[MEN573]
Advanced Control Systems I

Lecture 15 - Controllability and Observability of Continuous Time Systems

Associate Professor Joonbum Bae
Department of Mechanical Engineering
UNIST

Definition of controllability (CT)

Definition: The system

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

is said to be **<u>controllable</u>** if,

- for any **<u>initial</u>** state $x(0) = x_0$
  and any **<u>target</u>** state, x_1
- there exists a **<u>finite</u>** time $t_1 > 0$
  and a control function

$$\{u(t);\ t \in [0, t_1]\}$$

- that will transfer the state x_0 to $x(t_1) = x_1$

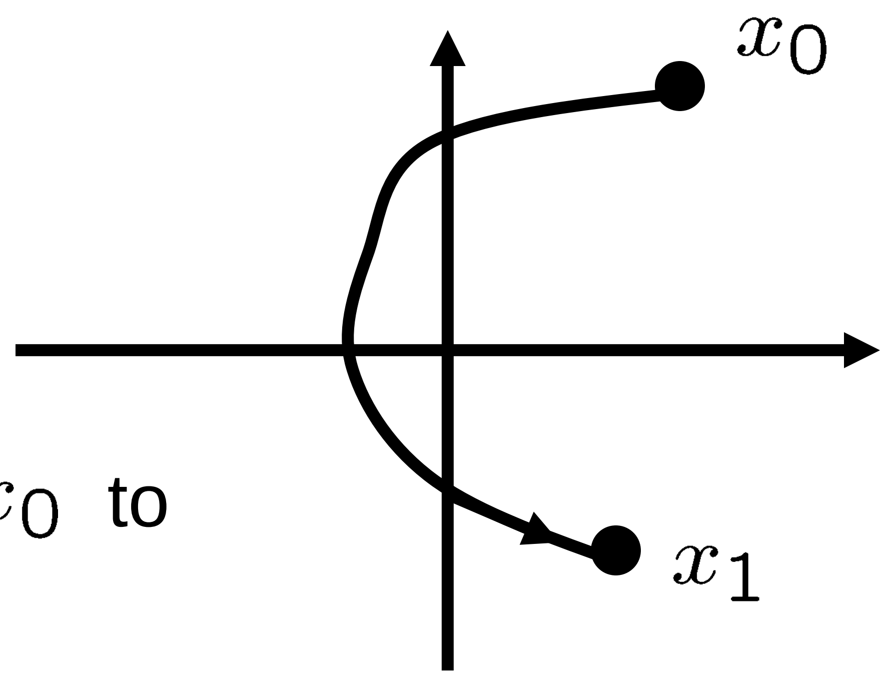
Definition of controllability (DT)

for any **initial** state $x(0) = x_0$

and any **target** state, x_1

there exists a **finite** time $t_1 > 0$

and a control function



that will transfer the state x_0 to
$x(t_1) = x_1$

Definition of controllability (DT)

Comments:

- The definition requires that both the initial state x_0 and the "target" state x_1 be ***arbitrary***.

- The definition requires the state to reach x_1 in a ***finite*** time t_1 and says nothing about what will happen to the state $x(t)$, for $t > t_1$

- It is not required that the state remains at x_1

Controllability Theorem

The following 3 statements are equivalent:

(a) The LTI system of order $\boldsymbol{n}$

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

is controllable.

Sometimes we simply state that the pair

$$\{A\,B\}$$

is controllable.

Controllability Theorem

The following 3 statements are equivalent:

(b) The controllability grammian

$$W_c(t_1) \;=\; \int_0^{t_1} e^{At} B B^T e^{A^T t}\, dt$$

is positive definite for all time $\boldsymbol{t_1 > 0}$

$$W_c(t_1) \succ 0 \qquad \forall t_1 > 0$$

Controllability Theorem

(c) The controllability matrix

$$P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

is rank $\boldsymbol{n}$.

