[MEN573] Advanced Control Systems I

Lecture 18 - State Observers and Observer State Feedback Control

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State Feedback

In the previous lecture we considered a nth order LTI system:

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad x(0) = x_o$$

$$y(t) = C\,x(t) \qquad y \in \mathcal{R}^m$$

Under state feedback control:

$$u(t) = -K\,x(t) + F\,v(t)$$

Problem:

- State $x(t)$ is generally not directly measurable.
- Only the output $y(t)$ is measurable.

State Observation

In this lecture we will discuss how to ***estimate*** the state $x(t)$ using:

- the output $y(t)$
- the input $u(t)$
- the model of the plant $\left\{ \begin{array}{rcl} \dot{x}(t) & = & A\,x(t) + B\,u(t) \\ y(t) & = & C\,x(t) \end{array} \right.$

Using model-based **state observers**

We will denote the estimate of the state by: $\widehat{x}(t)$

Open Loop State Observers

One way to estimate the state is to feed the same input to a "model" of the system:

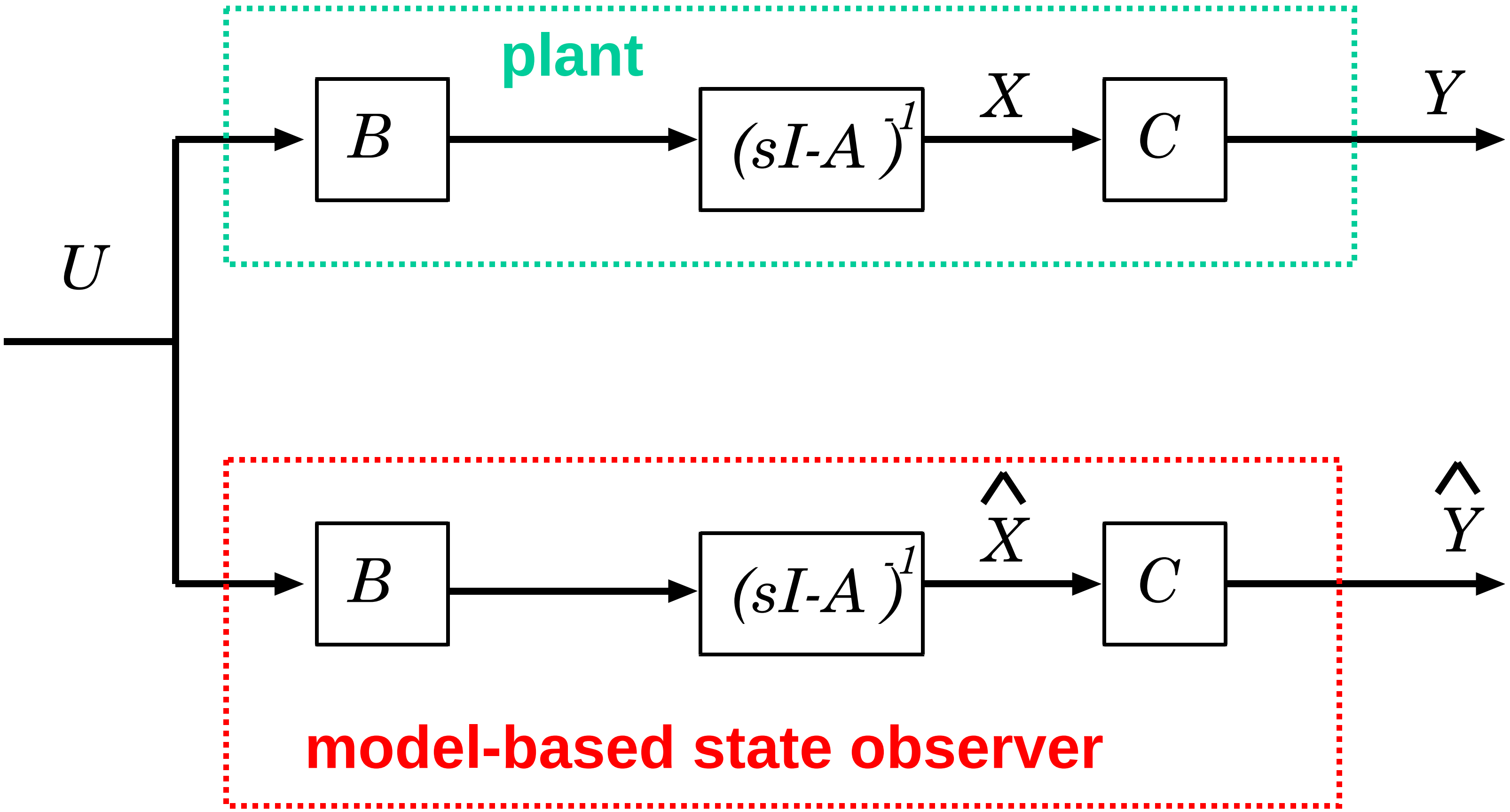
$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t) \qquad \hat{x}(0) = \hat{x}_o$$

$$\hat{y}(t) = C\,\hat{x}(t)$$

Where $\hat{x}(t)$ is the estimate of the true state $x(t)$

Open Loop State Observers

One way to estimate the state is to feed the same input to a "***model***" of the system:



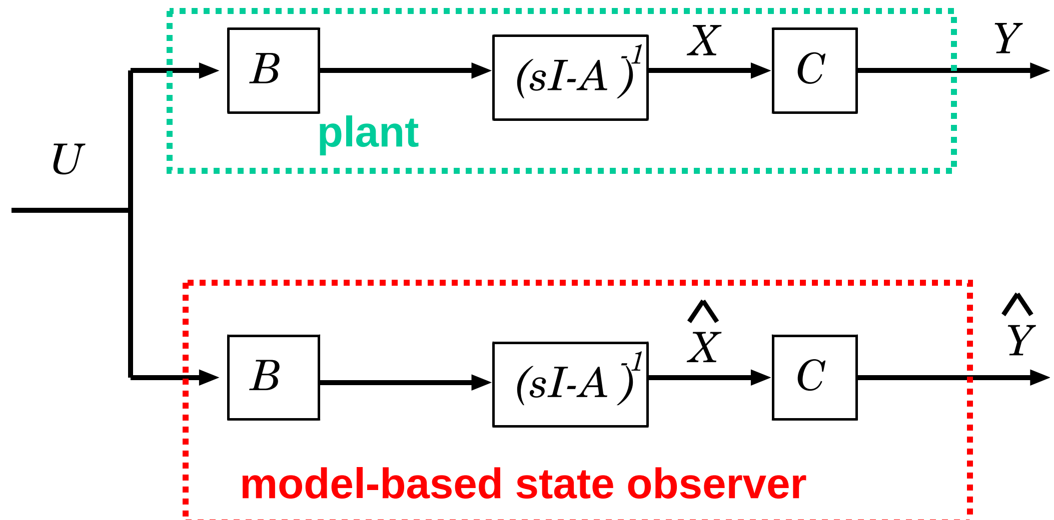
Open Loop State Observers

Problems:

- $x(0)$ is not known and $\hat{x}(0) \neq x(0)$

- $\hat{x}(t)$ may converge too slowly to $x(t)$

or not converge at all



Open Loop State Observers

Problems:

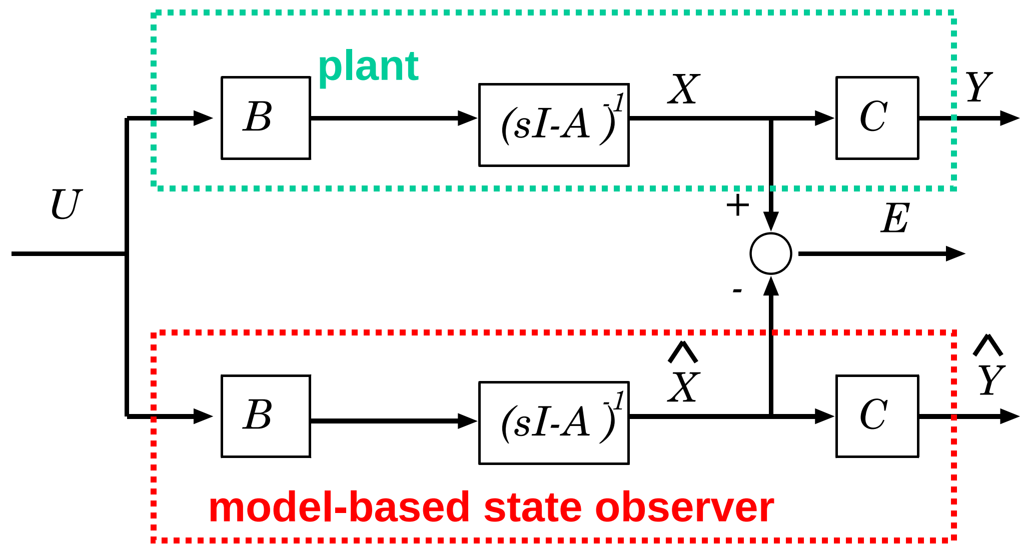
- $x(0)$ is not known and $\widehat{x}(0) \neq x(0)$

- $\widehat{x}(t)$ may converge too slowly to $x(t)$

or not converge at all

State estimation error:

$$e(t) = x(t) - \widehat{x}(t)$$



Open Loop State Estimation Dynamics

Subtract observer from actual state dynamics:

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad x(0) = x_o$$

$$-\dot{\hat{x}}(t) = -A\,\hat{x}(t) - B\,u(t) \qquad \hat{x}(0) = \hat{x}_o$$

$$\underbrace{\dot{x}(t) - \dot{\hat{x}}(t)}_{\dot{e}(t)} = A\underbrace{\left(x(t) - \hat{x}(t)\right)}_{e(t)} + B\underbrace{\left(u(t) - u(t)\right)}_{0}$$

$$\boxed{\dot{e}(t) = A\,e(t)}$$

$$\begin{aligned} e(0) &= e_o \\ &= x_o - \hat{x}_o \end{aligned}$$

Open Loop State Estimation Dynamics

Open loop state estimation error dynamics

$$\dot{e}(t) = A\,e(t) \qquad\qquad \begin{aligned} e(0) &= e_o \\ &= x_o - \widehat{x}_o \end{aligned}$$

The convergence of estimation error depends on the **eigenvalues** of the open-loop $\boldsymbol{A}$ matrix

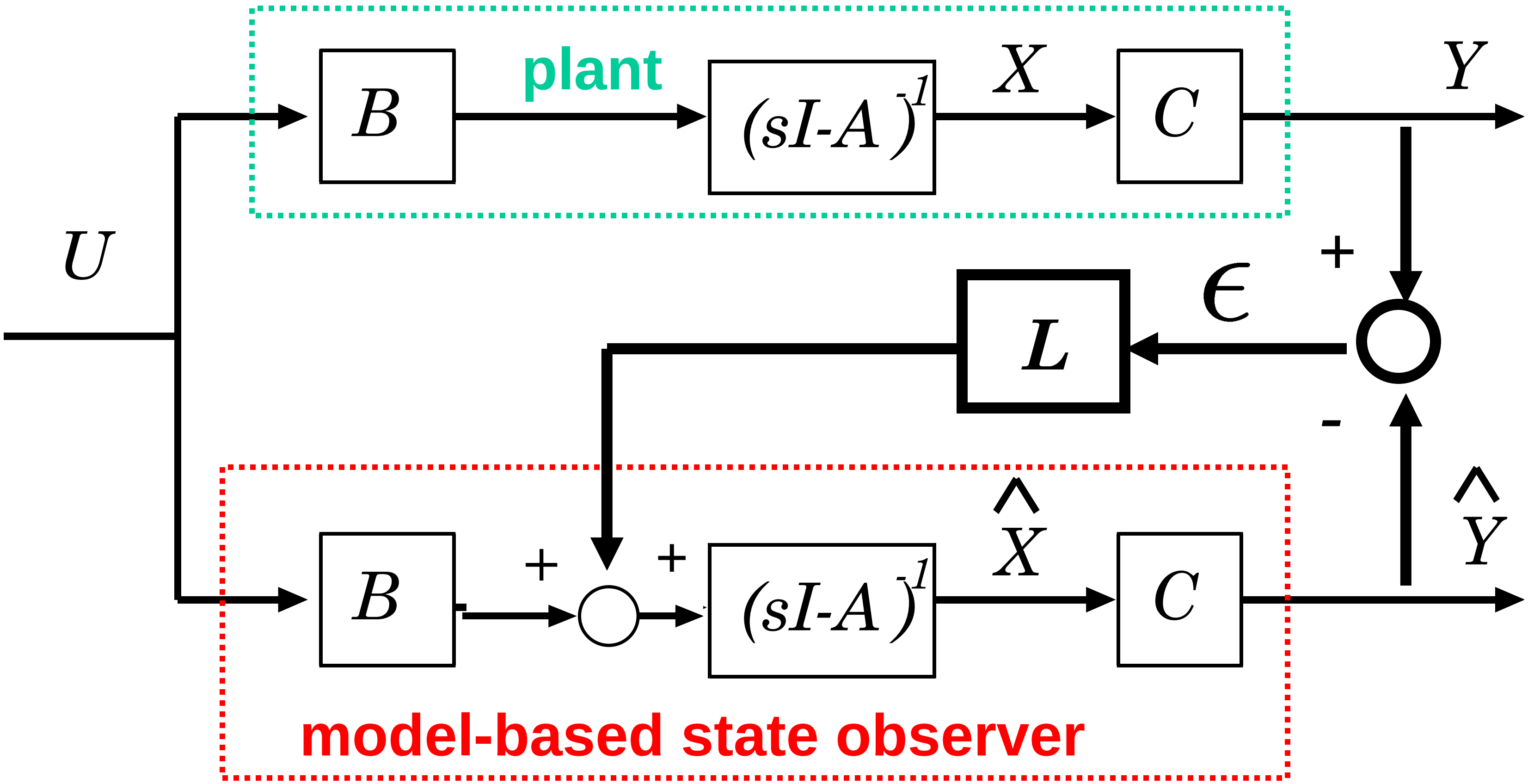
- $\boldsymbol{A}$ could have lightly damped - eigenvalues

