[MEN573]
Advanced Control Systems I

Lecture 5 – Mathematical Notation and Definitions and Vector Spaces

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Mathematical notation

- $x \in A$: x is an element of the set A
- $x \notin A$: x does not belong to A
- $B \subset A$: B is a subset of A
- $B \cap A$: intersection of B and A
- $B \cup A$: union of B and A

Mathematical notation

- $a \Rightarrow b$: a is true implies that b is true

  $(b$ not true implies a is not true$)$

- $a \Leftrightarrow b$: a is true iff (if and only if) b is true

- $\forall$: symbol “for all”

Mathematical notation

- $\boldsymbol{R}$ denotes the set of all real numbers
- $\boldsymbol{C}$ denotes the set of all complex numbers
- $\boldsymbol{R}_+ = \{\boldsymbol{x} \in \boldsymbol{R} : \boldsymbol{x} \geq \boldsymbol{0}\}$
- (i.e. the set of all nonnegative real numbers).
- $\boldsymbol{Z}$ denotes the set of all integers
- $\boldsymbol{Z}_+$ denotes the set of all nonegative integers.

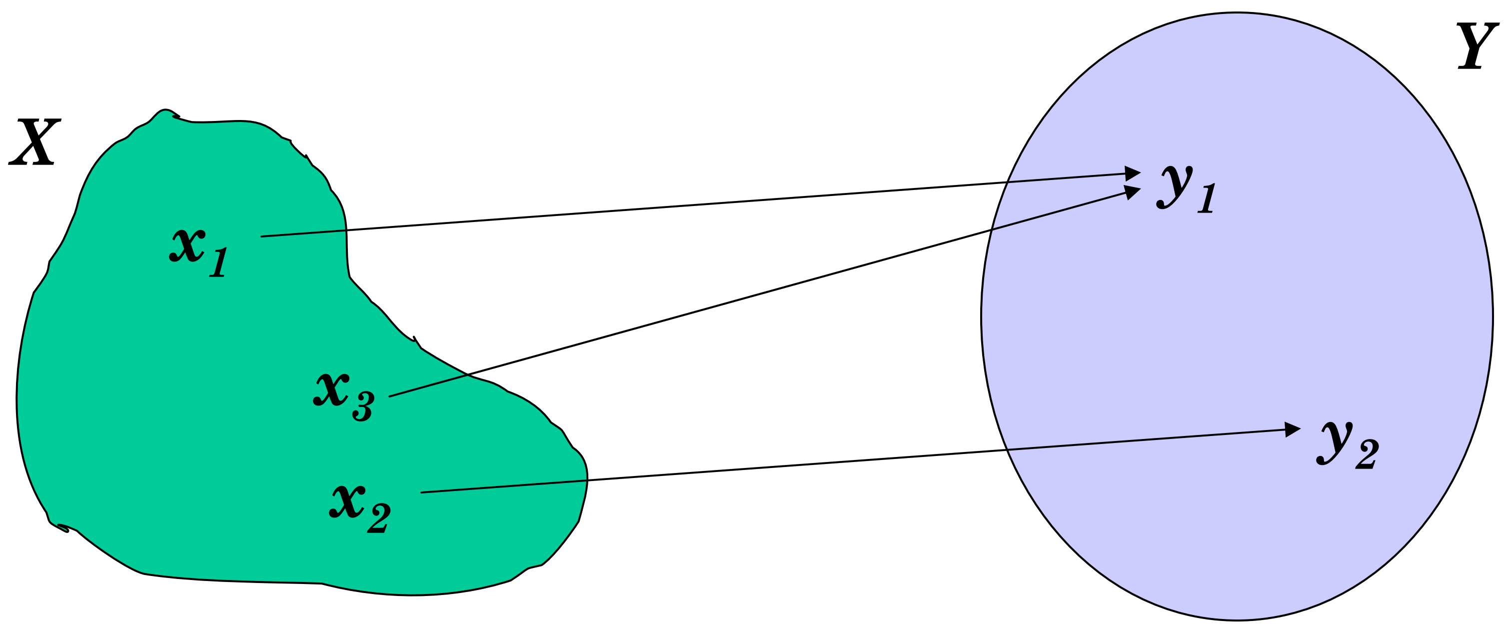
Functions

Given two sets X and Y, we denote a function f by

$$f : X \to Y$$

for every $x \in X$, f assigns ***one and only one*** element

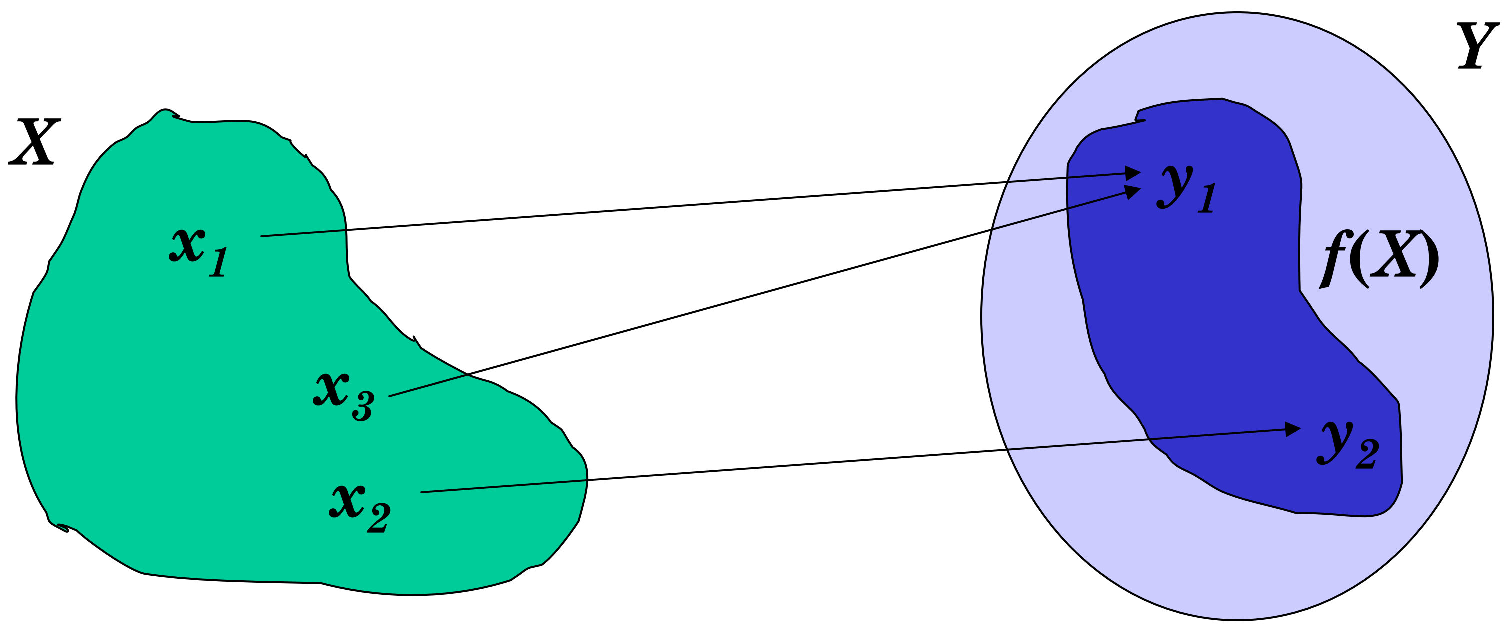
$f(x) \in Y$



Functions

- X is the ***domain*** of f

- $f(X) = \{ f(x) : x \in X \} \subset Y$ **is the *range* of *f*.**

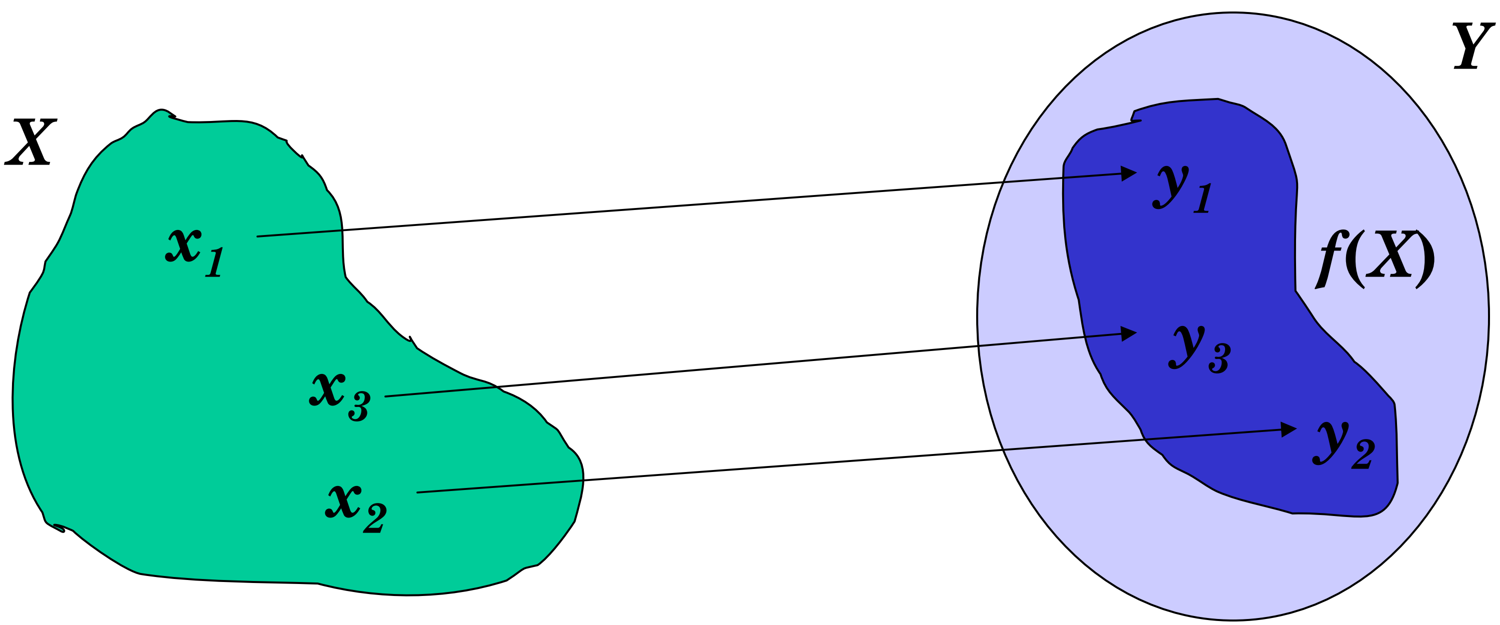


Functions

- $f : X \to Y$ is *one-to-one* iff

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

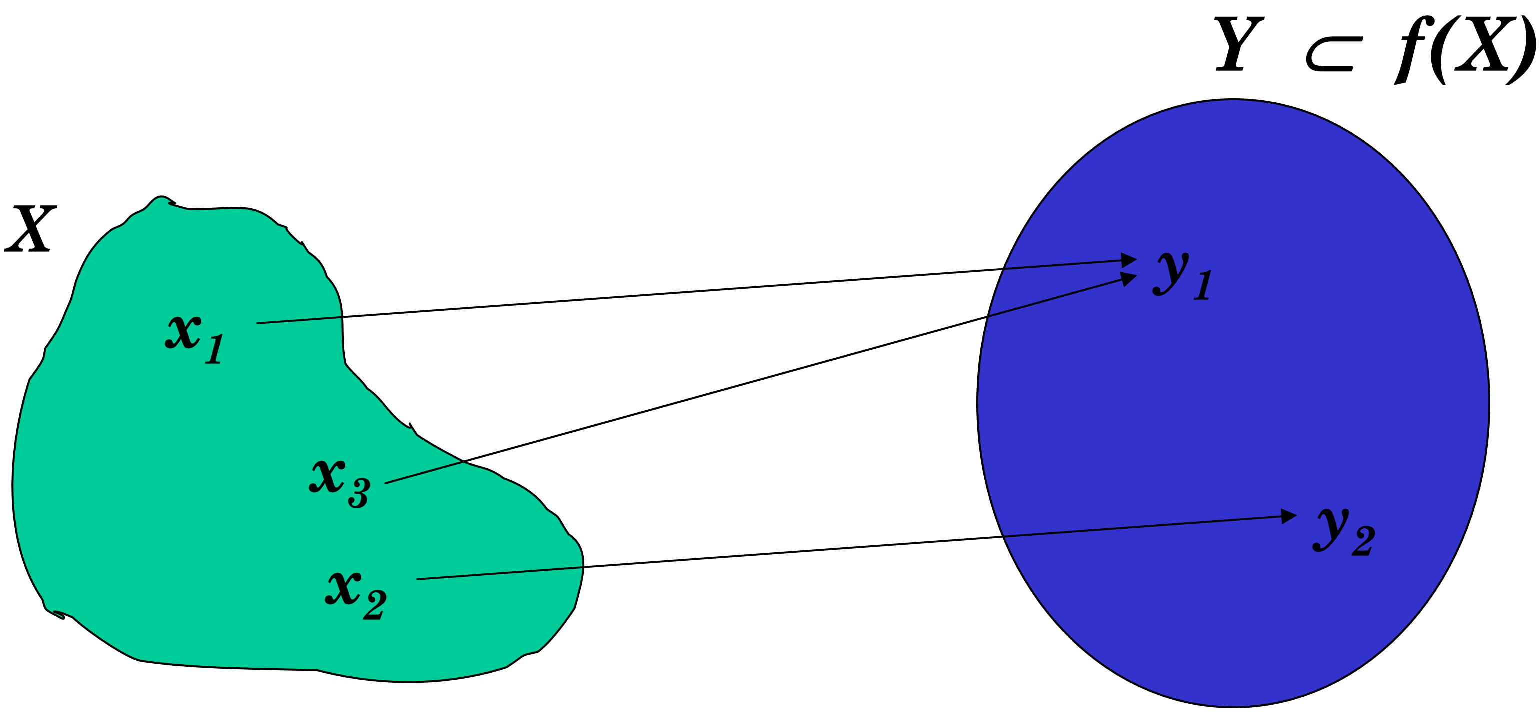
(or equivalently, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$).



