3}_{(A\,q_4 + a_3\,B)} + a_2\,B) + a_1\,B$$

$$\vdots$$

$$q_1 = A^{n-1}B + a_{n-1}\,A^{n-2}B + \cdots + a_1 B$$

Finding the transformation $\boldsymbol{Q}$

$$q_1 = A^{n-1}B + a_{n-1}A^{n-2}B + \cdots + a_1 B$$

Substitute this result into equation $(*)$

$$A\,q_1 + a_0\,B = 0 \qquad (*)$$

To obtain

$$A^n B + a_{n-1}A^{n-1}B + \cdots + a_1 AB + a_0 B = 0$$

This last equation is satisfied by the Cayley-Hamilton theorem.

Finding the transformation $\boldsymbol{Q}$

We need to show that the matrix $\boldsymbol{Q}$ is rank $\boldsymbol{n}$

The columns of $Q = \left[\begin{array}{cccc} q_1 & q_2 \cdots q_{n-1} & q_n \end{array}\right]$

can be expressed as follows:

$$
\begin{aligned}
q_n &= B \\
q_{n-1} &= A\,q_n + a_{n-1}\,q_n = A\,B + a_{n-1}\,B \\
q_{n-2} &= A\,q_{n-1} + a_{n-2}\,q_n = A^2 B + a_{n-1}\,AB + a_{n-2}B \\
&\vdots \\
q_1 &= A^{n-1}B + a_{n-1}\,A^{n-2}B + \cdots + a_1 B
\end{aligned}
$$

Finding the transformation $\boldsymbol{Q}$

Thus
$$Q = P\ N$$

where
$$N = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & a_{n-1} & 1 & 0 \\ a_3 & a_4 & \cdots & 1 & 0 & 0 \\ a_4 & a_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

Both matrices are nonsingular

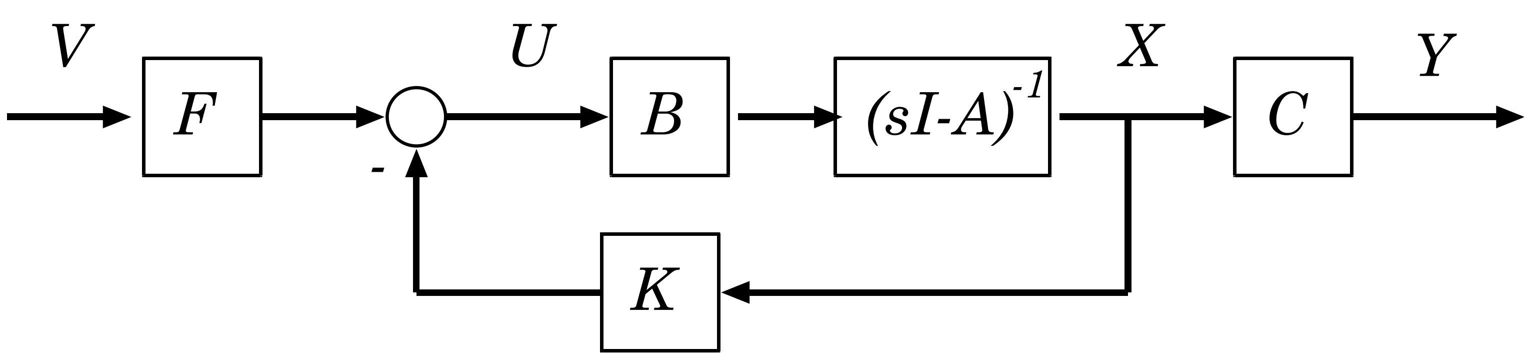
Selecting the constant $\boldsymbol{F}$

State variable feedback:

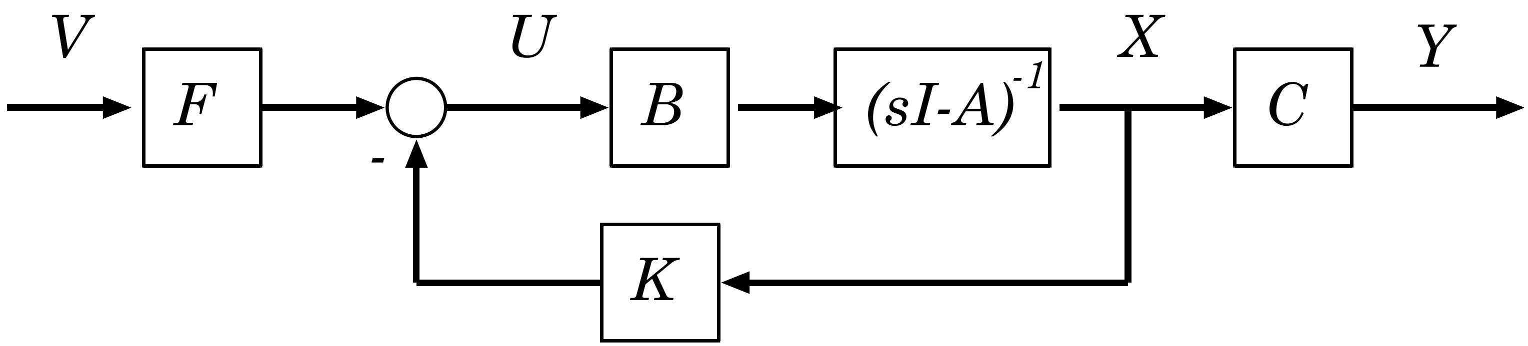
$$\boxed{u = -K\,x + F\,v}$$

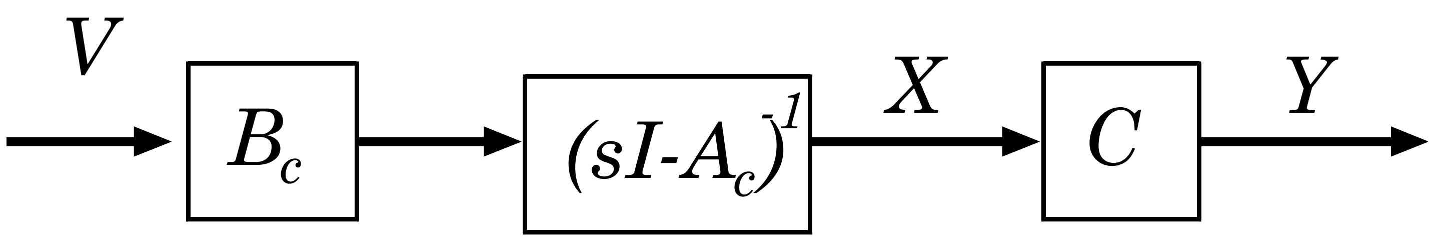
$$K \;\in\; \mathcal{R}^{n\times m}$$
$$F \;\in\; \mathcal{R}^{m\times m}$$

$v \;\in\; \mathcal{R}^{m}$ is the new exogenous input



Selecting the constant $\boldsymbol{F}$

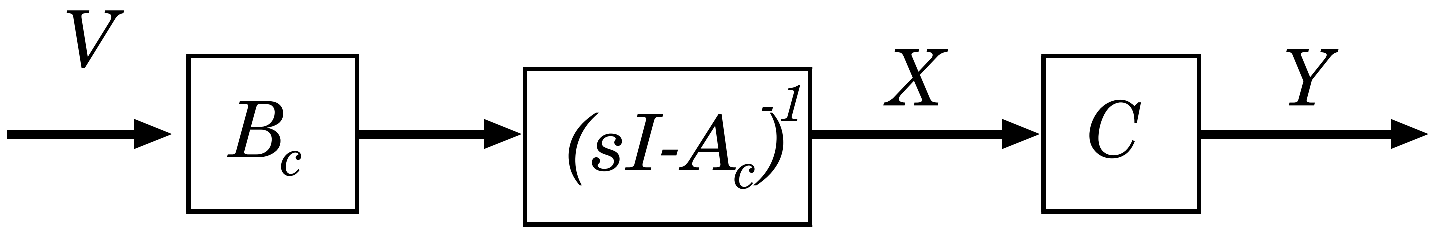




$$A_c = A - B\,K \qquad\qquad B_c = B\,F$$

Where $\boldsymbol{A_c}$ is assumed to be Hurwitz

Selecting the constant *F*



Assume that ***v(t)*** reaches some steady state

$$v_{ss} = \lim_{t\to\infty} v(t) \quad \Rightarrow \quad y_{ss} = \lim_{t\to\infty} y(t)$$

F can be selected to attain a desirable close loop DC gain

$$y_{ss} = -C\left(A - BK\right)^{-1} B\, F v_{ss}$$

State Feedback Control with I-action

$$\dot{x} = A\,x + B\,u \qquad y \in \mathcal{R}^m$$

$$y = C\,x \qquad u \in \mathcal{R}^m$$

(Notice that the input and output have the same dimension)

