Introduction to Control

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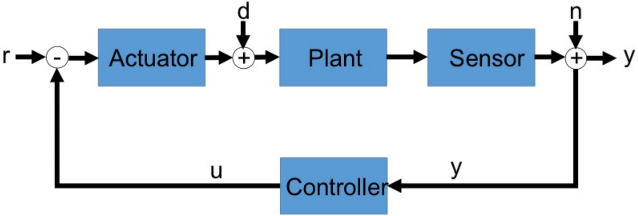
Frequency Domain Technique

February 26, 2018

Overview

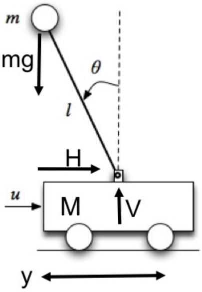
- Introduction to Control
- Frequency domain analysis

Introduction



- Before you design a control system, you must know the mathematical expression of the plant that you want to control, and need to understand its behavior with respect to time (or frequency)
- To model the plant, we need to understand some physical and engineering laws that we have learned
  - KCL, KVL, Ohm's law
  - Newton's law

Introduction: Pendulum

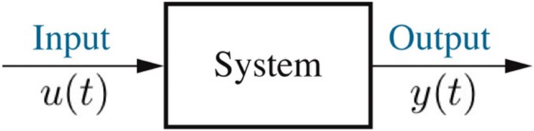


The equation of y and θ:

$$M\frac{d^2y}{dt^2} = u - H, \qquad H = m\frac{d^2}{dt^2}(y + \sin\theta)$$

$$mg - V = m\frac{d^2}{dt^2}(l\cos\theta), \quad mgl\sin\theta = ml^2\frac{d^2\theta}{dt^2} + m\frac{d^2y}{dt^2}l\cos(\theta)$$

Response by Convolution



The system satisfies the superposition property if

$$\begin{cases} u_1(t) \to y_1(t) \\ u_2(t) \to y_2(t) \end{cases}$$

then

$$\alpha_1 u_1(t) + \alpha_2 u_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

So it holds additivity and homogeniety.

- The system that holds the superposition property is called the linear system
- If the system does not hold the superposition property, the system is nonlinear

Response by Convolution

Example: Show that the superposition holds for the following system

$$\dot{y}(t) + ky(t) = u(t)$$

where y is output and u is input of the system

Let $u = \alpha_1 u_1 + \alpha_2 u_2$, and assume that $y = \alpha_1 y_1 + \alpha_2 y_2$. Then

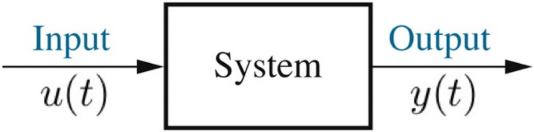
$$\dot{y} = \alpha_1 \dot{y}_1 + \alpha_2 \dot{y}_2$$

Hence,

$$\begin{aligned} \alpha_1 \dot{y}_1 + \alpha_2 \dot{y}_2 + k(\alpha_1 y_1 + \alpha_2 y_2) &= \alpha_1 u_1 + \alpha_2 u_2 \\ \Leftrightarrow \ \alpha_1(\dot{y}_1 + ky_1 - u_1) + \alpha_2(\dot{y}_2 + ky_2 - u_2) &= 0 \end{aligned}$$

Hence, if y_i is the solution of u_i, $i = 1, 2$, then the above equation is satisfied, and the response is the sum of the individual responses. The superposition property holds!!

Response by Convolution



The system is time invariant if $y(t)$ is the output caused by the input $u(t)$, then $y(t-\tau)$ will be the response to $u(t-\tau)$ with $t \geq \tau$.