- $\boldsymbol{A}$ may not be Hurwitz

Closed Loop State Observers

Key idea: Use the output estimation error signal

$$\epsilon(t) = y(t) - \widehat{y}(t)$$



Closed Loop State Observers

Use the output estimation error signal as feedback signal to the observer

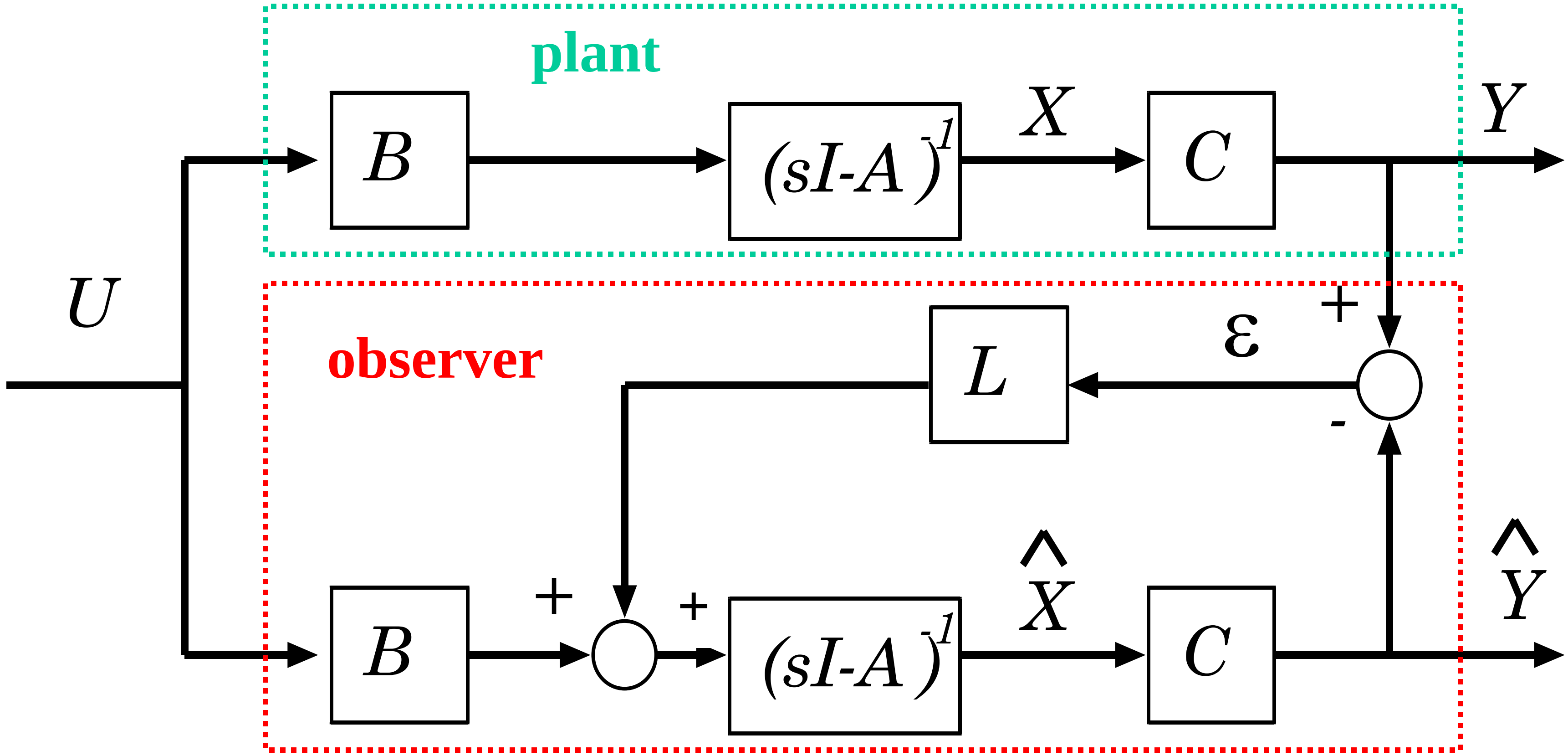
$$\epsilon(t) = y(t) - \hat{y}(t) \qquad \epsilon \in \mathcal{R}^m$$

Closed loop state observer:

$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t) + \underline{L\epsilon(t)}$$

$L \in \mathcal{R}^{n\times m}$: state estimation feedback gain

Closed Loop State Observer



$$\dot{\hat{x}}(t) \;=\; A\,\hat{x}(t) + B\,u(t) + L\epsilon(t)$$

Estimation error signals

The state estimation error (not accessible)

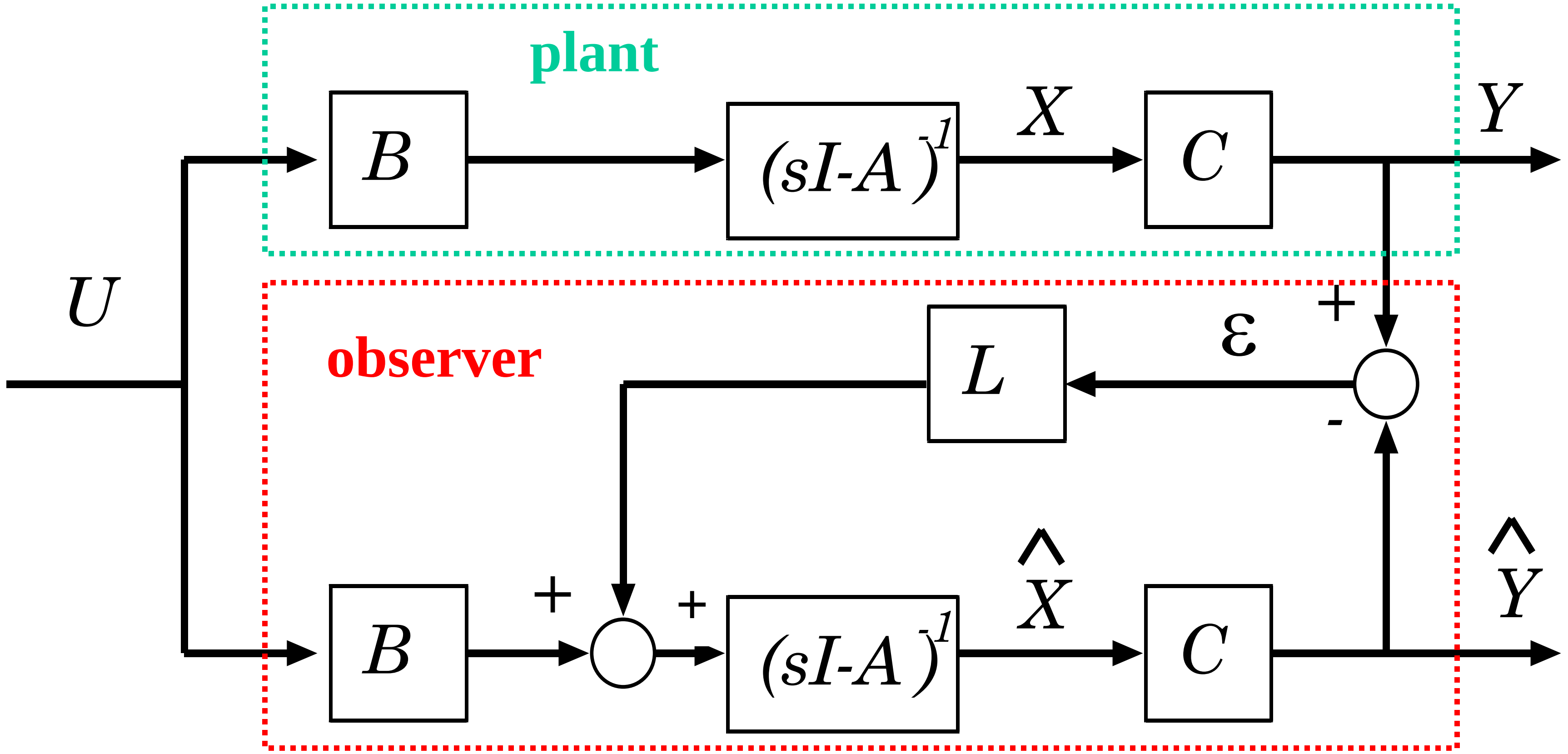
$$e(t) = x(t) - \widehat{x}(t)$$

The output estimation error (accessible)

$$\begin{aligned} \epsilon(t) &= y(t) - \widehat{y}(t) \\ &= Cx(t) - C\widehat{x}(t) \end{aligned}$$