Define $r \in \mathcal{R}^m$

to be a ***constant*** reference input,

representing the ***desired*** steady state value of $\boldsymbol{y}$,

State Feedback Control with I-action

$$\dot{x} = A\,x + B\,u \qquad y \in \mathcal{R}^m$$

$$y = C\,x \qquad u \in \mathcal{R}^m$$

Define the output error signal $e(t) \in \mathcal{R}^m$

$$e = y - r$$

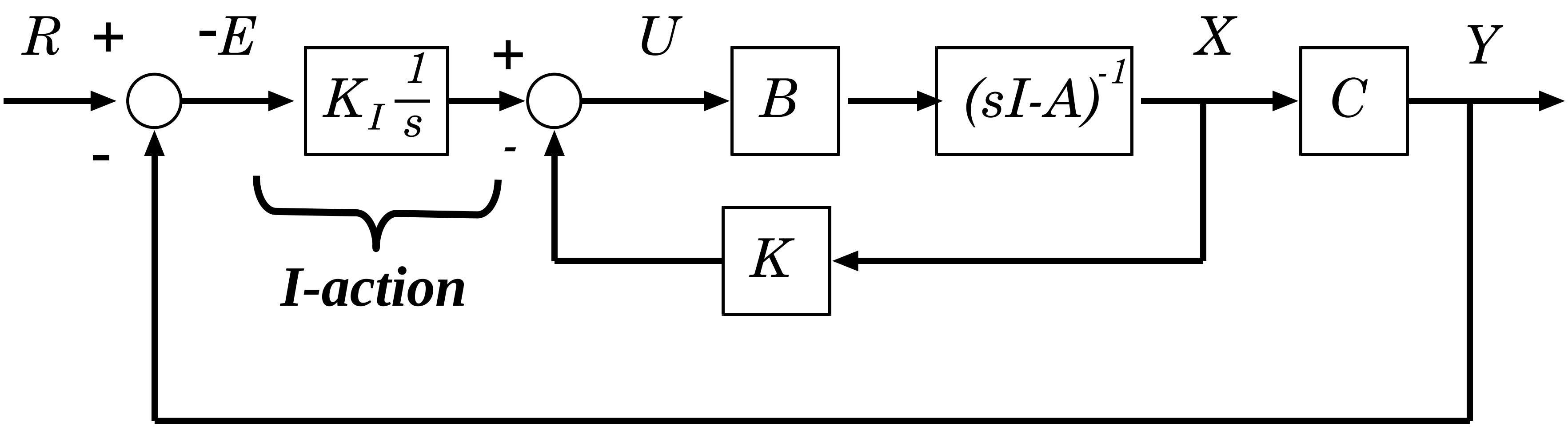
Notice:

$$\dot{e} = \dot{y} = C\dot{x}$$

State Feedback Control with I-action

$$\begin{aligned} u &= -K\,x + u_I \\ \dot{u}_I &= -K_I\,e \\ e &= y - r \end{aligned}$$

$$\begin{aligned} K &\in \mathcal{R}^{m\times n} \\ K_I &\in \mathcal{R}^{m\times m} \end{aligned}$$

