Linear System Theory

Jun Moon

Lecture 3: Existence and uniqueness of the solution

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Overview

- Nonlinear systems: existence and uniqueness of a solution of differential equations

One-dimensional LTI System

$$\frac{dx(t)}{dt} = Ax(t),\ x(0) = x_0,\ \ , t \in [0, T],\ A \in \mathbb{R}$$

- Is there a solution?
- What is the solution?
- If there is a solution, does it exist for all $t \in [0, T]$?
- Is there a unique solution?
- Does existence and uniqueness of the solution depend on the initial condition x_0?
- These properties also hold when A is a matrix, which will be studied later in this course

One-dimensional LTV System

$$\dot{x}(t) = A(t)x(t), \; x(0) = x_0, \; A(t) : [0, T] \to \mathbb{R}$$

- $A(t)$ is a function that is defined on $[0, T]$
- The solution does not exist if $A(t)$ is not continuous
- $A(t)$ is continuous on $[0, T] \Rightarrow$ there exists a constant $c \geq 0$ such that $|A(t)| \leq c$ for all $t \in [0, T]$ (why???), which implies

$$|A(t)x - A(t)y| \leq |A(t)||x - y| \leq c|x - y|$$

- It is Lipschitz!!!
- There exists a unique solution

Nonlinear Systems

$$\frac{dx(t)}{dt} = x^{1/3}(t),\ \ x(0) = 0,\ \ t \in [0, 1]$$

- Is there a solution? Yes
- What is the solution? $x(t) = 0$ and $x(t) = (2t/3)^{3/2}$
- If there is a solution, does it exist for all $t \in [0, T]$? Yes
- Is there a unique solution? No

Nonlinear Systems

$$\frac{dx(t)}{dt} = -x^2(t), \; x(0) = -1, \; t \in [0, 1]$$

- Is there a solution? Yes
- What is the solution?

$$x(t) = \frac{1}{t-1}$$

- If there is a solution, does it exist for all $t \in [0, 1]$? No
- finite escape time: the phenomenon that $x(t)$ escapes to infinity at a finite time (also called "conjugate point")
- In this case, $x(t)$ has a "finite escape time" at $t = 1$, since $\lim_{t \to 1} x(t) = \infty$

Nonlinear Systems

Dynamical system

$$\dot{x} = f(t, x), \; x(t_0) = x_0$$

- $x : [0, T] \to \mathbb{R}^n$: state ($x_0$: initial condition)
- $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$: vector field

- Question: What is the condition under which there exists a unique solution to the nonlinear system?

Nonlinear Systems

Definition

A function $f(t, x) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is *globally Lipschitz* uniformly in $t \in [0, T]$ if there exists a constant $L \geq 0$ such that for all $x, y \in \mathbb{R}^n$, (note that $x \in \mathbb{R}^n$, $\|x\|^2 = x^T x$)

$$\|f((t, x) - f(t, y)\| \leq L\|x - y\|$$

- f is Lipschitz uniformly in $t \in [0, T]$ if the above condition holds for any $x, y \in W \subset \mathbb{R}^n$
- A function that has infinite slope at some point is not Lipschitz

Question: Is $f(x)$ continuous?

- Is $f(x) = Ax$ Lipschitz? Yes (A: matrix or scalar)
- Is $f(x) = x^{1/3}$ Lipschitz?

$$f'(x) = \frac{1}{3}x^{\frac{-2}{3}} \to \infty \text{ as } x \to 0$$

Nonlinear Systems

Theorem: Local existence and uniqueness

Suppose that f is continuous in t, and $f(t,x)$ is Lipschitz uniformly in $t \in [0, T]$ for all $x, y \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$. Then, there exists some $\delta > 0$ such that the state equation $\dot{x}(t) = f(t,x)$ with $x(0) = x_0$ has a unique solution over $[0, \delta]$.

- f can be piecewise continuous in t

Nonlinear Systems

Theorem: Global existence and uniqueness

Suppose that f is continuous in t, and $f(t,x)$ is globally Lipschitz uniformly in $t \in [0, T]$. Then, the state equation $\dot{x}(t) = f(t,x)$ with $x(0) = x_0$ has a unique solution over $[0, T]$.

