Solution to the Selected Problems
Linear System Theory (ECE532)

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HW2

Problem 2 It can be seen that $f(x)$ is continuously differentiable, and we have

$$\| - x_1 + \frac{2x_2}{1+x_2^2} \| \leq k_1 + k\|x\|$$

Hence, we can show that it is Lipschitz continuous, and there exists a unique solution.

Problem 3 We can show that

$$\|f(x)\| = \frac{\|g(x)\|}{1+\|g(x)\|^2} \leq \frac{1}{2}$$

Hence, there exists a unique solution.

HW3

Problem 1 (a) $y_1, y_2 \in \mathcal{S}^\perp$

$$(\alpha_1 y_1 + \alpha_2 y_2)^T x = \alpha_1(y_1^T x) + \alpha_2(y_2^T x) = 0$$
$$\Rightarrow (\alpha_1 y_1 + \alpha_2 y_2) \in \mathcal{S}^\perp$$

(b) Assume that $v_1, ..., v_p$ are *not* linearly independent. Take $v_l \neq 0 \in \mathcal{S}^\perp$, $\alpha_l \neq 0$. Then

$$\sum_{i=1}^{k} \alpha_i v_i + \sum_{i=k+1, i\neq l}^{p} \alpha_i v_i + \alpha_l v_l = 0, \text{ zero vector}$$
$$\sum_{i=1}^{k} \alpha_i v_l^T v_i + \sum_{i=k+1, i\neq l}^{p} \alpha_i v_l^T v_i + \alpha_l v_l^T v_l = 0, \text{ scalar zero}$$
$$v_l^T \Big(\sum_{i=k+1, i\neq l}^{p} \alpha_i v_i + \alpha_l v_l \Big) = 0$$

This means that either $v_{k+1}, ..., v_p$ are linearly dependent, or v_l is 0, both of which are a contradiction of the fact that $v_{k+1}, ..., v_p$ is a basis of $\mathcal{S}^\perp$. The assumption is wrong, and therefore they must be linearly independent.

(c) Let $y \in \mathcal{S}^\perp$, and

$$A^T = \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_k^T \end{pmatrix}$$

$$A^T y = 0 \Rightarrow y \in N(A^T)$$

Since $y \in \mathcal{S}^\perp$, we have $\mathcal{S}^\perp \subset N(A^T)$. Also, take $y \in N(A^T)$

$$\begin{aligned} A^T y &= 0 \\ &\Rightarrow v_i y = 0,\ i = 1, ..., k \\ &\Rightarrow y \in \mathcal{S}^\perp \\ N(A^T) &\subset \mathcal{S}^\perp \end{aligned}$$

Hence, we have $N(A^T) = \mathcal{S}^\perp$. Then by the rank-nullity theorem

$$\begin{aligned} \dim(\mathbb{R}^n) &= n \\ &= \dim(N(A^T)) + \dim(R(A^T)) \\ &= nullity(A^T) + rank(A^T) \\ &\Rightarrow \dim(\mathcal{S}^\perp) = n - k \end{aligned}$$

(d) We first show that $R(A^T) \subset (N(A))^\perp$:

$$\begin{aligned} & y \in R(A^T),\ x \in N(A) \\ & \Rightarrow y = \alpha_1 v_1 + \cdots + \alpha_n v_n \\ & \Rightarrow y^T x = \alpha_1 v_1^T x + \cdots + \alpha_n v_n^T x = 0 \\ & \Rightarrow y \in (N(A))^\perp \\ & \Rightarrow R(A^T) \subset (N(A))^\perp \end{aligned}$$

We then show that $(N(A))^\perp \subset R(A^T)$. Assuming that $y \in (N(A))^\perp$, $y \notin R(A^T)$ and $x \notin N(A)$. $y \in (N(A))^\perp$ implies

$$y \in (N(A))^\perp \Rightarrow y^T x = \alpha_1 v_1^T x + \cdots + \alpha_n v_n^T x = 0$$

This is a contradiction, so $(N(A))^\perp \subset R(A^T)$. Hence, we have $(N(A))^\perp = R(A^T)$.

Problem 2 (a) The matrix has eigenvalue 2 with multiplicity 3. The eigenvectors of the matrix corresponding to eigenvalue 2 are obtained from the null space of the matrix $B - 2I$, and are found to be

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The generalized eigenvector is found from the null space of the matrix $(B - 2I)^2$, and is found to be

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(b) Since the matrix B has no eigenvalues at 0, it has full row rank and full column rank, and thus the range space is $\mathbb{R}^3$, and the null space is empty.

(c) No! Note that the matrix is diagonalizable if the matrix B has distinct eigenvalues or the dimension of its null space $(B - \lambda I)$ is equal to the multiplicity of eigenvalue λ for every eigenvalue λ. Since the dimension of the null space of $B - 2I$ is 2, the matrix is not diagonalizable.

