Linear System Theory

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Chapter 5: Stability

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Recap

- State space equation
- Linear Algebra
- Solutions of LTI and LTV system

Stability

Stability: fundamental objective of designing a control system

Unstable system: the state of the system diverges

What is stability? $\Rightarrow$ There are several notions of stability of the dynamical system

Stability

We will study..

- The state of the system is bounded for every bounded input: input-output (BIBO) stability
- The state of the system is bounded for any initial conditions: stability in the sense of Lyapunov
- The state of the system converges to zero as $t \to \infty$ for every initial condition: globally asymptotically stability
- BIBO and asymptotic stability of LTI systems

Stability

Example:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} x, \;\; x(0) = x_0$$

The system is unstable for nonzero initial conditions. That is

$$\lim_{t \to \infty} \|x(t)\| = \infty$$

Stability

Example:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x, \;\; x(0) = x_0$$

The system is asymptotically stable for every initial condition. That is

$$\lim_{t \to \infty} \|x(t)\| = 0$$

Stability

Example:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \; y = \begin{pmatrix} -2 & 1 \end{pmatrix} x$$

$$\det(sI - A) = s^2 - s - 2 = 0, \; s = -1, 2$$

$$G(s) = \frac{s-2}{s^2 - s - 2} = \frac{1}{s+1} \quad \text{unstable pole cancellation}$$

The system is stable for zero initial condition and bounded input (BIBO stable)

The system is unstable for non-zero initial condition when the input is zero or for some initial conditions with the bounded input

The issue of the pole-zero cancellation will be discussed in the section of controllability and observability

Stability in the Sense of Lyapunov

Definition (stability in the sense of Lyapynov)
$\Rightarrow$ Formal definition of Definition 5.1 of the textbook
$\Rightarrow$ The weakest stability concept

Consider a nonlinear system

$$\dot{x} = f(x), \ x(0) = x_0$$

We denote the solution by $x(t)$ or $\phi(t; x_0)$: the latter emphasizes the role of the initial condition.
We also note that x_e is an equilibrium if $f(x_e) = 0$.

Stability in the Sense of Lyapunov

We can assume that the equilibrium point is origin, i.e., $x_e = 0$

Consider the autonomous system:

$$\dot{x} = f(x)$$

W.O.G. we assume that the equilibrium point of $f(x)$ is $x_e = 0$, i.e., $f(x_e) = 0$.

- Assume that $x_e \neq 0$. Let $y = x - x_e$. Then

  $$\dot{y} = \dot{x} = f(x) = f(y + x_e) =: g(y)$$

  Hence $g(0) = 0$
- The linear system

  $$\dot{x} = Ax$$

  We can see that $x_e = 0$

Stability in the Sense of Lyapunov

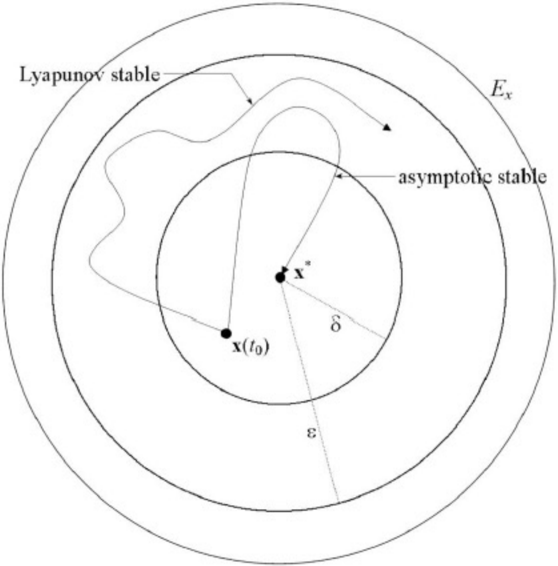
Definition (stability in the sense of Lyapynov)

The equilibrium x_e is stable in the sense of Lyapunov if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x_0 - x_e\| < \delta \;\Rightarrow\; \|\phi(t; x_0) - x_e\| < \epsilon, \;\forall t \geq 0$$

Note that the stability is defined with respect to the equilibrium point. If an initial condition is close to the equilibrium, then it will stay close forever.

