UNIST
Department of Mechanical Engineering

MEN 573: Advanced Control Systems I		**Spring, 2016**
Homework #8	Assigned: Monday, May 16, 2016	
Solution	Due: Wednesday, May 25, 2016 (in class)	

Problem 1.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x(k)$$

(a) $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$CA = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ $\quad CA^2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \text{rank}(Q) = 1 \neq 3$$

$\therefore$ The system is NOT observable.

$\Rightarrow$ The initial condition $x(0)$ can NOT be determined from outputs and inputs

(b) $x_1(0) = x_2(0) + v \quad , v \in R^3$

$$y(k) = C\{A^k x(0) + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j)\}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} x_1(0) + \begin{bmatrix} 0 & 0 & 0 \\ CB & 0 & 0 \\ CAB & CB & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x_1(0) + \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} (x_2(0) + v) + \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ u \\ u \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} v = 0 \Rightarrow v \text{ is null space of } Q. \quad \boxed{\therefore v = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}}$$

Problem 2.

$$G(z) = \frac{B(z)}{A(z)} = \frac{2z^2 - 0.2z - 0.12}{z^3 + 0.8z^2 + 0.17z + 0.01}$$

(a)
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.01 & -0.17 & -0.8 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_c} u(k)$$

$$y(k) = \underbrace{[-0.12 \quad -0.2 \quad 2]}_{C_c} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$C_cA_c = [0.02 \; -0.46 \; -1.8]$ $\quad C_cA_c^2 = [0.018 \quad 0.286 \quad 0.98]$

$$Q = \begin{bmatrix} C_c \\ C_cA_c \\ C_cA_c^2 \end{bmatrix} = \begin{bmatrix} -0.12 & -0.2 & -2 \\ -0.02 & -0.46 & -1.8 \\ 0.018 & 0.286 & 0.98 \end{bmatrix} \Rightarrow \text{rank}(Q) = 2 \neq 3.$$

$\boxed{\therefore \text{ The system is NOT observable.}}$

(b)
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} -0.6 & 1 & 0 \\ -0.17 & 0 & 1 \\ -0.01 & 0 & 0 \end{bmatrix}}_{A_o} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ -0.2 \\ -0.12 \end{bmatrix}}_{B_o} u(k)$$

$$y(k) = \underbrace{[1 \quad 0 \quad 0]}_{C_o} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$A_oB_o = [-1.8 \quad -0.46 \quad -0.02]^T \quad A_o^2B_o = [0.98 \quad 0.286 \quad 0.018]^T$

$$P = [B_o \quad A_oB_o \quad A_o^2B_o] = \begin{bmatrix} 2 & -1.8 & 0.98 \\ -0.2 & -0.46 & 0.286 \\ -0.12 & -0.02 & 0.018 \end{bmatrix} \Rightarrow \text{rank}(P) \doteq 2 \neq 3$$

$\boxed{\therefore \text{ The system is NOT controllable.}}$

(c)
$$G(z) = \frac{2(z-0.3)(z+0.2)}{(z+0.5)(z+0.2)(z+0.1)} = \frac{2z-0.6}{z^2+0.6z+0.05}$$

Controllable canonical form.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.05 & -0.6 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [-0.6 \quad 2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.6 & 2 \\ -0.1 & -1.8 \end{bmatrix} \Rightarrow \text{rank}(Q) = 2 \; : \text{full rank.}$$

$\boxed{\therefore \text{ The system is observable}}$

observable canonical form.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.6 & 1 \\ -0.05 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 2 \\ -0.6 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & -1.8 \\ -0.6 & -0.1 \end{bmatrix} \Rightarrow \text{rank}(P) = 2 : \text{full rank}$$

$\boxed{\therefore \text{The system is controllable.}}$

Problem 3.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -1.1 & -0.34 & -0.32 \\ 1 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u(k)$$

(a) $x_0 = [1 \ 1 \ 1]^T$

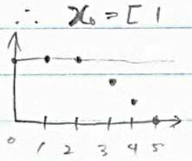
$$P = [B \ \ AB \ \ A^2B] = \begin{bmatrix} 2 & -2.2 & 1.74 \\ 0 & 2 & -2.2 \\ 0 & 0 & 0.5 \end{bmatrix} \Rightarrow \text{rank}(P) = 3 \ \therefore \text{full rank.}$$

$\therefore$ The system is controllable.

