Problems for MAE280A Linear Systems Theory, Fall 2015: due Friday October 23 before 16:00 at Martin's office or in class Thursday

Problem 1 [Hespanha 4.2]

Show that the $\mathcal{Z}$-transform of any output to the discrete-time LTI system

$$\begin{aligned} x^+ &= Ax + Bu, \\ y &= Cx + Du, \end{aligned}$$

is given by $\hat{y}(z) = \hat{\Psi}(z)x(0) + \hat{G}(z)\hat{u}(z)$, where $\hat{y}(z)$ and $\hat{u}(z)$ are the $\mathcal{Z}$-transforms of $\{y_k\}$ and $\{u_k\}$ respectively and

$$\begin{aligned} \hat{\Psi}(z) &:= C(zI - A)^{-1}z, \\ \hat{G}(z) &:= C(zI - A)^{-1}B + D. \end{aligned}$$

To compute the state response transform, take the $\mathcal{Z}$-transform of the system equation directly.

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ \sum_{j=0}^{\infty} z^{-j}x(j+1) &= A\sum_{j=0}^{\infty} z^{-j}x(j) + B\sum_{j=0}^{\infty} z^{-j}u(j) = A\hat{x}(z) + B\hat{u}(z). \end{aligned}$$

Now change the variable of summation in the left-hand side, $k := j + 1$.

$$\sum_{k=1}^{\infty} z^{-(k-1)}x(k) = z\sum_{k=1}^{\infty} z^{-k}x(k) = z\left[\sum_{k=0}^{\infty} z^{-k}x(k) - x(0)\right] = z\hat{x}(z) - zx(0).$$

Substitute this back in the orignal expression above.

$$\begin{aligned} z\hat{x}(z) - zx(0) &= A\hat{x}(z) + B\hat{u}(z), \\ (zI - A)\hat{x}(z) &= zx(0) + B\hat{u}(z), \\ \hat{x}(z) &= (zI - A)^{-1}x(0) + (zI - A)^{-1}B\hat{u}(z), \\ \hat{y}(z) &= C\hat{x}(z) = C(zI - A)^{-1}x(0) + C(zI - A)^{-1}B\hat{u}(z). \end{aligned}$$

Problem 2 [Hespanha 4.3]

Given a transfer fucntion $\hat{G}(s)$, let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be a realization for its transpose $\bar{G}(s) := \hat{G}(s)^T$. Show that (A, B, C, D), where $A := \bar{A}^T$, $B := \bar{C}^T$, $C := \bar{B}^T$, $D := \bar{D}^T$ is a realization for $\hat{G}(s)$.

We have

$$\hat{G}(s)^T = \bar{G}(s) = D + C(sI - A)^{-1}B.$$

Taking transposes and using the rules $(XY)^T = Y^T X^T$ and $(Z^{-1})^T = (Z^T)^{-1}$,

$$\hat{G}(s) = \bar{G}(s)^T = D^T + B^T(sI - A^T)^{-1}B^T,$$

and the answer follows.

Problem 3

I was browsing the interweb the other day and came across a very recent paper (after typing `mimo robot control model` into Google).

Mohamed Chemachema and Samir Zeglache, "Output feedback linearization based controller for a helicopter-like twin-rotor MIMO system", *Journal of Intelligent and Robotic Systems* (2015) 80:181-190.

This problem follows aspects of this paper. I proffer no comments on the quality of the paper. The paper itself is available from the UCSD Library via Roger and is in the October 2015 issue. The system in hand is depicted below in Figure 1.

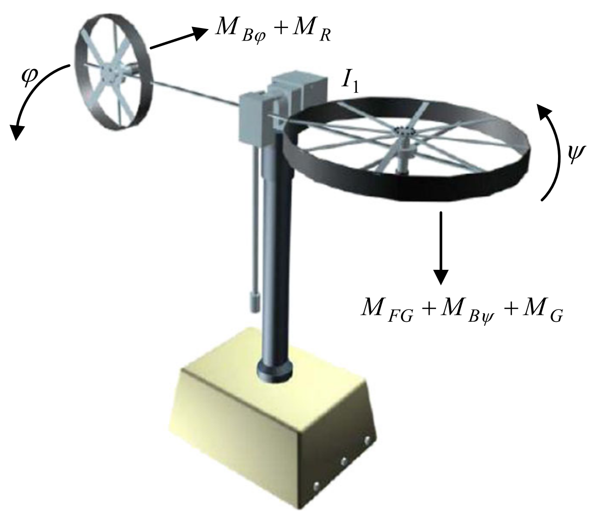

Figure 1: Twin-rotor MIMO system defining physical variables: pitch ψ, yaw φ, moments (not *momenta* as the paper suggests) M_{FG}, $M_{B\psi}$, $M_{B\varphi}$ M_G, M_R.