Stability in the Sense of Lyapunov



Stability in the Sense of Lyapunov

Back to the linear system
$\Rightarrow$ In the linear system x_e is an equilibrium if and only if $x_e \in N(A)$
$\Rightarrow$ The origin is the equilibrium point of the linear system (since $0 \in N(A)$)

Definition (Definition 5.1 of the text book)
The response of $\dot{x} = Ax$ is stable in the sense of Lyapunov if for any finite initial condition, the response (state) is bounded

Stability in the Sense of Lyapunov

Theorem (Generalized version of Theorem 5.4 (a) of the textbook)
The LTI system is stable in the sense of Lyapunov if and only if

- $Re(\lambda) \leq 0$ for every eigenvalue λ of A
- If $Re(\lambda) = 0$, where λ is an eigenvalue of A with multiplicity of m, then there exists m linearly independent corresponding eigenvectors.

Stability in the Sense of Lyapunov

Example

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$e^{A_1 t} = I, \; e^{A_2 t} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

The system with A_1 is stable in the sense of Lyapunov for any finite initial conditions

The system with A_2 is stable in the sense of Lyapunov with $x_0 = (\alpha, 0)^T$, but it is *not* stable with $x_0 = (\beta, \gamma)^T$, where $\gamma \neq 0$

The theorem means that when $Re(\lambda) = 0$, where λ is an eigenvalue of A with multiplicity of m, we need

$$nullity(\lambda I - A) = m$$

Globally Asymptotically Stability

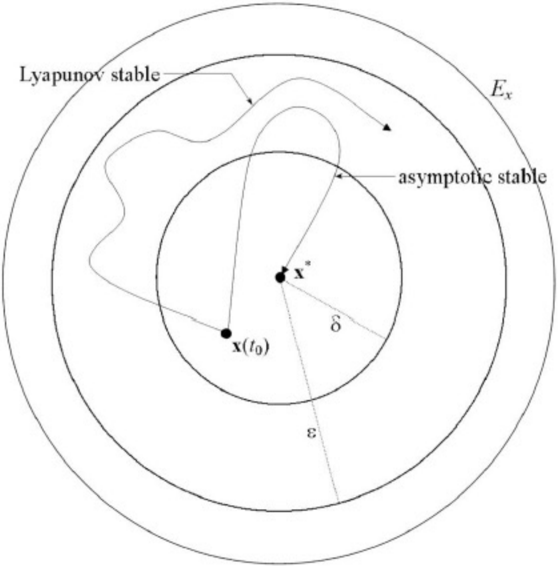
Stable in the sense of Lyapunov: does not guarantee the state converging to the equilibrium point (or to the origin for the LTI system): just guarantees boundedness of state

Definition: Asymptotic stability
An equilibrium x_e is globally asymptotic stable if for some $\|x_0 - x_e\| < \infty$, we have

$$\lim_{t \to \infty} \phi(x, x_0) = x_e$$

The stability is defined with respect to the equilibrium point

Stability in the Sense of Lyapunov



Globally Asymptotically Stability

For the LTI system, the origin is an equilibrium point. Hence, for asymptotic stability of the LTI system, we have $N(A) = \{0\}$. Note that by globally asymptotic stability, we cannot have more than one equilibrium.

Theorem (Theorem 5.4(b) of the text book)
An LTI system model is asymptotic stable if and only if $Re(\lambda) < 0$ for every eigenvalue λ of A

Lyapunov Direct Method (Second Method, page 170)

We study asymptotic stability for general nonlinear systems

Lyapunov Direct Method (Second Method, page 170)

Consider the function $V(x)$ defined over

$$V : \mathbb{R}^n \to \mathbb{R}$$

Note that

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) = \frac{d}{dt} V(\phi(t, x_0))\Big|_{t=0}$$

This means that if $\dot{V}$ is negative, V will decrease along the solution of the state equation.

Lyapunov Direct Method (Second Method, page 170)

A scalar-valued function $V : \mathbb{R}^n \to \mathbb{R}$ with continuous partial derivatives is called positive definite on an open region $\Omega \subset \mathbb{R}^n$ about the equilibrium point x_e if

$$V(x_e) = 0, \;\; V(x) > 0, \;\; \forall x \in \Omega, \;\; x \neq x_e$$

- Note that x_e can be origin
- Example

$$V(x) = x^T P x, \quad P = P^T > 0$$

Then $V(x) > 0$ for all $x \neq 0$.