Functions

- $f : X \to Y$ is *onto* iff

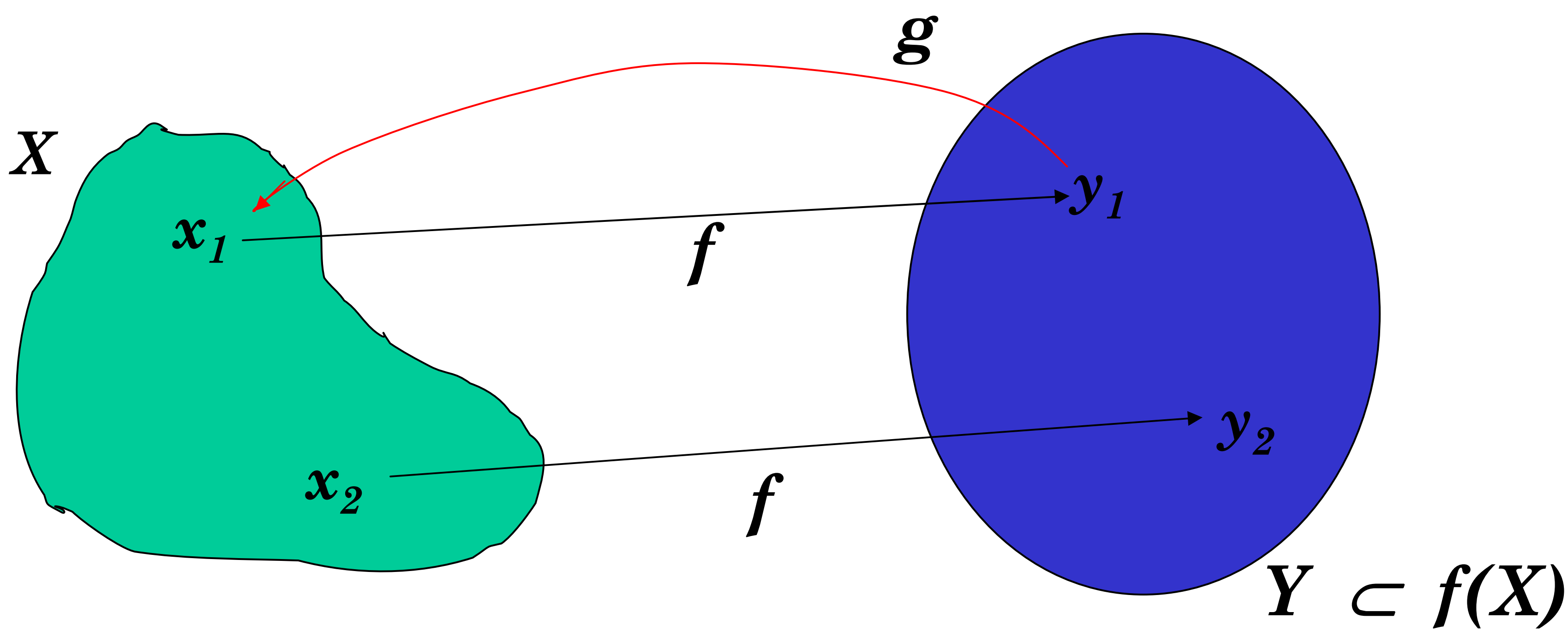
  $f(X) = Y$ (i.e. $Y \subset f(X)$ and $f(X) \subset Y$)



Functions

- $f : X \rightarrow Y$ is *onto* and *one to one* than it has an inverse

$g : Y \rightarrow X$ such that $g(f(x)) = x$



Fields

A *field* $\mathbf{F}$ is a set of elements and two binary operations:

addition (+) and *multiplication* (·) such that for all $\alpha, \beta, \gamma \in \mathbf{F}$:

1. **Closure**: $\alpha \cdot \beta \in \mathbf{F},\ \alpha + \beta \in \mathbf{F}$

2. **Commutativity**: $\alpha \cdot \beta = \beta \cdot \alpha,\ \alpha + \beta = \beta + \alpha$

3. **Associativity**:

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma,\ \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$

Fields

4. **Distribution**: $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$

5. **Identities**:
   1. Additive: $0 \in \mathbf{F}$ such that $\alpha + 0 = \alpha$
   2. Multiplicative: $1 \in \mathbf{F}$ such that $\alpha \cdot 1 = \alpha$

6. **Inverses**: *For all* $\alpha \in \mathbf{F}$ there exist
   1. Additive: $\alpha + (-\alpha) = 0$
   2. Multiplicative: $\alpha \cdot \alpha^{-1} = 1$

Examples of Fields

– $\boldsymbol{R}$ = the set of real numbers
– $\boldsymbol{C}$ = the set of complex numbers

– $\mathbf{Q}$ = the set of rational numbers

– $\boldsymbol{R}(\boldsymbol{s})$ = the set of rational functions in *s* with real coefficients

$$G(s) = \frac{s+1}{s^2+3s+2}$$

These are *not* fields:

– $\boldsymbol{R}[\boldsymbol{s}] =$ the set of polynomials in s with real coefficients. *Why?*
– $\boldsymbol{R}^{\mathbf{2x2}} =$ the set of real 2×2 matrices. *Why?*

Vector Spaces

A *vector space* $(\mathbf{V};\mathbf{F})$ is a set of *vectors* $\mathbf{V}$ together with a field $\mathbf{F}$

and two operations *vector-vector addition* (+) and *vector-scalar multiplication* $(\circ)$ such that for all $\alpha,\beta \in \mathbf{F}$ and all $v_1,\ v_2,\ v_3 \in \mathbf{V}$:

1. **Closure**: $v_1 + v_2 \in \mathbf{V},\ \ \alpha \circ v_1 \in \mathbf{V}$

2. **Commutativity**: $v_1 + v_2 = v_2 + v_1$

3. **Associativity**:

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

Vector Spaces

4. **Distribution**:

$$\alpha \circ (\beta \circ v_1) = (\alpha \cdot \beta) \circ v_1, \ \alpha \circ (v_1 + v_2) = \alpha \circ v_1 + \alpha \circ v_2$$

5. **Additive Identity**:

$$0 \in \mathbf{V} \text{ such that } v + 0 = v$$

6. **Additive Inverse**: *For all* $v \in \mathbf{V}$ there exist

$$v + (-v) = 0$$

Examples of Vector Spaces

– ***(R;R); (C; C)*** with addition and multiplication as defined in the field. (Any field is a vector space over itself).

– ***(R^n;R); (C^n; C)*** with component-wise addition and scalar multiplication.

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad v_i \in \mathcal{R} \qquad \text{or} \qquad v_i \in \mathcal{C}$$

Linear Independence and Dimension

- A set (possibly infinite) of vectors from $\mathbf{V}$

$$S = \{v_i \ : \ i \in I \subset Z\}$$

is called ***linearly dependent*** if there exist scalars α_i *not all zero* and only *finitely* many α_i being *nonzero* such that

$$\sum_{i \in I} \alpha_i v_i = 0$$

Otherwise, the set of vectors $\boldsymbol{S}$ is said to be ***linearly independent*.**

Linear Independence and Dimension

- The ***dimension*** of a vector space $\mathbf{V}$ is the ***maximal*** number of linearly independent vectors in $\mathbf{V}$.

