Kalman Filtering (I)

Autonomous Unmanned System

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Introduction

- **Filtering**
  - A filter extracts useful contents from the signal corrupted by noise
  - Filter design is focused on how to extract the necessary contents efficiently or optimally (e.g. high-pass or low-pass filter)
  - Kalman filter predicts the behaviour of a dynamic system when there exist uncertainties in the system and sensor noises

- **History**
  - Kalman filter (KF) was proposed by R.E. Kalman in the early 1960's for optimal state estimation of stochastic systems
  - Previous frequency-domain techniques (e.g., Wiener filtering) were not much successful (KF is time-domain)

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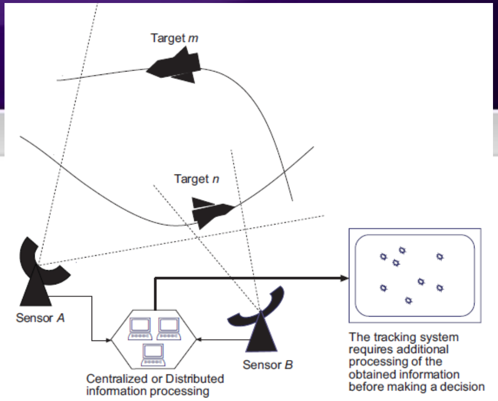
Introduction Example


Target *m*
Target *n*
Sensor *A*
Sensor *B*
Centralized or Distributed information processing
The tracking system requires additional processing of the obtained information before making a decision

- **1-D Position estimation of aircraft with a radar**
  - Prior knowledge
    - □ Motion of aircraft $\frac{dp(t)}{dt} = s(t), \quad \frac{ds(t)}{dt} = u(t)?$
    - □ Measurement equation (position radar) $z(t) = p(t) + v(t)$
  - Measurements: discrete position measurement $z_{t_0}, z_{t_1}, \cdots, z_{t_k}$

- **Questions**
  - How to get a good estimation of position with noisy measurements?
  - Can we get the estimate of the velocity?
  - Can we quantify how accurate the estimate is?
  - Is it possible to consider the sensor accuracy into estimation?
  - Can these be done optimally and in real time?

Introduction

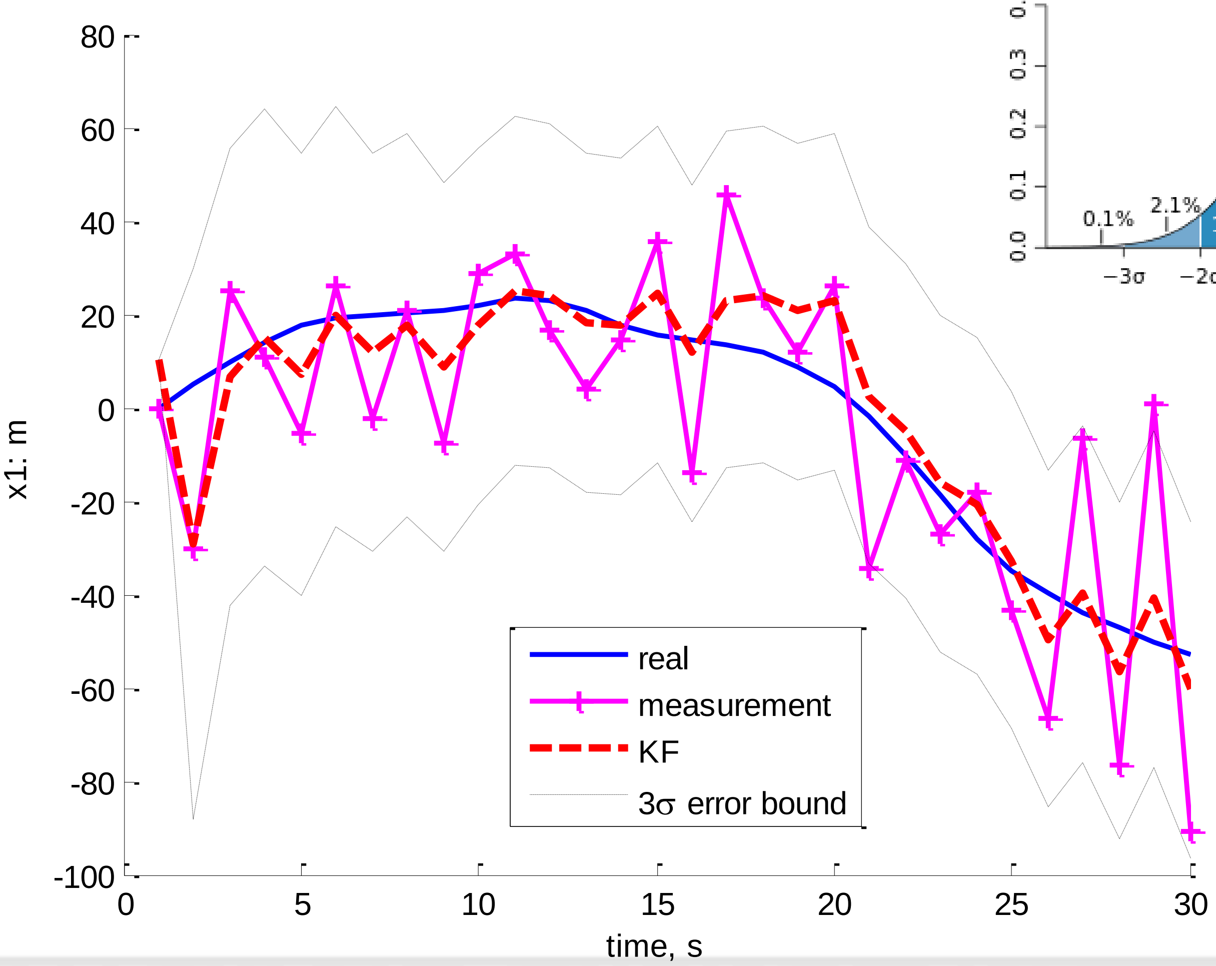
- **Kalman filter answers all questions! (Projection and Correction)**
  - Projection (Prediction)
    - Prediction of the next time state vector according to the model (system dynamics and statistical properties of noise); a priori value (not accurate as the system is uncertain)
    - How you think the system will evolve in time based on your theories and best knowledge so far

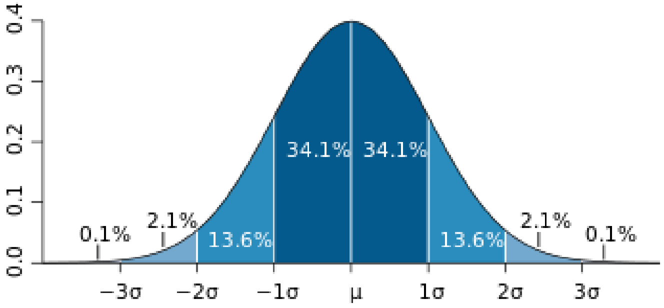
Introduction

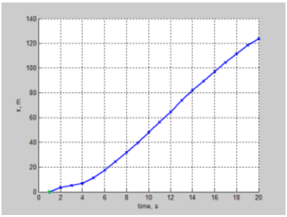
- **Kalman filter answers all questions!!!**
  - Correction (update)
    - Improve the estimation of the state vector according to the new measurement; a posteriori value
    - Predicted value is corrected by the new measurement
  - Recursive nature: optimal combination of the previous estimate/knowledge and the recent measurement recursively

    (Remind recursive Bayesian update!)