Lyapunov Direct Method (Second Method, page 170)

The Lyapunov stability theorem: one of the most important theorems for control systems.

It is a sufficient condition to verify stability for linear and nonlinear systems

The idea is to search for a "bowl-shaped" function V on the state space $\mathbb{R}^n$. If $V(x(t))$ is decreasing in time, then we can qualitatively describe the behavior of the trajectory $x(t)$ and determine if the system is stability.

Lyapunov Direct Method (Second Method, page 170)

Theorem: Lyapunov stability theorem
(not in the textbook, I am not going to prove it)

Suppose that $x_e \in \mathbb{R}^n$ is an equilibrium point, and that $\Omega \subset \mathbb{R}^n$ is an open set containing x_e. Then

(i) x_e is stable in the sense of Lyapunov if there exists a positive definite function V such that

$$\frac{d}{dt}V(x(t)) \leq 0, \text{ whenever } x(t) \in \Omega$$

Lyapunov Direct Method (Second Method, page 170)

(ii) x_e is globally asymptotic stable if there exists a positive definite function V such that $V(x) \to \infty$ as $x \to \infty$, and

$$\frac{d}{dt} V(x(t)) < 0, \ \forall x(t), \ x(t) \neq x_e$$

$V(t)$ is called a Lyapunov function. If V satisfies properties in (ii), V is radially unbounded

How to find $V(t)$ for the nonlinear system? $\Rightarrow$ There are no general methods. It is hard!!! However, for the linear system, we can find it easily.

Lyapunov Direct Method (Second Method, page 170)

Example: Consider the pendulum system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2, \quad a, b > 0$$

We set

$$V(x) = \frac{1}{2} x^T P x + a(1 - \cos x_1)$$

- $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} > 0$: positive definite
- P: $p_{11} > 0$ and $p_{11} p_{22} - p_{12}^2 > 0$

Lyapunov Direct Method (Second Method, page 170)

We have

$$\dot{V}(x) = a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 \\ + (p_{11} - p_{12}b)x_1x_2 + (p_{12} - p_{22}b)x_2^2$$

Let $p_{22} = 1$ and $p_{11} = bp_{12}$. Then from the constraint in P, $bp_{12} - p_{12}^2 = p_{12}(b - p_{12}) > 0$. Hence, $p_{12} \in (0, b)$. Let $p_{12} = b/2$. Then

$$\dot{V}(x) = -\frac{1}{2}abx_1 \sin x_1 - \frac{1}{2}bx_2^2$$

Note that $x_1 \sin x_1 > 0$ for all $|x_1| < \pi$. This implies

$$\dot{V}(x) < 0, \quad \forall x \in D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$$

Asymptotic stable (but not globally)

Lyapunov Direct Method (Second Method, page 170)

Example: Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -h(x_1) - ax_2, \ \ a > 0$$

- h: Lipschitz, $h(0) = 0$ and $yh(y) > 0$ for all y

Lyapunov Direct Method (Second Method, page 170)

Consider with $\delta > 0$ and $k \in (0, 1)$

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y) dy$$

- $V(x) > 0$
- $V(x) \to \infty$ as $\to \infty$: radially unbounded

Lyapunov Direct Method (Second Method, page 170)

We can show that

$$\dot{V}(x) = -a\delta(1-k)x_2^2 - a\delta k x_1 h(x_1) < 0, \quad \forall x \in \mathbb{R}^2$$

Note

- $k \in (0,1)$, $a > 0$, $\delta > 0$
- $yh(y) > 0$

Globally asymptotic stable

Lyapunov Direct Method (Second Method, page 170)

Consider the linear system

$$\dot{x} = Ax$$

We set

$$V(x) = x^T Px, \; P\text{: symmetric positive definite}$$

Note that $V(0) = 0$, $V(x) > 0$ for all $x \in \mathbb{R}^n$. Moreover, $V(x) \to \infty$ as $x \to \infty$.