(I.e. there are $\boldsymbol{n}$ linearly independent columns)

Remarks on Controllability Theorem

1. The controllable canonical pair

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is always controllable, since

$$P_c = \begin{bmatrix} B_c & A_c B_c & A_c^2 B_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & (-a_1 + a_2^2) \end{bmatrix}$$

is always full rank.

This result generalizes to an arbitrary order n

Controllability Grammian

Assume that the matrix $\boldsymbol{A}$ is Hurwitz.

Then, the asymptotic value of the controllability grammian

$$W_c \;=\; \lim_{t_1 \to \infty} W_c(t_1) = \int_0^{\infty} e^{At} B B^T e^{A^T t}\, dt$$

exists (all elements of $\boldsymbol{W_c}$ are bounded).

Controllability Grammian & Lyapunov Eq

Assume that the matrix $\boldsymbol{A}$ is Hurwitz.

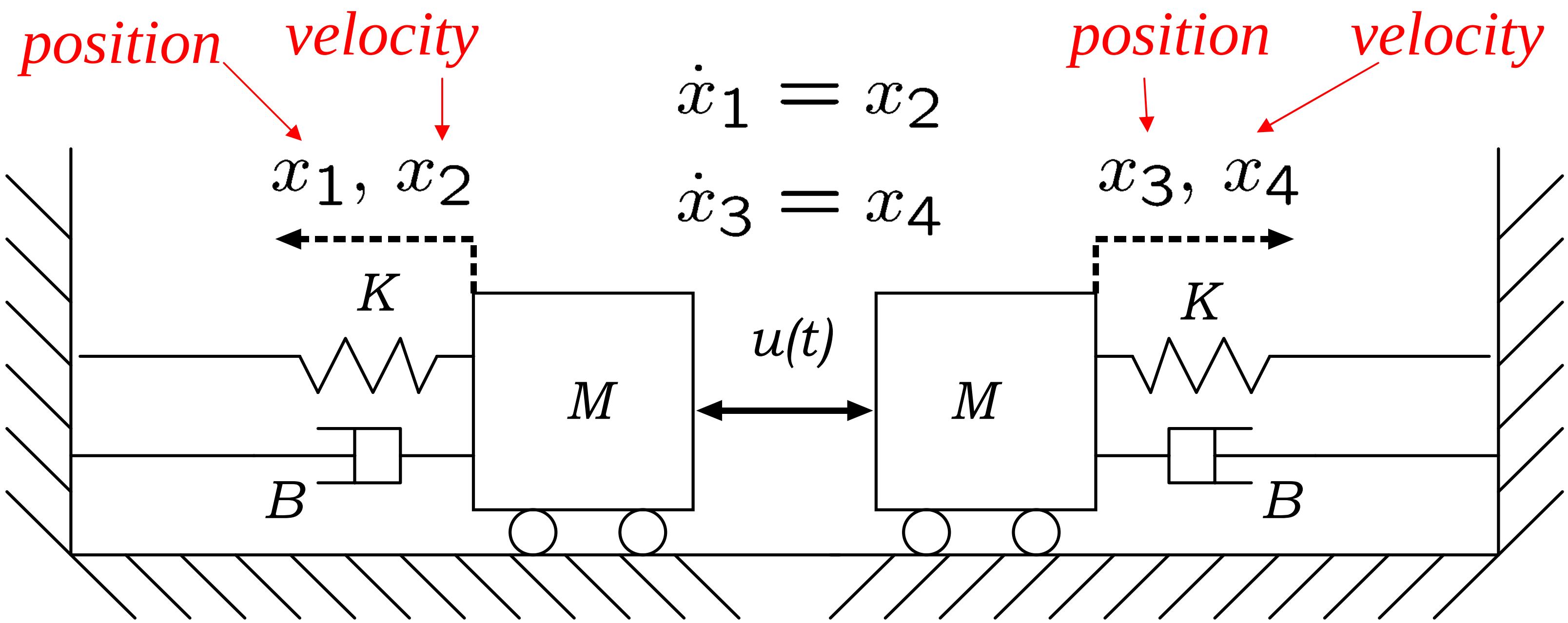
$$W_c = \int_0^\infty e^{At} B B^T e^{A^T t} \, dt$$

