Learning Networks

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April 2025

Abstract

This paper explores decision-making and learning dynamics in networks with incomplete information about their structure. Using an infinitely repeated local interaction game, we demonstrate that, at some point, the Bayesian Nash Equilibrium (BNE) under incomplete information is coincide to the Nash Equilibrium (NE) of the corresponding complete information game. This result establishes a theoretical bridge between incomplete and complete information network games, showing that, despite initial uncertainty, agents can act as if they know the true network structure through rational updating of beliefs. Furthermore, we identify conditions under which perfect learning occurs, provide bounds on learning times, and analyze how network topology and homophily influence the speed and quality of learning. Our findings contribute to the understanding of strategic behavior in networked systems and provide actionable insights for policymakers and researchers studying information diffusion and decision-making in social and economic networks.

JEL Classifications: C72, D81, D85

Keywords: Incomplete Information, Network Games, Network Uncertainty, Bayesian Learning

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1 Introduction

1.1 Overview

Networks play a crucial role as sources of information. Individuals communicate with those they are directly connected to, using this information to form opinions and make decisions. For instance, a firm's decision to adopt a new technology is influenced by its competitors' or collaborators' choices, just as adolescents' consumption of tobacco or alcohol is shaped by the consumption behaviors of their peers. While information and influence flow primarily through direct connections, indirect connections also significantly impact behavior. The agents who directly influence an individual are themselves influenced by others, meaning that individual behavior within a network inevitably depends on the entire architecture of that network.

A rational agent embedded in a network can recognize this. Faced with a decision, the agent may understand that the influence exerted upon it is shaped not only by the local neighborhood but also by the broader network structure. For example, when deciding whether to believe a rumor in a social network, knowing which actors have influenced the opinions of an agent's neighbors can provide valuable insight into whether the agent will be swayed by those opinions. Thus, as long as an agent can access information beyond its immediate connections, it can better inform its decisions.

However, one of the key challenges in utilizing the full network structure is the reality of incomplete information. In real-world social and economic networks, participants rarely have access to complete knowledge of the network's architecture. Firms know their own suppliers but may not know who their suppliers' suppliers are. Similarly, social media users know who their friends are but not who their friends' friends are. This prevalence of incomplete network information has been empirically demonstrated. For instance, Breza et al. (2018) found that when asked to identify their friends' friends, individuals in Indian villages struggled to do so accurately.

These stylized facts give rise to key questions that are the focus of this paper: Assuming that the structure is encoded in agents' behaviors, what would the learning dynamics look like if agents could observe their neighbors' actions over time? How players can learn the true network through the interaction with their neighbor?

We consider an environment in which a set of myopic but rational agents play an infinetly repeated local interaction game of incomplete information. Our research focuses on scenarios with local spillovers between players.

Our findings yield several important insights: We first show that, after learning ends, the Bayesian Nash Equilibrium (BNE) for arbitrary ex-ante distributions over graphs is exactly the same as the Nash Equilibrium (NE) under a complete information game. This discovery holds significant implications for predicting players' actions under incomplete information, suggesting that over time, play converges towards the Nash Equilibrium predicted by complete information. Additionally, this result serves as a bridge between incomplete and complete network games. Under rational Bayesian learning, after learning ends, a player will act as if they know the true network. This result aligns with some game theory papers(Jordan (1995) and Kalai and Lehrer (1995), as both our paper and theirs show the convergence of BNE and NE.

In terms of information aggregation, we have identified simple conditions under which all players possess knowledge of the realized network. Even if agents play the NE as if they know the realized graph, they may not know the realized graph. Additionally, we have established an upper bound for the learning time of the game, *i.e.*, the time required for all agents to learn about the structure. This insight offers valuable implications for understanding the stability and dynamics of information diffusion within networked systems. Furthermore, we show that how homophily affects learning time for a player.

This research contributes to the growing body of literature on Bayesian learning in networks, an area that has received comparatively less attention than DeGroot learning models. Additionally, our work aligns with and complements the existing literature. By providing a comprehensive analysis of decision-making and learning processes in a simple and intuitive way, our work offers both theoretical insights and practical implications for understanding complex social and economic systems. Our findings not only advance the theoretical underpinnings of network game theory but also provide valuable insights for policymakers, business leaders, and researchers seeking to understand and influence behavior in interconnected systems.

1.2 Related Literature

Bayesian learning within decision-making contexts has been extensively explored in the game theory literature. Much of the research focuses on analyzing the circumstances under which repeated pairwise communication among a finite group of individuals leads to consensus. Geanakoplos and Polemarchakis (1982), Aumann et al. (1995), and Gale and Kariv (2003) have investigated how two players adjust their posterior beliefs regarding the true state after observing each other's actions, demonstrating the convergence of posterior beliefs and equilibrium actions. However, these studies typically assume perfect monitoring of actions, whereas our model accounts for the limited observation capabilities of agents.

Considering Bayesian learning within a network presents added complexity. When agents can only observe the actions of their neighbors, perfect monitoring becomes impractical due to the limited information available. Agents must consider the network's connectivity and update their beliefs accordingly, rendering Bayesian learning within networks challenging. Consequently, much of the literature on learning and the evolution of behavior and opinions in social networks assumes agents with bounded rationality or non-Bayesian settings(Bala and Goyal (1998), Golub and Jackson (2010), Acemoglu et al. (2010), Jadbabaie et al. (2012), Frank and Neri(2021)).

Parikh and Krasucki (1990), Mueller-Frank (2013) provides relevant analysis of Bayesian learning under incomplete information within a repeated game framework. Their study reveals local indifference between connected agents, implying that after learning ends, any action an agent selects becomes optimal for all their neighbors. One of main ideas of these papers is imitation actions of connected players. After observing neighbors actions, players can follow the actions if it gives higher payoff. This is possible because of utility functions that have no externalities for all players, thereby overlooking potential positive or negative externalities in decision-making. In contrast, our Bayesian model incorporates such externalities under a more general utility function, yielding distinct results from local indifference due to varying optimal responses among agents based on connectivity.

Our equilibrium concept in a stage game is closely related to the work of Chaudhuri et al. (2024), who examine Bayesian Equilibrium when players can only observe their neighbors' actions under a linear-quadratic utility function (Ballester et al. (2006)). Extending this idea, we consider more general utility function and investigate how the Bayesian Equilibrium evolves over repeated games.

Our primary finding demonstrates that after the learning process ends, the Bayesian Nash Equilibrium (BNE) aligns precisely with the Nash Equilibrium under complete information. Jordan (1995) illustrates that Bayesian Nash Equilibria in incomplete repeated games asymptotically converge to the set of Nash Equilibria for complete repeated games for myopic players, with further extensions by Kalai and Lehrer (1995) study more sophisticated players while inducing the similar result.

However, our study has different points from previous work in several respects. Firstly, while prior studies assume perfect monitoring, we account for imperfect monitoring inherent in network settings where players can only observe their neighbors. Secondly, prior studies rely on the martingale convergence theorem, whereas we establish that after a specific time,

$t*$, the BNE at any stage $t \geq t^*$ precisely matches the NE without invoking the limit theorem.

Furthermore, Li and Tan (2020) have analyzed network structures conducive to players acquiring knowledge of the true state. In contrast, we identify another sufficient condition for perfect learning, focusing on the Nash Equilibrium under a complete network game rather than network structure.

The subsequent sections of this paper are organized as follows: Section 2 introduces network theory tools and defines the game setup. In Section 3, we present our main result that the BNE coincides with the NE after learning concludes. Also, we discuss conditions for perfect learning and the learning time. Section 4 shows the application of our model in a linear-quadratic game(Ballester et al., (2006)). Finally, section 5 provides concluding remarks. All proofs, as well as additional discussion on certain aspects of our model, are relegated to the appendix.

2 Model

2.1 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ denote the set of players. A network (or graph) g is the collection of all links that exist between players. In particular, $g = [g_{ij}] \in \mathbb{R}^{n \times n}$, where $g_{ij} \geq 0$ measures the extent to which agent i directly influences agent j. If $g = g^T$, then the network is undirected. If $g_{ij} \in \{0, 1\}$, $\forall i, j \in N$, it is unweighted.

The in-neighborhood of player i is the set of players that directly influence i and is denoted by: $N_i^{in}(g) = \{j \in N | g_{ji} > 0\}$. Similarly, the out-neighborhood of player i is the set of players that i directly influences and is denoted by: $N_i^{out}(g) = \{j \in N | g_{ij} > 0\}$. For an undirected network, the neighborhood set is denoted as: $N_i(g) = \{j \in N | g_{ij} > 0\} = \{j \in N | g_{ji} > 0\}$.

2.2 The Game

We study an infinitely repeated, incomplete information network game where each agent's knowledge about the network is restricted to the identity and the weights of their incoming links g_{ji}.[1] In the first stage, Nature, a non-strategic player, chooses a network out of a set containing networks on the number of vertices equal to the number of agents. The chosen network is drawn from an ex-ante distribution that is common knowledge among all agents.

[1]If we consider a undirected network, each player also can know who they can influence since $g_{ji} = g_{ij}$.

After Nature's selection, players observe their of their incoming links but do not know the full network structure beyond that. Using this information and Bayes' rule, agents update their beliefs about the network chosen by Nature and proceed to simultaneously exert action to maximize intermim payoff. Following action exertion in the first stage, players observe the action levels of their incoming neighbors. Since actions are informative about the linking profile of agents, observing these actions allows players to further update their beliefs about the true network architecture. With these revised beliefs, players proceed to exert actions in the second stage, and this process of belief updating and action exertion continues infinitely. It is assumed that players are myopic but rational. That is, we seek to characterize the learning process induced by the sequence of stage game equilibria. We proceed to formally describe the game.

2.2.1 Time

Time is discrete and indexed by $t \in \{1, 2, ...\}$.

2.2.2 Ex-Ante Beliefs

Denote by $\mathcal{G}$ a finite set of networks that Nature selects from whose cardinality is denoted by $|\mathcal{G}|$. For example, if this set contains all possible unweighted and undirected networks with n players, then its cardinality would be given by $|\mathcal{G}| = 2^{\frac{n(n-1)}{2}}$.

Let $p \in \Delta(\mathcal{G})$ denote a probability distribution over $\mathcal{G}$, with $\Delta(\mathcal{G})$ denoting the set of all probability distributions over $\mathcal{G}$. Nature's singular role in our game is to choose a specific network $g \in \mathcal{G}$ in the beginning of the first stage. This choice is made according to some $p \in \Delta(\mathcal{G})$, and these ex-ante beliefs are assumed to be common knowledge among players.

2.2.3 Types

We assume that agents can only identify the players by whom they are directly influenced. Stated differently, they can only observe their own column of the adjacency matrix corresponding to the network drawn by Nature. Moreover, as the game moves forward in time, by observing their in-neighbors behavior, agents will gain additional information about the true architecture of the network. Since players information change from period to period, we define types (private information) in a dynamic fashion. In particular, the type of a player in a particular period t is taken to be a set consisting of all those networks that rationalize all the information available to it. This is defined via the following notion of an indistinguishable set of networks.

Definition 1. In the counterfactual scenario in which Nature has selected network g, we denote by $T^t_{i(g)}$ the set of all networks that player i would not be able to distinguish from g in period t conditional on all its available information.

If g was selected by Nature, $T^t_{i(g)}$ consists of all those networks that player i would not be able to disqualify as having been drawn given the player's available information in period t. This set may consist of a single network if the agent learns the true network drawn by Nature. It may also consist of all possible networks in $\mathcal{G}$ if it cannot distinguish between any of them. More importantly, it is also possible that $T^t_{i(g)} = T^t_{i(g')}$ even if $g \neq g'$.

In our game, these indistinguishable sets of networks define player types. In particular, the type set of player i in period t is given by

$$T^t_i = \{T^t_{i(g)} | g \in \mathcal{G}\},$$

which is a collection of subsets of graphs. Thus, $T^t_i \subseteq \mathcal{P}(\mathcal{G})/\{\emptyset\}$, where $\mathcal{P}(\mathcal{G})$ is the power set of $\mathcal{G}$. Moreover, since $\mathcal{P}(\mathcal{G})/\{\emptyset\}$ is a finite collection, T^t_i is also finite for any $i \in N$ and any t. The period t type space of the game is

$$T^t = \times_{i\in N} T^t_i.$$

To distinguish between a possible type of a player and its realized type, we use notation I^t_i to denote the latter. For example, if Nature chooses $g^* \in \mathcal{G}$, then $I^t_i = T^t_{i(g^*)}$. Even though $I^t_i \in T^t_i$ and player i knows its private information I^t_i , this does not imply that it knows what g induces I^t_i. This is because it could be the case that $T^t_{i(g)} = T^t_{i(g^*)}$ with $g \neq g^*$.[2] As we will show, all players will be able to construct the type sets of others all agents and assign probabilities to all them being of particular types. However, realized type I^t_i for all $i \in N$ is private information.

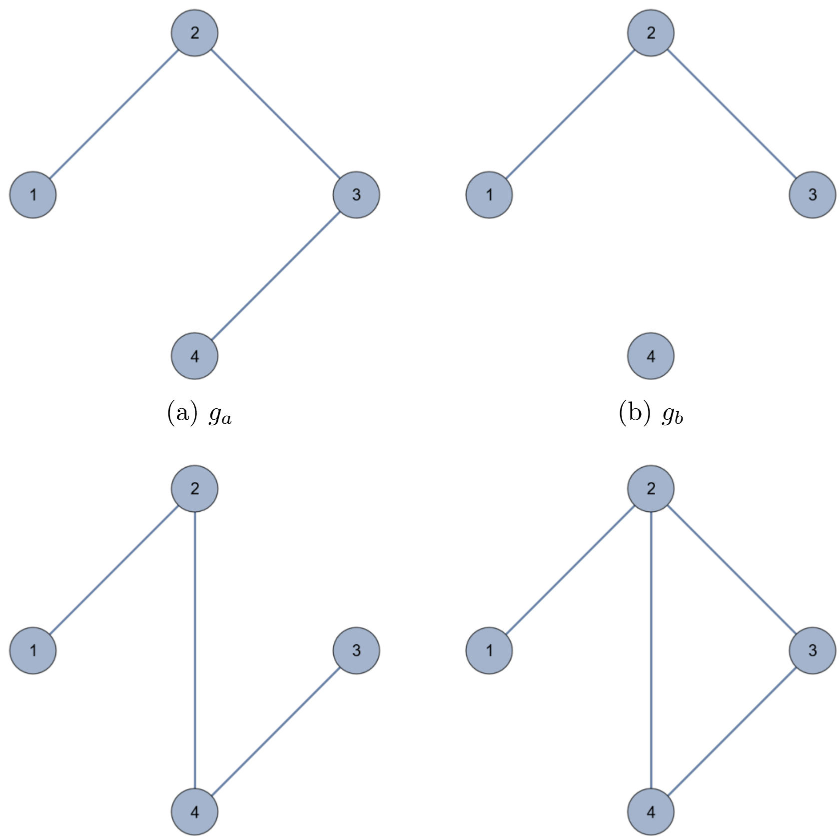
In an arbitrary period $t > 1$ the formal definition of $T^t_{i(g)}$ is provided in a later section as it is defined dynamically and employs equilibrium actions. However, during the first stage $t = 1$ no prior actions have been exerted, and hence the only source of information available to players is their realized in-neighborhood following Nature's draw. Recalling that player i can observe its incoming links, then the set of networks that are indistinguishable from g

[2] We explain some properties of a type in the later of this section.

for player i is given by:

$$T^1_{i(g)} = \left\{g' \in \mathcal{G} | g'_{ji} = g_{ji}, \forall j \in N\right\}.$$

Note that if we restrict attention only to unweighted and undirected networks, then $T^1_{i(g)} = \{g' \in \mathcal{G} | N_i(g') = N_i(g)\}$. To illustrate the construction of type sets in period $t = 1$, consider a 4-player game where there is uncertainty over a set of unweighted and undirected graphs $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$ as shown in Figure 1.[3]



(a) g_a (b) g_b

(c) g_c (d) g_d

Figure 1: A four player game with $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$

Consider player 1. In Figure 1, we have that $N_1(g_a) = N_1(g_b) = N_1(g_d) = N_1(g_d) = \{2\}$. Therefore, player 1's realized neighborhood is the same under any networks, hence the information it is able to extract from any of these networks is identical. This implies all $g \in \mathcal{G} = \{g_a, g_b, g_c, g_d\}$ are indistinguishable to player 1 regardless of which one is realized by Nature, i.e., $T^1_{1(g_a)} = T^1_{1(g_b)} = T^1_{1(g_c)} = T^1_{1(g_d)} = \{g_a, g_b, g_c, g_d\}$. This shows that $T^t_{i(g)}$ can be the same to $T^t_{i(g')}$ even if $g \neq g'$.