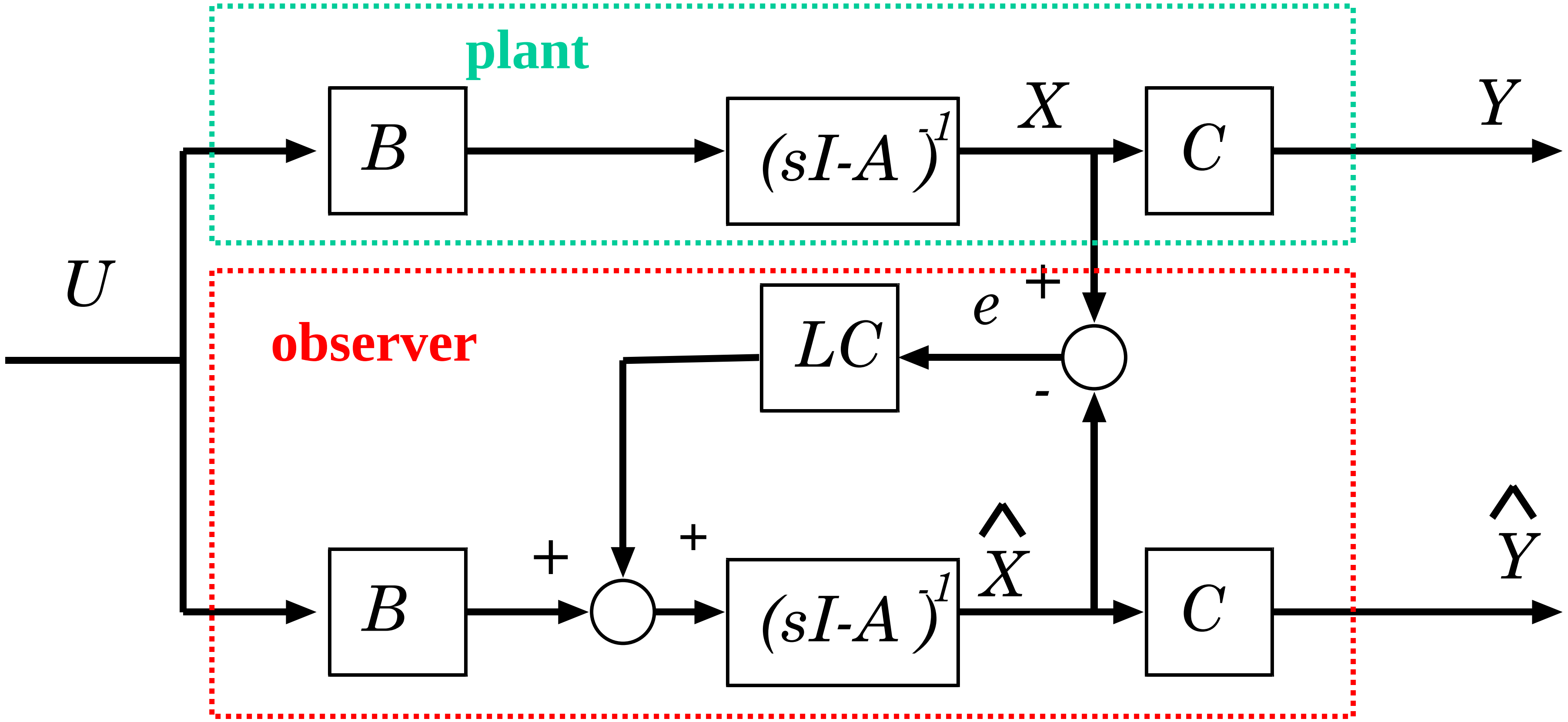
$$\epsilon(t) = C\,e(t)$$

Closed Loop State Observer



$$\dot{\hat{x}}(t) \;=\; A\,\hat{x}(t) + B\,u(t) + L\epsilon(t)$$

Closed Loop State Observer



$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t) + LC\,e(t)$$

Estimation error dynamics

Subtracting the observer from the actual system dynamics

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ -\dot{\hat{x}}(t) &= -A\,\hat{x}(t) - B\,u(t) - LCe(t) \end{aligned}$$

we obtain

$$\begin{aligned} \dot{e}(t) &= A\,e(t) - LC\,e(t) \\ &= \underbrace{[A - L\,C]}_{A_e}\,e(t) \end{aligned}$$

$$\begin{aligned} e(0) &= e_o \\ &= x_o - \hat{x}_o \end{aligned}$$

Estimation error dynamics

$$\dot{e}(t) \;=\; A_e\, e(t)$$

$$e = x - \widehat{x}$$

$$e(0) \;=\; e_o$$

where

observer gain matrix

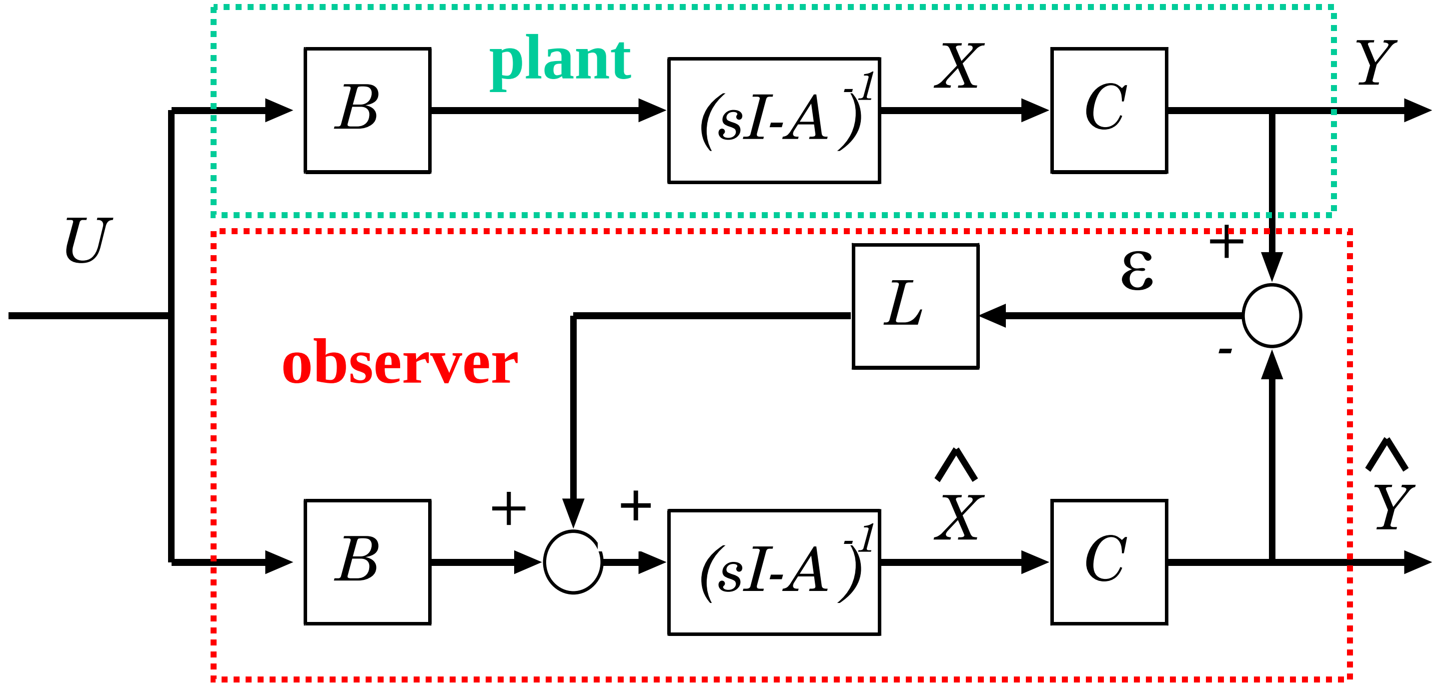
$$A_e = A - L\,C$$

Estimation closed loop characteristic polynomial:

$$A_e(s) \;=\; \mathrm{Det}\{[sI - A_e]\} = \mathrm{Det}\{[sI - A + L\,C]\}$$

$$= \; s^n + a_{e(n-1)} s^{n-1} + \cdots + a_{e0}$$

Closed Loop State Observer



Problem: Determine an estimation gain L

so that eigenvalues of $A_e = A - L\,C$

are placed at desired locations in the complex plane

Eigenvalue placement problem

Given a set of desired closed loop eigenvalues

$$\{\lambda_{e1},\, \lambda_{e2},\, \cdots\, ,\, \lambda_{en}\}$$

Find the observer feedback gain L such that

$$A_e = A - L\,C$$

and the closed loop characteristic polynomial satisfies:

$$\begin{aligned} A_e(s) &= \mathrm{Det}\{[sI - A + L\,C]\} \\ &= (s - \lambda_{e1})(s - \lambda_{e2}) \cdots (s - \lambda_{en}) \end{aligned}$$

Closed Loop State Observer

Theorem:

If the pair $\{A,\, C\}$ is observable, then the eigenvalues of

$$A_e = A - L\,C$$

can be *arbitrarily* assigned in the complex plane.

(complex roots must be accompanied by their complex conjugates – symmetry about real axis)

Proof outline

1. Convert the original realization to the **observable canonical** realization using a ***similarity transformation.***

2. Find the state observer gain matrix that will place the poles of the observable canonical realization to the desired location.

3. After the observer gain matrix is found, convert the system back to the original realization.

SISO Observable Canonical Form

In matrix form:

$$\frac{d}{dt}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}-a_2 & 1 & 0\\ -a_1 & 0 & 1\\ -a_0 & 0 & 0\end{bmatrix}}_{\bar{A}}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}b_2\\ b_1\\ b_o\end{bmatrix}}_{\bar{B}} u$$

$$y = \underbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}_{\bar{C}}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}$$

and

$$\bar{C}\,(s\,I - \bar{A}_o)^{-1}\,\bar{B} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Similarity transformation

Use the similarity transformation $\bar{x} = Q^{-1}\, x$

on

$$\begin{aligned} \dot{x} &= A\, x + B\, u \\ y &= C\, x \end{aligned}$$

to obtain

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\, \bar{x} + \bar{B}\, u \\ y &= \bar{C}\, \bar{x} \end{aligned}$$

$$\bar{A} = Q^{-1}\, A\, Q$$

$$\bar{B} = Q^{-1}\, B$$

$$\bar{C} = C\, Q$$

Single output (SO) observable canonical realization

Lemma: If the pair $\{A, C\}$ is observable,

there exists a similarity transformation matrix Q

such that

$$\bar{A} = Q^{-1} A Q \qquad \bar{C} = C Q$$

define the observable canonical pair

$$\bar{A} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_o & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \bar{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Computing the similarity transformation $\boldsymbol{Q}$

The observable similarity transformation is given by:

$$Q = P^{-1} \qquad \text{where,} \qquad P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}$$

and,

$$\boxed{\begin{aligned} p_1 &= C \\ p_{j+1} &= p_j\,A + a_{n-j}\,C \qquad j \in [1, n-1] \end{aligned}}$$

Computing the similarity transformation $\boldsymbol{Q}$

1. Compute the observability matrix of each pair:

Original pair

$$[A,\ C]$$

Observable canonical

$$[\bar{A},\ \bar{C}]$$

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$\bar{O} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix}$$

Computing the similarity transformation $\boldsymbol{Q}$

2. Since

$$\bar{A} = Q^{-1} A Q \qquad \bar{C} = C Q$$

Then

$$\bar{O} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C Q \\ C A Q \\ \vdots \\ C A^{n-1} Q \end{bmatrix} = O Q$$

Thus

$$\bar{O} = O Q \quad \Rightarrow \quad \boxed{Q = O^{-1} \bar{O}}$$

Single output observable canonical realization

$$\bar{e} = Q^{-1} e \; = Q^{-1}(x - \hat{x}) \; = \bar{x} - \hat{\bar{x}}$$

$$\frac{d}{dt}\bar{e} \;=\; \underbrace{\begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_o & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\bar{A}} \bar{e}$$

$$\epsilon \;=\; \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\bar{C}} \bar{e}$$

$$\bar{A}(s) \;=\; \mathsf{Det}\{[sI - \bar{A}]\} = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

Closed loop observable canonical dynamics

$$\frac{d}{dt}\bar{e}(t) = \underbrace{[\bar{A} - \bar{L}\,\bar{C}]}_{\bar{A}_e}\,\bar{e}(t)$$

$$\frac{d}{dt}\bar{e} = \left\{ \underbrace{\begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_o & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\bar{A}} - \underbrace{\begin{bmatrix} \bar{l}_1 \\ \bar{l}_2 \\ \vdots \\ \bar{l}_{n-1} \\ \bar{l}_n \end{bmatrix}}_{\bar{L}} \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\bar{C}} \right\} \bar{e}$$

Closed loop observable canonical dynamics

$$\frac{d}{dt}\bar{e}(t) = \underbrace{[\bar{A} - \bar{L}\,\bar{C}]}_{\bar{A}_e}\,\bar{e}(t)$$

$$\frac{d}{dt}\bar{e} = \left\{ \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_o & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} -\bar{l}_1 & 0 & 0 & \cdots & 0 \\ -\bar{l}_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\bar{l}_{n-1} & 0 & 0 & \cdots & 0 \\ -\bar{l}_n & 0 & 0 & \cdots & 0 \end{bmatrix} \right\} \bar{e}$$

Closed loop observable canonical dynamics

$$\frac{d}{dt}\bar{e}(t) = \underbrace{[\bar{A} - \bar{L}\,\bar{C}]}_{\bar{A}_e}\,\bar{e}(t)$$

$$\frac{d}{dt}\bar{e} = \begin{bmatrix} -(a_{n-1}+\bar{l}_1) & 1 & 0 & \cdots & 0 \\ -(a_{n-2}+\bar{l}_2) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_1+\bar{l}_{n-1}) & 0 & 0 & \cdots & 1 \\ \underbrace{-(a_o+\bar{l}_n)} & 0 & 0 & \cdots & 0 \end{bmatrix}\bar{e}$$

Observer gains l_i's can be chosen
to arbitrarily assign first column

Closed loop observable canonical dynamics

$$\bar{A}_e = \begin{bmatrix} -(a_{n-1}+\bar{l}_1) & 1 & 0 & \cdots & 0 \\ -(a_{n-2}+\bar{l}_2) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_1+\bar{l}_{n-1}) & 0 & 0 & \cdots & 1 \\ -(a_o+\bar{l}_n) & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} -a_{e(n-1)} & 1 & 0 & \cdots & 0 \\ -a_{e(n-2)} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{e1} & 0 & 0 & \cdots & 1 \\ -a_{eo} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Closed loop characteristic polynomial:

$$\begin{aligned} \bar{A}_e(s) &= \mathrm{Det}\{[sI - \bar{A}_e]\} \\ &= s^n + a_{e(n-1)}\, s^{n-1} + \cdots + a_{e0} \end{aligned}$$