(d) Yes! There exists a matrix T such that

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ B' = TBT^{-1}$$

Problem 3 (a) Note that $PM = MP$ implies

$$P^2 M = MP^2, \ldots, P^k M = MP^k$$

Then

$$\begin{aligned} e^{Pt}M &= \Big[\sum_{i=0}^{\infty} \frac{(Pt)^i}{i!}\Big] M \\ &= \sum_{i=0}^{\infty} \frac{P^i t^i M}{i!} \\ &= \sum_{i=0}^{\infty} \frac{P^i M t^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{M P^i t^i}{i!} \\ &= M\Big[\sum_{i=0}^{\infty} \frac{(Pt)^i}{i!}\Big] \\ &= Me^{Pt} \end{aligned}$$

(b) Note that from (a), $e^{Pt}M = Me^{Pt}$. Then we have

$$\begin{aligned} \dot{Q} =& \Big(\frac{d}{dt}e^{Pt}\Big)e^{Mt} + e^{Pt}\Big(\frac{d}{dt}e^{Mt}\Big) \\ &= Pe^{Pt}e^{Mt} + e^{Pt}Me^{Pt} \\ &= Pe^{Pt}e^{Mt} + Me^{Pt}e^{Pt} \\ &= (P+M)e^{Pt}e^{Mt} \\ &= (P+M)Q(t) \end{aligned}$$

(c) $\dot{Q} = (P+M)Q$ with $Q(0) = I$. Note that for $\dot{Q} = AQ$, we have $Q(t) = e^{At}I = e^{At}$. Hence,

$$Q(t) = e^{(P+M)t}Q(0) = e^{(P+M)t}$$

Problem 4 (a)

$$\Phi(t,s) = \begin{pmatrix} e^{-(t-s)} & \frac{1}{2}(e^{t+s} + e^{-t+3s}) \\ 0 & e^{-(t-s)} \end{pmatrix}$$

Note that $\Phi(t,s)$ depends on t and s only though their difference $t-s$. Reason: The system is time-varying.

(b) The solution is

$$x(t) = \Phi(t,0)x(0) = \begin{pmatrix} \frac{1}{2}(e^{t} + e^{-t}) \\ e^{-t} \end{pmatrix}$$

(c) The eigenvalues of $A(t)$ are -1 and -1 for all $t \geq 0$, which might indicates that the system is stable. However, since $A(t)$ is time varying, this conclusion does not necessarily follow.

(d) As $t \to \infty$, the first component $x(t)$ is unbounded, whereas the second component converges to zero. Note that $A(t)$ has eigenvalues that have negative real values, but the system does not converge to zero as $t \to \infty$.

HW4

Problem 3 Let $F(t) = e^{-At}Be^{At}$. We need to verify that $\frac{d}{dt}\Phi(t,s) = F(t)\Phi(t,s)$ and $\Phi(s,s) = I$. Then we have

$$\begin{aligned} \frac{d}{dt}\Phi(t,s) &= \frac{d}{dt} e^{-At} e^{(A+B)(t-s)} e^{As} \\ &= e^{-At}(-A)e^{(A+B)(t-s)}e^{As} + e^{-At}(A+B)e^{(A+B)(t-s)}e^{As} \\ &= F(t)\Phi(t,s) \\ \Phi(s,s) &= I \end{aligned}$$

Problem 4

$$\frac{d}{dt} e^{-A} e^{At} = A^{-1}Ae^{At} = e^{At}$$

Hence

$$\int_0^t e^{As} = (A^{-1}e^{At})|_0^t = A^{-1}[e^{At} - I]$$

Therefore

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Buds = e^{At}x_0 + A^{-1}[e^{At} - I]Bu$$

HW5

Problem 1 The characteristic equation is $\lambda^2 + (3-\alpha)\lambda + (1-3\alpha)$.

(a) The system is asymptotically stable if and only if $\alpha < 1/3$.

(b) $\alpha \leq 1/3$

When $\alpha \leq 1/3$, the poles are 0 and $-8/3$. Since 0 is a pole of multiplicity 1, its Jordan block has dimension 1. Hence, we need $\alpha \leq 1/3$.

Problem 2 The matrix A as given in the Jordan canonical form, which 2 Jordan blocks, if both α and β are nonzero; 3 Jordan blocks if either one is zero; and 4 Jordan blocks if both are zero. The eigenvalues are independent of α and β, and are 2 and 0, both with multiplicity 2.

(a) Since A has an eigenvalue of 0, the system is not asymptotically stable for any value of α and β.

(b)It is stable in the sense of Lyapunov if $\beta = 0$ (and no restriction on α). If $\beta \neq 0$, then there is a Jordan block of dimension 2 associated with the zero eigenvalue, which means that the third components of the state x becomes unbounded.

