[MEN573]
Advanced Control Systems I

Lecture 6 – Relation Between State Space Models and Transfer Functions

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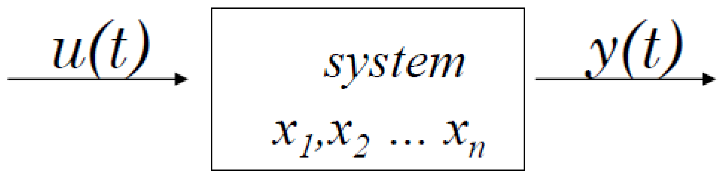
Transfer Functions and State Equations

- Given a state space model the corresponding transfer function model is **uniquely determined**, but given a transfer function model the choice of a state space model is **not unique**.

$$\begin{tikzcd} \text{State Space Model} \arrow[r, shift left, "\text{Unique}"] & \text{Transfer Function} \arrow[l, shift left, "\text{Non-unique}"] \end{tikzcd}$$

Continuous-time LTI
State Space Description



$$
\begin{aligned}
\frac{d}{dt}x(t) &= A\,x(t) + B\,u(t) \\
y(t) &= C\,x(t) + D\,u(t)
\end{aligned}
$$

- **Single Input and Single output (SISO) System**:

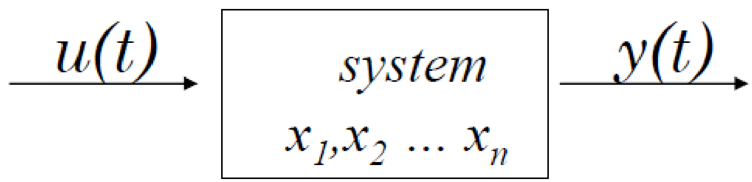
  $u(t) \in \mathcal{R}$ $\qquad$ $y(t) \in \mathcal{R}$

- State vector: $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}^T \in \mathcal{R}^n$

- State equation: $A \in \mathcal{R}^{n\times n}, \;\; B \in \mathcal{R}^{n\times 1}$

- Output equation $C \in \mathcal{R}^{1\times n}, \;\; D \in \mathcal{R}$

Continuous-time LTI
Input/output Description (forced response)



$u(t) \in \mathcal{R}$

$y(t) \in \mathcal{R}$

$$\boxed{\begin{aligned} y(t) &= (g * u)(t) \\ &= \int_0^t g(t-\tau)\, u(\tau)\, d\tau \end{aligned}}$$

$g(t)$ *System's impulse response*

Continuous-time LTI Transfer Function Description

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

- Using the Laplace transform

$$Y(s) = \mathcal{L}\{y(t)\} \quad U(s) = \mathcal{L}\{u(t)\}$$

$$G(s) = \mathcal{L}\{g(t)\}$$

- We obtain:

$$\boxed{Y(s) = G(s)U(s)}$$

From State Space to Transfer Function (CT, LTI, SISO)

$$\boxed{\begin{aligned}\frac{d}{dt}x(t) &= A\,x(t) + B\,u(t)\\ y(t) &= C\,x(t) + D\,u(t)\end{aligned}}$$

$$A \in \mathcal{R}^{n\times n},\ \ B \in \mathcal{R}^{n\times 1}$$
$$C \in \mathcal{R}^{1\times n},\ \ D \in \mathcal{R}$$

Taking the Laplace-transformation:

$$\begin{aligned} s\,X(s) - x(0) &= A\,X(s) + B\,U(s)\\ Y(s) &= C\,X(s) + D\,U(s)\end{aligned}$$

and assuming that: $x(0) = 0$

$$\begin{aligned} (sI - A)\,X(s) &= B\,U(s)\\ Y(s) &= C\,X(s) + D\,U(s)\end{aligned}$$

From State Space to Transfer Function (CT, LTI, SISO)

$$(sI - A)X(s) = BU(s)$$

$$Y(s) = C\,X(s) + DU(s)$$

- Solving:

$$X(s) = (sI - A)^{-1}BU(s)$$

$$Y(s) = C\,X(s) + DU(s)$$

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)}U(s)$$

From State Space to Transfer Function (CT, LTI, SISO)

- Given a set of matrices

$$A \in \Re^{n\times n},\ B \in \Re^{n\times 1},\ C \in \Re^{1\times n},\ D \in \Re$$

G(s) is obtained uniquely via

$$\boxed{G(s) \;=\; \left[\underbrace{C}_{1\times n}\,\underbrace{(s\,I - A)^{-1}}_{n\times n}\,\underbrace{B}_{n\times 1} + D\right] = \frac{B(s)}{A(s)}}$$

$$A(s) = s^n + a_{(n-1)}s^{(n-1)} + \cdots + a_0$$

$$B(s) = b_m s^m + b_{(m-1)}s^{(m-1)} + \cdots + b_0$$

$$m \le n\ (realizable)$$

From State Space to Transfer Function (CT, LTI, SISO)

- Remind

$$M^{-1} = \frac{1}{\det(M)} Adj(M)$$

$$Adj(M) = \{Cofactor\ matrix\ of\ M\}^T$$

- Example

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

$$\{\text{Cofactor matrix of } M\} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad c_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad c_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$c_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \quad c_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad c_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$c_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad c_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \quad c_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

From State Space to Transfer Function (CT, LTI, SISO)

- Transfer function

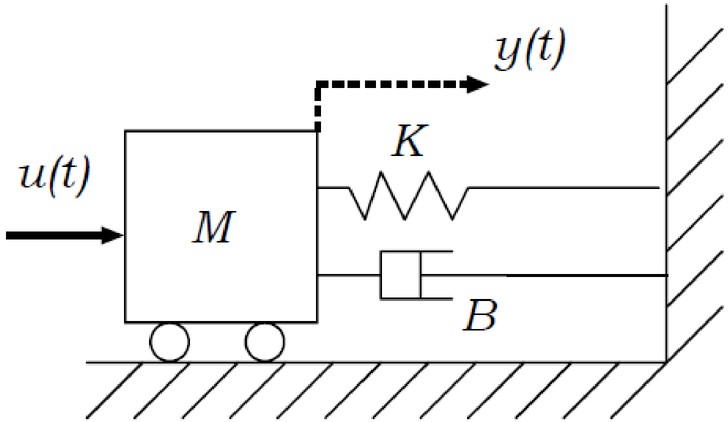
$$G(s) = \left[C(sI - A)^{-1} B + D \right] = \frac{B(s)}{A(s)}$$

$$A(s) = \det(sI - A)$$
$$= s^n + a_{(n-1)} s^{(n-1)} + \cdots + a_0$$

$$B(s) = (C\,Adj(sI - A)B + D\det(sI - A))$$
$$= b_m s^m + b_{(m-1)} s^{(m-1)} + \cdots + b_0$$

$$\Rightarrow \quad m = \begin{cases} n & if\ D \neq 0 \\ < n & if\, D = 0 \end{cases}$$

Example: Mass-spring-damper system



mass position

$$x(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} \in \mathcal{R}^2$$

mass velocity

$$\frac{d}{dt}\underbrace{\begin{bmatrix} p(t) \\ v(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} p(t) \\ v(t) \end{bmatrix}}_{x(t)}$$