Finding observer gain $\boldsymbol{L}$

Since $\bar{e} = Q^{-1} e \implies e = Q\,\bar{e}$

and
$$\frac{d}{dt}\bar{e}(t) \;=\; [\bar{A} - \bar{L}\,\bar{C}]\,\bar{e}(t)$$

$$\frac{d}{dt}e(t) \;=\; [\underbrace{Q\,\bar{A}Q^{-1}}_{A} - \underbrace{Q\,\bar{L}}_{L}\,\underbrace{\bar{C}Q^{-1}}_{C}]\,e(t)$$

$$\implies \quad L = Q\bar{L}$$

Closed loop observer eigenvalue placement: Procedure

1) Given desired close loop eigenvalues for the matrix $A_e = A - LC$:

$$\{\lambda_{e1},\, \lambda_{e2},\, \cdots\, ,\, \lambda_{en}\}$$

2) Compute $\left\{a_{e0},\, a_{e1},\, \cdots\, ,\, a_{e(n-1)}\right\}$ such that

$$\begin{aligned} A_e(s) &= s^n + a_{e(n-1)} s^{n-1} + \cdots + a_{e0} \\ &= (s - \lambda_{e1})(s - \lambda_{e2}) \cdots (s - \lambda_{en}) \end{aligned}$$

Closed loop observer eigenvalue placement: Procedure

3) Compute the observable similarity transformation $\boldsymbol{Q}$:

$$Q = P^{-1} \qquad \text{where,} \qquad P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}$$

and,

$$\boxed{\begin{aligned} p_1 &= C \\ p_{j+1} &= p_j A + a_{n-j} C \qquad j \in [1, n-1] \end{aligned}}$$

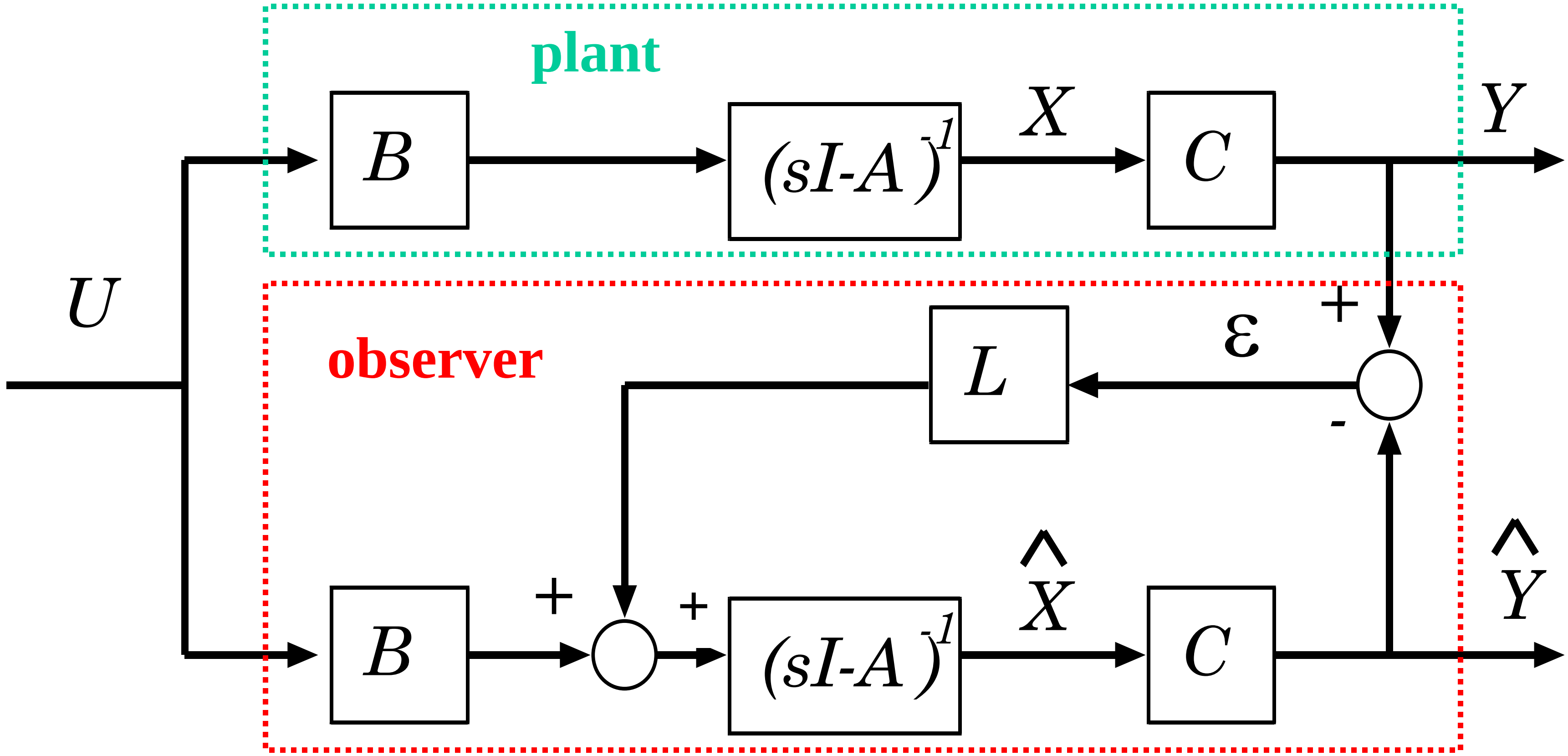
Closed loop observer eigenvalue placement: Procedure

4) Compute closed loop observer matrix $\boldsymbol{L}$:

$$\bar{L} = \begin{bmatrix} \bar{l}_1 \\ \vdots \\ \bar{l}_{n-1} \\ \bar{l}_n \end{bmatrix} = \begin{bmatrix} a_{e(n-1)} - a_{n-1} \\ \vdots \\ a_{e1} - a_1 \\ a_{e0} - a_0 \end{bmatrix}$$

$$\boxed{L = Q\,\bar{L} = Q \begin{bmatrix} a_{e(n-1)} - a_{n-1} \\ \vdots \\ a_{e1} - a_1 \\ a_{e0} - a_0 \end{bmatrix}}$$

Closed Loop State Observer



$$\dot{\hat{x}}(t) = A\,\hat{x}(t) + B\,u(t) + L\epsilon(t)$$

Observer state feedback control

Consider an nth order LTI CT system:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

