[MEN573] Advanced Control Systems I

Lecture 14 - Controllability and Observability of Discrete Time Systems

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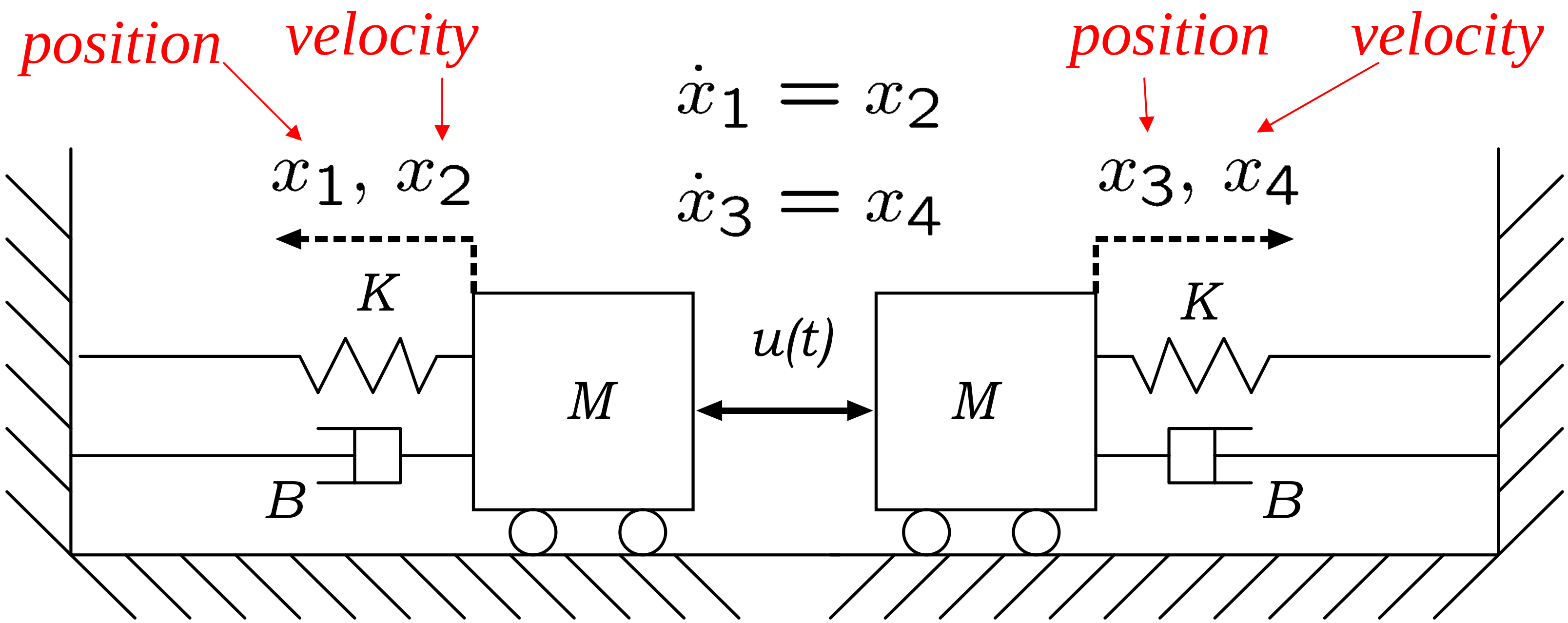
Controllability and Observability

These two important properties of dynamic systems are critical for the design and analysis of control systems:

- **Controllability:** determines if the system state can be arbitrarily steered by the controlling input.

- **Observability:** determines if the system state can be estimated from the measured output.

An uncontrollable system: Example



Assume that $x(0) = 0$

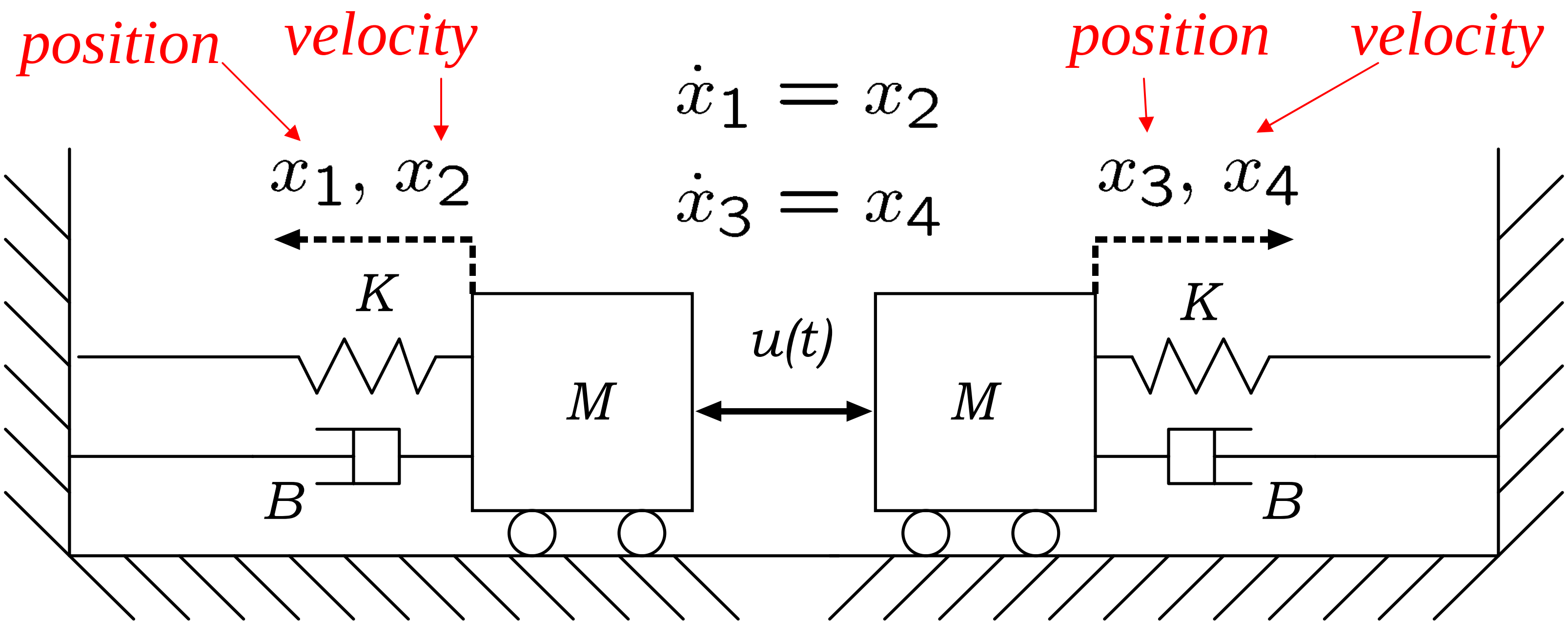
- Because of symmetry, no matter what the input is,

$$x_1(t) = x_3(t) \qquad \forall t \geq 0$$
$$x_2(t) = x_4(t)$$

State cannot be arbitrarily steered

An uncontrollable system: Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_3 = x_4$$



Assume that $x(0) = 0$

- It is **<u>not possible</u>** to make

$$x_1(t) \neq x_3(t)$$
$$x_2(t) \neq x_4(t)$$

State cannot be arbitrarily steered

Definition of controllability (DT)

