[MEN573]
Advanced Control Systems I

Lecture 10.2 – Discrete Time Models from Sampling Continuous Time Models

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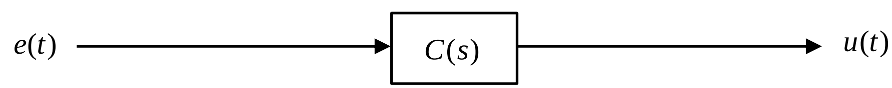
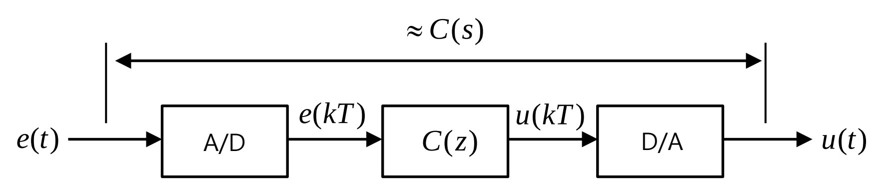
Department of Mechanical Engineering
UNIST

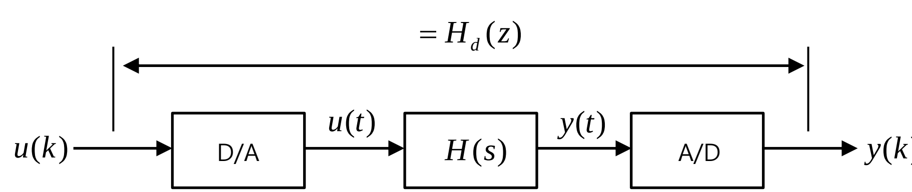
Design of Discrete Time Controller

- Indirect approach
  - Translating an existing continuous-time controller to a discrete-time controller using various approximations (emulation).



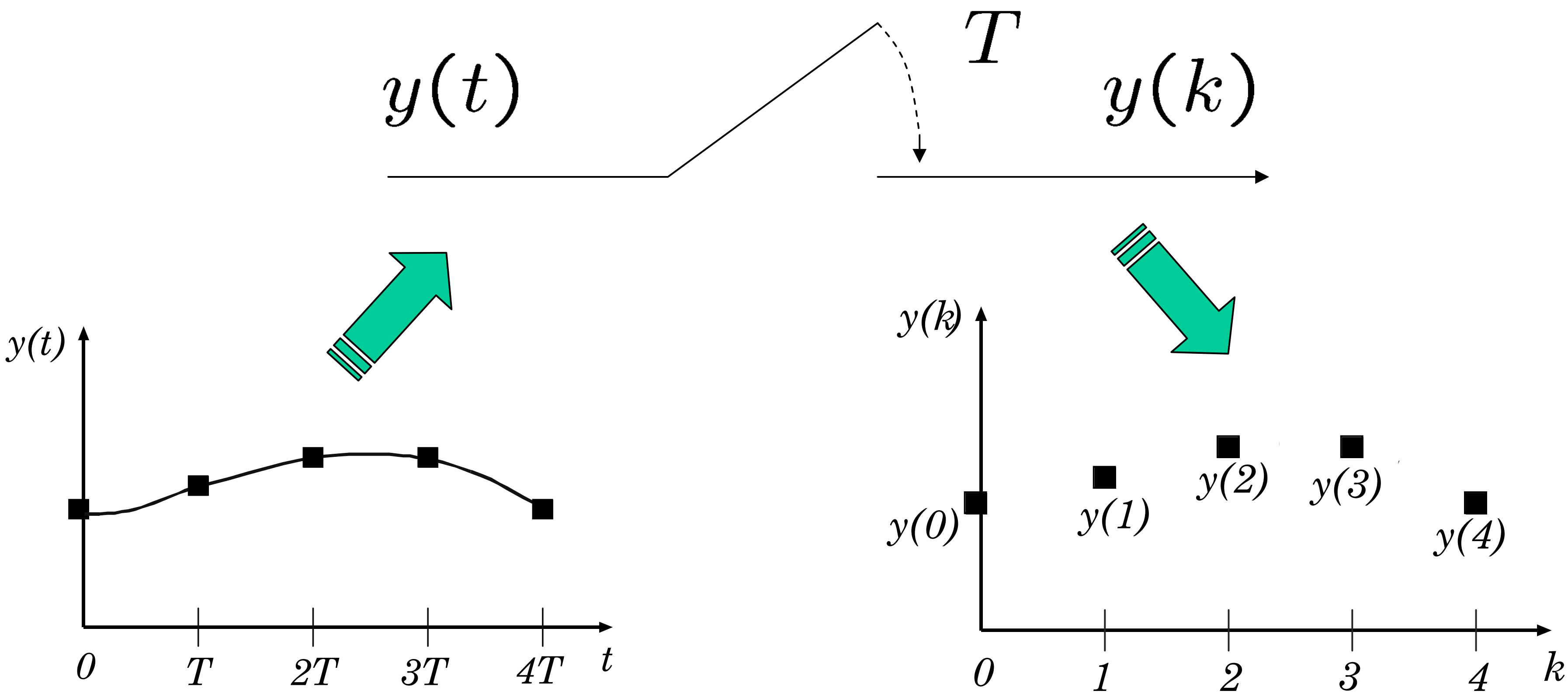


- Direct approach
  - Designing discrete-time controller in the state-space or transfer function domain. (discretized plant)



