[MEN573]
Advanced Control Systems I

Lecture 19 - Optimal Linear Quadratic Regulators (LQR) Continuous Time

Associate Professor Joonbum Bae
Department of Mechanical Engineering
UNIST

LQ optimal control problem

Consider an nth order LTI CT system:

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad\qquad x(0) = x_o$$

- Assume that the state vector is measurable.

- We are concerned with regulating the state around the origin,

i.e. bringing the state to the origin from an arbitrary $x(0) = x_o$

LQ optimal control problem

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t) \qquad\qquad x(0) = x_o$$

We want to obtain the optimal control, which minimizes:

$$J = \frac{1}{2}\,x^T(t_f)\,S\,x(t_f) + \frac{1}{2}\int_0^{t_f} \left\{ x^T\,Q\,x + u^T\,R\,u \right\}\,dt$$

$$S = S^T \succeq 0 \qquad\qquad Q = Q^T \succeq 0 \qquad\qquad R = R^T \succ 0$$

LQ optimal control problem

$$J = \frac{1}{2} x^T(t_f)\, S\, x(t_f) + \frac{1}{2}\int_0^{t_f} \left\{ x^T\, Q\, x + u^T\, R\, u \right\} dt$$

- t_f is the final time
- $\frac{1}{2} x^T(t_f)\, S\, x(t_f)$ penalizes the final state deviation from the origin
- $\frac{1}{2} x^T(t)\, Q\, x(t)$ penalizes the transient state deviation from the origin
- $\frac{1}{2} u^T(t)\, R\, u(t)$ penalizes the control effort

LQ optimal control problem

$$J = \frac{1}{2} x^T(t_f)\, S\, x(t_f) + \frac{1}{2} \int_0^{t_f} \left\{ x^T\, Q\, x + u^T\, R\, u \right\}\, dt$$

Notes:

- $\boldsymbol{S}$ and $\boldsymbol{Q}$ must be at least positive semi-definite
- $\boldsymbol{R}$ must be positive definite.
  - Without a weighting on $\boldsymbol{u(t)}$, the optimal control action would yield unbounded control magnitudes.
- All matrices (including $\boldsymbol{A}$ and $\boldsymbol{B}$) can be time varying.

LQ optimal control

$$u(t) = -K(t)\,x(t)$$

where:

$$K(t) = R^{-1}\,B^T\,P(t)$$

and $\boldsymbol{P}(t)$ satisfies the following ***Riccati*** ODE (or ***Riccati*** eq.):

$$-\frac{d}{dt}P = A^T\,P + P\,A + Q - P\,B\,R^{-1}\,B^T P$$

$$P(t_f) = S$$

LQ optimal control

$$u(t) \;=\; -K(t)\,x(t)$$

where:

$$K(t) \;=\; R^{-1}\,B^T\,P(t)$$

and $\boldsymbol{P}(t)$ satisfies the following ***Riccati*** ODE (or ***Riccati*** eq.):

$$-\frac{d}{dt}P \;=\; \underbrace{A^T\,P + P\,A + Q}_{\textit{Lyapunov Eq.}} - \underbrace{P\,B\,R^{-1}\,B^T P}_{\textit{additional term}}$$

LQ optimal control

$$-\frac{d}{dt}P = A^T P + P A + Q - P B R^{-1} B^T P$$

$$P(t_f) = S$$

- The ***Riccati*** ODE is integrated **backwards,**
  - – starting from the final time t_f , until $t = 0$

$$P(t) = P^T(t) \qquad P(t) \succeq 0$$

LQ optimal control

$$J = \frac{1}{2} x^T(t_f)\, S\, x(t_f) + \frac{1}{2} \int_0^{t_f} \left\{ x^T\, Q\, x + u^T\, R\, u \right\}\, dt$$

The minimum value of the quadratic cost function:

$$J^0(x(0)) = \frac{1}{2} x^T(0)\, P(0)\, x(0)$$

Integration of the Riccati ODE

The **backwards** integration of:

$$-\frac{d}{dt}P \;=\; A^T P + P\,A + Q - P\,B\,R^{-1}\,B^T P$$

$$P(t_f) \;=\; S$$

is numerically equivalent to the **forward** integration of:

$$\frac{d}{dt}P^* \;=\; A^T P^* + P^*\,A + Q - P^*\,B\,R^{-1}\,B^T P^*$$

$$P^*(0) \;=\; S$$

$$\boxed{P(t) = P^*(t_f - t)}$$

Numerical integration of the Riccati ODE

$$\frac{d}{dt}P^* \;=\; A^T P^* + P^* A + Q - P^* B R^{-1} B^T P^*$$

$$P^*(0) \;=\; S$$

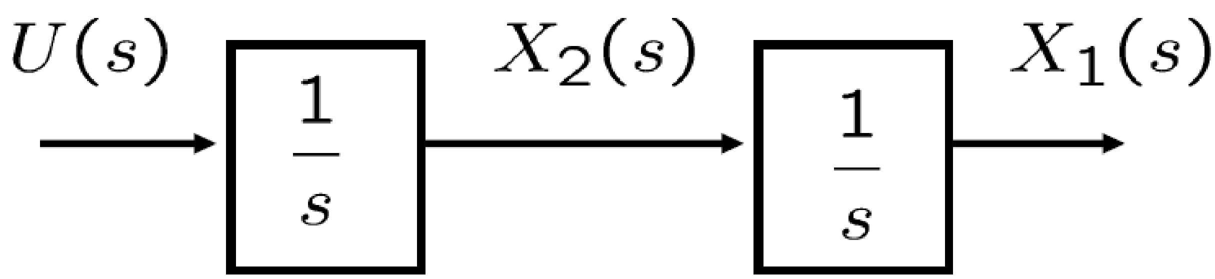
This ODE can be integrated using any of matlab's numerical integration functions (e.g. ODE45).