Definition: The system

$$x(k+1) = A\,x(k) + B\,u(k)$$

is said to be **<u>controllable</u>** if,

- for any **<u>initial</u>** state $x(0) = x_0$
  and any **<u>target</u>** state, x_1
- there exists a **<u>finite</u>** integer N
  and a control sequence

$$\{u(k);\ k \in [0, N]\}$$

- that will transfer the state x_0 to $x(N) = x_1$

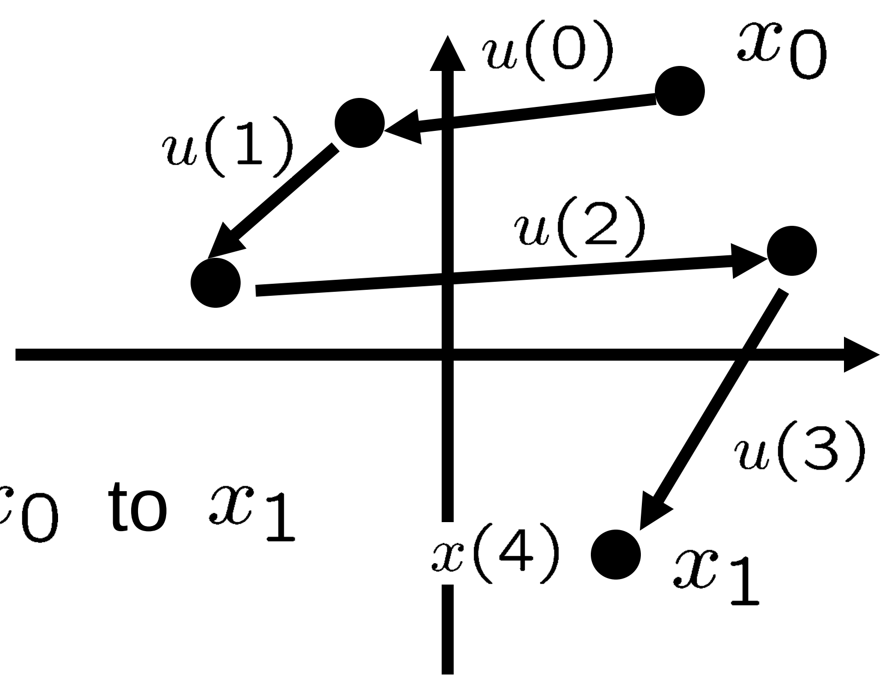
Definition of controllability (DT)

for any **initial** state $x(0) = x_0$

and any **target** state, x_1

there exists a **finite** integer N

and a control sequence



that will transfer the state x_0 to x_1

(in this example, N = 4)

Definition of controllability (DT)

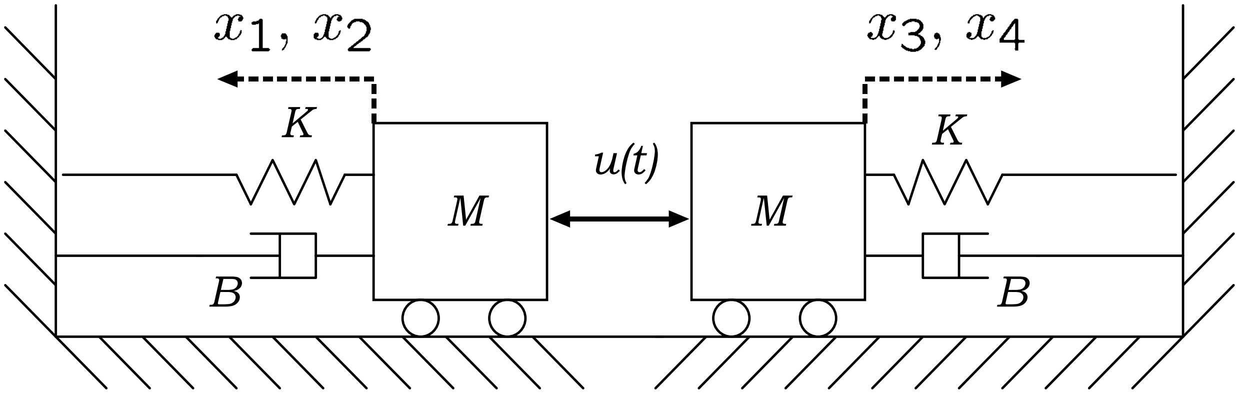
Comments:

- The definition requires that both the initial state x_0 and the “target” state x_1 be ***arbitrary***.

- The definition requires the state to reach x_1 in a ***finite*** number of steps N and says nothing about what will happen to the state $x(k)$, for $k > N$

- It is not required that the state remains at x_1 for $k > N$.

An uncontrollable system: example



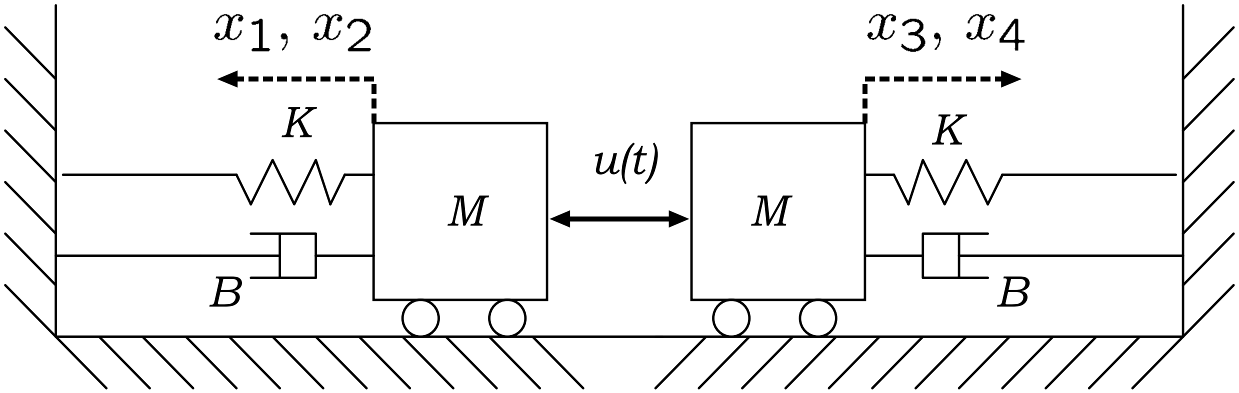
In this case, from $x(0) = 0$, we can only reach states that satisfy:

equal

$$x_1 = \begin{bmatrix} \boxed{x_{11}} & \boxed{x_{22}} & \boxed{x_{11}} & \boxed{x_{22}} \end{bmatrix}^T$$

equal

An uncontrollable system: example



The state

$$x_1 = \begin{bmatrix} x_{11} & x_{22} & 0 & 0 \end{bmatrix}^T \; x_{11} \neq 0, \, x_{22} \neq 0$$

can never be reached from $x(0) = 0$

Notation

The characteristic polynomial of a square matrix

$A \in \mathcal{R}^{n \times n}$ is:

$$\begin{aligned}\Delta(\lambda) &= \mathrm{Det}(\lambda I - A) \\ &= \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_o\end{aligned}$$

The eigenvalues of $\boldsymbol{A}$ are the roots of its characteristic equation

$$\Delta(\lambda) = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_o = 0$$

Cayley-Hamilton Theorem

Every matrix $A \in \mathcal{R}^{n \times n}$ **satisfies its own characteristic equation**. i.e.

$$\Delta(A) = 0$$

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_o I = 0$$

where

$$\Delta(\lambda) \;=\; \mathrm{Det}(\lambda I - A)$$

is the characteristic polynomial for the matrix A .