Sampler

Converts a time function into a sequence

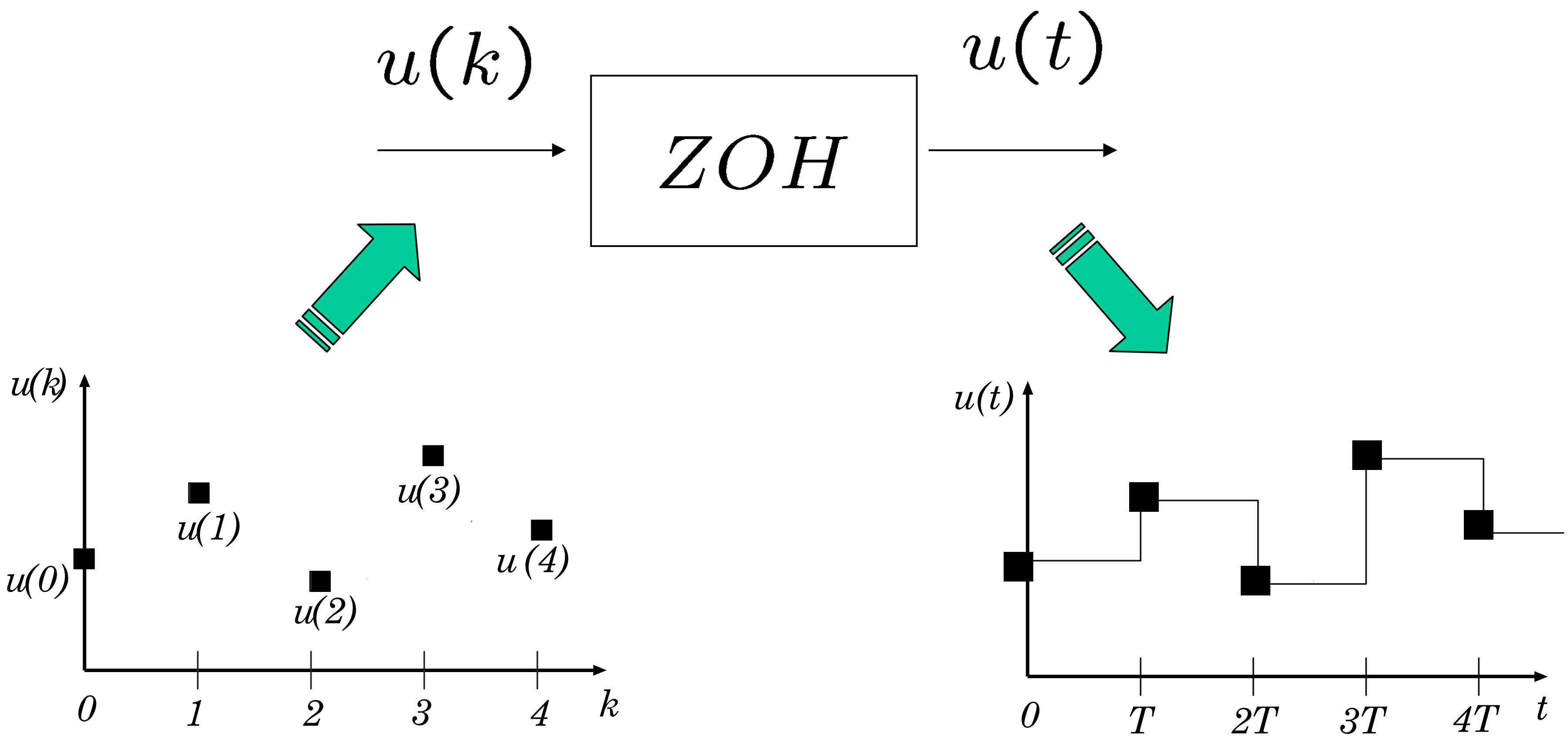


T : sampling time

$$y(k) = y(Tk)$$

Zero-order hold

Converts a sequence into a “stair-case” time function

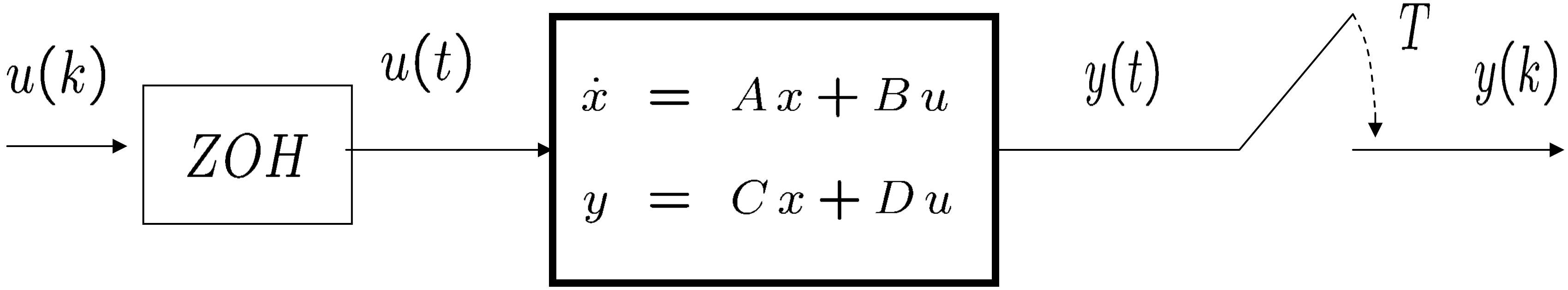


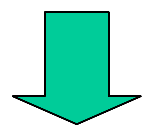
$$u(t) = u(k) \ \text{ for } t \in [kT, (k+1)T)$$

Discrete time models from continuous time models

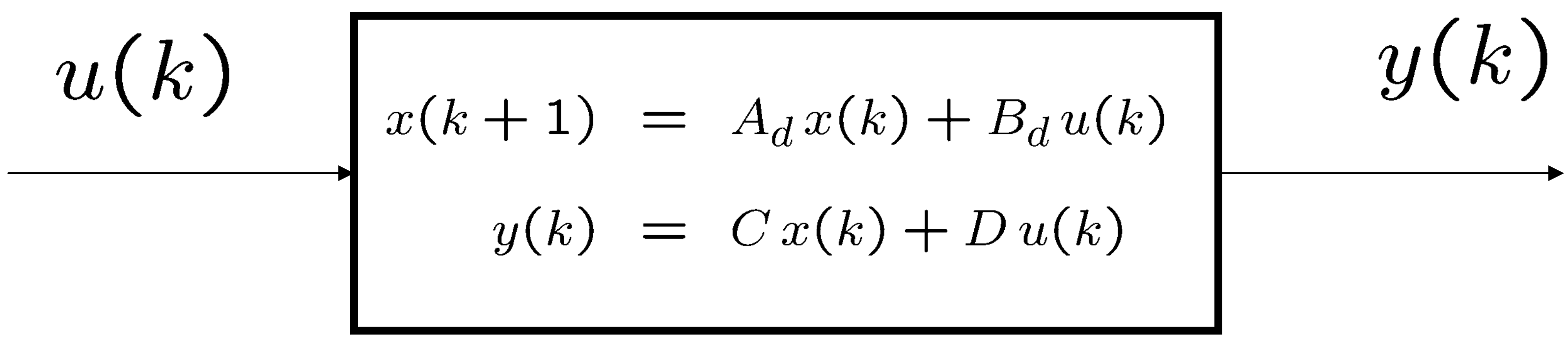
State space (SS) models:

continuous time

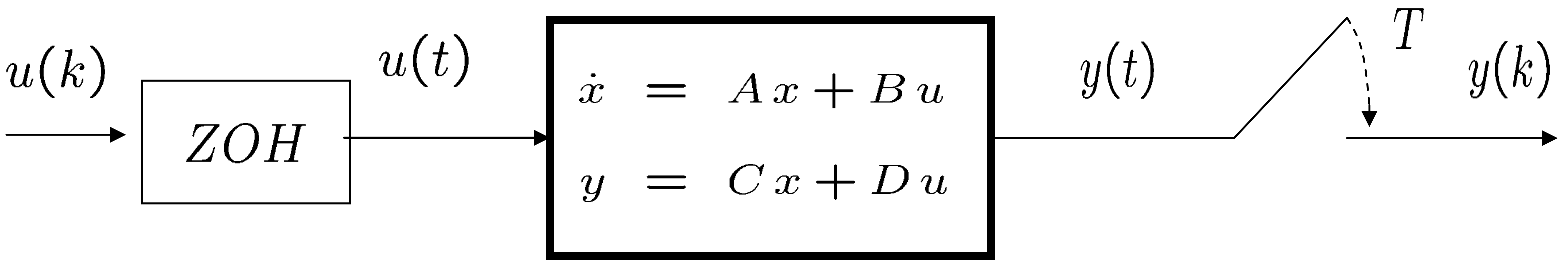


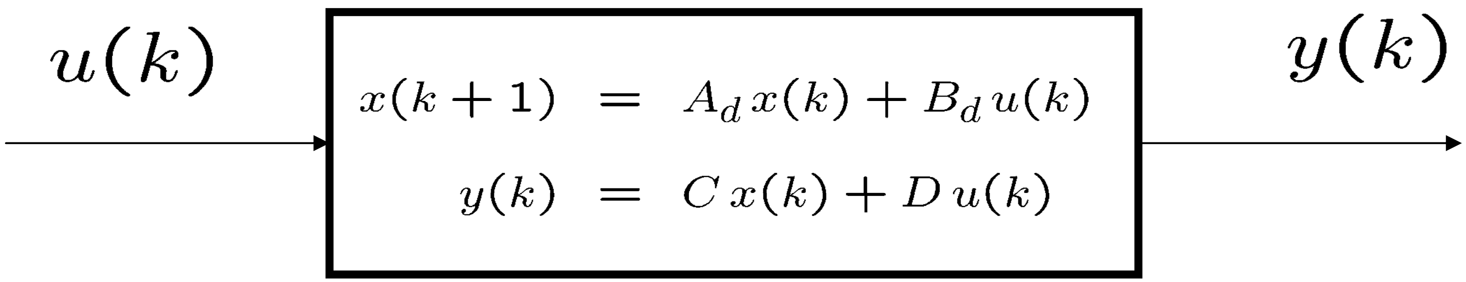


discrete time equivalent system



Discrete time SS models from continuous time models

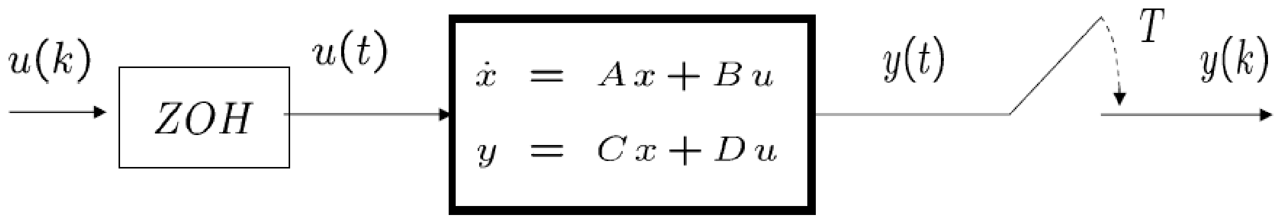




$$A_d = e^{AT} \qquad B_d = \int_0^T e^{At} dt\, B$$

T : sampling time

Discrete time SS models from continuous time models



$$u(k) \longrightarrow \boxed{\begin{aligned} x(k+1) &= A_d\, x(k) + B_d\, u(k) \\ y(k) &= C\, x(k) + D\, u(k) \end{aligned}} \longrightarrow y(k)$$

