[MEN573]
Advanced Control Systems I

Lecture 16
Kalman Canonical Decompositions

Associate Professor Joonbum Bae
Department of Mechanical Engineering
UNIST

Subspaces

Let $\boldsymbol{V}$ be a vector space and $\mathcal{F}$ a field.

A subset $\mathbf{W} \subset \mathbf{V}$

is called a ***subspace*** if $\boldsymbol{W}$ is itself a vector space.

Theorem: A set $\mathbf{W} \subset \mathbf{V}$ is a subspace if and only if it is close under vector addition and scalar multiplication.

i.e.

$$\boxed{\begin{array}{c} \alpha w_1 \in \mathbf{W},\ \ w_1 + w_2 \in \mathbf{W} \\ \\ \text{for all } \alpha \in \mathcal{F},\ \ w_1, w_2 \in \mathbf{W} \end{array}}$$

Span and orthogonal complement

Let $\mathcal{S} = \{v_1,\, v_2 \cdots\}$ be a set of vectors drawn from a subspace ***V***.

- $\mathsf{Span}\{\mathcal{S}\}$ is the set of all finite linear combinations of vectors in $\mathcal{S}$
- It is easy to verify that $\mathsf{Span}\{\mathcal{S}\}$ is a subspace of ***V***.
- Let ***W*** be a subspace of ***V***, then

$$\mathbf{W}^{\perp} = \{v \in \mathbf{V} \mid v^{*}w = 0\,,\ \forall w \in \mathbf{W}\}$$

is its orthogonal complement, which is also a subspace of ***V***.

Range Space and Null Space

Given the matrix $M \in \mathcal{R}^{m \times n}$

(m rows and n columns),

- Range space of $\boldsymbol{M}$

$$\mathcal{R}\{M\} = \text{Span}\{\text{ columns of } M\}$$

- Null space of $\boldsymbol{M}$

$$\mathcal{N}\{M\} = \{v \in \mathcal{R}^n \,;\; M\,v = 0\}$$

$$\mathcal{N}\{M\} = \mathcal{R}^{\perp}\{M^T\}$$

(orthogonal complement of $\mathcal{R}\{M^T\}$)

Controllable subspace

Consider an uncontrollable LTI discrete time system of order $\boldsymbol{n}$

$$x(k+1) = A\,x(k) + B\,u(k)$$

such that the controllability matrix

$$P = \begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix}$$

has rank $\mathsf{Rank}\{P\} = n_1 < n$

Controllable subspace

The controllable subspace $\mathcal{X}_c$ is the set of all vectors $x \in \mathcal{R}^n$ that can be reached from the origin ***0***.

Notice that the controllable subspace is equal to the range space of the controllability matrix,

$$\boxed{\mathcal{X}_c = \mathcal{R}\{P\}}$$

Note: For zero initial condition,

$$x(n) = A^n x_0 + P \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Controllable subspace

One can define the orthogonal complement of the controllable subspace.

$$\mathcal{X}_c^{\perp} = \mathcal{N}(P^T)$$

i.e. the set of all vectors that are orthogonal to the columns of the controllability matrix, and

$$\mathcal{R}^n = \mathcal{X}_c^{\perp} + \mathcal{X}_c$$

All vectors in $\mathcal{R}^n$ can be expressed as linear combinations of vectors in $\mathcal{X}_c$ and $\mathcal{X}_c^{\perp}$

Unobservable subspace

Consider an unobservable LTI discrete time system of order $\boldsymbol{n}$

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

such that the observability matrix

$$Q = \begin{bmatrix} C \\ C\,A \\ \vdots \\ C\,A^{n-1} \end{bmatrix}$$

has rank $\mathsf{Rank}\{Q\} = n_2 < n$

Unobservable subspace

The unobservable subspace $\mathcal{X}_{uo}$ is the set of all nonzero initial conditions $x(0) \in \mathcal{R}^n$ which produce a zero free response

$$y_{free}(k) = 0 \qquad \forall k \geq 0$$

Notice that the unobservable subspace is equal to the null space of the observability matrix,

$$\boxed{\mathcal{X}_{uo} = \mathcal{N}\{Q\}}$$

Note: $$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q} x_o$$

Unobservable subspace

One can define the orthogonal complement of the unobservable subspace.

