[MEN573]
Advanced Control Systems I

Lecture 10.1 – Discrete Time Models from Sampling Continuous Time Models

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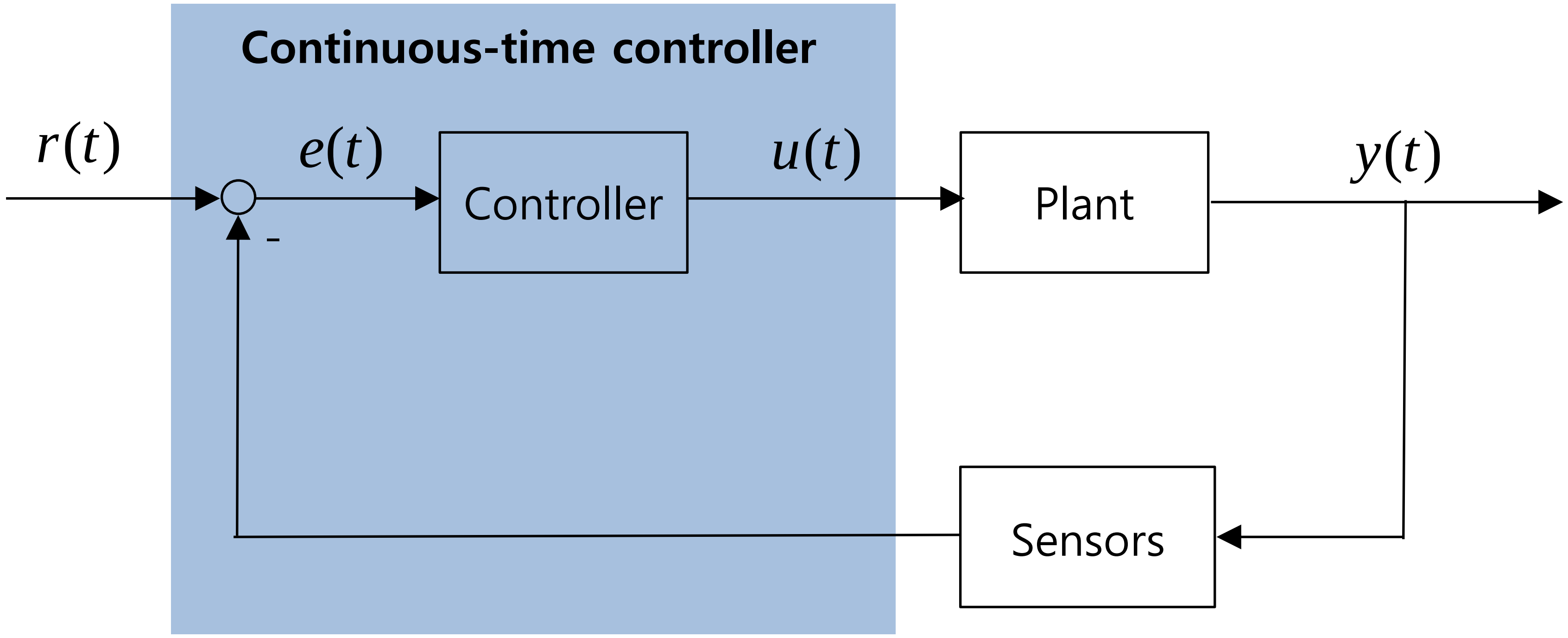
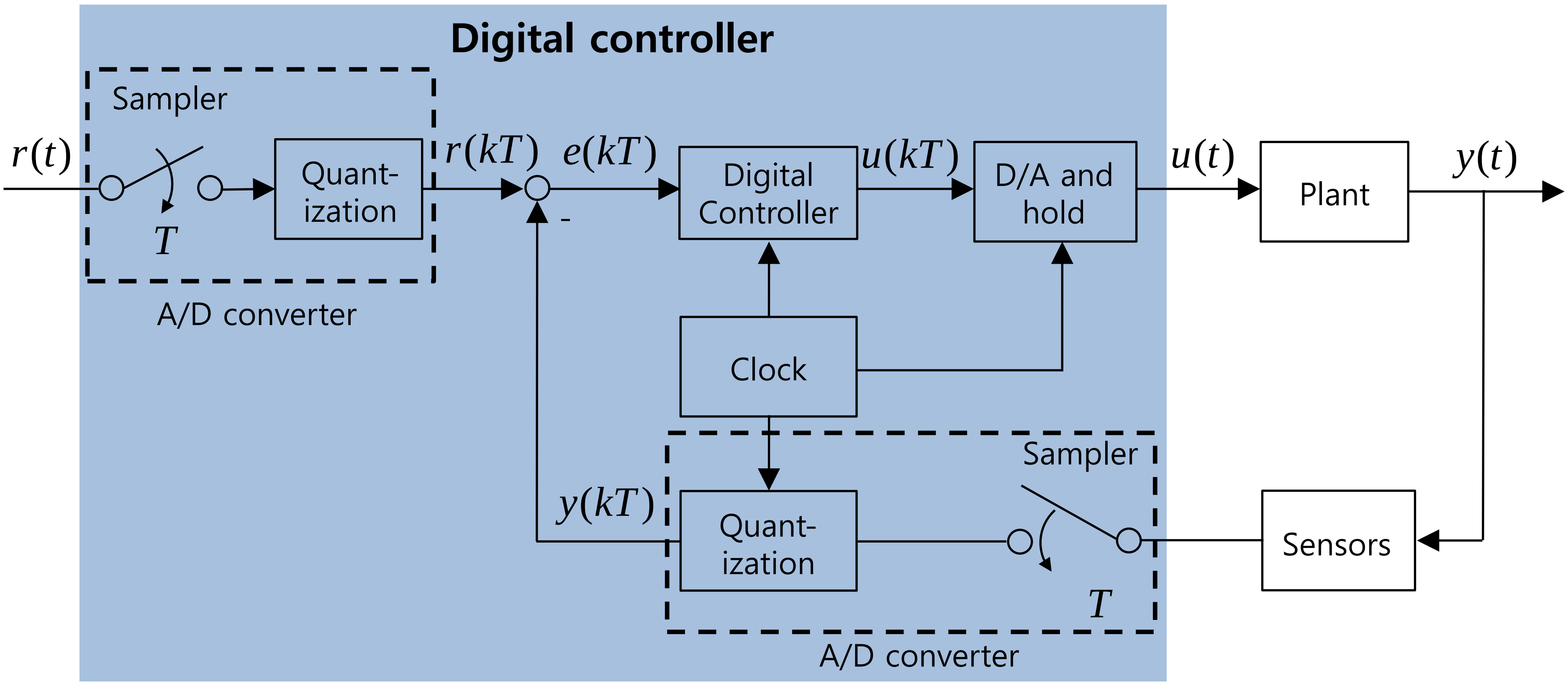
Continuous-Time Controller



