Linear System Theory

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Lecture 5

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Solutions of State Equations

LTI system

$$\dot{x}(t) = ax(t),\ \dot{x}(t) = Ax(t)$$

For $a \in \mathbb{R}$, $x(t) = e^{at}x(0)$. Also,

$$e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots$$

Solutions of State Equations

Hence, for $n \times n$ matrix A, $x(t) = e^{At}x(0)$, where

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Note that A and e^{At} commute ($Ae^{At} = e^{At}A$), since

$$\frac{de^{At}}{dt} = A + AAt + A\frac{At}{2!} + \cdots = A(I + At + \frac{A^2t^2}{2!}\cdots) = Ae^{At} = e^{At}A$$

Properties of e^{At}

- $e^{A0} = I$
- $(e^{At})^{-1} = e^{-At}$
- $e^{At}e^{As} = e^{A(t+s)}$ semi-group property

Solutions of State Equations

Note that e^{At} is an infinite series!!!!

By using the C-H theorem, there exists coefficients $\{\beta_0(t), ..., \beta_{n-1}(t)\}$ such that

$$\begin{aligned} e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\ &= \beta_0(t)I + \beta_1(t)A + \cdots + \beta_{n-1}(t)A^{n-1} \end{aligned}$$

It is a linear combination of $\{I, A, ..., A^{n-1}\}$

Solutions of State Equations

For $a, b \in \mathbb{R}$, $e^{at}e^{at} = e^{(a+b)t}$. How about $e^{At}e^{Bt} \stackrel{?}{=} e^{(A+B)t}$????
$\Rightarrow e^{At}e^{Bt} = e^{(A+B)t}$ if $AB = BA$ (A and B commute) $\Rightarrow$ HW

$$
\begin{aligned}
e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} \\
e^{Bt} &= I + Bt + \frac{B^2t^2}{2!} + \frac{B^3t^3}{3!} \\
e^{(A+B)t} &= I + (A+B)t + \frac{(A+B)^2t^2}{2!} + \frac{(A+B)^3t^3}{3!}
\end{aligned}
$$

General Solution to LTI Systems

LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Note that $e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$; hence,

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu(t)$$

$$\Rightarrow e^{-A\tau}x(t)\Big|_{\tau=0}^{t} = \int_0^t e^{-At}Bu(\tau)d\tau$$

$$\Rightarrow e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$\Rightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \text{ Solution of the LTI system}$$

Then

$$y(t) = Cx(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)$$

General Solution to LTI Systems

Note that with $X(s) = \int_0^\infty x(t)e^{-st}dt$, where $s = \sigma + j\omega$

$$X(s) = (sI - A)^{-1}BU(s), \;\; Y(s) = C(sI - A)^{-1}BU(s)$$
$$\Leftrightarrow G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

General Solution to LTI Systems

We need to compute e^{At}. There are five ways (the four ways are discussed in the textbook, page 106)

- Use Caley-Hamilton Theorem. That is, first compute eigenvalues of A. Next, find a polynomial $h(\lambda)$ of degree $n-1$ that equals $e^{\lambda t}$. Then $e^{At} = h(A)$.
- Use the Jordan form of A and Caley-Hamilton Theorem. Let $A = P\hat{A}P^{-1}$; then $e^{At} = Pe^{\hat{A}t}P^{-1}$, where $\hat{A}$ is the Jordan matrix of A.
- Use the power infinite-series.
- Use the inverse Laplace transformation. That is $\mathcal{L}(e^{At}) = (sI - A)^{-1}$, which implies $e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$.
- Use "expm" in MATLAB

Matrix Exponential

Example I: We want to compute $f(J) = e^{Jt}$ via the Caley-Hamilton Theorem

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

$$h(\lambda) = \beta_0 + \beta_1(\lambda - \lambda_1) + \beta_2(\lambda - \lambda_2)^2 + \beta_3(\lambda - \lambda_3)^3$$

$$\beta_0 = f(\lambda_1),\ \beta_1 = f'(\lambda_1),\ \beta_2 = \frac{f''(\lambda_1)}{2!},\ \beta_3 = \frac{f'''(\lambda_1)}{3!}.$$