[3]Thus, we can write $N^{in}_i(g) = N_i(g)$.

Now consider player 2. We have $N_2(g_a) = N_2(g_b) = \{1, 3\}$, $N_2(g_c) = \{1, 4\}$, and $N_2(g_d) = \{1, 3, 4\}$. This implies that $T^1_{2(g_a)} = T^1_{2(g_b)} = \{g_a, g_b\}$, $T^1_{2(g_c)} = \{g_c\}$, and $T^1_{2(g_d)} = \{g_d\}$. Consequently, player 2 will not be able to distinguish between g_a and g_b if either of them are realized. On the other hand, if either g_c or g_d are realized then player 2 would know the entire network since its neighborhood sets under both realizations are unique. Similar constructions can be made for players' 3 and 4 type sets.

We now show that all players can construct the type sets of all others. Consider the type $T^1_{2(g_a)} = \{g_a, g_b\}$ our example. Since all players know that Nature selects a network from $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$, and that in the first stage the only information available to players stems from their realized neighborhood, any player can conduct the counterfactual that if g_a is realized, then player 2 will be connected to players 1 and 3 and thus will not be able to distinguish between graphs g_a and g_b.

With regard to realized types, suppose Nature chooses g_a. In this case, the realized type of player 1 is $I^1_1 = \{g_a, g_b, g_c, g_d\} = T^1_{1(g_a)}$. Note, however, that $T^1_{1(g_a)} = T^1_{1(g_b)} = T^1_{1(g_c)} = T^1_{1(g_d)}$. Thus, player 1 cannot know which graph induced its realized type. Similarly, $I^1_2 = T^1_{2(g_a)} = T^1_{2(g_b)} = \{g_a, g_b\}$, $I^1_3 = T^1_{3(g_a)} = T^1_{3(g_d)} = \{g_a, g_d\}$, and $I^1_4 = T^1_{4(g_a)} = \{g_a\}$ which implies that players 2 cannot distinguish between g_a and g_b, player 3 cannot distinguish between g_a and g_d, but player 4 knows the true network.

In summary, with $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$ as shown in Figure 1, all players can construct their own as well as other players period 1 type sets as follows:

$$
\begin{aligned}
T^1_1 &= \{T^1_{1(g_a)}, T^1_{1(g_b)}, T^1_{1(g_c)}, T^1_{1(g_d)}\} = \{T^1_{1(g_a)}\} &&= \{\{g_a, g_b, g_c, g_d\}\}.\\
T^1_2 &= \{T^1_{2(g_a)}, T^1_{2(g_b)}, T^1_{2(g_c)}, T^1_{2(g_d)}\} = \{T^1_{2(g_a)}, T^1_{2(g_c)}, T^1_{2(g_d)}\} &&= \{\{g_a, g_b\}, \{g_c\}, \{g_d\}\}.\\
T^1_3 &= \{T^1_{3(g_a)}, T^1_{3(g_b)}, T^1_{3(g_c)}, T^1_{3(g_d)}\} = \{T^1_{3(g_a)}, T^1_{3(g_b)}, T^1_{3(g_c)}\} &&= \{\{g_a, g_d\}, \{g_b\}, \{g_c\}\}.\\
T^1_4 &= \{T^1_{4(g_a)}, T^1_{4(g_b)}, T^1_{4(g_c)}, T^1_{4(g_d)}\} = \{T^1_{4(g_a)}, T^1_{4(g_b)}, T^1_{4(g_c)}\} &&= \{\{g_a\}, \{g_b\}, \{g_c, g_d\}\}.
\end{aligned}
$$

2.2.4 Actions and Strategies

At each stage t, all players have the same action set A which can be discrete or continuous. Given their type sets, agents select actions based on choice correspondences

$$c^t_i : T^t_i \rightrightarrows A,$$

which assign a subset of A to each information set $T^t_{i(g)} \in T^t_i$. In period t, a pure strategy σ^t_i of player i is a function that assigns a single action to each information set

$$\sigma^t_i : T^t_i \to A.$$

That is $\sigma^t_i = (a^t_i(I^t_i = T^t_{i(g)}))_{T^t_{i(g)} \in T^t_i}$ with $\sigma^t_i(T^t_i) \in c^t_i(T^t_i)$. The choice correspondence assigns a set of optimal actions to each information set, and the pure strategy σ^t_i selects one of them. The strategies of all players, for any $t = 1, 2, \ldots,$ are assumed to be common knowledge.

2.2.5 Belief Updating

Given prior beliefs over graphs and a type $T^t_{i(g)}$, a posterior probability to a graph g' can be assigned according to Bayes' rule:

$$p(g'|T^t_{i(g)}) = \begin{cases} \frac{p(g')}{\sum_{g' \in T^t_{i(g)}} p(g')} & \text{if } g' \in T^t_{i(g)} \\ 0 & \text{otherwise.} \end{cases}$$

The denominator gives the total probability mass of graphs in $T^t_{i(g)}$, while the numerator is the probability of the specific graph g'. As an example, consider Figure 1 and suppose that Nature selects a graph according to the distribution $(p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$. Moreover, suppose g_a is realized so that $I^1_2 = T^1_{2(g_a)} = \{g_a, g_b\}$. Then, to g_a being realized by Nature, player 2 will assign:

$$p(g_a|I^1_2 = T^1_{2(g_a)}) = \frac{p(g_a)}{p(g_a) + p(g_b)} = \frac{1}{3}.$$

2.2.6 Local Spillovers, Expected Payoffs, and Equilibrium

The utility function of player i under a network g is denoted by

$$u_i(a_i, (a_j)_{j \in N^{in}_i(g)}, g).$$

The dependence on a_i captures how player $i's$ payoff affects on its own action, while the dependence on $(a_j)_{j \in N^{in}_i(g)}$ captures how its payoff explictly depends on those players' actions who directly influence it in g. The dependence on g captures the extent to which payoffs explictly depend of the architecture of the network itself.

Before we analyze the dynamics of BNE, we define a special type of network games. A given player's payoff depends on other players' actions, but only on those to whom the player is linked in the network. In fact, without loss of generality, the network can be taken to indicate the payoff interactions in the society(Jackson and Zenou, 2014). Following this idea, we consider network games where a player's payoff depends on the player's action and those who influence the player.

Assumption 1. Payoffs satisfy *local spillovers*. That is, $u_i(a_i, (a_j)_{j \in N_i^{in}(g)}, g) = u_i(a_i, (a_j)_{j \in N_i^{in}(g')}, g')$ when $g'_{ji} = g_{ji}, \forall i \in N$.

The definition implies that if the in-neighborhood of player i and the weights are the same under g and g', the payoff of i are the same under both graphs. Even if $g \neq g'$, when the neighborhood are the same, for player i, distinguishing g and g' does not matter. This is why we call this property as local spillover. The following example shows the widely used payoff function in network game contexts that satisfies local spillovers.[4]

Example 1. Consider the following linear-quadratic function of Ballester et al. (2006):

$$u_i(a_i, (a_j)_{j \in N_i(g)}, g) = a_i - \frac{1}{2}a_i^2 + \lambda a_i \sum_{j=1}^{n} g_{ij} a_j.$$

The first two terms in the utility function capture the direct benefit and cost to player i from exerting its own action. The third term captures local complementarities or substitutabilities with those agents that the player is connected to, with λ measuring the strength of this complementarity.

Now, we extend this payoff function under a complete network to an incomplete network with a repeated game. We assume that players ignore the future effects of their decisions. That is, they myopically exert the actions today based on current beliefs without regarding the effects of their actions on other players or future information availability. This can be the result of players heavily discounting the future.

At each stage t, all players have the same action set A and simultaneously exert action to maximize the expected payoff. Assuming player i's realized type is $T^t_{i(g)}$, these are given by:

$$E[u_i | I_i^t = T^t_{i(g)}] = \sum_{g' \in \mathcal{G}} p(g' | T^t_{i(g)}) u_i(a_i(I_i^t = T^t_{i(g)}), (a_j(I_j^t = T^t_{j(g')}))_{j \in N_i^{in}(g')}, g')$$

For each player i in stage t, a pure strategy σ_i^t maps each possible type to an action. That is, $\sigma_i^t = (a_i^t(I_i^t = T^t_{i(g)}))_{T^t_{i(g)} \in T_i^t}$. This is a simultaneous move game of incomplete information so we use Bayes-Nash as the equilibrium notion.

[4] Those examples consider undirected and unweighted graph, implying $g_{ij} = g_{ji} \in \{0, 1\}$.

Definition 2. The pure strategy profile $\sigma^{*t} = (\sigma_i^{*t}, \sigma_{-i}^{*t})$, where $\sigma_i^{*t} = (a_i^{*t}(I_i^t = T_{i(g)}^t))_{T_{i(g)}^t \in T_i^t}$ is a Bayesian-Nash equilibrium (BNE) of the t^{th} stage game if:

$$a_i^{*t}(I_i^t = T_{i(g)}^t) \in \underset{a_i^t(I_i^t = T_{i(g')}^t)}{argmax}\ E[u_i|I_i^t = T_{i(g)}^t], \forall i \in N,\ \forall T_{i(g)}^t \in T_i^t.$$

With type sets and strategies being common knowledge, all player can compute BNE action profiles of every stage game.

Considering multiple pure BNEs in the game, to ensure that players coordinate on a specific equilibrium sequence when multiple equilibria exist we assume that an equilibrium selection mechanism is in place. This mechanism could be based on pre-play communication, or public signals, ensuring that all players select the same equilibrium(Myerson, 1991). With the strategies and the equilibrium selection mechanism being common knowledge, every player knows which equilibrium sequence from the multiple pure BNEs set is being played in each stage. Without loss of generality, let a BNE sequence is selected and it is known for all players during this paper.[5]

Assumption 2. There exists unique pure BNE at each stage game.

2.2.7 Dynamic Type Updating

We assume that players can perfectly observe in-neighbors actions at the end of each period. Since actions are type dependent, this implies that by observing adjacent agents behavior, players can extract additional information about each others types and hence the true architecture of the network.

Definition 3. Suppose g has been realized by Nature so that, $I_j^t = T_{j(g)}^t$. The set of types that rationalize player $j's$ actions is given by

$$B^t(a_j^t(I_j^t = T_{j(g)}^t)) = \{T_{j(g')}^t \in T_j^t | a_j^t(I_j^t = T_{j(g')}^t) \in argmax_{a_j^t \in A} E[u_j|I_j^t = T_{j(g)}^t]\}.$$

Moreover, let $B_f^t(a_j^t(I_j^t = T_{j(g)}^t))$ denote the set where we merge all the elements of all subsets of $B^t(a_j^t(I_j^t = T_{j(g)}^t))$:[6]

$$B_f^t(a_j^t(I_j^t = T_{j(g)}^t)) = \bigcup_{T_{j(g')}^t \in B^t(a_j^t(I_j^t = T_{j(g)}^t))} T_{j(g')}^t.$$

[5]For the conditions for a unique equilibrium, please see E. Sadler(2024).

[6]For example, if $B^t(a_j^t(I_j^t = T_{j(g_a)}^t)) = \{T_{j(g_a)}^t, T_{j\{g_c\}}^t\}$ where $T_{j(g_a)}^t = \{g_a, g_b\}$ and $T_{j(g_c)}^t = \{g_c, g_d\}$, then $B_f^t(a_j^t) = \{g_a, g_b, g_c, g_d\}$.

Note that what player i observes is the action, not the type, and there may be more that one type that induces the action observed by the player. To state this formally, let $g, g' \in T^t_{i(g)}$ and suppose that $T^t_{j(g)} \neq T^t_{j(g')}$. If player i observes $a^t_j(I^t_j = T^t_{j(g)})$ and it holds that $a^t_j(I^t_j = T^t_{j(g)}) = a^t_j(I^t_j = T^t_{j(g')})$, then $T^t_{j(g)}, T^t_{j(g')} \in B^t(a^t_j)$. In this case, player i cannot know for certain the realized type of player j since both $T^t_{j(g)}$ and $T^t_{j(g')}$ would induce the same equilibrium action of player j.

Consider the incomplete linear-quadratic model as we presented in example 1 with Figure 1.[7] Let $(p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$ and $\lambda = 1/4$. In this case it can be shown that

$$\begin{aligned}
&a^1_2(I^1_2 = T^1_{2(g_a)}) = a^1_2(I^1_2 = T^1_{2(g_b)}) = 1.789\\
&a^1_2(I^1_2 = T^1_{2(g_c)}) = 1.877\\
&a^1_2(I^1_2 = T^1_{2(g_d)}) = 2.387.
\end{aligned}$$

Now suppose that g_a has been realized by Nature so that $I^1_2 = T^1_{2(g_a)} = \{g_a, g_b\}$ and $I^1_3 = T^1_{3(g_a)} = \{g_a, g_d\}$. Since players 2 and 3 are connected, player 3 observes that 2 exerts $a^1_2 = 1.789$. Therefore, for player 3, the set of types that rationalize player 2's action is $B^1(a^1_2(I^1_2 = T^1_{2(g_a)})) = \{T^1_{2(g_a)}, T^1_{2(g_b)}\} = \{\{g_a, g_b\}\}$ and hence $B^1_f(a^1_2(I^1_2 = T^1_{2(g_a)})) = \{g_a, g_b\}$. Consequently, after observing the action, player 3 knows that the set of networks that are possible from player 2's perspective are either g_a or g_b . In the second period $t = 2$, player 3 can update its belief about the true network by combining its original information I^1_3 with the information extracted from player 2's action, i.e., $I^2_3 = I^1_3 \cap B^1_f(a^1_2) = \{g_a, g_d\} \cap \{g_a, g_b\} = \{g_a\}$.

With this idea, we formally define a type set of each player.

Definition 4. The updating rule for an arbitrary $T^t_{i(g)}$ is given by:

$$T^t_{i(g)} = \begin{cases} \{g' \in \mathcal{G} | g'_{ji} = g_{ji}, \forall j \in N\} & t = 1 \\ T^{t-1}_{i(g)} \cap \left(\bigcap_{j \in N^{in}_i(g^{T^{t-1}_{i(g)}})} (B^{t-1}_f(a^{t-1}_j(I^{t-1}_j = T^{t-1}_{j(g)}))) \right) & t = 2, 3, 4, \dots \end{cases}$$

where

$$N^{in}_i(g^{T^{t-1}_{i(g)}}) = \left\{ j \in N | g'_{ji} > 0 \text{ and } g' \in T^{t-1}_{i(g)} \right\}$$

With the formal definition of types in hand, we proceed to discuss some of their properties.

Remark 1. $g \in T^t_{i(g)}$, for any $i \in N$, any $g \in \mathcal{G}$ and any $t = 1, 2, \dots$.

[7] Note that this linear-quadratic game has a unique BNE profile.

This follows directly form the definition of $T^t_{i(g)}$. From player $i's$ perspective, $T^t_{i(g)}$ consists of all those graphs that are indistinguishable from g in period t. Thus, the graph g itself should lie in $T^t_{i(g)}$. This implies that $T^t_{i(g)}$ is nonempty for any i, t, and g.