Example: Mass-spring-damper system

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\boxed{G(s) = C\,(s\,I - A)^{-1}\,B}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix}\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right]^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Example: Mass-spring-damper system

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m} s + \frac{k}{m}} \underbrace{\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}}_{\text{Adj}\left\{ \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix} \right\}}$$

Example: Mass-spring-damper system

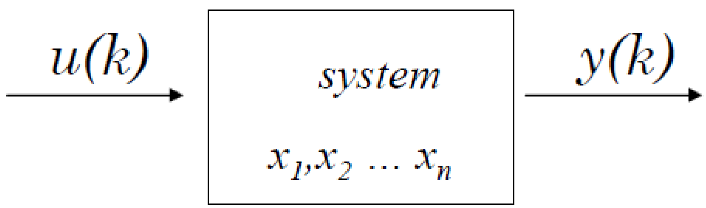
$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$G(s) = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}{s^2 + \frac{b}{m} s + \frac{k}{m}}$$

$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m} s + \frac{k}{m}}$$

$$\Rightarrow \quad Y(s) = \frac{1}{ms^2 + bs + k} U(s)$$

From State Space to Transfer Function (DT, LTI, SISO)



$$\boxed{\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}}$$

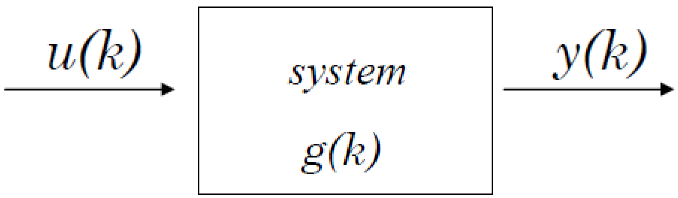
- **Single Input and Single output (SISO) System**:

$$u(k) \in \mathcal{R} \qquad\qquad y(k) \in \mathcal{R}$$

- State vector: $x(k) = \left[\; x_1(k) \;\; x_2(k) \;\; \cdots \;\; x_n(k) \;\right]^T \in \mathcal{R}^n$

- State equation: $A \in \mathcal{R}^{n\times n}, \;\; B \in \mathcal{R}^{n\times 1}$
- Output equation $C \in \mathcal{R}^{1\times n}, \;\; D \in \mathcal{R}$

From State Space to Transfer Function (DT, LTI, SISO)



$$\begin{aligned} y(k) &= (g * u)(k) \\ &= \sum_{j=0}^{k} g(k-j)\, u(j) \end{aligned}$$

(SISO System) $u(k) \in \mathcal{R} \quad y(k) \in \mathcal{R}$

Using the Z-transforms: $Y(z) = \mathcal{Z}\{y(k)\} \quad U(z) = \mathcal{Z}\{u(k)\}$

We obtain:

$$\boxed{Y(z) = G(z)\, U(z)}$$

$$G(z) = \mathcal{Z}\{g(k)\} = \frac{B(z)}{A(z)}$$

From State Space to Transfer Function (DT, LTI, SISO)

$$\boxed{\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}}$$

$$A \in \mathcal{R}^{n\times n}, \quad B \in \mathcal{R}^{n\times 1}$$
$$C \in \mathcal{R}^{1\times n}, \quad D \in \mathcal{R}$$

Taking the Z-transformation:

$$\begin{aligned} z\,X(z) - z\,x(0) &= A\,X(z) + B\,U(z) \\ Y(z) &= C\,X(z) + D\,U(z) \end{aligned}$$

and assuming that: $x(0) = 0$

we obtain

$$\boxed{Y(z) = \underbrace{\left[C\,(z\,I - A)^{-1}\,B + D\right]}_{G(z)} U(z)}$$

From State Space to Transfer Function (DT, LTI, SISO)

Given a set of matrices $A \in \mathcal{R}^{n\times n},\ \ B \in \mathcal{R}^{n\times 1}$

$C \in \mathcal{R}^{1\times n},\ \ D \in \mathcal{R}$

$G(z)$ is obtained ***uniquely*** via

$$\boxed{G(z) = \left[\underbrace{C}_{1\times n}\underbrace{(zI - A)^{-1}}_{n\times n}\underbrace{B}_{n\times 1} + D\right] = \frac{B(z)}{A(z)}}$$

$$\begin{aligned} A(z) &= \mathrm{det}(zI - A) \\ &= z^n + a_{(n-1)}z^{(n-1)} + \cdots a_o \end{aligned}$$

$$\boxed{m \leq n}$$

$$\begin{aligned} B(z) &= (C\,\mathrm{Adj}(zI - A)B + D\,\mathrm{det}(zI - A)) \\ &= b_m z^m + b_{m-1}z^{(m-1)} + \cdots b_o \end{aligned}$$

$$m = \begin{cases} n & \text{if } D \neq 0 \\ < n & \text{if } D = 0 \end{cases}$$

From Transfer Function to State Space

- Given a transfer function, the selection of state variables is **<u>not unique</u>**. This implies that there exist an infinite number of pairs of state and output equations.
- Given a $G(s)$, we cannot obtain a unique state realization $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$
- There are infinite sets of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ that produce the same $G(s)$

From Transfer Function to State Space

- Given a transfer function, $G(s)$

  We will now study the following state space realizations:

  1. Controllable canonical => controllability
  2. Observable canonical => observability
  3. Jordan form => distinct, repeated and complex poles

  that produce the same $G(s)$.

- These canonical forms are convenient in analysis and design of control systems.

From Transfer Function to State Space

- Instead of general n^{th} order systems, we will discuss a strictly causal third order transfer function expressed as

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{B(s)}{A(s)}$$

From Transfer Function to State Space

- What happens when $m = n$?

$$G'(s) = \frac{\beta_3 s^3 + \beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

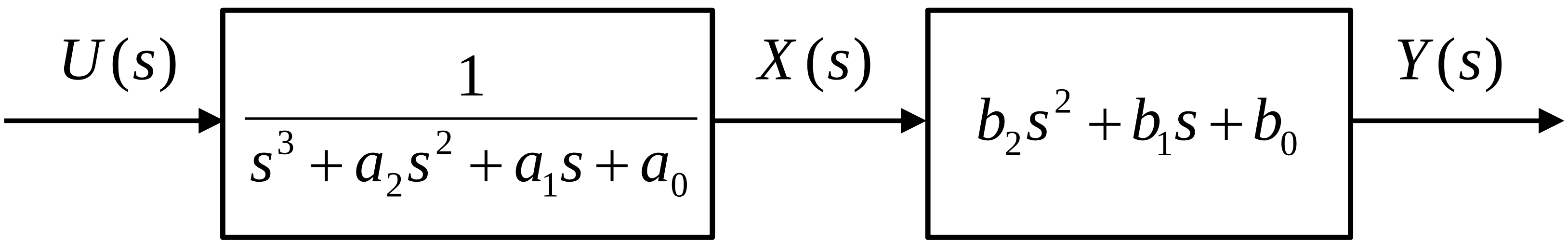
$$= \underbrace{\beta_3}_{D} + \underbrace{\frac{(\beta_2 - \beta_3 a_2)s^2 + (\beta_1 - \beta_3 a_1)s + (\beta_0 - \beta_3 a_0)}{s^3 + a_2 s^2 + a_1 s + a_0}}_{G(s) \ \text{where } b_i = \beta_i - \beta_3 a_i}$$