Examples

- In the vector space ***(R²;R)***,

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \; v_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

is a set of linearly dependent vectors because:

$$-v_1 - 2\,v_2 + v_3 = 0$$

Examples

- The vectors

$$v_1 = \begin{bmatrix} \frac{1}{s+1} \\ \\ \frac{1}{s+2} \end{bmatrix}, \quad v_2 = \begin{bmatrix} \frac{s+2}{s^2+4s+3} \\ \\ \frac{1}{s+3} \end{bmatrix}$$

are linearly dependent in $(\boldsymbol{R}^2(s), \boldsymbol{R}(s))$,

but linearly independent in $(\boldsymbol{R}^2(\mathrm{s}), \boldsymbol{R})$, *Why?*

Bases

A set $\boldsymbol{B}$ of ***linearly independent*** vectors in a vector space $\mathbf{V}$ is called a *basis* for $\mathbf{V}$ if:

- ***every*** vector in $\mathbf{V}$ can be ***uniquely*** expressed as a finite linear combination of vectors in $\boldsymbol{B}$**.**

Bases are ***not*** unique.

Bases

Example:

- The dimension of $(\boldsymbol{R}^n;\boldsymbol{R})$ is $\boldsymbol{n}$

- The set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ . \\ . \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ . \\ . \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} \right\}$$

qualifies as a basis for this vector space.

Dimension, span and bases

- *Let* $\mathbf{V}$ *be an* ***n****-dimensional vector space and*
- *Let* $\boldsymbol{B}$ *be a collection of vectors drawn from* $\mathbf{V}$*.*

$\boldsymbol{B}$ *is a basis if and only if* $\boldsymbol{B}$ *contains* **n** *linearly independent vectors.*

Dimension, span and bases

Let

$$S = \{v_i \ : \ i \in I \subset Z\}$$

be a set of vectors drawn from $\boldsymbol{V}$.

- The *span* of $\mathbf{S}$ is the set of ***all*** finite linear combinations of vectors in $\mathbf{S}$.
- We will denote this set $\mathcal{SP}(S)$

Dimension, span and bases

$$S = \{v_i \; : \; i \in I \subset Z\}$$

is a basis of $\boldsymbol{V}$ iff

- $\mathbf{S}$ is a linearly independent set, and
- $\mathrm{SP}(\mathbf{S}) = \boldsymbol{V}$.

Normed vector spaces

A *norm* on the vector space $(\boldsymbol{V}, \mathbf{F})$ is a function $\|\cdot\|: \boldsymbol{V} \to \boldsymbol{R}_+$ such that:

1. $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$

2. $\|\alpha v\| = |\alpha| \; \|v\|$ for all $\alpha \in \mathbf{F}, \; v \in \mathbf{V}$

3. $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

(triangle inequality

Examples

On $\boldsymbol{R^n}$ or $\boldsymbol{C^n}$:

1. 1- norm $$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

2. (Euclidean) $$\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2\right)^{\frac{1}{2}}$$

3. p – norm $$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}, \ 1 \le p < \infty$$

4. ∞ - norm $$\|v\|_\infty = \max_i \ |v_i|$$

Inner product spaces

Let V be a vector space on C^n .

Inner product : $<\cdot\,,\,\cdot> \;:\; V \times V \to \mathcal{C}$

1. $<v,w> = \overline{<w,v>}$ **(complex conjugate)**

2. $<v,\alpha w> = \alpha \;<v,w>$

3. $<v,w_1+w_2> = <v,w_1> \;+\; <v,w_2>$

4. $<v,v> \geq 0, \quad <v,v> = 0 \Longleftrightarrow v = 0$

Examples

1. In $\boldsymbol{R^n}$, $$<v, w> = v^T w$$

2. In $\boldsymbol{C^n}$, $$<v, w> = v^* w = \overline{v}^T w$$

Triangle inequality:

$$<v+w, v+w> \leq \left[<v,v>^{\frac{1}{2}} + <w,w>^{\frac{1}{2}} \right]^2$$

Inner product spaces

Let $\boldsymbol{V}$ be an inner product space. Then,

$$\|v\| \triangleq \langle v, v \rangle^{\frac{1}{2}}$$

qualifies as a norm on $\boldsymbol{V}$.

Schwartz inequality

$$|\ <v,w>\ |\ \le \|v\|\,\|w\|$$

1. In $\boldsymbol{R^n}$, $$|v^T w| \le \|v\|_2\,\|w\|_2$$

2. In $\boldsymbol{C^n}$, $$|v^* w| \le \|v\|_2\,\|w\|_2$$

$$v^* = \overline{v}^T$$ (complex conjugate transpose)

Inner product spaces

Two vectors $\boldsymbol{v}$, $\boldsymbol{w}$ are ***orthogonal*** if

$$< v, w > \; = 0$$

This is often written as

$$v \perp w$$

Inner product spaces

A set of vectors $\boldsymbol{S}$ is called *orthogonal* if

$$v \perp w \text{ for all } v, w \in \mathcal{S} \qquad v \neq w$$

it is ***orthonormal*** if, in addition, $\|v\|_2 = 1$ for all $\boldsymbol{v} \in \boldsymbol{S}$.

Linear operators

A linear operator is a mapping

$$\mathcal{A} : \mathbf{V} \longrightarrow \mathbf{W}$$

Such that for all $\boldsymbol{v}_1,\ \boldsymbol{v}_2 \in \boldsymbol{V}$ and all $\boldsymbol{\alpha} \in \boldsymbol{F}$:

1. (Additivity): $$\mathcal{A}(v_1 + v_2) = \mathcal{A}(v_1) + \mathcal{A}(v_2)$$

2. (Homogeneity): $$\mathcal{A}(\alpha\, v) = \mathcal{A}\,\phi(v)$$

Example: matrix multiplication

Let $\boldsymbol{v} \subset \boldsymbol{R^n}$ and $\boldsymbol{w} \subset \boldsymbol{R^m}$

$$w = \mathcal{A}(v) = A\,v$$

where

$$A \in \mathcal{R}^{m \times n} \qquad \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Example: Laplace Transform

$$F(s) = \mathcal{L}\{(f(t))\} \;\; = \;\; \int_0^{\infty} e^{-st} f(t) dt$$

Coordinate representation

- Given a linear operator

$$\mathcal{A} : \mathbf{V} \subset \mathcal{R}^n \rightarrow \mathbf{W} \subset \mathcal{R}^m$$