$$\mathcal{X}_{uo}^{\perp} = \mathcal{R}(Q^T)$$

i.e. the span of the rows of the observability matrix

$$\mathcal{R}^n = \mathcal{X}_{uo}^{\perp} + \mathcal{X}_{uo}$$

All vectors in $\mathcal{R}^n$ can be expressed as linear combinations of vectors in $\mathcal{X}_{uo}$ and $\mathcal{X}_{uo}^{\perp}$

Similarity transformations

Consider a LTI discrete time system of order $\boldsymbol{n}$

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

Define the similarity coordinate transformation

$$x'(k) = T^{-1}\,x(k)$$

The new state space realization is given by

$$\begin{aligned} x'(k+1) &= A'\,x'(k) + B'\,u(k) \\ y(k) &= C'\,x'(k) + D\,u(k) \end{aligned}$$

Similarity transformations

where

$$A' = T^{-1} A T \qquad B' = T^{-1} B$$
$$C' = C T$$

Facts:

$\boxed{\{ A, B \} \text{ is controllable } \textbf{ IFF } \{ A', B' \} \text{ is controllable}}$

$\boxed{\{ A, C \} \text{ is observable } \textbf{ IFF } \{ A', C' \} \text{ is observable}}$

Similarity transformations

Proof: (a similar proof can be done for observability)

The controllability matrix of the pair **{ *A, B* }** is

$$P = \begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix}$$

while the controllability matrix of the pair **{ *A', B'* }** is

$$P' = \left[B' \mid A'B' \mid \cdots \mid A'^{n-1}B' \right]$$

$$P' = \left[\overbrace{T^{-1}B} \mid \overbrace{T^{-1}A\underbrace{TT^{-1}}_{=I}} \overbrace{B} \mid \cdots \mid T^{-1}A^{n-1}\underbrace{TT^{-1}}_{=I}B \right]$$

$$P' = T^{-1}\left[B \mid AB \mid \cdots \mid A^{n-1}B \right]$$

$$= T^{-1}P$$

Similarity transformations

Thus, the controllability $\boldsymbol{P}$ matrix of the pair $\{\boldsymbol{A}, \boldsymbol{B}\}$

and the controllability matrix $\boldsymbol{P'}$ of the pair $\{\boldsymbol{A'}, \boldsymbol{B'}\}$

are related by

$$P' = T^{-1} P$$

Since $\boldsymbol{T}$ is rank $\boldsymbol{n}$, then

$$\boxed{\mathsf{Rank}\{P'\} = \mathsf{Rank}\{P\}}$$

Kalman canonical form (controllability)

Consider an **uncontrollable** LTI discrete time system of order ***n***

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) \\
y(k) &= C\,x(k) + D\,u(k)
\end{aligned}
$$

such that the controllability matrix

$$P = \left[B \;\; AB \;\; A^2B \;\; \cdots A^{n-1}B \right]$$

has rank $\boxed{\mathsf{Rank}\{P\} = n_1 < n}$

Kalman canonical form (controllability)

Define the following notation:

For the matrix $\boldsymbol{M}$,

$\{ \boldsymbol{m}_i \}$ is the set of all columns of $\boldsymbol{M}$

Kalman canonical form (controllability)

We now define the following similarity transformation matrix:

$$M = [\underbrace{m_1\, m_2 \cdots\ m_{n1}}_{M_c}\ \ \underbrace{m_{n1+1} \cdots m_n}_{M_{uc}}]$$

where

- The first $\boldsymbol{n_1}$ columns of $\boldsymbol{M}$ are $\boldsymbol{n_1}$ linearly independent columns of $\boldsymbol{P}$. Thus,

$$M_c \in \mathcal{R}^{n \times n1}$$

$$\boxed{\text{Rank}\{M_c\} = n_1 \qquad \{P_i\} \subset \mathcal{R}\{M_c\}}$$

Kalman canonical form (controllability)

$$M = [\underbrace{m_1\, m_2 \cdots\, m_{n1}}_{M_c}\ \underbrace{m_{n1+1} \cdots m_n}_{M_{uc}}]$$

- The remaining n-n_1 columns of M are selected so that M is rank n (i.e. invertible), I.e.