- f can be piecewise continuous in t
- LTI system: globally Lipschitz!!
- $\dot{x}(t) = x^{1/3}(t)$ dost not have a unique solution $\Rightarrow f(x) = x^{1/3}$ is not Lipschitz
- This is one of the main properties of LTI systems

Nonlinear Systems

- Main idea of the proof of the local existence and uniqueness

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0 \Leftrightarrow x(t) = x_0 + \int_0^t f(x(s))ds$$

- Let $(Px)(t) = x_0 + \int_0^t f(x(s))ds$
- If $x(t)$ satisfies the above relation, then $x(t)$ is a solution of the ODE
- Equivalently, if there exists $x(t)$ such that $x(t) = (Px)(t)$, then we are done
- $x(t) = (Px)(t) \Rightarrow$ fixed point!!!
- We can show that there exist a unique fixed point of $(Px)(t)$ under the Lipschitz condition

Fields

Fields

- A field $\mathbb{F}$ is an object that consisting of a set of elements, and two binary operations:
  addition $(+)$, multiplication $(\cdot)$ such that

Addition
(i) associative: $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ for all $\alpha,\beta,\gamma\in\mathbb{F}$
(ii) commutative: $\alpha+\beta=\beta+\alpha$ for all $\alpha,\beta\in\mathbb{F}$
(iii) there exists an identity element 0: $\alpha+0=\alpha$ for all $\alpha\in\mathbb{F}$
(iv) for all $\alpha\in\mathbb{F}$, there exists an inverse $-\alpha$ such that $\alpha+(-\alpha)=0$

Multiplication
(i) associative: $\alpha\cdot(\beta\cdot\gamma)=(\alpha\cdot\beta)\cdot\gamma$ for all $\alpha,\beta,\gamma\in\mathbb{F}$
(ii) commutative: $\alpha\cdot\beta=\beta\cdot\alpha$ for all $\alpha,\beta\in\mathbb{F}$
(iii) there exists an identity element 1: $\alpha\cdot 1=\alpha$ for all $\alpha\in\mathbb{F}$
(iv) for all $\alpha\in\mathbb{F}$, $\alpha\neq 0$, there exists an inverse α^{-1} such that
$\alpha\cdot\alpha^{-1}=\alpha^{-1}\cdot\alpha=1$

Fields

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

Example: Consider the set of all 2×2 matrices for the form
Matrix addition and multiplication

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \; x, y \in \mathbb{R}, \; 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; 1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The inverse always exists

How about the set of all 2×2 matrices?

Vector Space

- vector space = linear space = linear vector space
- A linear space over a field $\mathbb{F}$, $(\mathbb{V}, \mathbb{F})$, consists of a set $\mathbb{V}$ of vectors, a field $\mathbb{F}$, and two operations, vector addition and scalar multiplication
- The two operations satisfy

Addition
(a) addition: for all $x, y \in \mathbb{V}$, $x + y \in \mathbb{V}$
(b) associative: for all $x, y, z \in \mathbb{V}$, $(x + y) + z = x + (y + z)$
(c) commutative: for all $x, y \in \mathbb{V}$, $x + y = y + x$
(d) there exists a unique zero vector $0 \in \mathbb{V}$ such that $x + 0 = 0 + x = x$ for all $x \in \mathbb{V}$
(e) there exists a unique inverse $-x \in \mathbb{V}$ such that $x + (-x) = 0$ for all $x \in \mathbb{V}$

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Multiplication
(a) multiplication: for any $\alpha \in \mathbb{F}$ and $x \in \mathbb{V}$, $\alpha x \in \mathbb{V}$
(b) associative: for any $\alpha, \beta \in \mathbb{F}$ and $x \in \mathbb{V}$, $\alpha(\beta x) = (\alpha\beta)x$
(c) distributive w.r.t. scalar addition:
for any $\alpha \in \mathbb{F}$ and $x, y \in \mathbb{V}$, $\alpha(x + y) = \alpha x + \alpha y$
(d) distributive w.r.t. scalar multiplication
for any $\alpha, \beta \in \mathbb{F}$ and $x \in \mathbb{V}$, $(\alpha + \beta)x = \alpha x + \beta x$
(e) there exists a unique $1 \in \mathbb{F}$ such that for any $x \in \mathbb{V}$, $1x = x$
(f) there exists a unique $0 \in \mathbb{F}$ such that for any $x \in \mathbb{V}$, $0x = 0$