- Let $B = \{b_1, \cdots, b_n\}$ be a basis for ***V***

- Let $C = \{c_1, \cdots, c_m\}$ be a basis for ***W***

Coordinate representation

- $B = \{b_1, \cdots, b_n\}$ is a basis for ***V***

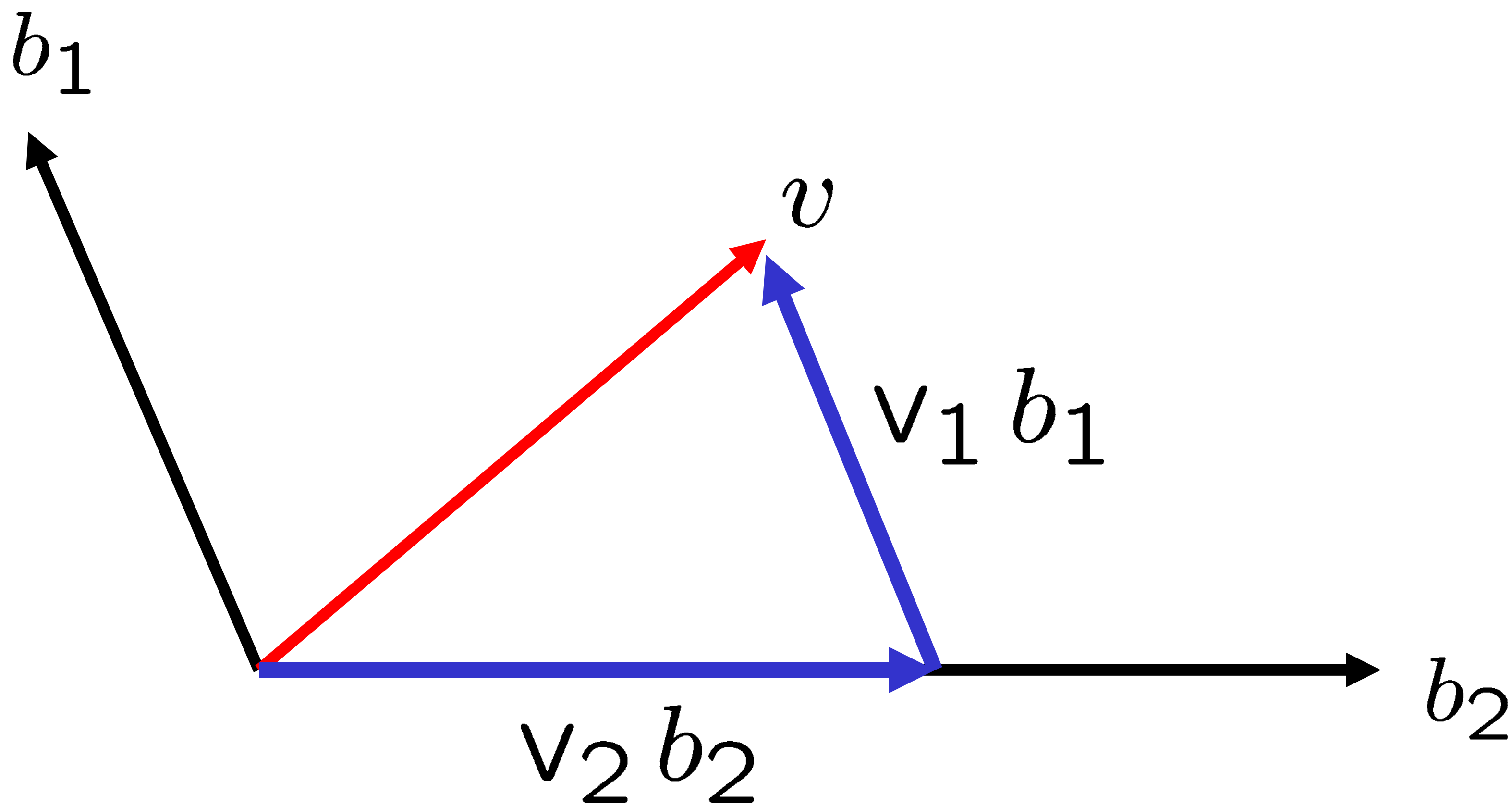
$$v \in V \qquad\qquad v = \sum_{j=1}^{n} \mathsf{v}_j\, b_j$$

- $C = \{c_1, \cdots, c_m\}$ is a basis for ***W***

$$w \in W \qquad\qquad w = \sum_{j=1}^{m} \mathsf{w}_j\, c_j$$

Coordinate representation

$$B = \{b_1,\, b_2\} \qquad v = \mathsf{v}_1\, b_1 + \mathsf{v}_2\, b_2$$

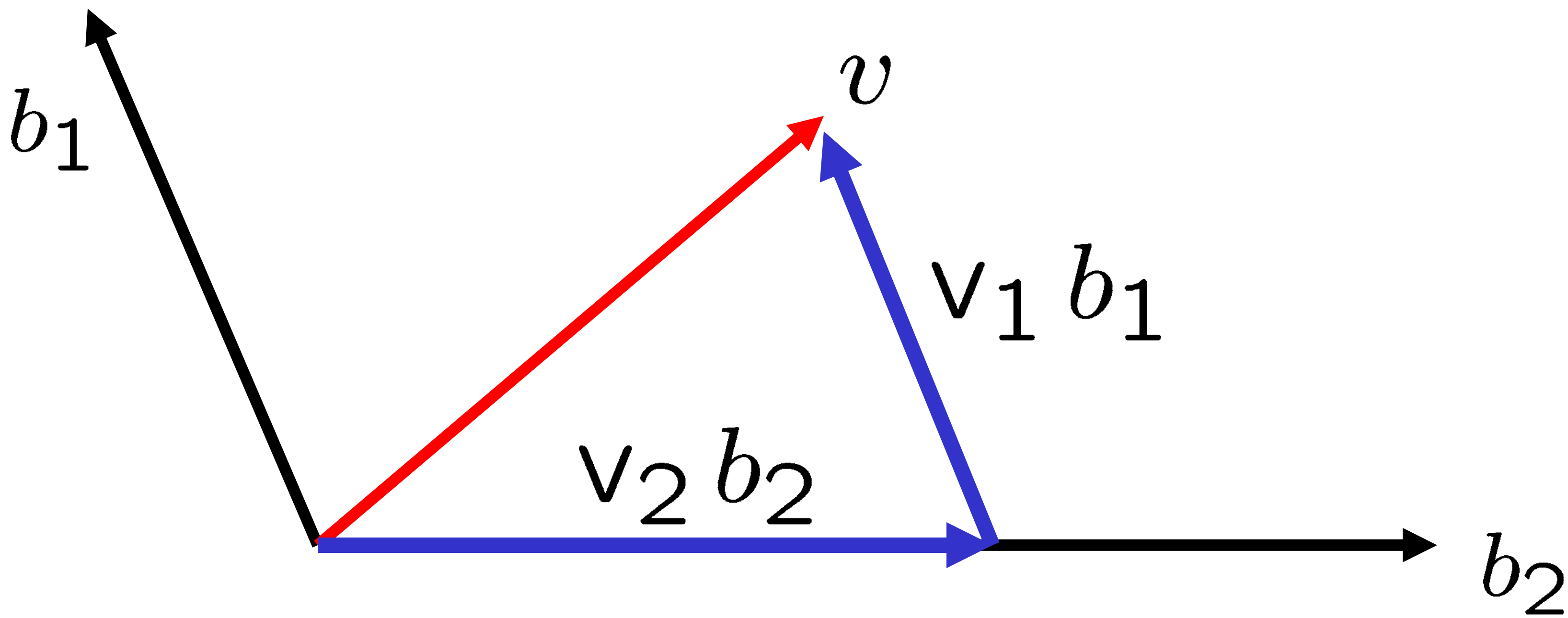


Coordinate representation

$$\mathrm{v} = \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix}$$

Coordinate representation of $\boldsymbol{v}$

In terms of the basis $\boldsymbol{B}$



Coordinate representation

- Example:

$$B = \left\{ b_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \end{bmatrix}, \cdots, b_n = \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} \right\}.$$

