Control Automático

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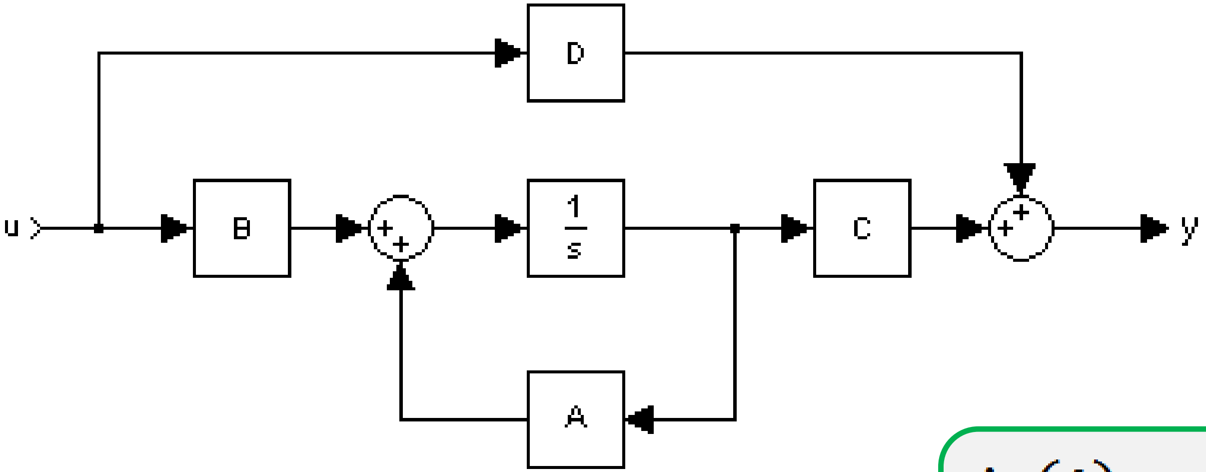
Sesión 2: Modelado
Espacio de Estados

Espacio de Estados



$$\dot{X} = f(x, t, u)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Donde ***x(t)*** *es el vector de estado*

u(t) es el vector de **entrada** de orden r x 1

y(t) el vector de **salida** de orden px1

A: **Matriz característica del sistema** de orden n x n .

B*:* **Matriz de entrada** de orden n x r

C*:* **Matriz de salida** de orden p x n

D*:* **Matriz de transferencia directa entrada- salida**, de orden p x r

Espacio de Estados Canónico

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2 y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt} = \dot{y}(t)$$

$$x_3(t) = \frac{d^2 y(t)}{dt^2} = \ddot{y}(t)$$

$$\vdots$$

$$x_{n-1}(t) = \frac{d^{n-2} y(t)}{dt^{n-2}} = \overset{n-2}{y}(t)$$

$$x_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}} = \overset{n-1}{y}(t)$$

$$\mathbf{x(t)} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}$$

Espacio de Estados Canónico

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

$$\dot{x}_1(t) = \frac{dy(t)}{dt} = \dot{y}(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{d^2y(t)}{dt^2} = \ddot{y}(t) = x_3(t)$$

$$\dot{x}_3(t) = \frac{d^3y(t)}{dt^3} = x_4(t)$$

$$\vdots$$

$$\dot{x}_{n-1}(t) = \frac{d^{n-1}y(t)}{dt^{n-1}} = x_n(t)$$

$$\dot{x}_n(t) = \frac{d^n y(t)}{dt^n}$$

$$\dot{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix}$$

Espacio de Estados Canónico

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

Conjunto de ecuaciones de primer orden

$$\dot{x}_1(t) = \frac{dy(t)}{dt} = \dot{y}(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{d^2y(t)}{dt^2} = \ddot{y}(t) = x_3(t)$$

$$\dot{x}_3(t) = \frac{d^3y(t)}{dt^3} = x_4(t)$$

$$\vdots$$

$$\dot{x}_{n-1}(t) = \frac{d^{n-1}y(t)}{dt^{n-1}} = x_n(t)$$

$$\dot{x}_n(t) = \frac{d^n y(t)}{dt^n} = -a_0x_1(t) - a_1x_2(t) - a_2x_3(t) - \cdots - a_{n-1}x_n(t) + u(t)$$