can be calculated as the solution of the following Lyapunov equation:

$$A\, W_c + W_c A^T = -B\, B^T$$

Moreover, $W_c \succ 0$ **<u>iff</u>** $\{A\, B\}$ is a controllable pair

An uncontrollable system: Example



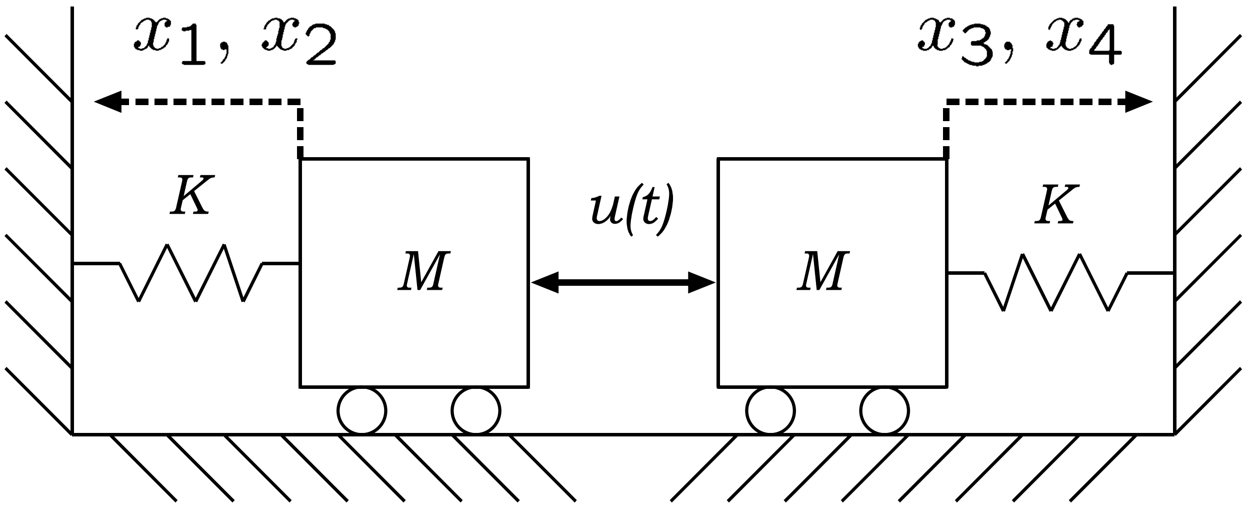
Assume that $x(0) = 0$

- Because of symmetry, no matter what the input is,

$$x_1(t) = x_3(t) \qquad \forall t \geq 0$$
$$x_2(t) = x_4(t)$$

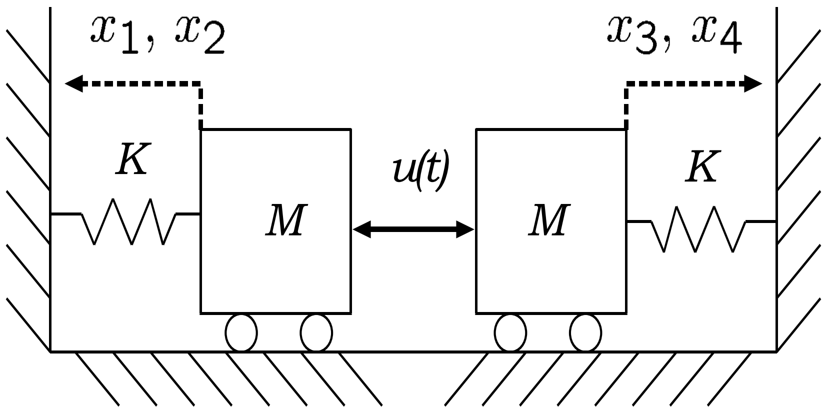
State cannot be arbitrarily steered

An uncontrollable system: example



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ \frac{1}{m_2} \end{bmatrix}}_{B} u$$