- Time-invariant $\Rightarrow$ The system characteristics do not change with respect to time shifting

Response by Convolution

Now, consider

$$\dot{y}_1(t) + k(t)(t)y_1(t) = u_1(t)$$

and

$$\dot{y}_2(t) + k(t)(t)y_2(t) = u_1(t-\tau)$$

Note that $k(t)$ varies over the time. Assume that $y_2(t) = y_1(t-\tau)$. Then

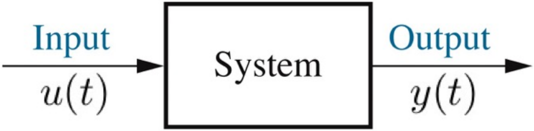
$$\frac{dy_1(t-\tau)}{dt} + k(t)y_1(t-\tau) = u_1(t-\tau)$$

Change of variable $\eta = t - \tau$ leads to

$$\frac{dy_1(\eta)}{dt} + k(\eta+\tau)y_1(\eta) = u_1(\eta)$$

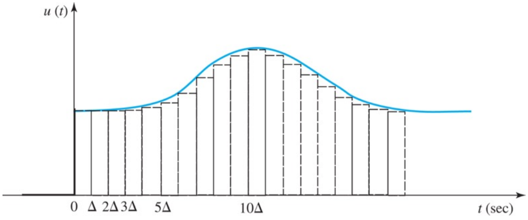
Hence, the system is not time invariant when $k(t)$ varies over the time. Note that the above system is time invariant when $k(t) = k$ for all t ($k(t)$ is a constant)

Response by Convolution



- If the system is linear (hence hold the superposition property), and time invariant, we call it the Linear Time Invariant (LTI) system

Response by Convolution



We note that any continuous signal $u(t)$ can be approximated by the rectangular pulse $p_\Delta(t)$

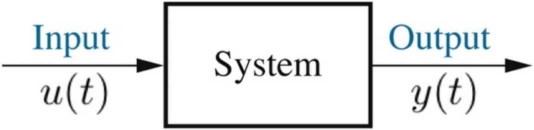
$$u(t) \approx \sum_i u(t_i) p_\Delta(t - t_i)\Delta$$

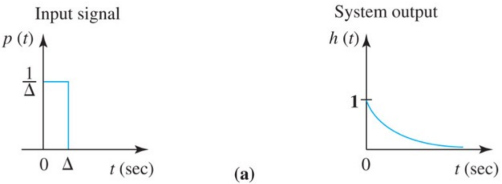
and as $\Delta \to 0$, we have

$$u(t) = \int_{\tau=0}^{\infty} u(\tau) p(t - \tau) d\tau$$

where $p(\tau)$ is the impulse signal

Response by Convolution

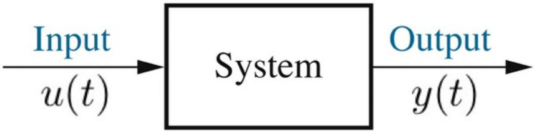


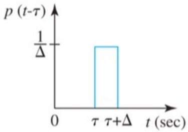


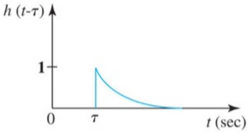
(a)

- Assume that $u(t)$ is a short pulse $p_\Delta(t)$ (rectangular pulse), i,e. $u(t) = p_\Delta(t)$, and the corresponding output is $y(t) = h(t)$
- Due to homogeniety, if the input is scaled by $u(t) = \alpha p_\Delta(t)$, then the output is also scaled, i.e., $y(t) = \alpha h(t)$

Response by Convolution

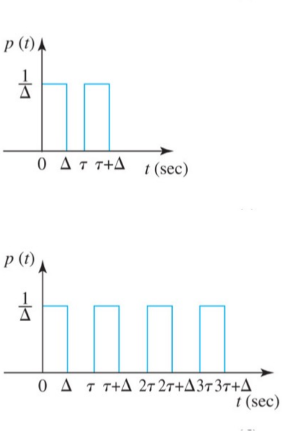






- If we delay the input by τ, i.e., $u(t) = p_\Delta(t - \tau)$, the output will be $y(t) = h(t - \tau)$
- Time invariance!!