$$P(t) = P^*(t_f - t)$$

LQ optimal control example

Double integrator

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



LQ optimal control example

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2} x^T(t_f)\, S\, x(t_f) + \frac{1}{2}\int_0^{t_f} \left\{ x^T Q\, x + R\, u^2 \right\} dt$$

with

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R \succ 0$$

Numerical integration

$$\frac{d}{dt}P^* \;=\; A^T P^* + P^* A + Q - P^* B R^{-1} B^T P^*$$

$$P(t) = P^*(t_f - t)$$

with

$$P^*(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R \succ 0$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Numerical integration

$$\frac{d}{dt}P^* \;=\; A^T P^* + P^* A + Q - P^* B R^{-1} B^T P^*$$

$$P(t) = P^*(t_f - t)$$

Define:

$$P^* = \begin{bmatrix} p_{11}^* & p_{12}^* \\ p_{12}^* & p_{22}^* \end{bmatrix}$$

LQ optimal control example

Plugging the matrix parameters into the RHS of the ODE

$$\frac{d}{dt}P^* \;=\; A^T P^* + P^* A + Q - P^* B R^{-1} B^T P^*$$

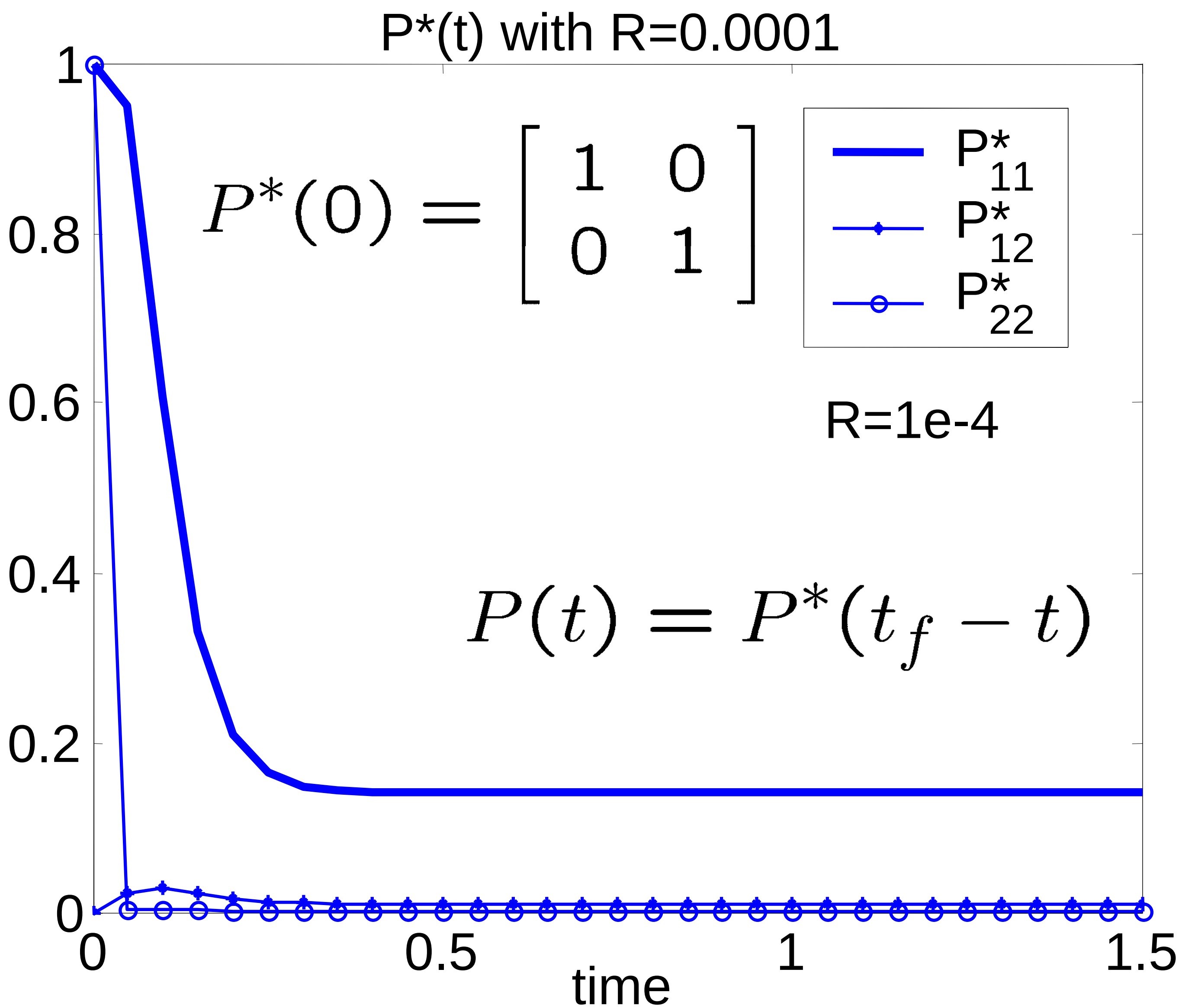
yields:

$$\frac{d}{dt}p^*_{11} \;=\; 1 - \frac{1}{R}{p^*_{12}}^2 \qquad\qquad p^*_{11}(0) = 1$$

$$\frac{d}{dt}p^*_{12} \;=\; p^*_{11} - \frac{1}{R}p^*_{12}\,p^*_{22} \qquad\qquad p^*_{12}(0) = 0$$

$$\frac{d}{dt}p^*_{22} \;=\; 2\,p^*_{12} - \frac{1}{R}{p^*_{22}}^2 \qquad\qquad p^*_{22}(0) = 1$$

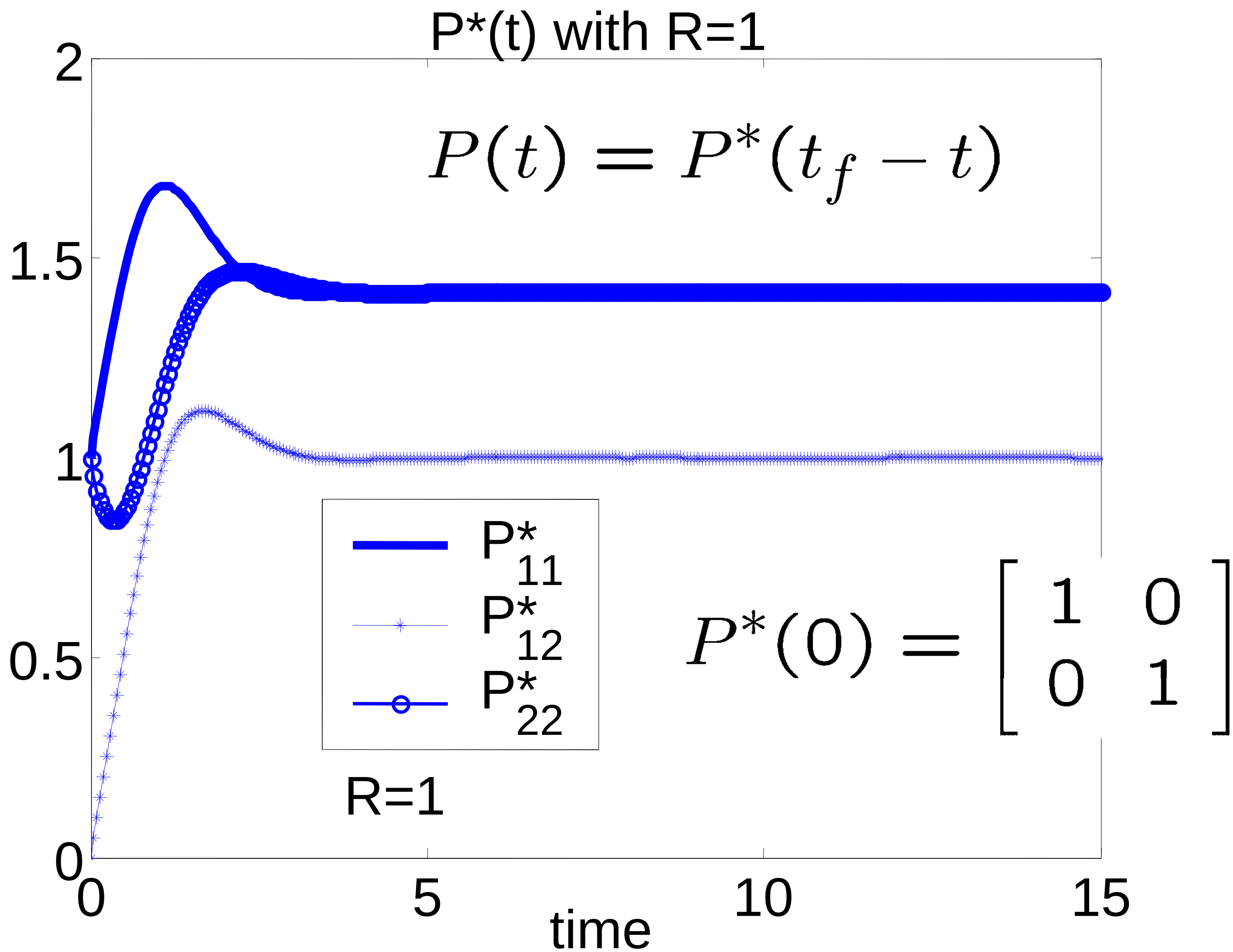
LQ optimal control example



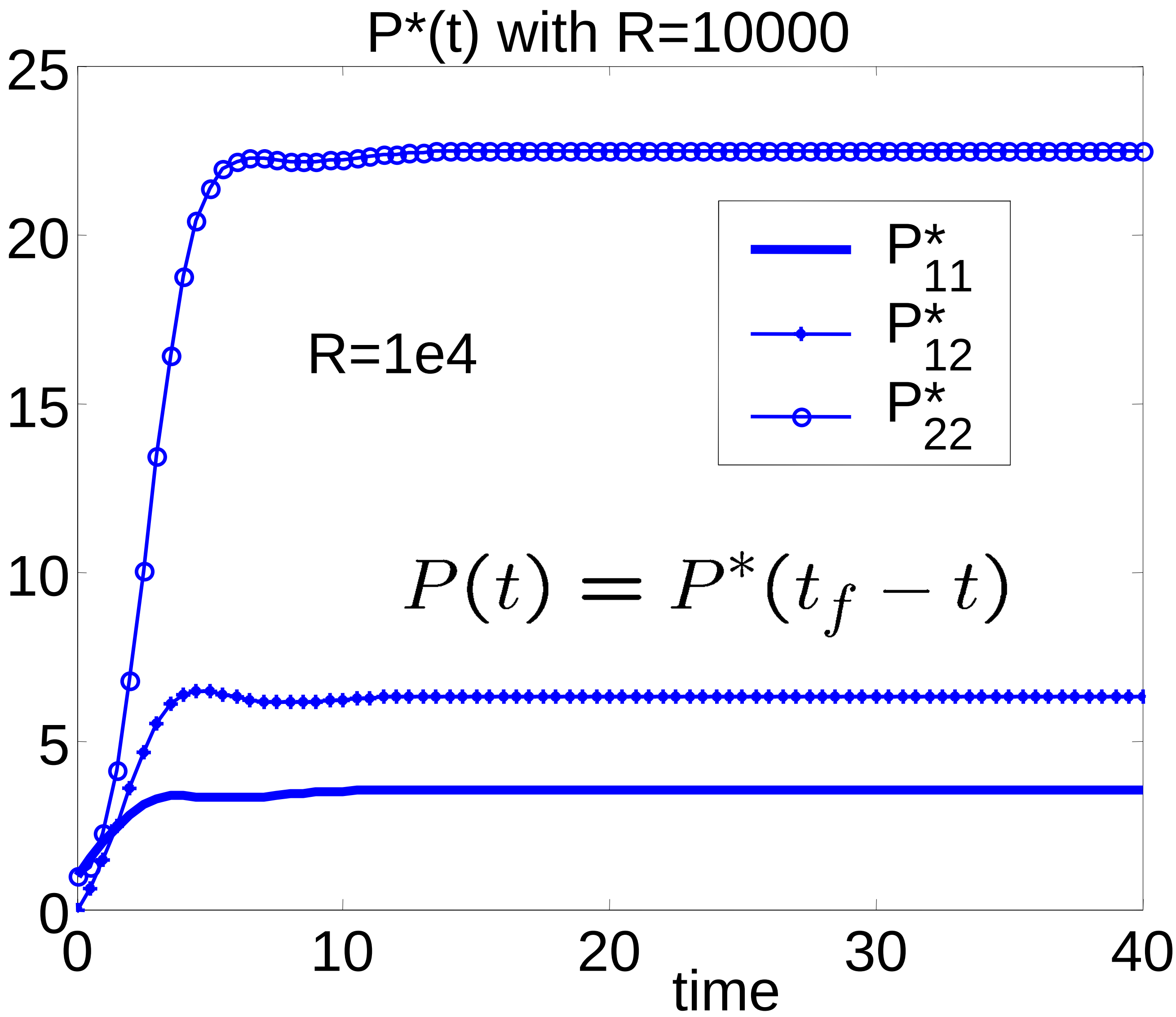
$$P^*(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P(t) = P^*(t_f - t)$$