- $\lambda_i(A_d)$: ith eigenvalue of A_d $\qquad$ $\lambda_i(A)$: ith eigenvalue of A

$$\boxed{\lambda_i(A_d) = e^{\lambda_i(A)T}}$$

Discrete time SS models from continuous time models

$u(t) = u(k)$ for $t \in [t_k, t_{k+1})$

$u(t)$ is **constant** for $t \in [t_k, t_{k+1})$

$$T = t_{k+1} - t_k$$

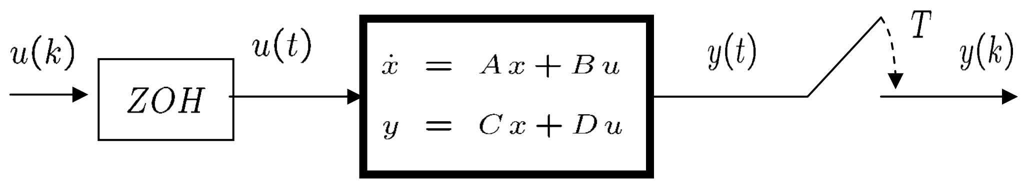
(sampling time)

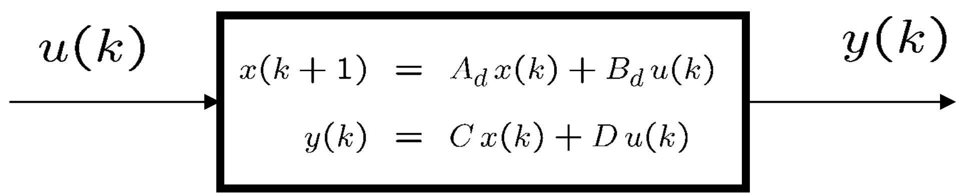
- Calculate $x(t_{k+1})$ given $x(t_k)$ and $u(k)$

$$\begin{aligned} x(t_{k+1}) &= e^{A(t_{k+1}-t_k)} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B\, u(\tau) d\tau \\ &= e^{AT} x(t_k) + \left\{ \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} d\tau\, B \right\} u(k) \\ &= e^{AT} x(t_k) + \left\{ \int_0^T e^{At} dt\, B \right\} u(k) \end{aligned}$$

(Change of variable) $\rightarrow$ $\left\{ \begin{array}{l} t = t_{k+1} - \tau\ ;\ dt = -d\tau \\ \tau = t_k \Rightarrow t = T \\ \tau = t_{k+1} \Rightarrow t = 0 \end{array} \right.$

Discrete time SS models from continuous time models





$$
\boxed{\begin{array}{c} u(t) = u(k) \;\; \text{for } t \in [t_k, t_{k+1}) \\ \begin{aligned} x(k+1) &= x(t_{k+1}) \\ x(k) &= x(t_k) \end{aligned} \end{array}}
$$

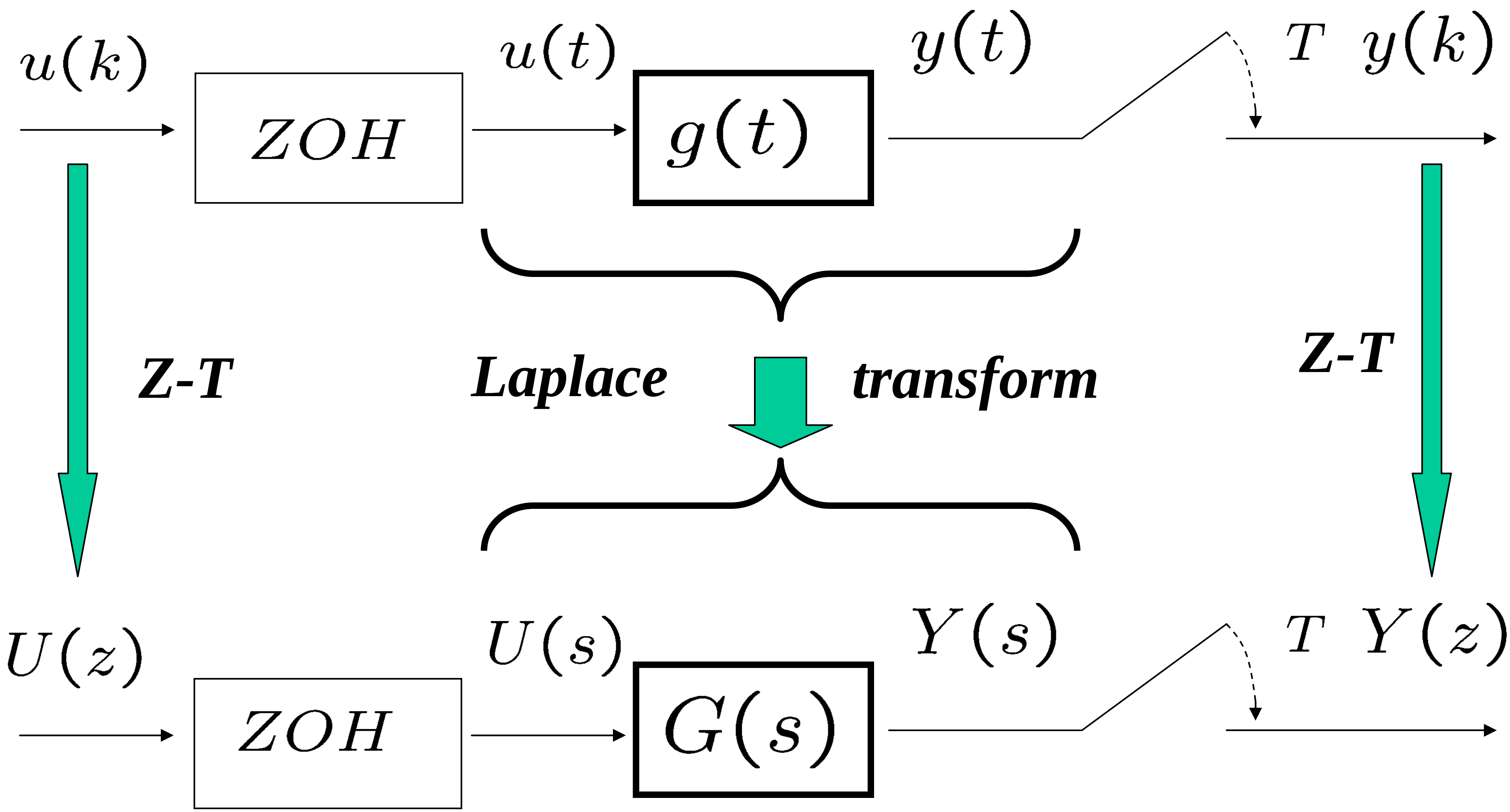
T : sampling time

$$
x(k+1) = \underbrace{e^{AT}}_{A_d} x(k) + \underbrace{\left\{ \int_0^T e^{At} dt \, B \right\}}_{B_d} u(k)
$$