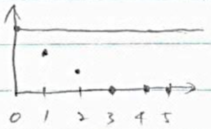
$\Rightarrow$ The system can reach at any states in 3 steps. ($<$ 5 steps)

$\therefore$ $x_0 = [1 \ 1 \ 1]^T$ can be transferred to the origin in 5 steps

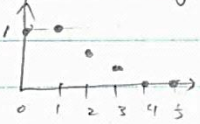
(b)



or



or



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There are many control sequences that transfer $x(0) = [1 \ 1 \ 1]^T$ to $x(5) = [0 \ 0 \ 0]^T$, since 5 steps are bigger than 3 steps, the exact steps.

I'll choose 2nd method. So, $u(3) = u(4) = 0. \Rightarrow x(5) = x(4) = x(3) = [0 \ 0 \ 0]^T$

$$x(3) = A^3x(0) + \sum_{j=0}^{2} A^{2-j} B u(j) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2)$$

$$= A^3x(0) + \underbrace{[B \ \ AB \ \ A^2B]}_{P} \underbrace{\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}}_{u}$$

$$u = -P^{-1}A^3x(0) = [0.04 \ \ 0.21 \ \ 0.88]^T$$

$\boxed{\therefore u(0) = 0.88, \ u(1) = 0.21, \ u(2) = 0.04, \ u(3) = 0, \ u(4) = 0}$

Problem 4.

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases}$$ $y(k) \in \mathbb{R}$, $u(k) \in \mathbb{R}$

$\Rightarrow$ controllable, observable.

$u(k) = -Kx(k) + v(k)$, $K \in \mathbb{R}^{1\times n}$: constant gain matrix, $v(k)$: new input

$$\begin{cases} x(k+1) = A_c x(k) + Bv(k) \\ y(k) = Cx(k) \end{cases}$$ $A_c = A - BK$

(a) $P = [B \ \ AB \ \ A^2B \ \cdots \ A^{n-1}B] \Rightarrow \text{rank}(P) = n$: controllable.

$A = A_c + BK$.

$P = [B \ \ (A_c+BK)B \ \ (A_c+BK)^2B \ \cdots \ (A_c+BK)^{n-1}B]$

$= [B \ \ (A_c+BK)B \ \ (A_c^2 + A_cBK + BKA_c + B^2K^2)B \ \cdots \ (A_c+BK)^{n-1}B]$

$= [B \ \ A_cB + KB\cdot B \ \ A_c^2B + KA_cB^2 + KBA_cB + K^2B^3 \ \cdots \ (A_c+BK)^{n-1}B]$

$= [B \ \ A_cB + KB\cdot B \ \ A_c^2B + KA_cB^2 + KB(A_c+KB)B \ \cdots \ (A_c+BK)^{n-1}B]$

$$= \underbrace{[B \ \ A_cB \ \ A_c^2B \ \cdots \ A_c^{n-1}B]}_{\hat{P}} \underbrace{\begin{bmatrix} I & KB & KAB & \cdots & KBA^{n-2}B \\ 0 & I & KB & \cdots & KBA^{n-1}B \\ 0 & 0 & I & \cdots & \vdots \\ \vdots & \vdots & 0 & \cdots & KB \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}}_{n}$$

rank $\Rightarrow n$. : full rank.

$\boxed{\therefore \text{rank}(\hat{P}) = n \Rightarrow \text{This system is controllable.}}$

Problem 5.

(a) $\cancel{G(s) = G_1(s)\,G_2(s)} = \dfrac{B_1(s)}{A_1(s)} \cdot \dfrac{B_2(s)}{A_2(s)}$

$\{A_1(s), B_1(s)\}$ and $\{A_2(s), B_2(s)\}$ are already coprime.

$\boxed{\therefore \{A_1(s), B_2(s)\} \text{ and } \{A_2(s), B_1(s)\} \text{ should be coprime.}}$

(b) $G(s) = G_1(s) + G_2(s) = \dfrac{B_1(s)}{A_1(s)} + \dfrac{B_2(s)}{A_2(s)} = \dfrac{B_1(s)A_2(s) + A_1(s)B_2(s)}{A_1(s)A_2(s)}$

$\boxed{\therefore \{A_1(s)A_2(s),\ A_1(s)B_2(s) + A_2(s)B_1(s)\} \text{ should be coprime}}$

Or you may refer another approach below.

(a) first rewrite eq$\underline{n}$s into series format:

$$\xrightarrow{u_1} \boxed{\text{sys 1}} \xrightarrow{y_1 \quad u_2} \boxed{\text{sys 2}} \xrightarrow{y_2}$$

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 \qquad \xrightarrow{y_1 = u_2} \qquad \dot{x}_2 = A_2 x_2 + B_1 (C_1 x_1 + D_1 u_1)$$

$$y_1 = C_1 x_1 + D_1 u_1 \qquad\qquad y_2 = C_2 x_2 + D_2 (C_1 x_1 + D_1 u_1)$$

Combine the state eq$\underline{n}$s:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}}_{\hat{A}_s} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_1 D_1 \end{bmatrix}}_{\hat{B}_s} u_1$$

$$y = \underbrace{\begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}}_{\hat{C}_s} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{D_2 D_1}_{\hat{D}_s} u_1$$

the series combination is controllable if $\{\hat{A}_s, \hat{B}_s\}$ is controllable

" observable if $\{\hat{A}_s, \hat{C}_s\}$ is observable.