An uncontrollable system: example



$$P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & \frac{1}{m_1} & 0 & -\frac{k_1}{m_1^2} \\ \frac{1}{m_1} & 0 & -\frac{k_1}{m_1^2} & 0 \\ 0 & \frac{1}{m_2} & 0 & -\frac{k_2}{m_2^2} \\ \frac{1}{m_2} & 0 & -\frac{k_2}{m_2^2} & 0 \end{bmatrix}$$

$$\det\{P\} = \frac{1}{m_1^2 m_2^2}\left(\frac{k_2}{m_2} + \frac{k_1}{m_1}\right)\left(\frac{k_2}{m_2} - \frac{k_1}{m_1}\right)$$

$$\det\{P\} \neq 0 \Leftrightarrow \frac{k_2}{m_2} \neq \frac{k_1}{m_1}$$

Definition of Observability (CT)

The LTI continuous time system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

is said to be observable if,

- for ***any*** initial state $x(0) = x_0$

there exists a finite time $t_1 > 0$ such that

- knowledge of the input and output time functions

$$\{u(t);\ t \in [0, t_1]\} \qquad \{y(t);\ t \in [0, t_1]\}$$

- is sufficient to determine the initial state x_0

Proof of Observability Theorem (a⇔b)

Notice that the response of

$$\begin{aligned}\dot{x}(t) &= A\,x(t) + B\,u(t)\\ y(t) &= C\,x(t) + D\,u(t)\end{aligned}$$

is composed of a response and a forced response:

$$y(t) = y_{free}(t) + \underline{y_{force}(t)}$$

$$y_{free}(t) = Ce^{At}x(0)$$ ***unknown***

$$\underline{y_{force}(t)} = C\int_0^t e^{A(t-\tau)}\,B\,u(\tau)d\tau + D\,u(t)$$

Determining the free response

$$y(t) \;=\; y_{free}(t) + y_{force}(t)$$

The forced response is entirely determined from the input function, which is **known**.

$$y_{force}(t) \;=\; C\int_0^t e^{A(t-\tau)}\,B\,u(\tau)d\tau + D\,u(t)$$

Thus, the free response output

$$y_{free}(t) = y(t) - y_{force}(t)$$

can be assumed to be measurable

Definition of Observability (CT)

Thus, without loss of generality,

The system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \qquad x(0) = x_o$$

is observable **<u>iff</u>**,

the free response system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) \\ y(t) &= C\,x(t) \end{aligned} \qquad x(0) = x_o$$

is observable

Observability Theorem

The following 3 statements are equivalent:

(a) The LTI system of order $\boldsymbol{n}$

$$\begin{aligned} \dot{x}(t) &= A\,x(t) \\ y(t) &= C\,x(t) \end{aligned}$$

is observable.

Sometimes we simply state that the pair

$$\{A\,C\}$$

is observable.

Observability Theorem

The following 3 statements are equivalent:

(b) The observability grammian

$$W_o(t_1) \;=\; \int_0^{t_1} e^{A^T t} C^T C e^{At}\, dt$$

is positive definite, for all finite time $\boldsymbol{t_1 > 0}$

$$W_o(t_1) \succ 0 \qquad \forall t_1 > 0$$

Observability Theorem

(c) The observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is rank $\boldsymbol{n}$.

(I.e. there are $\boldsymbol{n}$ linearly independent rows)

Remarks on Observability Theorem

1. The observable canonical pair

$$A_c = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_o & 0 & 0 \end{bmatrix} \; C_c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

is always observable, since

$$Q_c = \begin{bmatrix} C_c \\ C_c A_c \\ C_c A_c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ (-a_1 + a_2^2) & -a_2 & 1 \end{bmatrix}$$

is always full rank.