- For $v \in V$

$$v = v_1 \begin{bmatrix} 1 \\ 0 \\ . \\ . \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ . \\ . \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix}$$

Coordinate representations

$$\mathcal{A} : \mathbf{V} \subset \mathcal{R}^n \rightarrow \mathbf{W} \subset \mathcal{R}^m$$

- Suppose we express $\mathcal{A}(b_j)$ in the basis $\boldsymbol{C}$

$$\boxed{\mathcal{A}(b_j) = \sum_{i=1}^{m} \mathsf{a}_{ij}\, c_i} \qquad \mathsf{a}_{ij} \in \mathcal{R}$$

for each b_j in the basis B.

Coordinate representations

$$\mathcal{A} : \mathbf{V} \subset \mathcal{R}^n \rightarrow \mathbf{W} \subset \mathcal{R}^m$$

- Assume now that

$$w = \mathcal{A}(v)$$

$$w = \sum_{j=1}^{m} \mathsf{w}_j \, c_j \qquad v = \sum_{j=1}^{n} \mathsf{v}_j \, b_j$$

Coordinate representations

Then: $$w = \mathcal{A}(v)$$

$$\begin{aligned} w &= \mathcal{A}\left(\sum_{j=1}^{n} b_j \, \mathsf{v}_j\right) \\ &= \sum_{j=1}^{n} \mathcal{A}(b_j)\, v_j \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \mathsf{a}_{ij}\, c_i\right) \mathsf{v}_j \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \mathsf{a}_{ij}\mathsf{v}_j\right) c_i \end{aligned}$$

$$w = \sum_{i=1}^{m} \mathsf{w}_j\, c_j$$

Coordinate representations

Since

$$w = \mathcal{A}(v)$$

$$\sum_{i=1}^{m} \mathsf{w}_i \, c_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \mathsf{a}_{ij} \mathsf{v}_j \right) c_i$$

and $C = \{c_1 \cdots c_m\}$ is a basis

$$\boxed{\mathsf{w}_i = \sum_{j=1}^{n} \mathsf{a}_{ij} \mathsf{v}_j \,, \quad \text{for } i = 1 \cdots m}$$

Coordinate representations

Expanding each term in:

$$\mathsf{w}_i = \sum_{j=1}^{n} \mathsf{a}_{i,j}\mathsf{v}_j$$

Coordinate representation of $\boldsymbol{w}$ *w/r basis* $\boldsymbol{C}$

$$\underbrace{\begin{bmatrix} \mathsf{w}_1 \\ \vdots \\ \mathsf{w}_m \end{bmatrix}}_{\mathsf{w}} = \underbrace{\begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & \cdots & \vdots \\ \mathsf{a}_{m1} & \cdots & \mathsf{a}_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mathsf{v}_1 \\ \vdots \\ \mathsf{v}_n \end{bmatrix}}_{\mathsf{v}}$$

Coordinate representation of $\boldsymbol{v}$ *w/r basis* $\boldsymbol{B}$

Coordinate representations

Thus,

$$w = \mathcal{A}(v)$$

$$\underbrace{\begin{bmatrix} \mathsf{w}_1 \\ \vdots \\ \mathsf{w}_m \end{bmatrix}}_{\mathsf{w}} = \underbrace{\begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & \cdots & \vdots \\ \mathsf{a}_{m1} & \cdots & \mathsf{a}_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mathsf{v}_1 \\ \vdots \\ \mathsf{v}_n \end{bmatrix}}_{\mathsf{v}}$$

coordinate representation of $\mathcal{A}(\cdot)$ w/r to the bases ***B*** and ***C***

Coordinate representations

- v $\in \mathcal{R}^n$ is the coordinate representation of the vector v w/r to the basis ***B***
- w $\in \mathcal{R}^m$ is the coordinate representation of the vector $w = \mathcal{A}(v)$ w/r to the basis ***C***
- $A \in \mathcal{R}^{m \times n}$ is the coordinate representation of the linear map $\mathcal{A}(\cdot)$ w/r to the bases ***B*** and ***C***

$$\boxed{w = \mathcal{A}(v)}$$

$$\boxed{\mathrm{w} = A\,\mathrm{v}}$$

Coordinate representation

- Thus, the action of the linear operator $\mathcal{A}$

corresponds to a matrix multiplication as

$$\underbrace{\begin{bmatrix} \mathsf{w}_1 \\ \vdots \\ \mathsf{w}_m \end{bmatrix}}_{\mathsf{w}} = \underbrace{\begin{bmatrix} \mathsf{a}_{11} & \cdots & \mathsf{a}_{1n} \\ \vdots & \cdots & \vdots \\ \mathsf{a}_{m1} & \cdots & \mathsf{a}_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mathsf{v}_1 \\ \vdots \\ \mathsf{v}_n \end{bmatrix}}_{\mathsf{v}}$$

- The matrix A and vectors $\mathsf{w},\ \mathsf{v}$

depend on the particular choice of bases $\boldsymbol{B}$ *and* $\boldsymbol{C}$

Coordinate Transformations

- Let

$$B = \{b_1, \cdots, b_n\} \qquad \hat{B} = \left\{\hat{b}_1, \cdots, \hat{b}_n\right\}$$

be **<u>two</u>** bases for ***V*** and

$$\boxed{\hat{B} = B\,T}$$

$T \in \mathcal{R}^{n \times n}$ Is nonsingular

Coordinate representation of $v \in V$

- $\mathsf{v} \in \mathcal{R}^n$ is the coordinate representation of the vector v w/r to the basis B

$$v = B\,\mathsf{v}$$

- $\hat{\mathsf{v}} \in \mathcal{R}^n$ is the coordinate representation of the vector v w/r to the basis $\widehat{B}$

$$v = \widehat{B}\,\hat{\mathsf{v}}$$

Coordinate representation of $v \in V$

Since, $\boxed{\hat{B} = B\,T}$

$$v = B\,\mathsf{v} \qquad\qquad v = \hat{B}\,\hat{\mathsf{v}}$$

$$B\,\mathsf{v} = B\,T\,\hat{\mathsf{v}}$$

$$\boxed{\mathsf{v} = T\,\hat{\mathsf{v}}}$$

Coordinate Transformations

- $\mathsf{v} \in \mathcal{R}^n$ is the coordinate representation of the vector v w/r to the basis B

- $\hat{\mathsf{v}} \in \mathcal{R}^n$ is the coordinate representation of the vector v w/r to the basis $\widehat{B}$

$$\boxed{\widehat{B} = B\,T \qquad \Longleftrightarrow \qquad \mathsf{v} = T\,\hat{\mathsf{v}}}$$