- $D = \beta_3$ no matter what choice is made for A, B and C.
- Current output does not directly depend on current input.

$$y(t) = C\,x(t) + D\,u(t) \qquad\qquad y(k) = C\,x(k) + D\,u(k)$$

Controllable Canonical Form

- Define a fictitious intermediate variable, $x(t)$, as below.



$$\frac{d^3\,x(t)}{dt^3} + a_2\,\frac{d^2\,x(t)}{dt^2} + a_1\,\frac{d\,x(t)}{dt} + a_o\,x(t) = u(t)$$

$$y(t) = b_o\,x(t) + b_1\,\frac{d\,x(t)}{dt} + b_2\,\frac{d^2\,x(t)}{dt^2}$$

Controllable Canonical Form

- Now, define state variables

$$x_1(t) = x(t) \qquad x_2(t) = \frac{d\,x(t)}{dt} \qquad x_3(t) = \frac{d^2\,x(t)}{dt^2}$$

- Then, we can immediately write,

$$\begin{aligned}
\frac{d\,x_1(t)}{dt} &= x_2(t) \\
\frac{d\,x_2(t)}{dt} &= x_3(t) \\
\frac{d\,x_3(t)}{dt} &= -a_0\,x_1(t) - a_1\,x_2(t) - a_2\,x_3(t) + u(t) \\
y(t) &= b_o\,x_1(t) + b_1\,x_2(t) + b_2\,x_3(t)
\end{aligned}$$

Controllable Canonical Form

- State equations in the form given above are said to be in the **controllable canonical form**.
- Matrix representation is shown below.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_c} u$$

$$y = \underbrace{\begin{bmatrix} b_o & b_1 & b_2 \end{bmatrix}}_{C_c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

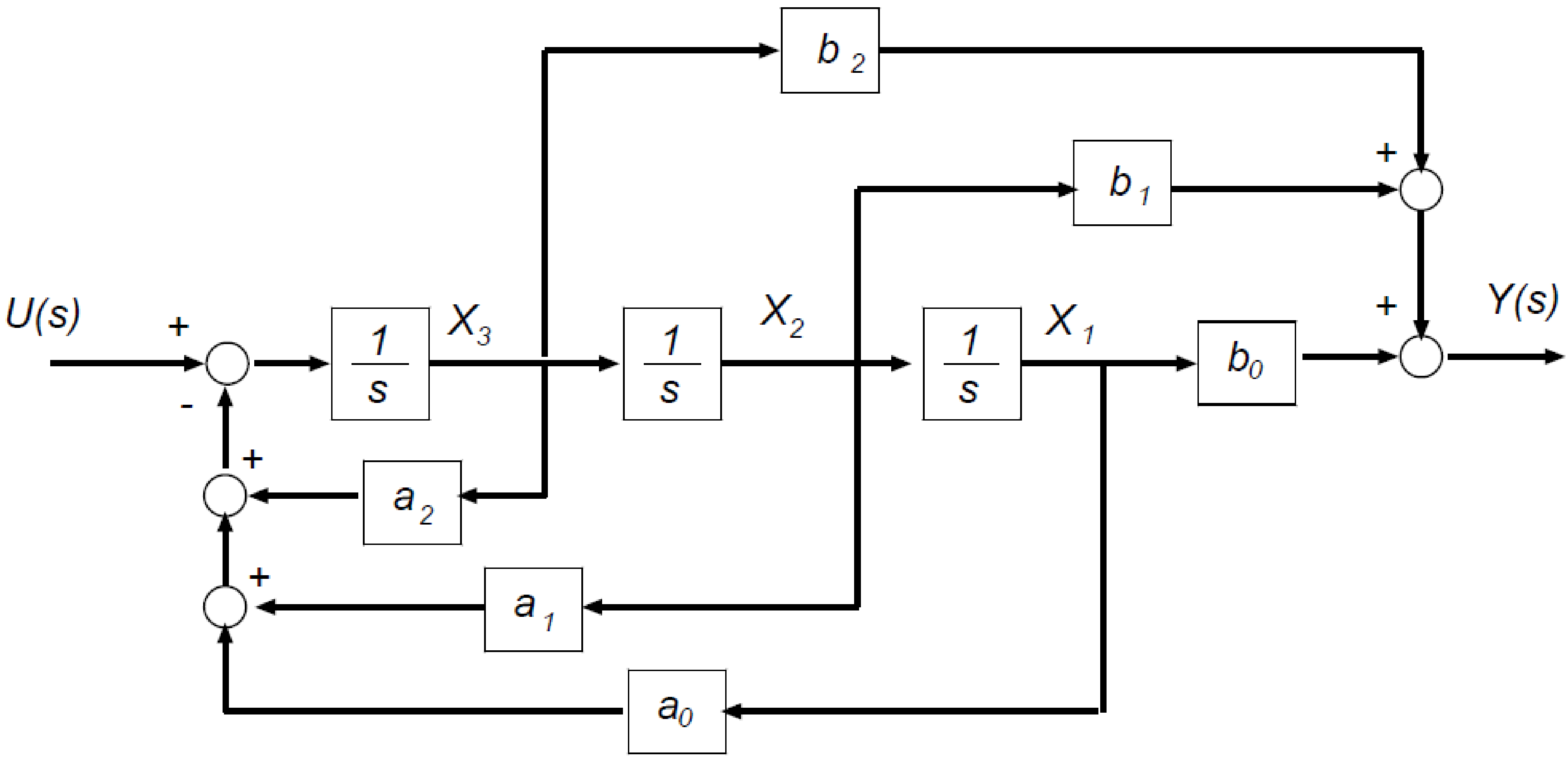
Controllable Canonical Form

- The transfer function by $A_c, B_c \text{ and } C_c$ is

$$C_c\,(s\,I - A_c)^{-1}\,B_c = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Controllable Canonical Form

- In block diagram,



Observable Canonical Form

- Expanding and dividing the transfer function by s^3.