- V: candidate Lyapunov function
- V: radially unbounded

Lyapunov Direct Method (Second Method, page 170)

For $\dot{x} = Ax$, we have

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= \frac{dV(x)}{dx}\frac{dx(t)}{dt} \\ &= 2x^T PAx = x^T(A^T P + PA)x\end{aligned}$$

since $x^T PAx = x^T A^T P^T x = x^T A^T Px$. Hence, $\dot{x} = Ax$ is asymptotically stable if for any given symmetric positive definite matrix Q, there exists a symmetric positive definite matrix P such that

$$\frac{d}{dt}V(x(t)) = x^T(A^T P + PA)x < 0 \Leftrightarrow A^T P + PA = -Q$$

Lyapunov Direct Method (Second Method, page 170)

The equation $A^T P + PA = -Q$ is known as the Lyapunov equation

Since the result holds for any $Q > 0$, we can simply use $Q = I$

The general Lyapunov stability theorem is a sufficient condition. Namely, if we can find a Lyapunov function $V(x)$ that satisfies $\frac{d}{dt}V(x(t)) < 0$, then the equilibrium point is globally asymptotically stable. But the converse it not generally true. However, for the LTI system, the converse is also true.

Lyapunov Direct Method (Second Method, page 170)

Theorem (Theorem 5.4(b) of the text book)
An LTI system model is asymptotic stable if and only if $Re(\lambda) < 0$ for every eigenvalue λ of A

Lyapunov Direct Method (Second Method, page 170)

Theorem (Theorems 5.5 and 5.6 of the textbook)

- The LTI system $\dot{x} = Ax$ is asymptotically stable if and only if for some $Q > 0$, there exists a solution $P > 0$ to the Lyapunov equation.
- Moreover, if the LTI system is asymptotically stable, then with a given $Q > 0$, the solution of the Lyapunov equation can be written as

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

Lyapunov Direct Method (Second Method, page 170)

Proof

The sufficiency follows from the Lyapunov stability theorem stated above.

The necessity (similar to the proof on page 171).

Suppose that the model is asymptotically stable. Then $Re(\lambda) < 0$ for any λ, so $e^{At}x \to 0$ as $t \to \infty$. Let

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt,$$

where $Q > 0$. Note that P is well defined, since $e^{At}x \to 0$ as $t \to \infty$.

Lyapunov Direct Method (Second Method, page 170)

Consider

$$x^T P x = \int_0^\infty x(t) Q x dt, \;\; x(0) = x$$

Note that

$$\frac{d}{dt} e^{A^T t} Q e^{At} = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A$$

integrating from 0 to s

$$e^{A^T t} Q e^{At} - e^{A^T 0} Q e^{A0}$$

$$= e^{A^T t} Q e^{At} - Q$$

$$= \int_0^s \frac{d}{dt} e^{A^T t} Q e^{At}$$

$$A^T \left(\int_0^s e^{A^T t} Q e^{At} dt \right) + \left(\int_0^s e^{A^T t} Q e^{At} dt \right) A$$

Lyapunov Direct Method (Second Method, page 170)

By taking $s \to \infty$, we have

$$-Q = A^T \left(\int_0^\infty e^{A^T t} Q e^{At} dt \right) + \left(\int_0^\infty e^{A^T t} Q e^{At} dt \right) A = A^T P + PA$$

Lyapunov Indirect Method (First Method)

The indirect method uses a linearized system with respect to the equilibrium point. This method verifies stability of the nonlinear system without the Lyapunov function. The method is useful when you can linearize the system.

Assume that the equilibrium point is $x_e = 0$, i.e., $f(0) = 0$. Suppose that

$$\dot{x} = f(x) \Rightarrow \text{linearization} \quad \dot{x} = Ax$$

Lyapunov Indirect Method (First Method)

Theorem (not in the textbook)
For the nonlinear system model above

- If the linearized system is asymptotically stable, then $x_e = 0$ is also asymptotically stable for the nonlinear system.
- If A has an eigenvalue with $Re(\Lambda) > 0$, then $x_e = 0$ is not stable in the sense of Lyapunov; hence, it is not asymptotically stable.
- If $Re(\lambda) \leq 0$ for all λ, but $Re(\lambda) = 0$ for some λ, then nothing can be said about the stability of the nonlinear system.