$T \in \mathcal{R}^{n \times n}$ Is nonsingular

Coordinate Transformations

- Let

$$C = \{c_1, \cdots, c_m\} \qquad \widehat{C} = \{\widehat{c}_1, \cdots, \widehat{c}_m\}$$

be two bases for $\boldsymbol{W}$ and let

$$\boxed{\widehat{C} = C\,R}$$

$R \in \mathcal{R}^{m \times m}$ Is nonsingular

Coordinate Transformations

- $\mathsf{w} \in \mathcal{R}^m$ is the coordinate representation of the vector w w/r to the basis C

- $\hat{\mathsf{w}} \in \mathcal{R}^m$ is the coordinate representation of the vector w w/r to the basis $\widehat{C}$

$$\boxed{\widehat{C} = C\,R \quad \Longleftrightarrow \quad \mathsf{w} = R\,\hat{\mathsf{w}}}$$

$R \in \mathcal{R}^{n \times n}$ Is nonsingular

Similarity Transformations $\mathcal{A} : \mathbf{V} \to \mathbf{W}$

- Let A be the matrix representation of $\mathcal{A}(\cdot)$

w/r to the bases B and C

- Let $\hat{A}$ be the matrix representation of $\mathcal{A}(\cdot)$

w/r to the bases $\hat{B}$ and $\hat{C}$

$$\boxed{\hat{A} = R^{-1} A T}$$

$$\hat{C} = C R$$

$$\hat{B} = B T$$

Similarity Transformations $\mathcal{A} : \mathbf{V} \rightarrow \mathbf{W}$

- Let A be the matrix representation of $\mathcal{A}(\cdot)$

w/r to the bases B and C

- Let $\widehat{A}$ be the matrix representation of $\mathcal{A}(\cdot)$

w/r to the bases $\widehat{B}$ and $\widehat{C}$

$$\boxed{\widehat{A} = R^{-1} A T}$$

$$\mathsf{w} = R\,\widehat{\mathsf{w}}$$

$$\mathsf{v} = T\,\widehat{\mathsf{v}}$$

Similarity Transformations

- $$w = A v \qquad\qquad \hat{w} = \hat{A}\hat{v}$$

$$w = R\hat{w} \qquad v = T\hat{v}$$

$$R\hat{w} = AT\hat{v} \qquad\qquad \hat{w} = \hat{A}\hat{v}$$

$$\boxed{\hat{w} = R^{-1}AT\hat{v} \iff \hat{w} = \hat{A}\hat{v}}$$

True for all vectors w and v

$$\boxed{R^{-1}AT = \hat{A}}$$

Similarity Transformations

We now consider linear maps

$$\mathcal{A} : \mathbf{V} \rightarrow \mathbf{V}$$

That are represented by **<u>square</u>** matrices

$$A \in \mathcal{R}^{n \times n} \qquad \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Similarity Transformations $\mathcal{A} : \mathbf{V} \rightarrow \mathbf{V}$

- Let A be the matrix representation of $\mathcal{A}(\cdot)$

w/r to the basis B

- Let $\hat{A}$ be the matrix representation of $\mathcal{A}(\cdot)$

w/r to the bases $\hat{B}$

$$\boxed{\hat{A} = T^{-1} A T}$$

$$\hat{B} = B T$$

$$\mathsf{v} = T \hat{\mathsf{v}}$$

Matrix properties

Let A and B be matrices of compatible dimensions

(a) $(A^*)^* = A$

(b) $(A+B)^* = A^* + B^*$

(c) $(AB)^* = B^* A^*$

(d) If A is nonsingular, then $(A^*)^{-1} = (A^{-1})^*$

Where $A^* = (\overline{A})^T$ is the complex conjugate transpose

Null Space

Let $\mathcal{A} : \mathbf{V} \longrightarrow \mathbf{W}$ be a linear operator

- **Null space of A:** is the set

$$\mathcal{N}(A) = \{x \in \mathbf{V} : \ \mathcal{A}(x) = 0\}$$

Range Space

Let $\mathcal{A} : \mathbf{V} \longrightarrow \mathbf{W}$ be a linear operator

- **Range space of A:** is the set

$$\mathcal{R}(\mathcal{A}) = \{y \in \mathbf{W} : \ \mathcal{A}(x) = y \text{ for some } x \in \mathbf{V}\}$$

$$\mathcal{R}(\mathcal{A}) = \mathcal{A}(\mathbf{V}) \subset \mathbf{W}$$

Range and Null Space

Let $\mathcal{A} : \mathbf{V} \subset \mathcal{R}^n \longrightarrow \mathbf{W} \subset \mathcal{R}^m$

be a linear operator represented by the matrix multiplication

$$w = A\,v$$

Range and Null Space

$\mathcal{R}(\mathcal{A})$ is the set of all linear combinations of the columns of *A*

$$\mathcal{R}(\mathcal{A}) = \mathcal{SP}\{\text{columns of } A\}$$

span

$\mathcal{N}(\mathcal{A})$ is the set of all linearly independent vectors

$$A\,x = 0$$

Subspaces

Let ***V*** be a vector space.

A subset $\mathbf{W} \subset \mathbf{V}$ is a **subspace** if it is also a vector space,

$$\alpha_1 v_1, + \alpha_2 v_2 \in \mathbf{W}$$

$\forall v_1,\, v_2 \in \mathbf{W}$ $\forall \alpha_1,\, \alpha_2 \in \mathbf{F}$ (Field)

Subspaces

Let $\boldsymbol{V}$ be a vector space and

$$S = \{v_i \ : \ i \in I \subset Z\}$$

be a set of vectors drawn from $\boldsymbol{V}$ vector

Then, the span of $\boldsymbol{S}$ is a subspace of $\boldsymbol{V}$.

$\mathcal{SP}(S)$: span of $\boldsymbol{S}$

Orthogonal complement

Let $\mathcal{S}$ be a subspace of a vector space $\boldsymbol{V}$.

- The *orthogonal complement of* $\mathcal{S}$ is the set $\mathcal{S}^{\perp}$ defined by

$$\mathcal{S}^{\perp} = \{v \in \mathbf{V} : \ v \perp \mathcal{S}\}$$

- I.e. $\forall\, v \in \mathcal{S}^{\perp}$ and $\forall\, w \in \mathcal{S}$ then

$$v^{*}w = 0$$

Orthogonal complement

Let $\boldsymbol{V}$ be a **finite dimensional** vector space

$\mathcal{S}$ is subspace of $\boldsymbol{V}$ and $\mathcal{S}^{\perp}$ its orthogonal complement.