Vector Space

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Example: $(\mathbb{F}^n, \mathbb{F})$ where $\mathbb{F}^n = \mathbb{F} \times \cdots \times \mathbb{F}$

Example: $(\mathbb{R}^n, \mathbb{R})$, $(\mathbb{C}^n, \mathbb{C})$, $(\mathbb{C}^n, \mathbb{R})$

Example: $(\mathbb{R}, \mathbb{C})$ is not a vector space! (why?) $(1 + i)1 = 1 + i \notin \mathbb{R}$

Example: a continuous function $f : [t_0, t_1] \to \mathbb{R}^n$, the set of such functions, $(C([t_0, t_1], \mathbb{R}^n), \mathbb{R})$, is a linear space

Norm and Sequences

- Normed vector space: A vector space X is a normed vector space if there exists $\|\cdot\| : X \to R$ such that
  - $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$
  - $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$
  - $\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$
- Convergence: A sequence $\{x_n\} \in X$ converges to x if

$$\|x_n - x\| \to 0 \quad \text{as } n \to \infty$$

- Cauchy sequence
  - A sequence $\{x_n\} \in X$ is said to be a Cauchy sequence if

$$\|x_n - x_m\| \to 0 \quad \text{as } n, m \to 0$$

Norm and Sequences

Example: normed vector space

- $\mathbb{R}^n$ with the Euclidian norm $|\cdot|$
- $L_p(a, b, \mathbb{R}^n)$ space with $p \geq 1$ and

  $$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p},$$

  where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$ (note that when $p = 2$, $L_2(a, b, \mathbb{R}^n)$ is an inner product space)
- $C(a, b, \mathbb{R}^n)$: the space of $\mathbb{R}^n$-valued continuous functions on $[a, b]$ with

  $$\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|$$

Norm and Sequences

Banach space (complete normed vector space)

- A normed vector space X is complete if every Cauchy sequence in X converges to the limit in X.
- A complete normed vector space is called a Banach space
- A complete inner product space is called a Hilbert space
- Any Hilbert space is a Banach space

Example

- $\mathbb{R}^n$ with $|\cdot|$: Hilbert space hence Banach space (what is inner product?)
- $L_p(a, b, \mathbb{R}^n)$ with $\|\cdot\|_p$ for $p \geq 1$: Banach space
- When $p = 2$, $L_2(a, b, \mathbb{R}^n)$ is a Hilbert space
- $C(a, b, \mathbb{R}^n)$ with $\|\cdot\|_\infty$: Banach space

Contraction Mapping

Contraction mapping theorem (Banach fixed point theorem)

Let $S \subset X$ be the closed subset of a Banach space X, and $\mathcal{T} : S \to S$, and $\mathcal{T}$ satisfies

$$\|\mathcal{T}(x) - \mathcal{T}(y)\| \leq \rho\|x - y\|, \ \forall x, y \in S, \ \rho < 1.$$

Then

- $\mathcal{T}$ has a unique fixed point, i.e., $\exists$ a unique x^* such that $\mathcal{T}(x^*) = x^*$
- $\mathcal{T}^n(x_1) \to x^*$ as $n \to \infty$ for any $x_1 \in S$
- Equivalently, let $x_{n+1} = \mathcal{T}(x_n)$ with $x_1 \in S$. Then $\lim_{n\to\infty} x_n = x^*$

Note

- The condition is called contraction
- The function that is contraction is Lipschitz. But not vice-versa.

Proof of Contraction Mapping Theorem

Select $x_1 \in S$. Then

$$
\begin{aligned}
|x_{n+1} - x_n| &= |\mathcal{T}(x_n) - \mathcal{T}(x_{n-1})| \\
&\leq \rho|x_n - x_{n-1}| \leq \rho^2|x_{n-1} - x_{n-2}| \leq \cdots \leq \rho^{n-1}|x_2 - x_1|
\end{aligned}
$$

Hence

$$
\begin{aligned}
|x_{n+r} - x_n| &\leq |x_{n+r} - x_{n+r-1}| + \cdots + |x_{n+1} - x_n| \\
&\leq [\rho^{n+r-2} + \rho^{n+r-3} + \cdots + \rho^{n-1}]|x_2 - x_1| \\
&\leq \rho^{n-1} \sum_{i=1}^{\infty} \rho^t |x_2 - x_1| = \frac{\rho^{n-1}}{1-\rho}|x_2 - x_1|
\end{aligned}
$$

Note that as $n \to \infty$, the RHS converges to zero. This means that $\{x_n\}$ is a Cauchy sequence. Since X is a Banach space (complete normed vector space), there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Proof of Contraction Mapping Theorem

Since $S \subset X$ is closed, and we assumed that $\mathcal{T} : S \to S$, and $\{x_n\}$ is a sequence in S, $x^* \in S$.