Proof of the Cayley-Hamilton Theorem

Notice that

$$\Delta(A) \;=\; (\lambda_1 I - A)(\lambda_2 I - A)\cdots(\lambda_n I - A)$$

where λ_i is the i-th eigenvalue of A

If some eigenvalues are repeated, we can re-write

$$\Delta(A) = (\lambda_1 I - A)^{m_1} \cdots (\lambda_p I - A)^{m_p}\,,$$

$$m_1 + \cdots + m_p = n$$

Proof of the Cayley-Hamilton Theorem

- Let v_1 be the eigenvector associate with the repeated eigenvalue λ_1

Since $(\lambda_1 I - A)\, v_1 = 0$

$$\boxed{\Delta(A)v_1 \;=\; (\lambda_p I - A)^{m_p} \cdots (\lambda_1 I - A)^{m_1}\, v_1 = 0}$$

Proof of the Cayley-Hamilton Theorem

- Let v_2 be a generalized eigenvector, defined as

$$(\lambda_1 I - A)\, v_2 = -v_1$$

since, $$(\lambda_1 I - A)^{m_1}\, v_2 = -(\lambda_1 I - A)^{m_1 - 1} v_1 = 0$$

$$\boxed{\Delta(A) v_2 \;=\; (\lambda_p I - A)^{m_p} \cdots (\lambda_1 I - A)^{m_1}\, v_2 = 0}$$

Proof of the Cayley-Hamilton Theorem

Thus, defining the nonsingular matrix

$$T = [v_1 \, v_2 \, \cdots \, v_n]$$

formed by the eigenvectors and generalized eigenvectors of A

we obtain,

$$\boxed{\Delta(A)\,T \;=\; 0}$$

which in turn implies that,

$$\boxed{\Delta(A) \;=\; 0}$$

Q.E.D

Cayley-Hamilton Theorem

- According to the C-H theorem,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_o I = 0$$

Multiplying by a matrix $\boldsymbol{B}$ on the right, we obtain

$$A^n B + a_{n-1}A^{n-1}B + \cdots + a_1 AB + a_o B = 0$$

Cayley-Hamilton Theorem

$$A^n B + a_{n-1} A^{n-1} B + \cdots + a_1 AB + a_o B = 0$$

which means that the vectors formed by the columns of

$$A^n B,\ A^{n-1} B,\ \cdots,\ AB,\ B$$

are **linearly dependent**.

Thus, we get a corollary on the next page.

Corollary of the C-H Theorem

If there are ***m*** linearly independent vectors in the columns of

$$A^{n-1}B,\ A^{n-2}B,\ \cdots,\ A^2B,\ AB,\ B$$

$$m \le n \qquad\qquad A \in \mathcal{R}^{n\times n} \qquad B \in \mathcal{R}^{n\times r}$$

Then, there will <u>still be</u> ***m*** linearly independent vectors in the columns of

$$\underbrace{A^nB,}_{\textit{Adding these columns does not help}}\ A^{n-1}B,\ A^{n-2}B,\ \cdots,\ A^2B,\ AB,\ B$$

Controllability Theorem

The following 3 statements are equivalent:

(a) The LTI system of order $\boldsymbol{n}$

$$x(k+1) = A\,x(k) + B\,u(k)$$

is controllable.

Sometimes we simply state that the pair

$$\{A\,B\}$$

is controllable.

Controllability Theorem

The following 3 statements are equivalent:

(b) The controllability grammian

$$W_c(m) \;=\; \sum_{k=0}^{m} A^k B B^T (A^T)^k$$

is positive definite, for some finite integer $\boldsymbol{m = k_1}$

$$W_c(k_1) \succ 0$$

Controllability Theorem

(c) The controllability matrix

$$P = \left[B \;\; AB \;\; A^2B \;\; \cdots A^{n-1}B \right]$$

is rank $\boldsymbol{n}$.

(I.e. there are $\boldsymbol{n}$ linearly independent columns)

Controllability matrix

The controllability matrix

$$P = \left[B \;\; AB \;\; A^2B \;\; \cdots A^{n-1}B \right]$$

is particularly useful in determining the controllability of the pair

$$\{A \, B\}$$

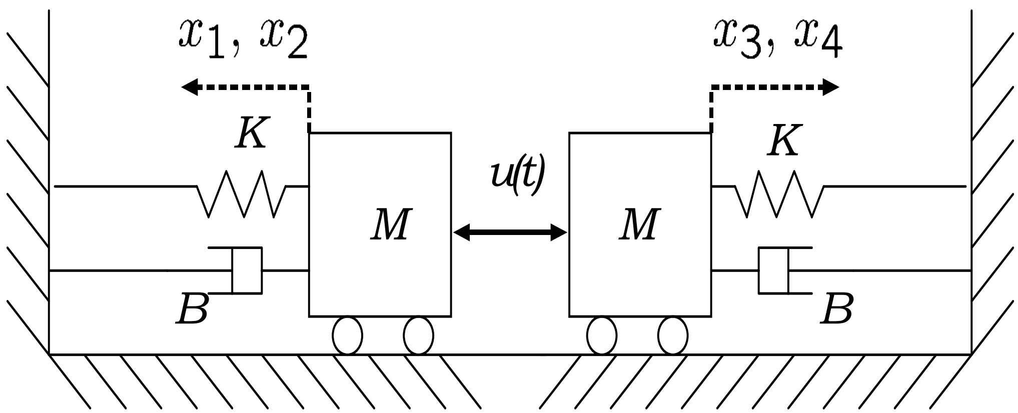
Example: controllability matrix

Consider the pair

$$A = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Discrete-time sampled model:

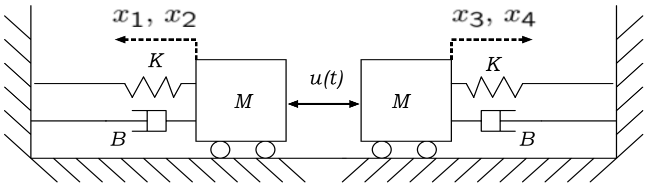
$\boldsymbol{A}$ is Schur



Example: controllability matrix

Given the pair

$$A = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}$$

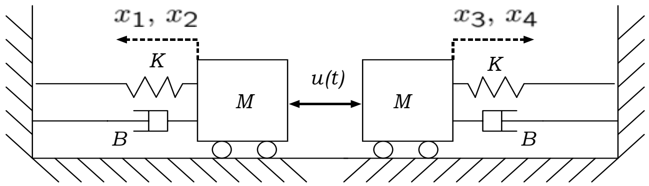


$$\begin{array}{c} P = \begin{bmatrix} 0.3000 & 0.2800 & -0.0072 & -0.0953 \\ 0.4000 & -0.2980 & -0.2311 & 0.0227 \\ 0.3000 & 0.2800 & -0.0072 & -0.0953 \\ 0.4000 & -0.2980 & -0.2311 & 0.0227 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ B & AB & A^2B & A^3B \end{array} \end{array}$$

Example: controllability matrix

Given the pair

$$A = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}$$



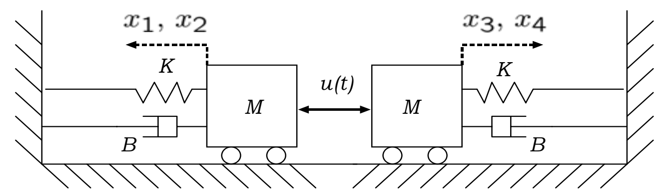
$$P = \begin{bmatrix} 0.3000 & 0.2800 & -0.0072 & -0.0953 \\ 0.4000 & -0.2980 & -0.2311 & 0.0227 \\ 0.3000 & 0.2800 & -0.0072 & -0.0953 \\ 0.4000 & -0.2980 & -0.2311 & 0.0227 \end{bmatrix}$$

$\text{rank}(P) = 2$ ⟹ ***System is not controllable***

Example: controllability matrix

Given the pair

$$A = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}$$



Matlab commands:

P = ctrb(A,B)

$$P = \begin{bmatrix} 0.3000 & 0.2800 & -0.0072 & -0.0953 \\ 0.4000 & -0.2980 & -0.2311 & 0.0227 \\ 0.3000 & 0.2800 & -0.0072 & -0.0953 \\ 0.4000 & -0.2980 & -0.2311 & 0.0227 \end{bmatrix}$$

R= rank(P)

R = 2

Controllability matrix

Notice that the controllability matrix may not be square.

