[MEN573]
Advanced Control Systems I

Lecture 9 – Solution Matrix via Inverse Laplace and Z- Transforms

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Continuous time n-th order system

$$\boxed{\frac{d}{dt}x(t) = A\,x(t) + B\,u(t)}$$

$$\begin{aligned} x &\in \mathcal{R}^n \\ A &\in \mathcal{R}^{n\times n} \end{aligned}$$

- **Solution**:

$$\boxed{x(t) = \underbrace{e^{A(t)}\,x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{A(t-\tau)}\,B\,u(\tau)d\tau}_{\text{forced response}}}$$

Continuous time n-th order system in the Laplace domain

$$\boxed{sX(s) - x(0) \;=\; A\,X(s) + B\,U(s)}$$

$$\begin{aligned} X(s) &= \mathcal{L}\{x(t)\} \in \mathcal{C}^n \\ U(s) &= \mathcal{L}\{u(t)\} \end{aligned}$$

- **Solution**:

$$\boxed{X(s) \;=\; \underbrace{(sI - A)^{-1}\,x(0)}_{\text{free response}} + \underbrace{(sI - A)^{-1}\,B\,U(s)}_{\text{forced response}}}$$

Continuous time n-th order system in Laplace domain

Comparing both solutions:

$$x(t) = \underbrace{e^{A(t)}\,x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{A(t-\tau)}\,B\,u(\tau)d\tau}_{\text{forced response}}$$

$$X(s) = \underbrace{(sI - A)^{-1}\,x(0)}_{\text{free response}} + \underbrace{(sI - A)^{-1}\,B\,U(s)}_{\text{forced response}}$$

Continuous time n-th order system in Laplace domain

Comparing both solutions:

$$\mathcal{L}\left\{e^{At}\right\} = (sI - A)^{-1}$$

$$\mathcal{L}\underbrace{\left\{\int_o^t e^{A(t-\tau)}Bu(\tau)d\tau\right\}}_{e^{At} * Bu(t)} = (sI - A)^{-1} B\,U(s)$$

(convolution)

Example: Solution Matrix via Laplace Domain

- Complex eigenvalues

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} (s-\sigma) & -\omega \\ \omega & (s-\sigma) \end{bmatrix}^{-1}$$

$$= \frac{1}{(s-\sigma)^2 + \omega^2} \begin{bmatrix} (s-\sigma) & \omega \\ -\omega & (s-\sigma) \end{bmatrix}$$

Example: Solution Matrix via Laplace Domain

$$(sI - A)^{-1} = \frac{1}{(s-\sigma)^2 + \omega^2} \begin{bmatrix} (s-\sigma) & \omega \\ -\omega & (s-\sigma) \end{bmatrix}$$

Using table look-up

$$\mathcal{L}\{e^{\sigma t}\cos(\omega t)\} = \frac{s-\sigma}{(s-\sigma)^2 + \omega^2} \qquad \mathcal{L}\{e^{\sigma t}\sin(\omega t)\} = \frac{\omega}{(s-\sigma)^2 + \omega^2}$$

$$\Downarrow$$

$$e^{At} = e^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

Discrete time n-th order system

$$\boxed{x(k+1) \;=\; A\,x(k) + B\,u(k)}$$

$$\begin{aligned} x &\in \mathcal{R}^{n} \\ A &\in \mathcal{R}^{n\times n} \end{aligned}$$

- **Solution**:

$$\boxed{x(k) \;=\; \underbrace{A^{k}\,x(0)}_{\text{free response}} + \underbrace{\sum_{j=0}^{(k-1)} A^{(k-1-j)}\,B\,u(j)}_{\text{forced response}}}$$

$$A^{k} = \underbrace{A\cdots A}_{k\ \text{times}}$$

Discrete time n-th order system in the Z-domain

$$\boxed{zX(z) - zx(0) \;=\; A\,X(z) + B\,U(z)}$$

$$X(z) \;=\; \mathcal{Z}\{x(k)\} \in \mathcal{C}^n$$
$$U(z) \;=\; \mathcal{Z}\{u(k)\}$$

- **Solution**:

$$\boxed{X(z) \;=\; \underbrace{(zI - A)^{-1} z\, x(0)}_{\text{free response}} + \underbrace{(zI - A)^{-1}\, B\, U(z)}_{\text{forced response}}}$$

