HW8: Linear System Theory (ECE532)

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Due Date: May 30 (Wed) at the beginning of the class.

Problem 1:

In this question, we will prove the Kalman decomposition in a different way.

Definition: Given $S \subset V$, where V is a (finite-dimensional) vector space and S is its subspace. Let $A : V \to V$ be a linear operator (Note that any linear operator in a finite-dimensional space can be represented by a matrix). We say that S is A-invariant (with respect to V) if

$$AS \subseteq S.$$

Remark 1: If $S = \emptyset$, then S is A-invariant. Also, $S = V$ is A-invariant. We can also show that the range and null spaces of A are also A-invariant.

Remark 2: Suppose that W is a subspace of $\mathbb{R}^n$ with dimension $r < n$, and A is $n \times n$ dimensional. If W is A-invariant, then we can show that there exists a nonsigular T such that A can be written as

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{21} \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{r \times r}$ and $A_{21} \in \mathbb{R}^{(n-r)\times(n-r)}$. Also, $TW =$ range space of $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, where I_r is an $r \times r$ identity matrix.

Consider the linear system

$$\dot{x} = Ax + Bu,\ \ y = Cx.$$

Recall the controllability subspace

$$\mathcal{C}_{AB} = \text{range (column) space of } \mathcal{C},$$

where $\mathcal{C}$ is the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}.$$

Recall also the unobservable subspace

$$\mathcal{N}_{CA} = \text{null space of } \mathcal{O},$$

where $\mathcal{O}$ is the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Solve the following problems

- Show that $\mathcal{C}_{AB}$ and $\mathcal{N}_{CA}$ are subspaces of $\mathbb{R}^n$, which are A-invariant.
- Show that $\mathcal{C}_{AB} \cap \mathcal{N}_{CA}$ is A-invariant.
- Prove the Kalman decomposition. Namely there exists a nonsingular T such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 & A_6 & 0 \\ A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & A_7 & 0 \\ 0 & 0 & A_8 & A_9 \end{bmatrix}, \ TB = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$$
$$CT^{-1} = \begin{bmatrix} C_1 & 0 & C_2 & 0 \end{bmatrix}$$

  where

  - (C_1, A_1) observable
  - (A_1, B_1) controllable
  - $(\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix})$ controllable
  - $(\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} A_1 & A_6 \\ 0 & A_7 \end{bmatrix})$ observable

  To show this, you need to A-invariant properties of $\mathcal{C}_{AB}$ and $\mathcal{N}_{CA}$, and the decomposition matrix T that decomposes the state space equation into the four independent subspaces: controllable and observable, controllable and unobservable, uncontrollable and observable, and uncontrollable and unobservable.

Problem 2: Solve the following problems. In some problems (design problems), you need to solve the problem by your hands and using MATLAB ("place" or "acker")

- Problem 8.14 of the textbook
- Problem 8.7 of the textbook
- Problem 8.19 (full observer design only) of the textbook
- Problem 8.5 of the textbook
- Problem 8.9 of the textbook
- Problem 8.15 of the textbook
- Problem 8.8 of the textbook

Problem 3

Consider the harmonic oscillator with position measurement

$$\ddot{x} + x = u, \ y = x$$

- Obtain the state-space representation of the system

- Show that it cannot asymptotically stabilized by static output feedback of the form $u = ky$ (Note that $y = x$)
- Find a dynamic output feedback (controller and estimator) that asymptotically stabilizes the system.

Problem 4

Consider the open-loop transfer function system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2}.$$

(i) Obtain the state space equation

(ii) Discretize the system using zero-order hold via MATLAB with sampling $20Hz$.

(iii) Find the feedback gain K of the discretized system (ii) so that the control close-loop poles have natural frequency $\omega_n = 10rad/sec$ and the damping ratio $\zeta = 0.7$.

(iv) Find the estimator gain L so that $\omega_n = 20rad/sec$ and the damping ratio $\zeta = 0.7$

(v) Determine the transfer function of the closed-loop system obtained from (iii) and (iv).

(vi) Using MATLAB, obtain the time response when the input is the unit step

(vii) What is the steady state error? Propose the method to reduce the steady state error.