Response by Convolution



- as a consequence of superposition and time invariance of the system

Response by Convolution

By using the superposition and time-invariance properties,

$p_\Delta(t) \to h(t)$ the output of the rectangular pulse by the definition

$p_\Delta(t - t_i) \to h(t - t_i)$ time invariance

$p_\Delta(t - t_i)u(t_i)\Delta \to h(t - t_i)u(t_i)\Delta$ homogeneity

$\sum_{i=0} p_\Delta(t - t_i)u(t_i)\Delta \to \sum_{i=0} h(t - t_i)u(t_i)\Delta$ additivity

Response by Convolution

Hence, the output $y(t)$ from the input $u(t)$ can be approximated by

$$y(t) \approx \sum_{i=0} h(t - t_i)u(t_i)\Delta,$$

and by taking $\Delta \to 0$,

$$y(t) = \int_{\tau=0}^{\infty} h(t - \tau)u(\tau)d\tau$$

This is the convolution integral

- $h(t)$ is called the impulse response, since if $u(t)$ is impulse, then $y(t) = h(t)$.

Response by Convolution

$$y(t) = \int_{\tau=0}^{\infty} h(t-\tau)u(\tau)d\tau$$

Notice that the limits of the integral are at infinity. This means that either or both h and u are nonzero for negative time. Note that if h has values for the negative time, it means that the system starts before the input is applied, that is, the future output affects the current output.

Response by Convolution

Definition (Causality): The present value of the output signal depends only on the present or past values of the input signal.

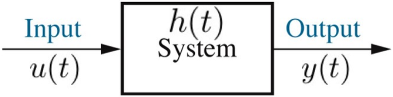
Most engineering systems are causal systems, in which case

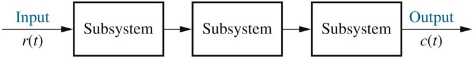
$$h(t) = 0 \ \forall t < 0 \quad \Leftrightarrow h(t-\tau) = 0 \ \forall \tau > t$$

Hence,

$$\begin{aligned} y(t) &= \int_{\tau=0}^{t} h(t-\tau)u(\tau)d\tau \\ &= \int_{\tau=0}^{t} h(\tau)u(t-\tau)d\tau \text{ by change of variables} \end{aligned}$$

Response by Convolution

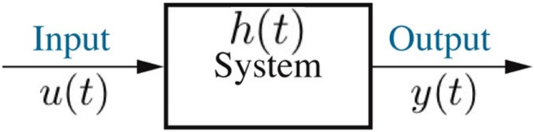




$$y(t) = \int_{\tau=0}^{t} h(t-\tau)u(\tau)d\tau = \int_{\tau=0}^{t} h(\tau)u(t-\tau)d\tau$$

- Mostly, the computation of the convolution integral is very challenging. You must sweep all the previous responses to get the current output.
- Moreover, if the system consists of multiple subsystems, then the computation of the convolution integral is even more challenging
- But we know, the convolution becomes multiplication via Laplace transform theory
- This is our central motivation studying linear system theory

The Transfer Function

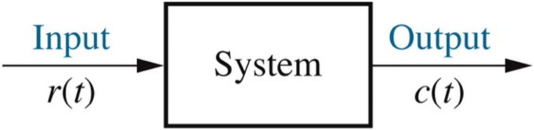


- The transfer function from the input and output is defined by

$$H(s) = \frac{Y(s)}{U(s)}$$

- This is a very important definition, since we will analyze the system by using its transfer function (see Section 2.3)
- If $U(s) = 1$, then $H(s) = Y(S)$. It means that $H(s)$ is the Laplace transformation of the system when the input is impulse
- Hence, $H(s) = \mathcal{L}[h(t)]$, and $h(t)$ is called the impulse respose

The Transfer Function



Suppose that the input $r(t)$ and output $c(t)$ behavior of the system is described by the following ODE with the zero initial condition:

$$a_2 \frac{dc^2(t)}{dt^2} + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_1 \frac{dr(t)}{dt} + b_0 r(t)$$
$$\Rightarrow (a_2 s^2 + a_1 s + a_0) C(s) = (b_1 s + b_0) R(s)$$

Hence, the transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

Therefore, the transfer function $H(s)$ is the ratio of the Laplace transform of the output to the input of the system with zero initial conditions