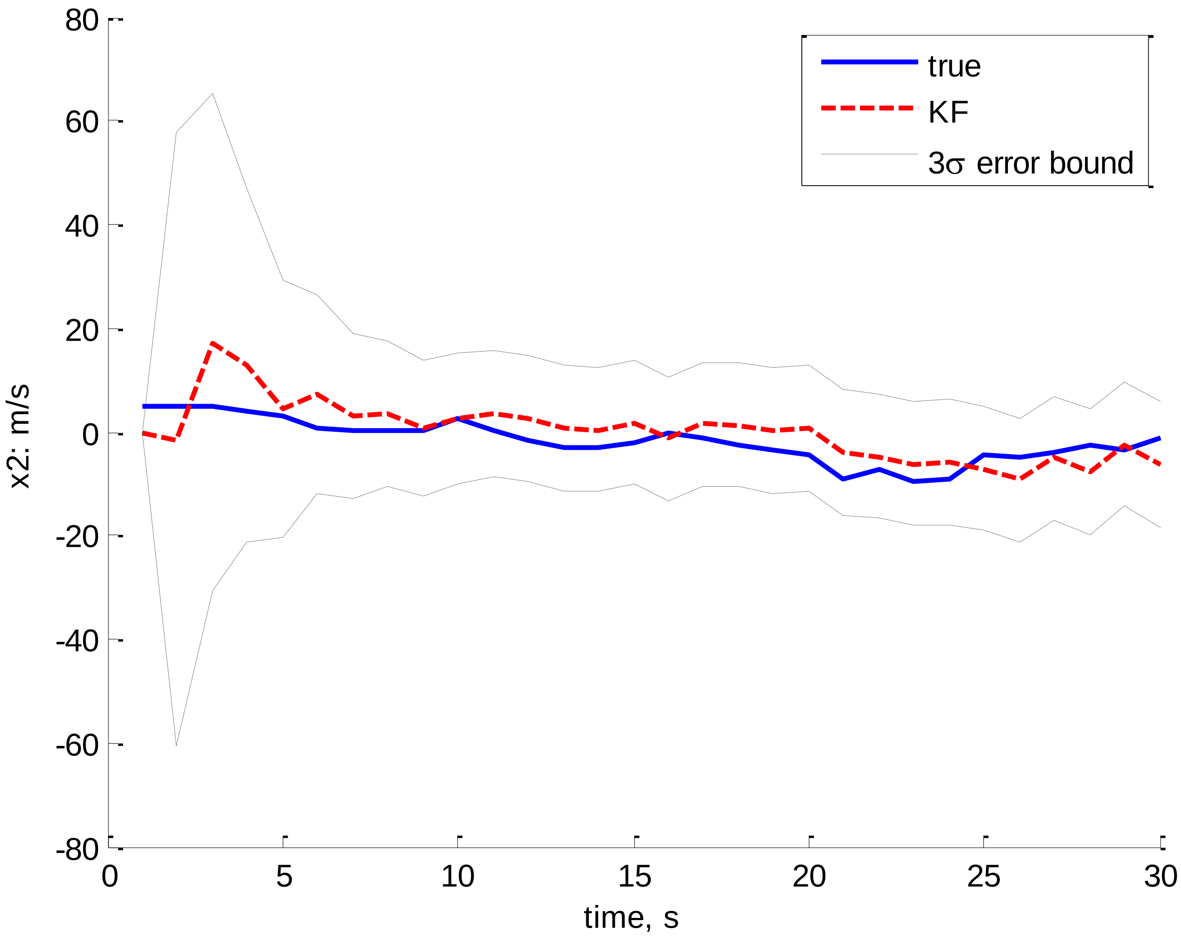
Position Estimation by KF







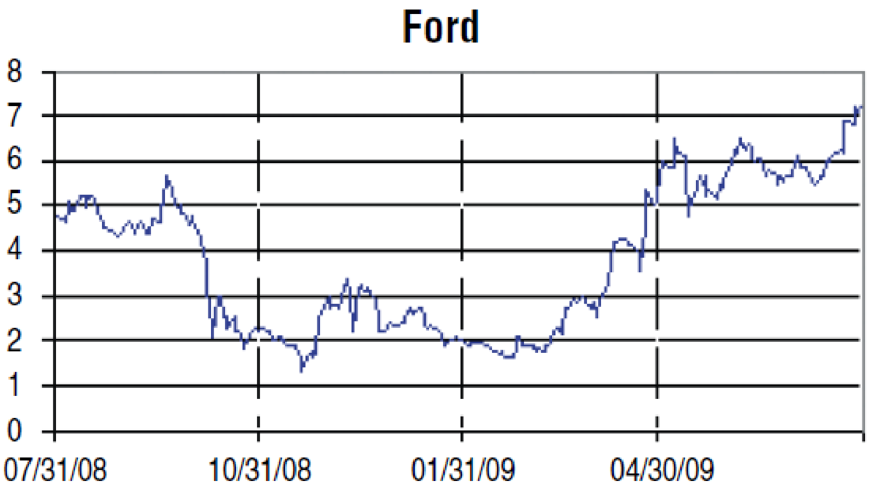
Velocity Estimation by KF

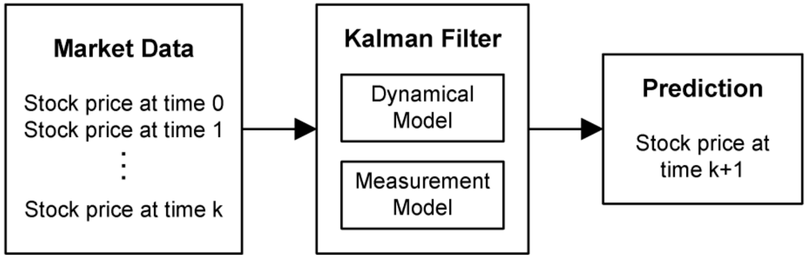


Recap of the Kalman Filter

- **Applications**
  - Avionics (guidance, navigation and control of vehicles)
    - Integrated/aided Navigation (e.g., GPS aided INS system)
    - Attitude and orbit determination of aircraft/spacecraft
  - Air-traffic control (aircraft trajectory)
  - Surveillance (moving target tracking)
  - Missile guidance / Fire control
  - Motion capture system
  - Stock-market analysis
  - Other signal processing and econometrics (which needs prediction and estimation in an uncertain and noisy environment)