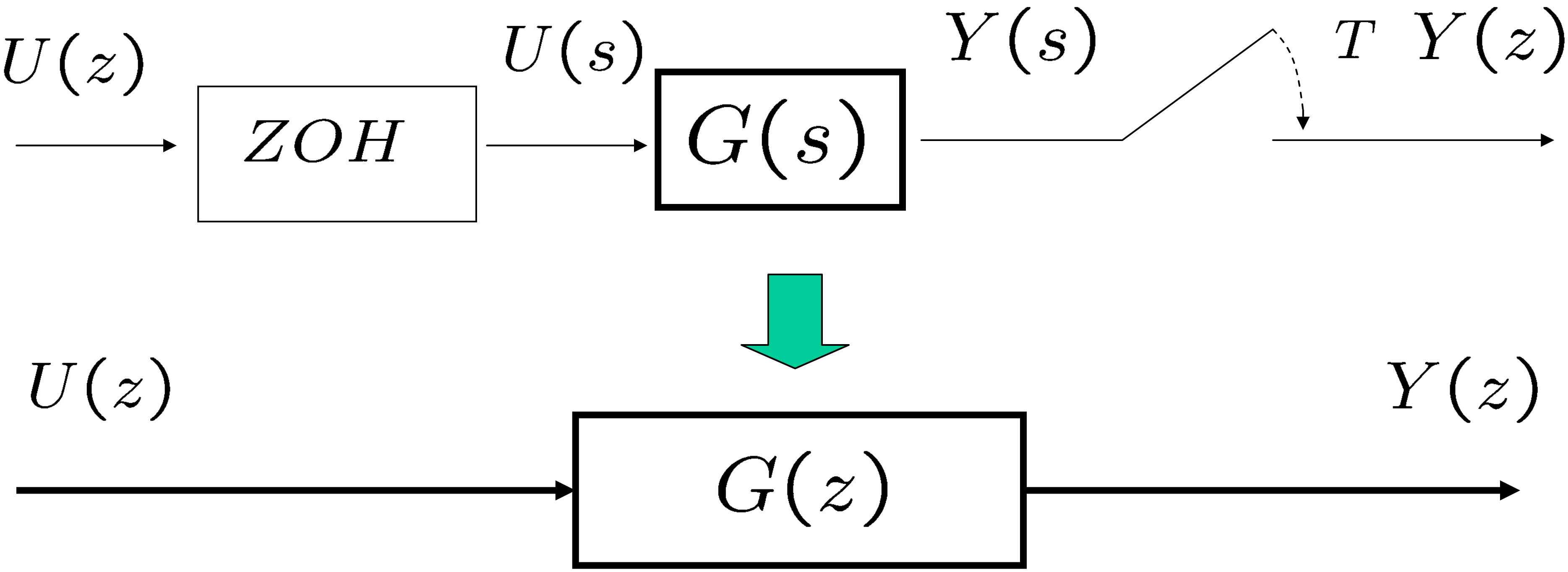
$$
\boxed{A_d = e^{AT} \qquad\qquad B_d = \int_0^T e^{At} dt \, B}
$$

Discrete time models from continuous time models

Input/output (IO) models:

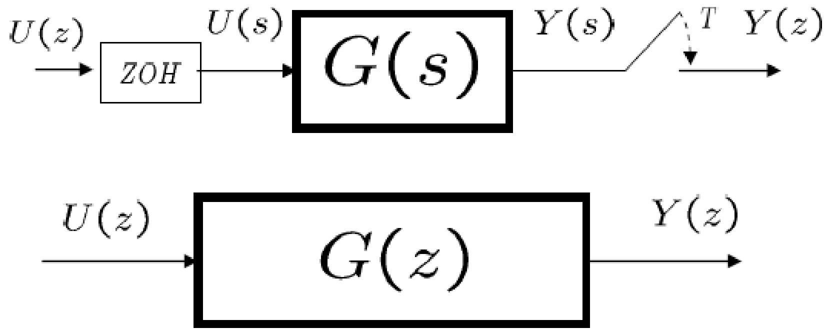


Discrete time from continuous time I/O models



$$\boxed{G(z) = \left(1 - z^{-1}\right)\, \mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}}$$