This result generalizes to an arbitrary order n

Observability Grammian

Assume that the matrix $\boldsymbol{A}$ is Hurwitz.

Then, the asymptotic value of the observability grammian

$$W_o = \lim_{t_1 \to \infty} W_o(t_1) = \int_0^{\infty} e^{A^T t}\, C^T\, C\, e^{At}\, dt$$

exists (all elements of $\boldsymbol{W_c}$ are bounded).

Observability Grammian & Lyapunov Eq

Assume that the matrix $\boldsymbol{A}$ is Hurwitz.

$$W_o \;=\; \int_0^{\infty} e^{A^T t}\, C^T\, C\, e^{At}\, dt$$

can be calculated as the solution of the following Lyapunov equation:

$$A^T W_o + W_o A = -C^T C$$

Moreover, $W_o \succ 0$ **<u>iff</u>** $\{A\, C\}$ is an observable pair

Remarks on Observability Theorem

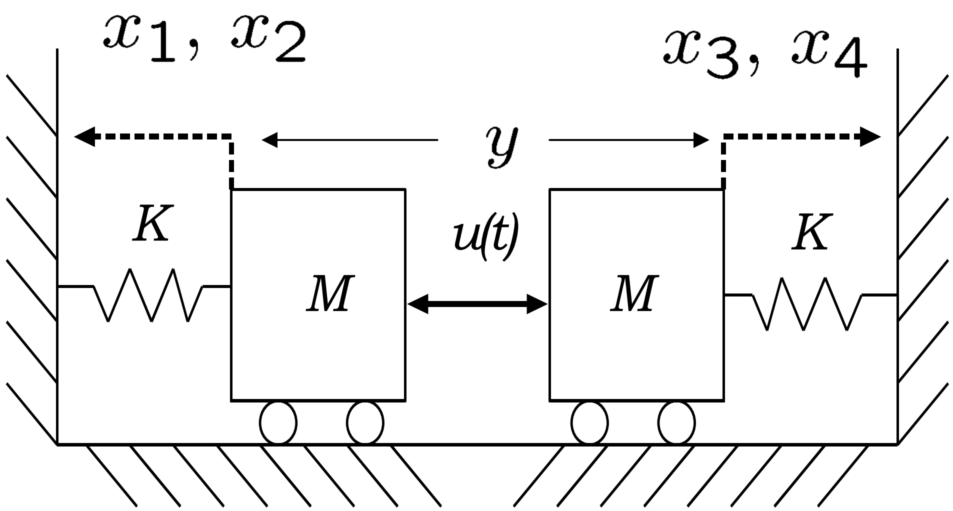
The observability results are dual of the controllability results in the following sense:

The pair $\{A\,, C\}$ is observable if and only if

the pair $\left\{A^T\,, C^T\right\}$ is controllable.

We will often use the duality between observability and controllability in deriving future results.