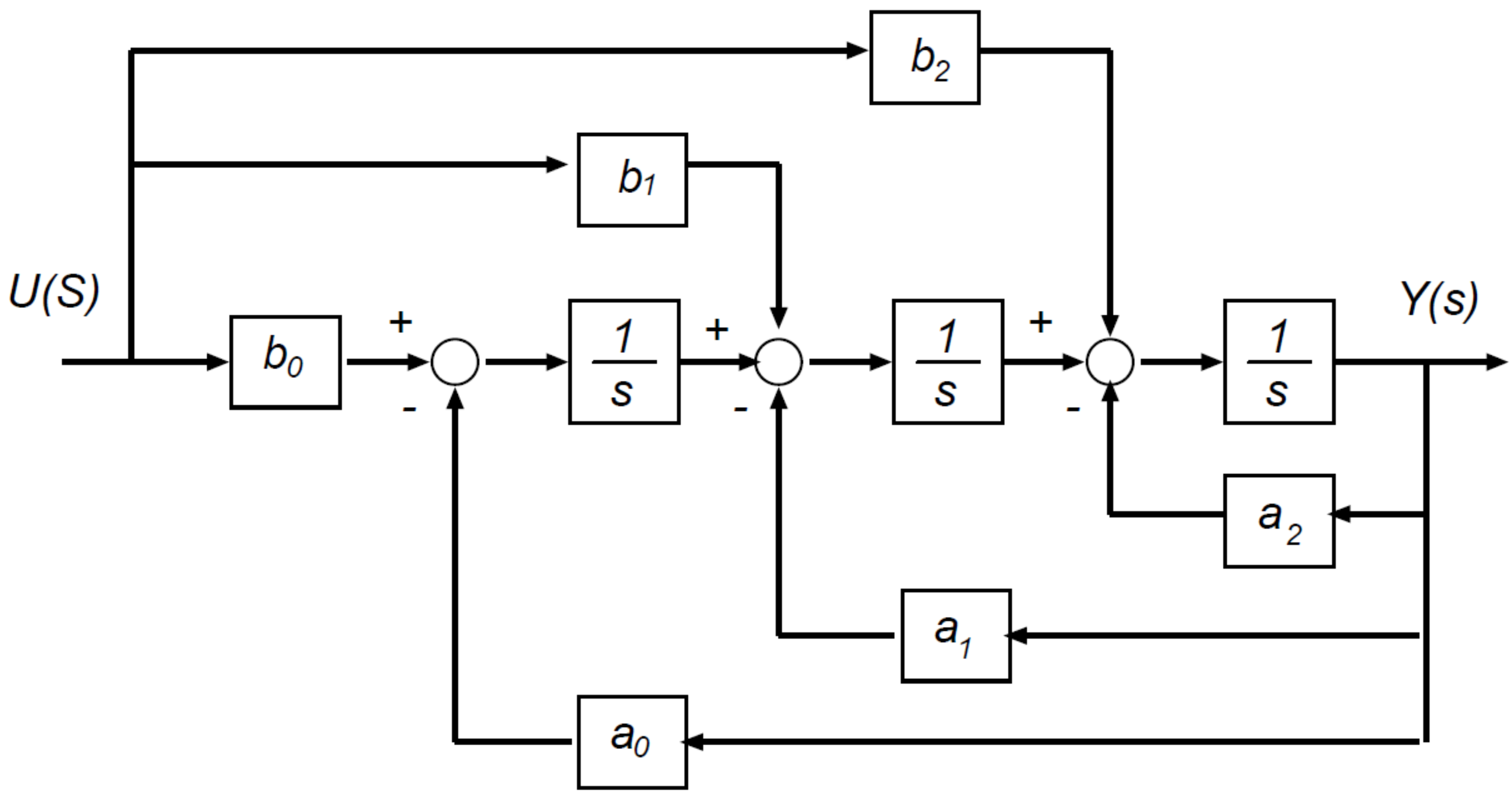
$$Y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

$$(1 + a_2\frac{1}{s} + a_1\frac{1}{s^2} + a_0\frac{1}{s^3})\,Y(s) = (b_2\frac{1}{s} + b_1\frac{1}{s^2} + b_0\frac{1}{s^3})\,U(s)$$

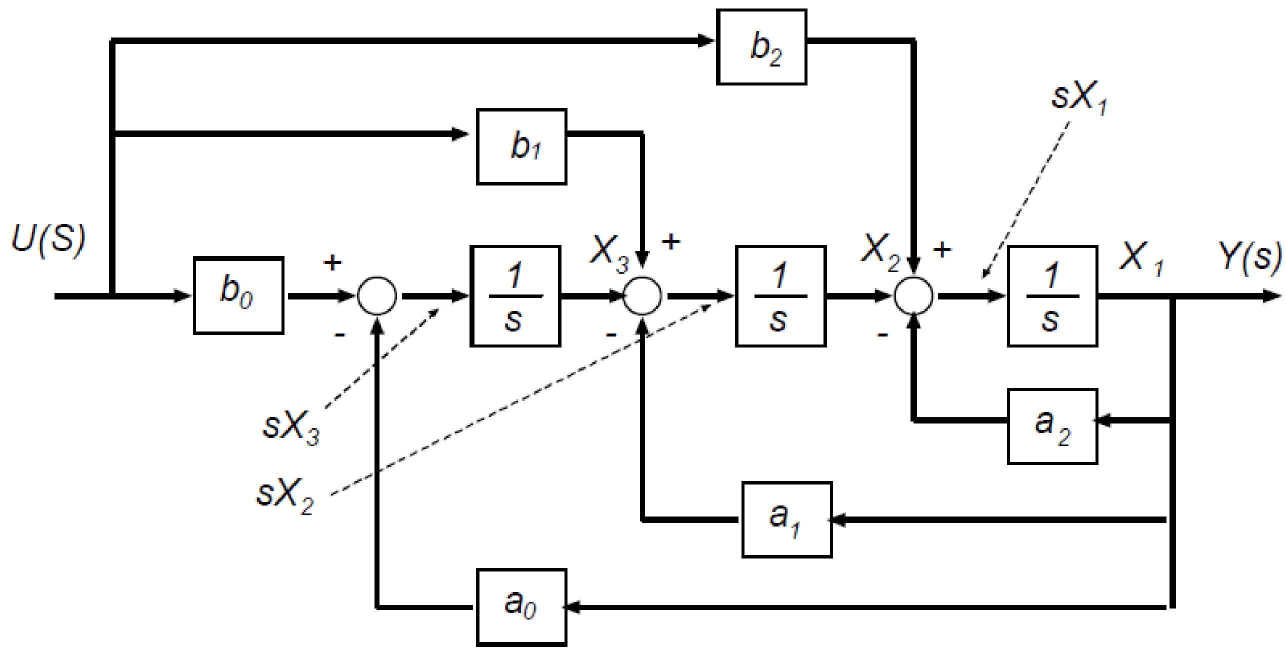
$$\Rightarrow \boxed{\begin{aligned} Y(s) \;=\; & -a_2\frac{1}{s}Y(s) - a_1\frac{1}{s^2}Y(s) - a_0\frac{1}{s^3}Y(s) \\ & +b_2\frac{1}{s}U(s) + b_1\frac{1}{s^2}U(s) + b_0\frac{1}{s^3}U(s) \end{aligned}}$$

Observable Canonical Form

- In block diagram,



Observable Canonical Form



$$Y = X_1$$

$$\begin{aligned} sX_1 &= -a_2 X_1 + X_2 + b_2 U \\ sX_2 &= -a_1 X_1 + X_3 + b_1 U \\ sX_3 &= -a_0 X_1 + b_0 U \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x}_1 &= -a_2 x_1 + x_2 + b_2 u \\ \dot{x}_2 &= -a_1 x_1 + x_3 + b_1 u \\ \dot{x}_3 &= -a_0 x_1 + b_0 u \end{aligned}$$

Observable Canonical Form

- State equations in the form given above are said to be in the **observable canonical form**.
- Matrix representation is shown below.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}}_{A_o} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} b_2 \\ b_1 \\ b_o \end{bmatrix}}_{B_o} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C_o} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observable Canonical Form

- The transfer function by $A_o, B_o\ and\ C_o$ is

$$C_o\,(s\,I - A_o)^{-1}\,B_o = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Jordan (Diagonal) Form

- Assume that the transfer function can be expanded

$$\frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \frac{K_3}{s - p_3}$$

where p_i's are the poles of the transfer function.