(b) parallel case

$$\dot{x}_1 = A_1 x_1 + B_1 u \qquad\qquad y = y_1 + y_2 = C_1 x_1 + D_1 u + C_2 x_2 + D_2 u$$

$$\dot{x}_2 = A_2 x_2 + B_2 u$$

then
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{\hat{A}_p} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{\hat{B}_p} u$$

$$y = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{\hat{C}_p} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{(D_1 + D_2)}_{\hat{D}_p} u$$

then the parallel combination is controllable if $\{\hat{A}_p, \hat{B}_p\}$ is controllable

" observable if $\{\hat{A}_p, \hat{C}_p\}$ is observable.

note: if either system is not controllable/observable then either combination is not respectively controllable/observable.

However $\{A_1, B_1\}$ and $\{A_2, B_2\}$ controllable (or observable counterpart) is not sufficient for the combination to be controllable (or observable).

For instance, look at the parallel case:

$$P_p = \left[\hat{B}_p \quad \hat{A}_p \hat{B}_p \dots \quad \hat{A}_p^{2n-1} \hat{B}_p \right]$$

then $$P_p = \left[\begin{array}{c|c|c|c|c} B_1 & A_1 B_1 & A_1^2 B_1 & & A_1^{2n-1} B_1 \\ \dots & \dots & \dots & \dots & \dots \\ B_2 & A_2 B_2 & A_2^2 B_2 & \dots & A_2^{2n-1} B_2 \end{array} \right]$$

thus if either $\{A_1, B_1\}$ or $\{A_2, B_2\}$ is not controllable, then P_p will not have rank $2n$.

But $\{A_1, B_1\}$ and $\{A_2, B_2\}$ controllable $\not\Rightarrow$ P_p is controllable.

look at Q_p as well :

$$Q_p = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_1A_1 & C_2A_2 \\ \hline \vdots & \vdots \\ \hline C_1A_1^{2n-1} & C_2A_2^{2n-1} \end{array}\right]$$

the similar argument can be made about observability.

Problem 6.

(a) $P = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 3 & 1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow \text{rank}(P) = 2.$

$$\boxed{X_c = R\{P\} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}\right\}}$$ ✓

(b) $Q = \begin{bmatrix} 2 & 1 & 3 \\ -3 & 1 & -2 \\ 7 & 1 & 8 \end{bmatrix} \Rightarrow \text{rank}(Q) = 2$

$$\boxed{X_{uo} = N\{Q\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}}$$ ✓

(c) $M = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ Kalman canonical controllability transformation.

$\Rightarrow \bar{x} = M^{-1}x$

$\underbrace{\text{cols } 1,2}_{\text{lin. indep. col. of } P}$, $\underbrace{\text{col } 3}_{\text{To make rank}(M)=3}$

$A_c = M^{-1}AM = \left[\begin{array}{cc|c} 0 & 1 & -1.25 \\ 1 & 0 & 1.75 \\ \hline 0 & 0 & -2 \end{array}\right] \Rightarrow A_{uc} = -2$: stable. ✓

$\boxed{\therefore \text{ Stabilizable}}$

(d) $O = \left[\begin{array}{ccc} 2 & 1 & 3 \\ -3 & 1 & -2 \\ \hline 1 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \} \text{ lin. indep. row. of } Q \\ \} \text{ To make rank}(O) = 3 \end{array}$

Kalman canononical observability transformation

$A_o = OAO^{-1} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 2 & -1 & 0 \\ \hline 0.4 & -0.4 & -1 \end{array}\right] \Rightarrow A_{uo} = -1$: stable. ✓

$\boxed{\therefore \text{ Detectable}}$ ✓

(e) $G(s) = \dfrac{2s^2+6s+4}{s^3+2s^2-s-2} = \dfrac{2\cancel{(s+1)}\cancel{(s+2)}}{(s-1)\cancel{(s+1)}\cancel{(s+2)}} = \boxed{\dfrac{2}{s-1}}$

$-1, -2$: pole-zero cancellation (by A_{uc}, A_{uo}) ✓

$\boxed{\therefore \text{ pole} : 1, \text{ no zero.}}$ ✓