Discrete time n-th order system in the Z-domain

Comparing both solutions:

$$x(k) = \underbrace{A^k x(0)}_{\text{free response}} + \underbrace{\sum_{j=0}^{(k-1)} A^{(k-1-j)} B\, u(j)}_{\text{forced response}}$$

$$X(z) = \underbrace{(zI - A)^{-1} z\, x(0)}_{\text{free response}} + \underbrace{(zI - A)^{-1} B\, U(z)}_{\text{forced response}}$$

Discrete time n-th order system in the Z-domain

Comparing both solutions:

$$\mathcal{Z}\left\{A^k\right\} = z\left(zI - A\right)^{-1}$$

$$\underbrace{\mathcal{Z}\left\{\sum_{j=0}^{k-1} A^{(k-1-j)}\, B\, u(j)\right\}}_{A(k-1) * Bu(k)} = (zI - A)^{-1}\, B\, U(z)$$

(convolution)

Example: Solution Matrix via Z-Domain

- Complex eigenvalues

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$z(zI - A)^{-1} \;=\; z \begin{bmatrix} (z-\sigma) & -\omega \\ \omega & (z-\sigma) \end{bmatrix}^{-1}$$

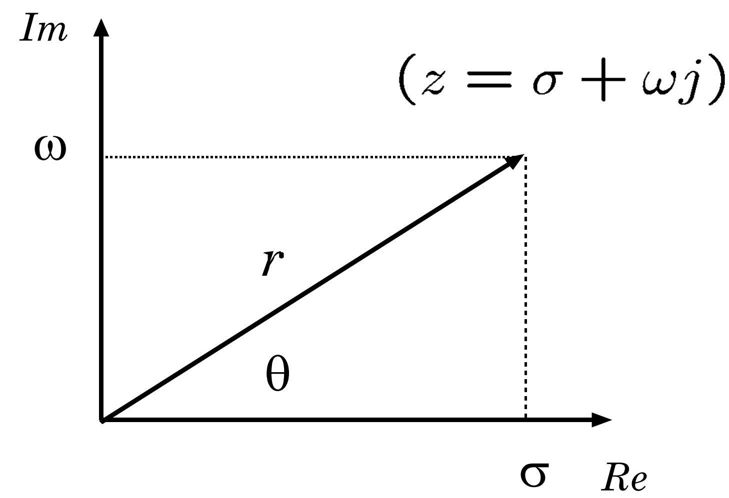
$$= \;\frac{z}{(z-\sigma)^2 + \omega^2} \begin{bmatrix} (z-\sigma) & \omega \\ -\omega & (z-\sigma) \end{bmatrix}$$

Example: Solution Matrix via Z-Domain

$$z(zI - A)^{-1} = \frac{z}{(z-\sigma)^2 + \omega^2} \begin{bmatrix} (z-\sigma) & \omega \\ -\omega & (z-\sigma) \end{bmatrix}$$

$$= \frac{z}{z^2 - 2r\cos(\theta)z + r^2} \begin{bmatrix} (z - r\cos(\theta)) & r\sin(\theta) \\ -r\sin(\theta) & (z - r\cos(\theta)) \end{bmatrix}$$

$$\sigma = r\cos(\theta)$$
$$\omega = r\sin(\theta)$$



Example: Solution Matrix via Z-Domain

$$z(zI - A)^{-1} = \frac{z}{z^2 - 2r\cos(\theta)z + r^2}\begin{bmatrix} (z - r\cos(\theta)) & r\sin(\theta) \\ -r\sin(\theta) & (z - r\cos(\theta)) \end{bmatrix}$$

Using table look-up

$$\mathcal{Z}\{r^k\cos(\theta k)\} = \frac{z(z - r\cos(\theta))}{z^2 - 2r\cos(\theta)z + r^2} \qquad \mathcal{Z}\{r^k\sin(\theta k)\} = \frac{zr\sin(\theta)}{z^2 - 2r\cos(\theta)z + r^2}$$

$$A^k = r^k\begin{bmatrix} \cos(\theta k) & \sin(\theta k) \\ -\sin(\theta k) & \cos(\theta k) \end{bmatrix}$$