- It is assumed that the poles are distinct.

$$p_1 \neq p_2 \neq p_3$$

Jordan (Diagonal) Form

- Coefficients in partial fraction expansion are obtained as follows.

$$B(s) = K_1(s-p_2)(s-p_3) + K_2(s-p_1)(s-p_3) + K_3(s-p_1)(s-p_2)$$

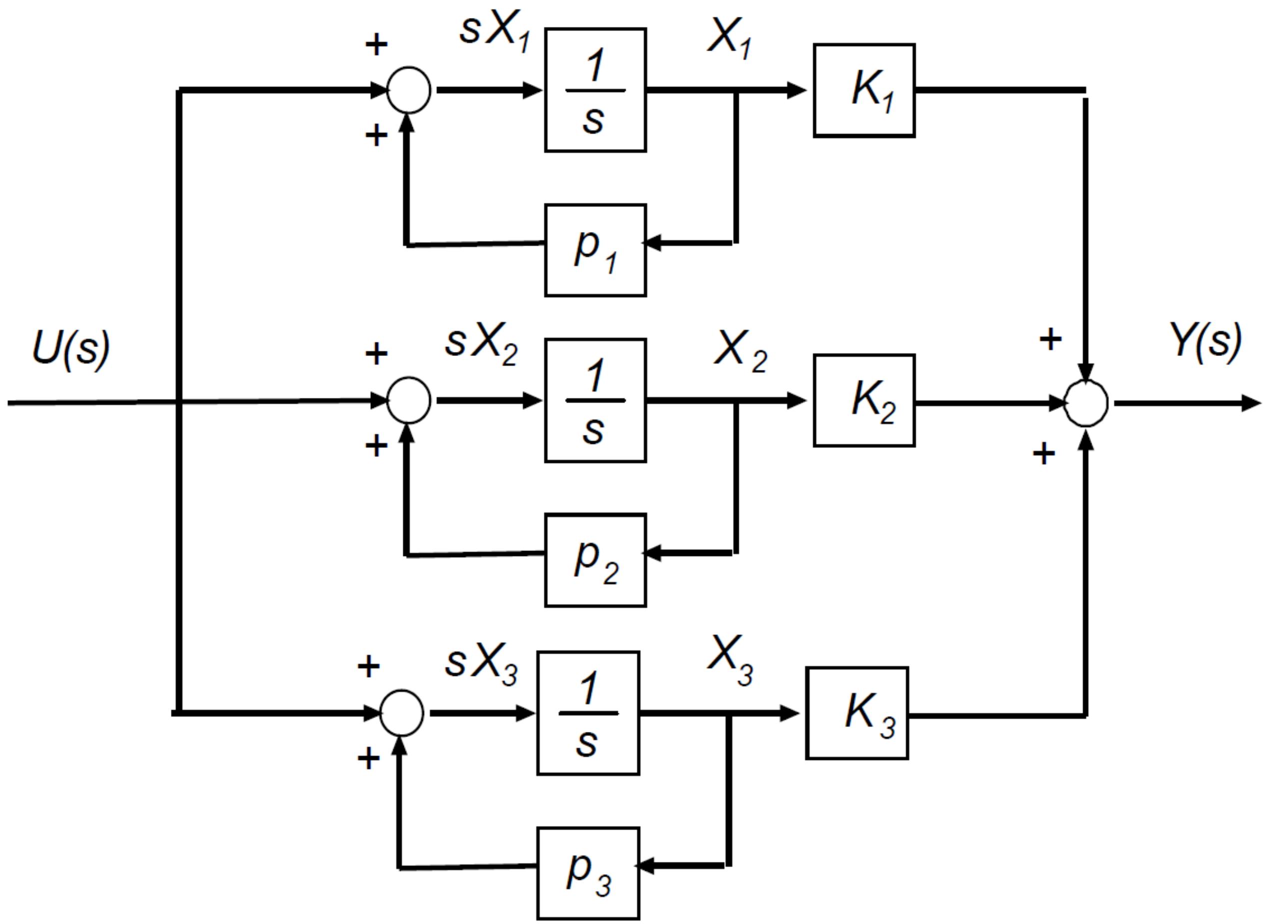
$$B(p_1) = K_1(s-p_2)(s-p_3)$$

$$K_1 = \frac{B(p_1)}{(p_1-p_2)(p_1-p_3)}$$

$$\Rightarrow \quad K_i = \frac{B(p_i)}{\prod_{j\neq i}^{3}(p_i-p_j)}$$

Jordan (Diagonal) Form

- In block diagram,



Note that each block represents a first order system.

Jordan (Diagonal) Form

- In matrix form,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}}_{A_d} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B_d} u$$

$$y = \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{C_d} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- The transfer function by $A_d, B_d \textit{ and } C_d$ is

$$C_d\,(s\,I - A_d)^{-1}\,B_d = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Jordan (Non-diagonal) Form

- Assume that $p_1 \neq p_m$ and $p_2 = p_3 = p_m$, i.e.,

$$s^3 + a_2 s^2 + a_1 s + a_0 = (s - p_1)(s - p_m)^2$$

- The transfer function can be expanded

$$\frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{K_1}{s - p_1} + \frac{K_2}{(s - p_m)^2} + \frac{K_3}{s - p_m}$$

Jordan (Non-diagonal) Form

- K_1 and K_2 can be determined as shown below.

$$B(s) = K_1(s-p_m)^2 + K_2(s-p_1) + K_3(s-p_1)(s-p_m)$$

$$\Rightarrow \quad K_1 = \frac{B(p_1)}{(p_1-p_m)^2} \qquad K_2 = \frac{B(p_m)}{p_m-p_1}$$