LQ optimal control example

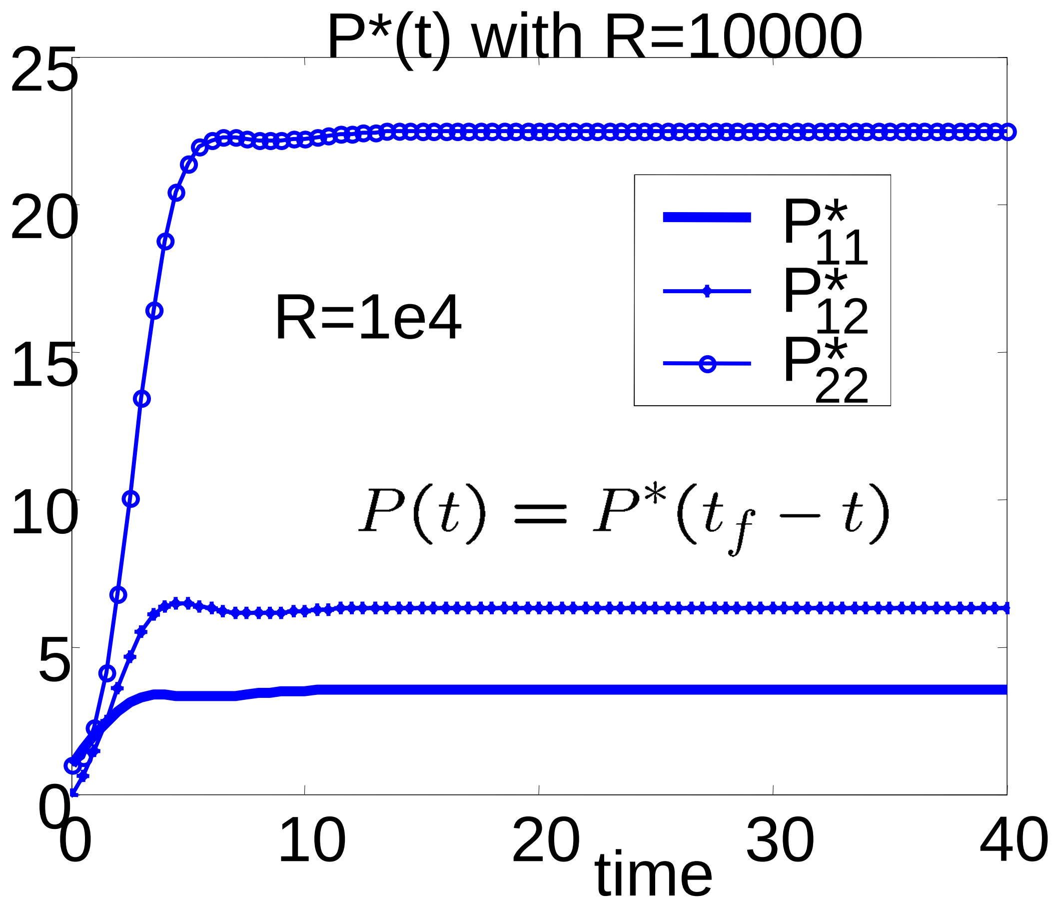


LQ optimal control example



LQ optimal control example

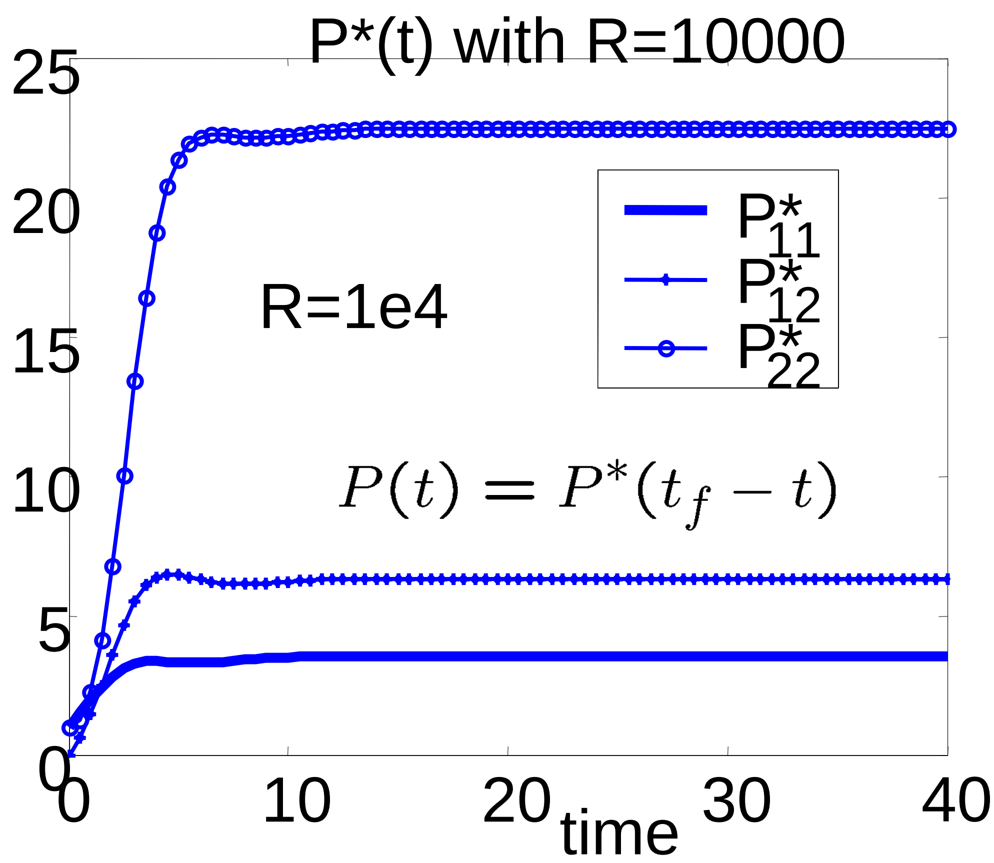
1. If the final time t_f is relatively large, $P(t)$ will be stationary from $t = 0$ until t approaches t_f