Remark 2. For any $g, g' \in T^t_{i(g)}$, $g'_{ji} = g_{ji}, \forall j \in N$ and $N^{in}_i(g) = N^{in}_i(g')$, $\forall i \in N, \forall t = 1, 2, 3, \ldots$

Suppose $g, g' \in T^t_{i(g)}$. Since by construction $T^t_{i(g)}$ is non-increasing in t, it then follows that $g, g' \in T^1_{i(g)}$. Recalling that $T^1_{i(g)} = \{g' \in \mathcal{G} | g'_{ji} = g_{ji}, \forall j \in N\}$, $g'_{ji} = g_{ji}$ for any $j \in N$. So, the in-neighbors of player i under g and g' must be same and. Thus, we can write $g_{ij}^{I^t_i = T^t_{i(g)}} = g_{ij} = g'_{ij}$ so that $N^{in}_i(g^{T^t_{i(g)}}) = N^{in}_i(g) = N^{in}_i(g')$ for any $g, g' \in T^t_{i(g)}$. The type updating rule with $t \geq 2$ can thus be rewritten as

$$T^t_{i(g)} = T^{t-1}_{i(g)} \cap \left(\bigcap_{j \in N^{in}_i(g)} (B^{t-1}_f(a^{t-1}_j(I^{t-1}_j = T^{t-1}_{j(g)}))) \right).$$

Lastly, recall that I^t_i denotes the realized type of player i. As mentioned previously, even though the player knows its realized type, it may not know what graph induced this type. This is because there may be $g, g' \in \mathcal{G}$ such that $I^t_i = T^t_{i(g)} = T^t_{i(g')}$. The following lemma and remark show this formally.

Lemma 1. *For any $g, g' \in \mathcal{G}$, either $T^t_{i(g)} = T^t_{i(g')}$ or $T^t_{i(g)} \cap T^t_{i(g')} = \emptyset$, for any $i \in N$ and any $t = 1, 2, 3, \ldots$.*

Remark 3. For any $i \in N$, if $g, g' \in T^t_{i(g)}$, then $T^t_{i(g)} = T^t_{i(g')}$ for any $t = 1, 2, 3, \ldots$.

Suppose $g \neq g'$. When $g' \in T^t_{i(g)}$,then $g' \in T^t_{i(g)} \cap T^t_{i(g')}$,implying that $T^t_{i(g)} \cap T^t_{i(g')} \neq \emptyset$. Thus, by lemma 1, $T^t_{i(g)} = T^t_{i(g')}$. As an example, suppose that $I^t_i = \{g_a, g_b, g_c\}$. If g_a was selected by Nature, then this realized type was generated by $I^t_i = T^t_{i(g_a)}$. However, since $g_a, g_b, g_c \in T^t_{i(g_a)}$, then $T^t_{i(g_a)} = T^t_{i(g_b)} = T^t_{i(g_c)}$. Thus, player i cannot know which network induced its private information I^t_i. Nonetheless, and as is stated in the following remark, the network selected by Nature must always belongs to players realized types.

Remark 4. Let g^* be the realized graph. Then, $g^* \in I^t_i, \forall t = 1, 2, 3, \ldots$ and $\forall i \in N$.

In overall, $T^t_{i(g)}$ is non-increasing in t. Furthermore, since $\mathcal{G}$ is finite, $T^t_{i(g)}$ is also finite. Therefore, $T^t_{i(g)}$ is a nonempty, non-increasing and finite set for any $i \in N$, for any $g \in \mathcal{G}$, and for any $t = 1, 2, 3, \ldots$.

2.3 Example of the Learning Process

We now illustrate the learning process using our example 1 with the linear-quadratic expected utility function under figure 1. We set $(p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$, $\lambda = \frac{1}{4}$, and it is assumed that g_a has been selected by Nature.

The BNE action profile of player 2 is given by:

$$
\begin{aligned}
a_2^1(I_2^1 = T_{2(g_a)}^1) &= a_2^1(I_2^1 = T_{2(g_b)}^1) = 1.789 \\
a_2^1(I_2^1 = T_{2(g_c)}^1) &= 1.877 \\
a_2^1(I_2^1 = T_{2(g_d)}^1) &= 2.387
\end{aligned}
$$

Since g_a has been realized, player 1 will obserse its sole neighbor's, player 2's, action to be $a_2^1(I_2^1 = T_{2(g_a)}^1) = 1.789$. Thus, the set of player 2 types that rationalize player 1's observation are $B^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{T_{2(g_a)}^1, T_{2(g_b)}^1\}$, with a corresponding set of networks $B_f^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{g_a, g_b\}$. Using this information in period $t = 2$, player 1 updates its beliefs regarding the true network realized by Nature according to

$$
I_1^2 = I_1^1 \cap B_f^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{g_a, g_b, g_c, g_d\} \cap \{g_a, g_b\} = \{g_a, g_b\}.
$$

Even though player 1 can only observe the action of player 2, it can still compute the BNE action profile of all players. This allows it to consistently construct what the type sets of all players in period $t = 2$ will be. To see this, first note that with its updated realized type $I_1^2 = \{g_a, g_b\}$, player 1 knows that the network selected by Nature is either g_a or g_b. Note that in both of these networks, player 2 is connected to player 1 and player 3. Therefore, player 1 knows that player 2 will observe 1's and 3's actions. The BNE action profile of player 3 is given by:

$$
\begin{aligned}
a_3^1(I_3^1 = T_{3(g_a)}^1) &= a_3^1(I_3^1 = T_{3(g_d)}^1) = 2.041 \\
a_3^1(I_3^1 = T_{3(g_b)}^1) &= 1.447 \\
a_3^1(I_3^1 = T_{3(g_c)}^1) &= 1.498
\end{aligned}
$$

In the scenario in which g_a is the true network, player 1 knows that player 2 will observe player 3's action to be $a_3^1(I_3^1 = T_{3(g_a)}^1) = 2.041$. In this case, player 1 knows that the corresponding type of player 2, $T_{2(g_a)}$ will be updated in stage $t = 2$ as follows:

$$\begin{aligned} T^2_{2(g_a)} &= T^1_{2(g_a)} \cap B^1_f(a^1_3(I^1_3 = T^1_{3(g_a)})) \cap B^1_f(a^1_1(I^1_1 = T^1_{1(g_a)})) \\ &= \{g_a, g_b\} \cap \{g_a, g_d\} \cap \{g_a, g_b, g_c, g_d\} = \{g_a\}. \end{aligned}$$

On the other hand, in the scenario in which g_b is the true network, player 1 knows that player 2 will observe $a^1_3(I^1_3 = T^1_{3(g_b)}) = 1.447$, and not $a^1_3(I^1_3 = T^1_{3(g_a)}) = 2.041$. The set of networks that would rationalize player 3's action in this case would therefore be $B^1_f(a^1_3(I^1_3 = T^1_{3(g_b)})) = \{g_b\}$. Thus, player 1 can infer that player 2's type $T_{2(g_b)}$, will be updated as follows:

$$\begin{aligned} T^2_{2(g_b)} &= T^1_{2(g_b)} \cap B^1_f(a^1_3(I^1_3 = T^1_{3(g_b)})) \cap B^1_f(a^1_1(I^1_1 = T^1_{1(g_b)})) \\ &= \{g_a, g_b\} \cap \{g_b\} \cap \{g_a, g_b, g_c, g_d\} = \{g_b\} \end{aligned}$$

In this fashion, player 1 can determine how all of its neighbor's period $t = 1$ types will be updated next period. This process, however, can be performed even if players are not connected. To see this, consider player 4's BNE action profile:

$$\begin{aligned} &a^1_4(I^1_4 = T^1_{4(g_a)}) = 1.51 \\ &a^1_4(I^1_4 = T^1_{4(g_b)}) = 1 \\ &a^1_4(I^1_4 = T^1_{4(g_c)}) = a^1_4(I^1_4 = T^1_{4(g_d)}) = 1.994 \end{aligned}$$

Recall that $I^2_1 = T^2_{1(g_a)} = \{g_a, g_b\}$. In the scenario in which g_a is the true network, player 3 will be connected to 2 and 4. Therefore, player 1 knows that player 3 will observe $a^1_4 = 1.51$ and $a^1_2 = 1.789$. Player 1 can thus infer that player 3 will update its information according to:

$$\begin{aligned} T^2_{3(g_a)} &= T^1_{3(g_a)} \cap B^1_f(a^1_2(I^1_2 = T^1_{2(g_a)})) \cap B^1_f(a^1_4(I^1_4 = T^1_{4(g_a)})) \\ &= \{g_a, g_d\} \cap \{g_a, g_b\} \cap \{g_a\} = \{g_a\} \end{aligned}$$

A similar calculation holds for $T^2_{3(g_b)}$ in the scenario in which g_b is the true network. Thus, even if players are not connected, each player can infer any other player's type updating rule by fixing networks in their own realized type. This is because they know what each player will observe in the reference frames of these fixed networks, and consequently how types will be updated. In sum, the type sets of each player at the second stage are common knowledge and are given by: [8]

$$
\begin{aligned}
&T_1^2 = \{T_{1(g_a)}^2, T_{1(g_b)}^2, T_{1(g_c)}^2, T_{1(g_d)}^2\} = \{T_{1(g_a)}^2, T_{1(g_c)}^2, T_{1(g_d)}^2\} = \{\{g_a, g_b\}, \{g_c\}, \{g_d\}\}.\\
&T_2^2 = \{T_{2(g_a)}^2, T_{2(g_b)}^2, T_{2(g_c)}^2, T_{2(g_d)}^2\} = \{\{g_a\}, \{g_b\}, \{g_c\}, \{g_d\}\}.\\
&T_3^2 = \{T_{3(g_a)}^2, T_{3(g_b)}^2, T_{3(g_c)}^2, T_{3(g_d)}^2\} = \{\{g_a\}, \{g_b\}, \{g_c\}, \{g_d\}\}.\\
&T_4^2 = \{T_{4(g_a)}^2, T_{4(g_b)}^2, T_{4(g_c)}^2, T_{4(g_d)}^2\} = \{\{g_a\}, \{g_b\}, \{g_c\}, \{g_d\}\}.
\end{aligned}
$$

Furthermore, since we are assuming that g_a has been selected by Nature, the realized types of each player are $I_1^2 = T_{1(g_a)}^2 = \{g_a, g_b\}$, $I_2^2 = T_{2(g_a)}^2 = \{g_a\}$, $I_3^2 = T_{3(g_a)}^2 = \{g_a\}$, and $I_4^2 = T_{4(g_a)}^2 = \{g_a\}$. This implies that at the beginning of the second stage, all players except for player 1 know the true network.

At this point, it is important to note that even if a player knows the true network, this does not imply that will necessarily exert the complete information action. This is because, it still internalizes the uncertainty of others. To see this, recall that g_a is the true network so that player 3's realized type at stage 2 is $I_3^2 = T_{3(g_a)}^2 = \{g_a\}$. Player 3 knows that since g_a is the true network, player 1 will observe player 2's action so that its type will be updated according to:

$$
T_{1(g_a)}^2 = T_{1(g_a)}^1 \cap B_f^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{g_a, g_b, g_c, g_d\} \cap \{g_a, g_b\} = \{g_a, g_b\}.
$$

Hence, even if it knows the true network, player 3 knows that player 1 is still uncertain about player 3's types. This is because under g_a, $T_{3(g_a)}^2 = \{g_a\}$, while and under g_b, $T_{3(g_b)}^2 = \{g_b\}$. Thus, both g_a and g_b are still possible from player 1's perspective and player 3 knows this. The fact that individuals internalize the uncertainty is reflected on equilibrium actions since agents maximize interim expected payoffs.

Lastly, with the type space in period 2 being common knowledge, all players can compute the corresponding BNE. The BNE action profile of player 2 in the second stage is given by:

$$
\begin{aligned}
a_2^2(I_2^2 = T_{2(g_a)}^2) = 1.813\\
a_2^2(I_2^2 = T_{2(g_b)}^2) = 1.716\\
a_2^2(I_2^2 = T_{2(g_c)}^2) = 1.818\\
a_2^2(I_2^2 = T_{2(g_c)}^2) = 2.486
\end{aligned}
$$

Player 1 being connected to player 2 observes that $a_2^2(I_2^2 = T_{2(g_a)}^2) = 1.813$. With the

[8] Note that $T_{1(g_a)}^2 = T_{1(g_b)}^2$.

information stemming from its neighbor's period $t = 2$ actions, player 1 can can further update its type as follows:

$$\begin{aligned} I_1^3 = T_{1(g_a)}^3 &= T_{1(g_a)}^2 \cap B_f^2(a_2^2(I_2^2 = T_{2(g_a)}^2)) \\ &= \{g_a, g_b\} \cap \{g_a\} = \{g_a\}. \end{aligned}$$

Thus, at beginning of the third stage, all players know the true network.

3 Convergence and Information Aggregation

3.1 Convergence of BNE

Recall that we are interested in the learning dynamics induced by the myopic sequence of stage game equilibria.

In this section, we examine the convergence properties induced by the myopic sequence of stage game Bayes-Nash equilibria. This convergence bridges the gap between incomplete and complete information behavior, offering insights into how beliefs and actions stabilize over time. We assume that both Nash and Bayes-Nash equilibria of each stage game exist.[9] We start by formally defining the end of the learning process.

Definition 5. We say that learning ends if there exists a t^* such that for any $t \geq t^*$, $T_{j(g)}^t = T_{j(g)}^{t^*}$, for all $j \in N$ and $g \in \mathcal{G}$.

In other words, learning ends if there is no player that can update any type beyond stage t^*. This includes players realized types $I_i^t = T_{i(g^*)}^t$, where $g^* \in \mathcal{G}$, is the graph selected by Nature. The following proposition guarantees that learning will end at a finite stage of the game.

Proposition 1. *Let assumption 1 and 2 be satisfied. There exists a finite t^* such that learning ends.*

Recall that each stage game BNE can be computed by all players, implying that all players can consistently update the type space of the game every period. As a consequence of this, learning ending is common knowledge for all players and all players know when no other player can update its beliefs further. Next, we state an important property of BNE actions once learning has ended.

[9] For the conditions under which the equilibrium exists, please see Jackson and Zenou(2015).

Lemma 2. *Let assumption 1 and 2 be satisfied. Suppose that learning ends at t^*. For any $t \geq t^*$, and for any $g', g'' \in T^t_{i(g)}$, $a^t_j(I^t_j = T^t_{j(g')}) = a^t_j(I^t_j = T^t_{j(g'')})$, $\forall\ i \in N$, $\forall\ j \in N^{in}_i(g)$, and $\forall\ g \in \mathcal{G}$.*

Suppose learning ends at t^* and $t \geq t^*$. If $g', g'' \in T^t_{i(g)}$, then networks g' and g'' are indistinguishable to g for player i after learning has ended. For those networks networks, there exist associated types of an in-neighbor player j, $T^t_{j(g')}$ and $T^t_{j(g'')}$ such that $p(g'|I^t_i = T^t_{i(g)}) > 0$ and $p(g''|I^t_i = T^t_{i(g)}) > 0$.[10] That is, from player $i's$ perspective whose type includes networks g' and g'', both $T^t_{j(g')}$ and $T^t_{j(g'')}$ are possible types for player j. The lemma states that if the actions associated with types $T^t_{j(g')}$ and $T^t_{j(g'')}$ were different, then player i could still learn. Indeed, if $a^t_j(I^t_j = T^t_{j(g')}) \neq a^t_j(I^t_j = T^t_{j(g'')})$ and $a^t_j(I^t_j = T^t_{j(g')})$ is observed, player i would be able to infer that g'' can not induce player $j's$ realized type. This would imply that $g'' \notin T^{t+1}_{i(g)}$, leading to further learning.

It is important to note that the lemma does not imply that all actions for all types of a particular player are identical after learning ends. Instead, it is a restriction on the equilibrium actions of player j contingent on types that are induced by networks in a specific type of its neighbor i after learning has ended. That is, a neighbor j's types that remain possible from player i's perspective must lead to the same action. If they didn't, player i would continue learning, contradicting the assumption that learning has ended.