- Assume that $\boldsymbol{B}$ has $\boldsymbol{2}$ columns

$$B = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \in \mathcal{R}^{n\times 2}$$

- Then $\boldsymbol{P}$ has $\boldsymbol{2n}$ columns

$$P = \Big[\underbrace{b_1\ b_2}_{B} \mid \underbrace{Ab_1\ Ab_2}_{AB} \mid \underbrace{A^2b_1\ A^2b_2}_{A^2B} \mid \cdots \underbrace{A^{n-1}b_1\ A^{n-1}b_2}_{A^{n-1}B} \Big] \in \mathcal{R}^{n\times 2n}$$

We need to find $\boldsymbol{n}$ linearly independent (LI) columns out of $\boldsymbol{2n}$

Controllability Theorem

The pair $\{A \, B\}$ is controllable iff

(b) The controllability grammian

$$W_c(m) \;=\; \sum_{k=0}^{m} A^k B B^T (A^T)^k$$

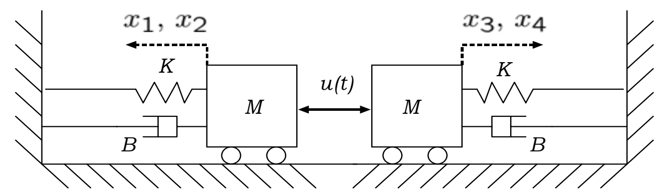
is positive definite, for some finite integer $\boldsymbol{m = k_1}$

$$W_c(k_1) \succ 0$$

Example: controllability grammian

Given the pair

$$A = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}$$



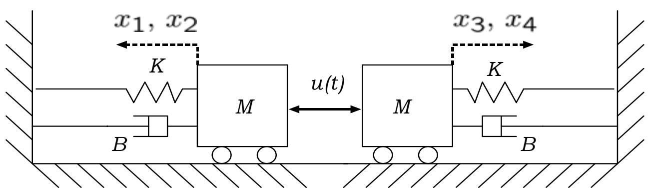
$$W_c(0) \;=\; B\,B^T \succeq 0$$

$$= \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix} \begin{bmatrix} 0.3 & 0.4 & 0.3 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.09 & 0.12 & 0.09 & 0.12 \\ 0.12 & 0.16 & 0.12 & 0.16 \\ 0.09 & 0.12 & 0.09 & 0.12 \\ 0.12 & 0.16 & 0.12 & 0.16 \end{bmatrix}$$

Example: controllability grammian

Given the pair

$$A = \begin{bmatrix} 0.4 & 0.4 & 0 & 0 \\ -0.9 & -0.07 & 0 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & -0.9 & -0.07 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix}$$



$$W_c(1) \;=\; B\,B^T + AB\,(AB)^T \;\succeq 0$$

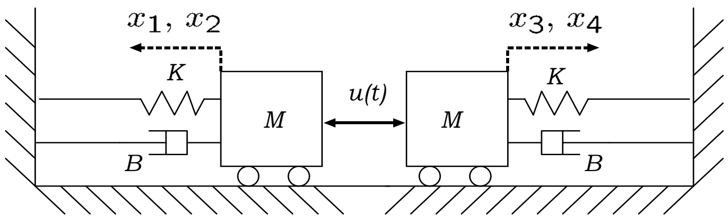
$$= \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \\ 0.4 \end{bmatrix} \begin{bmatrix} 0.3 & 0.4 & 0.3 & 0.4 \end{bmatrix} + \begin{bmatrix} 0.28 \\ -0.298 \\ 0.28 \\ -0.298 \end{bmatrix} \begin{bmatrix} 0.28 & -0.298 & 0.28 & -0.298 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1684 & 0.0366 & 0.1684 & 0.0366 \\ 0.0366 & 0.2488 & 0.0366 & 0.2488 \\ 0.1684 & 0.0366 & 0.1684 & 0.0366 \\ 0.0366 & 0.2488 & 0.0366 & 0.2488 \end{bmatrix}$$

Example: controllability grammian

Given the pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$



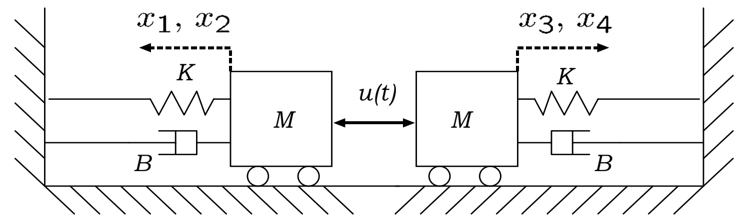
$$W_c(3) \;=\; B\,B^T + AB\,(AB)^T + A^2B\,(A^2B)^T + A^3B\,(A^3B)^T$$

$$W_c(3) = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \begin{bmatrix} B^T \\ (AB)^T \\ (A^2B)^T \\ (A^3B)^T \end{bmatrix}$$

Example: controllability grammian

Given the pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$



$$W_c(3) = \underbrace{\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} B^T \\ (AB)^T \\ (A^2B)^T \\ (A^3B)^T \end{bmatrix}}_{P^T}$$

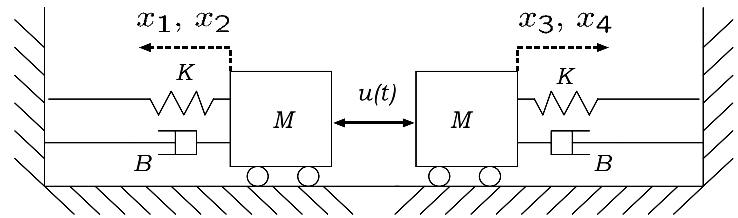
controllability matrix

controllability matrix transposed

Example: controllability grammian

Given the pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$



$$W_c(3) = P\,P^T \succeq 0$$

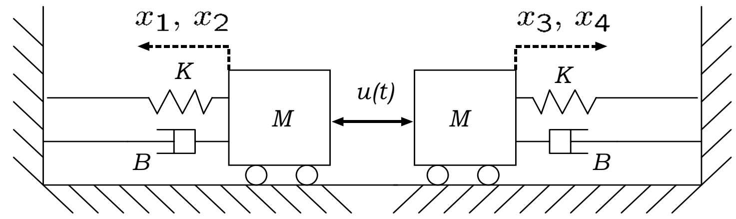
since

$$\text{rank}(P) = 2$$

Example: controllability grammian

Given the pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$



Moreover, according to the controllability theorem, for this system (notice that $\boldsymbol{A}$ is Schur):

$$W_c(\infty) = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k \succeq 0$$

$$= \begin{bmatrix} 0.2158 & 0.0396 & 0.2158 & 0.0396 \\ 0.0396 & 0.3753 & 0.0396 & 0.3753 \\ 0.2158 & 0.0396 & 0.2158 & 0.0396 \\ 0.0396 & 0.3753 & 0.0396 & 0.3753 \end{bmatrix}$$

Proof of Controllability Theorem

1) (c) implies (b):

Assume that the controllability matrix

$$P = \begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix}$$

is rank $\boldsymbol{n}$

We will show that

$$W_c(n-1) \succ 0$$

and, as a consequence,

$$W_c(k_1) \succ 0 \qquad \forall k_1 \geq n-1$$

Proof of Controllability Theorem

1) (c) implies (b):

the controllability matrix for ***n-1*** is

$$W_c(n-1) = \sum_{k=0}^{n-1} A^k B B^T (A^T)^k$$

$$= B\,B^T + (AB)\,(AB)^T + \cdots + (A^{n-1}B)\,(A^{n-1}B)^T$$

$$= \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix}}_{P^T}$$

controllability matrix

controllability matrix transposed

Proof of Controllability Theorem

1) (c) implies (b):

the controllability matrix for ***n-1*** is

$$W_c(n-1) = \sum_{k=0}^{n-1} A^k B B^T (A^T)^k$$

$$= P\,P^T \succ 0$$

Since $\boldsymbol{P}$ is rank n

Proof of Controllability Theorem

2) (b) implies (a):

Assume that the controllability grammian for ***n-1*** is positive definite

$$W_c(n-1) \ = \ P\,P^T \ \succ 0$$

We will show that:

- Given any $x(0) = x_0$ and final state x_1
- We can find a control sequence $\{u(0),\, u(1),\, \cdots\,,\, u(n-1)\}$
- That will take the $x_o \longrightarrow x_1$ in ***n*** steps

Proof of Controllability Theorem

Given any $x(0) = x_0$ the state $x(n)$ is given by

$$x(n) = A^n x_0 + \underbrace{\sum_{k=0}^{n-1} A^{n-1-k} B\, u(k)}$$

expanding,

$$x(n) = A^n x_0$$

$$\underbrace{+ \; B\, u(n-1) + AB\, u(n-2) + \cdots + A^{n-1} B\, u(0)}$$

$$\underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{P} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}$$

controllability matrix $\longrightarrow P$

Proof of Controllability Theorem

2) (b) implies (a) (continued):

$$x(n) = A^n x_0 + P \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Assume that we want $x(n) = x_1$

Thus, we want

$$P \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix} = x_1 - A^n x_0$$

Proof of Controllability Theorem

2) (b) implies (a) (continued):

$\boldsymbol{P}$ is rank $\boldsymbol{n}$ $\Longrightarrow$ $\boxed{W_c(n-1) \;=\; \left\{PP^T\right\} \succ 0}$

Since we want

$$P \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = x_1 - A^n x_0$$

Choose:

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = P^T \left[PP^T\right]^{-1} (x_1 - A^n x_0)$$

Proof of Controllability Theorem

2) (b) implies (a) (continued):

We have

$$x(n) = A^n x_0 + P \underbrace{\begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}}_{P^T \left[PP^T\right]^{-1} (x_1 - A^n x_0)}$$

$$x(n) = A^n x_0 + P\, P^T \left[PP^T\right]^{-1} (x_1 - A^n x_0)$$

Proof of Controllability Theorem

2) (b) implies (a) (continued):

$$x(n) = A^n x_0 + \underbrace{P\,P^T\,\left[PP^T\right]^{-1}}_{I}\,(x_1 - A^n x_0)$$

$$x(n) = \cancel{A^n x_0} + x_1 - \cancel{A^n x_0}$$

$$x(n) = x_1$$

Q.E.D

Proof of Controllability Theorem

3) a) implies c):

Assume that the pair $\{A\ B\}$ is controllable

We need to show that the controllability matrix

$$P = \left[B\ \ AB\ \ A^2B\ \ \cdots A^{n-1}B\right]$$

<u>must be</u> rank $\boldsymbol{n}$

Proof of Controllability Theorem

3) We will prove a) $\Longrightarrow$ **c) by proving that:**

not c) $\Longrightarrow$ **not a),** i.e.

Assume that the controllability matrix

$$P = \begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix}$$

is not rank $\boldsymbol{n}$**,** but rank $\boldsymbol{m} < \boldsymbol{n}$.

We need to show that the pair $\{A\ B\}$

is not controllable

Proof of Controllability Theorem

3) **not c)** implies **not a):**

Assume that $\mathrm{rank}(P) < n$

Then, given $\boldsymbol{x(0) = x_o}$, the state $\boldsymbol{x(n)}$ is:

$$x(n) = A^n x_0 + P \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Proof of Controllability Theorem

3) **not c)** implies **not a):**

$$x(n) = A^n x_0 + P \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Is it possible to find a vector $\begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$

So that $x(n) = x_1$?

Proof of Controllability Theorem

3) **not c)** implies **not a):**

Is it possible to find a vector $\begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$

that solves $P \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix} = x_1 - A^n x_0$

when x_1 and x_0 are arbitrary?

Proof of Controllability Theorem

3) not c) implies **not a):**

Because $\text{rank}(P) < n$

It is **not possible** to find a vector $\begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$ that solves

$$P \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix} = x(1) - A^n x_0$$

when $x_1 - A^n x_0 \not\in \text{Range}(P)$

Proof of Controllability Theorem

3) not c) implies **not a):**

If it is not possible to transfer to $\boldsymbol{x_1}$ in $\boldsymbol{n}$ steps, is it possible to do so in $\boldsymbol{n+1}$ time steps?

At time $\boldsymbol{n+1}$

$$[P \;\; A^n B \;] \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{bmatrix} = \underbrace{x(n+1)}_{= x_1\,?} - A^{n+1} x_0$$

Proof of Controllability Theorem

3) **not c)** implies **not a):**

Is it possible to find a vector $\begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$

that solves

$$[P \;\; A^n B \;] \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{bmatrix} = x_1 - A^{n+1} x_0$$

when $x(1)$ and $x(0)$ are arbitrary?

Proof of Controllability Theorem

3) **not c)** implies **not a):**

The Corollary of the Cayley Hamilton theorem says

$$\text{rank}\left([P \;\; A^n B \;]\right) = \text{rank}\left(P\right)$$

therefore,

$$\text{rank}\left([P \;\; A^n B \;]\right) = m < n$$

Proof of Controllability Theorem

3) **not c)** implies **not a):**

Because $\mathrm{rank}([P \;\; A^n B \;]) < n$

It is **not possible** to find a vector $\begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$ that solves

$$[P \;\; A^n B \;] \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{bmatrix} = x_1 - A^{n+1} x_0$$

when $x_1 - A^n x_0 \notin \mathrm{Range}([P \;\; A^n B \;])$

Proof of Controllability Theorem

3) not c) implies **not a):**

For the same reason as shown on the previous pages,

- if it is not possible to transfer to x_1 in n steps,
- it is not possible to do so in $n+l$ steps ($l > 1$).

→ **The system is not controllable. Q.E.D**

Remarks on Controllability Theorem

1. If a discrete time LTI system of order n is controllable, it can reach any arbitrary target state from an arbitrary initial condition in n steps.

2. The conditions in the theorem only give a “yes” or “no” answer to the question of controllability.

3. No statement is provided regarding the “degree of controllability”, or whether it is difficult or easy to control the system.

Remarks on Controllability Theorem

Example: The following two pairs are both controllable:

a) $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $P_1 = \begin{bmatrix} B_1 & A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $A_2 = \begin{bmatrix} 0 & 0.01 \\ 0 & 1 \end{bmatrix}$ $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $P_2 = \begin{bmatrix} B_2 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0.01 \\ 1 & 1 \end{bmatrix}$

Both, $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ have rank 2.

Thus, both can reach the target state in two steps.

Remarks on Controllability Theorem

a) $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad P_1 = \begin{bmatrix} B_1 & A_1 B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $A_2 = \begin{bmatrix} 0 & 0.01 \\ 0 & 1 \end{bmatrix} \; B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} B_2 & A_2 B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0.01 \\ 1 & 1 \end{bmatrix}$

However, the control action required to go from $[0,0]^T$ to $[1,1]^T$ is quite different:

a) $\{u(0),\, u(1)\} = \{1, 1\}$

b) $\{u(0),\, u(1)\} = \{100, -99\}$

Remarks on Controllability Theorem

The controllable canonical pair

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is always controllable, since

$$P_c = \begin{bmatrix} B_c & A_c B_c & A_c^2 B_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & (-a_1 + a_2^2) \end{bmatrix}$$

is always full rank.

This result generalizes to an arbitrary order n

Controllability Grammian

Assume that the matrix $\boldsymbol{A}$ is Schur.

Then, the asymptotic value of the controllability grammian

$$W_c = \lim_{k_1 \to \infty} W_c(k_1) = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

exists (all elements of $\boldsymbol{W_c}$ are bounded).