An unobservable system: example

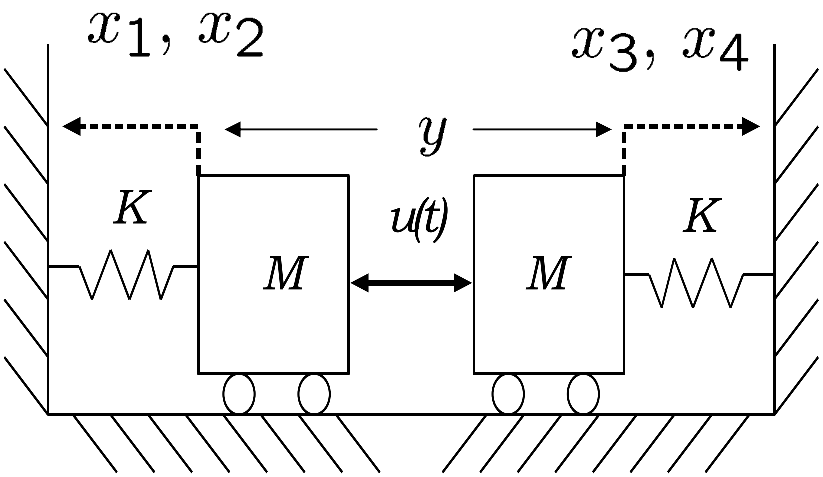


Output:
the distance between the two masses

$$y = x_1 + x_3 = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}}_{C} x$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ \frac{1}{m_2} \end{bmatrix}}_{B} u$$

An unobservable system: example



$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} C \\ C\,A \\ C\,A^2 \\ C\,A^3 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{k_1}{m_1} & 0 & -\frac{k_2}{m_2} & 0 \\ 0 & -\frac{k_1}{m_1} & 0 & -\frac{k_2}{m_2} \end{bmatrix}$$

$$\det\{Q\} = \left(\frac{k_2}{m_2} - \frac{k_1}{m_1}\right)^2 \qquad \det\{Q\} \neq 0 \Leftrightarrow \frac{k_2}{m_2} \neq \frac{k_1}{m_1}$$

Observability Theorem

The following 3 statements are equivalent:

(a) The LTI system of order $\boldsymbol{n}$

$$
\begin{aligned}
\dot{x}(t) &= A\,x(t) \\
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\end{aligned}
$$

is observable.

Sometimes we simply state that the pair

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is positive definite, for all finite time $\boldsymbol{t_1 > 0}$

$$W_o(t_1) \succ 0 \qquad\qquad \forall t_1 > 0$$

Observability Theorem

(c) The observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is rank $\boldsymbol{n}$.

(I.e. there are $\boldsymbol{n}$ linearly independent rows)

Proof of Observability Theorem (b⇔c)

(c) implies (b):

We will show that not (b) $\Rightarrow$ not (c)

Assume that the observability grammian

$$W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At}\, dt$$

is **not** positive definite for any $\boldsymbol{t_1 > 0}$

We will now show that the observability matrix is not rank $\boldsymbol{n}$

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Proof of Observability Theorem (b⇔c)

not (b) $\Rightarrow$ **not (c)**

Assume that the observability grammian

$$W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At} \, dt$$

is **not** positive definite for any $\boldsymbol{t_1 > 0}$

Then, there exists a vector $v \in \mathcal{R}^n$ such that

$$v^T W_o(t_1) \, v = 0$$

Proof of Observability Theorem (b⇔c)

not (b) $\Rightarrow$ **not (c)**

Define the function $g(t) = Ce^{At}v$ and notice that

$$
\begin{aligned}
v^T W_o(t_1)\, v &= v^T \int_0^{t_1} e^{A^T t} C^T C e^{At}\, dt\, v \\
&= \int_0^{t_1} \underbrace{v^T e^{A^T t} C^T}_{g^T(t)} \underbrace{C e^{At} v}_{g(t)}\, dt \\
&= \int_0^{t_1} \|g(t)\|_2^2 dt
\end{aligned}
$$

Proof of Observability Theorem (b⇔c)

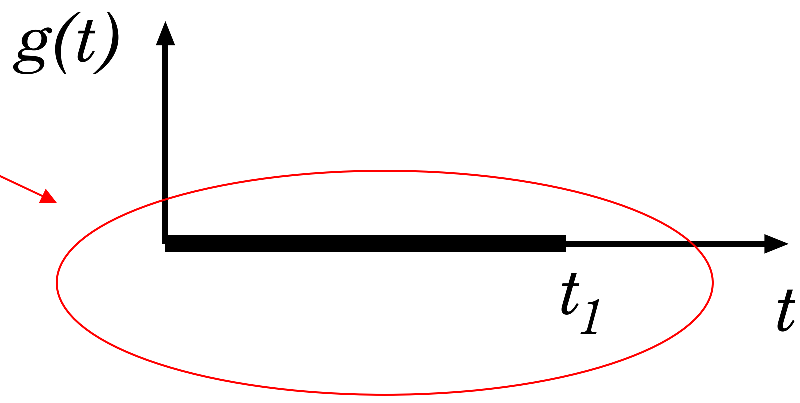
not (b) $\Rightarrow$ **not (c)**

Thus

$$v^T W_o(t_1)\, v \;=\; 0 \Leftrightarrow \; g(t) = 0 \;\; \forall t \in [0, t_1]$$