Discrete-Time Controller



Design of Discrete Time Controller

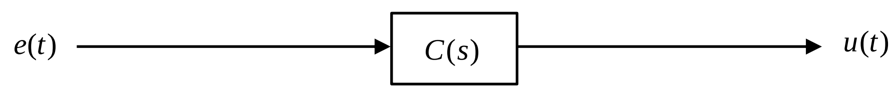
- Indirect design approach
  - Design a continuous time (analog) controller in the continuous time domain, and redesign the continuous time controller to a digital controller.
  - GOOD
    - 1) rich knowledge on the design of controllers in the continuous time domain
    - 2) far easier to understand the dynamics of and performance specification for the plant in the continuous time domain than in the discrete time domain.
  - BAD
    - 1) the redesigned controller is an approximation of the continuous time controller, which affects the performance of the digital control system
    - 2) the computational delay and a data hold will decrease the phase margin unless they are properly taken into consideration

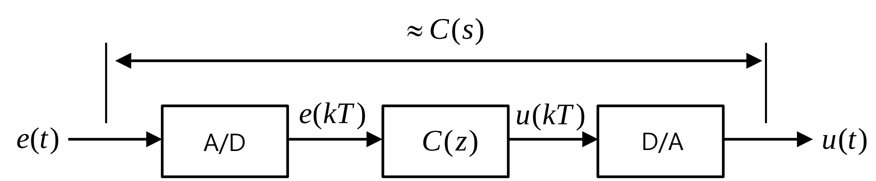
Design of Discrete Time Controller

- Direct design approach
  - Design a discrete time (digital) controller in the discrete time domain based on a discrete time model of the controlled plant.
  - GOOD
    - Zero order hold (ZOH) equivalent of the plant is an exact model of the controlled plant => NO approximation
  - BAD
    - Discrete control theory is required.

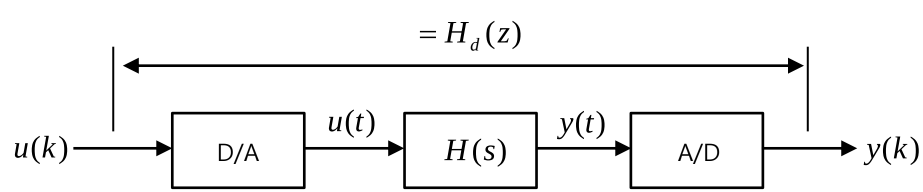
Design of Discrete Time Controller

- Indirect approach
  - Translating an existing continuous-time controller to a discrete-time controller using various approximations (emulation).