Discrete time IO models, an example

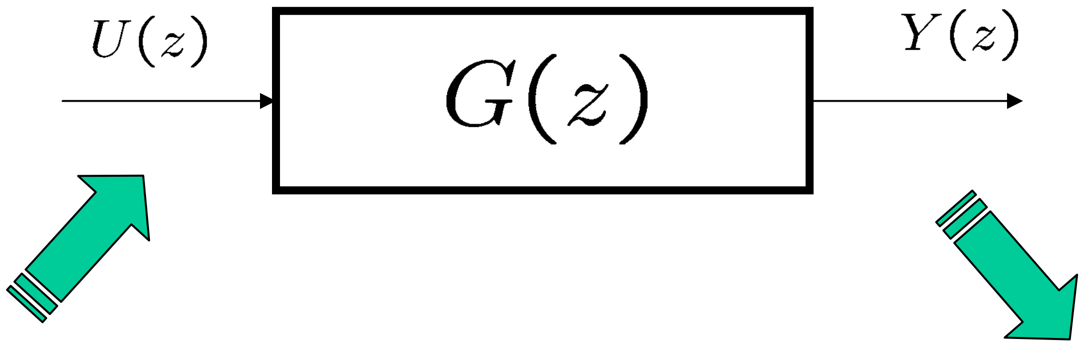


$$G(s) = \frac{k_p}{\tau s + 1} \qquad G(z) = \frac{b_1}{z - p}$$

$$p = e^{-T/\tau} \qquad b_1 = k_p(1 - p)$$

Matlab: c2d

Discrete time IO models from continuous time models



$$U(z) = 1$$

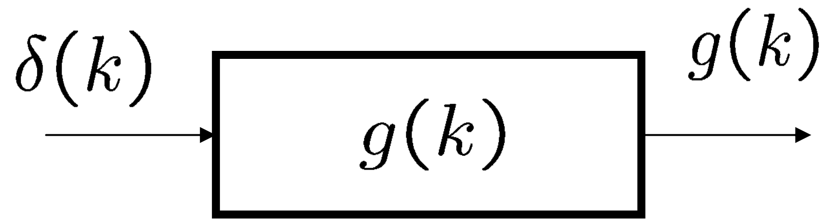
$$Y(z) \;=\; G(z)$$

$$u(k) = \delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Pulse sequence

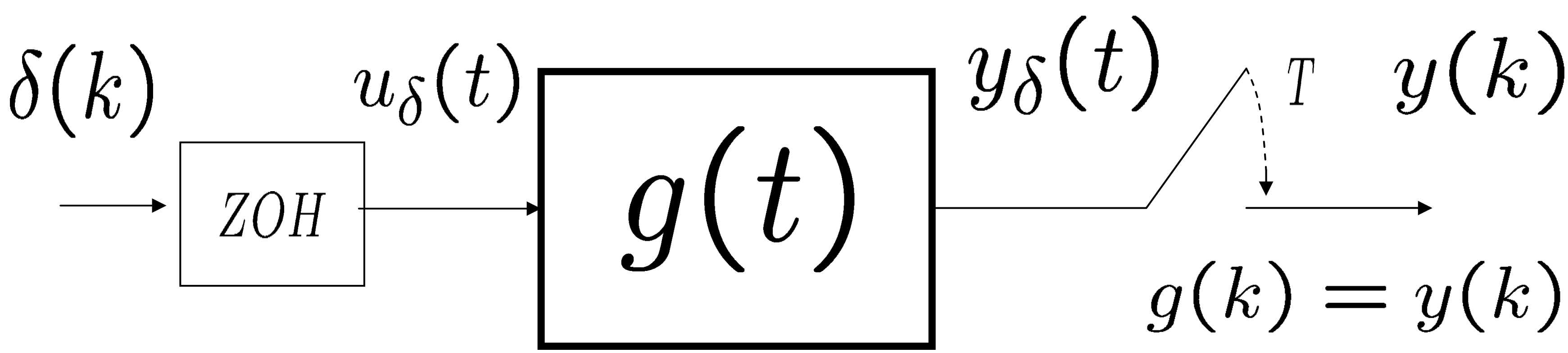
$$U(z) = \mathcal{Z}\{\delta(k)\} = 1$$

Discrete time IO models from continuous time models



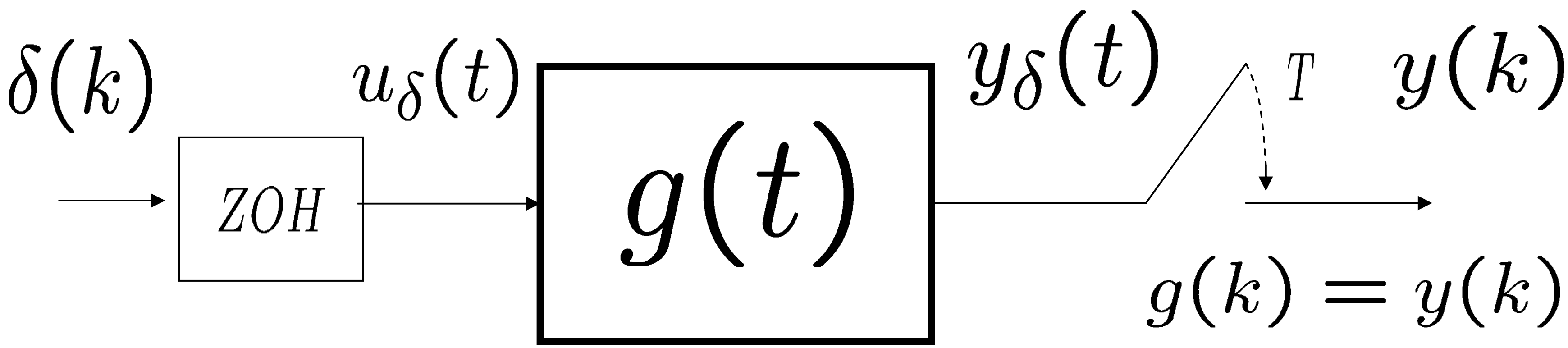
Then: $y(k) = g(k)$ and $G(z) = \mathcal{Z}\{y(k)\}$

Time domain:

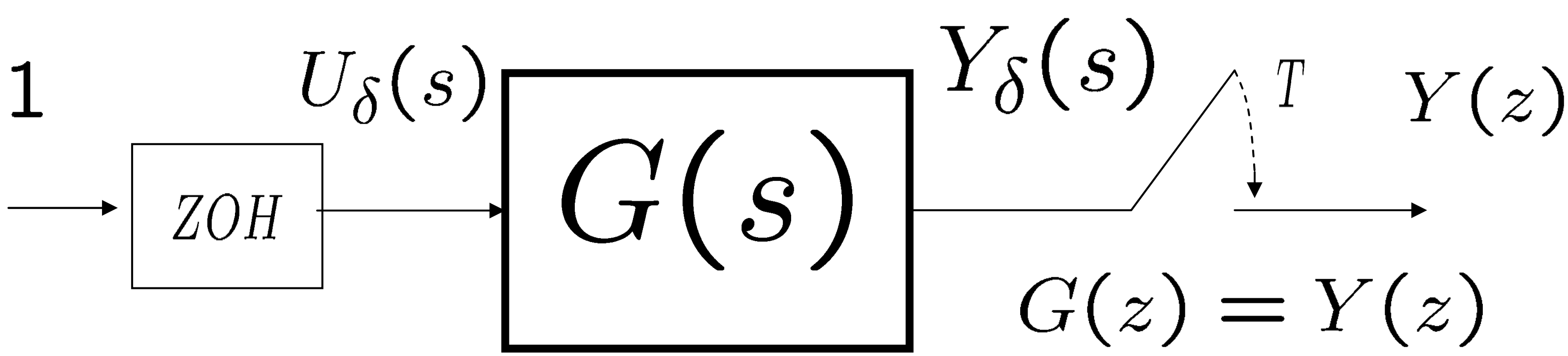


Discrete time IO models from continuous time models

Time domain:

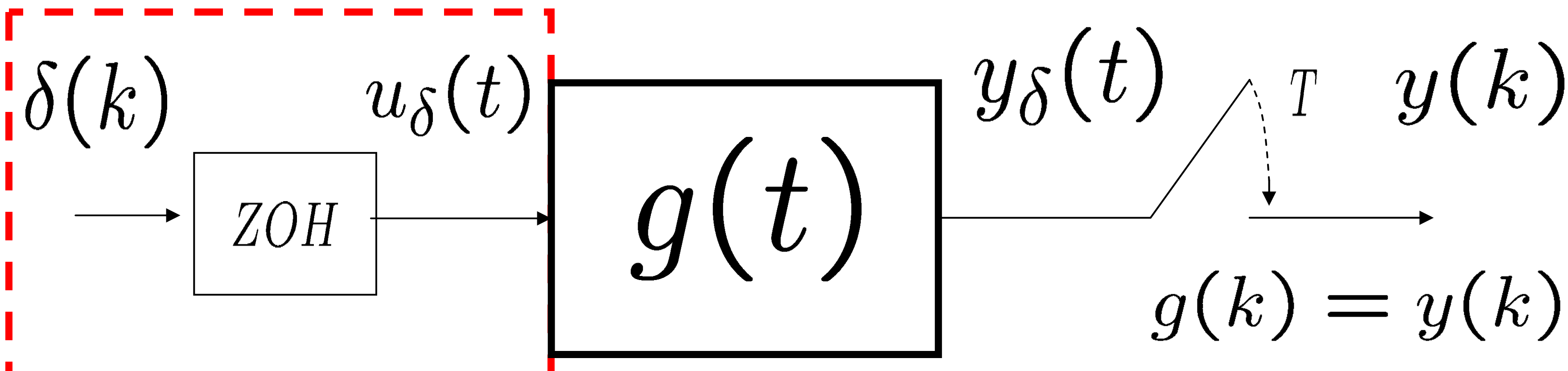


z and s domains:

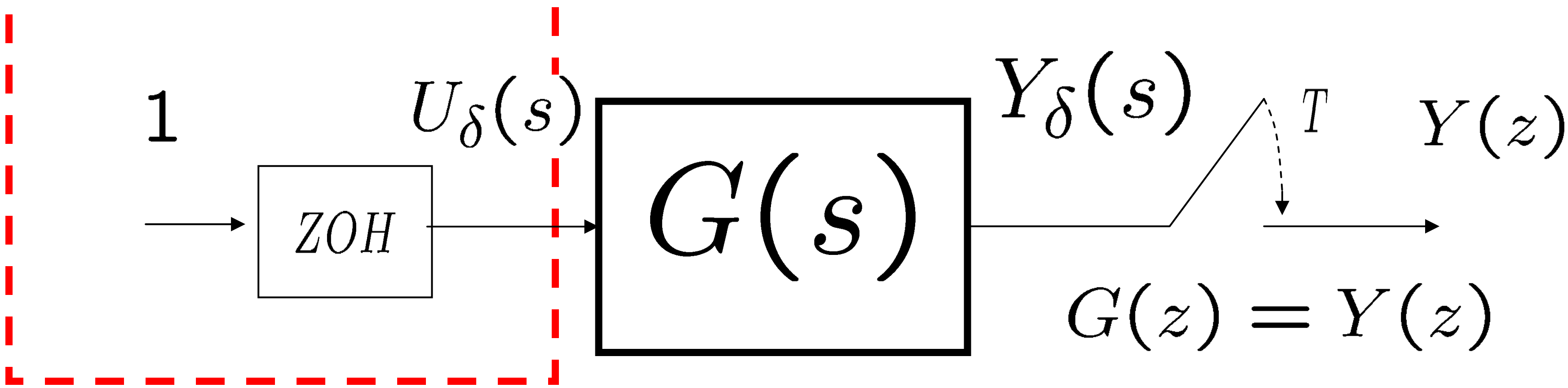


We now analyze each component:

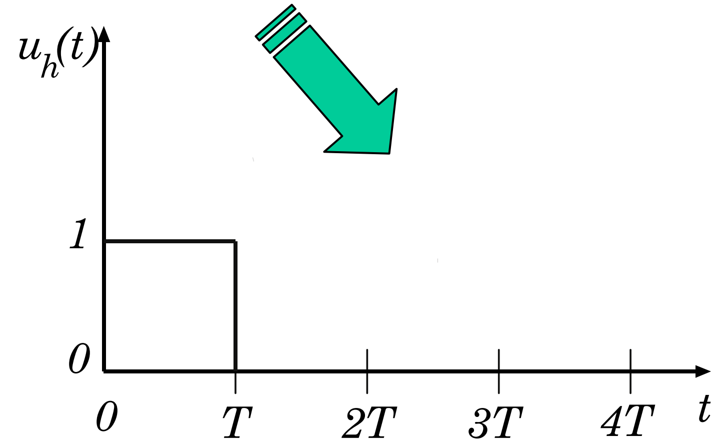
Time domain:



Z - Transform domain:



Discrete time IO models from continuous time models



Unit pulse

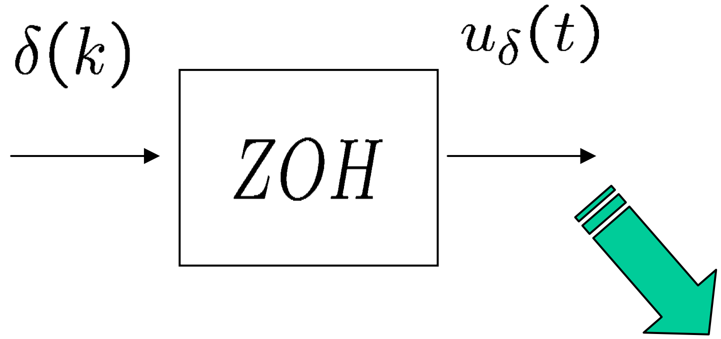
$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{u_\delta(t) \;=\; 1(t) - 1(t-T)}$$

Unit step

$$1(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Discrete time IO models from continuous time models

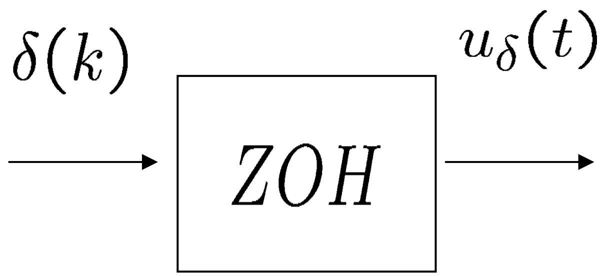


$$u_\delta(t) \;=\; 1(t) - 1(t-T)$$

$$U_\delta(s) \;=\; \mathcal{L}\{u_\delta(t)\} = \mathcal{L}\{1(t) - 1(t-T)\}$$

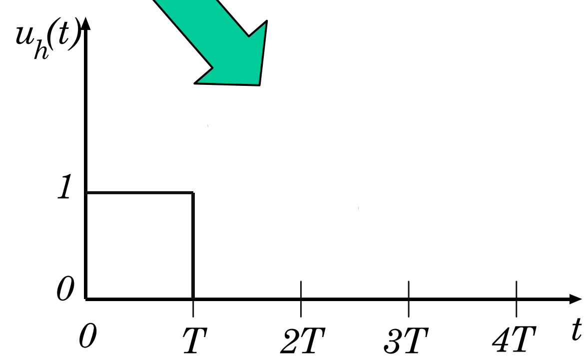
$$\boxed{U_\delta(s) \;=\; (1 - e^{-sT})\frac{1}{s}}$$

Discrete time IO models from continuous time models



Unit pulse sequence

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$



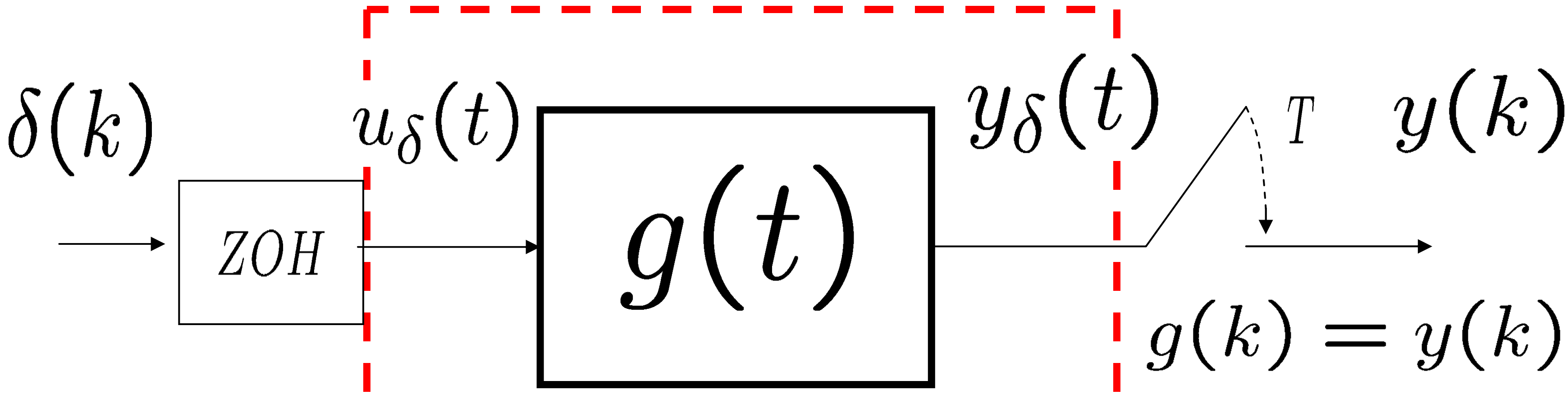
$$u_\delta(t) = 1(t) - 1(t-T)$$

$$U(z) = 1$$

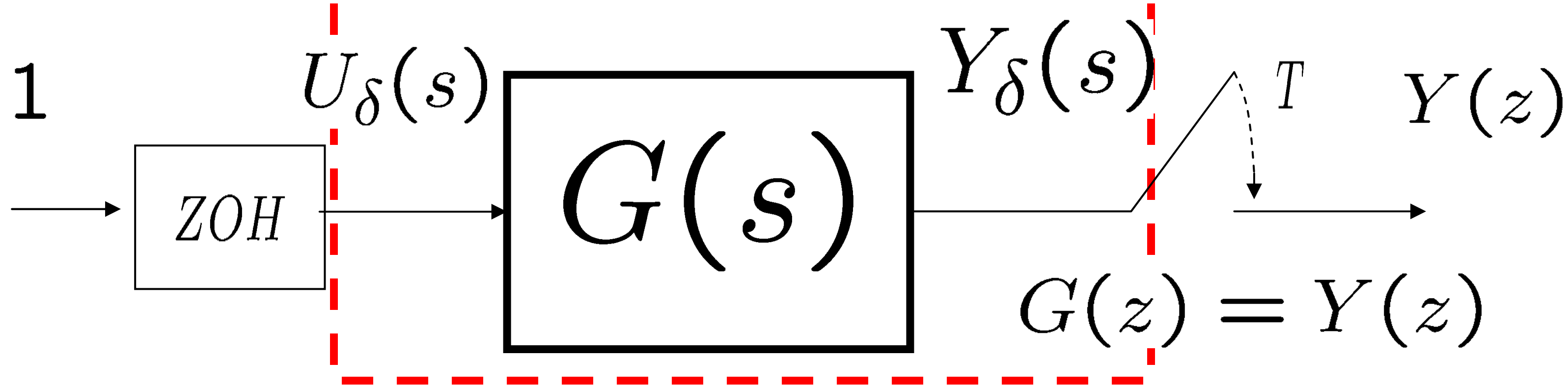
$$U_\delta(s) = (1 - e^{-sT})\frac{1}{s}$$

We now analyze each component:

Time domain:



Z - Transform domain:



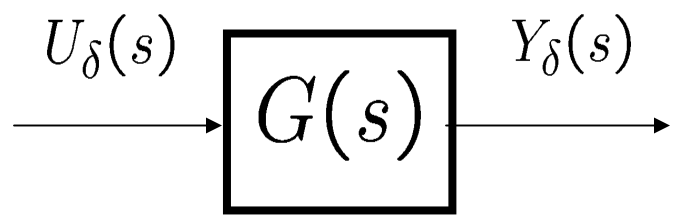
Discrete time IO models from continuous time models

$$\begin{tikzcd} {} \arrow[r, "U_\delta(s)"] & \boxed{G(s)} \arrow[r, "Y_\delta(s)"] & {} \end{tikzcd}$$

$$U_\delta(s) \;=\; (1 - e^{-sT})\frac{1}{s} \qquad\qquad Y_\delta(s) \;=\; G(s)\,(1 - e^{-sT})\frac{1}{s}$$

$$Y_\delta(s) \;=\; \underbrace{\frac{G(s)}{s}}_{Y_s(s)} - e^{-sT}\underbrace{\frac{G(s)}{s}}_{Y_s(s)} \qquad\qquad Y_s(s) \;=\; \frac{G(s)}{s}$$