Matrix Exponential

Hence,

$$f(J) = f(\lambda_1)I + \frac{f'(\lambda_1)}{1!}(J-\lambda_1 I) + \frac{f''(\lambda_1)}{2!}(J-\lambda_1 I)^2 + \frac{f'''(\lambda_1)}{3!}(J-\lambda_1 I)^3$$

$$e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & t^3 e^{\lambda_1 t}/3! \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! \\ 0 & 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & e^{\lambda_1 t} \end{pmatrix}$$

Matrix Exponential

Example II: We want to compute $f(J) = e^{Jt}$ via the Caley-Hamilton Theorem

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

Matrix Exponential

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t}/2! & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_2 t} & te^{\lambda_2 t} \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{pmatrix}$$

Discrete-Time LTI System

How to discretize? $\Rightarrow$ Sampling $t = kT$, where $k = 0, 1, \ldots$ and T is sampling rate

Detailed derivation: Textbook

Discrete-time LTI system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ x(1) &= Ax(0) + Bu(0) \\ x(2) &= Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1) \end{aligned}$$

Hence,

$$x(k) = A^k x(0) + \sum_{m=0}^{n-1} A^{k-1-m} Bu(m)$$

Equivalence System: Similarity Transformation

Let P be an $n \times n$ real nonsingular matrix and let $z = Px$. Then the state-space equation

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t), \;\; y(t) = \bar{C}z(t)$$

where

$$\bar{A} = PAP^{-1}, \;\; \bar{B} = PB, \;\; \bar{C} = CP^{-1}$$

is said to be equivalent to the system

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cz(t)$$

and $z = Px$ is called an equivalence transformation.

We have

$$\begin{aligned}\det(\lambda I - A) &= \det(\lambda I - P\bar{A}P^{-1}) \\ &= \det(P)\det(\lambda I - \bar{A})\det(P)^{-1} = \det(\lambda I - \bar{A})\end{aligned}$$

Realizations

Definition

A rational (transfer) function $\hat{g}(s)$ is said to be proper if $\hat{g}(\infty) < \infty$. If $\hat{g}(\infty) = 0$, then $\hat{g}(s)$ is strictly proper. A rational matrix $\hat{G}(s)$ is said to be proper if $\hat{G}(\infty) < \infty$. If $\hat{G}(\infty) = 0$, then $\hat{G}(s)$ is strictly proper.

- We consider the LTI system
- $G(t)$ is an impulse response of $\hat{G}(s)$

Realizations

Definition:
A matrix transfer function $\hat{G}(s)$ is realizable if there exists a finite-dimensional state-space equation, i.e., $\{A, B, C, D\}$ such that

$$\hat{G}(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

- $\{A, B, C, D\}$ is a realization of $G(s)$
- Realization is not unique, since for the SISO case, $\hat{g}(s) = \hat{g}^T(s)$. For the general matrix case, the realization is not also unique. See page 130 and Problem 4.13 of the textbook.

Realizations

Theorem (Generalized version of Theorem 4.2 of the textbook)

- A matrix transfer function $\hat{G}(s)$ is realizable by a finite-dimensional LTI system if and only if $\hat{G}(s)$ is a proper rational matrix function
- (same as Theorem 4.2 of the textbook) For a SISO system, $\hat{g}(s)$ is realizable by a finite-dimensional SISO LTI system if and only if $\hat{g}(s)$ is a proper rational transfer function

Note that

- If $\hat{G}(s)$ is strictly proper, then $D = 0$
- If $\hat{g}(s)$ is strictly proper, then $D = 0$

Realizations

Example (Example 4.5.2): SISO LTI system

$$\hat{g}(s) = \frac{4s + 3}{40s^3 + 30s^2 + 9s + 3}$$

Example (Example 4.5.1): SISO LTI system

$$\hat{g}(s) = \frac{3s^4 + 5s^3 + 24s^2 + 23s - 5}{2s^4 + 6s^3 + 15s^2 + 12s + 5}$$

Realizations

Example (Example 4.5.3): MIMO LTI system

$$\hat{G}(s) = \begin{pmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{-12}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{pmatrix}$$

$$= \frac{1}{s^3 + 4.5s^2 + 6s + 2} \begin{pmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Realizations