Since $g(t) = Ce^{At}\, v$ is an analytical function of time,

$$\begin{aligned} g(0) &= 0 \\ \dot{g}(0) &= 0 \\ \ddot{g}(0) &= 0 \\ &\vdots \\ \left.\frac{d^{n-1}}{dt^{n-1}} g(t)\right|_0 &= 0 \end{aligned}$$



Proof of Observability Theorem (b⇔c)

not (b) $\Rightarrow$ **not (c)**

$$g(t) = Ce^{At}\,v$$

$$\begin{array}{rclcl}
g(0) & = & 0 & \Rightarrow & C\,v = 0 \\
\dot{g}(0) & = & 0 & \Rightarrow & C\,A\,v = 0 \\
\ddot{g}(0) & = & 0 & \Rightarrow & C\,A^2\,v = 0 \\
 & \vdots & & & \\
\left.\dfrac{d^{n-1}}{dt^{n-1}} g(t)\right|_0 & = & 0 & \Rightarrow & C\,A^{n-1}\,v = 0
\end{array}$$

$$\boxed{\Rightarrow\; Q\,v = 0}$$

Thus, since $v \in \mathcal{R}^n$ and

$$Q\,v = 0 \qquad \Rightarrow \qquad \boxed{Q \text{ is not rank } n}$$

Proof of Observability Theorem (b⇔c)

(b) implies (c):

We will show that not (c) $\Rightarrow$ not (b)

Assume that the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is **not** rank $\boldsymbol{n}$.

We will show that the observability grammian

$$W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At} \, dt$$

is **not** positive definite for any $t_1 > 0$

Proof of Observability Theorem (b⇔c)

not (c) $\Rightarrow$ **not (b)**

Assume that the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is not rank $\boldsymbol{n}$.

Then, there exists a vector $v \in \mathcal{R}^n$ such that

$$Q\,v = 0$$

Proof of Observability Theorem (b⇔c)

not (c) $\Rightarrow$ **not (b)**

We will now show that, for any $t_1 > 0$,

$$v^T W_o(t_1)\, v = 0$$

where
$$W_o(t_1) \;=\; \int_0^{t_1} \underbrace{e^{A^T t} C^T}_{H^T(t)} \underbrace{C e^{At}}_{H(t)} \, dt$$

Consider a Taylor series expansion of the function

$$H(t) = Ce^{At}$$

Proof of Observability Theorem (b⇔c)

not (c) $\Rightarrow$ **not (b)**

$$
\begin{aligned}
H(t) &= Ce^{At} \\
&= \underbrace{C + CAt + \frac{1}{2}CA^2t^2 + \cdots + \frac{1}{(n-1)!}CA^{n-1}t^{n-1}}_{H_1(t)} \\
&\quad + \underbrace{\frac{1}{n!}\,CA^nt^n + \cdots}_{H_2(t)} .
\end{aligned}
$$

Notice that $H_1(t)$ can be expressed as

$$
\begin{aligned}
H_1(t) &= \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(n-1)!}t^{(n-1)} \end{bmatrix} Q \\
&= f_1^T(t)Q
\end{aligned}
$$