Discrete time IO models from continuous time models



$$U_\delta(s) = (1 - e^{-sT})\frac{1}{s}$$

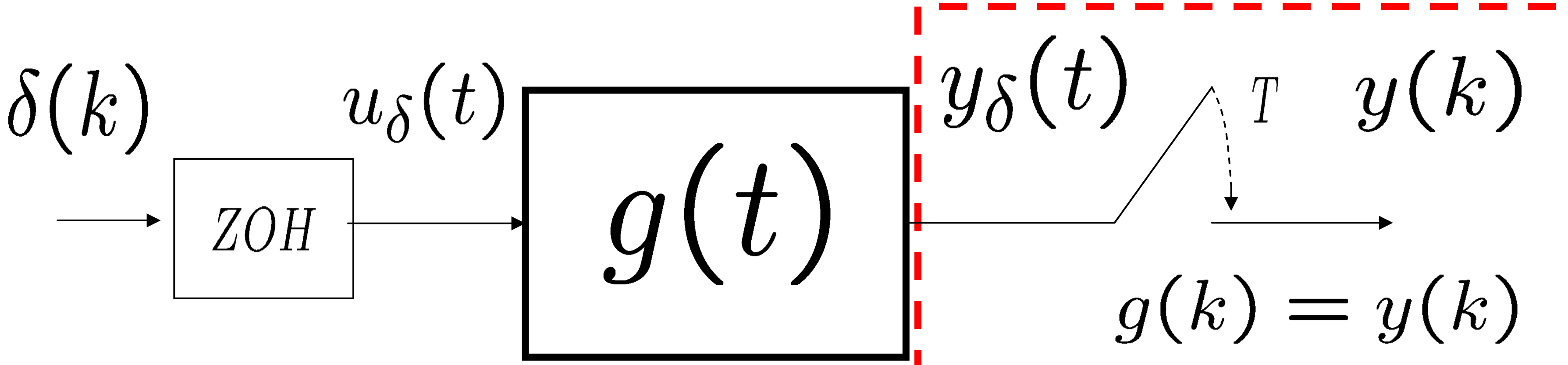
$$Y_\delta(s) = Y_s(s) - e^{-sT} Y_s(s)$$

$$Y_s(s) = \frac{G(s)}{s}$$

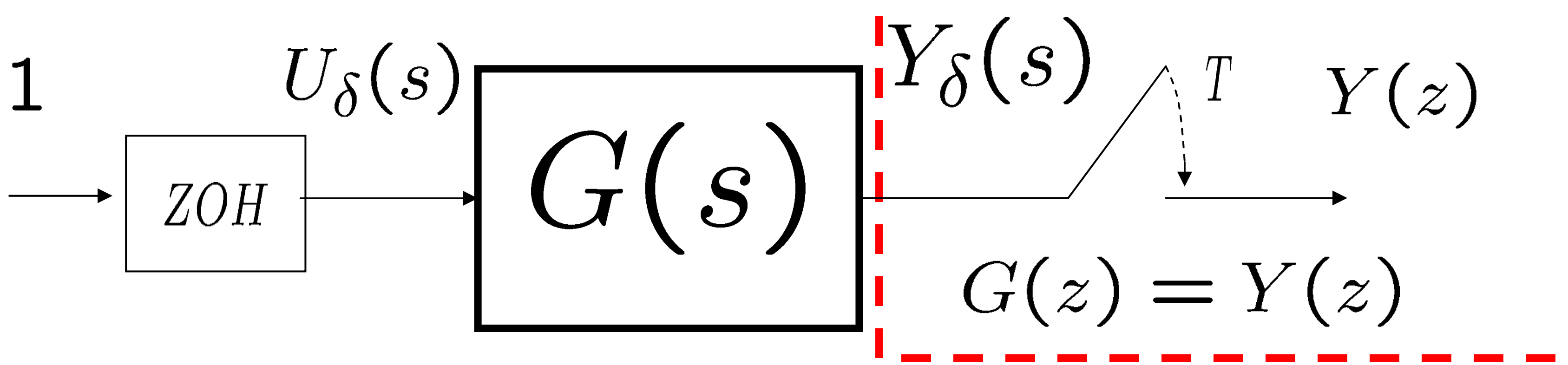
$$\boxed{\begin{aligned} y_\delta(t) &= y_s(t) - y_s(t-T) \\ y_s(t) &= \mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \end{aligned}}$$

We now analyze each component:

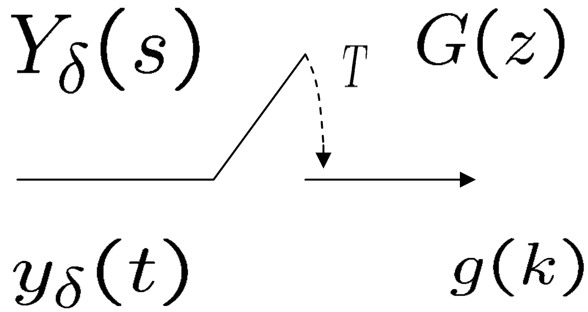
Time domain:



Z - Transform domain:



Discrete time models from continuous time models



$$y_\delta(t) \;=\; y_s(t) - y_s(t-T)$$

$$g(k) \;=\; y_s(k) - y_s(k-1)$$

$$G(z) \;=\; Y_s(z) - z^{-1}Y_s(z)$$

$$G(z) \;=\; (1 - z^{-1})\,Y_s(z)$$

Discrete time models from continuous time models

$$Y_\delta(s) \qquad G(z)$$

$$y_\delta(t) \xrightarrow{T} g(k)$$

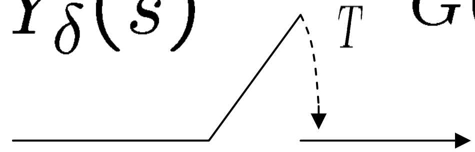
$$y_\delta(t) = y_s(t) - y_s(t-T)$$

$$G(z) = (1 - z^{-1})\, Y_s(z)$$

$$y_s(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] \qquad Y_s(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}$$

Discrete time models from continuous time models

$Y_\delta(s)$ $G(z)$



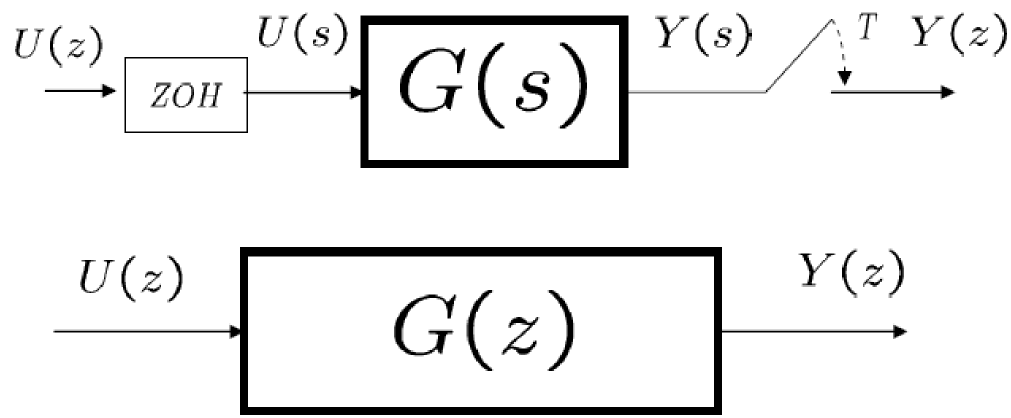
$y_\delta(t)$ $g(k)$

$$y_\delta(t) = y_s(t) - y_s(t-T)$$

$$G(z) = (1-z^{-1})\,\mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}$$

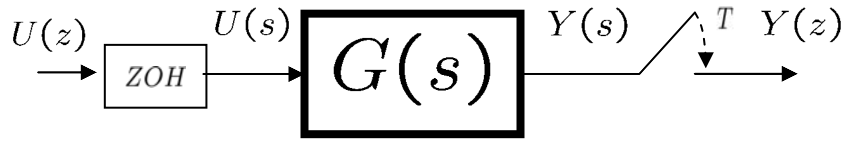
Discrete time models from continuous time models