We show that $x^* = \mathcal{T}(x^*)$. Since $\mathcal{T}$ is contraction, we have

$$\begin{aligned} |x^* - \mathcal{T}(x^*)| &\leq |x^* - x_n| + |x_n - \mathcal{T}(x^*)| \\ &= |x^* - x_n| + |\mathcal{T}(x_{n-1}) - \mathcal{T}(x^*)| \\ &\leq |x^* - x_n| + \rho|x_{n-1} - x^*| \end{aligned}$$

Note that when $n \to \infty$, we have $|x_{n-1} - x^*| \to 0$ and $|x^* - x_n| \to 0$. Hence, $|x^* - \mathcal{T}(x^*)| \to 0$ as $n \to \infty$. This implies $x^* = \mathcal{T}(x^*)$.

Now, we show uniqueness. Suppose that there exists $y^* \neq x^*$ such that $\mathcal{T}(y^*) = y^*$. Then

$$|x^* - y^*| = |\mathcal{T}(x^*) - \mathcal{T}(y^*)| \leq \rho|x^* - y^*|, \ \rho < 1$$

which is a contradiction.

Nonlinear Systems

Theorem: Local existence and uniqueness

Suppose that f is continuous in t, and $f(t,x)$ is Lipschitz uniformly in $t \in [0, T]$ for all $x, y \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$. Then, there exists some $\delta > 0$ such that the state equation $\dot{x}(t) = f(t,x)$ with $x(0) = x_0$ has a unique solution over $[0, \delta]$.

- f can be piecewise continuous in t

Nonlinear Systems

Theorem: Global existence and uniqueness

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- f can be piecewise continuous in t
- LTI system: globally Lipschitz!!
- $\dot{x}(t) = x^{1/3}(t)$ dost not have a unique solution $\Rightarrow f(x) = x^{1/3}$ is not Lipschitz
- This is one of the main properties of LTI systems

Proof of Local Existence and Uniqueness

We have

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0 \Leftrightarrow x(t) = x_0 + \int_0^t f(x(s))ds$$

Let

$$x(t) = x_0 + \int_0^t f(x(s))ds = (Px)(t)$$

Let

$$X = C(0,\delta),\ \|\cdot\|_C = \sup_{t\in[0,\delta]} |\cdot|$$

$$S = \{x \in X \mid \|x - x_0\|_C \leq r\}$$

- r radius of the ball in the statement of theorem
- $0 < \delta \leq T$ constant to be chosen
- Note: $[0,\delta] \subseteq [0,T]$
- $|\cdot|$: norm on $\mathbb{R}^n$
- $\|\cdot\|_C$: norm on X

Proof of Local Existence and Uniqueness

Claim 1: $(Px)(t) : S \to S$

Proof: Note that $(Px)(t) : X \to X$, and we have

$$h = \max_{t \in [0,\delta]} |f(t, x_0)| < \infty \quad \text{(why???)}$$

$$|x(t) - x_0| \le r, \quad \forall x \in S, \ t \in [0, \tau] \quad \text{(why???)}$$

Since f is Lipschitz on B with radius r, for any $x \in S$ and $t \in [0, \tau]$,

$$\begin{aligned}
|(Px)(t) - x_0| &\le \int_0^t \Big[|f(s, x(s)) - f(s, x_0)| + |f(s, x_0)| \Big] \\
&\le \int_0^t \Big[L|x(s) - x_0| + h \Big] ds \le \tau(Lr + h) \\
&\Rightarrow \|(Px) - x_0\|_C = \sup_{t \in [0,\tau]} |(Px)(t) - x_0| \le \delta(Lr + h)
\end{aligned}$$

By choosing $\delta \le r/(Lr + h)$, we have $\|(Px) - x_0\|_C \le r$

Proof of Local Existence and Uniqueness

Claim 2: $(Px)(t)$ is contraction

Proof: Since f is Lipschitz, we can show that

$$|(Px)(t) - (Py)(x)| \leq \int_0^t L|x(s) - y(s)|ds \leq \int_0^t dsL\|x - y\|_C$$

Hence

$$\|(Px) - (Py)\| \leq L\delta\|x - y\|_C \leq \rho\|x - y\|_C,\ \delta \in (0, \frac{\rho}{L})$$

Choosing $\rho < 1$ and $\delta \leq \frac{\rho}{L}$, P is contraction.