Predicting Market Data with KF




Market Data

Stock price at time 0
Stock price at time 1
⋮
Stock price at time k

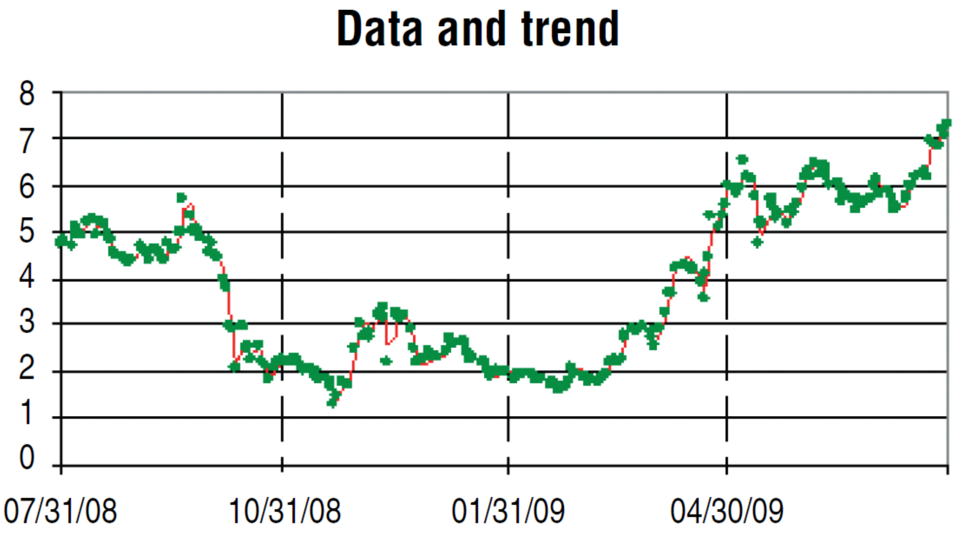
Kalman Filter

Dynamical Model

Measurement Model

Prediction

Stock price at time k+1

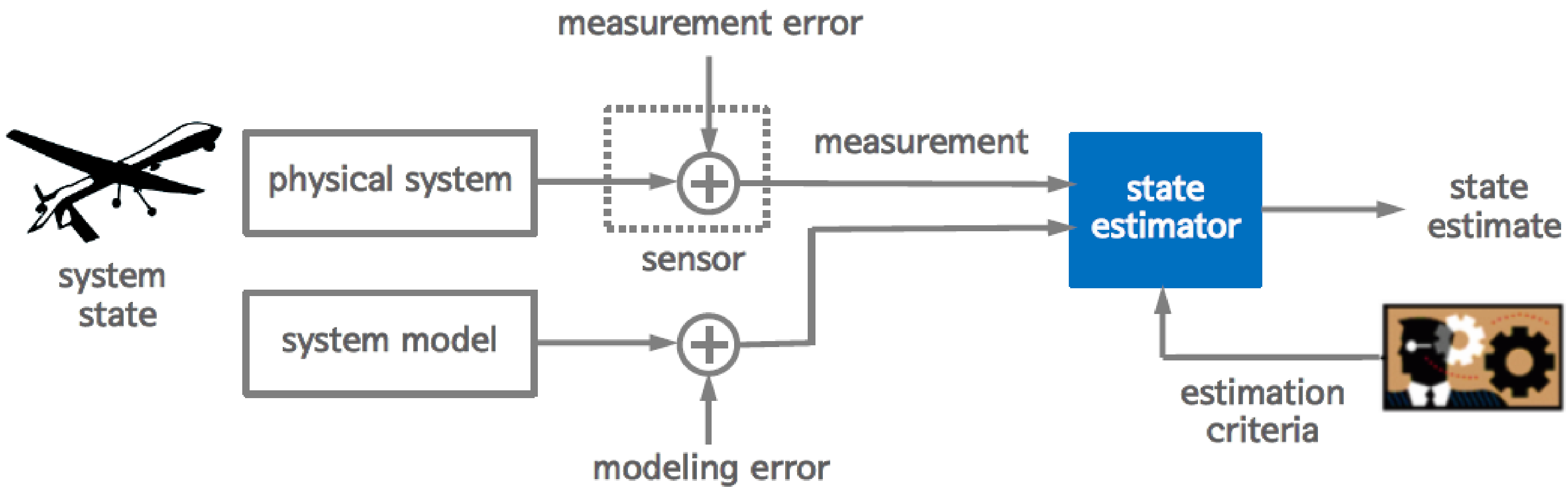


Kalman filter to forecast stock price movements by combining the daily price change model and stock data

$$x_{k|k-1} = 3(x_{k-1} - x_{k-2}) + x_{k-3}$$

$$x_{k|k} = x_{k|k-1} + K_k(z_k - x_{k|k-1})$$

R. Martinelli & N. Rhoads, Predicting Market Data Using the KF, *Stocks & Commodities*, 28(1), 2010.


measurement error
physical system
measurement
state estimator
state estimate
sensor
system state
system model
estimation criteria
modeling error

State Estimation

- **Definition**
  - Estimation of the state vector of a dynamic system
  - Three types of information are necessary for state estimation:
    - System model (a priori knowledge on the system)
    - Measurement data
    - Estimation criteria

Predict with a system model & update by combining a priori and measurements optimally!

- **System Model**
  - State equation (process/system dynamics) in a state-space form
  - Measurement equation
  - Noise characteristics

$$\underline{x}_{k+1} = F_k \underline{x}_k + G_k \underline{w}_k$$

$$\underline{z}_{k+1} = H_{k+1} \underline{x}_{k+1} + \underline{v}_{k+1}$$

System Model Example

- **1-D Constant velocity model with noisy acceleration**

$$x_{1_{k+1}} = x_{1_k} + T x_{2_k} + \frac{T^2}{2} w_k, \quad N \sim (0, \sigma_w^2)$$

$$x_{2_{k+1}} = x_{2_k} + T w_k$$

$$z_{k+1} = x_{1_{k+1}} + v_k, \quad N \sim (0, \sigma_v^2)$$

$$\Rightarrow$$

$$\underline{x}_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \underline{x}_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} w_k$$

$$z_{k+1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}_k + v_k$$

Recall State-Space!

$$\underline{x}_{k+1} = F_k \underline{x}_k + G_k \underline{w}_k$$

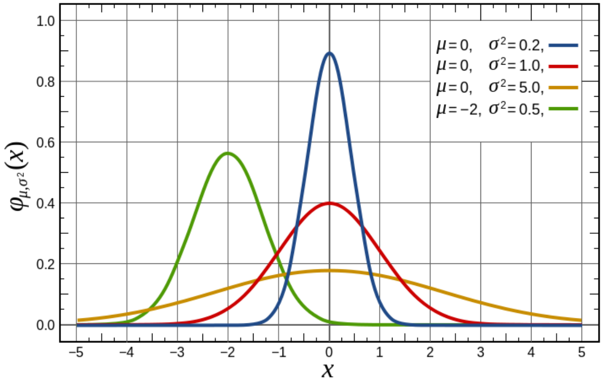
$$\underline{z}_{k+1} = H_{k+1} \underline{x}_{k+1} + \underline{v}_k$$

Noise Modelling and Error

- **Assumption for KF: white Gaussian noise**
  - White noise: serially uncorrelated random variables with zero mean and finite variance
  - Gaussian: normal distribution

$$\underline{w}_k \sim N(0, Q_k)$$

$$\underline{v}_k \sim N(0, R_k)$$



$$E[\underline{w}_k] = \underline{0}, \quad E\left[\underline{w}_k \underline{w}_i^T\right] = \begin{cases} Q\ (i=k) \\ [0]\ (i \neq k) \end{cases}$$

$$E[\underline{v}_k] = \underline{0}, \quad E\left[\underline{v}_k \underline{v}_k^T\right] = \begin{cases} R\ (i=k) \\ [0]\ (i \neq k) \end{cases}$$

$$E(\underline{w}_k \underline{v}_i^T) = [0]$$

- **Error**
  - Estimation error $\underline{e}_k \equiv \underbrace{\underline{x}_k}_{\text{True state}} - \underbrace{\hat{\underline{x}}_k}_{\text{Estimate}}$
  - Error covariance $\underline{P}_k \equiv E[(\underline{x}_k - \hat{\underline{x}}_k)(\underline{x}_k - \hat{\underline{x}}_k)^T] = E[\underline{e}_k \underline{e}_k^T]$