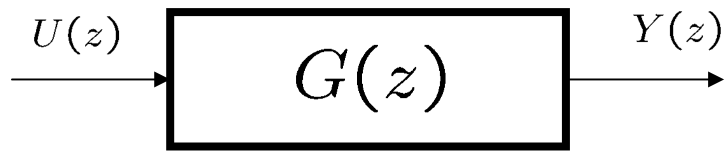
Input/output models:



$$\boxed{G(z) = \left(1 - z^{-1}\right)\ \mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}}$$

Discrete time IO models, an example





$$0 \leq L < T$$

L: plant time delay

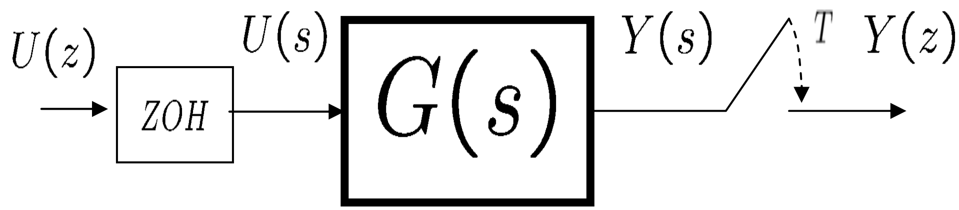
$$G(s) = \frac{k_p \, e^{-sL}}{\tau s + 1},$$

τ: plant time constant

$$G(z) = k_p \frac{(1 - pd)z + p(d - 1)}{z(z - p)}$$

$$p = e^{-T/\tau} \quad d = e^{L/\tau}$$

Discrete time IO models, an example



$$G(s) = \frac{k_p e^{-sL}}{\tau s + 1},$$

$0 \leq L < T$

L: plant time delay

τ : plant time constant

$$y_s(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{s}\right]$$

Step response

$$y_s(t) = k_p \left(1 - e^{-(t-L)/\tau}\right) 1(t-L)$$

Discrete time IO models, an example

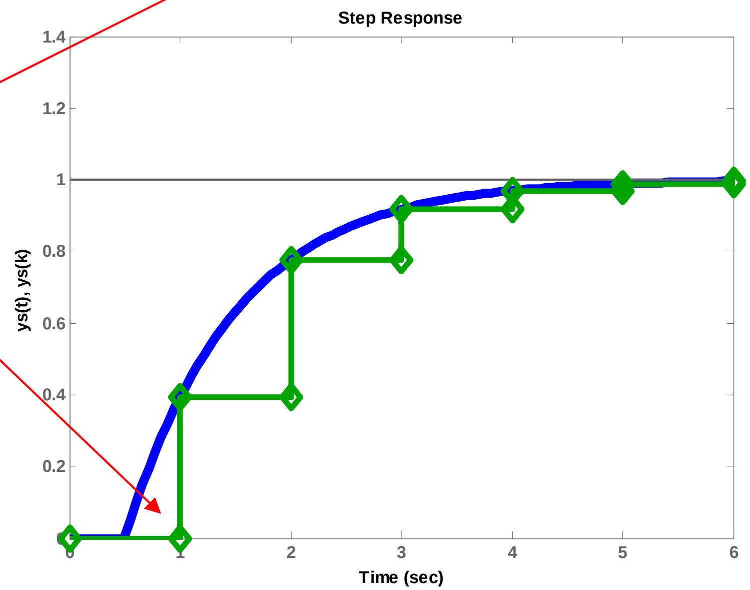
Continuous-time step response:

$$y_s(t) \;=\; k_p\left(1 - e^{-(t-L)/\tau}\right) 1(t-L)$$

Sampled step response:

$$\boxed{y_s(k) = k_p\left(1(k-1) - e^{L/\tau}\, e^{-kT/\tau} 1(k-1)\right)}$$

Notice the “one-step” sample delay



Discrete time IO models, an example

$$y_s(k) = k_p \left(1(k-1) - e^{L/\tau}\, e^{-kT/\tau} 1(k-1)\right)$$

Taking Z-transforms,

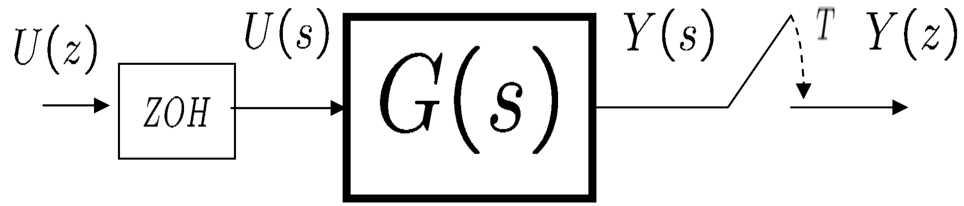
$$Y_s(z) = k_p \left(\mathcal{Z}\{1(k-1)\} - e^{L/\tau}\, \mathcal{Z}\{e^{-kT/\tau} 1(k-1)\}\right)$$

$$Y_s(z) \;=\; k_p \left(\frac{z^{-1}}{1 - z^{-1}} - \frac{pd\, z^{-1}}{1 - pz^{-1}}\right)$$

$d = e^{L/\tau}$

$p = e^{-T/\tau}$

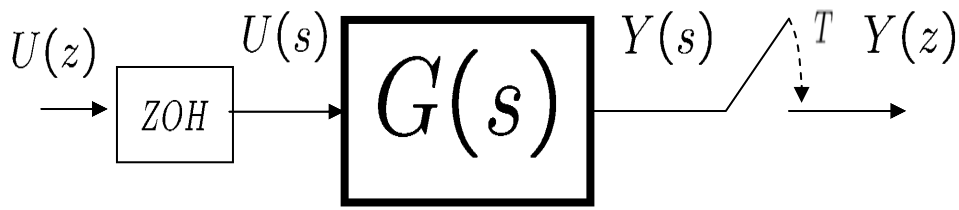
Discrete time IO models, an example



$$Y_s(z) \;=\; k_p \left(\frac{z^{-1}}{1 - z^{-1}} - \frac{pd\, z^{-1}}{1 - pz^{-1}} \right)$$

$$\boxed{Y_s(z) \;=\; k_p \frac{(1-pd)z + p(d-1)}{(z-1)(z-p)} \qquad p = e^{-T/\tau} \qquad d = e^{L/\tau}}$$

Discrete time IO models, an example



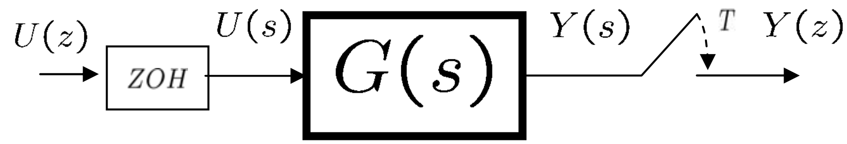
$$Y_s(z) = k_p \frac{(1-pd)z + p(d-1)}{(z-1)(z-p)}$$

$$G(z) = (1-z^{-1})Y_s(z)$$

$$\boxed{G(z) = k_p \frac{(1-pd)z + p(d-1)}{z(z-p)}}$$

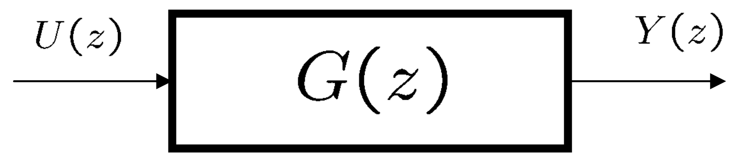
Discrete time IO models, an example

$$G(s) \;=\; \frac{k_p\, e^{-sL}}{\tau s + 1}, \;\; 0 \leq L < T$$



L: plant time delay

τ : plant time constant



T: sampling time

$$Y_s(z) \;=\; k_p \frac{(1-pd)z + p(d-1)}{(z-1)(z-p)}$$

$$\boxed{p = e^{-T/\tau}} \quad \boxed{d = e^{L/\tau}}$$

$$G(z) \;=\; (1 - z^{-1}) Y_s(z)$$

$$\boxed{G(z) \;=\; k_p \frac{(1-pd)z + p(d-1)}{z(z-p)}}$$