LQ optimal control example

As $t_f \to \infty, P(t)$ becomes stationary

$$P(t) \to P_+$$

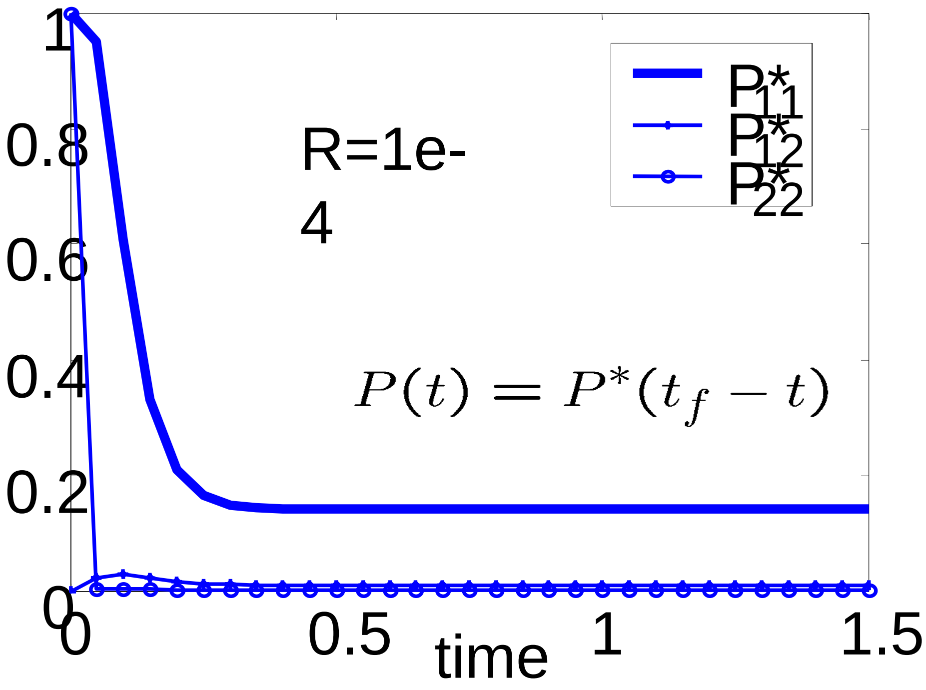


LQ optimal control example

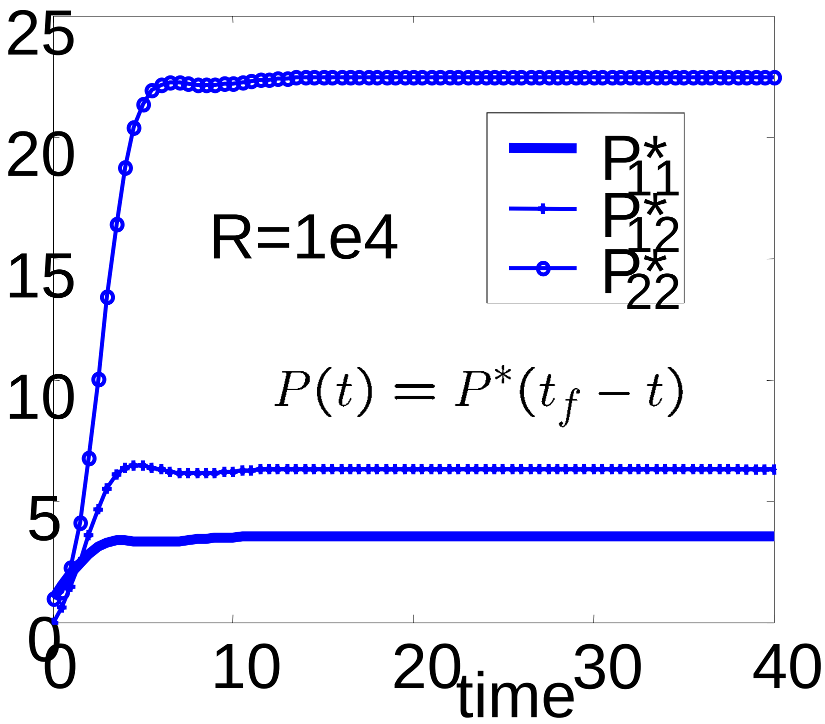
A large input weighting, $\boldsymbol{R}$ results in a longer transient.

A small input weighting, $\boldsymbol{R}$ results in a shorter transient.