KF maintains:

i) $E[\underline{e}_k] = 0 \; or \; E[\hat{\underline{x}}_k] = \underline{x}_k$ Unbiased estimation

ii) Minimum variance $\underline{P}_k$

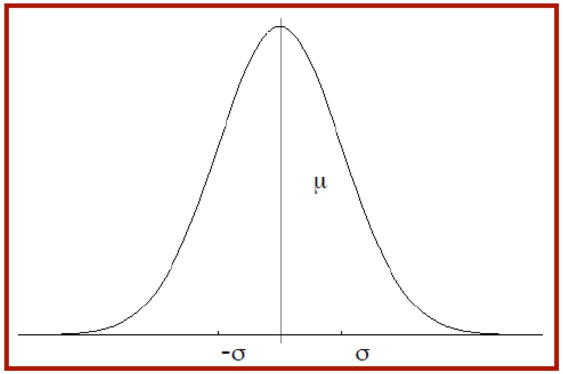
Noise Modelling and Error

Gaussians

$$p(x) \sim N(\mu,\sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

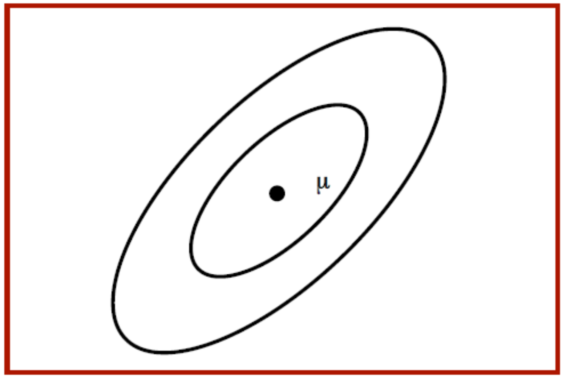
Univariate



$$p(\mathbf{x}) \sim N(\mu,\Sigma):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}\left|\Sigma\right|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^t\Sigma^{-1}(\mathbf{x}-\mu)}$$

Multivariate



Noise Modelling and Error

- **Mean & covariance matrix of process noise**

$$E[\underline{w}_k] = \underline{0}, \quad Q = E\left[\underline{w}_k \underline{w}_k{}^T\right] = \begin{pmatrix} E\left[w_{1_k}{}^2\right] & 0 & \cdots & 0 \\ 0 & E\left[w_{2_k}{}^2\right] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E\left[w_{n_k}{}^2\right] \end{pmatrix}$$

- **Mean & covariance matrix of measurement noise**

$$E[\underline{v}_k] = \underline{0}, \quad R = E\left[\underline{v}_k \underline{v}_k{}^T\right] = \begin{pmatrix} E\left[v_{1_k}{}^2\right] & 0 & \cdots & 0 \\ 0 & E\left[v_{2_k}{}^2\right] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E\left[v_{n_k}{}^2\right] \end{pmatrix}$$

Estimation Criteria for the KF

- The Kalman filter algorithm can be derived by several mathematical concepts, resulting in the same algorithm for linear Gaussian cases.

  - Bayesian Approach (conditional density function)
  - Orthogonal Projection
  - Minimum Variance Filter
  - Weighted Least-Square Estimation (deterministic concept)
  - Maximum-a-Posterior (MAP) Estimation

Estimation Criteria for the KF

- Conditional mean (Bayesian Approach):

$$\hat{x} = E[x \mid z] = \int \xi \, f_{x|z}(\xi \mid \zeta) \, d\xi$$

- Minimum mean square estimate(MMSE):

$$\min_{\xi} J = E[(x-\xi)^T W (x-\xi)]$$

$\Rightarrow$ the conditional mean is the MMSE for any pdf.

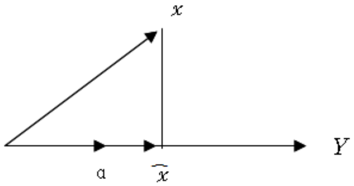
- Minimum variance linear unbiased estimate:

$$\min_{\xi} J = E[\mathrm{tr}\{(x-\xi)(x-\xi)^T\}]$$

$\Rightarrow$ identical to the conditional mean for linear Gaussian cases

- Orthogonal projection:

$\Rightarrow$ a linear filter which gives the projection of x on the subspace spanned by the measurements



Estimation Criteria for the KF

- Maximum a posteriori(MAP) estimate:

$$\min_{\xi} J = f_{x|z}(\xi \mid \eta) \qquad \Rightarrow \qquad \hat{x} = \text{mode of the cond. pdf}$$

- Maximum likelihood estimate(MLE):

$$\min_{\xi} J = L(\xi, \eta)$$

  - MLE depends on the likelihood function *L*.
  - Classical likelihood function is given as

$$L(\xi, \eta) = f_{z|x}(\eta \mid \xi)$$

1-D Example 1

- **Equation of a constant x (no dynamics and no uncertainty)**
  - Measurement (sensor) equation

$$x_{k+1} = F_k x_k + G_k w_k = x_k$$

$$z_1 = x + v, \;\; v \sim N(0, P)$$

  - Given z_1 is your measurement, the best estimate of x, denoted as $\hat{x}$

$$\hat{x} = z_1$$

  - Estimation error and its covariance: $e = x - \hat{x}$

$$E[e] = [x - (x + v)] = E[v] = 0$$

$$E[e^2] = E[(x - x - v)^2] = E[v^2] = P$$

1-D Example 2 (1)

- **Estimation of x from two sensor measurements**
  - Now, new measurement z_2

$$z_1 = x + v_1,\ \ v_1 \sim N(0, P)$$

$$\boxed{z_2 = x + v_2,\ \ v_2 \sim N(0, R)}$$

  - Then, how to combine z_1 and z_2 ?

$$\boxed{\hat{x} = k_1 z_1 + k_2 z_2}$$

⇨ The gain k_1 and k_2 should be determined with some optimal sense

1-D Example 2 (2)

$$z_1 = x + v_1,\ \ v_1 \sim N(0,P)$$
$$z_2 = x + v_2,\ \ v_2 \sim N(0,R)$$

- **Determination of k_1 and k_2**
  - Requirements:
    - □ Unbiased estimation error: $E[x-\hat{x}]=0 \quad (E[\hat{x}]=x)$
    - □ Minimum error variance: $E[(x-\hat{x})^2]$ is minimal
  - Unbiased estimator

$$E[\hat{x}] = E[k_1(x+v_1)+k_2(x+v_2)] = (k_1+k_2)x \Rightarrow k_1+k_2=1$$

  - Minimum variance estimator

$$\min E[e^2] = k_1^2P + k_2^2R = k_1^2P + (1-k_1)^2R \Rightarrow k_1 = \frac{R}{P+R},\ k_2 = \frac{P}{P+R}$$

1-D Example 2 (3)

- **Optimal estimate**

$$\hat{x} = k_1 z_1 + k_2 z_2 = \frac{R}{P+R} z_1 + \frac{P}{P+R} z_2$$

$$= z_1 + \boxed{\frac{P}{P+R}}(z_2 - z_1) = z_1 + \boxed{K}(z_2 - z_1)$$

New measurement

previous est.