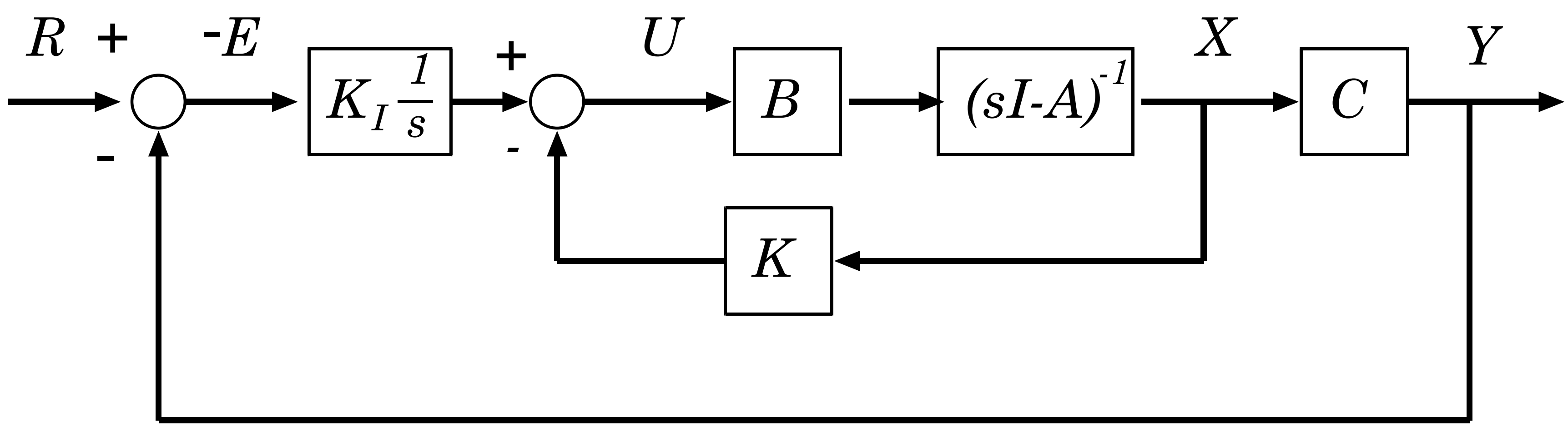


State Feedback Control with I-action

We would like to find the gains

$$K \in \mathcal{R}^{m \times n}$$
$$K_I \in \mathcal{R}^{m \times m}$$

so that eigenvalues of the closed loop system are placed at desired locations.



Augmented Systems

Assume that
$$\tilde{u} \;=\; \dot{u}$$

Is the ***"new"*** control input

differentiating,
$$u \;=\; -Kx + u_I$$
$$\dot{u} \;=\; -K\dot{x} + \underbrace{\dot{u}_I}_{-K_I\,e}$$

$$\tilde{u} \;=\; \dot{u} = -K\dot{x} - K_I e$$

Augmented Systems

Define:

1) Augmented state: $\tilde{x} = \begin{bmatrix} e \\ \dot{x} \end{bmatrix} \in \mathcal{R}^{m+n}$

2) Control input to the augmented system:

$$\tilde{u} = -K_I e - K\dot{x}$$

$$\tilde{u} = -\tilde{K}\,\tilde{x} \qquad \tilde{K} = \begin{bmatrix} K_I & K \end{bmatrix}$$

Augmented Systems

1) Augmented state: $\tilde{x} = \begin{bmatrix} e \\ \dot{x} \end{bmatrix} \in \mathcal{R}^{m+n}$

2) Dynamics

$$\dot{e} \;=\; C\dot{x}$$

$$\frac{d}{dt}\dot{x} = A\,\dot{x} + B\,\underbrace{\dot{u}}_{\tilde{u}}$$

Augmented Systems

1) Augmented state: $$\tilde{x} = \begin{bmatrix} e \\ \dot{x} \end{bmatrix} \in \mathcal{R}^{m+n}$$

2) Dynamics

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}\begin{bmatrix} e \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix}\tilde{u}$$

$$\tilde{u} = -\tilde{K}\,\tilde{x}$$

State Feedback Control with I-action

Equivalent augmented system is given by:

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} e \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\tilde{B}} \tilde{u}$$

$$\tilde{y} = \dot{y} = \underbrace{\begin{bmatrix} 0 & C \end{bmatrix}}_{\tilde{C}} \underbrace{\begin{bmatrix} e \\ \dot{x} \end{bmatrix}}_{\tilde{x}}$$

Control law:

$$\tilde{u} = -\tilde{K}\,\tilde{x}$$

State Feedback Control with I-action

Notice that the equivalent augmented system is order $\boldsymbol{n+m}$:

$$\frac{d}{dt}\tilde{x} \;=\; \tilde{A}\,\tilde{x} + \tilde{B}\,\tilde{u} \qquad\qquad \tilde{u} \;=\; -\tilde{K}\,\tilde{x}$$

$$\tilde{A} = \begin{array}{cl} \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix} & \begin{array}{l} \} \; m \text{ rows} \\ \} \; n \text{ rows} \end{array} \\ \begin{array}{cc} \underbrace{}_{m \text{ columns}} & \underbrace{}_{n \text{ columns}} \end{array} & \end{array} \qquad\qquad \tilde{B} = \begin{array}{cl} \begin{bmatrix} 0 \\ B \end{bmatrix} & \begin{array}{l} \} \; m \text{ rows} \\ \} \; n \text{ rows} \end{array} \\ \underbrace{}_{m \text{ columns}} & \end{array}$$