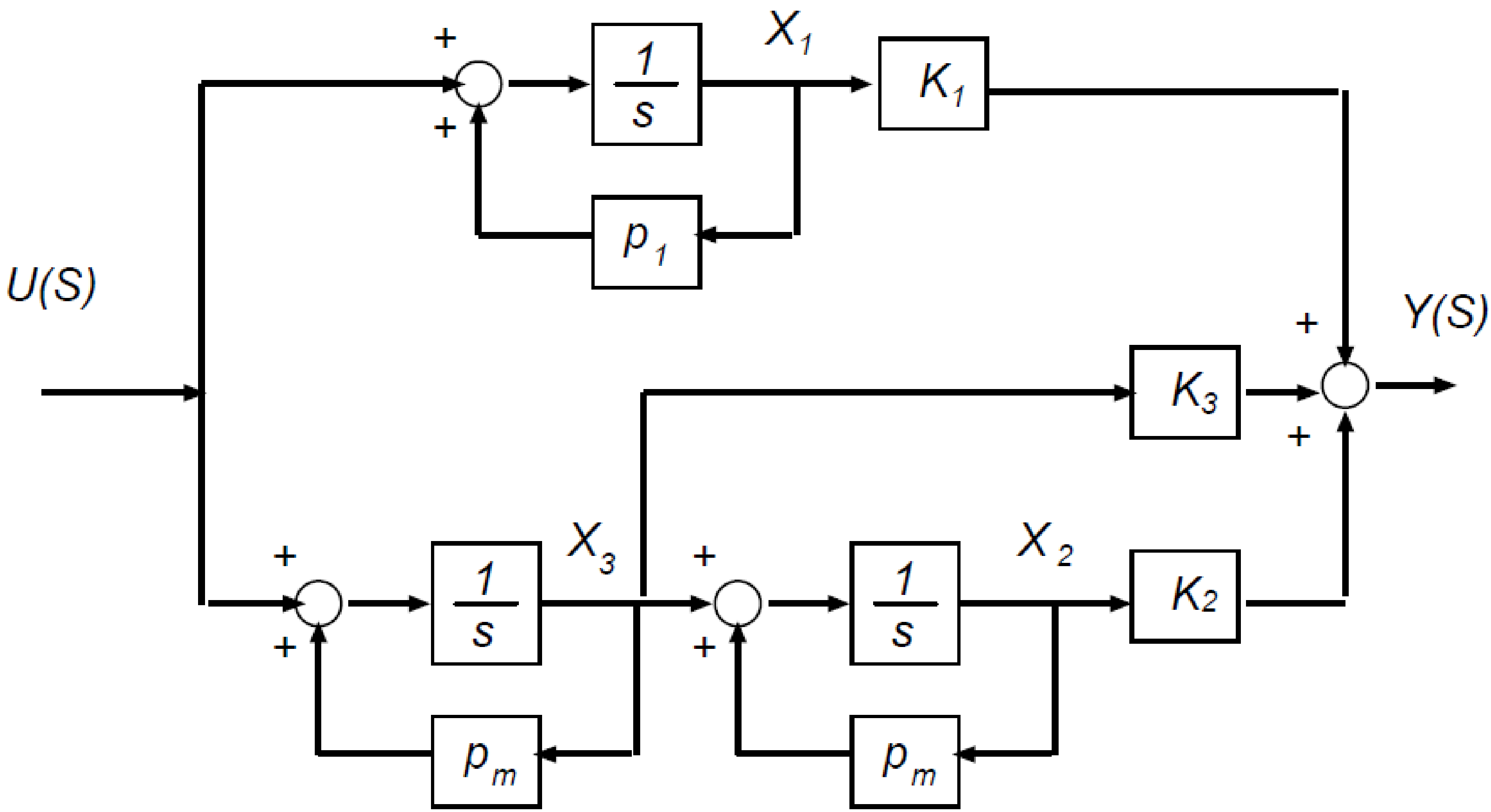
- K_3 can be determined in several ways. One way is,

$$\frac{dB(s)}{ds} = 2K_1(s-p_m) + K_2 + K_3[(s-p_m)+(s-p_1)]$$

$$\Rightarrow \quad K_3 = \frac{\left.\frac{dB(s)}{ds}\right|_{s=p_m} - K_2}{p_m-p_1}$$

Jordan (Non-diagonal) Form

- In block diagram,



Jordan (Non-diagonal) Form

- In matrix form,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_m & 1 \\ 0 & 0 & p_m \end{bmatrix}}_{A_j} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{B_j} u$$

$$y = \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{C_j} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- The transfer function by $A_j, B_j \, and \, C_j$ is

$$C_j \left(s\,I - A_j\right)^{-1} B_j = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Complex Conjugate Poles

- When the transfer function has complex poles, its expansion can consist of first order blocks for real poles and second order blocks for complex poles.
- Such expansions can be converted to state and output equations in modified canonical form.

Complex Conjugate Poles

- Assume there are one real pole and two complex conjugate poles.

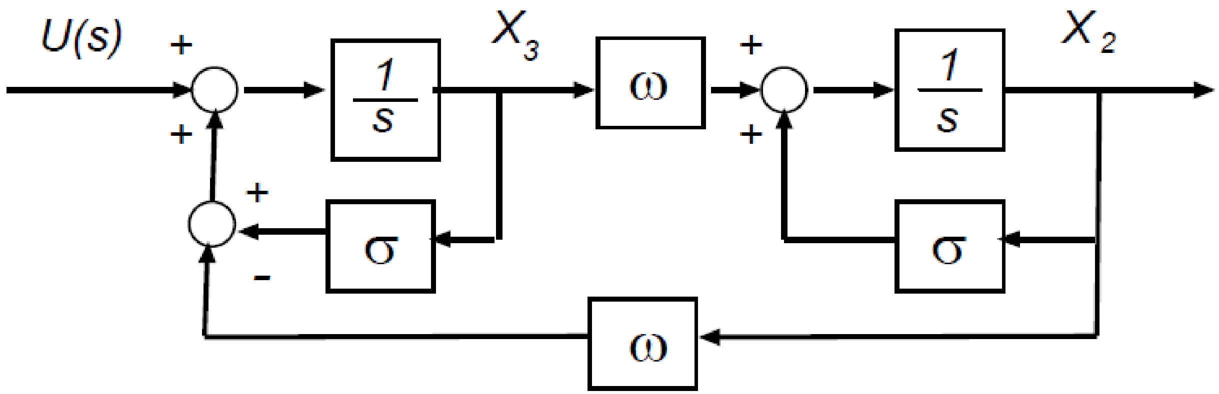
$$p_1,\ \sigma,\ \omega \in \mathcal{R}, \qquad p_2 = \sigma + j\omega,\ \ p_3 = \sigma - j\omega$$

$$s^3 + a_2 s^2 + a_1 s + a_0 = (s - p_1)\left((s-\sigma)^2 + \omega^2\right)$$

- The transfer function can be expanded

$$\frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{K_1}{s - p_1} + \frac{\alpha s + \beta}{(s-\sigma)^2 + \omega^2}$$

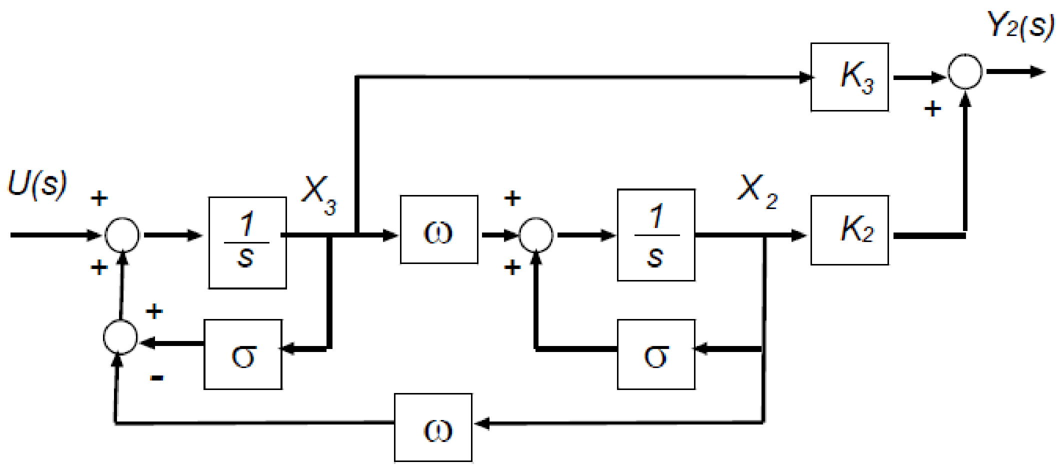
Complex Conjugate Poles



$$\frac{d}{dt}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}_{A_o} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\boxed{\det\left(s\,I - A_o\right) = \left((s-\sigma)^2 + \omega^2\right)}$$