Theorem 1. *Let assumption 1 and 2 be satisfied. Suppose that Nature has selected g and that learning ends at t^*. Then, $a^{t^*}_i(I^{t^*}_i = T^{t^*}_{i(g)}) = a^c_i(g), \forall i \in N$ and $\forall g \in \mathcal{G}$, where $a^c_i(g)$ be NE of player i in a complete game under the graph g.*

The theorem says that after learning ends, BNE actions corresponding to realized types are the same as equilibrium actions when there is no uncertainty over the network. As discussed earlier, after learning ends, the possible types of player j from player i's perspective result in identical actions, preventing further learning. Thus, even if player i does not know player j's exact type, the identical actions across possible types from i's view resolve the uncertainty.

Our model departs from the insights of Parikh and Krasucki (1990) and Mueller-Frank (2013) by accounting for network externalities within a more general utility framework. While those studies reveal that connected agents become locally indifferent after learning ends—making imitation an optimal strategy since any observed action by a neighbor is equally rewarding—our approach demonstrates that when positive or negative externalities are present, the optimal response becomes heterogeneous. In our framework, the payoff

[10] This is because $g' \in T^t_{i(g)}$ and $g'' \in T^t_{i(g)}$.

of an agent is intricately linked to the actions of others through network effects, so even after learning ends, players may select distinct actions based on their connectivity and the external impacts of those actions. This key distinction underscores that the uniformity in decision-making predicted by the earlier literature does not hold in settings where network externalities play a critical role.

Consequently, the incomplete network game behaves as if it were a complete information game, implying our theorem makes a bridge between complete information and incomplete information network games. However, we do not say that all player know the exact network structure. We are saying the action is the same as the complete Nash Equilibrium. We show sufficient conditions for the perfect learning in the next section.

Concluding our convergence result, we would like to say that our result can be proved under the situation where players can only know their realized ex-post payoff and update the information set while using this ex-post payoff information, not observing neighbors' actions. The key idea is similar to our lemma and theorem. By excluding some graphs that give different ex-post payoff value that a player gets from its information set, players can learn about the realized graph. After learning ends, all possible graphs should induce the same ex-post payoff if not the player can keep learning like our lemma 2. Using the similar idea of the theorem, we can get the same result.[11]

3.2 Information Aggregation

In this section, we show when players can know the realized network and the bounds for the learning time.

3.2.1 Perfect Learning

We begin by defining the concept of perfect learning for a player within the network.

Definition 6. Learning is perfect for player i if $I_i^{t^*} = \{g\}$, after learning ends at t^*, where g is the realized graph.

This definition implies that once learning ends, player i has complete knowledge of the true network structure.

The following theorem shows when perfect learning happens for a player.

[11]Note that observing neighbors equilibrium actions is one way of knowing the ex-post payoff.

Theorem 2. *Let Assumption 1 and 2 be satisfied. Suppose a graph g is realized. Let $a^c_j(g)$ be the Nash-Equilibrium actions in a complete network game under the graph g. If there exists player j of player i's in-neighbor such that $a^c_j(g)$ is all different for any $g \in \mathcal{G}$, or $a^c_i(g)$ is all different for any $g \in \mathcal{G}$, then player i can achieve the perfect learning.*

This theorem implies that if my equilibrium can be distinguished uniquely or one of my in-neighbor shows distinguish actions, then by using this information, player i can learn the realized network. From theorem 1, we know, after learning ends, the graphs in $T^t_{i(g)}$ give the same equilibrium action as a complete network game. If there is a graph $g' \in T^1_{i(g)}$ which gives different Nash equilibrium value of i to g, we can know that g' will be excluded during the learning.

Thus, if no graph gives the same Nash equilibrium values as the realized graph at some time t, then the agent will know the true graph since I^t_i is a non-increasing set. From this theorem, we can know when all players know the true graph. This observation can be interpreted that if a network game's equilibrium is determined by a particular network feature or property, then comparing that feature across different graphs can serve as a tool for pinpointing perfect learning.

For instance, suppose the equilibrium action $a^c_i(g)$ is determined by the degree of i for a network game. If $g', g \in I^{t^*}_i$, then the degree of player i is the same under g and g'. This equivalence implies that understanding the equilibrium action can reveal the underlying structural similarities between graphs once learning has ended. We will discuss this in the later section while studying a linear-quadratic game.

Next, we study which graphs can be survived during the learning process. Let $\mathcal{G}^n$ contain all possible unweighted and undirected networks with n players. The uniform distribution p gives the probability $\frac{1}{|\mathcal{G}\}}$ for any $g \in \mathcal{G}$. Let $f : N \to N$ be a bijection that makes isomorphism between g and g'.[12]

Proposition 2. *Consider $\mathcal{G}^n$ with a uniform prior distribution p. Let $g \in \mathcal{G}^n$ be realized and learning ends at t^*. Then, if there exists an isomorphic graph $g' \neq g$ with $f(i) = i$ and $f(j) = j, \forall j \in N_i(g)$, then $g' \in T^{t^*}_{i(g)}$.*

The idea of the proposition is that symmetry may not help for learning. If two graphs g and g' are isomorphic each other, then the information flow during the learning process may be similar under both network structures, implying that player i may not distinguish those graphs.

[12] $g_{kl} = g'_{f(k)f(l)}, \forall k, l \in N.$

3.2.2 Speed of Learning

Now, we consider the learning time of the game. We show that how a diamater of a graph is related to the learning time of a game. Under the learning in a network, each player can observe their in-neighbors' actions at stage 1 and use their type information. Simultaneously, my in-neighbors observe their in-neighbors' actions and update types. Thus, at the second stage, observing my in-neighbors' actions can give information of my in-neighbors' in-neighbors' information because my in-neighbors' actions at stage 2 reflect the type information that uses my in-neighbors' in-neighbors' information. Following this idea, we can get the following result.

Note that the distance between two agents in the network is the length of the shortest path connecting them. Let $d(g)$ be the largest diameter in all components under the network g. Define $d(\mathcal{G}) = \max_{g \in \mathcal{G}} \{d(g)\}$.

Theorem 3. *Let assumption 1 and 2 be satisfied. The learning ends in stage* $t^* \leq d(\mathcal{G}) + 1$.

The intuition behind this theorem is that at stage t, a player gets all possible information from t-distance players and updates it at the $t+1$ stage. Consider how a player i updates the information set under a graph g. Note that there is no player whose distance from i is greater than $d(g)$. Thus, at the $(d(g)+1)$ stage, player i uses all possible information; thus, after that, learning stops. Also, the uniqueness condition implies that observing a player's action can extract all possible types about that player. Since there is only one possible BNE action in a game, players do not need to consider the case where some BNE actions may not informative. So, observing player j's action at period t can give all information from j at period t.

Consider our game with n-players. Note that for any graph with n-nodes, the diameter of the graph is less than n. The maximum diameter of a graph with n-nodes is $n-1$. Thus, we can say that learning stops at least n-stage.

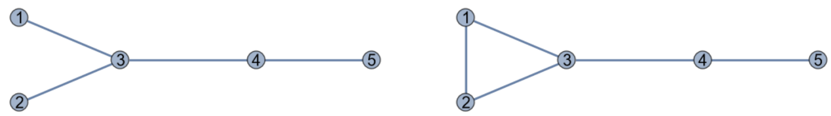
Corollary 1. *The learning ends in stage* $t^* \leq n - 1$.

Thus, the upper bound for learning time is linear in the number of players.

Now, we consider individual upper bound for a learning time. Let $t(i, g)$ be a learning end time of player i under the realization of graph g. Let $d_g(i, j)$ be a distance from j to i under the graph g with $d_g(i, i) = 0, \forall i \in N$ and $d_g(i, j) = 0$ if there is no path from j to i. Then, the upper bound of learning time of player i under the realization g follows:

Corollary 2. $t(i, g) \leq \max_{g' \in T^1_{i(g)}} \max_{j \in N} d_{g'}(i, j) + 1$.

The example below with undirected and unweighted graphs shows how the network structure influences to the learning time.



(a) g_a (b) g_b

Figure 2: $\mathcal{G} = \{g_a,g_b\}$

Note that the diameter of graphs is 3 since the shortest path is from player 1 to 5.[13] Without loss of generality, let g_a be realized. Then, $T^1_{i(g_a)} = \{g_a\}$ with $i = 1, 2$, and $T^1_{j(g_a)} = T^1_{j(g_b)} = \{g_a, g_b\}$ with $j = 3, 4, 5$. Suppose $a^1_1(I^1_1 = T^1_{1(g_a)}) \neq a^1_1(I^1_1 = T^1_{1(g_b)})$. Then, player 3 would observe $a^1_1(I^1_1 = T^1_{1(g_a)})$. Using $B^1(a^1_1(I^1_1 = T^1_{1(g_a)})) = \{T^1_{1(g_a)}\}$, player 3 can update its own type from $T^1_{3(g_a)} = \{g_a, g_b\}$ to $T^2_{3(g_a)} = \{g_a\}$. Now, consider player 4 who can observe player 3 and 5. After observing $a^1_3(I^1_3 = T^1_{3(g_a)})$, player 4 cannot update its type because $B^1(a^1_3(I^1_3 = T^1_{3(g_a)})) = \{T^1_{3(g_a)}\} = \{\{g_a, g_b\}\}$. However, at the second stage, if $a^2_3(I^2_3 = T^2_{3(g_a)}) \neq a^2_3(I^2_3 = T^2_{3(g_a)})$, then player 4, as player 3 did at the first stage, can update the type to $T^3_{4(g_a\}} = \{g_a\}$. Finally, player 5 can update its type after observing $a^3_4(I^3_4 = T^3_{4(g_a)})$ and learning ends at stage 4. From the corollary and this example, we can know the maximum distance in a graph from player i's perspective is important in the learning time.

The degree of a player does not always guarantee faster learning time as we will show this in the later section. However, considering the fact that a higher degree may decrease the distance from other players, we show that under Erdos-Renyi random graph process a degree of an agent plays a role in learning time. Let $\mathcal{G}^{er}(n, p)$ be the set of graphs produced by n-players and each edge is included in the graph with probability p. Let $|N_i(g)| = d_i(g)$ be the degree of player i under the graph g. Recall that $g' \in T^1_{i(g)}$ iff $N_i(g) = N_i(g')$, implying if $g' \in T^1_{i(g)}$, then $d_i(g) = d_i(g')$, where $d_i(g)$ be a degree of player i under the graph g. Consider the upper bound of $t(i, g), \max_{g' \in T^1_{i(g)}} \max_{j \in N} d_{g'}(i, j) + 1$.

Remark 5. *prob*$\{\max_{g' \in T^1_{i(g)}} \max_{j \in N} d_{g'}(i, j) + 1 \leq k\}$ is approximately proportional to $d_i(g)$ under $\mathcal{G}^{er}(n, p)$,.

[13] Formally, $d(g_a) = 3$ and $d(g_b) = 3$.

This is because

$$\begin{aligned}
&prob\{\max_{g' \in T^1_{i(g)}} \max_{j \in N} d_{g'}(i,j) + 1 \leq k\} \\
\Leftrightarrow &prob\{\max_{j \in N} d_{g'}(i,j) \leq k-1, \forall g' \in T^1_{i(g)}\} \\
\Leftrightarrow &\prod_{g' \in T^1_{i(g)}} prob\{\max_{j \in N} d_{g'}(i,j) \leq k-1 | d_i(g)\} \\
\Leftrightarrow &(prob\{\max_{j \in N} d_g(i,j) \leq k-1 | d_i(g)\})^{|T^1_{i(g)}|}.
\end{aligned}$$

Since E-R process is independent, we can put out the term $\forall g' \in T^1_{i(g)}$ by using the production term. Note that $\max_{j \in N} d_g(i,j)$ is related to the diameter of player i in the graph. When player i has the degree $d_i(g)$, from Jackson (2008), the amount of players that player i can approximately reach out to at a l-th distance is

$$\sum_{m=1}^{l} d_i(g) \left(\frac{d^2 - d}{d} \right)^{m-1},$$

where d is an expected degree of a player under E-R process.

Let $\frac{\sum_{m=1}^{k-1} d_i(g) \left(\frac{d^2-d}{d} \right)^{m-1}}{n-1}$ be the ratio of players that player i can reach out with distance $k-1$. Then, we can approximate $prob\{\max_{j \in N} d_g(i,j) \leq k-1 | d_i(g)\}$ as follow:

$$prob\{\max_{j \in N} d_g(i,j) \leq k-1 | d_i(g)\} \approx \begin{cases} \frac{d_i(g) \sum_{m=1}^{k-1} \left(\frac{d^2-d}{d} \right)^{m-1}}{n-1} & if \quad \frac{\sum_{m=1}^{k-1} d_i(g) \left(\frac{d^2-d}{d} \right)^{m-1}}{n-1} \leq 1 \\ 1 & otherwise \end{cases}$$

Thus, we can know the probability is proportional to the degree of i under the graph g, $d_i(g)$.

3.2.3 Learning in Homophily

In real-world networks, such as social media, homophily often leads to groups where members are more similar than different. This similarity shapes interactions, creating clusters that can either accelerate or restrict information flow depending on network structure. This phenomenon can significantly shape information flow, as homophily may facilitate rapid internal learning while potentially restricting information exchange with other groups. Investigating how homophily influences learning times in network games is essential because it can reveal whether such clusters enhance or impede the overall dissemination of information within a networked community. In this section, we show how homophily affects learning time of a player.

Define the set $l(g) = \{j \in N | g \notin I_j^1\}$. $l(g)$ represents the set of players who can distinguish the graph g at the first stage. The shortest learning time for player i regarding a graph g, denoted by $s_i(g)$, as follows:

$$s_i(g) = \begin{cases} \min_{j \in l(g)} d'_g(i,j) + 1 & \text{if } d'_g(i,j) \text{ is finite} \\ 1 & \text{if } \min_{j \in l(g)} d'_g(i,j) \text{ is infinite} \end{cases},$$

where $d'_g(i,j)$ is the distance from j to i under the graph g with $d'_g(i,i) = 0, \forall i \in N$, and $d'_g(i,j) = \infty$ when there is no path from j to i. Note that there is a difference between $d_g(i,j)$ and $d'_g(i,j)$. $d_g(i,j) = 0$ when there is no path from j to i, defined when we consider the upper bound of the learning time.

The value $s_i(g)$ reflects the minimal time required for player i to learn about g. If $g \notin I_i^1$, then $s_i(g) = 1$ since $d'_g(i,i) = 0$ as player i can exclude g at the first stage. However, if $g \in I_i^1$ but player j with $g \notin I_j^1$ who has a path to i exists, then player i may exclude g at some point. For example, let j influence k and k influence i. Suppose $a_j^1(I_j^1 = T_{j(g^*)}^1) \neq a_j^1(I_j^1 = T_{j(g)}^1)$, where g^* is the realized graph. Then, after observing $a_j^1(I_j^1 = T_{j(g^*)}^1)$, player k can exclude g at the second stage. Similarly, let $a_k^2(I_k^2 = T_{k(g^*)}^2) \neq a_k^2(I_k^2 = T_{k(g)}^2)$. Then, player i can distinguish $g^*$and g at the third stage. However, if $a_j^1(I_j^1 = T_{j(g^*)}^1) = a_j^1(I_j^1 = T_{j(g)}^1)$ or $a_k^2(I_k^2 = T_{k(g^*)}^2) = a_k^2(I_k^2 = T_{k(g)}^2)$, the learning time of i regarding g may increase that is why we interpret $s_i(g)$ as the possible shortest learning time for player i regarding graph g. If there is no player j with a path to i for which $g \notin I_j^1$, player i can never exclude g during the learning process. Thus, $g \in I_i^{t^*}$, when learning ends at t^*. In this case, we can consider the learning time regarding g to be 1, since after the initial stage, player i cannot distinguish the graph g. There is no learning time regarding g.

