CSE530: Algorithms & Complexity
Notes on Lecture 6: Introduction to Linear Programming

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We show how to formulate examples 1–4 as linear program.

1 Elections

We want to minimize the total cost, so it will be our objective function:

$$c(x) = r + g + f + t$$

where r, g, f and t are the thousands of dollars spent on advertising for building roads, gun control, fart subsidies and gasoline taxes, respectively. As we need to earn 50,000 urban votes, we have the constraint that

$$-2r + 8g + 10t \geqslant 50$$

The other constraints are obtained in the same way, and we obtain the linear program:

$$\begin{array}{rrcl} \text{minimize} & r + g + f + t & & \\ \text{subject to} & -2r + 8g + 10t & \geqslant & 50 \\ & 5r + 2g & \geqslant & 100 \\ & 3r - 5g + 10f - 2t & \geqslant & 25 \end{array}$$

2 Fitting a line

We use as variables the coefficients a and b of the line equation, as well as n auxiliary variables $\varepsilon_1, \dots, \varepsilon_n$ that model the "error" made at each point. The goal is to minimize $\sum_i \varepsilon_i$, so we have the following program

$$\begin{array}{rll} \text{minimize} & \sum_i \varepsilon_i & \\ \text{subject to} & ax_i + b - y_i \leqslant \varepsilon_i, & i = 1, \dots, n \\ & -ax_i - b + y_i \leqslant \varepsilon_i, & i = 1, \dots, n \end{array}$$

where the two constraints $ax_i + b - y_i \leqslant \varepsilon_i$ and $-ax_i - b + y_i \leqslant \varepsilon_i$ encode the fact that $|ax_i + b - y_i| \leqslant \varepsilon_i$.

3 The shortest path problem

For each vertex $v \in V$, we denote by d_v the length of the shortest path from s to v. So $d_s = 0$, and for all $v \neq s$,

$$d_v = \min_{u:(u,v)\in E} d(u) + \ell(u,v)$$

where the minimum is achieved for the vertex u preceding v in a shortest path from s to v. Then we can reformulate the shortest path problem using the following linear program, where the variables are the n shortest path lengths d_v.

$$\begin{array}{rll} \text{maximize} & d_t & \\ \text{subject to} & d_v - d_u \leqslant \ell(u,v) & \text{for all } (u,v) \in E \\ & d_s = 0 & \end{array} \tag{1}$$

We can even solve the single-source shortest path problem, where the goal is to compute the shortest path lengths $d(s,v)$ for all v using a slightly modified version of the above LP:

$$\begin{array}{rll} \text{maximize} & \sum_{v\in V} d_v & \\ \text{subject to} & d_v - d_u \leqslant \ell(u,v) & \text{for all } (u,v) \in E \\ & d_s = 0 & \end{array} \tag{2}$$

Proof of correctness. We now prove that LP (1) and (2) return the correct answer. So the optimal value d_t^* should be the length of the shortest path $s \rightsquigarrow t$, and for (2), all d_v^* should also be shortest path lengths to the corresponding vertex v. We assume that all nodes in V are reachable from s, as otherwise the problem is not well defined.

We first prove that (2) is correct, as it will help in proving that (1) is also correct. Let d_v^*, $v \in V$ be an optimal solution to (2), and we will show that d_v^* is the length of the shortest path to v. Let p_v be the shortest path to v, and $\ell(p_v)$ be its length. We split the proof into two lemmas.

Lemma 1. *In the optimal solution to* (2)*, we have* $d_v^* \leqslant \ell(p_v)$*.*

Proof. Let $(s, u_1, \dots, u_k, v)$ be the sequence of vertices of p_v. The constraints of the LP ensure that

$$\begin{aligned} d_{u_1}^* - d_s^* &\leqslant \ell(s, u_1) \\ d_{u_2}^* - d_{u_1}^* &\leqslant \ell(u_1, u_2) \\ \vdots \qquad & \quad \vdots \\ d_v^* - d_{u_k}^* &\leqslant \ell(u_k, t) \end{aligned}$$

Summing up these inequalities, we obtain $d_v^* - d_s^* \leqslant \ell(p_v)$, and since $d_s^* = 0$, it follows that $d_v^* \leqslant \ell(p_v)$. □

Lemma 2. *In the optimal solution to* (2)*, we have* $\ell(p_v) \leqslant d_v^*$*.*

Proof. Let w be a node in $V \setminus \{s\}$. The variable d_w is subject to one or several constraints of the form $d_w - d_u \leqslant \ell(u,w)$. In the optimal solution, at least one of these constraints must be tight, which means that $d_w^* - d_u^* = \ell(u,w)$, because if it were not the case, we could increase the value of d_w^* and get a feasible solution achieving a larger value of the objective function.

Now we show how to construct a path from s to v whose length is d_v^*. As p_v is a shortest path to v, it implies that $\ell(p_v) \leqslant d_v^*$, thus completing the proof. We trace the path backwards from v. There is a node w_1 such that the constraint involving w_1 and v is tight, and hence $d_v^* - d_{w_1}^* = \ell(w_1, v)$. Similarly, there is a vertex w_2 such that $d_{w_1}^* - d_{w_2}^* = \ell(w_2, w_1)$. When we repeat this process, $d_{w_i}^*$ decreases strictly. But each d_u^* must be nonnegative, as otherwise we could increase it without violating any constraint. So in the end, we get $w_k = s$, and we have constructed a path backwards from s to v whose length is d_v^*. □

Lemma 1 and 2 show that LP (2) solves the single source shortest path problem. We now briefly explain how this proof extends to LP (1), so the optimal value $\hat{d}_t$ for (1) is the length of the shortest path to t. Lemma 1 applies in exactly the same way to LP (1). So we have $\hat{d}_t \leqslant \ell(p_v)$, and thus $\hat{d}_t \leqslant d_t^*$, where d_t^* is the value of d_t in the optimal solution to LP (2). As the two linear programs have the same constraints, and hence same feasible region, it means that the optimal solution to (2) is feasible for (1), and since $\hat{d}_t \leqslant d_t^*$, then we must have $\hat{d}_t = d_t^*$.

4 Maximum flow

The variables of the LP are just the flows $f(u, v)$ for all edge $(u, v) \in E$. On the other hand, the capacities $c(u, v)$ are parameters of the LP, so we regard them as constants. We will denote $x_{uv} = f(u, v)$ to avoid confusion. Then the capacity constraint, skew symmetry, and flow conservation are linear constraints. So the maximum flow problem is the following LP:

$$
\begin{array}{lll}
\text{maximize} & \sum_{v \in V} x_{sv} & \\
\text{subject to} & x_{uv} \leqslant c(u, v) & \text{for all } u, v \in V \\
& x_{uv} = -x_{vu} & \text{for all } u, v \in V \\
& \sum_{v \in V} x_{uv} = 0 & \text{for all } u \in V \setminus \{s, t\}
\end{array}
$$