HW2: Linear System Theory (ECE532)

Instructor: Jun Moon

Due Date: March 19, Monday

Reading Assignment: Read Chapters 1 and 2 of the textbook.

Note: You must use LaTeXto write your homework.

Problem 1

Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on $W \subset \mathbb{R}^n$, then $f(x)$ is *uniformly* continuous on W.

- A function $f : W \to \mathbb{R}^n$ is uniformly continuous on W if

$$\begin{aligned}&\text{for each } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that}\\ &x, y \in W \ \& \ |x - y| < \delta \ \rightarrow \ |f(x) - f(y)| < \epsilon\end{aligned}$$

Problem 2

Show that

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{2x_2}{1 + x_2^2}, \ x_1(0) = a \in \mathbb{R}\\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1 + x_1^2}, \ x_2(0) = b \in \mathbb{R}\end{aligned}$$

has a unique solution defined for all $t \geq 0$. Obtain the solution via the successive approximation (Picard iteration) method using MATLAB.

Problem 3

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable for all $x \in \mathbb{R}^n$, and define $f(x)$ by

$$f(x) = \frac{1}{1 + g^T(x)g(x)} g(x)$$

Show that $\dot{x} = f(x)$ with $x(0) = x_0$ has a unique solution for all $t \geq 0$. Obtain the solution via the successive approximation (Picard iteration) method using MATLAB.

Problem 4

Consider

$$\dot{x} = f(t, x), \ x(t_0) = x_0$$

Suppose that $f(t, x)$ is continuous in t, Lipschitz in x, and satisfies a linear growth:

$$|f(t, x)| \leq k_1 + k_2|x|, \ k_i \geq 0, \ i = 1, 2, \ \forall t \geq t_0, \ x \in \mathbb{R}^n$$

- Show that the solution satisfies

$$|x(t)| \leq |x_0| \exp[k_2(t - t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t - t_0)] - 1\}$$

  for all $t \geq t_0$ for which the solution exists

- Can the solution have a finite escape time?

Problem 5

Prove the Gronwall-Bellman Inequality discussed in class for the following two special cases:

- If $\lambda(t) = \lambda$ is a constant, then

$$y(t) \leq \lambda \exp\Big[\int_a^t \mu(\tau) d\tau\Big]$$

- If $\lambda(t) = \lambda$ and $\mu(t) = \mu \geq 0$ are constants, then

$$y(t) \leq \lambda \exp[\mu(t - a)]$$

Problem 6

Consider the fixed point equation

$$x(t) = \frac{1}{2}t^3 + \alpha \sin \pi x(t)$$

defined over the interval $t \in [-2, 2]$, with $x \in C[-2, 2]$, where C is the space of $\mathbb{R}$-valued continuous functions on $[-2, 2]$, and α a positive constant (a parameter). For what values of α does there exist a unique continuous function $x(\cdot)$ on $[-2, 2]$ which solves the fixed point equation. Show (prove) that for these values of α a solution indeed exists and is unique.

Hint: Use a contraction mapping type argument, applied to a subset of $C[-2, 2]$, which comprises all uniformly bounded functions, such as functions satisfying the bound $|x(t)| \leq \beta$, for some β.