$$M_{uc} \in \mathcal{R}^{n\times(n-n1)}$$

$$\boxed{\mathsf{Rank}\{M_{uc}\} = n - n_1}$$

$$\boxed{\mathcal{R}\{M_c\} + \mathcal{R}\{M_{uc}\} = \mathcal{R}^n}$$

Kalman canonical form (controllability)

The Kalman canonical basis are given by

$$\boxed{\bar{x} = \begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix} = M^{-1}x}$$

where $\bar{x}_c \in \mathcal{R}^{n_1}$ $\bar{x}_{uc} \in \mathcal{R}^{n-n_1}$ and

$$\boxed{\begin{aligned} \begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}_{\bar{B}} u(k) \\ y(k) &= \underbrace{\begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k) \end{aligned}}$$

Kalman canonical form (controllability)

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}_{\bar{B}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

Notice that, in the Kalman canonical basis,

$$\begin{bmatrix} \bar{x}_c \\ 0 \end{bmatrix} \in \mathcal{X}_c \qquad \begin{bmatrix} 0 \\ \bar{x}_{uc} \end{bmatrix} \in \mathcal{X}_c^{\perp}.$$

Kalman canonical form (controllability)

The n_1 th order subsystem

$$\begin{aligned} \bar{x}_c(k+1) &= \bar{A}_c\,\bar{x}_c(k) + \bar{B}_c\,u(k) \\ y(k) &= \bar{C}_c\,\bar{x}_c(k) + D\,u(k) \end{aligned}$$

is controllable; i.e. the pair $\{\bar{A}_c,\ \bar{B}_c\}$ is controllable; i.e. defining,

$$\bar{P} = \left[\bar{B}_c\ \ \bar{A}_c\bar{B}_c\ \ \cdots \bar{A}_c^{n1-1}\bar{B}_c\right]$$

$$\text{Rank}\{\bar{P}\} = n_1$$

Stabilizability

The system is stabilizable if **all** its unstable modes (if any) are controllable.

i.e., either the system is controllable, or in the Kalman canonical realization,

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}_{\bar{B}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

$\bar{A}_{uc}$ has no unstable modes

Kalman canonical form (controllability)

Notice that

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

where $\boxed{x \in \mathcal{R}^n}$ and

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}_{\bar{B}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

where $\boxed{\bar{x}_c \in \mathcal{R}^{n_1}}$ $\bar{x}_{uc} \in \mathcal{R}^{n-n_1}$

Kalman canonical form (controllability)

Assume for simplicity a SISO system,

$$G(z) = \frac{Y(z)}{U(z)}$$

Then, $$G(z) \;=\; C[zI - A]^{-1}B + D = \frac{B(z)}{A(z)}$$

Also,

$$\begin{aligned} G(z) &= \bar{C}[zI - \bar{A}]^{-1}\bar{B} + D \\ &= \bar{C}_c[zI - \bar{A}_c]^{-1}\bar{B}_c + D \\ &= \frac{\bar{B}_c(z)}{\bar{A}_c(z)} \end{aligned}$$

Kalman canonical form (controllability)

Thus,

$$G(z) \;=\; C[zI - A]^{-1}B + D = \frac{B(z)}{A(z)}$$

$$G(z) \;=\; \bar{C}_c[zI - \bar{A}_c]^{-1}\bar{B}_c + D = \frac{\bar{B}_c(z)}{\bar{A}_c(z)}$$

where $A(z) = \det[zI - A]$ is $\boldsymbol{n}$ th order

and $\bar{A}_c(z) = \det[zI - \bar{A}_c]$ is $\boldsymbol{n_1}$ th order

$A(z),\, B(z)$ are not co-prime and pole-zero cancellation takes place

Kalman canonical form (observability)

Consider an unobservable LTI discrete time system of order $\boldsymbol{n}$

$$
\begin{aligned}
x(k+1) &= A\,x(k) + B\,u(k) \\
y(k) &= C\,x(k) + D\,u(k)
\end{aligned}
$$

such that the observability matrix

$$
Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}
$$

has rank $\mathsf{Rank}\{Q\} = n_2 < n$

Kalman canonical form (observability)