Proof of Local Existence and Uniqueness

This implies that if

$$\delta \leq \min\{T, \frac{r}{Lr+h}, \frac{\rho}{L}\}, \ \rho < 1$$

then

- $P : S \to S$
- P is contraction

Therefore, there exists a unique fixed point x^*; hence, the state equation admits a unique solution on S

Proof of Local Existence and Uniqueness

Finally, we need to show that with $\delta \leq \min\{T, \frac{r}{Lr+h}, \frac{\rho}{L}\}$ and $\rho < 1$, the solution exists in X.

Claim: If the solution exists in X, then the solution must be in $S = \{x \in X \mid \|x - x_0\|_C \leq r\}$ for $t \in [0, \delta]$.

Proof: Assume that there exists μ such that

$$|x(\mu) - x_0| = r$$

On the other hand, for $t \leq \mu$,

$$|x(t) - x_0| \leq \int_0^t \Big[L|x(s) - x_0| + h\Big] ds \leq \int_0^t (Lr + h) ds$$

Hence

$$r = |x(\mu) - x_0| \leq (Lr + h)\mu \;\Rightarrow\; \mu \geq \frac{r}{Lr + h} \geq \delta$$

Note that $t \in [0, \delta]$. This implies that the solution cannot leave the set S for $t \in [0, \delta]$.

Proof of Global Existence and Uniqueness

In this theorem, note that since we have Global Lipschitz property, the radius r is sufficiently large. This implies $\frac{r}{Lr+h} > \frac{\rho}{L}$, and

$$\delta \leq \min\{T, \frac{\rho}{L}\}, \ \rho < 1$$

- If $T \leq \frac{\rho}{L}$, then choose $\delta = T$, and we are done
- Otherwise, set $\delta \leq \frac{\rho}{L}$, solve the problem for $[0, \delta]$, then continue $[\delta, 2\delta], \ldots, [T - \delta, T]$

This completes the proof.

Back to Examples

- LTI system satisfies the Global Lipschitz condition
- LTV system satisfies the Global Lipschitz condition, provided that $A(t)$ is bounded
- This is why we study linear system theory

Gronwall-Bellman Inequality

Let $\lambda : [a, b] \to \mathbb{R}$ be continuous and $\mu : [a, b] \to \mathbb{R}$ be continuous and nonnegative. If a function $y : [a, b] \to \mathbb{R}$ satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s) y(s) ds$$

for $t \in [a, b]$, then on the same interval

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\Big[\int_s^t \mu(\tau) d\tau\Big] ds$$

Proof of Gronwall-Bellman Inequality

Proof: Let $z(t) = \int_a^t \mu(s)y(s)ds$, and

$$v(t) = z(t) + \lambda(t) - y(t) \geq 0$$

Note that z is differentiable, and

$$\dot{z}(t) = \mu(t)y(t) = \mu(t)\Big[z(t) + \lambda(t) - v(t)\Big], \;\; z(a) = 0$$

The solution of z can be written as

$$z(t) = \int_a^t \phi(t,s)\Big[\mu(s)\lambda(s) - \mu(s)v(s)\Big]ds,$$

where

$$\phi(t,s) = \exp\Big[\int_s^t \mu(\tau)d\tau\Big]$$

This is the state transition matrix, and we will study this later

Proof of Gronwall-Bellman Inequality

Note that $\mu(t) \geq 0$ and $v(t) \geq 0$ for all $t \in [a, b]$, and $\phi(t, s) \geq 0$ for all $t, s \in [a, b]$. Hence

$$\int_a^t \phi(t, s)\mu(s)v(s)ds \geq 0$$

This implies

$$z(t) \leq \int_a^t \phi(t, s)\mu(s)\lambda(s)ds$$

Since $y(t) \leq \lambda(t) + z(t)$, this completes the proof.

Summary

- Existence and uniqueness
- The motivation studying linear system theory
- Next class
  - Linear algebra