Complex Conjugate Poles



$$\frac{d}{dt}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}}_{A_o}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y_2 = \begin{bmatrix} K_2 & K_3 \end{bmatrix}\begin{bmatrix} x_2 & x_3 \end{bmatrix}^T$$

$$\frac{Y_2(s)}{U(s)} = \begin{bmatrix} K_2 & K_3 \end{bmatrix}[sI - A_o]^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Complex Conjugate Poles

$$\frac{Y_2(s)}{U(s)} = \begin{bmatrix} K_2 & K_3 \end{bmatrix} \begin{bmatrix} (s-\sigma) & -\omega \\ \omega & (s-\sigma) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

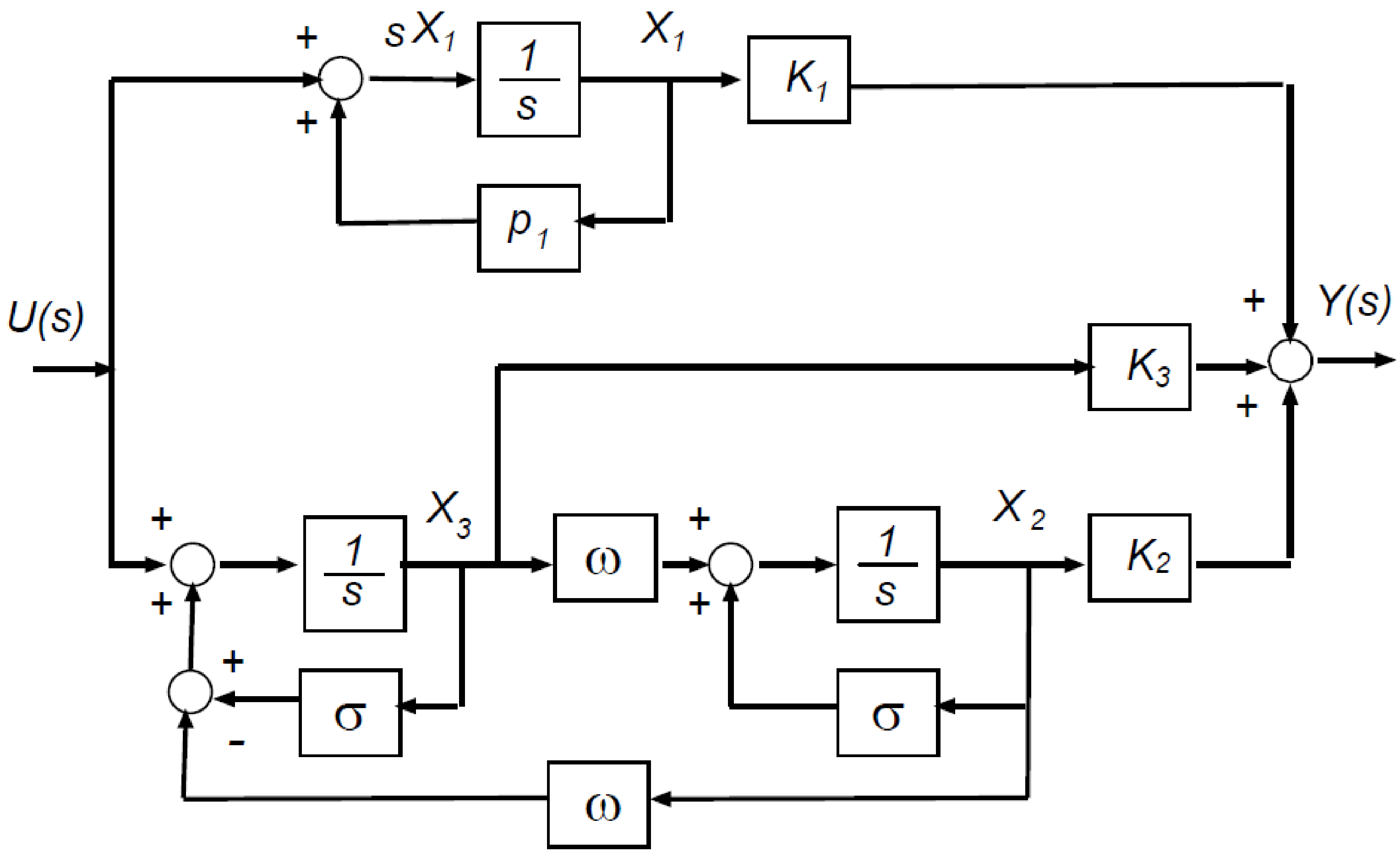
$$\frac{Y_2(s)}{U(s)} = \frac{K_2\, w + K_3\,(s-\sigma)}{\left((s-\sigma)^2 + \omega^2\right)}$$

$$= \frac{\alpha\, s + \beta}{(s-\sigma)^2 + \omega^2}$$

$$\boxed{\begin{aligned} K_2 &= \frac{\beta + \alpha\sigma}{\omega} \\ K_3 &= \alpha \end{aligned}}$$

Complex Conjugate Poles

- In block diagram,



Complex Conjugate Poles

- In matrix form,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$K_2 = \frac{\beta + \alpha\sigma}{\omega}$$

$$K_3 = \alpha$$

Summary - State Space Realization

- Given a transfer function, $G(s)$, there are infinite sets of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

  that produce the same $G(s)$
- We have obtained the following state space realizations:
  1. Controllable canonical
  2. Observable canonical
  3. Jordan forms for distinct, repeated and complex poles respectively.

Summary - State Space Realization

- Given

$$G(s)=\frac{B(s)}{A(s)}=\frac{b_2 s^2+b_1 s+b_0}{s^3+a_2 s^2+a_1 s+a_0}$$

- Controllable canonical

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_c} u$$

$$y = \underbrace{\begin{bmatrix} b_o & b_1 & b_2 \end{bmatrix}}_{C_c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Summary - State Space Realization

- Given

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

- Observable canonical

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}}_{A_o} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} b_2 \\ b_1 \\ b_o \end{bmatrix}}_{B_o} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C_o} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Summary - State Space Realization

- Given

$$G(s)=\frac{B(s)}{A(s)}=\frac{K_1}{s-p_1}+\frac{K_2}{s-p_2}+\frac{K_3}{s-p_3}$$

- Jordan form (distinct poles)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}}_{A_d} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B_d} u$$

$$y = \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{C_d} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Summary - State Space Realization

- Given

$$G(s) = \frac{B(s)}{A(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{(s - p_m)^2} + \frac{K_3}{s - p_m}$$

- Jordan form (2 repeated poles)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_m & 1 \\ 0 & 0 & p_m \end{bmatrix}}_{A_j} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{B_j} u$$

$$y = \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{C_j} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Summary - State Space Realization

- Given

$$G(s) = \frac{B(s)}{A(s)} = \frac{K_1}{(s-p_m)^3} + \frac{K_2}{(s-p_m)^2} + \frac{K_3}{s-p_m}$$

- Jordan form (3 repeated poles)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} p_m & 1 & 0 \\ 0 & p_m & 1 \\ 0 & 0 & p_m \end{bmatrix}}_{A_j} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_j} u$$

$$y = \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{C_j} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Summary - State Space Realization

- Given

$$G(s)=\frac{B(s)}{A(s)}=\frac{K_1}{s-p_1}+\frac{\alpha s+\beta}{(s-\sigma)^2+\omega^2}$$

- Jordan form (2 complex poles)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$K_2 = \frac{\beta+\alpha\sigma}{\omega}$$

$$K_3 = \alpha$$

State Space Realizations in Discrete-time

- Given a transfer function, $G(z)$, there are infinite sets of

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

  that produce the same $G(z)$
- We have obtained the following state space realizations:
  1. Controllable canonical
  2. Observable canonical
  3. Jordan forms for distinct, repeated and complex poles respectively.

  as in the continuous-time case.