The Transfer Function

The general transfer function has the following form

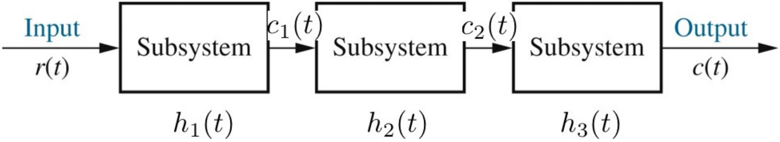
$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

Definition: The transfer function

- is proper if $n \geq m$
- is strictly proper if $n > m$
- is biproper if $n = m$
- is improper if $n < m$

For the causal system, we must have $n \geq m$. In this course, we will consider the proper system only

The Transfer Function



For a cascading system, we can show that the transfer function of the overall system is

$$H(s) = H_1(s)H_2(s)H_3(s) = \frac{C(s)}{R(s)}$$

where

$$H_1(s)\frac{C_1(S)}{R(S)} \quad H_2(s)\frac{C_2(S)}{C_1(S)} \quad H_1(s)\frac{C(S)}{C_2(S)}$$

The Transfer Function

Example: Find the transfer function of

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \quad H(s) = \frac{1}{s+2}$$

If the input $r(t)$ is the step input, i,e. $r(t) = 1$ when $t \geq 0$, then $R(s) = \frac{1}{s}$. In this case, we have

$$C(s) = H(s)R(s) = \frac{1}{s(s+2)} = \frac{0.5}{2} - \frac{0.5}{s+2} \quad c(t) = 0.5 - 0.5e^{-2t}$$

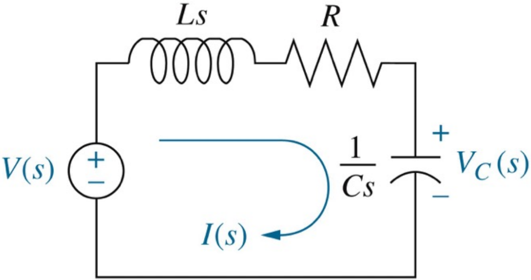
Modeling in Frequency Domain: Electric Networks

- Objective: Obtain the transfer function of various electric circuits
- Ohm's law, KCL, KVL

Component	Voltage-current	Current-voltage	Voltage-charge
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$

Modeling in Frequency Domain: Electric Networks

Example: RCL network with KVL, Find the transfer function
$H(s) = \frac{V(S)}{I(S)}$

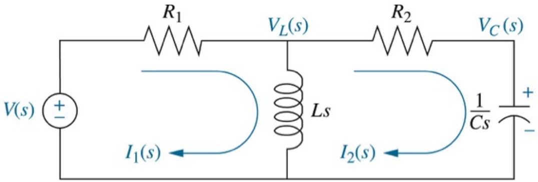


Modeling in Frequency Domain: Electric Networks

$$
\begin{aligned}
& L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \\
\Rightarrow \quad & sLI(s) + RI(s) + \frac{1}{Cs}I(s) = V(s) \\
\Leftrightarrow \quad & H(s) = \frac{V(S)}{I(S)} = \frac{1}{Ls + R + \frac{1}{Cs}} = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}
\end{aligned}
$$

Modeling in Frequency Domain: Electric Networks

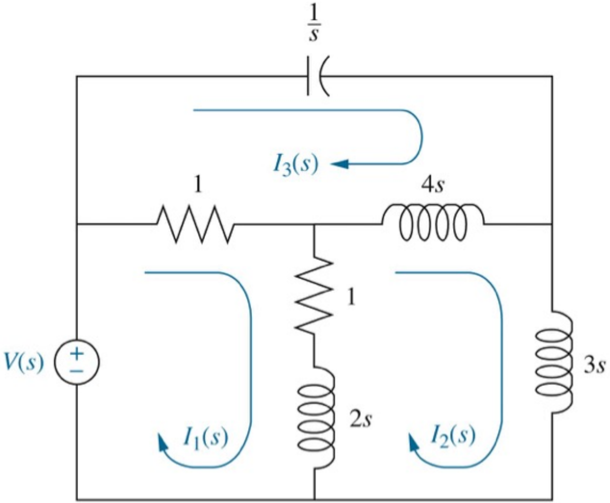
Example: RCL network, Find the transfer function $H(s) = \frac{I_2(S)}{V(S)}$



