UNIST
Department of Mechanical Engineering

MEN 573: Advanced Control Systems I		**Spring, 2016**
Homework #7	Assigned: Wednesday, May 4, 2016	
Solution	Due: Monday, May 18, 2016 (in class)	

Problem 1.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$J = \int_0^\infty \{2x_1^2(t) + q x_2^2(t)\}\, dt = \int_0^\infty x^T(t) \begin{bmatrix} 2 & 0 \\ 0 & q \end{bmatrix} x(t)\, dt$$

Let $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 0 \\ 0 & q \end{bmatrix}$ s.t. satisfy Lyapunov equatio

$A^T P + PA = -Q$. Then,

$$J = \int_0^\infty x^T(t) Q x(t)\, dt = -\int_0^\infty x^T(t)(A^T P + PA)\, x(t)\, dt$$

$$= -\int_0^\infty \dot{V}(x)\, dt \qquad \left(V(x) = x^T P x \right)$$

$$= -\{V(\infty) - V(0)\} = x^T(0) P x(0) - x^T(\infty) P x(\infty)$$

Since, the system is asymptotically stable, $x(\infty) = 0$. Thus,

$$J = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = P_{11}$$

From $A^T P + PA = -Q$,

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -q \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2P_{11} - 2P_{12} & P_{11} - 2P_{12} - P_{22} \\ P_{11} - 2P_{12} - P_{22} & 2P_{12} - 2P_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -q \end{bmatrix}$$

$$\Rightarrow \begin{cases} P_{11} + P_{12} = 1 \\ P_{11} - 2P_{12} - P_{22} = 0 \\ P_{12} - P_{22} = -\frac{q}{2} \end{cases} \Rightarrow \begin{cases} P_{11} = \frac{6+q}{6} \\ P_{12} = \frac{2-q}{6} \\ P_{22} = \frac{2+3q}{6} \end{cases}$$

$$\boxed{\therefore J = \frac{6+q}{6}}$$

Problem 2.

$\dot{x} = Ax$, $A \in \mathbb{R}^{n \times n}$

$\bar{A} = A + \lambda_p I$, $\lambda_p \in R$, I : identity matrix.

(a) Char. eqn. of A : $\det(\lambda_{Ai} I - A) = 0$. $\Rightarrow \det\{(\lambda_{Ai} + \lambda_p)I - (A + \lambda_p I)\} = 0$.

Let $\bar{A} = A + \lambda_p I$, then $\lambda_{\bar{A}i} = \lambda_{(A+\lambda_p I)i} = \lambda_{Ai} + \lambda_p$

$\therefore \lambda_{\bar{A}i} = \lambda_{Ai} + \lambda_p$

(b) Lyapunov like eqn. $\bar{A}P + PA^T = -Q \Rightarrow (A + \lambda_p I)P + P(A + \lambda_p I)^T = -Q$

$\Rightarrow AP + PA^T = -Q - 2\lambda_p P = -(Q + 2\lambda_p P)$

let $\sigma = 2\lambda_p$, then $AP + PA^T = -(Q + \sigma P)$.

If $\forall Q \succ 0$, $\exists P$ (P is unique, $P \succ 0$), then the eigenvalues ($\lambda_{\bar{A}i} = \lambda_{Ai} + \lambda_p$) $\bar{A}$ ($= A + \lambda_p I$) have negative real parts. (If we let $B = A^T$, then $B^T P + PB = -(Q + \sigma P)$: Lyapunov

And eigenvalue of $B (= A^T)$ = e.val of A ..

$\Rightarrow Re\{\lambda_{\bar{A}i}\} = Re\{\lambda_{Ai} + \lambda_p\} < 0$.

$\Rightarrow Re\{\lambda_{Ai}\} < -Re\{\lambda_p\}$

$\therefore$ Check Lyapunov like eqn. $\bar{A}P + PA^T = -Q$, it means $AP + PA^T = -(Q + \sigma P)$.
If the eigenvalues of $\bar{A}$ have negative real parts, the eigenvalues of A have real parts smaller than $-\lambda_p$.

Problem 3.

3. $D_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $D_A(P) = A^T P A - P$, $A \in \mathbb{R}^{n\times n}$

(a) $D_A(\alpha_1 P_1 + \alpha_2 P_2) = A^T(\alpha_1 P_1 + \alpha_2 P_2)A - (\alpha_1 P_1 + \alpha_2 P_2)$

$= \alpha_1(A^T P_1 A - P_1) + \alpha_2(A^T P_2 A - P_2)$

$= \alpha_1 D_A(P_1) + \alpha_2 D_A(P_2)$ ✓

$\therefore D_A(\alpha_1 P_1 + \alpha_2 P_2) = \alpha_1 D_A(P_1) + \alpha_2 D_A(P_2)$ $\Rightarrow$ linear function.