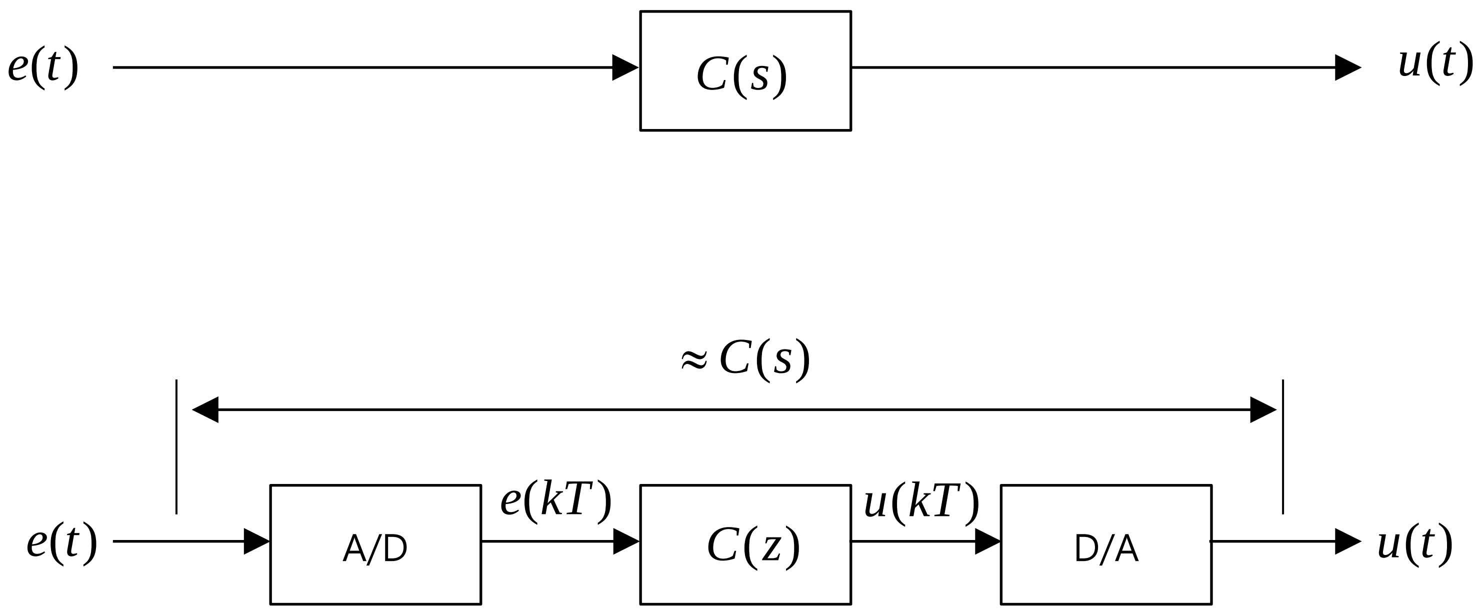


- Direct approach
  - Designing discrete-time controller in the state-space or transfer function domain. (discretized plant)



Approximating Continuous-Time Controllers (Emulation)

- The technique to “translate” analog designs into digital designs is important in the early days of digital control.



Numerical Integration (Approximate Differentiation)

- What is the equivalent the differential operator (d/dt or s) in terms of the shift operator (z)?
- Let's start with a single integrator system.

$$\dot{u}(t) = e(t) \qquad \text{or} \qquad C(s) = \frac{U(s)}{E(s)} = \frac{1}{s}$$

- The solution to the system is

$$u(t) = u(t_0) + \int_{t_0}^{t} e(\tau)d\tau$$

- At the sample instants

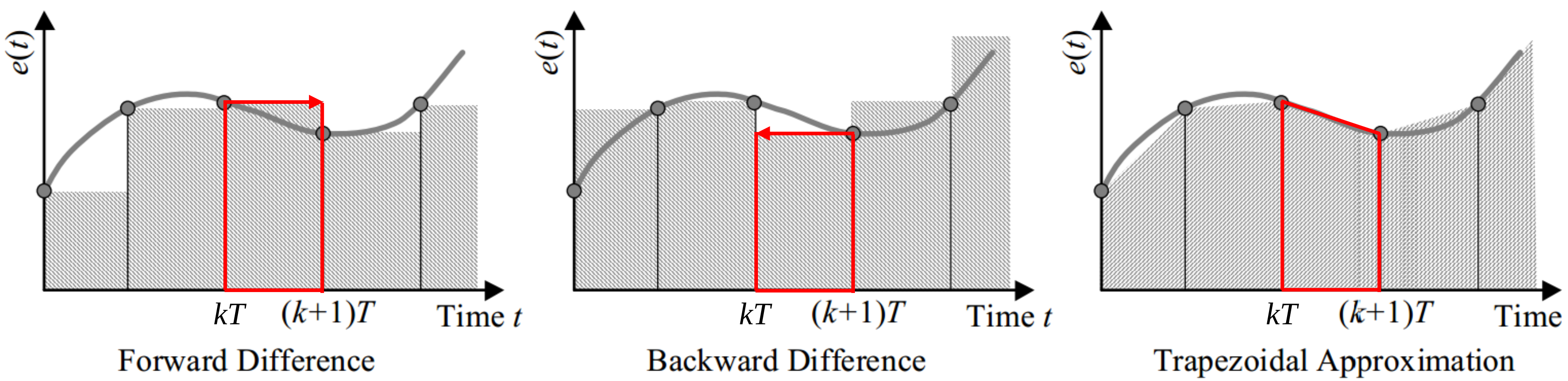
$$u\big((k+1)T\big) = u(kT) + \int_{kT}^{(k+1)T} e(\tau)d\tau$$

Numerical Integration (Approximate Differentiation)

- Forward difference $$u(k+1) \approx u(k) + e(k) \cdot T$$

- Backward difference $$u(k+1) \approx u(k) + e(k+1) \cdot T$$

- Trapezoid Approximation (Bilinear transformation, Tustin's approximation) $$u(k+1) \approx u(k) + \frac{[e(k+1) + e(k)]}{2} \cdot T$$