State Space Realizations in Discrete-time

- The procedures for finding state space realizations in discrete time is very similar to the continuous-time cases.
- The only difference is that we use:

$$\boxed{\mathcal{Z}\left\{x(k+n)\right\} = z^n X(z)}$$

instead of

$$\mathcal{L}\left\{\frac{d^n}{dt^n}x(t)\right\} = s^n X(s)$$

while still assuming zero state initial conditions.

State Space Realizations in Discrete-time

- Given

$$G(z) = \frac{B(z)}{A(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- Controllable canonical

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_c} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} b_o & b_1 & b_2 \end{bmatrix}}_{C_c} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

State Space Realizations in Discrete-time

- Given

$$G(z) = \frac{B(z)}{A(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- Observable canonical

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}}_{A_o} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} b_2 \\ b_1 \\ b_o \end{bmatrix}}_{B_o} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C_o} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

State Space Realizations in Discrete-time

- Given

$$G(z) = \frac{B(z)}{A(z)} = \frac{K_1}{z - p_1} + \frac{K_2}{z - p_2} + \frac{K_3}{z - p_3}$$

- Jordan form (distinct poles)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}}_{A_d} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B_d} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{C_d} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

State Space Realizations in Discrete-time

- Given

$$G(z) = \frac{B(z)}{A(z)} = \frac{K_1}{z-p_1} + \frac{K_2}{(z-p_m)^2} + \frac{K_3}{z-p_m}$$

- Jordan form (2 repeated poles)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_m & 1 \\ 0 & 0 & p_m \end{bmatrix}}_{A_j} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{B_j} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}}_{C_j} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

State Space Realizations in Discrete-time

- Given

$$G(z)=\frac{B(z)}{A(z)}=\frac{K_1}{(z-p_m)^3}+\frac{K_2}{(z-p_m)^2}+\frac{K_3}{z-p_m}$$

- Jordan form (3 repeated poles)

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\underbrace{\begin{bmatrix}p_m & 1 & 0\\0 & p_m & 1\\0 & 0 & p_m\end{bmatrix}}_{A_j}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{B_j}u(k)$$

$$y=\underbrace{\begin{bmatrix}K_1 & K_2 & K_3\end{bmatrix}}_{C_j}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

State Space Realizations in Discrete-time

- Given

$$G(s)=\frac{B(s)}{A(s)}=\frac{K_1}{s-p_1}+\frac{\alpha s+\beta}{(s-\sigma)^2+\omega^2}$$

- Jordan form (2 complex poles)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$K_2 = \frac{\beta+\alpha\sigma}{\omega}$$

$$K_3 = \alpha$$

Discrete-time Controllable Canonical Form

- Consider the transfer function such that

$$G(z) = \frac{B(z)}{A(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- Find the controllable state space realization

$$x(k+1) = A_c x(k) + B_c u(k)$$

$$y(k) = C_c x(k)$$

so that

$$C_c (zI - A_c)^{-1} B_c = \frac{B(z)}{A(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

Discrete-time Controllable Canonical Form

- Let
$$Y(z) = G(z)U(z)$$
$$G(z) = \frac{B(z)}{A(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- Define:
$$X(z) = \frac{1}{A(z)}U(z)$$

so that
$$Y(z) = B(z)X(z)$$

Discrete-time Controllable Canonical Form

From

$$\boxed{X(z) = \frac{1}{A(z)} U(z)}$$

We obtain:

$$\begin{aligned} A(z)X(z) &= U(z) \\ (z^3 + a_2 z^2 + a_1 z + a_0)X(z) &= U(z) \end{aligned}$$

Remembering that, for zero initial conditions,

$$\mathcal{Z}\left\{x(k+n)\right\} = z^n X(z)$$

We obtain:

$$\boxed{x(k+3) + a_2\, x(k+2) + a_1\, x(k+1) + a_o\, x(k) = u(k)}$$

Discrete-time Controllable Canonical Form

From

$$x(k+3)+a_2\,x(k+2)+a_1\,x(k+1)+a_o\,x(k)=u(k)$$

Define the state variables :

$$x_1(k)=x(k) \qquad x_2(k)=x(k+1) \qquad x_3(k)=x(k+2)$$

And notice that $x(k+3)=x_3(k+1)$

Thus, the above equation can be rewritten in matrix form

Discrete-time Controllable Canonical Form

$$x(k+3)+a_2\,x(k+2)+a_1\,x(k+1)+a_o\,x(k) = u(k)$$

$$x_1(k) = x(k) \qquad x_2(k) = x(k+1) \qquad x_3(k) = x(k+2)$$

Is equivalent to :

$$\begin{aligned}
x_1(k+1) &= x_2(k)\\
x_2(k+1) &= x_3(k)\\
x_3(k+1) &= -a_0\,x_1(k) - a_1\,x_2(k) - a_2\,x_3(k) + u(k)
\end{aligned}$$

$$x(k) = x_1(k)$$

Discrete-time Controllable Canonical Form

From $$\boxed{Y(z) = B(z)\,X(z)}$$

We obtain: $$Y(z) = (b_0 + b_1 z + b_2 z^2)X(z)$$

and, for zero initial conditions,

$$y(k) = b_o\,x(k) + b_1\,x(k+1) + b_2\,x(k+2)$$

$$\boxed{y(k) = b_o\,x_1(k) + b_1\,x_2(k) + b_2\,x_3(k)}$$

Discrete-time Controllable Canonical Form

$$x(k+3)+a_2\,x(k+2)+a_1\,x(k+1)+a_o\,x(k) = u(k)$$

$$y(k) = b_o\,x_1(k) + b_1\,x_2(k) + b_2\,x_3(k)$$

Is equivalent to :

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= -a_0\,x_1(k) - a_1\,x_2(k) - a_2\,x_3(k) + u(k) \end{aligned}$$

$$y(k) = b_o\,x_1(k) + b_1\,x_2(k) + b_2\,x_3(k)$$

Discrete-time Controllable Canonical Form

In matrix form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}_{A_c} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_c} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} b_o & b_1 & b_2 \end{bmatrix}}_{C_c} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

and

$$C_c\,(z\,I - A_c)^{-1}\,B_c = \frac{B(z)}{A(z)} = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$