Part (i): Determine what the authors decide to use for their state variables and make their substitutions to arrive at state-variable realization (12). Comment specifically on the mixed time-domain and frequency-domain (s-domain) approach and their consideration of M_1, M_2 and M_R as independent control inputs. Also comment on the linearity and/or time-invariance of this state-variable realization.

The authors choose as state $\begin{bmatrix}\psi & \dot{\psi} & \phi & \dot{\phi}\end{bmatrix}^T$, where ψ is the pitch angle and ϕ is the yaw angle. The authors also mix time domain (differential equation descriptions) with frequency domain (Laplace variable descriptions). Particularly, in the description of the motor dynamics between applied voltages u_1 and u_2 to motor torques τ_1 and τ_2, they leave the description as a transfer function to be dealt with later on. Probably, this is justified since the motor time constants are fast compared to the pitch and yaw time constants. But they never actually deal with this properly, which is one of my complaints. They could introduce additional state variables τ_1 and τ_2 to capture these dynamics. This would yield a sixth order state vector and would leave us with dealing with the quadratic nonlinearities. I do not find their approach problematic at this stage though – it is quite systematic.

Part (ii): Consider the feedback linearization approach embodied in (13-14) and show that, indeed, we arrive at (15), which is now LTI, but the feedback controller used to achieve this is nonlinear. Continue with the substitution from (16) to show that this choice of v_v and v_h yields a stable system. [They use the word "Hurwitz" to describe a continuous-time LTI system (or its characteristic polynomial) with all its poles/roots in the open left half-plane.] Can you interpret what the signals x_{1d} and x_{3d} are in this framework?

The substitution of (13) and (14) into (15) achieves two distinct tasks: it renders the state equations linear by cancelling the nonlinearities (a property called feedback linearization) and it decouples the dynamics of yaw from the those of pitch (called decoupling) by cancelling the terms M_R. The resulting system is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}0&0&0&0\\1&1&0&0\\0&0&0&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}v_v\\ \ddot{x}_{1d}\\ v_h\\ \ddot{x}_{3d}\end{bmatrix}.$$

This is a linear system, which is unstable (two double eigenvalues at $s=0$ as is evident since the matrix is in Jordan form) and has four control inputs. The controller expressed in (13-14) contains nonlinear terms, sine, cosine, quotient, product, in the state variables.

When we continue with the state variable feedback from (16), we arrive at

$$\dot{x} = \begin{bmatrix}0&1&0&0\\-\lambda_1&-\lambda_2&0&0\\0&0&0&1\\0&0&-\lambda_1&-\lambda_2\end{bmatrix}x + \begin{bmatrix}0\\ \ddot{x}_{1d}+\lambda_2\dot{x}_{1d}+\lambda_1 x_{1d}\\0\\ \ddot{x}_{3d}+\lambda_2\dot{x}_{3d}+\lambda_1 x_{3d}\end{bmatrix}.$$

The matrix above is block diagonal and has characteristic equation $(s^2+\lambda_2 s+\lambda_1)^2=0$. Evidently, λ_1 and λ_2 may be selected to make this characteristic equation Hurwitz. So they have used this second round of feedback to stabilize the system rendered linear by feedback linearization. In Figure 3 we see the left sets of blocks achieves this.

As is apparent from the discussion, the intention is to define vertical (pitch) and horizontal (yaw) tracking error signals $e_{v,}=x_1-x_{1d}$ and $e_h=x_3-x_{3d}$. Then we find from the above equations that $\ddot{e}_v+\lambda_1\dot{e}_v+\lambda_2 e_v=0$ and $\ddot{e}_h+\lambda_1\dot{e}_h+\lambda_2 e_h=0$. Since λ_1 and λ_2 are chosen to make the roots of $s^2+\lambda_1 s+\lambda_2$ stable, this implies that $e_v\to 0$ and $e_h\to 0$ as $t\to\infty$. Or, $x_1\to x_{1d}$ and $x_3\to x_{3d}$. From a control theoretic standpoint, this is problematic, since no assumption is made on the reference trajectories, other than the implicit assumption that they are twice differentiable. Since the system states are physical quantities and the references are just signals, it would be more appropriate to ensure that the signals were realistic for the aircraft.