Kalman gain: determines how much we rely on the new meas. or previous est.

$$K = \frac{P}{P+R} \qquad R\uparrow \rightarrow K\downarrow$$

$$\hat{P} = E[e^2] = \left(\frac{R}{P+R}\right)^2 P + \left(1-\frac{R}{P+R}\right)^2 R = (1-K)P$$

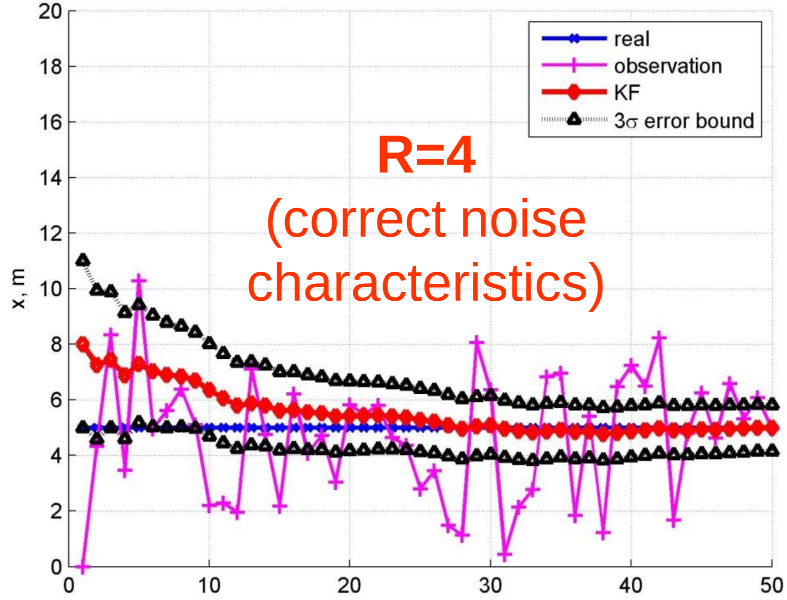
Error covariance: how reliable the estimation is

1-D Example 3

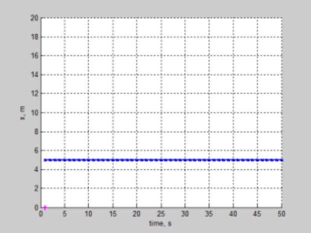
- **Sequential estimation**
  - What if we have another measurement?
    - ❑ A priori estimate: $N(\hat{x}, \hat{P})$ (best knowledge before processing z_3)
    - ❑ Measurement: $z_3 = x + v_3,\ \ v_3 \sim N(0, R_1)$
  - Optimal estimate

$$\hat{x} = z_1 + \frac{P}{P+R}(z_2 - z_1) = z_1 + K(z_2 - z_1)$$
$$\hat{P} = (1-K)P$$

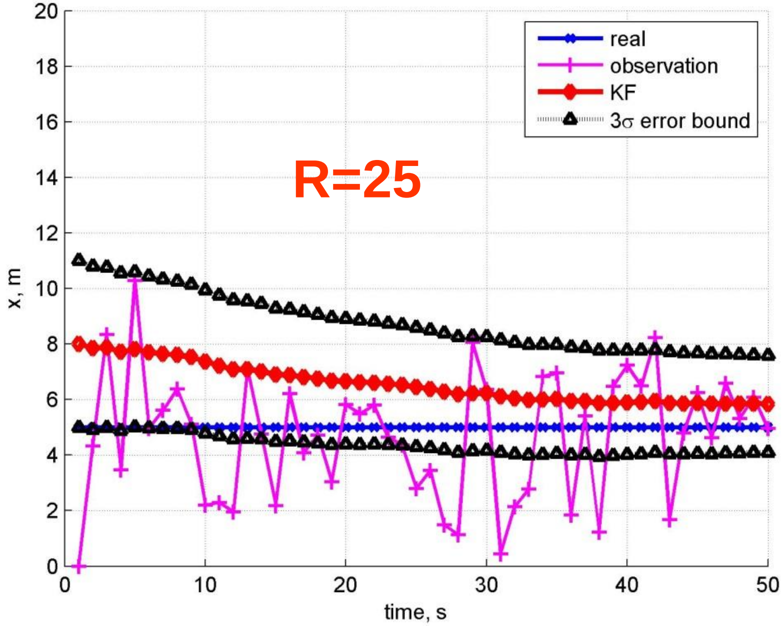
$$\hat{x}^{+} = \hat{x} + \frac{\hat{P}}{\hat{P}+R_1}(z_3 - \hat{x}) = \hat{x} + K^{+}(z_3 - \hat{x})$$
$$\hat{P}^{+} = (1-K^{+})\hat{P}$$



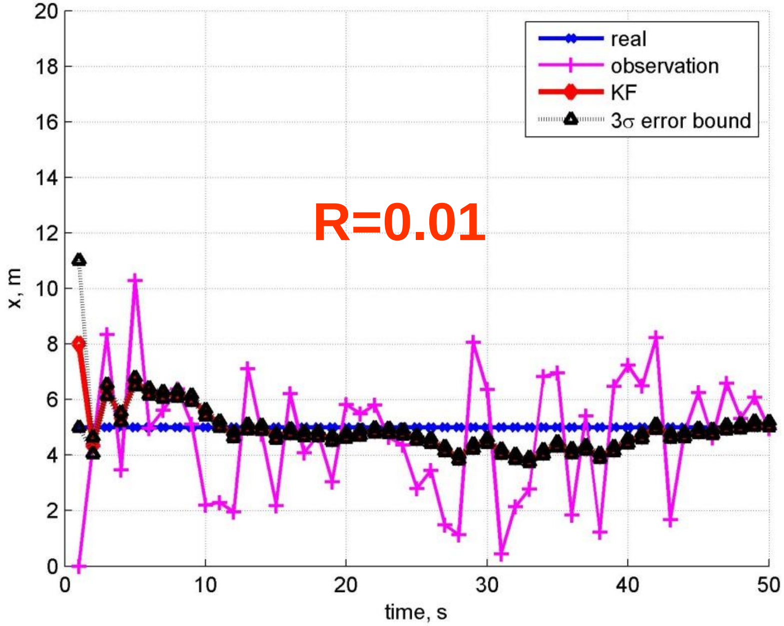
R=4
(correct noise characteristics)



$$\hat{x}^{+} = \hat{x} + \frac{\hat{P}}{\hat{P}+R}(z_{new} - \hat{x}) = \hat{x} + K^{+}(z_{new} - \hat{x})$$



R=25



R=0.01

Extension to the General Case

- **Kalman filter algorithm**

A priori estimate: $N(\underline{\bar{x}}, \bar{P})$

Measurement: $z = H\underline{x} + v, \;\; v \sim N(0, R)$

$$\underline{x} = [x_1 \quad x_2]^T, \quad x_1 : pos \, / \, x_2 : vel$$

$$z = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v = x_1 + v$$

A posteriori estimate: $N(\hat{x}, \hat{P})$

$$K = \bar{P}H^T(H\bar{P}H^T + R)^{-1}$$

$$\underline{\hat{x}} = \underline{\bar{x}} + K(z - H\underline{\bar{x}})$$

$$\hat{P} = (I - KH)\bar{P}$$

$$K = \bar{P}(\bar{P} + R)^{-1}$$

$$\hat{x} = \bar{x} + K(z - \bar{x})$$

$$\hat{P} = (I - K)\bar{P}$$

Dynamic Model of Interest

- **Discrete LTI (Linear Time Invariant) model**

$$\underline{x}_{k+1} = F_k \underline{x}_k + G_k \underline{w}_k$$

$$\underline{x}_{k+1}(t) = \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} w_k$$