Then:

- $$\mathbf{V} = \mathcal{S} + \mathcal{S}^{\perp}$$

- $$\mathcal{S} = (\mathcal{S}^{\perp})^{\perp}$$

Orthogonal complement

Let

$$\mathbf{V} = \mathcal{S} + \mathcal{S}^{\perp}$$

Any vector $v \in \mathbf{V}$ can be expressed as:

$$v = v_{\mathcal{S}} + v_{\mathcal{S}^{\perp}}$$

$$v_{\mathcal{S}} \in \mathcal{S} \qquad\qquad v_{\mathcal{S}^{\perp}} \in \mathcal{S}^{\perp}$$

Range and Null Space

Let $A \in \mathcal{C}^{m \times n}$

(a) $\mathcal{R}^{\perp}(A) = \mathcal{N}(A^*)$

(b) $\mathcal{C}^m = \mathcal{R}(A) + \mathcal{N}(A^*)$

(c) $\mathcal{N}(A^* A) = \mathcal{N}(A)$

(d) $\mathcal{R}(AA^*) = \mathcal{R}(A)$

Range and Null Space

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Lets verify:

$$\boxed{\text{(a) } \mathcal{R}^{\perp}(A) = \mathcal{N}(A^T)}$$

$$\mathcal{R}(A) = \mathcal{SP}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \Rightarrow \qquad \mathcal{R}^{\perp}(A) = \emptyset$$

$$A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \mathcal{N}(A^T) = \emptyset$$

Prove:

$$\text{(a) } \mathcal{R}^{\perp}(A) = \mathcal{N}(A^*)$$

Part I:

$$\mathcal{N}(A^*) \subset \mathcal{R}^{\perp}(A) :$$

$$\forall\, n \in \mathcal{N}(A^*) \;\Rightarrow A^* n = 0 \;\Rightarrow n^* A = 0$$

Therefore, $\boldsymbol{n}$ is orthogonal to all columns of $\boldsymbol{A}$

$$n \in \mathcal{R}^{\perp}(A)$$

Prove: (a) $\mathcal{R}^{\perp}(A) = \mathcal{N}(A^*)$

Part II: $\boxed{\mathcal{R}^{\perp}(A) \subset \mathcal{N}(A^*) :}$

Let $\boldsymbol{a_i}$ be the ith column of $\boldsymbol{A}$

$$\forall\, r \in \mathcal{R}^{\perp}(A) \Rightarrow r^* \sum_{i=1}^{n} \alpha_i\, a_i = 0 \Rightarrow$$

$$r^* \underbrace{\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}}_{A} = 0 \Rightarrow A^* r = 0$$

$$\Rightarrow r \in \mathcal{N}(A^*)$$

Range and Null Space

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Lets verify:

$$\boxed{\text{(b) } \mathcal{R}^2 = \mathcal{R}(A) + \mathcal{N}(A^T)}$$

$$\mathcal{R}(A) = \mathcal{SP}(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) \qquad \mathcal{N}(A^T) = \emptyset$$

$$\mathcal{R}^2 = \mathcal{SP}(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

Prove: (b) $\mathcal{C}^m = \mathcal{R}(A) + \mathcal{N}(A^*)$

Remember that $\boldsymbol{R(A)}$ is the span of the columns of $\boldsymbol{A}$

- $\boldsymbol{R(A)}$ is a subspace of $\boldsymbol{C^m}$
- by property (a) $\mathcal{N}(A^*) = \mathcal{R}^\perp(A)$
- (b) is proven from $\mathcal{C}^m = \mathcal{R}(A) + \mathcal{R}^\perp(A)$

Prove: (c) $\mathcal{N}(A^*A) = \mathcal{N}(A)$

Part I:

$$\boxed{\mathcal{N}(A) \subset \mathcal{N}(A^*A) :}$$

Let $n \in \mathcal{N}(A) \Rightarrow An = 0$

Therefore: $A^*(An) = A^*0 = 0$

$$\Rightarrow\ n \in \mathcal{N}(A^*A)$$

Verify: $\mathcal{N}(A^T A) \subset \mathcal{N}(A)$

Let $$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathcal{N}(A) = \mathcal{SP}(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix})$$

$$n = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$A^T A \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = A^T (A \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}) = A^T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Prove: (c) $\mathcal{N}(A^*A) = \mathcal{N}(A)$

Part II: $\boxed{\mathcal{N}(A^*A) \subset \mathcal{N}(A):}$ *(by contradiction)*

Assume $m \in \mathcal{N}(A^*A)$ but $m \notin \mathcal{N}(A)$

$A^*Am = 0$ $\qquad\qquad$ $Am \neq 0$

Define $w = Am$,

Prove: (c) $\mathcal{N}(A^*A) = \mathcal{N}(A)$

$$\boxed{w = Am} \qquad A^*w = 0$$

$$A^*w = 0 \Rightarrow w \in \mathcal{N}(A^*)$$

By (b) $\Rightarrow w \in \mathcal{R}^{\perp}(A) \Rightarrow \boxed{w \notin \mathcal{R}(A)}$

Contradiction: $w = Am \Rightarrow w \in \mathcal{R}(A)$

Prove: (d) $\mathcal{R}(AA^*) = \mathcal{R}(A)$

Notice that by property (a):

$$\mathcal{R}^{\perp}(A) = \mathcal{N}(A^*)$$

$$\mathcal{R}^{\perp}(AA^*) = \mathcal{N}(AA^*)$$

Therefore, by property (c)

$$\mathcal{N}(AA^*) = \mathcal{N}(A^*) \Rightarrow \mathcal{R}^{\perp}(AA^*) = \mathcal{R}^{\perp}(A)$$

Prove:

(d) $\mathcal{R}(AA^*) = \mathcal{R}(A)$

Since,

$$\mathcal{R}^{\perp}(AA^*) = \mathcal{R}^{\perp}(A)$$

Taking orthogonal complements of both finite dimensional subspaces,

$$(\mathcal{R}^{\perp}(AA^*))^{\perp} = (\mathcal{R}^{\perp}(A))^{\perp}$$

$$\mathcal{R}(AA^*) = \mathcal{R}(A)$$

Verify:

(d) $\mathcal{R}(AA^*) = \mathcal{R}(A)$

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad AA^* = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\mathcal{R}(A) = \mathcal{SP}(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \mathcal{R}^2$$

$$\mathcal{R}(AA^*) = \mathcal{SP}(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \mathcal{R}^2$$

Rank and Nullity of a matrix

Let $A \in \mathcal{C}^{m \times n}$ (m rows and n columns)

- **The *rank* of a matrix A is the dimension of $\boldsymbol{R(A)}$,**

i.e. the number of linearly independent vectors that can be extracted from the columns of A.

- **The *nullity* of a matrix A is the dimension of $\boldsymbol{N(A)}$,**

i.e. n – {number of linearly independent vectors that can be extracted from the rows of A}.

Rank and Nullity of a matrix

Let $A \in \mathcal{C}^{m\times n}$ and $B \in \mathcal{C}^{n\times r}$

(a) $\text{rank}(A) = \text{rank}(A^*)$

(b) $\text{rank}(A) \leq \min\{m, n\}$

(c) $\text{rank}(A) + \text{nullity}(A^*) = m$

(d) $\text{rank}(A^*) + \text{nullity}(A) = n$

(e) $\text{rank}(A) = \text{rank}(AA^*) = \text{rank}(A^*A)$

(f) (Sylvester's inequality)

$$\text{rank}(A)+\text{rank}(B)-n \leq \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$