Part (iii): Equations (18-19) are the same as (13-14) with the state values replaced by state estimates from an *observer*. We will consider observers later in the course. For our purposes, we will stick to (13-14) for M_1 and M_2 and assume that the state is fully measured. The authors use (20-21) to invert the relations (2) and (8). Comment on their choice of which of the two solutions should be used, i.e. the choice of sign on the square root. Next they decide on "inverting the dynamics of motor models" in (22-23) to yield the voltage signals, u_1 and u_2, to be applied to the two motors. Comment on this final choice with reference to the simulation results presented in Section 5.

The quadratic relationship between motor torque and thrust is first asserted in (2), $M_1=a_1\tau_1^2+b_1\tau_1$, and its partner (8). What does this mean? The parameters are given as $a_1=0.0135$ and $b_2=0.0924$ (without units) but presumably so that M_1 is given in Nm. This describes a parabola passing through zero and with its other real root at $\tau=-4.5$. It makes no sense to apply (2) for negative torques and so the pitch and yaw torques are always positive. Indeed, the M_1 versus τ_1 curve is monotonically increasing for $\tau_1>0$. There is a mistake in (20-21), which should be

$$\tau_1=-\frac{b_1}{2}+\sqrt{\frac{b_1^2}{4}+a_1M_1},\quad \tau_2=-\frac{b_2}{2}+\sqrt{\frac{b_2^2}{4}+a_2M_2}.$$

Evidently, this $\tau_{1,2}$ is positive and the other solution is negative and beyond the validity of (2). The entire discussion surrounding (20-21) is surreal and misdirected.

The inversion of the motor dynamics is where the whole control design moves into cloud-cuckoo-land. Equations (22-23) suggest differentiating the signal τ_i derived the "measured" signals, ψ and ϕ. The result is that we see violent transients for the voltage signals even in the case of minor perturbations to the measured angles. See particularly Figure 10 but also the others. The make dramatic changes to motor torques by changing the voltage one needs a very large range voltage supply – I used to use a 75V supply to control a small 100mA stepper motor fast. In this case of controlling an aircraft, this is a really silly way to go about the control design. Even in simulation it misbehaves badly. It would never work in practice ... and presumably it never did.

If I were to pursue this control design, I would start with their feedback linearization and decoupling approach and then run simulation to determine how large and how violent the corresponding control signals are likely to be. I would have a design process for the realistic selection of λ_1 and λ_2, which would be distinct between the horizontal and vertical loops, because they are sized differently. I would look at linearizing (or piecewise linearizing) the quadratic nonlinearity (which I still do not understand, although it might capture a deadband. Who knows?) andI would conduct a design using six state variables, which would ensure that τ_i and u_i has sensible magnitudes.

Part (iv): The complete control design is depicted in their Figure 3 on page 185. We see a number of aspects of control design: derivation of the state space model for a MIMO system, construction of a feedback controller from fictitious inputs M_1 and M_2 which cancels the nonlinearity of the system yielding an LTI problem, stabilizing linear pole-placement control v_v and v_h, state estimation using an observer, and system inversion. Would you buy a control system from these people? Justify your decision.

There is no way that I would buy a controller from these characters. There approach is simplistic and accounts very poorly for model mismatch, even in their simulation examples. The differentiation in the controller is my core concern .. along with their failure to conduct the arithmetic accurately. Not only would I not buy their controller. I also would not ride in their helicopter.

This should be a warning to you to be skeptical in reading technical papers. You already have the skills to sniff out poor material.

[Unbonus (no marks just a penalty if you do not do it)] Problem 4 [The Volkswagen problem]

In MATLAB and SIMULINK run through the example *MIMO control of a diesel engine* located at:

http://www.mathworks.com/help/robust/examples/mimo-control-of-diesel-engine.html

Follow through the example to explore the use of state space models for MIMO analysis and design. [It worked well for me, although the randomized initialization gave answers different from the web page. To view the output from the simulation: first bring the *scope* to the foreground by double clicking it, then right-click it to set it to *autoscale*, then run the simulation by clicking the *play* button in Simulink.] Click on boxes in SIMULINK to discover how they are used.

Summarize the objectives and outcomes of each step of modeling and control design.

Yes.