$x_1 : pos \; / \; x_2 : vel$

- **State and error covariance prediction**

$$\hat{\underline{x}}_{k|k} \qquad P_{k|k} \qquad \Rightarrow$$

$$\hat{\underline{x}}_{k+1|k} = F_k \hat{\underline{x}}_{k|k}$$

$$P_{k+1|k} = E[(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1|k})(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1|k})^T]$$

$$= F_k P_{k|k} F_k^T + G_k Q_k G_k^T$$

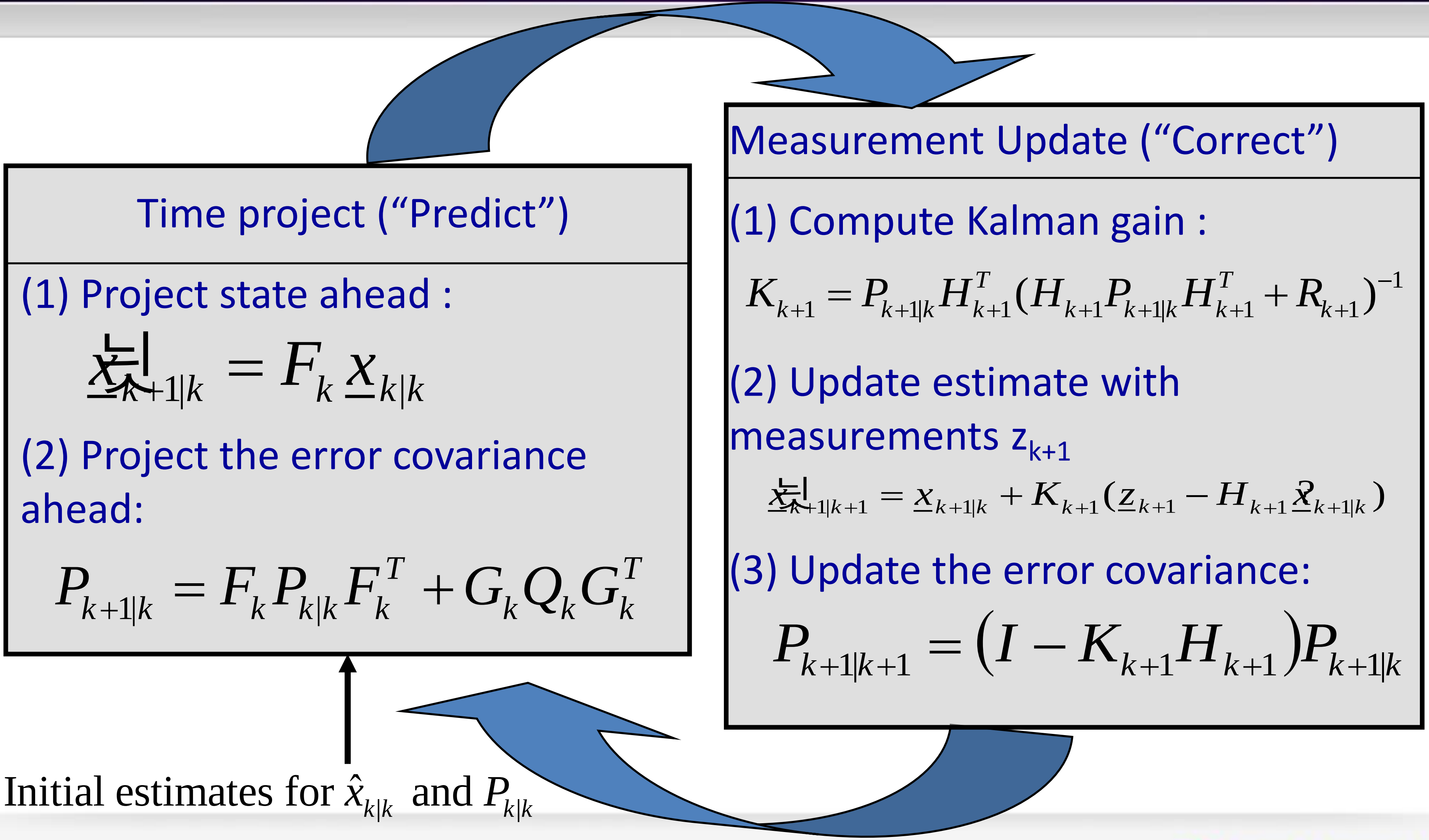
$$k \mid k \rightarrow k+1 \mid k \rightarrow k+1 \mid k+1$$

A priori

Time prediction

Correction

Description of Kalman Filter



Time project ("Predict")

(1) Project state ahead :

$$\underline{\hat{x}}_{k+1|k} = F_k \underline{x}_{k|k}$$

(2) Project the error covariance ahead:

$$P_{k+1|k} = F_k P_{k|k} F_k^T + G_k Q_k G_k^T$$

Measurement Update ("Correct")

(1) Compute Kalman gain :

$$K_{k+1} = P_{k+1|k} H_{k+1}^T (H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1})^{-1}$$

(2) Update estimate with measurements z_{k+1}

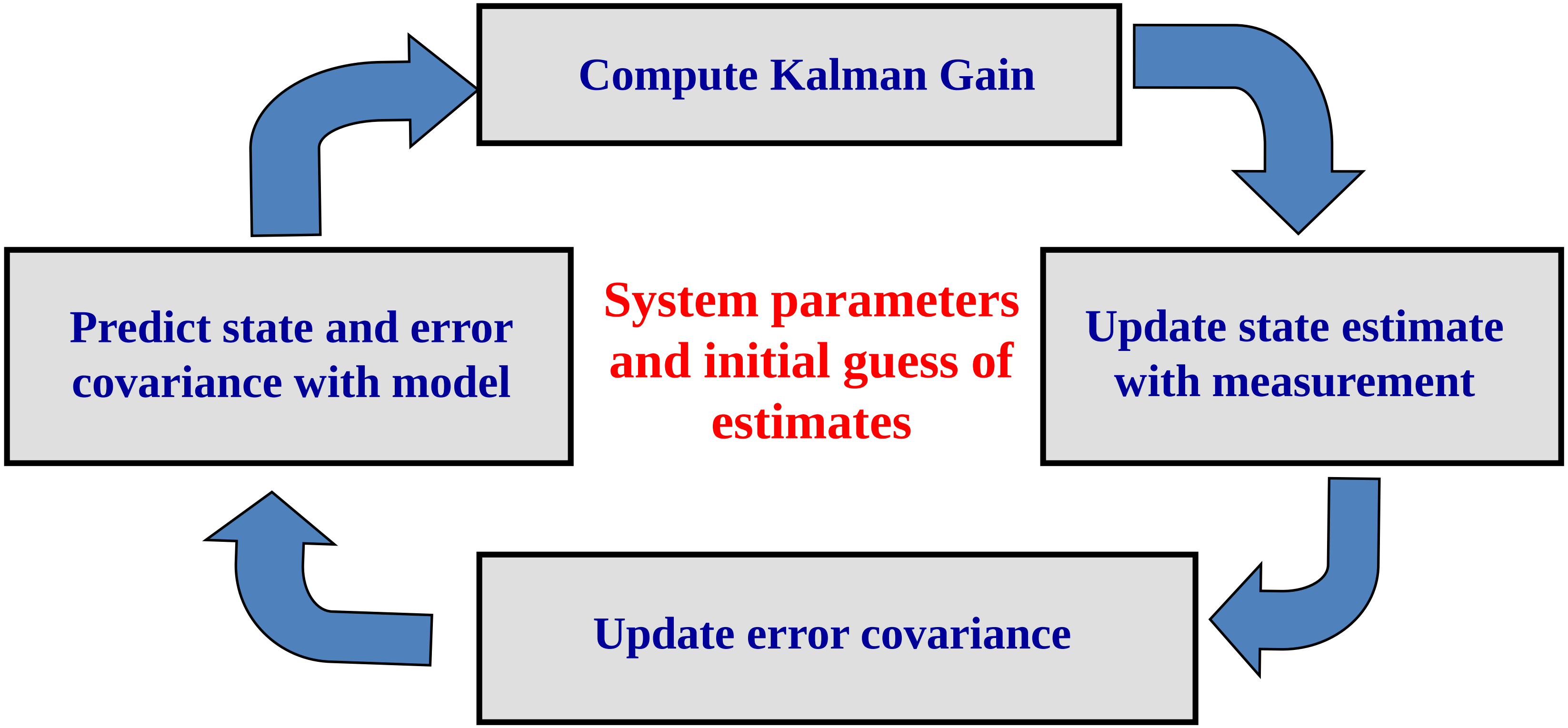
$$\underline{\hat{x}}_{k+1|k+1} = \underline{x}_{k+1|k} + K_{k+1}(\underline{z}_{k+1} - H_{k+1}\underline{\hat{x}}_{k+1|k})$$