Controllability Grammian & Lyapunov Eq

Assume that the matrix $\boldsymbol{A}$ is Schur.

$$W_c = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

can be calculated as the solution of the following Lyapunov equation:

$$A\, W_c A^T - W_c = -B\, B^T$$

Moreover, $W_c \succ 0$ **iff** $\{A\, B\}$ is a controllable pair

Definition of Observability (DT)

The LTI discrete time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

is said to be observable if,

for ***any*** initial state $x(0) = x_0$

there exists a finite integer N such that

knowledge of the input and output sequences

$$\{u(k);\ k \in [0, N]\} \qquad \{y(k);\ k \in [0, N]\}$$

over the interval $[0, N]$

is sufficient to determine the initial state x_0

Definition of Observability (DT)

Notice that **only** the output $\boldsymbol{y(k)}$ is measured

and, the initial state x_0 is unknown at $\boldsymbol{k = 0}$.

If the system is observable, after collecting

$$\{u(0),\, u(1),\, \cdots\, u(N)\} \quad \textit{input sequence}$$

$$\{y(0),\, y(1),\, \cdots\, y(N)\} \quad \textit{output sequence}$$

for some finite $\boldsymbol{N}$,

we are able to determine the initial state x_0

Determining the free response

Notice that the response of

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned} \qquad x(0) = x_o$$

is composed of a free response and a forced response:

$$y(k) = y_{free}(k) + y_{force}(k)$$

$$y_{free}(k) = C A^k x(0)$$ ***unknown***

$$y_{force}(k) = C \sum_{j=0}^{k-1} A^{k-1-j} B\,u(j) + D\,u(j)$$

Determining the free response

$$y(k) \;=\; y_{free}(k) + y_{force}(k)$$

The forced response is entirely determined from the input sequence, which is **known**.

$$y_{force}(k) \;=\; C \sum_{j=0}^{k-1} A^{k-1-j} \, B \, u(j) + D \, u(j)$$

Thus, the free response output

$$y_{free}(k) = y(k) - y_{force}(k)$$

can be assumed to be measurable

Determining the free response

Thus, without loss of generality,

The system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned} \qquad x(0) = x_o$$

is observable **iff**,

the free response system

$$\begin{aligned} x(k+1) &= A\,x(k) \\ y(k) &= C\,x(k) \end{aligned} \qquad x(0) = x_o$$

is observable

Definition of Observability (DT)

The LTI discrete time system

$$\begin{aligned} x(k+1) &= A\,x(k) \\ y(k) &= C\,x(k) \end{aligned} \qquad x(0) = x_o$$

is said to be observable if,

for ***any*** initial state $x(0) = x_0$ (unknown)

there exists a finite integer N such that

knowledge of the output sequence

$$\{y(0),\, y(1),\, \cdots,\, y(N)\}$$

is sufficient to determine the initial state $x(0) = x_0$

Observability Theorem

The following 3 statements are equivalent:

(a) The LTI system of order $\boldsymbol{n}$

$$\begin{aligned} x(k+1) &= A\,x(k) \\ y(k) &= C\,x(k) \end{aligned}$$

is observable.

Sometimes we simply state that the pair

$$\{A\,C\}$$

is observable.

Observability Theorem

The following 3 statements are equivalent:

(b) The observability grammian

$$W_o(m) \;=\; \sum_{k=0}^{m} (A^T)^k \, C^T C \, A^k$$

is positive definite, for some finite integer $\boldsymbol{m = k_1}$

$$W_o(k_1) \succ 0$$

Observability Theorem

(c) The observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is rank $\boldsymbol{n}$.

(I.e. there are $\boldsymbol{n}$ linearly independent rows)

Observability matrix

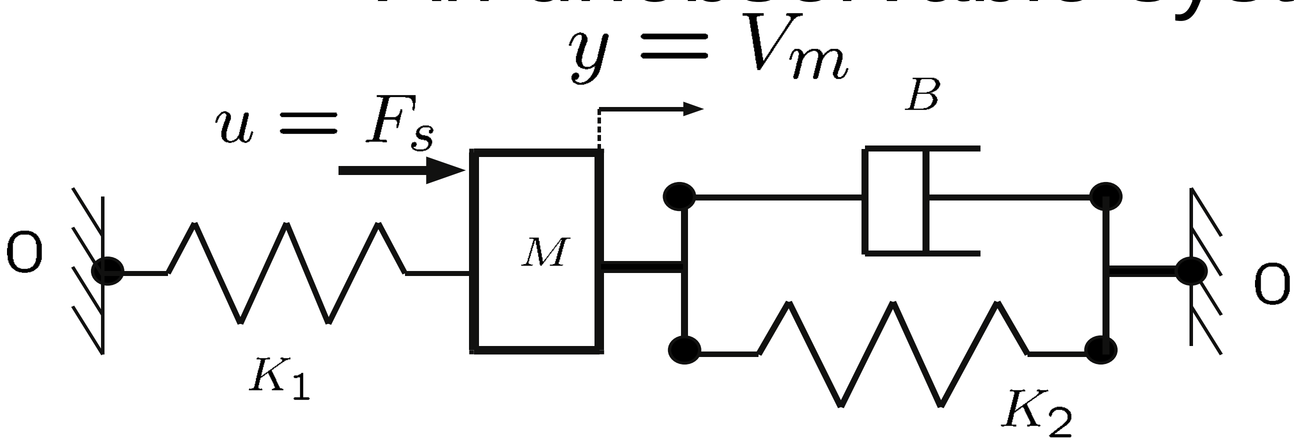
The observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is particularly useful in determining the observability of the pair

$$\{A\, C\}$$

An unobservable system

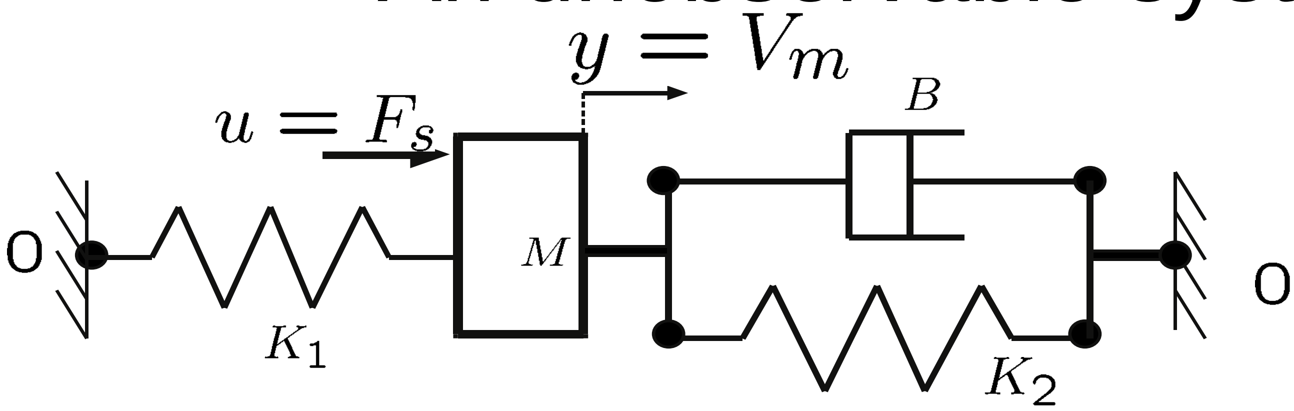


- State and output equations (CT)

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ f_2 \\ f_4 \end{bmatrix} = \begin{bmatrix} -\frac{B}{M} & \frac{-1}{M} & \frac{-1}{M} \\ K_1 & 0 & 0 \\ K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ f_2 \\ f_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{M} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ f_2 \\ f_4 \end{bmatrix}$$

An unobservable system



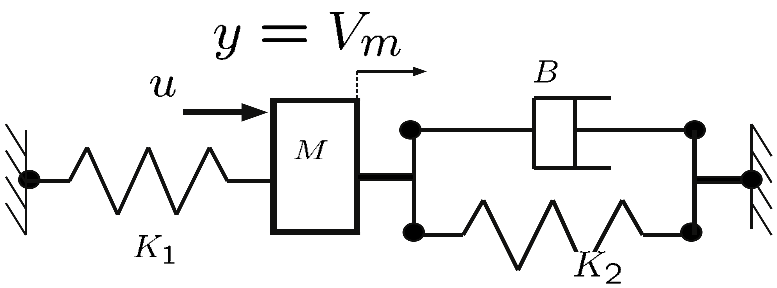
- Autonomous state and output equations (DT)

$$\begin{bmatrix} v_1(k+1) \\ f_2(k+1) \\ f_4(k+1) \end{bmatrix} = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix} \begin{bmatrix} v_1(k) \\ f_2(k) \\ f_4(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(k) \\ f_2(k) \\ f_4(k) \end{bmatrix}$$