Using this framework, we derive the following proposition. We define a set $E_i(g) = \{g' \in I_i^1 | a_i^c(g') \neq a_i^c(g) \text{ or } a_j^c(g') \neq a_j^c(g), \text{ for a player } j \in N_i(g)\}$. When a graph g is realized, we can certainly know that player i can exclude the graphs in $E_i(g)$ during the learning time by our main result.[14] If $E_i(g)$ is non-empty, player i will exclude a graph during the learning time. If $E_i(g)$ is empty, player i may not learning during the learning process. But it may be also possible that player i can exclude a graph g' that $a_i^c(g') = a_i^c(g)$ or $a_j^c(g') = a_j^c(g)$, for all player $j \in N_i(g)$ during the learning process. This is because we do not know that whether such a graph can be excluded or not during the learning process. From this

[14]Recall that after learning ends, $a_i^t(I_i^t = T_{i(g)}^t) = a_i^c(g)$.

idea, we construct the lower bound of learning time.

By this proposition, the shortest learning time for player i depends on proximity to agents with distinct information:

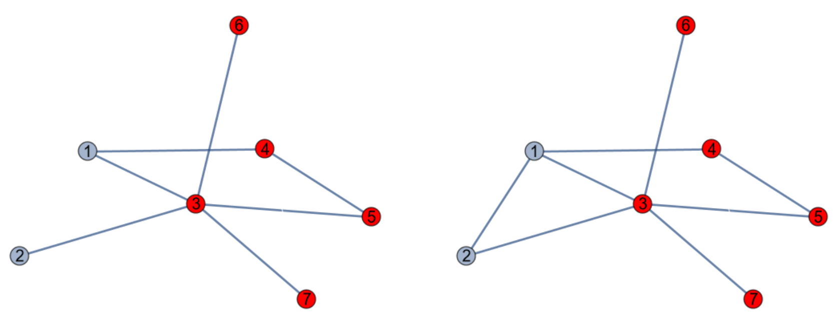
Proposition 3. *Under the assumption 1 and 2, the lower bound for learning time of player i under g is*

$$\begin{cases} \max_{g' \in E_i(g)} s_i(g') & \textit{if } E_i(g) \textit{ is non-empty} \\ 1 & \textit{if } E_i(g) \textit{ is empty} \end{cases},$$

where g is the realized graph.

This proposition implies that the closer an agent is to those with different information, the faster they learn. Furthermore, it illustrates homophily effect within the network. Consider the definition of a type. A type of player i is a set of possible graphs from i's perspective. If player i is surrounded by the neighbors whose private type is equal to i's type, *i.e.*, $I_i^1 = I_j^1, \forall j \in N_{i(g)}^{in}$ with a realization of g. This means that all neighbors have the same idea about the true graph or state. Then, observing the in-neighbors' actions may not help for update the information because $I_j^1 = T_{j(g)}^1 = I_i^1 \in B^1(a_j^1(I_j^1 = T_{j(g)}^1)), \forall j \in N_{i(g)}^{in}$, implying that $I_i^2 = I_i^1 \cap \left(\bigcap_{j \in N_{i(g)}^{in}} B_f^1(a_j^1(I_j^1 = T_{j(g)}^1)) \right) = I_i^1$. If my in-neighbors' in-neighobrs also have the same opinion about the true network, the learning time will increase more. This shows that the homophily about the true network will increase the learning time.

Consider the following examples: Figure 3 and 4.[15]

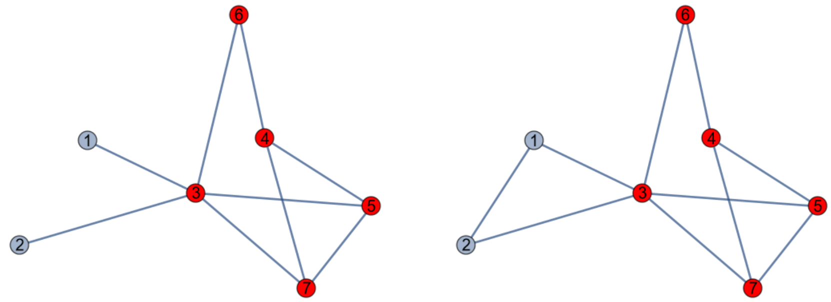


(a) g_a (b) g_b

Figure 3: $\mathcal{G} = \{g_a, g_b\}$ with low homphily

In $\mathcal{G}'$, the nodes in the same homophily group (specifically players 3, 4, 5, 6, and 7) have denser connections than in $\mathcal{G}$, indicating stronger within-group connectivity. Despite this

[15]We consider unweighted and undirected graphs.



(a) g'_a (b) g'_b

Figure 4: $\mathcal{G}' = \{g'_{a,}g'_b\}$ with high homphily

difference, both $\mathcal{G}$ and $\mathcal{G}'$ have the same network diameter of 3. Assume g_a and g'_a are realized under $\mathcal{G}$ and $\mathcal{G}'$ respectively. Players other than 1 and 2 cannot distinguish between g_a and g_b(or g'_a and g'_b) initially, as they lack direct access to distinguishing information. Assuming the actions from different types are different, the learning ends at stage 3 for low homophily group, but the learning time for the high homophily group is 4, where Player 4's learning time is particularly delayed.

This is because due to the high homophily, player 4 relies on a more indirect information flow, needing updates from out-group players like Player 1 or 2 who possess distinguishing information g'_a and g'_b. Due to the high homophily, player 4's in-group connections are less diverse, as the in-neighbors share similar perspectives on the true network with types $I_k^1 = \{g'_a, g'_b\}$ with $k = 3, 4, 5, 6, 7$. Consequently, the path to meaningful information is longer than in the low homophily case.

This result aligns with Golub and Jackson (2012). They showed that under DeGroot learning, the time to convergence increases with homophily. Under Bayesian learning in a network, similar dynamics arise: if player i and their neighbors share perspectives, then i has fewer opportunities to eliminate potential states in I_i^{t+1}. Therefore, under Bayesian learning, we derive a similar intuition to DeGroot learning regarding the relationship between learning time and homophily.

4 Application: Linear-Quadratic Game

In this section, we consider a linear-quadratic payoff function in an incomplete network game (Ballester et al. (2006)), as shown in our example 1. In period t, the incomplete network game possesses a unique Bayesian Nash Equilibrium (BNE), whose characterization has been extensively studied in Chaudhuri et al. (2024). A key insight is that the equilibrium action in the complete network game coincides with the Katz–Bonacich centrality of the network g, that is

$$a_i^c(g) = KB_i(g), \forall i \in N,$$

where $KB_i(g)$ is the Katz-Bonacich centrality of player i under the graph g.

4.1 Equilibrium Characterization

Recalling our main result, we can know that $a_i^{t^*}(I_i^{t^*} = T_{i(g)}^{t^*}) = a_i^c(g) = KB_i(g), \forall i \in N$ when learning ends at t^*. This tells us that, following the end of the learning phase, each agent's equilibrium action equals its Katz–Bonacich centrality. This observation is pivotal because it allows us to pinpoint the moment at which each player perfectly knows the realized network structure.

Proposition 4. *If there exists some $k \geq 1$ such that player i has different k^{th} order walks under g and g', then for almost all values of λ, such that $0 < \lambda < 1/\max\{\rho(g), \rho(g')\}$, where $\rho(g)$ is the spectral radius of g, then the KB centrality of player i is different in those two graphs.*

The intuition behind Proposition 4 is that the equilibrium (Katz–Bonacich centrality) is sensitive to differences in k-order walks for any $k \geq 1$. If within the set $T_{i(g)}^1$ the graphs exhibit different walks at k-length relative to g, then player i can perfectly learn or infer the true network structure. Conversely, if $g, g' \in I_i^{t^*}$ after learning ends, then both graphs have the same k-walks for any $k \geq 1$ for almost all λ. This means that, from player i's perspective, these graphs are indistinguishable based solely on their k-th order walk for any $k \geq 1$.

4.2 Isomorphism Network

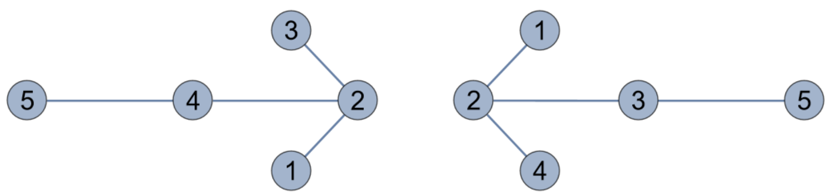
We explore the effects of isomorphic networks on learning time. Let $\mathcal{G}^{iso}$ represent a set of networks that are isomorphic to each other. Define $N_i^d(g)$ as the set of neighbors of player i at distance d with $d \geq 1$. For example, if $i \sim j \sim k$, then $j \in N_i^1(g)$ and $j, k \in N_i^2(g)$.

Definition 7. Player i is said to be in the $(d+1)$-th order isomorphism within a network $g \in \mathcal{G}^{iso}$, if, for any $g' \in I_i^1 = T_{i(g)}^1$, there exists a bijection f that preserves the isomorphism between any two graph g and g' , such that, for any $j \in N_i^d(g)$, $f(j) = j$.[16]

[16] To account for i itself, we refer to this as the $(d+1)$-th order rather than $d-$th order isomorphism. The special case where $f(i) = i$ may correspond to the1-th order isomorphism.

Proposition 5. *If player i be in the n-th order isomorphism within a network $g \in \mathcal{G}^{iso}$ with $n \geq 2$, then player i cannot learn in the game.*

The inability to update arises from the symmetrical structure of the n-th order isomorphism. Player i's new information flows from its neighbors, but in an n-th order isomorphic network, the information available to i's neighbor is symmetric and provides no differentiation for i. Even if player i is in the first order isomorphism with in a network, meaning that $f(i) = i$, neighbors of i can give a new information that may make an update of I_i^1. The following example shows this property.



(a) g_a (b) g_b

Figure 5: $\mathcal{G}^{iso} = \{g_a, g_b\}$

Consider two graphs g_a and g_b, which are isomorphic via the bijection f defined as $f(1) = 1$, $f(2) = 2$, $f(3) = 4$, $f(4) = 3$, and $f(5) = 5$ where $g_{a,ij} = g_{b,f(i)f(j)}$.

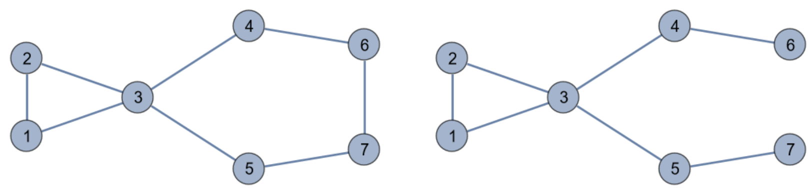
Let g_a be realized. Player 1's initial type is $I_1^1 = T^1_{1(g_a)} = \{g_a, g_b\}$ and player 1 is in the second order isomorphism. This is because the only neighbor of player 1 is 2 and $f(2) = 2$. From the proposition, player 1 cannot learn anything throughout the game due to the symmetry in environment of its neighbor 2.

In contrast, consider player 2 with $I_2^1 = T^1_{2(g_a)} = \{g_a, g_b\}$. Although player 2's initial type is similar to player 1's, player 2 can distinguish g_a and g_b after observing player 3 and 4's actions at the second period. This distinction arises because players 3 and 4 occupy assymetry positions in each graph, leading to different action values based on their respective positions, as established by our lemma.

Also, the following proposition shows how the graph set $\mathcal{G}^{iso}$ determines the learning time for players.

Proposition 6. *Under $\mathcal{G}^{iso}$, the learning time for any player is at most $t = 2$.*

Thus, if a network set $\mathcal{G}$ is constructed by symmetric topology graphs with isomorphism, players can only learn one time or less. Also, we can think that during the learning process, all possible graphs from any player's perspective are isomorphic each other, learning ends



(a) g_a (b) g_b

Figure 6: $\mathcal{G} = \{g_a, g_b\}$

soon. For example, consider a set of core-periphery graphs with k-core players. Since any two graphs in this set are isomorphic each other, we can know that the maximum learning time under this set will be 2. Actually, the periphery players know the exact network at period 1 and the core players can know the true network after observing core and periphery players' actions.

4.3 Nested Split Graph

At first glance, a player's degree might seem crucial for faster learning, as a higher degree indicates more connections and, potentially, greater access to information as we shown in our remark 5. A player with a higher degree is likely to have shorter paths to other players compared to one with a lower degree as we showed in section 3. However, a higher degree does not always ensure faster learning. The following example shows this.

Even if player 3 has the highest degree in both graphs, player 4 and 5 learn the realized graph faster than player 3. Thus, a higher degree does not always ensure faster learning. However, some network structures ensure faster learning time for players who have higher degree than other players.

We investigate how a set of nested split graphs influences learning time and information aggregation. Nested split graphs encompass many common network structures, such as the star network. As their name suggests, these graphs exhibit a nested neighborhood structure, meaning that the set of neighbors for each agent is fully contained within the set of neighbors of every higher-degree agent. This unique property, along with their well-defined topological characteristics and structured adjacency matrices, makes nested split graphs particularly interesting for studying information dynamics.

We demonstrate how a specific type of player, referred to as a core player who has the highest degree under a realized graph g, can leverage its position in the network to enhance learning. Let $\mathcal{G}^{nest}$ be an arbitrary set of nested split graphs with n-players.

Definition 8. We say that a game has no initial information advantage if there is no player i who ends learning at the initial period, *i.e.*, for all $i \in N$, $T^{t^*}_{i(g)} \subset T^1_{i(g)}$ when learning ends at t^* under the realization of g.

The introduction of the definition is crucial because it establishes a baseline condition for the learning process. Without this condition, some players might already possess complete knowledge of the network structure at the outset, eliminating the need for learning altogether. By assuming no initial information advantage, we ensure that all players start with incomplete information and must engage in the learning process. This setup allows us to study how network structure and player positions influence learning dynamics in a meaningful way.

If $T^1_{i(g)}$ is a singleton, player i knows the exact network from the outset, leaving no room for learning. However, when the graph set $\mathcal{G}$ includes numerous and complex graphs, players may initially struggle to distinguish between them. In such cases, the game can be said to have no initial information advantage.

Proposition 7. *Suppose there is no initial information advantage in the game. Under a $\mathcal{G}^{nest}$ set, the learning time of core players is shorter than that of other players, and they achieve perfect learning.*

This proposition highlights the advantage of core players, who benefit from their higher degree and central position in the network. Their privileged location enables them to learn faster and attain perfect information, underscoring the importance of network structure in learning dynamics.

5 Concluding Remark

In this paper, we have examined how rational agents learn and make decisions in networks with incomplete information, addressing a key gap in the literature on Bayesian learning in networked environments. Our findings reveal that, over time, agents' behavior aligns with the predictions of complete information games, thereby demonstrating the robustness of Bayesian learning mechanisms in such settings. By establishing conditions for perfect learning and quantifying learning times, we have provided valuable insights into how network structure—such as isomorphism, topology, and homophily—shapes information diffusion and equilibrium behavior. These results have significant implications for understanding decision-making in complex networks and designing interventions to optimize information flow.

Furthermore, we investigated a linear–quadratic game and demonstrated that a network feature—specifically, the k-th order walks—can facilitate perfect learning and help characterize the relationship between graphs once learning concludes. Also, we have showed the learning processes depending on specific network structures, isomorphism and nested-split graphs.

Future work may extend our model to include more nuanced network features, such as dynamic link formation or agent heterogeneity, to further enrich the analysis of learning in networked environments.