Numerical Integration
(Approximate Differentiation)

- Forward difference
$$C(z) = \frac{U(z)}{E(z)} = \frac{T}{z-1} = \frac{Tz^{-1}}{1-z^{-1}}$$

- Backward difference
$$C(z) = \frac{U(z)}{E(z)} = \frac{Tz}{z-1} = \frac{T}{1-z^{-1}}$$

- Trapezoid Approximation
$$C(z) = \frac{U(z)}{E(z)} = \frac{T}{2}\frac{z+1}{z-1} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

Numerical Integration (Approximate Differentiation)

- Forward difference $\quad s \rightarrow \dfrac{z-1}{T} \quad$ i.e. $\quad C(z) = C(s)\big|_{s \rightarrow \frac{z-1}{T}}$

- Backward difference $\quad s \rightarrow \dfrac{z-1}{Tz} \quad$ i.e. $\quad C(z) = C(s)\big|_{s \rightarrow \frac{z-1}{Tz}}$

- Trapezoid Approximation $\quad s \rightarrow \dfrac{T}{2}\dfrac{z-1}{z+1} \quad$ i.e. $\quad C(z) = C(s)\big|_{s \rightarrow \frac{2}{T}\frac{z-1}{z+1}}$

Example

- Using the three approximation methods to find the discrete-time equivalent of a lead compensator.

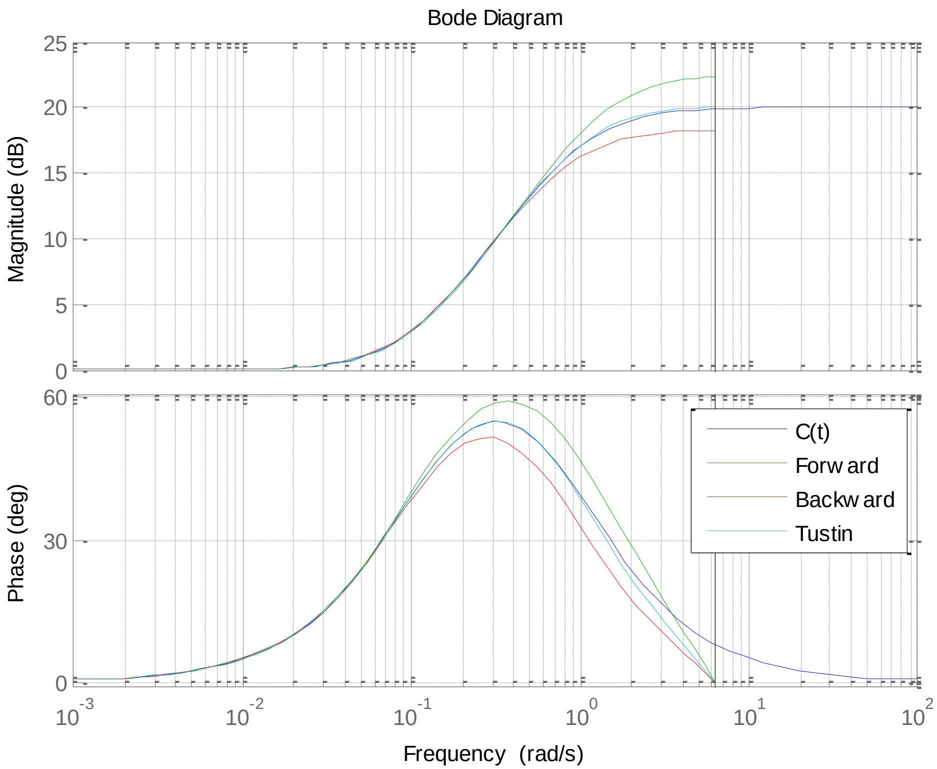
$$C(s) = \frac{10s+1}{s+1}$$

$$\textit{Forward Difference}: \quad C(z) = C(s)\Big|_{s \to \frac{z-1}{T}} = \frac{10z-(10-T)}{z-(1-T)}$$