State Feedback Control For Augmented System

Given the augment system under state variable feedback:

$$\frac{d}{dt}\tilde{x} \;=\; \tilde{A}\,\tilde{x} + \tilde{B}\,\tilde{u} \qquad\qquad \tilde{u} \;=\; -\tilde{K}\,\tilde{x}$$

Find the state variable feedback gain :

$$\tilde{K} = \left[\begin{array}{cc} K_I & K \end{array}\right]$$

So that $\tilde{A}_c = \tilde{A} - \tilde{B}\tilde{K}$ is Hurwitz and

its eigenvalues are placed at prescribed locations

State Feedback Control with I-action

If:

1) The pair $\{A,\ B\}$ is controllable

2) A is nonsingular

3) $C\,A^{-1}\,B$ is nonsingular,

then

the pair $\{\tilde{A},\ \tilde{B}\}$ is also controllable and the eigenvalues of $\tilde{A}_c$ can be set arbitrarily with

$$\tilde{u} = -\tilde{K}\,\tilde{x}$$

Asymptotic convergence

Selecting the gain $\tilde{K} = \begin{bmatrix} K_I & K \end{bmatrix}$ so that

$$\tilde{A}_c = \tilde{A} - \tilde{B}\tilde{K}$$

is Hurwitz, assures that

$$\left.\begin{aligned} \lim_{t\to\infty} \begin{bmatrix} e(t) \\ \dot{x}(t) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ e &= y - r \end{aligned}\right\} \Rightarrow \boxed{y_{ss} = \lim_{t\to\infty} y(t) = r}$$

State Feedback Control with I-action

Proof:

1) Since the $\{A,\, B\}$ is controllable, then

$$P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is rank n.

2) Compute the controllability matrix of the augmented system up to $\boldsymbol{n}$*

$$\tilde{P}(n) = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n}\tilde{B} \end{bmatrix}$$

* Since the augmented system is (n+m)-th order, we are supposed to go up to $\tilde{A}^{n+m-1}\tilde{B}$, we do not need to do so in this example.

State Feedback Control with I-action

- Since

$$\tilde{A} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}$$

$$\begin{aligned} \tilde{P}(n) &= \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^n\tilde{B} \end{bmatrix} \\ &= \begin{bmatrix} 0 & CB & CAB & \cdots & CA^{n-1}B \\ B & AB & A^2B & \cdots & A^nB \end{bmatrix} \end{aligned}$$

State Feedback Control with I-action

3) Since $\boldsymbol{A}$ is nonsingular,

$$M = \begin{bmatrix} -I_m & CA^{-1} \\ 0 & A^{-1} \end{bmatrix} \in \mathcal{R}^{(m+n)\times(m+n)}$$

is also nonsingular

3) Now compute the matrix

$$\begin{aligned} \tilde{P}_e &= M\,\tilde{P}(n) \\ &= \left[\begin{array}{c|cccc} CA^{-1}B & 0 & 0 & 0 & 0 \\ A^{-1}B & B & AB & \cdots & A^{n-1}B \end{array}\right] \end{aligned}$$

State Feedback Control with I-action

5) Since $C\,A^{-1}\,B$ is nonsingular, (i.e. rank $\boldsymbol{m}$),

Therefore, $$\text{Rank}\left\{\begin{bmatrix} CA^{-1}B \\ A^{-1}B \end{bmatrix}\right\} = m$$

and

$$\tilde{P}_e = \begin{bmatrix} \underbrace{\begin{matrix} CA^{-1}B \\ A^{-1}B \end{matrix}}_{\boldsymbol{m}\text{ indep columns}} & \begin{matrix} | \\ | \end{matrix} & \underbrace{\begin{matrix} 0 & 0 & 0 & 0 \\ B & AB & \cdots & A^{n-1}B \end{matrix}}_{\boldsymbol{n}\text{ indep columns}} \end{bmatrix}$$

$$\boxed{\text{Rank}\{\tilde{P}_e\} = m + n}$$

State Feedback Control with I-action

$$\tilde{P}_e \;=\; M\,\tilde{P}(n)$$

and $$\text{Rank}\{\tilde{P}_e\} \;=\; \text{Rank}\{M\} = m + n$$

$\Rightarrow$ $$\text{Rank}\{\tilde{P}(n)\} = m + n$$

$\Rightarrow$ The pair $\{\tilde{A},\ \tilde{B}\}$ is controllable.