We now define the following similarity transformation matrix:

$$O = \begin{bmatrix} O_o \\ O_{uo} \end{bmatrix} \begin{array}{l} \} \, n_2 \text{ rows} \\ \} \, (n - n_2) \text{ rows} \end{array}$$

where

- The first $\boldsymbol{n_2}$ rows of $\boldsymbol{O}$ are $\boldsymbol{n_2}$ linearly independent rows of $\boldsymbol{Q}$. Thus,

$$O_o \in \mathcal{R}^{n_2 \times n}$$

$$\boxed{\text{Rank}\{O_o\} = n_2 \qquad\qquad \mathcal{R}\{O_o^T\} = \mathcal{R}\{Q^T\}}$$

Kalman canonical form (observability)

$$O = \begin{bmatrix} O_o \\ O_{uo} \end{bmatrix} \begin{array}{l} \} \; n_2 \text{ rows} \\ \} \; (n - n_2) \text{ rows} \end{array}$$

The remaining ***n-n₂*** rows of ***O*** are selected so that ***O*** is rank ***n*** (i.e. invertible)**.** I.e.

$$O_{uo} \in \mathcal{R}^{(n-n2)\times n}$$

$$\boxed{\text{Rank}\{O_{uo}\} = n - n_2} \qquad \mathcal{R}\{O_o^T\} + \mathcal{R}\{O_{uo}^T\} = \mathcal{R}^n$$

Kalman canonical form (observability)

The Kalman canonical basis are given by

$$\boxed{\bar{x} = \begin{bmatrix} \bar{x}_o \\ \bar{x}_{uo} \end{bmatrix} = Ox}$$

where $\bar{x}_o \in \mathcal{R}^{n_2}$ $\bar{x}_{uo} \in \mathcal{R}^{n-n_2}$ and

$$\boxed{\begin{aligned} \begin{bmatrix} \bar{x}_o(k+1) \\ \bar{x}_{uo}(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix}}_{\bar{B}} u(k) \\ y(k) &= \underbrace{\begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + Du(k) \end{aligned}}$$

Kalman canonical form (observability)

$$\begin{bmatrix} \bar{x}_o(k+1) \\ \bar{x}_{uo}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix}}_{\bar{B}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + Du(k)$$

Notice that, in the Kalman canonical basis,

$$\begin{bmatrix} 0 \\ \bar{x}_{uo} \end{bmatrix} \in \mathcal{X}_{uo} \qquad \begin{bmatrix} \bar{x}_o \\ 0 \end{bmatrix} \in \mathcal{X}_{uo}^{\perp}.$$

Kalman canonical form (observability)

The $\boldsymbol{n_2}$ th order subsystem

$$\begin{aligned} \bar{x}_O(k+1) &= \bar{A}_O \bar{x}_O(k) + \bar{B}_O u(k) \\ y(k) &= \bar{C}_O \bar{x}_O(k) + D u(k) \end{aligned}$$

is observable,

i.e. the pair $\{\bar{A}_O,\ \bar{C}_O\}$ is observable.

Detectability

The system is detectable if **all** its unstable modes (if any) are observable.

i.e., either the system is observable, or in the Kalman canonical realization,

$$\begin{bmatrix} \bar{x}_o(k+1) \\ \bar{x}_{uo}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix}}_{\bar{B}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + Du(k)$$

$\boxed{\bar{A}_{uo} \quad \text{has no unstable modes}}$

Kalman canonical form (observability)

Assume for simplicity a SISO system,

$$G(z) = \frac{Y(z)}{U(z)}$$

Then,

$$G(z) \;=\; C[zI - A]^{-1}B + D = \frac{B(z)}{A(z)}$$

Also,

$$\begin{aligned} G(z) &= \bar{C}[zI - \bar{A}]^{-1}\bar{B} + D \\ &= \bar{C}_o[zI - \bar{A}_o]^{-1}\bar{B}_o + D \\ &= \frac{\bar{B}_o(z)}{\bar{A}_o(z)} \end{aligned}$$

Kalman canonical form (observability)