Input-Output Stability: BIBO Stability

We have seen the stability concepts for the zero input.
Consider the SISO system with the zero initial condition

$$\dot{x} = Ax + Bu, \;\; y = Cx, \;\; x(0) = 0$$
$$\hat{g}(s) = C(sI - A)^{-1}B$$

A input $u(t)$ is bounded if $|u(t)| \leq u_m$ for all $t \geq 0$.

Input-Output Stability: BIBO Stability

Definition (Input-output (BIBO) Stability for a SISO system (page 153))
A state space model with input u and output y is bounded input and bounded output (BIBO) stable if for any for any bounded input, the output $y(t)$ is bounded $t \geq 0$, i.e., $|y(t)| \leq y_m$ for all $t \geq 0$.

Note that for the LTI system, the asymptotic stability of $\dot{x} = Ax$ implies BIBO stability. However, the converse is not true, since pole-zero cancellation may occur: Think about the example on page 4 of the slides (or page 161 of the textbook).

Input-Output Stability: BIBO Stability

Theorem (Theorem 5.1 of the textbook)
A SISO system is BIBO stable if and only if there exists a constant $M \geq 0$ such that

$$\int_0^\infty |g(t)|dt = \int_0^\infty |Ce^{At}B|dt \leq M$$

Input-Output Stability: BIBO Stability

Proof (sufficiency, necessity: see the textbook)
Note that

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \\ &= \int_0^t Ce^{As}Bu(t-s)ds, \text{ change variable } s = t-\tau \end{aligned}$$

Then

$$\begin{aligned} |y(t)| = |\int_0^t Ce^{As}Bu(t-s)ds| &\leq \int_0^t |Ce^{As}B||u(t-s)|ds \\ &\leq \int_0^\infty |Ce^{As}B||u(t-s)|ds \\ &\leq Mu_m =: y_m \end{aligned}$$

Input-Output Stability: BIBO Stability

Theorem (Theorem 5.3 and Corollary 5.3 of the textbook)

Theorem 5.3: A SISO system is with proper rational transfer function is BIBO stable if and only if every pole has a negative real part.

Corollary 5.3: A SISO system is with proper rational transfer function, $g(t)$, is BIBO stable if and only if $\lim_{t\to\infty} g(t) = 0$.

$\Leftrightarrow \int_0^\infty |Ce^{At}B|dt \leq M$ if and only if every eigenvalue of A has a negative real part, or $\lim_{t\to\infty} Ce^{At}B = 0$

Input-Output Stability: BIBO Stability

BIBO stability for the MIMO system: the definition is the same, but we need norm $\| \cdot \|$

Theorem (Theorems 5.M.1 and 5.M.2 of the textbook)

- A MIMO system with $G(t) = [g_{ij}(t)]$ is BIBO stable if and only if $\int_0^\infty |g_{ij}(t)|dt \leq M_{ij}$ for all i and j
- A MIMO system with proper transfer function matrix is BIBO stable if and only if every pole of the transfer function has a negative real part.

Input-Output Stability: BIBO Stability

Hence, for the BIBO stability of the MIMO system, we need

$$\int_0^{\infty} \|Ce^{At}B\| dt \leq M$$

Discrete-Time LTI System

Discrete-Time system

$$x(k+1) = Ax(k) + Bu(k), \; y(k) = Cx(k)$$

BIBO stability: page 162

The system is asymptotically stable if and only if

- Every eigenvalue of A is inside of the unit circle
- For any $Q > 0$, there exists a $P > 0$ that is a solution to the Lyapunov equation

$$A^T PA + P = -Q, \; P = \sum_{k=0}^{\infty} (A^T)^k QA^k$$

Discrete-Time LTI System

Note that if there exists a Lyapunov function $V(x(k))$ such that

$$V(x(k+1)) - V(x(k)) < 0,$$

then the discrete-time system is stable. Hence, $V(x) = x^T P x$, where $P > 0$, we have

$$V(x(k+1)) - V(x(k)) = A^T PA + P < 0 \Leftrightarrow A^T PA + P = -Q$$

Conclusions

Stability

- BIBO stability
- Asymptotic stability
- Lyapunov stability theorems

Next

- Controllability and observability