(3) Update the error covariance:

$$P_{k+1|k+1} = \left(I - K_{k+1} H_{k+1}\right) P_{k+1|k}$$

Initial estimates for $\hat{x}_{k|k}$ and $P_{k|k}$

Structure of KF



Compute Kalman Gain

Update state estimate with measurement

Update error covariance

Predict state and error covariance with model

System parameters and initial guess of estimates

KF Algorithms

- STEP 1: Set parameters of model & noise (F, G, H, Q, R)
- STEP 2: Initialise $P_{0|0}$ and $\underline{x}_{0|0}$ (Initial guess)
- STEP 3: Calculate $\underline{x}_{1|0}$ and $P_{1|0}$
- STEP 4: Calculate K_1
- STEP 5: Take measurement $\underline{z}_1$ and calculate $x_{1|1}$
- STEP 6: Calculate $P_{1|1}$
- STEP 7: Return to STEP 3 and repeat the cycle recursively

$$\underline{x}_{k+1} = F_k \underline{x}_k + G_k \underline{w}_k$$

$$\underline{z}_{k+1} = H_{k+1} \underline{x}_{k+1} + \underline{v}_{k+1}$$

KF Example

- **1D target tracking**

$$x_{1_{k+1}} = x_{1_k} + T x_{2_k} + \frac{T^2}{2} w_k, \quad N \sim (0, \sigma_w^2)$$

$$x_{2_{k+1}} = x_{2_k} + T w_k$$

$$z_{k+1} = x_{1_{k+1}} + v_k, \quad N \sim (0, \sigma_v^2)$$

$$\Longrightarrow$$

$$\underline{x}_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \underline{x}_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} w_k$$

$$z_{k+1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}_k + v_k$$

- **True IC**

$$x_{1_0} = 0\,m$$

$$x_{2_0} = 5\,m/s$$

Recall State-Space!

$$\underline{x}_{k+1} = F_k\,\underline{x}_k + G_k\,\underline{w}_k$$

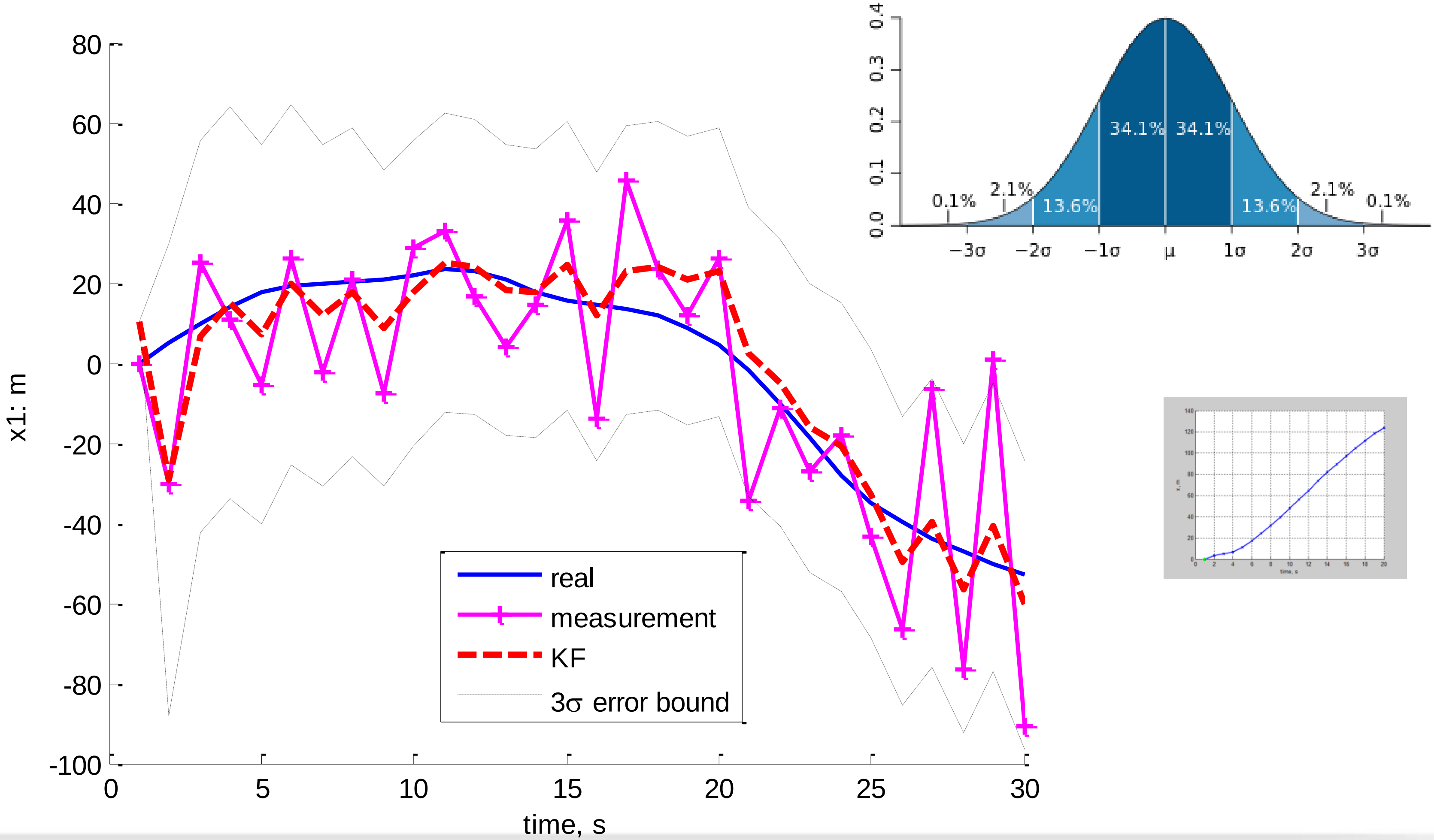
$$\underline{z}_{k+1} = H_{k+1}\underline{x}_{k+1} + \underline{v}_k$$