Appendix A: Proofs

Proof of Lemma 1

We prove the result by induction. Start by recalling the definition of $T^t_{i(g)}$

$$T^t_{i(g)} = \begin{cases} \{g' \in \mathcal{G} | g'_{ji} = g_{ji}, \forall j \in N\} & t = 1 \\ T^{t-1}_{i(g)} \cap \left(\bigcap_{j \in N_i(g)} (B^{t-1}_f (a^{t-1}_j (I^{t-1}_j = T^{t-1}_{j(g)}))) \right) & t = 2, 3, 4, \ldots \end{cases}$$

First, consider $T^1_{i(g)}$ and $T^1_{i(g')}$. Note that, at stage 1, types are determined by in-neighborhoods and the weight from these neighbors. That is, if $g, g' \in T^1_{i(g)}, g'_{ji} = g_{ji}, \forall j \in N$. Suppose that $T^1_{i(g)} \neq T^1_{i(g')}$ and there exists g'' such that $g'' \in T^1_{i(g)} \cap T^1_{i(g')}$. Then, it has to be the case that $g''_{ji} = g_{ji}$, for any $j \in N$ since $g'' \in T^1_{i(g)}$. Similarly, $g''_{ji} = g'_{ji}$, for any $j \in N$ since $g'' \in T^1_{i(g')}$. This implies that $g''_{ji} = g_{ji} = g'_{ji}, \forall j \in N$. Thus, if $g_a \in T^1_{i(g)}$, then $g_{a_{ji}} = g_{ji} = g''_{ji} = g'_{ji}, \forall j \in N$, implying $g_{a_{ji}} \in T^1_{i(g')}$. Similarly, if $g_b \in T^1_{i(g')}$, then $g_b \in T^1_{i(g)}$, which implies $T^1_{i(g)} = T^1_{i(g')}$. And this is a contradiction.

Now, assume that either $T^t_{i(g)} = T^t_{i(g')}$ or $T^t_{i(g)} \cap T^t_{i(g')} = \emptyset, \forall i \in N$. If $T^t_{i(g)} \cap T^t_{i(g')} = \emptyset$, then the intersection of $T^{t+1}_{i(g)}$ and $T^{t+1}_{i(g')}$ is also empty since $T^t_{i(g)}$ is non-increasing in $t, \forall g \in \mathcal{G}$.

Now suppose that $T^t_{i(g)} = T^t_{i(g')}$ and that there exists a g'' such that $g'' \in T^{t+1}_{i(g)} \cap T^{t+1}_{i(g')}$ and $T^{t+1}_{i(g)} \neq T^{t+1}_{i(g')}$. Note that the in-neighborhood of i under type $T^t_{i(g)}$ must be the same as its in-neighborhood under $T^t_{i(g')}$ since $T^t_{i(g)} = T^t_{i(g')}$.

Let i and j be connected and suppose that the graph g has be realized by Nature. Note that for any $g_a \in T^{t+1}_{i(g)}$, it has to be the case that $a^t_j(I^t_j = T^t_{j(g_a)}) = a^t_j(I^t_j = T^t_{j(g)})$. If not, such a g_a cannot be in $T^{t+1}_{i(g)}$. Similarly, for any $g_b \in T^{t+1}_{i(g')}$, $a^t_j(I^t_j = T^t_{j(g_b)}) = a^t_j(I^t_j = T^t_{j(g')})$. Since $g'' \in T^{t+1}_{i(g)} \cap T^{t+1}_{i(g')}$, $a^t_j(I^t_j = T^t_{j(g)}) = a^t_j(I^t_j = T^t_{j(g'')}) = a^t_j(I^t_j = T^t_{i(g')})$. Thus, for any $g_a \in T^{t+1}_{i(g)}$, $g_a \in T^{t+1}_{i(g')}$ since $g_a \in T^t_{i(g)} = T^t_{i(g')}$ and it induces the same action as type $T^{t+1}_{i(g')}$. Similarly, if $g_b \in T^{t+1}_{i(g')}$, then $g_b \in T^{t+1}_{i(g)}$, which implies $T^{t+1}_{i(g)} = T^{t+1}_{i(g')}$ and it is a contradiction.

■

Proof of Proposition 1

Recall that for any $i \in N$, any $g \in \mathcal{G}$, and any t we have $g \in T^t_{i(g)}$ so that $T^t_{i(g)}$ is non-empty. Next, from the definition of type updating we have $T^{t+1}_{i(g)} = T^t_{i(g)} \cap \left(\bigcap_{j \in N^{in}_i(g)} B^t_f(a^t_j(I^t_j = T^t_{j(g)})) \right)$ implying that $T^t_{i(g)}$ is non-increasing in t. Moreover, note that since $\mathcal{G}$ is finite, $T^t_{i(g)}$ is also finite. To prove the result, suppose that such a t^* does not exist and consider the infinite sequence defined by player i's dynamic type updates: $\{T^1_{i(g)}, T^2_{i(g)}, \ldots, \}$. Since there is no t^* such that for all $t \geq t^*, T^t_{i(g)} = T^{t^*}_{i(g)}$, and $T^t_{i(g)}$ is non-increasing in t, then there is a strictly decreasing subsequence $\{T^{t_1}_{i(g)}, T^{t_2}_{i(g)}, \dots\}$ satisfying $T^{t_1}_{i(g)} \supset T^{t_2}_{i(g)} \supset \dots$. However, since the cardinality of $\mathcal{G}$ is finite, $T^{t_k}_{i(g)}$ must also be finite and we know that $T^{t_k}_{i(g)}$ is nonempty for any $k \in \mathbb{N}$. Hence, such a subsequence $\{T^{t_k}_{i(g)}\}_{k=1}^{\infty}$ satisfying the decreasing property cannot exist, so there must exist a t^* such that for any $t \geq t^*, T^t_{i(g)} = T^{t^*}_{i(g)}, \forall i \in N, \forall g \in \mathcal{G}$. The same argument applies for players realized types I^t_i.

■

Proof of Lemma 2

Let $g', g'' \in T^t_{i(g)}$ for some $t \geq t^*$ with $g' \neq g''$. Note that the in-neighborhood of i is the same under both g' and g'' since $g', g'' \in T^t_{i(g)}$. Assume that $T^t_{j(g')} \neq T^t_{j(g'')}$ and $a^t_j(I^t_j = T^t_{j(g')}) \neq a^t_j(I^t_j = T^t_{j(g'')})$. Recall that that player i can only observe its in-neighbor j's action, without knowing its true type at t. Without loss of generality, suppose that player i observes $a^t_j(I^t_j = T^t_{j(g'')})$. This implies that $T^t_{j(g')} \notin B^t(a^t_j(I^t_j = T^t_{j(g'')}))$, and hence $g' \notin B^t_f(a^t_j(I^t_j = T^t_{j(g'')}))$ as $g' \in T^t_{j(g')}$. By the definition of $T^{t+1}_{i(g)} = T^t_{i(g)} \cap \left(\bigcap_{j\in N^{in}_i(g)}(B_f(a^t_j(I^t_j = T^t_{j(g)})))\right)$, it follows that $g' \notin T^{t+1}_{i(g)}$. Moreover, since $g' \in T^t_{i(g)}$, then learning can happen, and hence we arrive at a contradiction. On the other hand, if $T^t_{j(g')} = T^t_{j(g'')}$, then $a^t_j(I^t_j = T^t_{j(g')}) = a^t_j(I^t_j = T^t_{j(g'')})$.

■

Proof of Theorem 1

Suppose learning ends at t^* and $t \geq t^*$. Consider a BNE profile $(a^t_i(I^t_i = T^t_{i(g)}))_{i\in N, g\in\mathcal{G}}$. By the locality property and the remark 2, we can rewrite

$$\begin{aligned}
E[u_i|I^t_i = T^t_{i(g)}] &= \sum_{g'\in\mathcal{G}} p(g'|T^t_{i(g)}) u_i(a_i(I^t_i = T^t_{i(g)}), (a_j(I^t_j = T^t_{j(g')}))_{j\in N^{in}_i(g')}, g') \\
&= \sum_{g'\in\mathcal{G}} p(g'|T^t_{i(g)}) u_i(a_i(I^t_i = T^t_{i(g)}), (a_j(I^t_j = T^t_{j(g')}))_{j\in N^{in}_i(g)}, g).
\end{aligned}$$

By lemma 2 we can rewrite

$$\begin{aligned}
&\sum_{g'\in\mathcal{G}} p(g'|T^t_{i(g)}) u_i(a_i(I^t_i = T^t_{i(g)}), (a_j(I^t_j = T^t_{j(g')}))_{j\in N^{in}_i(g)}, g) \\
&= \sum_{g'\in\mathcal{G}} p(g'|T^t_{i(g)}) u_i(a_i(I^t_i = T^t_{i(g)}), (a_j(I^t_j = T^t_{j(g)}))_{j\in N^{in}_i(g)}, g) \\
&= u_i(a_i(I^t_i = T^t_{i(g)}), (a_j(I^t_j = T^t_{j(g)}))_{j\in N_i(g)}, g) \sum_{g'\in\mathcal{G}} p(g'|T^t_{i(g)}) \\
&= u_i(a_i(I^t_i = T^t_{i(g)}), (a_j(I^t_j = T^t_{j(g)}))_{j\in N^{in}_i(g)}, g),
\end{aligned}$$

implying that $(a^t_i(I^t_i = T^t_{i(g)}))_{i\in N}$ is a NE of complete network game under the graph g.

Let g^* be the realized graph. Then, $I_i^t = T_{i(g^*)}^t$ and $a_i^t(I_i^t = T_{i(g^*)}^t)$ is the NE action of player i under the graph g^*. Assume $g \in T_{i(g^*)}^t$, implying $T_{i(g)}^t = T_{i(g^*)}^t$. Thus, $a_i^t(I_i^t = T_{i(g)}^t) = a_i^t(I_i^t = T_{i(g^*)}^t)$.

■

Proof of Theorem 2

Let learning ends at t^*. First, assume that there exists player j of player i's in-neighbor such that $a_j^c(g)$ is all different for any $g \in \mathcal{G}$ with a realization of g. Suppose player i fails to learn perfectly. Then, there exists $g' \neq g$ such that $g' \in I_i^{t^*}$. Since $g' \in I_i^{t^*}$,$a_j^{t^*}(I_j^{t^*} = T_{j(g)}^{t^*}) = a_j^{t^*}(I_j^{t^*} = T_{j(g')}^{t^*})$ by lemma 2. By theorem 1, $a_j^{t^*}(I_j^{t^*} = T_{j(g)}^{t^*}) = a_j^c(g)$ and $a_j^{t^*}(I_j^{t^*} = T_{j(g')}^{t^*}) = a_j^c(g')$. Since $a_j^c(g)$ is all different for any $g \in \mathcal{G}$, $a_j^c(g) \neq a_j^c(g')$, implying a contradiction.

Now, assume that $a_i^c(g)$ is all different for any $g \in \mathcal{G}$. Suppose player i fails to learn perfectly. Then, there exists $g' \neq g$ such that $g' \in I_i^{t^*}$. Then, $T_{i(g)}^{t^*} = T_{i(g')}^{t^*}$ implying $a_i^{t^*}(I_i^{t^*} = T_{i(g)}^{t^*}) = a_i^{t^*}(I_i^{t^*} = T_{i(g')}^{t^*})$. By theorem 1, $a_i^{t^*}(I_i^{t^*} = T_{i(g)}^{t^*}) = a_i^c(g)$ and $a_i^{t^*}(I_i^{t^*} = T_{i(g')}^{t^*}) = a_i^c(g')$. Since $a_i^c(g)$ is all different for any $g \in \mathcal{G}$,$a_i^c(g) \neq a_i^c(g')$, implying a contradiction.

■

Proof of Proposition 2

Let g and g' be isomorphic graphs with a bijective f that makes isomorphism between g and g' with $f(j) = j, \forall j \in N_i(g) \cup \{i\}$. Thus, $g_{ij} = g'_{f(i)f(j)} = g'_{ij}, \forall j \in N_i(g) \cup \{i\}$. Then, $g' \in T_{i(g)}^1$ since $g'_{ij} = g_{ij}, \forall j \in N$. Because we consider all possible graphs and uniform distribution over the graphs, the updating behavior(actions, type sets) under g and g' are the same if we ignore the node labeling. Because of $f(j) = j, \forall j \in N_i(g) \cup \{i\}$ and two graphs are isomorphic, $a_j^t(I_j^t = T_{j(g)}^t) = a_j^t(I_j^t = T_{j(g')}^t)$ for any $j \in N_i(g) \cup \{i\}$ and $t = 1, 2, 3, \ldots$.

■

Proof of Theorem 3

Assume a graph g is realized. Let the diameter of g be α. Without loss of generality, let $\alpha = d(i, j_\alpha)$. So, $i \sim j_1 \sim j_2 \sim \cdots \sim j_{\alpha-1} \sim j_\alpha$. Each j_d, $d = 1, 2, \ldots, \alpha$, represents a player whose distance from i is d.

Note a player's action at stage 1 based on her initial type. After observing $a_{j_1}^1$, player i can update her type using her neighbor's action. Also, player j_1 can update her type observing $a_{j_2}^1$. Thus, we know each player uses his neighbor's information in the first stage. At stage 2, player i observes $a_{j_1}^2$ which reflects j_1's neighbor's information, j_2. Thus, by using $a_{j_1}^2$, i can update her information using her neighbor and the neighbor's neighbor's information. In this way, after stage α, player i uses all the player's information under a graph g. Since we consider an incomplete game, by considering all graphs using $d(\mathcal{G})$, we can know learning ends before $(d(\mathcal{G})+1)$-stage.

■

Proof of Proposition 3

Let the set $E_i(g)$ be empty. Then, there are two possible cases. First, $I_i^1 = \{g\}$, which implies that the learning ends at stage 1. Second, there is a graph $g' \in I_i^1$ but $g' \notin E_i(g)$ with $g' \neq g$. In this case, there are two possibilities. First, player i can distinguish g at some point. Then, the learning time is longer than 1 since $g \in I_i^1$. Second, the player i cannot distinguish g even if learning ends. Then, the learning ends at stage 1 for the player 1. Thus, when $E_i(g)$ is empty, the lower bound for learning time is 1.

Before considering the case where $E_i(g)$ is non-empty, there is a possibility that player i cannot distinguish $g' \in E_i(g)$ even after observing $a_j^t(I_j^t = T_{j(g)}^t)$ even if a neighbor j distinguishes a graph g at t. Consider the following situation. Let $g' \in I_i^1$ and $g' \neq g$. Let $g' \notin I_j^1 = T_{j(g)}^1$ so that $T_{j(g')}^1 \cap T_{j(g)}^1 = \emptyset$. If $B^1(a_j^t(I_j^t = T_{j(g)}^t)) = \{T_{j(g')}^1, T_{j(g)}^1\}$, then $g' \in I_i^2$. This is because the action is not a one-to-one function, so there is a possibility that some types would induce the same equilibrium action. But, in the next stage, if $B^2(a_j^2(I_j^2 = T_{j(g)}^2)) = \{T_{j(g)}^2\}$, then $g \notin I_i^3$, which implies that even if my neighbor can distinguish the graph that I cannot distinguish, the learning time for that graph may take more than 1. Thus, $s_i(g)$ is the minimum learning time regarding the graph $g' \in E_i(g)$ in a general case.

Now, let $E_i(g)$ be non-empty. Let $g' \in E_i(g)$. Then, we can know g' will be excluded during the learning process. As we have seen, $s_i(g)$ is the minimum learning time for the graph g. So, the $max_{g \in E_i(g^*)} s_i(g)$ is the minimum learning time for those excluded graphs.[17]

■

Proof of Proposition 4

Lemma 3. *Consider a matrix* $\mathbf{A}$ *of dimension* $n \times n$ *such that the entries of the matrix are polynomials of degree* 1*, i.e.*

$$\mathbf{A} = \begin{pmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \dots & \dots & \dots \\ a_{n1}(x) & \dots & a_{nn}(x) \end{pmatrix}$$

where $a_{ij}(x) \in P_1$*. Then* $det(\mathbf{A}) = p(x) \in P_n$*.*

[17] As we mentioned, even if my neighbor can distinguish the graph that I cannot distinguish, learning about that graph may not happen in the next stage. The learning time for that graph may take more than 1. That is why we use the minimum learning time.

Proof of Lemma 3

We will prove this by the first principle of mathematical induction. Consider the case for $n = 2$. Then

$$\mathbf{A} = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}$$

and

$$det(\mathbf{A}) = a_{11}(x)\, a_{22}(x) - a_{12}(x)\, a_{21}(x)$$

Since $a_{ij}(x) \in P_1 \Rightarrow a_{ij}(x)\, a_{kl}(x) \in P_2$. Thus, $det(\mathbf{A}) = p(x) \in P_2$.