Example (Example 4.5.3): Two-input Two-Output LTI system

$$\hat{G}(s) = \begin{pmatrix} \frac{\frac{4s-10}{2s+1}}{\frac{1}{(2s+1)(s+2)}} & \frac{\frac{3}{s+2}}{\frac{s+1}{(s+2)^2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\frac{-12}{2s+1}}{\frac{1}{(2s+1)(s+2)}} & \frac{\frac{3}{s+2}}{\frac{s+1}{(s+2)^2}} \end{pmatrix}$$

$$= \frac{1}{s^3 + 4.5s^2 + 6s + 2} \begin{pmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{s^3 + 4.5s^2 + 6s + 2} \left(\begin{pmatrix} -6 & 3 \\ 0 & 1 \end{pmatrix} s^2 + \begin{pmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{pmatrix} s + \begin{pmatrix} -24 & 3 \\ 1 & 0.5 \end{pmatrix} \right)$$

$$+ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Realizations

Strictly proper transfer function

$$\hat{g}(s) = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5}$$

- Controllable canonical form

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_5 & -a_4 & -a_3 & -a_2 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} b_4 & b_3 & b_2 & b_1 \end{pmatrix} x(t)$$

- Observable canonical form

$$\dot{x}(t) = A^T x(t) + C^T u(t), \; y(t) = B^T x(t)$$

- Modal form: Similarity transfomration

General Solution to LTV Systems

One-dimensional LTV system

$$
\begin{aligned}
&x(t) = a(t)x(t),\ a(t)\text{: continuous}\\
&x(t) = e^{\int_0^t a(\tau)d\tau} x(0)\\
&\text{check: } \frac{d}{dt} e^{\int_0^t a(\tau)d\tau} = a(t)e^{\int_0^t a(\tau)d\tau} = e^{\int_0^t a(\tau)d\tau} a(t)
\end{aligned}
$$

LTI approach holds!!!

General Solution to LTV Systems

Let's see general LTV systems

$$\dot{x}(t) = A(t)x(t)$$

$A(t)$ needs to be a continuous function (why????)

Can we do?

$$x(t) = e^{\int_0^t A(\tau)d\tau} x(0)$$

General Solution to LTV Systems

No!! Consider

$$
\begin{aligned}
e^{\int_0^t A(\tau)d\tau} &= I + \int_0^t A(\tau)d\tau + \frac{1}{2}\int_0^t A(\tau)d\tau \int_0^t A(\tau)d\tau + \cdots \\
\frac{d}{dt}e^{\int_0^t A(\tau)d\tau} &= A(t) + \frac{1}{2}A(t)\int_0^t A(s)ds + \frac{1}{2}\int_0^t A(s)dsA(t) + \cdots \\
&\neq A(t)e^{\int_0^t A(s)ds}
\end{aligned}
$$

The LTI approach does not hold for LTV systems. Namely, we cannot extend the solution of the scalar time-varying equations to the matrix case, and must use a different approach.

General Solution to LTV Systems

Note that, since $A(t)$ is continuous, we must have a unique solution of $x(t)$ with a given initial condition (why??? think about the Lipschitz condition discussed in class)

We can have n linearly independent initial conditions, $x_i(0)$, $i = 1, 2, ..., n$, and for each $x_i(0)$, there exists a unique solution, denoted by $x_i(t)$, $i = 1, 2, ..., n$. Then due to the linearity, $x_i(t)$, $i = 1, 2, .., n$ form an n-dimensional vector space over $\mathbb{R}$. That is, $x_i(t)$, $i = 1, 2, .., n$, are linearly independent. Let

$$X(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \; X(0) = (x_1(0), \ldots, x_n(0))$$

General Solution to LTV Systems

Then we can show that

$$\dot{X}(t) = A(t)X(t)$$

$X(t)$ is called a fundamental matrix. Note that $X(t)$ is not unique, since we can choose n linearly independent initial conditions arbitrarily. We also note that $X(t)$ is always invertible for all t, since $x_i(t)$, $i = 1, 2, ..., n$ are linearly independent.

General Solution to LTV Systems

Definition (Definition 4.2 of the textbook):
Let $X(t)$ be a fundamental matrix of $\dot{x} = A(t)x$. Then

$$\Phi(t, 0) := X(t)X^{-1}(0)$$

is called the state transition matrix of $\dot{x} = A(t)x$. The state transition matrix is also the unique solution of

$$\frac{d}{dt}\Phi(t, 0) = A(t)\Phi(t, 0),$$

where $\Phi(0, 0) = I$.