- **Noise**

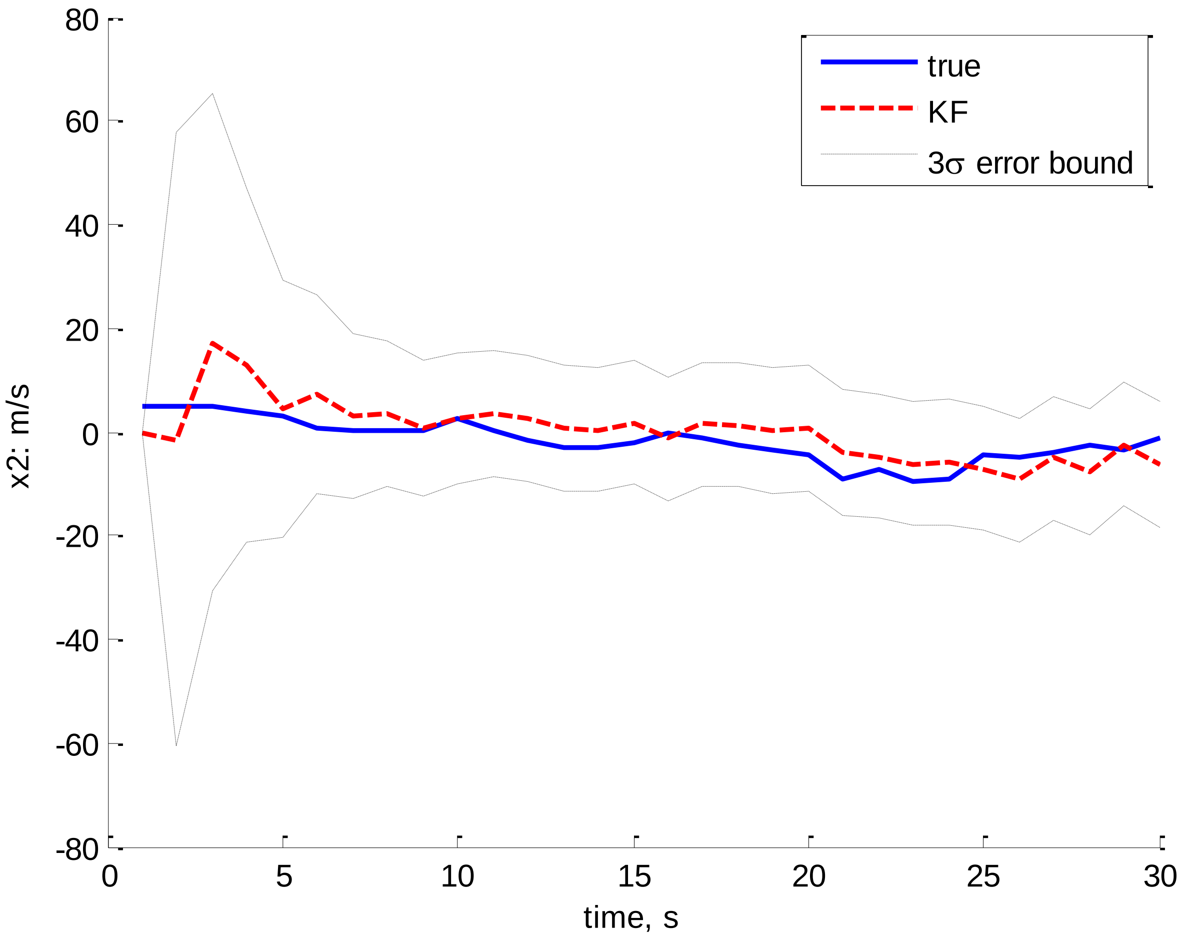
$$\sigma_w = 2\,m/s^2 \quad \Longrightarrow \quad Q = 2^2$$

$$\sigma_v = 20\,m \qquad R = 20^2$$

Position Estimation by KF



Velocity Estimation by KF



General Discussion

Observations from the 1-D KF equations

- Error covariance: implies a posteriori state estimate is always better than a priori state estimate

$$\frac{P_{k+1|k+1}}{P_{k+1|k}} = 1 - K_{k+1} = \frac{R_{k+1}}{P_{k+1|k} + R_{k+1}} < 1$$

$$K_{k+1} = P_{k+1|k}\left(P_{k+1|k} + R_{k+1}\right)^{-1}$$
$$\underline{\hat{x}}_{k+1|k+1} = \underline{\hat{x}}_{k+1|k} + K_{k+1}\left(\underline{z}_{k+1} - \underline{\hat{x}}_{k+1|k}\right)$$
$$P_{k+1|k+1} = \left(1 - K_{k+1}\right)P_{k+1|k}$$

- In case of $R \rightarrow \infty : K \rightarrow 0,\ \underline{\hat{x}}_{k+1|k+1} \rightarrow \underline{\hat{x}}_{k+1|k}$

  as the measurement noise becomes very intense, we do not use the measurement much to update the estimate as it is unreliable

- In case of $R \rightarrow 0 : K \rightarrow 1,\ \underline{\hat{x}}_{k+1|k+1} \rightarrow \underline{z}_{k+1}$

  as the measurement noise is low, we rely more on the new measurement

- How about Q? (system uncertainty)

  (as Q increases, P & K increases)

$$\underline{\hat{x}}_{k+1|k} = F_k \underline{\hat{x}}_{k|k} + G_k \underline{u}_k$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + G_k Q_k G_k^T$$

KF Derivations and Variations

- **Kalman filter algorithm derivation**
  - Bayesian approach (conditional density function)
  - Orthogonal projection
  - Minimum variance filter
  - Weighted least-square estimation (deterministic concept)

⇨ resulting in the same KF algorithm for a linear and Gaussian system

- **Variations**
  - Extended Kalman Filter (EKF) / Unscented Kalman Filter (UKF)
  - Particle Filter (Sequential Monte Carlo methods)
  - Sensor fusion techniques (how to combine multiple sensor measurement systematically in an optimal sense)

KF Design Procedure

- **Develop process/error models for relevant systems**
- **Determine measurement equations**
  - Position (radar, radio, GPS, laser), velocity (doppler), or altitude
- **Investigate the measurement and process noise characteristic**
- **Discretise the error state space models**

$$\dot{x}(t) = Ax(t) \Leftrightarrow x_{k+1} = e^{AT} x_k$$

$$where\ e^{AT} = I + AT + \frac{(AT)^2}{2} + \cdots$$

- **Design Kalman filtering algorithm**
- **Simulation**
- **Validation**

Recap of the Kalman Filter

- **KF: a recursive linear state estimator**
  - The gain of KF is **optimally** chosen s.t.

$$\underline{\hat{x}}_k \text{ minimises } E[(\underline{x}_k - \underline{\hat{x}}_k)(\underline{x}_k - \underline{\hat{x}}_k)^T]$$

- **Why popular?**
  - Incorporate noise effects (both measurement and modelling)
  - Recursive computational structure (real time application)
  - Explicit description of process and observations allows different sensor models to be incorporated within the basic KF algorithm

- **KF performs better than any other linear filter**
  - In the right situations: Gaussian noise and accurate modelling