P*(t) with R=0.0001

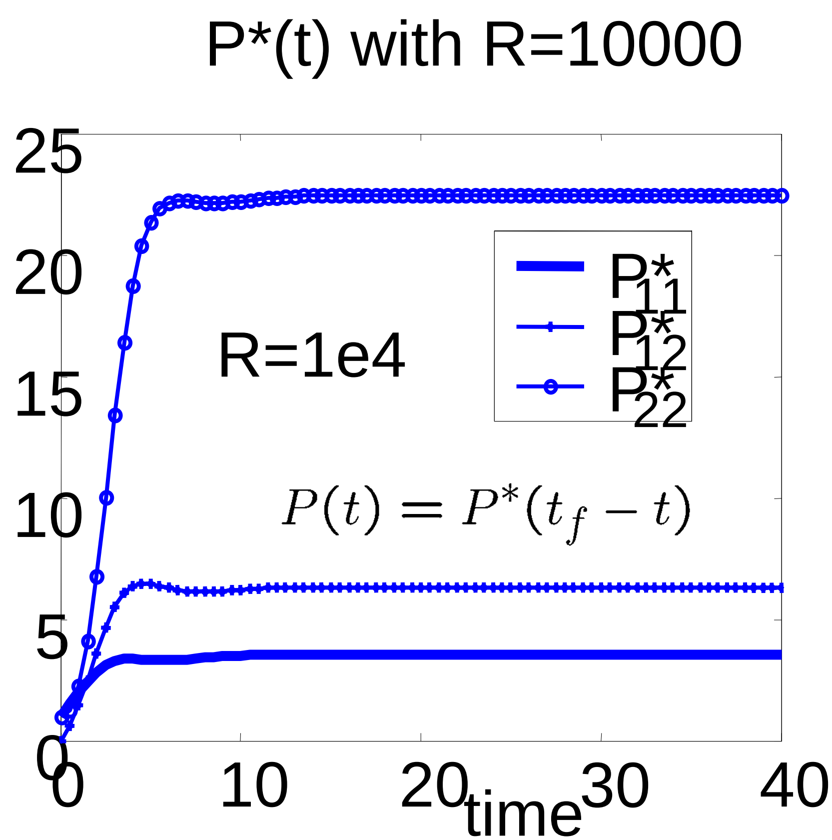


P*(t) with R=10000



LQ optimal control example

$P(t_f) \ = \ S$ becomes irrelevant as $t_f \rightarrow \infty$



LQ optimal control example

1. If the final time t_f is relatively large, $P(t)$ will be stationary from $t = 0$ until t approaches t_f

2. As $t_f \to \infty$, P becomes stationary $P(t) \to P_+$

3. A large input weighting, $\boldsymbol{R}$ results in a longer transient.

4. A small input weighting, $\boldsymbol{R}$ results in a shorter transient.

5. $P(t_f) = S$ becomes irrelevant as $t_f \to \infty$

Infinity horizon LQ optimal regulator

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t) \qquad\qquad x(0) = x_o$$

We want to obtain the optimal control, which minimizes:

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T C^T C\, x + u^T R\, u \right\} dt$$

$$Q = C^T C \succeq 0 \qquad\qquad R = D^T D \succ 0$$

Infinity horizon LQ optimal regulator

Conditions for the existence of a LQR solution:

1. The pair $\{A, B\}$ must be controllable or stabilizable.

- This condition is necessary for $\boldsymbol{J}$ to be bounded and for $\boldsymbol{P}(t)$ to converge to a stationary value, P, when integrated backwards from any arbitrary $P(t_f)$

Infinity horizon LQ optimal regulator

2. The pair $\{A, C\}$ must be observable or detectable.

- This condition assures that the optimal closed loop system is asymptotically stable.

- If $\{A, C\}$ is observable, then $P \succ 0$

- If $\{A, C\}$ is detectable, but not observable, then

  $P \succeq 0$

Infinity horizon LQ optimal regulator

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t) \qquad\qquad x(0) = x_o$$

We want to obtain the optimal control, which minimizes:

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T\, C^T\, C\, x + u^T\, R\, u \right\}\, dt$$

$$Q = C^T\, C \succeq 0 \qquad\qquad R = R^T \succ 0$$

LQ optimal regulator

$$u(t) \;=\; -K\,x(t)$$

where:

$$K \;=\; R^{-1}\,B^T\,P$$

and $\boldsymbol{P}$ satisfies the following ***Algebraic Riccati Equation*** (ARE) :

$$0 \;=\; A^T P + P A + C^T C - P B R^{-1} B^T P$$

LQ optimal regulator

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T\, C^T\, C\, x + u^T\, R\, u \right\}\, dt$$

Optimal Cost:

$$J^o = \frac{1}{2}\, x(0)^T\, P\, x(0)$$

$$0 \;=\; A^T\, P + P\, A + C^T\, C - P\, B\, R^{-1}\, B^T P$$

LQ optimal regulator

Theorem

If $\{A, B\}$ is controllable and $\{A, C\}$ observable

$$A_c = A - B\,K \qquad\qquad K \; = \; R^{-1}\,B^T\,P$$

is Hurwitz, and satisfies the following Lyapunov equation

$$A_c^T\,P + P\,A_c = -C^T\,C - P\,B\,R^{-1}\,B^T\,P$$

Solution of the ARE

There exists a technique for obtaining the solution $\boldsymbol{P}$ of the Algebraic Riccati Equation (ARE)

$$0 \;=\; A^T P + P A + C^T C - P B R^{-1} B^T P$$

in close form

Solution of the ARE

$$0 \;=\; A^T P + P A + C^T C - P B R^{-1} B^T P$$

1) Define the Hamiltonian matrix:

$$H = \begin{bmatrix} A & -B R^{-1} B^T \\ -C^T C & -A^T \end{bmatrix} \in \mathcal{R}^{2n \times 2n}$$

2) Compute its first ***n*** eigenvalues ($\mathrm{Re}(\lambda_i) < 0$):

$$\{\lambda_1, \lambda_2, \cdots, \lambda_n \,|\, \lambda_{n+1}, \cdots, \lambda_{2n}\}$$