General Solution to LTV Systems

Properties of the state transition matrix

- $\Phi(t,t) = I$
- $\Phi^{-1}(t,0) = (X(t)X^{-1}(0))^{-1} = X(0)X^{-1}(t) = \Phi(0,t)$
- $\Phi(t,0) = \Phi(t,s)\Phi(s,0)$ (semi-group property)

General Solution to LTV Systems

The properties of the state transition matrix can be extended to when the initial time is t_0

- $\Phi(t,t) = I$
- $\Phi^{-1}(t,t_0) = (X(t)X^{-1}(t_0))^{-1} = X(t_0)X^{-1}(t) = \Phi(t_0,t)$
- $\Phi(t,t_0) = \Phi(t,s)\Phi(s,t_0)$ (semi-group property)

General Solution to LTV Systems

For LTI system, the state transition matrix $\Phi(t, t_0)$ can be viewed as $\Phi(t, t_0) = e^{A(t-t_0)} = e^{At}e^{-At_0} = X(t)X^{-1}(t_0)$

$$\frac{d}{dt}e^{A(t-t_0)} = Ae^{A(t-t_0)}, \;\; e^{A(t_0-t_0)} = I$$

- $\Phi(t, t) = e^{A(t-t)} = I$
- $\Phi^{-1}(t, t_0) = (e^{A(t-t_0)})^{-1} = e^{-A(t-t_0)} = e^{A(t_0-t)} = \Phi(t_0, t)$
- $\Phi(t, t_0) = \Phi(t, s)\Phi(s, t_0) = e^{A(t-t_0)} = e^{A(t-s)}e^{A(s-t_0)}$

General Solution to LTV Systems

Example (Examples 4.6.1 and 4.6.2 of the textbook)

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} x(t), \; \dot{x}_1(t) = 0, \; \dot{x}_2(t) = tx_1(t)$$

$$x_1(t) = x_1(0), \; \dot{x}_2(t) = tx_1(0),$$

$$x_2(t) = \int_0^t \tau x_1(0) d\tau + x_2(0) = 0.5t^2 x_1(0) + x_2(0)$$

Hence, we have

$$x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x(t) = \begin{pmatrix} 1 \\ 0.5t^2 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow x(t) = \begin{pmatrix} 1 \\ 0.5t^2 + 2 \end{pmatrix}$$

$$X(t) = \begin{pmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{pmatrix}, \; X(t) \text{ is invertible for all } t!!!!$$

General Solution to LTV Systems

Example (Examples 4.6.1 and 4.6.2 of the textbook)

$$X(t) = \begin{pmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{pmatrix}, \; X(t) \text{ is invertible!!!!}$$

$$X(0) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \; X^{-1}(0) = \begin{pmatrix} 1 & -0.5 \\ 0 & 0.5 \end{pmatrix}$$

$$\Phi(t, 0) = X(t)X^{-1}(0)$$

General Solution to LTV Systems

The solution to the LTI system

$$\dot{x}(t) = A(t)x(t), \;\; x(t) = \Phi(t,0)x(0)$$

This can be checked easily via the definition of the state transition matrix
Also,

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
x(t) &= \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau \\
&= \Phi(t,0)\Big(x_0 + \int_0^t \Phi(0,\tau)B(\tau)u(\tau)d\tau\Big)
\end{aligned}$$

Note that $\Phi(t,0)$ is the state transition matrix of $\dot{x}(t) = A(t)x(t)$. We now check the above solution

General Solution to LTV Systems

Note that

$$
\begin{aligned}
x(0) &= \Phi(0,0)x_0 + \int_0^0 \Phi(0,\tau)B(\tau)u(\tau)d\tau = x_0 \\
\frac{d}{dt}x(t) &= \frac{d}{dt}\left(\Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau\right) \\
&= A(t)\Phi(t,0)x_0 + \int_0^t A(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + B(t)u(t) \\
&= A(t)\left(\Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau\right) + B(t)u(t) \\
&= A(t)x(t) + B(t)u(t)
\end{aligned}
$$

Hence, we have the solution.

General Solution to LTV Systems

The LTV system for the discrete-time case is fairly easy, since we can just enumerate the state transition matrix $\Phi(k, 0) = A(k)A(k-1)\cdots A(0)$. The detailed discussion is provided in the textbook (page 140).

Conclusions

- Solutions of LTI, LTV systems
- State transition matrix and matrix exponential
- Caley-Hamilton theorem
- Next class: Lyapunov stability of nonlinear systems