$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s) \quad \text{1st loop}$$

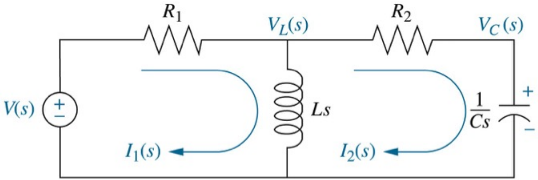
$$LsI_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - LsI_1(s) = 0 \quad \text{2nd loop}$$

Modeling in Frequency Domain: Electric Networks



Modeling in Frequency Domain: Electric Networks

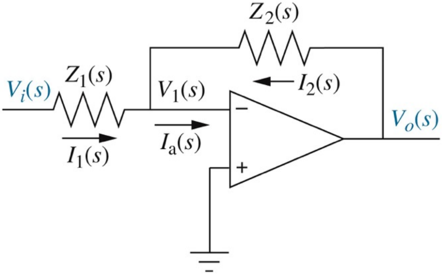
Example: RCL network, Find the transfer function $H(s) = \frac{V_C(S)}{V(S)}$ (KCL)



$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0 \quad \text{1st loop}$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0 \quad \text{2nd loop}$$

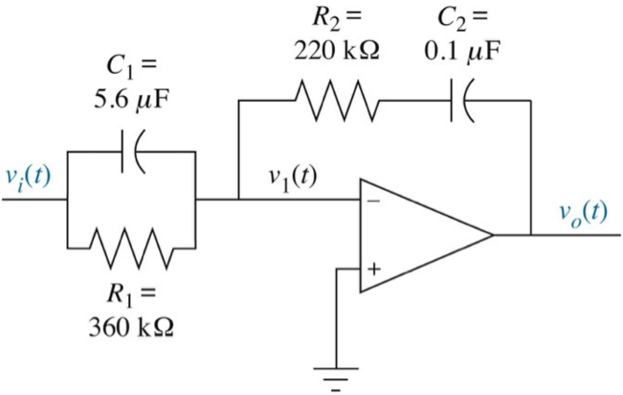
Modeling in Frequency Domain: Electric Networks



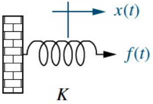
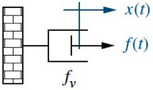
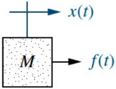
- Negative feedback OP-AMP circuit
- $I_1 = -I_2$, $I_1 = V_i/Z_1$, $-I_2 = -V_0/Z_2$
- The transfer function from V_i to V_0 is

$$\frac{V_0(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

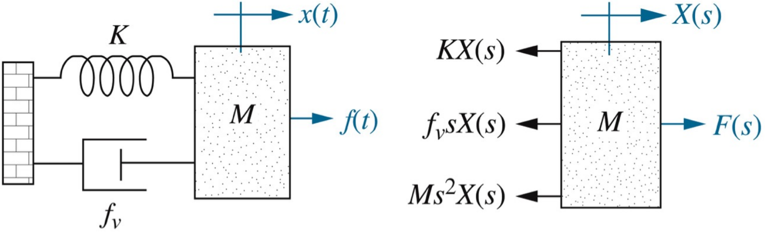
Modeling in Frequency Domain: Electric Networks



Modeling in Frequency Domain: Mechanical Systems

<table>
<tr><th>Component</th><th>Force-velocity</th><th>Force-displacement</th></tr>
<tr><td>Spring
</td><td>$f(t) = K\int_0^t v(\tau)d\tau$</td><td>$f(t) = Kx(t)$</td></tr>
<tr><td>Viscous damper
</td><td>$f(t) = f_v v(t)$</td><td>$f(t) = f_v \dfrac{dx(t)}{dt}$</td></tr>
<tr><td>Mass
</td><td>$f(t) = M\dfrac{dv(t)}{dt}$</td><td>$f(t) = M\dfrac{d^2x(t)}{dt^2}$</td></tr>
</table>