Proof of Observability Theorem (b$\Leftrightarrow$c)

not (c) $\Rightarrow$ **not (b)**

By the Cayley-Hamilton Theorem, the term $\boldsymbol{H_2(t)}$

$$H_2(t) \;=\; \frac{1}{n!}\,CA^n t^n + \frac{1}{(n+1)!}\,CA^{(n+1)}\,t^n + \cdots$$

which contains terms of the form $CA^k\,,\;\; k \geq n$

must also be expressed as

$$H_2(t) \;=\; f_2^T(t)Q$$

Proof of Observability Theorem (b⇔c)

not (c) $\Rightarrow$ **not (b)**

Therefore,

$$\begin{aligned} H(t) &= Ce^{At} = \left\{ f_1^T(t) + f_2^T(t) \right\} Q \\ &= f^T(t)\, Q \end{aligned}$$

and

$$\begin{aligned} W_o(t_1) &= \int_0^{t_1} H^T(t)\, H(t)\, dt \\ &= Q^T \left\{ \int_0^{t_1} f(t) f^T(t) dt \right\} Q \end{aligned}$$

Thus,

$$\boxed{Q\, v = 0 \quad \Longrightarrow \quad v^T\, W_o(t_1)\, v = 0}$$

Proof of Observability Theorem (b⇔a)

(b) implies (a):

Assume that the observability grammian is positive definite for any $t_1 > 0$

$$W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At}\, dt > 0$$

We will show that the system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) \qquad\qquad x(0) = x_0 \\ y(t) &= C\,x(t) \end{aligned}$$

Is observable.

Proof of Observability Theorem (b⇔a)

(b) implies (a):

Let

$$\begin{aligned} \dot{x}(t) &= A\,x(t) \qquad\qquad x(0) = x_0 \\ y(t) &= C\,x(t) \end{aligned}$$

Therefore,

$$y(t) = C\,e^{At}\,x(0)$$

And assume that the observability grammian is positive definite for any $t_1 > 0$

$$W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At}\,dt > 0$$

Proof of Observability Theorem (b⇔a)

(b) implies (a): Computing,

$$\begin{aligned}
\int_0^{t_1} e^{A^T t}\, C^T\, y(t) dt &= \int_0^{t_1} e^{A^T t}\, C^T\, C\, e^{At}\, x(0)\, dt \\
&= \left\{ \int_0^{t_1} e^{A^T t}\, C^T\, C\, e^{At} dt \right\} x(0) \\
&= W_o(t_1)\, x(0)
\end{aligned}$$

Thus,

$$\boxed{x(0) = W_o^{-1}(t_1) \int_0^{t_1} e^{A^T t}\, C^T\, y(t) dt}$$

Proof of Observability Theorem (b$\Leftrightarrow$a)

(a) implies (b):

we will show that not (b) $\Rightarrow$ not (a)

Assume that the observability grammian is not positive define for any $t_1 > 0$

$$W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At}\, dt \geq 0$$

We will show that the pair $[A,C]$ is not observable, I.e.

$$\begin{aligned} \dot{x}(t) &= A\,x(t) \qquad\qquad x(0) = x_0 \\ y(t) &= C\,x(t) \end{aligned}$$

is unobservable.

Proof of Observability Theorem (b⇔a)

not (b) $\Rightarrow$ **not (a)**

Assume that the observability grammian is not positive define for any $\boldsymbol{t_1 > 0}$

Then, there exists a vector $v \in \mathcal{R}^n$ such that

$$v^T W_o(t)\, v \;=\; 0 \qquad \forall t \geq 0$$

As we have already shown,

$$v^T W_o(t)\, v \;=\; 0 \Leftrightarrow \boxed{C\, e^{At}\, v = 0 \;\; \forall\, t \geq 0}$$

Proof of Observability Theorem (b⇔a)

not (b) $\Rightarrow$ **not (a)**

Therefore, if

$$y(t) = C\,e^{At}\,x_1(0)$$

Then,

$$y(t) = C\,e^{At}\,\underbrace{\{x_1(0) + v\}}_{x_2(0)}$$

Thus, two different initial conditions produce the same output and cannot cannot be distinguished.