**UNIST**
**Department of Mechanical Engineering**

| **MEN 573: Advanced Control Systems I** | **Spring, 2016** |
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| **Homework #10** | Assigned: Sunday, June 5, 2016 |
| **Solution** | You don't have to submit this HW. The solution will be uploaded at June 12, 2016. |

**Problem 1.**

(a) combining terms

$$J = \int_0^\infty \left\{ x_1^2(t) + x_2^2(t) + s x_1^2(t) + 2s\, x_1(t)\, x_2(t) + s x_2^2(t) + \rho\left(u_1^2(t) + u_2^2(t)\right) \right\} dt$$

$$= \int_0^\infty \left\{ \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} 1+s & -s \\ -s & 1+s \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right\} dt$$

for $u(t) = -Kx(t)$ $\quad K = R^{-1}B^T P$ where $P$ solves the ARE:

$$0 = A^T P + PA + C^T C - PBR^{-1}B^T P$$

there are several ways to solve for $K$:

1. >> $[K, P, E] = \mathrm{lqr}(sys, Q, R)$
2. >> $[P, E, K] = \mathrm{care}(A, B, Q, R)$
3. using the Hamilton matrix : $P = X_2 X_1^{-1}$
4. solving analytically by taking advantage of the form of $Q$ and $R$.

for $s=0$: $\quad K = \begin{bmatrix} 0.618 & 0 \\ 0 & 0.618 \end{bmatrix} \Rightarrow A-BK = \begin{bmatrix} -111.8034 & 0 \\ 0 & -111.8034 \end{bmatrix}$

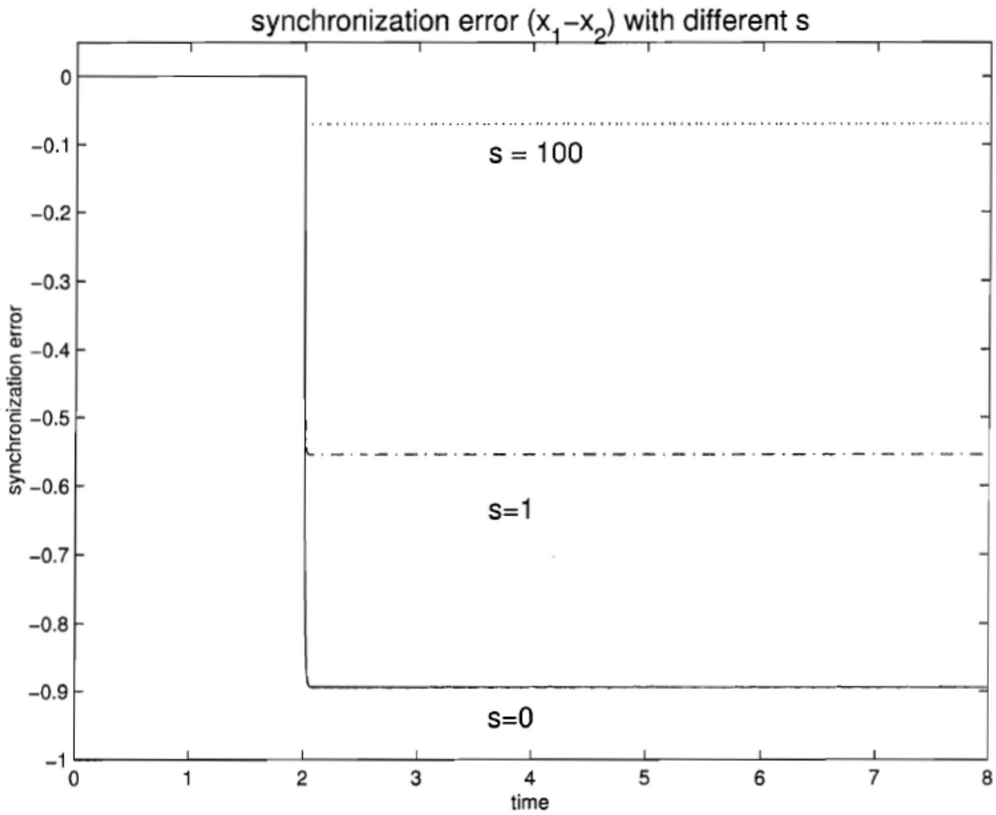
$s=1$: $\quad K = \begin{bmatrix} 0.9604 & -.3424 \\ -.3424 & .9604 \end{bmatrix} \Rightarrow A-BK = \begin{bmatrix} -146.04 & 34.237 \\ 34.237 & -146.04 \end{bmatrix}$

$s=100$ $\quad K = \begin{bmatrix} 7.15 & -6.53 \\ -6.53 & 7.15 \end{bmatrix} \Rightarrow A-BK = \begin{bmatrix} -765 & 653 \\ 653 & -765 \end{bmatrix}$

(b) we get, using the feedback law:

$$\dot{x} = (A-BK)x(t) + Bd(t)$$

$$y = x_1(t) - x_2(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$



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clc;
A=[-50 0;0 -50];
B=[100 0;0 100];
C=[1 -1];
s=100;
rho=1;
Q=[1+s -s;-s 1+s];
R=rho*eye(2);
% compute K analytically.  NOT a very good method. eigenvalues must be
% carefully chosen so that X1 is not singular.