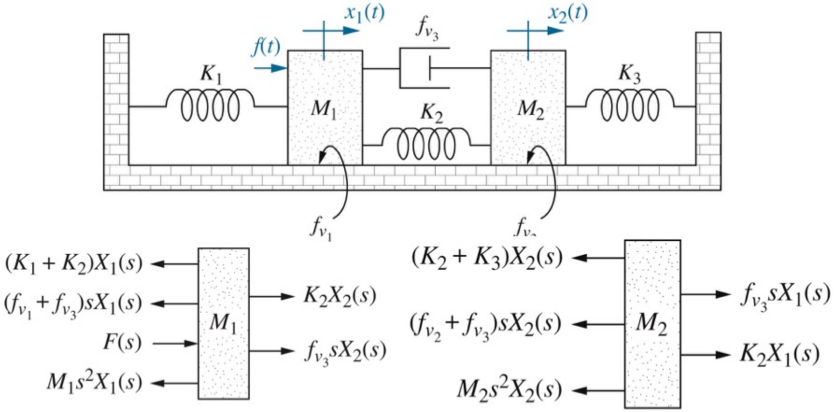
Modeling in Frequency Domain: Mechanical Systems



$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

Modeling in Frequency Domain: Mechanical Systems



Modeling in Frequency Domain: Mechanical Systems

Motion equation

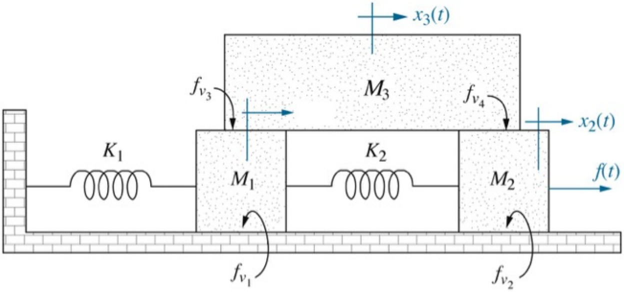
$$\Big(M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)\Big)X_1(s) - \Big(f_{v_3} s + K_2\Big)X_2(s) = F(s)$$

$$-\Big(f_{v_3} s + K_2\Big)X_1(s) + \Big(M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)\Big)X_2(s) = 0$$

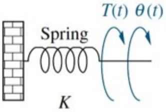
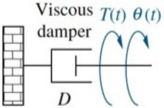
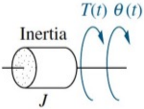
Transfer function

$$G(s) = \frac{X_2(s)}{F(s)}$$

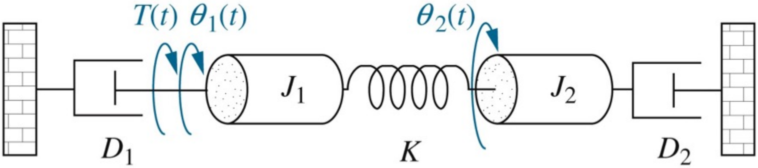
Modeling in Frequency Domain: Mechanical Systems



Modeling in Frequency Domain: Rotational Systems

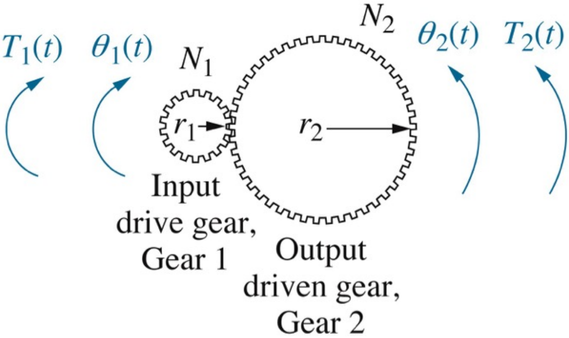
Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Modeling in Frequency Domain: Rotational Systems



$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s)$$
$$-K_1\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