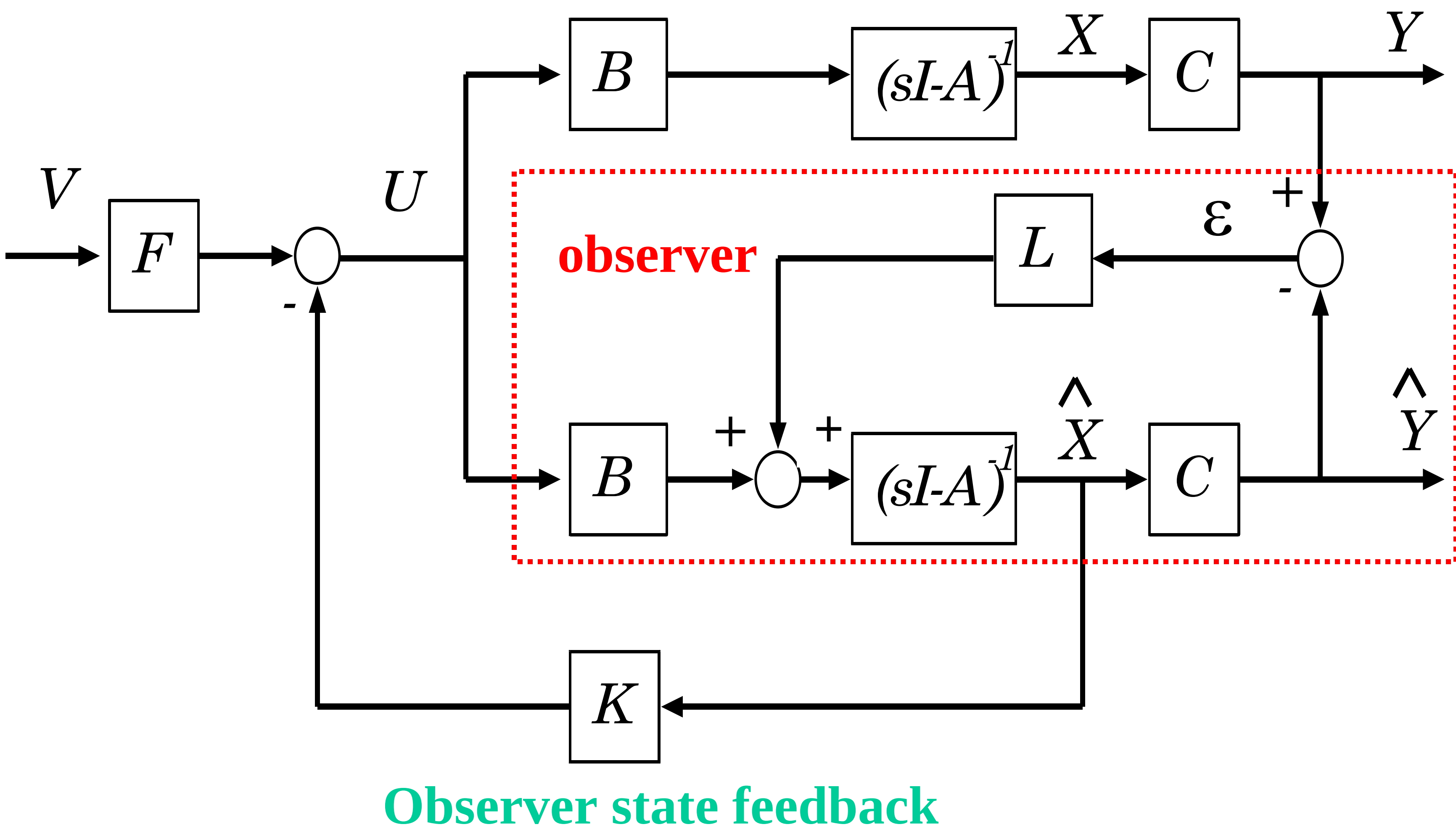
$$y(t) = C\,x(t)$$

Under ***<u>state estimate</u>*** feedback:

$$u = -K\,\underline{\widehat{x}} + F\,v$$

where $\widehat{x}(t)$ is the estimate of the state $x(t)$

Observer state feedback control



Observer state feedback control

We now have a 2n-th order system:

$$\begin{aligned}
\dot{x}(t) &= A\,x(t) + B\,\underline{u(t)} \\
\dot{\hat{x}}(t) &= A\,\hat{x}(t) + B\,u(t) + L\underline{\epsilon(t)}
\end{aligned}$$

where

$$\begin{aligned}
\underline{u(t)} &= -K\,\hat{x}(t) + F\,v(t) \\
\underline{\epsilon(t)} &= C\,\{x(t) - \hat{x}(t)\}
\end{aligned}$$

Observer state feedback control

Substitute control law

$$\dot{x}(t) \;=\; A\,x(t) + B\,u(t)$$

$$\dot{\widehat{x}}(t) \;=\; A\,\widehat{x}(t) + B\,u(t) + L\epsilon(t)$$

$$u(t) \;=\; -K\,\widehat{x}(t) + F\,v(t)$$

Observer state feedback control

Substitute error equation

$$\dot{x} = A\,x + B\,[-K\,\widehat{x} + F\,v]$$

$$\dot{\widehat{x}} = A\,\widehat{x} + B\,[-K\,\widehat{x} + F\,v] + L\epsilon$$

$$\epsilon(t) = C\,\{x(t) - \widehat{x}(t)\}$$

Observer state feedback control

$$\dot{x} \;=\; A\,x + B\,[-K\,\widehat{x} + F\,v]$$

$$\dot{\widehat{x}} \;=\; A\,\widehat{x} + B\,[-K\,\widehat{x} + F\,v] + LC\,[x - \widehat{x}]$$

Rearrange:

$$\dot{x} \;=\; A\,x - BK\,\widehat{x} + BF\,v$$

$$\dot{\widehat{x}} \;=\; LC\,x + [A - BK - LC]\,\widehat{x} + BF\,v]$$

Observer state feedback control

$$\dot{x} \;=\; A\,x - BK\,\hat{x} + BF\,v$$

$$\dot{\hat{x}} \;=\; LC\,x + [A - BK - LC]\,\hat{x} + BF\,v]$$

Equations above can be expressed in matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & (A - LC - BK) \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} BF \\ BF \end{bmatrix} v$$

Note: The overall system has 2n eigenvalues.

Observer state feedback control

In order to determine the 2n eigenvalues, we use the following similarity transformation:

$$T_s = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix}$$

so that,

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ \widehat{x} \end{bmatrix}$$

$$e(t) = x(t) - \widehat{x}(t)$$

state estimation error vector

Observer state feedback control

Applying the similarity transformation:

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} \overbrace{(A-BK)}^{A_c} & BK \\ 0 & \underbrace{(A-LC)}_{A_e} \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} BF \\ 0 \end{bmatrix} v$$

Separation Principle

The 2n-th observer state feedback system

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} (A-BK) & BK \\ 0 & (A-LC) \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} BF \\ 0 \end{bmatrix} v$$

has 2n eigenvalues:

- n eigenvalues are the eigenvalues of

$$A_c = A - B\,K$$

- n eigenvalues are the eigenvalues of

$$A_e = A - L\,C$$

Separation Principle

Thus, the state observer feedback design can be accomplished in two steps:

1) Find the state feedback gain $\boldsymbol{K}$ to place the n eigenvalues of

$$A_c = A - B\,K$$

2) Find the observer gain $\boldsymbol{L}$ to place the n eigenvalues of

$$A_e = A - L\,C$$

Discrete Time State Observers

Consider an nth order LTI DT system:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

Assume that the state vector **is not** measurable and only the output is measurable.