H=[A -B*R^-1*B';-Q -A'];
[t,v]=eig(H);
X1=t([1:2],[1:2]);
X2=t([3:4],[1:2]);
P=X2*X1^-1;
K=R^-1*B'*P;

% really easy way
[P2,L,K2]=care(A,B,Q,R);
Ac=A-B*K2;

% (b) simulating with a disturbance
sys = ss(Ac,B,C,0);

%create time vectors
time = [1:0.001:8];
time1=[1:0.001:2];
time2=[2.001:.001:8];

% disturbance
d1=zeros(1,length(time));
d2=[zeros(1,length(time1)) ones(1,length(time2))];
d=[d1;d2];

y=lsim(sys,d,time);
```

**Problem 2.**

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

we use the control law $u(t) = -Kx(t)$ that minimizes

$$J = \int_0^\infty \{ y^2(t) + Ru^2(t) \} dt = \int_0^\infty \{ x_1^2(t) + Ru^2(t) \} dt$$

(a) $eig(A) = 0, 1$ (diagonal)

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{observable} \Rightarrow \text{detectable}$$

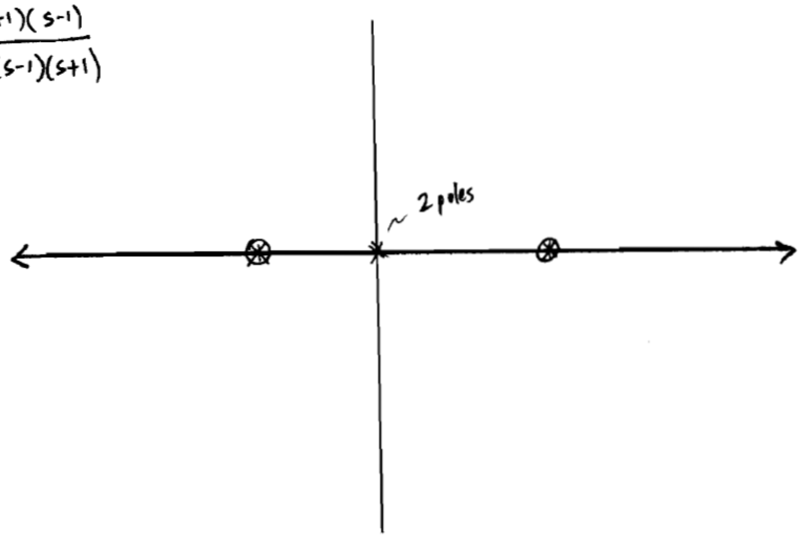
$$P = [B \ \ AB] = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow \text{controllable} \Rightarrow \text{stabalizable}$$

(b) $$G(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2 - s}$$

$A(s) = s(s-1)$ $\qquad$ $A(-s) = (-s)(-s-1) = s(s+1)$ $\qquad$ $\left.\begin{matrix} n = 2 \\ m = 1 \end{matrix}\right\}$ $\underline{odd} = n - m = 1$

$B(s) = s+1$ $\qquad$ $B(-s) = -s+1 = (-1)(s-1)$ $\qquad$ asymptotes: $\frac{\ell\pi}{1} = 0, \pi,$

$$\frac{B(s)B(-s)}{A(s)A(-s)} = \frac{(s+1)(s-1)}{s^2(s-1)(s+1)}$$



(c) if $R \to \infty$: $\quad \dfrac{A_c(s)\,A_c(-s)}{A(s)\,A(-s)} = 1 - \dfrac{1}{R}\,\dfrac{B(s)\,B(-s)}{A(s)\,A(-s)} = 1$

thus $A_c(s)$ is the stable O.L. poles: $\dfrac{1}{s(s+1)} = A_c(s)$

(d) if $R \to 0$ $\quad \lambda_{c_1} = -1 \,,\ \lambda_{c_2} = -\infty$

(e) if one Evalue is at $\lambda_{c_1} = -2$, the other must be at $\lambda_{c_2} = -1$ for pole zero cancellation.

then $$\frac{(s+2)\cancel{(s+1)}\cdot(s-2)\cancel{(s-1)}}{s\cancel{(s+1)}\cdot s\cancel{(s-1)}} = 1 - \frac{1}{R}\,\frac{\cancel{(s+1)}\cancel{(s-1)}}{s^2\cancel{(s-1)}\cancel{(s+1)}}$$

$$(s+2)(s-2) = s^2 - \frac{1}{R}$$

$$Rs^2 - 1 = R(s^2 - 4)$$

$$4R = 1$$

$$\underline{R = \frac{1}{4}}$$

find $K$ by pole-placement

$$A_c(s) = (s+2)(s+1) = s^2 + 3s + 2$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \\ -k_1 & 1-2k_2 \end{bmatrix}$$

$$\text{eig}(A-BK) = (\lambda + k_1)(\lambda - 1 + 2k_2) - (k_1)(-1+k_2)$$

$$= \lambda^2 - \lambda + \lambda 2k_2 + 2k_1 - k_1 + 2k_1k_2 + k_1 - k_1k_2$$

$$= \lambda^2 + \lambda(-1+2k_2+k_1) + k_1k_2$$

equating $(A-BK) = A_c$

$$\left.\begin{aligned} k_1k_2 &= 2 \\ k_1 + 2k_2 &= 4 \end{aligned}\right\}$$

$$(4-2k_2)k_2 = 2$$

$$2k_2^2 - 4k_2 + 2 = 0$$

$$k_2^2 - 2k_2 + 1 = 0$$

$$k_2 = 1, 1 \Rightarrow k_1 = 2$$

$$\boxed{K = [2 \quad 1]}$$