Observability matrix example

Given the pair



$$A = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

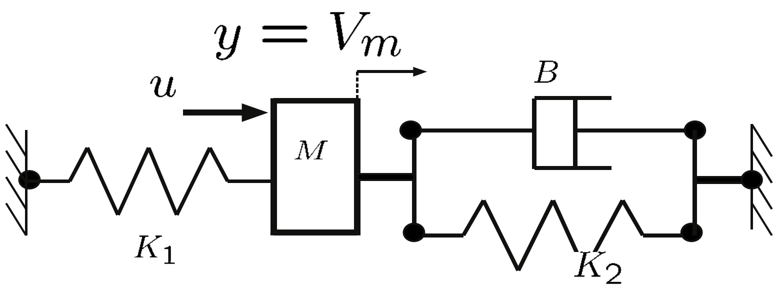
$$Q = \begin{bmatrix} 1 & 0 & 0 \\ -0.24 & -0.36 & -0.36 \\ -0.35 & 0.04 & 0.04 \end{bmatrix} \begin{array}{l} \Leftarrow C \\ \Leftarrow CA \\ \Leftarrow CA^2 \end{array}$$

Observability matrix example

Given the pair

$$A = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

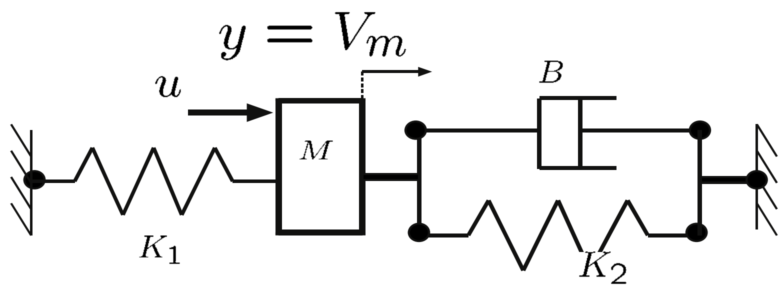


$$Q = \begin{bmatrix} 1 & 0 & 0 \\ -0.24 & -0.36 & -0.36 \\ -0.35 & 0.04 & 0.04 \end{bmatrix}$$

$\mathrm{rank}(Q) = 2$ ⇨ ***System is unobservable***

Example: observability matrix

Given the pair



$$A = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Matlab commands:

P = obsv(A,C)

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ -0.24 & -0.36 & -0.36 \\ -0.35 & 0.04 & 0.04 \end{bmatrix}$$

R= rank(Q)

R = 2

Observability Theorem

The following 3 statements are equivalent:

(b) The observability grammian

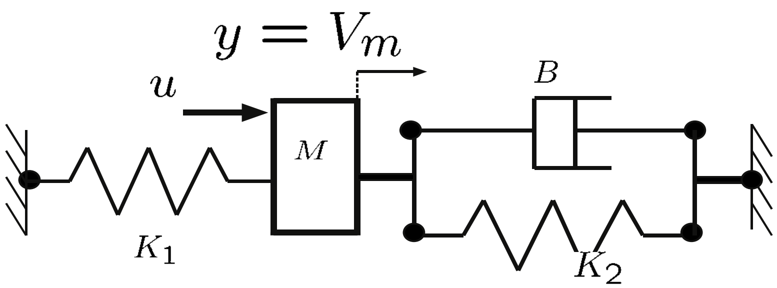
$$W_o(m) \;=\; \sum_{k=0}^{m} (A^T)^k \, C^T C \, A^k$$

is positive definite, for some finite integer $\boldsymbol{m = k_1}$

$$W_o(k_1) \succ 0$$

Example: observability grammian

Given the pair



$$A = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix}$$

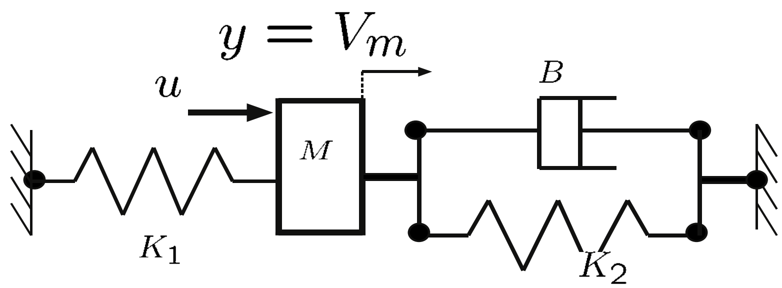
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$W_o(0) \;=\; C^T C \succeq 0$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: observability grammian

Given the pair



$$A = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$W_o(1) \;=\; C^T C + (CA)^T CA \qquad \succeq 0$$

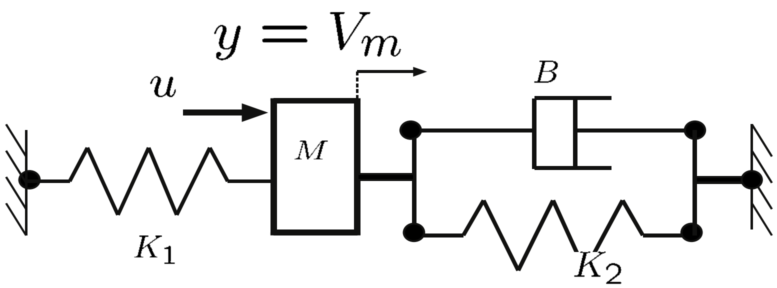
$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -0.24 \\ -0.36 \\ -0.36 \end{bmatrix} \begin{bmatrix} -0.24 & -0.36 & -0.36 \end{bmatrix}$$

Example: observability grammian

Given the pair

$$A = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

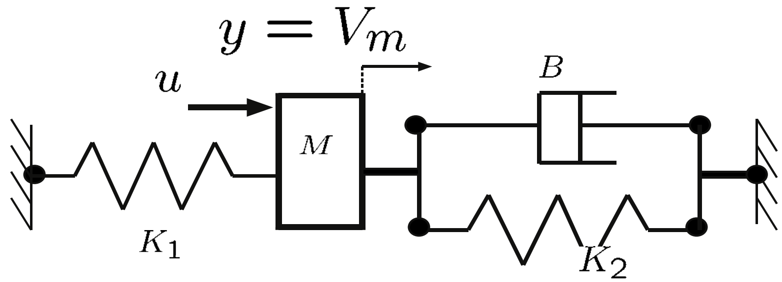


$$W_o(2) \;=\; C^T C + (CA)^T CA + (CA^2)^T CA^2$$

$$W_o(2) \;=\; \begin{bmatrix} C^T & (CA)^T & (CA^2)^T \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

Example: observability grammian

Given the pair



$$A = \begin{bmatrix} -0.24 & -0.36 & -0.36 \\ 0.36 & 0.71 & -0.29 \\ 0.73 & -0.58 & 0.42 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$W_o(2) = \underbrace{\begin{bmatrix} C^T & (CA)^T & (CA^2)^T \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}}_{Q}$$

$$W_o(2) = Q^T Q \succeq 0 \qquad \text{rank}(Q) = 2$$

Controllability and Observability Duality

The observability results are *duals* of the controllability results in the following sense:

The pair $\{A\,,C\}$ is observable **iff**

the pair $\left\{A^T\,,C^T\right\}$ is controllable.

We will often use the duality between observability and controllability in deriving future results.

