Linear System Theory

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State space model

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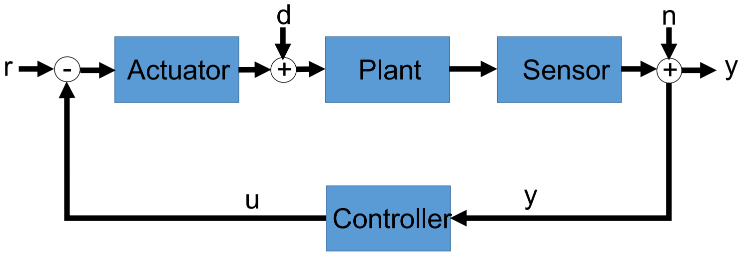
Outline

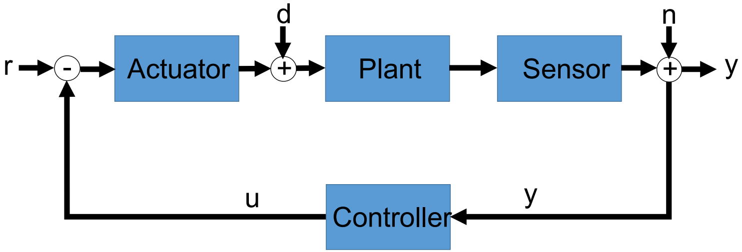
- Frequency domain to state space

Structure of Control Systems



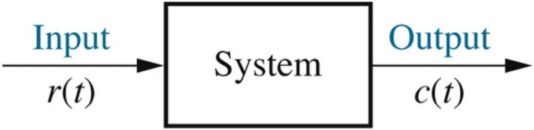
- Block
  - Plant: system that needs to be controlled (ODE or difference equations) $\Rightarrow$ motor, aircraft, pendulum, etc
  - Actuator/sensor
  - Controller: controller that controls the plant

Structure of Control Systems



- Signal
  - Input (r): reference signal
  - Output (y): sensor signal
  - Noise (n) / Disturbance (d): unwanted signal (need to reduce their effect)
  - Control (u): control signal (we need to design)
- Structure: Open loop / Feedback

State space system



Suppose that the input $r(t)$ and output $c(t)$ behavior of the system is described by the following ordinary differential equation with the zero initial condition:

$$a_2 \frac{dc^2(t)}{dt^2} + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_0 r(t)$$

Let

$$c(t) = x_1(t)$$
$$\frac{dc(t)}{dt} = \dot{c}(t) = x_2(t) = \dot{x}_1(t)$$

State space system

Let

$$c(t) = x_1(t) \quad \dot{c}(t) = x_2(t) = \dot{x}_1(t)$$

Then

$$a_2 \frac{dc^2(t)}{dt^2} + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_0 r(t)$$

$$a_2 \dot{x}_2(t) + a_1 x_2(t) + a_0 x_1(t) = b_0 r(t)$$

$$\Leftrightarrow \quad \dot{x}_2(t) = -\frac{a_0}{a_2} x_1(t) - \frac{a_1}{a_2} x_2(t) + \frac{b_0}{a_2} r(t)$$

State space system

$$c(t) = x_1(t) \qquad \dot{c}(t) = x_2(t) = \dot{x}_1(t)$$

$$\dot{x}_2(t) = -\frac{a_0}{a_2}x_1(t) - \frac{a_1}{a_2}x_2(t) + \frac{b_0}{a_2}r(t)$$

Let

$$x(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$$

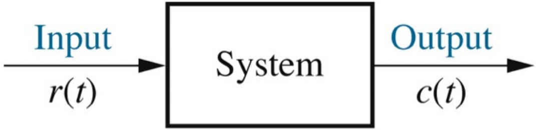
Then

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{a_0}{a_2} & -\frac{a_1}{a_2} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{b_0}{a_2} \end{pmatrix} r(t), \quad x(0)$$

$$c(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

- This is the state space representation of the ODE

State space system



Let's do one more example:

$$a_2 \frac{dc^2(t)}{dt^2} + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_0 r(t) + b_1 \frac{dr(t)}{dt}$$

The transfer function of the above system

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

State space system

The transfer function can be decomposed into

$$G(s) = \frac{C(s)}{R(s)} = \frac{C(s)}{X(s)}\frac{X(s)}{R(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0},$$

where

$$\frac{X(s)}{R(s)} = \frac{1}{a_2 s^2 + a_1 s + a_0} \qquad \frac{C(s)}{X(s)} = b_1 s + b_0$$

The state space system is

$$\begin{aligned}
\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ -\frac{a_1}{a_2} & -\frac{a_0}{a_2} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{a_2} \end{pmatrix} r(t), \quad x(0) \\
c(t) &= \begin{pmatrix} b_1 & b_1 \end{pmatrix} x(t)
\end{aligned}$$

State space system

Every LTI system can be described by the state space system with the following general form:

$$\dot{x}(t) = Ax(t) + Bu(t) + Dd(t), \; x(0)$$
$$y(t) = Cx(t) + Eu(t) + Fn(t)$$

- $t \geq 0$: time
- $x \in \mathbb{R}^n$: state
- $u \in \mathbb{R}^m$: control
- $d \in \mathbb{R}^l$: disturbance
- $y \in \mathbb{R}^p$: (sensor) output
- $n \in \mathbb{R}^q$: noise
- A: $n \times n$ system matrix
- B: $n \times m$ input matrix
- D: $n \times l$ disturbance matrix
- C: $p \times n$ output matrix
- E: $p \times m$ feedthrough matrix
- F: $p \times q$ noise matrix

State space system

The simplified state space equation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad y(t) = Cx(t) + Eu(t)$$

- $x \in \mathbb{R}^n$: state
- $u \in \mathbb{R}^m$: control
- $y \in \mathbb{R}^p$: (sensor) output

What is "state"?