Denote by $\Delta_{ij}^{\mathbf{A}}$ the $(i,j)^{th}$ minor of $\mathbf{A}$ which is the determinant of the $(n-1)\times(n-1)$ matrix obtained by removing the i^{th} row and j^{th} column of $\mathbf{A}$. Then the determinant of $\mathbf{A}$ can be written as, $det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}(x)\, \Delta_{1j}^{\mathbf{A}}$. Suppose $det(\mathbf{A}) = p(x) \in P_n$ holds true for $n = m$. Next, consider the case for $n = m+1$. Then $\Delta_{ij}^{\mathbf{A}}$ is the determinant of a $m \times m$ matrix whose entries are linear polynomials as well. And hence $\Delta_{ij}^{\mathbf{A}} = p(x) \in P_m$. And since, $a_{ij}(x) \in P_1$ we will have that $a_{ij}(x)\, \Delta_{ij}^{\mathbf{A}} \in P_{m+1} \Rightarrow \sum_{j=1}^{n} (-1)^{1+j} a_{1j}(x)\, \Delta_{1j}^{\mathbf{A}} = det(\mathbf{A}) \in P_{m+1}$. Thus, by the first principle of mathematical induction, $det(\mathbf{A}) = p(x) \in P_n$.

■

Corollary 3. *The Katz-Bonacich centrality of a player i in a connected graph $\mathbf{g}$, can be written as $b_i = \frac{p(\lambda)}{q(\lambda)}$ where, $p(\lambda) \in P_{n-1}$ and $q(\lambda) \in P_n$ and $p(0) = q(0) = 1$.*

Proof of Corollary 3

The Katz-Bonacich centralities of all players are given by

$$\mathbf{b} = (I - \lambda \mathbf{g})^{-1} \cdot \mathbf{1}$$

and hence $b_i = \sum_{j \in N} m_{ij}$ where $\mathbf{M} = \mathbf{L}^{-1} = (I - \lambda \mathbf{g})^{-1}$. We can compute

$$\mathbf{M} = \frac{adj(\mathbf{L})}{det(\mathbf{L})}$$

and, denote by $\Delta_{ij}^{\mathbf{L}}$ the $(i,j)^{th}$ minor of $\mathbf{L}$ which is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and j^{th} column of $\mathbf{L}$. Then $adj(\mathbf{L}) = \left[(-1)^{i+j} \Delta_{ji}^{\mathbf{L}}\right]_{1 \le i,j \le n}$ and $det(\mathbf{L}) = \sum_{j=1}^{n} (-1)^{1+j} l_{1j} \Delta_{1j}^{\mathbf{L}}$. Also,

$$l_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\lambda g_{ij} & \text{otherwise} \end{cases}$$

and thus, $l_{ij} = l_{ij}(\lambda) \in P_1$ for all $1 \le i, j \le n$. Then from the previous lemma, $det(\mathbf{L}) \in P_n$ and $\Delta_{ij}^{\mathbf{L}} \in P_{n-1}$. Also,

$$b_i = \frac{1}{det(\mathbf{L})} \cdot \sum_{j=1}^{n} (-1)^{i+j} \Delta_{ji}^{\mathbf{L}}$$

where $det(\mathbf{L}) \in P_n$ and $\sum_{j=1}^{n} (-1)^{i+j} \Delta_{ji}^{\mathbf{L}} \in P_{n-1}$. Thus, the Katz-Bonacich centrality of a player i can be written as

$$b_i(\lambda) = \frac{p(\lambda)}{q(\lambda)}$$

with $p(\lambda) \in P_{n-1}$ and $q(\lambda) \in P_n$.

■

The Newmann series expansion of the K-B centrality of player i in a realized graph $\mathbf{g}$ is given by

$$\begin{aligned} b_i &= \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_k \in N} \lambda^k g_{ij_1} g_{j_1 j_2} \ldots g_{j_{k-1} j_k} \\ &= \sum_{k=0}^{\infty} w_k^i \lambda^k \end{aligned}$$

where, $w_0^i = 1$ and $w_k^i = \sum_{j_1,\ldots,j_k \in N} g_{ij_1} g_{j_1 j_2} \ldots g_{j_{k-1} j_k}$ for all $k \ge 1$. From the corollary above, $b_i(\lambda) = \frac{p(\lambda)}{q(\lambda)}$ and hence

$$\begin{aligned} \sum_{k=0}^{\infty} w_k^i \lambda^k &= \frac{p(\lambda)}{q(\lambda)} \\ \Rightarrow q(\lambda) \sum_{k=0}^{\infty} w_k^i \lambda^k &= p(\lambda) \end{aligned}$$

Let, $p(\lambda) = \sum_{k=0}^{p} \alpha_k \lambda^k$ and $q(\lambda) = \sum_{k=0}^{q} \beta_k \lambda^k$ with $\alpha_0 = \beta_0 = 1$. Comparing the coefficients of λ^k for any $k \ge 1$ we get

$$\sum_{s=0}^{k} \beta_s w_{k-s}^i = \alpha_k \tag{1}$$

where $\alpha_s = 0$ for all $s > p$ and $\beta_s = 0$ for all $s > q$.

For $0 < \lambda < 1/\max\{\rho(g), \rho(g')\}$, suppose the K-B centrality of player i in two different graphs g and g' be $b_i(\lambda) = p(\lambda)/q(\lambda)$ and $\tilde{b}_i(\lambda) = \tilde{p}(\lambda)/\tilde{q}(\lambda)$ respectively.

Player i will have the same K-B centrality in both the graphs, when $b_i(\lambda) = \tilde{b}_i(\lambda)$, i.e the values of λ for which $p(\lambda)/q(\lambda) = \tilde{p}(\lambda)/\tilde{q}(\lambda)$.

If $deg\left(p\left(\lambda\right)\right) \neq deg\left(\tilde{p}\left(\lambda\right)\right)$ or $deg\left(q\left(\lambda\right)\right) \neq deg\left(\tilde{q}\left(\lambda\right)\right)$ then it would imply that $p\left(\lambda\right) \not\equiv \tilde{p}\left(\lambda\right)$ or $q\left(\lambda\right) \not\equiv \tilde{q}\left(\lambda\right)$ respectively. As a result $p\left(\lambda\right)/q\left(\lambda\right) = \tilde{p}\left(\lambda\right)/\tilde{q}\left(\lambda\right)$ will have at most

$\max\left\{deg\left(p\left(\lambda\right)\right) + deg\left(\tilde{q}\left(\lambda\right)\right), deg\left(\tilde{p}\left(\lambda\right)\right) + deg\left(q\left(\lambda\right)\right)\right\}$ many solutions of λ, which is a null set. And hence, our claim holds true.

Next we consider the case when $deg\left(p\left(\lambda\right)\right) = deg\left(\tilde{p}\left(\lambda\right)\right) = p$ and $deg\left(q\left(\lambda\right)\right) = deg\left(\tilde{q}\left(\lambda\right)\right) = q$. Suppose $w_k \neq \tilde{w}_k$[18] for some $k \geq 1$.

We will prove our claim using the second principle of mathematical induction. Consider the case when $k = 1$. Then $\beta_0, \tilde{\beta}_0$, α_1, β_1 and $\tilde{\alpha}_1, \tilde{\beta}_1$ must satisfy

$$\beta_0 w_1 + \beta_1 = \alpha_1 \Rightarrow w_1 = \frac{\alpha_1 - \beta_1}{\beta_0}$$
$$\tilde{\beta}_0 \tilde{w}_1 + \tilde{\beta}_1 = \tilde{\alpha}_1 \Rightarrow \tilde{w}_1 = \frac{\tilde{\alpha}_1 - \tilde{\beta}_1}{\tilde{\beta}_0}$$

Let $\beta_0 = \tilde{\beta}_0$. Since, $w_1 \neq \tilde{w}_1$ that implies $w_1 - \tilde{w}_1 \neq 0 \Rightarrow (\alpha_1 - \tilde{\alpha}_1) \neq \left(\beta_1 - \tilde{\beta}_1\right)$. Thus, it must be the case that $\alpha_1 \neq \tilde{\alpha}_1$ or $\beta_1 \neq \tilde{\beta}_1$. Which implies that $p\left(\lambda\right) \not\equiv \tilde{p}\left(\lambda\right)$ or $q\left(\lambda\right) \not\equiv \tilde{q}\left(\lambda\right)$. As a result $p\left(\lambda\right)/q\left(\lambda\right) = \tilde{p}\left(\lambda\right)/\tilde{q}\left(\lambda\right)$ will have at most $p + q$ many solutions of λ, which is a null set. And hence, our claim holds true.

Next, let $\beta_0 \neq \tilde{\beta}_0$. Then, $q(\lambda) \neq \tilde{q}(\lambda)$,implying that $p\left(\lambda\right)/q\left(\lambda\right) = \tilde{p}\left(\lambda\right)/\tilde{q}\left(\lambda\right)$ will have at most $p + q$ many solutions of λ, which is a null set.

Next, we assume that the claim holds true for all $1 \leq k \leq m$ for some $m > 1$. Consider $k = m + 1$. Then $\alpha_{m+1}, \tilde{\alpha}_{m+1}$ and $\beta_s, \tilde{\beta}_s$ for all $s \in \{1, 2, \ldots, m\}$ must satisfy

$$\beta_0 w_{m+1} + \sum_{s=1}^{m+1} \beta_s w_{m+1-s} = \alpha_{m+1}$$
$$\tilde{\beta}_0 \tilde{w}_{m+1} + \sum_{s=1}^{m+1} \tilde{\beta}_s \tilde{w}_{m+1-s} = \tilde{\alpha}_{m+1}$$

If there exists at least one $s \in \{1, 2, \ldots, m\}$ such that $w_s \neq \tilde{w}_s$ then by induction hypothesis our claim holds true. Hence, we consider the case when $w_s = \tilde{w}_s$ for all $s \in \{1, 2, \ldots, m\}$.

If $\beta_0 \neq \tilde{\beta}_0$. Then, $q(\lambda) \neq \tilde{q}(\lambda)$,implying that $p\left(\lambda\right)/q\left(\lambda\right) = \tilde{p}\left(\lambda\right)/\tilde{q}\left(\lambda\right)$ will have at most $p+q$ many solutions of λ, which is a null set.

[18] We replace w_k^i with w_k whenever the context is clear.

Now, suppose $\beta_0 = \tilde{\beta}_0$. Since, $w_{m+1} \neq \tilde{w}_{m+1}$ we would have that

$$\begin{aligned}
&\beta_0(w_{m+1} - \tilde{w}_{m+1}) \neq 0 \\
&\Rightarrow (\alpha_{m+1} - \tilde{\alpha}_{m+1}) \neq \sum_{s=1}^{m+1} \left(\beta_s - \tilde{\beta}_s\right) w_{m+1-s}
\end{aligned}$$

It must be the case that $\alpha_{m+1} \neq \tilde{\alpha}_{m+1}$ or $\beta_s \neq \tilde{\beta}_s$ for some $s \in \{1, 2, \ldots, m+1\}$. Which implies that $p(\lambda) \not\equiv \tilde{p}(\lambda)$ or $q(\lambda) \not\equiv \tilde{q}(\lambda)$. As a result $p(\lambda)/q(\lambda) = \tilde{p}(\lambda)/\tilde{q}(\lambda)$ will have at most $p+q$ many solutions of λ, which is a null set. Which in turn implies that, the set of values for which $b_i(\lambda) = \tilde{b}_i(\lambda)$ holds true is a null set. And hence, our claim holds true.

Thus, by the Second Principle of mathematical induction, if there exists some $k \geq 1$ such that player i has different k^{th} order walks under $\mathbf{g}$ and $\mathbf{g}'$, i.e. $w_k \neq \tilde{w}_k$ then for *almost all* values of λ, such that $0 < \lambda < 1/\max\{\rho(g), \rho(g')\}$, the K-B centrality of player i is different in those two graphs, i.e. $b_i(\lambda) \neq \tilde{b}_i(\lambda)$.

■

Proof of Proposition 5

We first show that the second order isomorphism is enough condition for no learning. After then, we show under higher order isomorphism.

Suppose learning can happen for player i. Without loss of generality, let $g' \in I_i^1 = T_{i(g)}^1$ with $g' \neq g$ but $g' \notin I_i^2 = T_{i(g)}^2$. Thus, there exists a player $j \in N_i(g)$ such that $a_j^1(I_j^1 = T_{j(g)}^1) \neq a_j^1(I_j^1 = T_{j(g')}^1)$. Let $KB_j(g)$ be the Katz-Bonacich centrality of player j under a network g. Let f be a bijection that keeps isomorphism between g and g' implying that for any $k, l \in N$, $g_{kl} = g'_{f(k)f(l)}$.

Note that

$$\begin{aligned}
KB_j(g) =& 1 + \lambda \sum_{k \in N} g_{jk} + \lambda^2 \sum_{k \in N} g_{jk} \sum_{l \in N} g_{kl} + \ldots \\
KB_j(g') =& 1 + \lambda \sum_{k \in N} g'_{jk} + \lambda^2 \sum_{k \in N} g'_{jk} \sum_{l \in N} g'_{kl} + \ldots
\end{aligned}$$

Since $g_{jk} = g'_{f(j)f(k)} = g'_{jf(k)}, \forall k \in N$, with $f(j) = j$, $\sum_{k \in N} g_{jk} = \sum_{k \in N} g'_{f(j)f(k)} = \sum_{k \in N} g'_{jf(k)} = \sum_{k \in N} g'_{jk}$. The last equation is from the fact that f is a bijection.

Note that

$$\sum_{k \in N} g'_{jk} \sum_{l \in N} g'_{kl} = \sum_{f(k) \in N} g'_{jf(k)} \sum_{f(l) \in N} g'_{f(k)f(l)} = \sum_{k \in N} g_{jk} \sum_{l \in N} g_{kl}.$$

Repeating this way, we can know that $KB_j(g) = KB_j(g')$ when $f(j) = j$.

The following lemma shows that $a_j^t(I_j^t = T_{j(g)}^t) = KB_j(g)$ under $\mathcal{G}^{iso}$ for any $t = 1, 2, 3, \dots$.