Problem 3 Note that

$$(A + \mu I)^T P + P(A + \mu I) = PA + A^T P + 2\mu P = -Q$$

Therefore, if we can show that $(A + \mu I)$ has all its eigenvalues in the left half part, then $\dot{x} = (A + \mu I)x$ is asymptotically stable and we can use the theorem discussed in class to derive the existence of a solution $P = P^T > 0$ to the above equation.

If λ' is an eigenvalue of $(A + \nu I)$, then

$$0 = \det(A + \mu I - \lambda' I) = \det(A - (\lambda' - \mu)I)$$

which implies $\lambda = \lambda' - \mu$ is an eigenvalue of A. Since $Re(\lambda) < -\mu < 0$ (from the statement of the question), we conclude that $Re(\lambda') < 0$. Hence, we have the first part of the question.

Problem 4 (a) Note that

$$\dot{V} = x^T \begin{pmatrix} 0 & -1 \\ -1 & 6 \end{pmatrix} x,$$

and this is not negative definite, and in fact, is not even non-positive definite. Thus, this function tells us nothing about the stability of the system, and is not a Lyapunov function.

(b)We can see that

$$P = \frac{1}{4} \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} = P^T > 0$$

Hence, the system is asymptotically stable.

(c) For the modified system, we first find the new equilibrium point by setting $\dot{x} = 0$. This produces the equilibrium $\bar{x} = -A^{-1}b = (2, 1)^T$. The Lyapunov function for this new system is then obtained by simply shifting the coordinate in the Lyapunov function for the old system. Hence,

$$V(x) = (x - \bar{x})^T P(x - \bar{x}),$$

where P is given in part (b).

HW6

Problem 3

1. Note that

$$x(t) = \int_0^t e^{-(t-\tau)} u d\tau = -u(1 - e^{-t})$$

So to have $x(t) = 1$, we just need to choose $u = 1/(e^{-t} - 1)$.

2. The control is not unique. The easiest thing to see is that you could wait to turn on the control until some time t_0 still using a constant control. The control you now use would simply have to be larger than before:

$$x(t) = \int_{t_0}^{t} e^{-(t-\tau)} u d\tau = -u(1 - e^{t_0 - t})$$

So to have $x(t) = 1$, we need to choose

$$u = \begin{cases} \frac{1}{e^{t_0 - t} - 1} & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$

3. The problem is no longer solvable. To see why, consider the extreme cases where $u = 1$ and $u = -1$:

$$\begin{aligned} u = 1 \quad x(t) = 1 - e^{-t} &\;\Rightarrow\; 0 \leq x(t) \leq 1 \\ u = -1 \quad x(t) = e^{-t} - 1 &\;\Rightarrow\; -1 \leq x(t) \leq 0 \end{aligned}$$

In either case, $x(t)$ will never reach $x = 1$.

4. The problem is no longer solvable. To see why, consider the extreme cases where $u = 1$ and $u = -1$:

$$\begin{aligned} u = 1 \quad x(t) = e^{t} - 1 &\;\Rightarrow\; x(t) = 1,\; t = \ln 2 \\ u = -1 \quad x(t) = 1 - e^{-t} &\;\Rightarrow\; -\infty \leq x(t) \leq 0 \end{aligned}$$

When $u = 1$, $x = 1$ when $t = \ln 2$, which implies that $x = 1$ can be reached at some point in time, but not necessarily at any time.

Problem 4 (a) The system is controllable.

(b) By changing A to

$$\begin{pmatrix} -1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the system is not controllable.

Problem 5 The controllability matrix can be computed by

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4 \end{pmatrix}$$

Obviously, the system is not controllable. The transformation matrix that we can use

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Hence,

$$\bar{A} = P^{-1}AP = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}, \quad \bar{B} = P^{-1}B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvalue 2 is the controllable one.

HW7

Problem 1 (a) Since there is a pole-zero cancellation, the minimal realization is

$$\dot{x} = 2x + u,\ y = x$$

(b)

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u,\ y = (1\ 0\ 0)x$$

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u,\ y = (0\ 0\ 1)x$$

HW8

Problem 3 Using the output feedback, $u = -Ky$, we have

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1-K & 0 \end{pmatrix} x$$

Note that the eigenvalues of the above matrix are $\lambda = \pm\sqrt{1+K}$. Hence, we cannot stabilize the system.

We now deisgn an observer

$$\dot{\hat{x}} = (A + LC)\hat{x} - LCx + BK\hat{x}.$$

We choose the pole $\{-10, -10\}$ for the estimator. Then $L = (-20\ \ -101)^T$.

Moreover, with the pole $\{-1-1\}$ for the pole-placement, we have $K = (-2\ 0)$.