State space system

Definition of the state

⇒ The state is a vector that consists of state variables.

Definition of the state variable

⇒ The smallest set of linearly independent system variables such that the values of the members of the set at time 0 along with known input completely determine the value of all system variables for all $t \geq 0$

We use "state" and "state variables" interchangeability in control theory

State space system

Better definition of the state

$\Rightarrow$ The state of a dynamical system (plant or model) that completely characterizes the motion of a system for the purpose of predicting future motion

Easy definition: The state describes the behavior of the dynamical system (plant)

Examples of the state

- Electrical systems: current, voltage, charge
- Mechanical systems: position, velocity, acceleration
- Economics: output of the employer's effort, customer's perference

State space system

Selection of the state

- Each state variable must be independent: each state variable cannot be represented by a linear combination of other state variables
- Minimum dimension of the state vector $\Leftrightarrow$ The minimum number of state variables

From Time Domain to Frequency Domain

Laplace transformation of the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; x(0) = 0$$
$$y(t) = Cx(t)$$

Let $X(s) = \int_0^\infty x(t)e^{-st}dt$, $s = \sigma + j\omega$

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1}BU(s)$$

Then, since $Y(s) = CX(s)$

$$Y(s) = C(sI - A)^{-1}BU(s) \Rightarrow G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

$G(s)$: transfer function from input to output
$g(t)$: inverse Laplace transformation of $G(s) \Rightarrow$ impulse response

Discrete-Time LTI Systems

- discrete-Time LTI System

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) + Dd(t) \\
y(t) &= Cx(t) + Eu(t) + Fn(t)
\end{aligned}
$$

- $t \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$
- difference equation (first-order recursive equation)
- x, y, u are sequences
- sampled system: $x(t) := x(tT)$ (T: sampling period)

From Time Domain to Frequency Domain

z-transformation of the LTI system

$$x(t+1) = Ax(t) + Bu(t), \; x(0) = 0$$
$$y(t) = Cx(t)$$

Let $X(z) = \sum_{t=0}^{\infty} x(t) z^{-t}$, $z = e^{sT}$, $s = \sigma + j\omega$, $T > 0$ (sampling)

$$zX(s) - zx(0) = AX(z) + BU(z)$$
$$X(z) = (zI - A)^{-1} BU(z)$$

Then, since $Y(z) = CX(z)$

$$Y(z) = C(zI - A)^{-1} BU(z) \Rightarrow G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1} B$$

$G(z)$: transfer function from input to output
$g(t)$: inverse z-transformation of $G(z)$ $\Rightarrow$ impulse response

Continuous-Time Linear Time-Varying (LTV) System

$$\dot{x}(t) = \frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + D(t)d(t)$$
$$y(t) = C(t)x(t) + E(t)u(t) + F(t)n(t)$$

- $t \geq 0$: time
- $x \in \mathbb{R}^n$: state
- $u \in \mathbb{R}^m$: control
- $d \in \mathbb{R}^l$: disturbance
- $y \in \mathbb{R}^p$: (sensor) output
- $n \in \mathbb{R}^q$: noise
- $A(t)$: $n \times n$ system matrix
- $B(t)$: $n \times m$ input matrix
- $D(t)$: $n \times l$ disturbance matrix
- $C(t)$: $p \times n$ output matrix
- $E(t)$: $p \times m$ feedthrough matrix
- $F(t)$: $p \times q$ noise matrix

LTV and LTI Systems

- LTV system: constants are *time-varying*
- LTI system: constants are *time-invariant*
- LTI system can be converted into the transfer function via the Laplace (or z) transformation
- LTV and LTI systems: first-order ODE (first-order recursive equation)

LTV and LTI Systems

- state x: position, velocity, acceleration, etc, which capture the behavior of the system
- scalar (one-dimensional) u and y: single-input-single-output (SISO) system
- In this course, we consider continuous-time LTV and LTI systems when $D = E = F = 0$ (system without disturbance, noise and feedthrough terms)

Nonlinear Systems

- continuous-time nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t), d(t)), \;\; y(t) = g(t, x(t), u(t), n(t))$$

- discrete-time nonlinear system

$$x(t+1) = f(t, x(t), u(t), d(t)), \;\; y(t) = g(t, x(t), u(t), n(t))$$

- Example: $\dot{x}(t) = x^2(t)$, $\dot{x}(t) = \cos(t)$

- Linear system can be obtained by *linearization* of a nonlinear system
  $\Rightarrow$ next class

Nonlinear Systems

- continuous-time nonlinear system

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- Linear system can be obtained by *linearization* of a nonlinear system
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Why Study Linear Systems?

- Linear system is a special case of nonlinear systems
- Why do we study linear systems?
- If you do not understand linear systems, you cannot understand nonlinear systems
- Nonlinear system
  - Existence of solution?
  - Hard to analyze its dynamic behavior
  - Hard to see its input/output characteristics

Why Study Linear Systems?

- Linear system
  - Solution always exists
  - System characteristics depend on coefficients of the system
  - Computationally inexpensive
  - Easy to implement (real-time systems)
  - Linear algebra is the most effective tool
  - Many applications can be represented by linear systems (circuits, aircraft, missile, communication, traffic, guidance, economics)

Continuous-Time LTI Systems

Linear system in this course

- Lumped system $\Leftrightarrow$ finite-dimensional system where the state space is finite dimensional

- Distributed system $\Leftrightarrow$ infinite-dimensional system (on some function spaces)
  - Delay system
  - PDE
  - Will NOT be covered in this lecture

Next Class

- Existence and uniqueness of the solution of ODEs
- Linearization of nonlinear systems