Solution of the ARE

- The first n eigenvalues of $\boldsymbol{H}$ are the eigenvalues of

$$A_c = A - B\,K \qquad \text{where} \qquad K \;=\; R^{-1}\,B^T\,P$$

and are all in the open left complex plane, $\mathrm{Re}(\lambda_i) < 0$
(i.e. asymptotically stable)

- The remaining eigenvalues of $\boldsymbol{H}$ satisfy:

$$\lambda_{n+i} = -\lambda_i \qquad\qquad i = 1,\,\cdots\,,\,n$$

Solution of the ARE

3) For each eigenvalue of ***H*** in the open left complex plane, compute its associated eigenvector :

$$\mathrm{Re}(\lambda_i) < 0 \qquad\qquad i = 1, \cdots, n$$

$$H \underbrace{\begin{bmatrix} f_i \\ g_i \end{bmatrix}}_{v_i} = \lambda_i \underbrace{\begin{bmatrix} f_i \\ g_i \end{bmatrix}}_{v_i}$$

$$f_i,\ g_i \in \mathcal{R}^n$$

Solution of the ARE

4) Define the following two $n \times n$ matrices:

$$X_1 = \left[\begin{array}{cccc} f_1 & f_2 & \cdots & f_n \end{array} \right]$$

$$X_2 = \left[\begin{array}{cccc} g_1 & g_2 & \cdots & g_n \end{array} \right]$$

5) Finally, $\boldsymbol{P}$ is computed as follows:

$$\boxed{P = X_2 X_1^{-1}}$$

Solution of the ARE using matlab

- Matlab command **`care`**: (Continuous ARE)

$$\boxed{[\,P,\,\Lambda,\,K\,] = \texttt{care}(A,\;B,\;C^T\,C,\;R)}$$

Solution $\boldsymbol{P}$ of $\;0 \;=\; A^T\,P + P\,A + C^T\,C - P\,B\,R^{-1}\,B^T P$

$$\Lambda = \mathrm{Diag}(\lambda_1,\,\cdots\,,\,\lambda_n) \qquad \mathrm{Re}(\lambda_i) < 0$$

eigenvalues of $\;A_c = A - B\,K$

$$K \;=\; R^{-1}\,B^T\,P$$

LQ optimal regulator

Other matlab commands:

- **lqr** : $$\boxed{[\,K,\ P,\ \Lambda\,] = \mathtt{lqr}(A,\ B,\ C^T C,\ R)}$$

- **lqry** : $$\boxed{[\,K,\ P,\ \Lambda\,] = \mathtt{lqry}(\mathtt{sys},\ Q_y,\ R)}$$

sys:

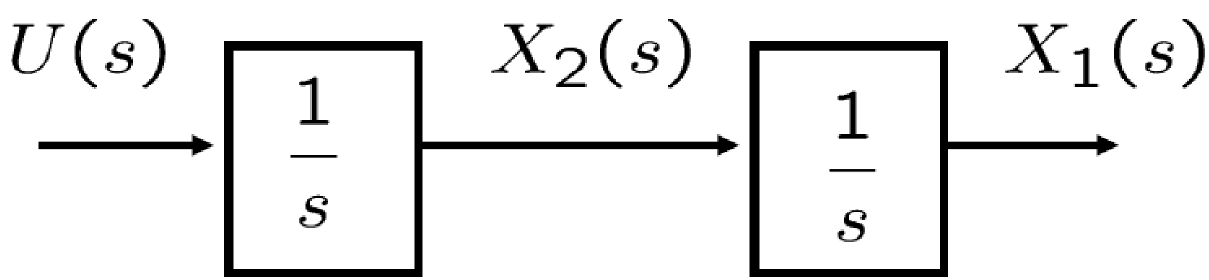
$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u \end{aligned}$$

$$J = \frac{1}{2}\int_0^\infty \left\{ y^T\,Q_y\,y + u^T\,R\,u \right\}\,dt$$

LQ optimal control example

Double integrator

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



LQR – double integrator

Double integrator

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$J = \frac{1}{2}\int_0^\infty \left\{ x^T C^T C\, x + R\, u^2 \right\} dt$$

with

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad\qquad R > 0$$

LQR – double integrator

Solving the ARE analytically

$$0 = A^T P + P A + C^T C - P B R^{-1} B^T P$$

yields:

$$P = \begin{bmatrix} \sqrt{2}\, R^{1/4} & R^{1/2} \\ R^{1/2} & \sqrt{2}\, R^{3/4} \end{bmatrix}$$

$$A_c = A - B K = \begin{bmatrix} 0 & 1 \\ -R^{-1/2} & -\sqrt{2} R^{-1/4} \end{bmatrix}$$

Eigenvalues:

$$\lambda_{1,2} = -\frac{1}{\sqrt{2}\, R^{1/4}} \pm \frac{1}{\sqrt{2}\, R^{1/4}} j$$

LQR – double integrator

Close loop eigenvalues a a function of $\boldsymbol{R}$

$$\lambda_{1,2} = -\frac{1}{\sqrt{2}\,R^{1/4}} \pm \frac{1}{\sqrt{2}\,R^{1/4}}\,j$$