(b) $A^T v_i = \lambda_i v_i$, $V_k = v_i v_j^T$

$D_A(V_k) = D_A(v_i v_j^T) = A^T(v_i v_j^T)A - (v_i v_j^T)$

$= (A^T v_i)(v_j^T A) - (v_i v_j^T)$

$= (\lambda_i v_i)(\lambda_j v_j^T) - (v_i v_j^T)$

$= (\lambda_i \lambda_j)(v_i v_j^T) - (v_i v_j^T)$ ✓

$= (\lambda_i \lambda_j - 1)(v_i v_j^T)$

$= \mu_k V_k$. where $\mu_k = \lambda_i \lambda_j - 1$

$\boxed{\therefore \mu_k = \lambda_i \lambda_j - 1}$

(c) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0.3 \end{bmatrix}$

(i) $\det(\lambda I - A^T) = \begin{vmatrix} \lambda & 0 \\ -1 & \lambda - 0.5 \end{vmatrix} = \lambda^2 - 0.3\lambda = \lambda(\lambda - 0.5) = 0.$

$\begin{matrix} 0 & 0 \\ -1 & -0.5 \end{matrix}$ $\quad$ $\begin{matrix} 0.5 & 0 \\ -1 & 0 \end{matrix}$

$\lambda_1 = 0$, $\lambda_2 = 0.5$ $\Rightarrow$ $v_1 = [1\ -2]^T$, $v_2 = [0\ \ 1]^T$

From the result of (b), $\mu_k = \lambda_i \lambda_j - 1$

$\therefore \mu_1 = \lambda_1 \lambda_1 - 1 = -1$ $\quad$ $V_1 = v_1 v_1^T = \begin{bmatrix} 1 \\ -2 \end{bmatrix}[1\ -2] = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

$\mu_2 = \lambda_1 \lambda_2 - 1 = -1$ $\quad$ $V_2 = v_1 v_2^T = \begin{bmatrix} 1 \\ -2 \end{bmatrix}[0\ \ 1] = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$

$\mu_3 = \lambda_2 \lambda_1 - 1 = -1$ $\quad$ $V_3 = v_2 v_1^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[1\ -2] = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$

$\mu_4 = \lambda_2 \lambda_2 - 1 = -\frac{3}{4}$ $\quad$ $V_4 = v_2 v_2^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0\ \ 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) $A^T P A - P = -Q$

$\Rightarrow A^T [P_1 \ P_2] A - P = -Q$

$\Rightarrow [A^T P_1 \ \ A^T P_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - [P_1 \ \ P_2] = -[q_1 \ \ q_2]$

$\Rightarrow \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \begin{bmatrix} A^T P_1 \\ A^T P_2 \end{bmatrix} - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$\Rightarrow \left\{ \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} - I \right\} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = -\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$\Rightarrow \{ (A^T \otimes I)(I \otimes A^T) - I \} P = -Q \quad \Rightarrow D_A P = -Q \quad$ where $D_A = (A^T \otimes I)(I \otimes A^T) - I$

$\therefore D_A = (A^T \otimes I)(I \otimes A^T) - I$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0.25 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0.5 & 0.5 & -0.75 \end{bmatrix}$$

(iii) Using MATLAB, get eigenvalue & eigenvector of D_A.

eigenvalue matrix : $\begin{bmatrix} -0.75 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ eigenvector matrix : $\begin{bmatrix} 0 & 0.2425 & 0 & 0 \\ 0 & 0 & 0.4472 & 0 \\ 0 & 0 & 0 & 0.4472 \\ 1 & -0.9701 & -0.8944 & -0.8944 \end{bmatrix}$

$\therefore$ The eigenvalues of D_A is same with those of the linear function $\mathcal{D}_A(\cdot)$.

$\mu_1 = \mu_2 = \mu_3 = -1. \quad \Rightarrow D_A \cdot U_k = (-1) U_k$

$\Rightarrow (\mu_k I - D_A) U_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -0.5 & -0.5 & -0.25 \end{bmatrix} U_k = 0. \Rightarrow U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -4 \end{bmatrix}, U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$

$\mu_4 = -0.75 \quad \Rightarrow D_A \cdot U_4 = (-0.75) U_4.$

$\Rightarrow (\mu_4 I - D_A) U_4 = \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ -1 & -0.5 & -0.5 & 0 \end{bmatrix} U_4 = 0 \quad \Rightarrow U_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore U_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -4 \end{bmatrix}, U_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}, U_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(same with the result of MATLAB, there is just scalar multiplication)

(d) $D_A(P) = Q$ has a unique solution. $\Rightarrow D_A$ is invertible.

$\Rightarrow$ all eigenvalues of D_A are non-zero.

From the result of (b), $\mu_k = \lambda_i \lambda_j - 1 \neq 0 \Rightarrow \lambda_i \lambda_j \neq 1$.

No need in this proof.

Let $A v_i = \lambda_i v_i$,

$A^T P A - P = -Q \Rightarrow v_i^T (A^T P A - P) v_i = -v_i^T Q v_i$

$\Rightarrow \lambda_i^T \lambda_i v_i^T P v_i - v_i^T P v_i = -v_i^T Q v_i$

$\Rightarrow \lambda_i^T \lambda_i - 1 = -\dfrac{v_i^T Q v_i}{v_i^T P v_i} < 0 \Rightarrow |\lambda_i^T \lambda_i| \leq |\lambda_i||\lambda_i| < 1 \Rightarrow |\lambda_i| < 1$

$\therefore \lambda_i \lambda_j \neq 1$, $|\lambda_i| < 1$.