We will denote the estimate of the state by: $\widehat{x}(k)$

A-priori State Observers

Use the output estimation error signal

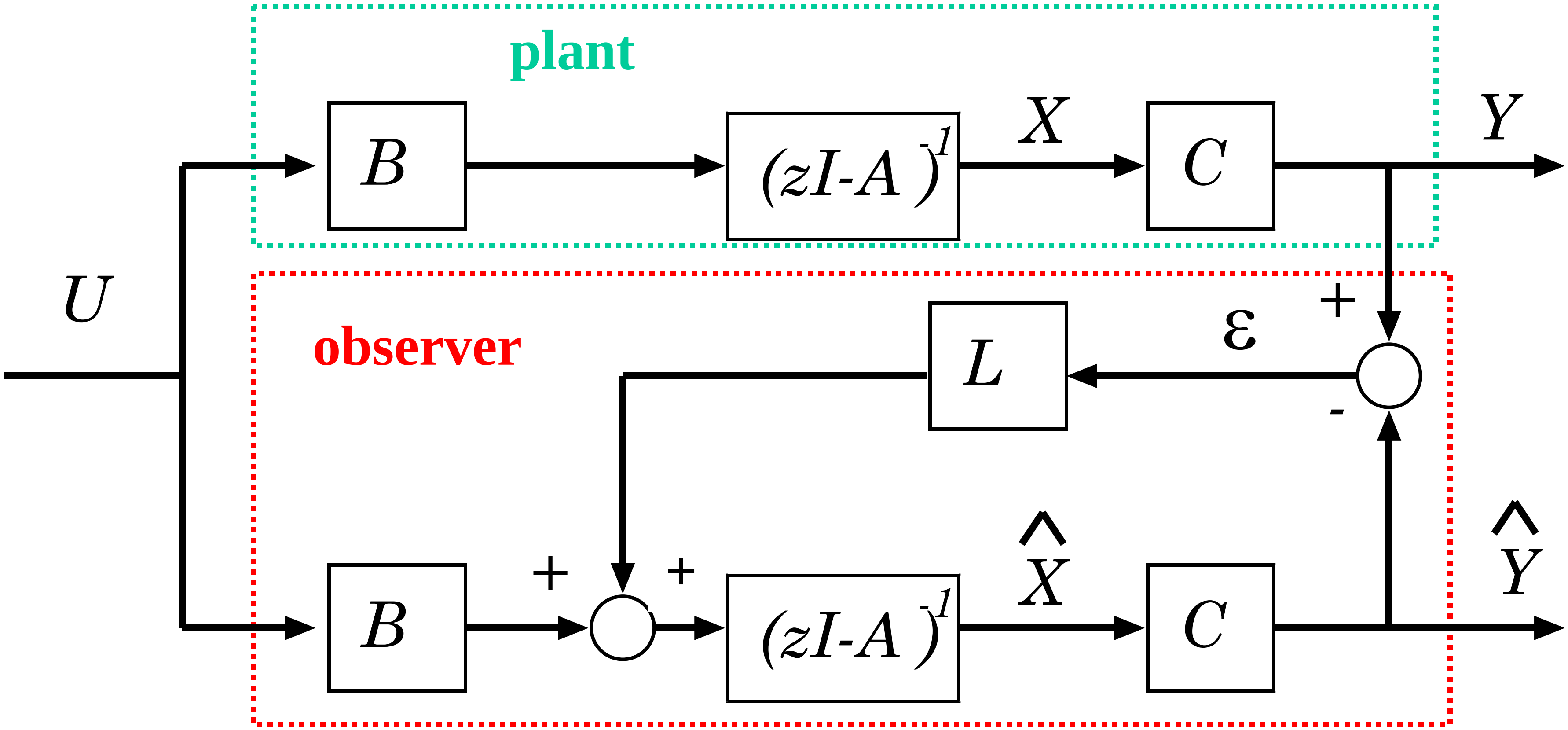
$$\epsilon(k) = y(k) - \hat{y}(k) \qquad \epsilon \in \mathcal{R}^m$$

Closed loop a-priori state observer:

$$\hat{x}(k+1) \;=\; A\,\hat{x}(k) + B\,u(k) + L\epsilon(k)$$

$L \in \mathcal{R}^{n \times m}$: observer gain

A-priori State Observer



$$\hat{x}(k+1) \;=\; A\,\hat{x}(k) + B\,u(k) + L\epsilon(k)$$

A-priori Estimation error dynamics

Subtracting the observer from the actual state dynamics

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) \\
-\hat{x}(k+1) &= -A\,\hat{x}(k) - B\,u(k) - L\,\underbrace{C e(k)}_{\epsilon(k)}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
e(k+1) &= A\,e(k) - LC\,e(k) \\
&= \underbrace{[A - L\,C]}_{A_e}\,e(k)
\end{aligned}
\qquad e(0) = e_o
$$

A-priori estimation error dynamics

$$e(k+1) = A_e\, e(k)$$

$$e = x - \hat{x}$$

$$e(0) = e_o$$

where

$$A_e = A - L\,C$$

Closed loop characteristic polynomial:

$$\begin{aligned} A_e(z) &= \mathrm{Det}\{[zI - A_e]\} = \mathrm{Det}\{[zI - A + L\,C]\} \\ &= z^n + a_{e(n-1)} z^{n-1} + \cdots + a_{e0} \end{aligned}$$

A-priori State Observers

Notice that the latest value of $\boldsymbol{y}(k)$ is not used to compute $\widehat{x}(k)$

$$\widehat{x}(k+1) = A\,\widehat{x}(k) + B\,u(k) + L\epsilon(k)$$

$$\begin{aligned}\underline{\widehat{x}(k)} &= A\,\widehat{x}(k-1) + B\,u(k-1) \\ &\quad + L\underbrace{\epsilon(k-1)}_{[\underline{y(k-1)} - C\,\widehat{x}(k-1)]}\end{aligned}$$

A-posteriori State Observers

Predictor: (a-priori)

$$\widehat{x}^{o}(k) \;=\; A\,\widehat{x}(k-1) + B\,u(k-1)$$

Corrector: (a-posteriori)

$$\underline{\widehat{x}(k)} \;=\; \widehat{x}^{o}(k) + L\,[\,\underline{y(k)} - C\,\widehat{x}^{o}(k)\,]$$

A-posteriori state observer dynamics

$$\widehat{x}(k) \;=\; \widehat{x}^o(k) + L\,[\,y(k) - C\,\widehat{x}^o(k)\,]$$

$$\widehat{x}(k+1) \;=\; \widehat{x}^o(k+1) + L\,[\,y(k+1) - C\,\widehat{x}^o(k+1)\,]$$

$$\widehat{x}^o(k+1) \;=\; A\,\widehat{x}(k) + B\,u(k)$$

$$\boxed{\begin{aligned} \underline{\widehat{x}(k+1)} \;&=\; [I - LC]\,A\,\widehat{x}(k) + [I - LC]\,Bu(k) \\ &\quad + L\,\underline{y(k+1)} \end{aligned}}$$

A-posteriori state observer dynamics

Subtracting the observer from the actual state dynamics

$$x(k+1) \;=\; A\,x(k) + B\,u(k)$$

$$-\hat{x}(k+1) \;=\; -[I - LC]\,A\,\hat{x}(k) - [I - LC]\,Bu(k)$$

$$- L\,C\underbrace{x(k+1)}_{[A\,x(k) + B\,u(k)]}$$

$$\boxed{e(k+1) \;=\; [I - L\,C]\,A\,e(k)}$$

A-posteriori estimation error dynamics

$$e(k+1) \;=\; [I - L\,C]\,A\,e(k)$$

$$e(k) \;=\; x(k) - \widehat{x}(k)$$

$$A_e = [I - L\,C]\,A$$

$$\boxed{A_e = A - L\,C' \qquad\qquad C' = C\,A}$$

A-posteriori State Observer

Theorem:

If the pair $\{A,\, C'\}$ is observable, $C' = C\,A$

the eigenvalues of

$$A_e = A - L\,C'$$

can be *arbitrarily* assigned in the complex plane.

(complex roots must be accompanied by their complex conjugates – symmetry about real axis)

A-posteriori State Observer

Lemma:

$\{A,\, C'\}$ is observable if

- $\{A,\, C\}$ is observable
- A is nonsingular.

Remarks:

1. $\boldsymbol{A}$ is always nonsingular if the discrete time model is derived from a continuous time model:

– using zero-order holder and a sampler, i.e.

$$A = e^{A_c T}$$

$\boldsymbol{T}$ is the sampling time

2. In discrete time, eigenvalues at the origin do not cause stability problems.

Remarks:

3. If $\boldsymbol{A}$ has eigenvalues at the origin, $\boldsymbol{A}$ is singular

Then

$$A_e = A - L\,C' \;\; = [I - L\,C]\,A$$

- has the same eigenvalues at the origin as $\boldsymbol{A}$

- remaining eigenvalues can be arbitrarily placed in the complex plane, if $[\,\boldsymbol{A}, \boldsymbol{C}\,]$ is observable.