Thus,

$$G(z) \;=\; C[zI-A]^{-1}B + D = \frac{B(z)}{A(z)}$$

$$G(z) \;=\; \bar{C}_c[zI-\bar{A}_o]^{-1}\bar{B}_o + D = \frac{\bar{B}_o(z)}{\bar{A}_o(z)}$$

where $A(z) = \det[zI-A]$ is $\boldsymbol{n}$ th order

and $\bar{A}_o(z) = \det[zI-\bar{A}_o]$ is $\boldsymbol{n_2}$ th order

$A(z),\, B(z)$ are not co-prime and pole-zero cancellation takes place

Kalman canonical form (controllability)

$$\begin{aligned} \begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}_{\bar{B}} u(k) \\ y(k) &= \underbrace{\begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k) \end{aligned}$$

Notice that, in the Kalman canonical basis,

$$\begin{bmatrix} \bar{x}_c \\ 0 \end{bmatrix} \in \mathcal{X}_c \qquad \begin{bmatrix} 0 \\ \bar{x}_{uc} \end{bmatrix} \in \mathcal{X}_c^{\perp} \,.$$

Kalman Canonical form (controllability)

Proof:

Because the matrix $M = [M_c \;\; M_{uc}]$ is invertible and

$$\{P_i\} \subset \mathcal{R}\{M_c\}$$

All columns of P are linear combinations of the columns of M_c and vice versa.

All columns of M_{uc} are linearly independent from the columns of P

Kalman Canonical form (controllability)

Remember that:

$$\bar{A} = M^{-1} A M \qquad \bar{B} = M^{-1} B$$

$$\bar{C} = C M$$

We want to show that:

1) $$\bar{B} = M^{-1} B = \begin{bmatrix} \bar{B}_c \\ \bar{B}_2 \end{bmatrix} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

2) $$\bar{A} = M^{-1} A M = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{uc} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}$$

Kalman Canonical form (controllability)

Write:

$$M^{-1} = \begin{array}{l} \left[\begin{array}{c} M_{inv-c} \\ M_{inv-uc} \end{array} \right] \end{array} \begin{array}{l} \} \, n_1 \text{ rows} \\ \} \, (n - n_1) \text{ rows} \end{array}$$

$$M = [\quad \underbrace{M_c}_{n_1 \, columns} \quad \underbrace{M_{uc}}_{n-n_1 \, columns} \quad]$$

Then:

$$M^{-1}M \; = \; I \; \Rightarrow$$

$$M^{-1}M_c = \left[\begin{array}{c} I_{n1} \\ 0 \end{array} \right] \begin{array}{l} \} \, n_1 \text{ rows} \\ \} \, (n - n_1) \text{ rows} \end{array}$$

Kalman Canonical form (controllability)

$$M^{-1}M_c = \begin{bmatrix} I_{n1} \\ 0 \end{bmatrix} \begin{matrix} \}\, n_1 \text{ rows} \\ \}\, (n-n_1) \text{ rows} \end{matrix}$$

For any $m \in Span\{M_c\}$

$$M^{-1}m = \begin{bmatrix} \times \\ 0 \end{bmatrix} \begin{matrix} \}\, n_1 \text{ rows} \\ \}\, (n-n_1) \text{ rows} \end{matrix}$$

Now, note that each column of *B* and each column of AM_c belongs to $Span\{M_c\}$

Kalman Canonical form (controllability)

For any $m \in Span\{M_c\}$

$$M^{-1}m = \begin{bmatrix} \times \\ 0 \end{bmatrix} \begin{array}{l} \} \, n_1 \text{ rows} \\ \} \, (n - n_1) \text{ rows} \end{array}$$

Now, note that each column of $\boldsymbol{B}$ and each column of $\boldsymbol{AM_c}$ belongs to $Span\{M_c\}$

1) $\bar{B} = M^{-1} B = \begin{bmatrix} \bar{B}_c \\ \bar{B}_2 \end{bmatrix} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$

2) $\bar{A} = M^{-1} A M = \begin{bmatrix} M^{-1}AM_c & M^{-1}AM_{uc} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}$

Kalman canonical form (controllability)

$$\boxed{\begin{aligned} \begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}_{\bar{B}} u(k) \\ y(k) &= \underbrace{\begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k) \end{aligned}}$$

Notice that, in the Kalman canonical basis,

$$\boxed{\begin{bmatrix} \bar{x}_c \\ 0 \end{bmatrix} \in \mathcal{X}_c \qquad \begin{bmatrix} 0 \\ \bar{x}_{uc} \end{bmatrix} \in \mathcal{X}_c^{\perp}\ .}$$

Kalman canonical form (controllability)

The $\boldsymbol{m}$ th order subsystem

$$\begin{aligned} \bar{x}_c(k+1) &= \bar{A}_c\,\bar{x}_c(k) + \bar{B}_c\,u(k) \\ y(k) &= \bar{C}_c\,\bar{x}_c(k) + D\,u(k) \end{aligned}$$

is controllable, i.e. the pair $\{\bar{A}_c,\ \bar{B}_c\}$ is controllable.