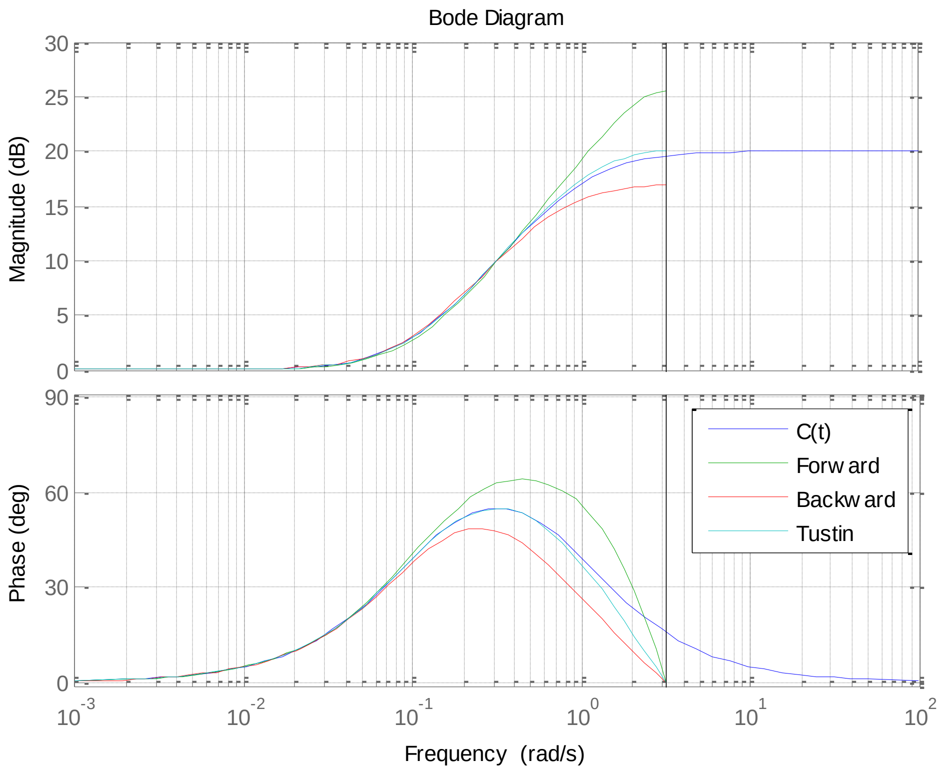
$$\textit{Backward Difference}: \quad C(z) = C(s)\Big|_{s \to \frac{z-1}{Tz}} = \frac{(10+T)z-10}{(1+T)z-1}$$

$$\textit{Trapezoidal Approximation}: \quad C(z) = C(s)\Big|_{s \to \frac{2}{T}\frac{z-1}{z+1}} = \frac{(20+T)z-(20-T)}{(2+T)z-(2-T)}$$

Example



T=0.5



T=1

Stability

- As an example, for forward difference approximation,

$$\mathrm{Re}(s) = \mathrm{Re}\left(\frac{z-1}{T}\right) < 0 \qquad \Rightarrow \qquad \mathrm{Re}(z-1) < 0$$

Let $z = \sigma + j\omega$

$$\mathrm{Re}(\sigma + j\omega - 1) = \sigma - 1 < 0 \qquad \Rightarrow \qquad \sigma < 1$$

Stability

Forward rule	Backward rule	Trapezoid rule
$s \leftarrow \dfrac{z-1}{T}$	$s \leftarrow \dfrac{z-1}{Tz}$	$s \leftarrow \dfrac{2}{T}\dfrac{z-1}{z+1}$
$z = 1 + Ts$	$z = \dfrac{1}{1-Ts}$	$z = \dfrac{1+Ts/2}{1-Ts/2}$