HW3: Linear System Theory (ECE532)

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Due Date: April 2 at the begging of the class

Reading Assignment: Read the textbook on linear algebra (e.g. Linear Algebra and Its Applications by G. Strang).

Problem 1: The two vectors, x and y, in $\mathbb{R}^n$ are said to be *orthogonal* if $x^T y = 0$ (page 60 of the textbook). Given a k-dimensional subspace $\mathcal{S}$ in $\mathbb{R}^n$, we define its orthogonal set as

$$S^{\perp} := \{y \in \mathbb{R}^n : \ x^T y = 0, \text{for all } x \in \mathcal{S}\}.$$

- Show that $S^{\perp}$ is a subspace
- Prove that if $v_1, ..., v_k$ is a basis for $\mathcal{S}$, and $v_{k+1}, ..., v_p$ is a basis for $\mathcal{S}^{\perp}$, then the vectors $v_1, ..., v_p$ are linearly independent
- Prove that $\dim(\mathcal{S}^{\perp}) = n - k$ and thus that $\mathcal{S} + \mathcal{S}^{\perp} = \mathbb{R}^n$.
- Given a real $n \times m$ matrix A, show that

  $$R(A^T) = (N(A))^{\perp}.$$

  Namely, the range space of A^T is equal to the orthogonal set of the null space $N(A)$.

Problem 2: Given a matrix

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- Determine its eigenvalues and a set of normalized eigenvectors or normalized generalized eigenvectors (see page 60 of the textbook for the definition of the normalized eigenvectors).
- If B is a matrix representation of the linear operator $\mathcal{B} : \mathbb{R}^3 \to \mathbb{R}^3$ (with respect to the natural basis), determine the null space and the range space of $\mathcal{B}$.
- Can you diagonalize B? Justify your answer
- Is B *similar* to the matrix

  $$B' = \begin{pmatrix} 2 & \alpha & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

  where $\alpha \neq 0$ is a given real number? If your answer is the affirmative, find the corresponding *similarity transformation*.

Problem 3: Prove that $e^{(P+M)t} = e^{Pt}e^{Mt}$ if $PM = MP$ (P and M commute). Assuming that $PM = MP$, use the following steps:

- Show that e^{Pt} and M commute;
- by taking derivatives directly, show that the time derivative of $Q(t) = e^{Pt}e^{Mt}$ satisfies

$$\dot{Q}(t) = (P + M)e^{Pt}e^{Mt};$$

- since $Q(t)$ satisfies $\dot{Q}(t) = (P + M)Q(t)$ with $Q(0) = I$, show that $Q(t) = e^{(P+M)t}$.

Problem 4: You are given a LTV system

$$\dot{x}(t) = A(t)x(t),\ x(0) = (1, 1)^T,\ A(t) = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix}$$

- Obtain a closed-form expression for the state transition matrix, $\Phi(t, s)$, corresponding to A
- Obtain an expression for the solution of the differential equation above for all $t \geq 0$
- Obtain the eigenvalues of matrix $A(t)$ for all $t \geq 0$. Based on this information, would you expect the system trajectories to be bounded for all $t \geq 0$
- Now, use the solution obtained in (ii) to deduce whether the system states remain bounded as $t \to \infty$. Any surprises?

Problem 5:

Let V be the set of all $n \times n$ matrices with real entires with the *matrix addition* and *scalar multplication* given by component-wise addition and multiplication. Prove that $(V, \mathbb{R})$ is a vector space.

Problem 6:

Suppose W is a subspace of a finite-dimensional vector space V. For some $v \in V$ and $v \notin W$, set $X = \text{span}(W \cup \{v\})$. Prove that $\dim(X) = \dim(W) + 1$.