Proof:

$$\bar{A} = M^{-1}\,A\,M, \quad \bar{B} = M^{-1}\,B,$$

Since

$$\text{Rank}\{P\} = n_1$$

and

$\bar{P} = M^{-1}\,P$ and $\boxed{\text{Rank}\{\bar{P}\} = n_1}$

Kalman canonical form (controllability)

Computing $\bar{P} = \left[\begin{array}{cccc} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{array} \right]$

we obtain

$$\bar{P} = \left[\begin{array}{cccc} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c{}^{n-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 \end{array} \right]$$

Since $\bar{A}_c \in \mathcal{R}^{n1 \times n1}$,by the Cayley-Hamilton theorem,

$$\mathsf{Span}\{\bar{P}\} = \mathsf{Span}\left\{ \left[\begin{array}{cccc} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c{}^{n1-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 \end{array} \right] \right\}$$

Kalman canonical form (controllability)

Thus,

$$\text{Span}\{\bar{P}\} = \text{Span}\left\{\begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c{}^{n1-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right\}$$

implies

$$\boxed{\text{Rank}\{\bar{P}\} = \text{Rank}\left\{\underbrace{\begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c{}^{n1-1}\bar{B}_c \end{bmatrix}}_{\bar{P}_c}\right\}}$$

where $\bar{P}_c$ is the controllability matrix of the pair $\{\bar{A}_c, \bar{B}_c\}$

Thus, since $\text{Rank}\{\bar{P}\} = n_1$, then $\text{Rank}\{\bar{P}_c\} = n_1$

and the pair $\{\bar{A}_c, \bar{B}_c\}$ is controllable.

Remarks

- The decompositions in this lecture applies to both discrete time systems and continuous time systems.

- The controllable subsystem of the controllability Kalman canonical form may be decomposed to the observable subsystem and unobservable subsystem.

Kalman Canonical Form (Theorem K-3)

An n-th order discrete time system

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

may be transformed to

$$\begin{bmatrix} \bar{x}_{cuo}(k+1) \\ \bar{x}_{co}(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_{cuo} & \bar{A}_{12} & \bar{A}_{13} \\ 0 & \bar{A}_{co} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_{cuo}(k) \\ \bar{x}_{co}(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_{cuo} \\ \bar{B}_{co} \\ 0 \end{bmatrix}}_{\bar{B}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 0 & \bar{C}_{co} & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_{cuo}(k) \\ \bar{x}_{co}(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

x_{cuo} : *controllable and unobservable*

x_{co} : *controllable and observable*

x_{uc} : *uncontrollable*

Kalman Canonical Form (Theorem K-3)

$$\begin{bmatrix} \bar{x}_{cuo}(k+1) \\ \bar{x}_{co}(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_{cuo} & \bar{A}_{12} & \bar{A}_{13} \\ 0 & \bar{A}_{co} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{uc} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} \bar{x}_{cuo}(k) \\ \bar{x}_{co}(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_{cuo} \\ \bar{B}_{co} \\ 0 \end{bmatrix}}_{\bar{B}} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 0 & \bar{C}_{co} & \bar{C}_{uc} \end{bmatrix}}_{\bar{C}} \begin{bmatrix} \bar{x}_{cuo}(k) \\ \bar{x}_{co}(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

The transfer function represents the controllable and observable portion: i.e.

$$G(z) = C[zI - A]^{-1}B + D = \frac{B(z)}{A(z)}$$

$$G(z) = \bar{C}_{co}[zI - \bar{A}_{co}]^{-1}\bar{B}_{co} + D = \frac{\bar{B}_{co}(z)}{\bar{A}_{co}(z)}$$