Lemma 4. *Under $\mathcal{G}^{iso}$, $a_j^t(I_j^t = T_{j(g)}^t) = KB_j(g)$ for any $j \in N$ with $t = 1, 2, 3, \dots$.*

Proof of Lemma 4

Note that when $g' \in I_j^t = T_{j(g)}^t$, by the definition of a type set, $N_j(g) = N_j(g')$, implying $g_{jk} = g'_{jk}, \forall k \in N$. Let $f_{g'}$ be a bijection between g and g' for any $g' \in \mathcal{G}^{iso}$ with $g_{ij} = g'_{f_{g'}(i) f_{g'}(j)}$

Note that from the FOC of player j,

$$\begin{aligned}
a_j^t(I_j^t = T_{j(g)}^t) =& 1 + \lambda \sum_{g' \in \mathcal{G}^{iso}} \sum_{k \in N} g'_{jk} a_k^t(I_k^t = T_{k(g')}^t) p(g' | I_j^t = T_{j(g)}^t) \\
=& 1 + \lambda \sum_{k \in N} g_{jk} \sum_{g' \in \mathcal{G}^{iso}} a_k^t(I_k^t = T_{k(g')}^t) p(g' | I_j^t = T_{j(g)}^t).
\end{aligned}$$

By using the FOC of $a_k^t(I_k^t = T_{k(g')}^t) = 1 + \lambda \sum_{g'' \in \mathcal{G}^{iso}} \sum_{l \in N} g''_{kl} a_l^t(I_l^t = T_{l(g'')}^t) p(g'' | I_k^t = T_{k(g')}^t)$,

$$\begin{aligned}
& 1 + \lambda \sum_{k \in N} g_{jk} \sum_{g' \in \mathcal{G}^{iso}} a_k^t(I_k^t = T_{k(g')}^t) p(g' | I_j^t = T_{j(g)}^t) \\
=& 1 + \lambda \sum_{k \in N} g_{jk} + \lambda^2 \sum_{k \in N} g_{jk} \sum_{g' \in \mathcal{G}^{iso}} \sum_{g'' \in \mathcal{G}^{iso}} \sum_{l \in N} g''_{kl} a_l^t(I_l^t = T_{l(g'')}^t) p(g'' | I_k^t = T_{k(g')}^t) p(g' | I_j^t = T_{j(g)}^t) \\
=& 1 + \lambda \sum_{k \in N} g_{jk} + \lambda^2 \sum_{k \in N} g_{jk} \sum_{g' \in \mathcal{G}^{iso}} \sum_{g'' \in \mathcal{G}^{iso}} \sum_{l \in N} g''_{kl} p(g'' | I_k^t = T_{k(g')}^t) p(g' | I_j^t = T_{j(g)}^t) + \dots \\
=& 1 + \lambda \sum_{k \in N} g_{jk} + \lambda^2 \sum_{k \in N} g_{j f_{g''}(k)} \sum_{g' \in \mathcal{G}^{iso}} \sum_{g'' \in \mathcal{G}^{iso}} \sum_{l \in N} g''_{f_{g''}(k) f_{g''}(l)} p(g'' | I_{f_{g''}(k)(g')}^t = T_{f_{g''}(k)(g')}^t) p(g' | I_j^t = T_{j(g)}^t) + \dots \\
=& 1 + \lambda \sum_{k \in N} g_{jk} + \lambda^2 \sum_{k \in N} g_{jk} \sum_{g' \in \mathcal{G}^{iso}} \sum_{g'' \in \mathcal{G}^{iso}} \sum_{l \in N} g_{kl} p(g'' | I_k^t = T_{k(g')}^t) p(g' | I_j^t = T_{j(g)}^t) + \dots \\
=& 1 + \lambda \sum_{k \in N} g_{jk} + \lambda^2 \sum_{k \in N} g_{jk} \sum_{l \in N} g_{kl} \sum_{g' \in \mathcal{G}^{iso}} p(g' | I_j^t = T_{j(g)}^t) \sum_{g'' \in \mathcal{G}^{iso}} p(g'' | I_k^t = T_{k(g')}^t) + \dots \\
=& 1 + \lambda \sum_{k \in N} g_{jk} + \lambda^2 \sum_{k \in N} g_{jk} \sum_{l \in N} g_{kl} + \dots.
\end{aligned}$$

By repeating this process, we can get the result. We just relabel indices of g and g'. This is possible because f is a bijection and the summation term consider all players. Repeating this process, we can get the result.

■

Following the lemma, $a_j^1(I_j^1 = T_{j(g)}^1) = KB_j(g)$ and $a_j^1(I_j^1 = T_{j(g')}^1) = KB_j(g')$. Since $a_j^1(I_j^1 = T_{j(g)}^1) = KB_j(g) = a_j^1(I_j^1 = T_{j(g')}^1) = KB_j(g')$ under $f(j) = j$, contradiction happens.

Thus, under the second order isomorphism, player i cannot learn anything. Under the n-th order isomorphism with $n > 2$, the second order isomorphism is still applied so that player i cannot still make an update.

■

Proof of Proposition 6

Let $g \in \mathcal{G}^{iso}$ be realized. Consider $T_{i(g)}^1$. First, if $T_{i(g)}^1$ is a singleton, learning ends at period 1. Now, suppose there exists $g' \neq g$ in $T_{i(g)}^1$. Let $j \in N_i(g)$. If $N_i(g)$ is empty, player i cannot update anymore because there is no resource for the information. By the lemma, $a_j^1(I_j^1 = T_{j(g)}^1) = KB_j(g)$ and $a_j^1(I_j^1 = T_{j(g')}^1) = KB_j(g')$. If $KB_j(g) \neq KB_j(g')$, $g' \notin T_{i(g)}^2$,implying learning can happen. Let $g'' \in T_{i(g)}^2$ with $g'' \neq g'$ and $g'' \neq g$. Since $g'' \in T_{i(g)}^2$, for any $j \in N_{i(g)}$, $KB_j(g) = KB_j(g'')$. Since $KB_j(g'') = a_j^t(I_j^t = T_{j(g'')}^t)$ with $t = 1, 2, \ldots,$ $a_j^t(I_j^t = T_{j(g'')}^t) = a_j^t(I_j^t = T_{j(g)}^t)$ for any t, implying player i cannot distinguish g'' in the game. Thus, learning cannot happen after period 2.

■

Proof of Proposition 7

Note that a core player in $\mathcal{G}^{nest}$ are connected to all other players except itself. Recall that under theorem 3 and Corollary 3, we show that, at the first period, each player observes and extracts all possible information from the neighbors. Since the core player is connected to all other players except itself, after observing the first period actions of neighbors, the core player can extract all possible information, implying learning ends at the second period. Since there is no other players who learn at the initial stage, the core player learns faster than other players.

Now, let's consider perfect learning. Suppose a core player i cannot achieve perfect learning. Let $g \in \mathcal{G}^{nest}$ be realized and $g' \in I_i^t$, where $g' \neq g$ for any $t = 1, 2, \ldots.$ Since player i is connected to all other players, $a_j^t(I_j^t = T_{j(g)}^t) = a_j^t(I_j^t = T_{j(g')}^t), \forall j \in N, \forall t = 1, 2, \ldots.$[19] If not, player i can exclude g' at some point. Since, after learning ends, $a_j^t(I_j^t = T_{j(g)}^t) = a_j^c(g), \forall j \in N, \forall g \in \mathcal{G}$, we can know that $a_j^c(g) = a_j^c(g'), \forall j \in N$, implying $KB_j(g) = KB_j(g')$.

From Konig et al., (2014) proposition 1, under a nested split graph, $KB_j(g) > KB_l(g)$ if and only if $d_j(g) > d_l(g)$,where $d_j(g)$ is the degree of j under the graph g. Suppose

[19] Since $g' \in I_i^t = T_{i(g)}^t$, $T_{i(g)}^t = T_{i(g')}^t$, implying $a_i^t(I_i^t = T_{i(g)}^t) = a_i^t(I_i^t = T_{i(g')}^t)$.

$d_j(g) > d_l(g)$ for any $j, l \in N$. Then, $KB_j(g) > KB_l(g)$, implying $KB_j(g') > KB_l(g')$. Thus, $d_j(g') > d_l(g')$. So, degree ordering between players under g is still preserved under g' and viceversa for any players.

Let the graph g distinct degrees as follows $d_{(1)}(g) < d_{(2)}(g) < \cdots < d_{(k)}(g)$. Define $D_i(g) = \{n \in N | d_n(g) = d_{(i)}(g)\}$. Then, the set-valued vector $D(g) = \{D_1(g), D_2(g), \ldots, D_k(g)\}$ is called the degree partition of g (Konig et al., (2014)). Let the graph g' also distinct degrees as follows $d_{(1)}(g') < d_{(2)}(g') < \cdots < d_{(k')}(g')$ and $D(g') = \{D_1(g'), D_2(g'), \ldots, D_{k'}(g')\}$. Since the ordering is preserved, the number of partitions are the same. Thus, we can rewrite $k' = k$.

Without loss of generality, let $l \in D_i(g)$. Then $d_k(g) > \cdots > d_{(i+1)}(g) > d_l(g) > d_{(i-1)}(g) > \cdots > d_{(1)}(g)$. Then, $d_k(g') > \cdots > d_{(i+1)}(g') > d_l(g') > d_{(i-1)}(g') > \cdots > d_{(1)}(g')$. Thus, $D_i(g) = D_i(g'), \forall i = 1, 2, \ldots, k$, implying $D(g) = D(g')$. Considering the definition and construction of a nested split graph form (Mahadev and Peled, 1995), $g = g'$.

■

Appendix B: Bayesian Learning vs Myopic Learning

In this section, we compare our Bayesian learning model with a myopic learning model under a linear–quadratic framework. Consider n-players.

Definition 9. Let g be the realized graph and a_i^1 be the initial value for player i. Myopic learning is defined by the recursive process:

$$a_i^t = 1 + \lambda \sum_{j=1} g_{ij} a_j^{t-1}, \quad t = 2, 3, \dots,$$

which can be expressed in vector form as:

$$\mathbf{a}^t = \mathbf{1}_n + \lambda \mathbf{g} \mathbf{a}^{t-1},$$

where $\mathbf{a}^t = (a_1^t, \dots, a_n^t)^T$. With the initial condition $\mathbf{a}^1 = \mathbf{1}_n$, iterating the process yields:

$$\mathbf{a}^t = \mathbf{1}_n + \lambda \mathbf{g} \mathbf{a}^{t-1} = \mathbf{1}_n + \lambda \mathbf{g} \mathbf{1}_n + \lambda^2 \mathbf{g}^2 \mathbf{1}_n + \cdots + \lambda^{t-1} \mathbf{g}^{t-1} \mathbf{1}_n.$$

Thus,

$$\lim_{t\to\infty} \mathbf{a}^t = \mathbf{1}_n + \lambda \mathbf{g} \mathbf{1}_n + \lambda^2 \mathbf{g}^2 \mathbf{1}_n + \cdots = (\mathbf{I} - \lambda \mathbf{g})^{-1} \mathbf{1}_n = \mathbf{a}^c = \mathbf{KB},$$

where $\mathbf{a}^c = (a_1^c(g), \dots, a_n^c(g))$ denotes the equilibrium under complete information, and $\mathbf{KB}$ is the vector of Katz–Bonacich centralities.

Recall that in the linear-quadratic model, after learning ends at t^*, $a_i^t(I_i^t = T_{i(g)}^t) = a_i^c(g) = KB_i(g)$, for any $t \geq t^*$. Thus, if time goes to infinity, both our Bayesian learning and myopic learning yield the same equilibrium outcome. However, notable differences exist in the convergence process.

To differentiate our Bayesian learning from myopic learning, denote the equilibrium action of our model as $\mathbf{a}^t = (a_1^t(I_1^t = T_{1(g)}^t), \dots, a_1^t(I_n^t = T_{n(g)}^t))$, the equilibrium of myopic learning as $\mathbf{b}^t$, and the complete learning as $\mathbf{a}^c$.

First, compare the residuals. For any t, $\mathbf{a}^c - \mathbf{b}^t$ is

$$\mathbf{a}^c - \mathbf{b}^t = \lambda^t \mathbf{g}^t \mathbf{1}_n + \lambda^{t+1} \mathbf{g}^{t+1} \mathbf{1}_n + \cdots = \lambda^t \mathbf{g}^t (\mathbf{I} - \lambda \mathbf{g})^{-1} \mathbf{1}_n.$$

In contrast, for our Bayesian model, our theorem 3 shows that learning is complete before

$t \leq n$:

$$\mathbf{a}^c - \mathbf{a}^n = \mathbf{0}_n.$$

Hence, at $t = n$ the myopic learning residual is

$$\mathbf{a}^c - \mathbf{b}^n = \lambda^n \mathbf{g}^n (\mathbf{I} - \lambda \mathbf{g})^{-1} \mathbf{1}_n,$$

while for Bayesian learning the residual is zero.

Since after the learning stage, Bayesian learning exactly recovers the complete-information equilibrium, it does not rely on an asymptotic process. In fact, the upper bound for the learning time in the Bayesian model is always lower than n and independent of the convergence threshold ϵ.

Remark 6. For any $\epsilon < \|(\lambda \mathbf{g})^n (\mathbf{I} - \lambda \mathbf{g})^{-1} \mathbf{1}_n\|_2$, $\mathbf{a}^t$ converges to $\mathbf{a}^c$ faster than $\mathbf{b}^t$.

In other words, as we require a more precise approximation of the equilibrium(for smaller ϵ) the convergence time $t(\epsilon)$ for myopic learning increases, whereas Bayesian learning converges in a fixed number of steps. Thus, for any given error threshold, the equilibrium action under Bayesian learning is achieved more rapidly, reaching the complete information outcome by stage $t = n$.

The graphs below compare convergence times for both learning processes. We generate Erods-Renyi random graphs with varying the number of nodes and edge probability p. For each n, we compute the diameter of the largest connected component and the convergence time for the myopic learning algorithm using $\lambda = \frac{1}{2n-2}$ and the error threshold $\epsilon = 10^{-3}$. When we calculate the convergence time, we use the 2-norm and find the smallest iteration count t such that this norm is less than a predefined error threshold.

This process is repeated 1000 times for each n, and we record the average diameter and convergence time. Since the diameter plus one is an upper bound for the learning time in our Bayesian model, the graphs below display both the average diameter (plus one) and the average convergence time for myopic learning. In the graphs, the circled line represents the average diameter plus one, and the squared line represents the average convergence time.

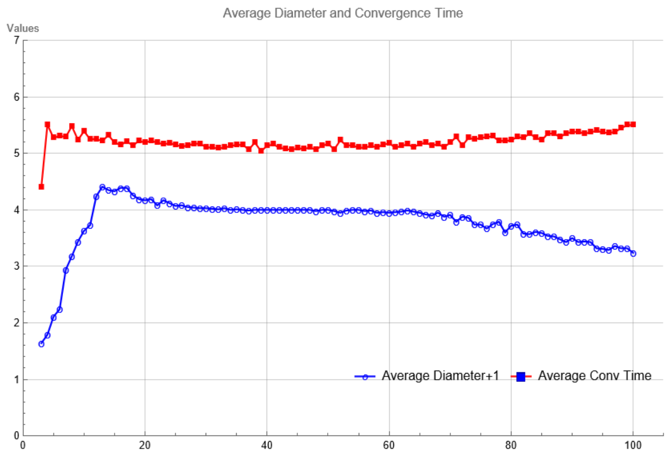


Figure 7: Average Diameter and Average Convergence Time for each n when p=0.3

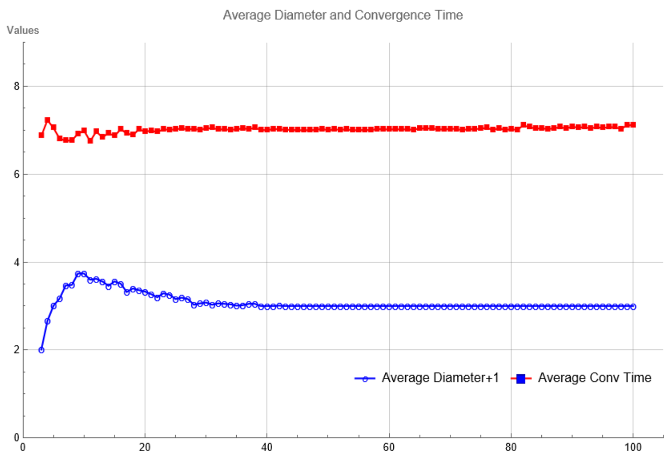


Figure 8: Average Diameter and Average Convergence Time for each n when p=0.5

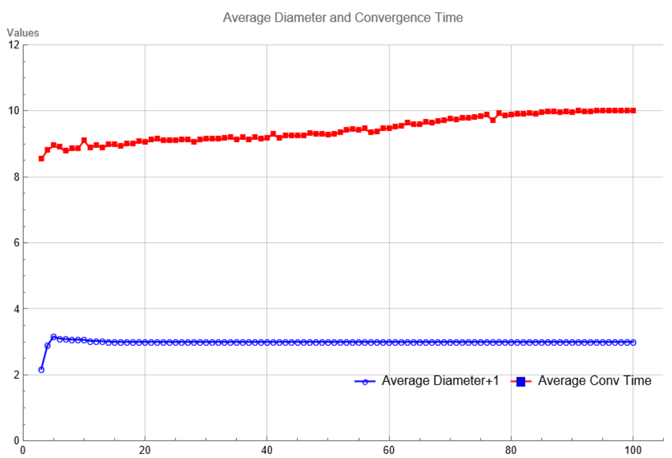


Figure 9: Average Diameter and Average Convergence Time for each n when p=0.7

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