Controllability and Observability Duality

Example:

The pair $\{A\,, C\}$ is observable **<u>iff</u>**

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ is rank n

The pair $\left\{A^T\,, C^T\right\}$ is controllable **<u>iff</u>**

$$P_o = \begin{bmatrix} C^T & A^T C^T & \cdots & (A^{n-1})^T C^T \end{bmatrix}$$ is rank n

Controllability and Observability Duality

Example:

Since,

$$P_o^T = \begin{bmatrix} C^T & A^T C^T & \cdots & (A^{n-1})^T C^T \end{bmatrix}^T = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = Q$$

The pair $\{A, C\}$ is observable **<u>iff</u>**

the pair $\{A^T, C^T\}$ is controllable.

Controllability and Observability Duality

Example:

The pair $\left\{A^T , C^T\right\}$ is controllable **iff** the controllability grammian

$$W_c(n-1) \;=\; \sum_{k=0}^{n-1} (A^T)^k\, C^T C\, A^k \succ 0$$

However,

$$W_c(n-1) \;=\; W_o(n-1)$$

which is the observability grammian of the pair $\{A\,, C\}$

Proof of the observability theorem

Most of the results in observability theorem can be proven using the proof of the controllability theorem and utilizing duality:

The pair $\{A, C\}$ is observable **<u>iff</u>**

the pair $\left\{A^T, C^T\right\}$ is controllable.

Proof of Observability Theorem

We will prove: (b) implies (a):

Assume that the observability grammian is positive definite for ***n-1***

$$W_o(n-1) \;=\; \sum_{k=0}^{n-1}(A^T)^k C^T C A^k = \left\{Q^T Q\right\} \succ 0$$

We will show that

$$\begin{aligned} x(k+1) &= A\,x(k) \qquad\qquad x(0) = x_0 \\ y(k) &= C\,x(k) \end{aligned}$$

is observable

Proof of Observability Theorem

We will prove: (b) implies (a):

Assume that the observability grammian is positive definite for ***n-1***

$$W_o(n-1) \;=\; Q^T Q > 0$$

We will show that we can determine $x(0) = x_0$ from

$$Y_{n-1} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} \leftarrow$$ ***Collection of output measurements***

(b) implies (a)

Notice that, since

$$x(k+1) = A\,x(k) \quad \Rightarrow \quad x(k) = A^k\,x_o$$

$$y(k) = C\,x(k) \quad \Rightarrow \quad y(k) = C\,A^k\,x_o$$

Then

$$Y_{n-1} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} Cx(0) \\ Cx(1) \\ \vdots \\ Cx(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_o$$

(b) implies (a):

$$Y_{n-1} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q} x_o$$

$$Y_{n-1} = Q\, x_o$$

Multiplying this equation on the left by Q^T

$$Q^T Y_{n-1} = \underbrace{Q^T Q}_{W_o(n-1)\ \succ\ 0} x_o$$

(by assumption)

(b) implies (a):

$$Q^T Y_{n-1} = Q^T Q\, x_o$$

We can determine $\boldsymbol{x_o}$ **uniquely** from

$$x_o = \left\{Q^T Q\right\}^{-1} Q^T Y_{n-1}$$

Q.E.D

Note that the observability matrix $\boldsymbol{Q}$ may not be square.

Proof of Observability Theorem

We will now prove: (a) implies (c):

The pair $\{A, C\}$ is observable

$$\Rightarrow \quad Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ has rank $\boldsymbol{m} < \boldsymbol{n}$

By proving that **NOT (a) implies NOT (c):**

$$\text{rank}(Q) = m < n \quad \Rightarrow \quad \{A, C\}$$

is **<u>not</u>** observable

Not (c) implies Not (a):

Assume that

$$\mathsf{rank}(Q) = \mathsf{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = m < n$$

⇨ the null space of $\boldsymbol{Q}$ contains $\boldsymbol{n} - \boldsymbol{m}$ independent vectors

$$\mathcal{N}(Q) = \{v \in \mathcal{R}^n \ : \ Q\,v = 0\}$$

Not (c) implies Not (a):

Given an initial condition $x(0) = x_0$ and a set of output measurements:

$$Y_{n-1} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} Cx(0) \\ Cx(1) \\ \vdots \\ Cx(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_o$$

$$Y_{n-1} = Q\, x_o$$

Not (c) implies Not (a):

Given an initial condition $x(0) = x_0$ and a set of output measurements:

$$Y_{n-1} = Q\, x_o$$

However,

$$Y_{n-1} = Q\,(x_o + v)$$

For any vector $\boldsymbol{v}$ in the null space of $\boldsymbol{Q}$

$$\mathcal{N}(Q) = \{v \in \mathcal{R}^n \;:\; Q\,v = 0\}$$

Not (c) implies Not (a):

Given an initial condition $x(0) = x_0$ and a set of output measurements:

$$Y_{n-1} = Q\, x_o$$

However,

$$Y_{n-1} = Q\, \underbrace{(x_o + v)}$$

Another possible initial condition $\longrightarrow$ $\bar{x}_0 \neq x_0$

The initial state cannot be determined **<u>uniquely</u>** from $\boldsymbol{n}$ output observations

Not (c) implies Not (a):

What happens if we add an additional output measurement?

$$Y_n = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \\ y(n) \end{bmatrix} = \begin{bmatrix} Cx(0) \\ Cx(1) \\ \vdots \\ Cx(n-1) \\ Cx(n) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \\ CA^n \end{bmatrix} x_o$$

$$Y_n = \begin{bmatrix} Q \\ CA^n \end{bmatrix} x_o$$

Not (c) implies Not (a):

The Cayley-Hamilton corollary states that:

$$\operatorname{rank}\left(\begin{bmatrix} Q \\ CA^n \end{bmatrix}\right) = \operatorname{rank}(Q) = m < n$$

Therefore

$$\mathcal{N}\begin{bmatrix} Q \\ CA^n \end{bmatrix} = \mathcal{N}(Q)$$

$$\Rightarrow \qquad Y_n = \begin{bmatrix} Q \\ A^n C \end{bmatrix} x_o = \begin{bmatrix} Q \\ A^n C \end{bmatrix} (x_o + v)$$

$$\forall v \in \mathcal{N}(Q)$$

Not (c) implies Not (a):

Adding $y(n)$ to the measurement set does not help: i.e.

The initial state cannot be uniquely determined from Y_n

For the same reason, adding y_{n+l} $(l > 0)$ will not help to eliminate the null space of $\boldsymbol{Q}$.

$\Rightarrow$ The system is not observable.

Q.E.D

Remarks on Observability Theorem

1. If a discrete time LTI system of order n is observable, the initial state can be determined after observing n output sequences.

2. The conditions in the theorem only give a "yes" or "no" answer to the question of observability.

No statement is provided regarding the "degree of observability".

Remarks on Observability Theorem

3. The observable canonical pair

$$A_o = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_o & 0 & 0 \end{bmatrix} \quad C_o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

is always observable, since

$$Q_o = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ (-a_1 + a_2^2) & -a_2 & 1 \end{bmatrix}$$

is always full rank.

This result generalizes to an arbitrary order n

Observability Grammian

Assume that the matrix $\boldsymbol{A}$ is Schur.

Then, the asymptotic value of the controllability grammian

$$W_o = \lim_{k_1 \to \infty} W_o(k_1) = \sum_{k=0}^{\infty} (A^k)^T C^T C A^k$$

exists (all elements of $\boldsymbol{W_o}$ are bounded).

Controllability Grammian & Lyapunov Eq

Assume that the matrix $\boldsymbol{A}$ is Schur.

$$W_o = \sum_{k=0}^{\infty} (A^k)^T C^T C A^k$$

It can be calculated as the solution of the following Lyapunov equation:

$$A^T W_o A - W_o = -C^T C$$

Moreover, $W_o \succ 0$ **iff** $\{A\, C\}$ is an observable pair