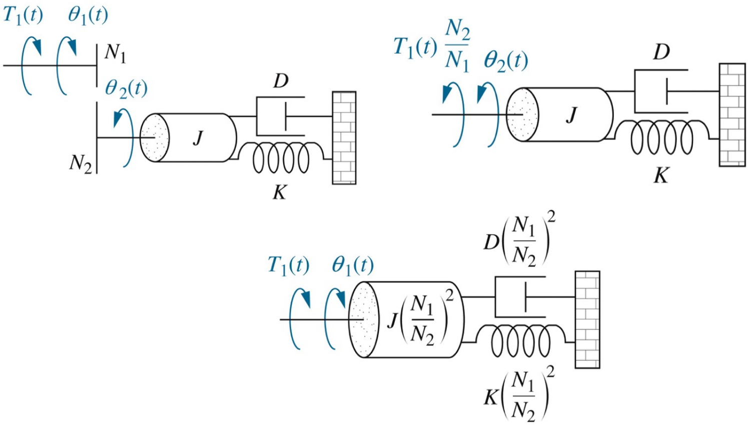
Modeling in Frequency Domain: Systems with Gears



- Circumference distance: $r_1\theta_1 = r_2\theta_2$
- Energy equivalence: $T_1\theta_1 = T_2\theta_2$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1} = \frac{r_2}{r_1} = \frac{\omega_1}{\omega_2}$$

Modeling in Frequency Domain: Systems with Gears



Modeling in Frequency Domain: Systems with Gears

$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

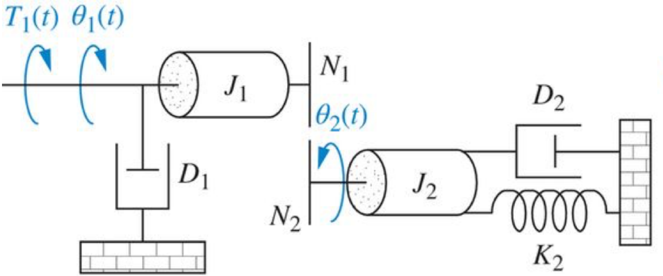
$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

$$\left[J\left(\frac{N_1}{N_2}\right)^2 a^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

$$\boxed{\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2}$$

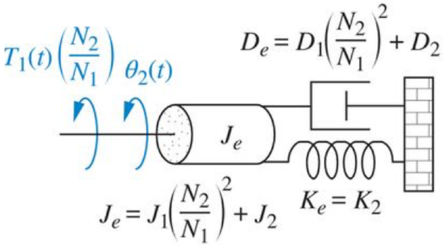
Modeling in Frequency Domain: Systems with Gears

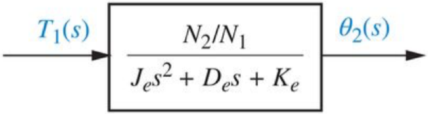
Example: Equivalent transfer function of $\theta_2(s)/T_1(s)$



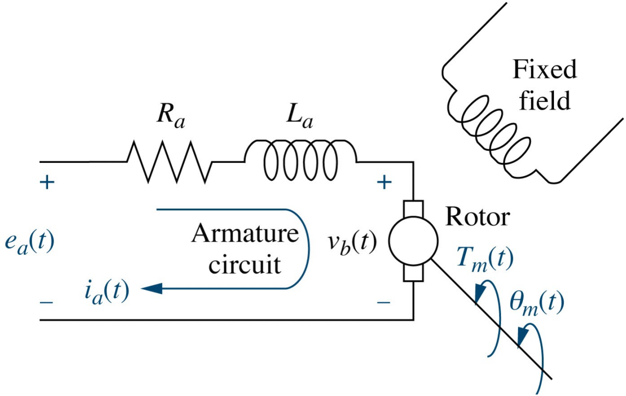
Modeling in Frequency Domain: Systems with Gears

Example: Equivalent transfer function of $\theta_2(s)/T_1(s)$

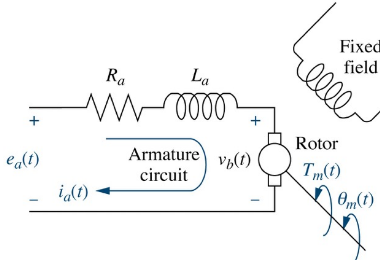


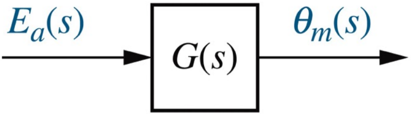


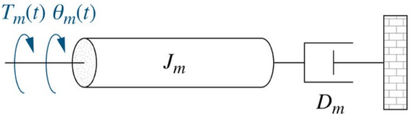
Modeling in Frequency Domain: Electromech. Systems