$\Rightarrow$ Same with the theorem in the top of page ME230-77.

(e) $A \in \mathbb{R}^{n\times n}$ is Schur. $\Leftrightarrow \forall Q \in \mathbb{R}^{n\times n}$, Q: sym, PD, bounded element, $\exists P$, P: sym, PD, bounded elements s.t. P is the unique sol. of $A^T P A - P = -Q$

$(\Rightarrow)$ · A is Schur $\Rightarrow |\lambda_i(A)| < 1$

$\lambda_i(A^T)\lambda_j(A) \leq |\lambda_i(A^T)||\lambda_i(A)| < 1 \Rightarrow \lambda_i(A^T)\cdot\lambda_i(A) \neq 1$

$\Rightarrow \lambda_i(A)\cdot\lambda_j(A) \neq 1$

From the result of (d), there is the unique solution of $A^T P A - P = -Q$

· And since $A^T P A - P = -Q$, if Q is bounded, P is bounded.

· $(A^T P A - P = -Q)^T \Rightarrow A^T P^T A - P^T = -Q^T = -Q$

Since P is unique, $P = P^T$ (symmetric)

No need in this proof.

$(\Leftarrow)$ $A^T P A - P = -Q \Rightarrow D_A P = -Q$

Since P is unique. $\Rightarrow D_A$ is invertible. $\Rightarrow$ all eval of D_A are non-zero.

eval of D_A: μ_k, evalue of A: λ_i

$\mu_k = \lambda_i \lambda_j - 1 \neq 0 \Rightarrow \lambda_i \lambda_j \neq 1 \Rightarrow |\lambda_i| > 1$ or $|\lambda_i| < 1$.

If P, Q are PD. Let $A v_i = \lambda_i v_i$

$A^T P A - P = -Q \Rightarrow v_i^T (A^T P A - P) v_i = -v_i^T Q v_i$

$\Rightarrow \lambda_i^T \lambda_i v_i^T P v_i - v_i^T P v_i = -v_i^T Q v_i$

$\Rightarrow \lambda_i^T \lambda_i - 1 = -\dfrac{v_i^T P v_i}{v_i^T Q v_i} < 0 \Rightarrow |\lambda_i^T \lambda_i| \leq |\lambda_i||\lambda_i| < 1 \Rightarrow |\lambda_i| < 1$.

$\Rightarrow A$ is Schur.

Problem 4.

$\nearrow$ Schur

1. $A \in \mathbb{R}^{n\times n}$, $\forall Q \succ 0$, Lyapunov eqn. $A^TPA - P = -Q \Rightarrow P = \sum_{k=0}^{\infty} (A^k)^T Q A^k$

$x(k+1) = Ax(k)$, $x(0) = x_0 \in \mathbb{R}^n$, $V(x) = x^T P x$, $P \succ 0$

(a) $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$

$= x^T(k+1) P x(k+1) - x^T(k) P x(k)$

$= \{Ax(k)\}^T P \{Ax(k)\} - x^T(k) P x(k)$

$= x^T(k)(A^TPA - P)x(k)$

$= -x^T(k) Q x(k) \quad \cdots ①$

$$\frac{V(x(k+1)) - V(x(k))}{V(x(k))} = -\frac{x^T(k) Q x(k)}{x^T(k) P x(k)} = -\frac{(\lambda_Q)_{min}}{(\lambda_P)_{max}} = -\alpha, \quad \alpha > 0.$$

$\Rightarrow V(x(k+1)) = (1-\alpha) V(x(k))$

Since $V(k+1)$ must be positive definite unless $x(k+1) = 0.$, $(1-\alpha) \geq 0$

$\Rightarrow 0 < \alpha \leq 1 \Rightarrow 0 \leq (1-\alpha) < 1$

$\Rightarrow V(x(k)) \leq (1-\alpha)^k V(0)$

$\therefore \lim_{k\to\infty} V(x(k)) = 0.$

(b) From ①, $V(x(k+1)) - V(x(k)) = -x^T(k) Q x(k)$

$\Rightarrow V(x(0)) - V(x(1)) = x^T(0) Q x(0)$

$V(x(1)) - V(x(2)) = x^T(1) Q x(1)$

$\vdots \qquad \vdots$

Suming up both sides of above equations,

$$V(x(0)) - V(x(\infty)) = \sum_{k=0}^{\infty} x^T(k) Q x(k)$$

Since $V(x(\infty)) = 0$, $V(x(0)) = \sum_{k=0}^{\infty} x^T(k) Q x(k)$

(c) $x(k) = A^k x_0$, $V(x(0)) = x_0^T P x_0$

From the result of (b),

$$x_0^T P x_0 = \sum_{k=0}^{\infty} x^T(k) Q x(k) = \sum_{k=0}^{\infty} (A^k x_0)^T Q (A^k x_0) = \sum_{k=0}^{\infty} x_0^T (A^k)^T Q A^k x_0$$

$$\therefore P = \sum_{k=0}^{\infty} (A^k)^T Q A^k$$