Modeling in Frequency Domain: Electromech. Systems



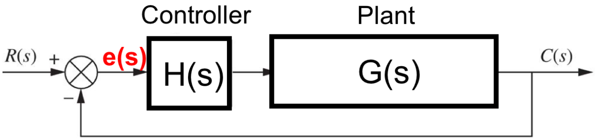




$$G(s) = \frac{\theta(s)}{E_a(s)}$$

Control Design via Frequency Domain Analysis

Motivation: Feedback Control System Design



Goal of control system design

- Stability
- Robustness
- Performance

Stability: Stability Concepts

The system's response can be decomposed as

$$c(t) = c_{forced}(t) + c_{natural}(t)$$

- c_{forced}: response with respect to the input
- $c_{natural}$: response with respect to the initial condition

Stability: Stability Concepts

Asymptotic stability (zero input and nonzero initial condition)

- A system is asymptotically stable if $\lim_{t \to \infty} c_{natural}(t) = 0$ for all initial conditions
- A system is unstable if $\lim_{t \to \infty} c_{natural}(t) = \infty$ for some initial conditions
- A system is marginally stable (stable in the sense of Lyapunov) if $c_{natural}(t)$ is bounded for all initial conditions

Asymptotic stability considers system response with respect to the initial condition

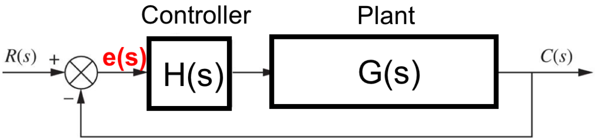
Stability: Stability Concepts

BIBO stability (for the forced response (or the total response))

- A system is BIBO stable if every bounded input excites the bounded output
- A system is unstable in the BIBO sense if any bounded output excites the unbounded output

BIBO stability considers system response with respect to the input

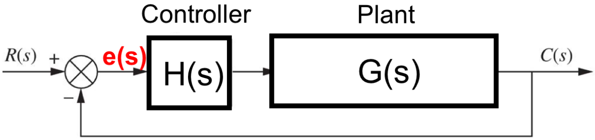
Motivation: Feedback Control System Design



Designing the feedback control system for $G(s)$

- Stability!!
- We want to improve the transient response (fast rising time and peak time, settling time, less overshoot)
- We want to improve the steady state response (reduce steady state error)
- We need to design $H(s)$ to meet design specifications
- Two major approach: Root locus and Bode plot

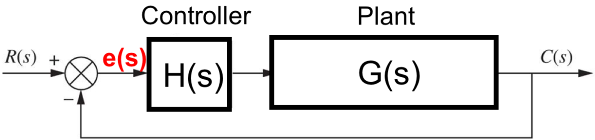
Motivation: Feedback Control System Design



How to improve the transient response?

- Add an ideal derivative s in the feedback system (control the slope of the response)
- Add a lead compensator
- Zero: determines the amplitude and the transient response of the system

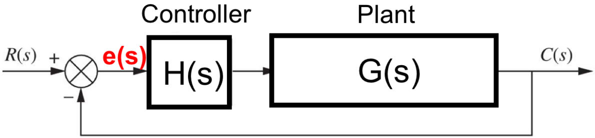
Motivation: Feedback Control System Design



How to improve the steady state response?

- Add an integrator: increase the type of the open-loop system $G(s)$
- It reduces the steady state error

PID Control



Proportional-Integral-Derivative (PID) control

$$H(s) = K_p + \frac{K_I}{s} + K_D s$$

- The input of the controller $H(s)$ is the error signal, $e(s) = R(s) - Y(s)$
- Unknown K_p, K_I, K_D: PID gains
- The derivative control is rarely used (why?)