Espacio de Estados Canónico

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2 y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

Conjunto de ecuaciones de primer orden expresadas en forma matricial

$$\dot{\mathbf{x}}(\mathbf{t}) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

La ecuación de salida sería: $y(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathbf{x}(\mathbf{t})$

Espacio de Estados Canónico

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\mathbf{x(t)} = \begin{bmatrix} \mathit{0} & I \\ & a \end{bmatrix}\mathbf{x(t)} + \begin{bmatrix} \mathit{0} \\ 1 \end{bmatrix}\mathbf{u(t)}$$
$$\mathbf{y(t)} = \begin{bmatrix} 1 & \mathit{0} \end{bmatrix}\mathbf{x(t)}$$

0 = vector de ceros de orden o , según sea el caso.
I = Matriz identidad de orden n-1
a = Vector de coeficientes de la ecuación diferencial de orden

Espacio de Estados Canónico

$$\dot{x}_1(t) = \frac{dy(t)}{dt} = \dot{y}(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{d^2y(t)}{dt^2} = \ddot{y}(t) = x_3(t)$$

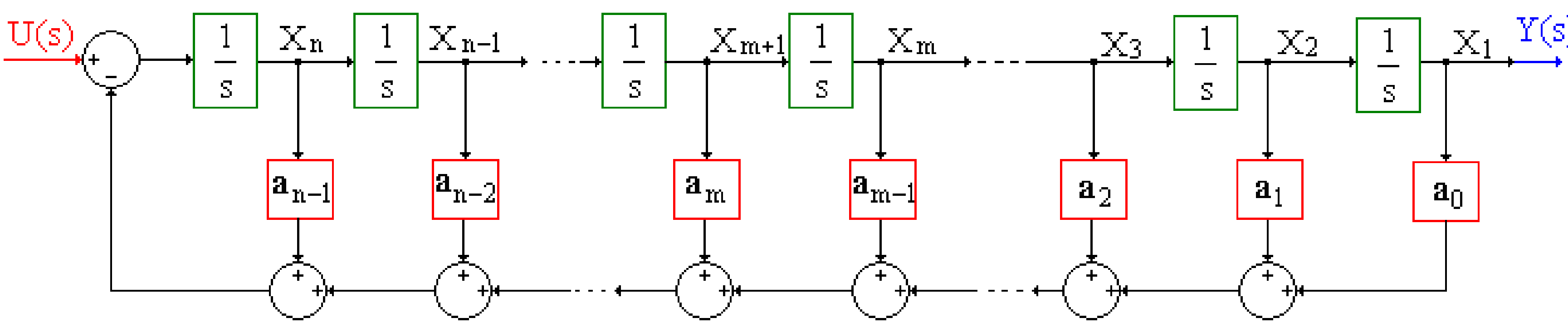
$$\dot{x}_3(t) = \frac{d^3y(t)}{dt^3} = x_4(t)$$

$$\vdots$$

$$\dot{x}_{n-1}(t) = \frac{d^{n-1}y(t)}{dt^{n-1}} = x_n(t)$$

$$\dot{x}_n(t) = \frac{d^ny(t)}{dt^n} = -a_0x_1(t) - a_1x_2(t) - a_2x_3(t) - \cdots - a_{n-1}x_n(t) + u(t)$$

$$\boxed{\frac{d^ny(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = u(t)}$$



Espacio de Estados Canónico

Con derivadas de la entrada

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = c_m\frac{d^m u(t)}{dt^m} + c_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + c_1\frac{du(t)}{dt} + c_0$$

$$U(s) \longrightarrow \boxed{\frac{c_m s^m + c_{m-1}s^{m-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0}} \longrightarrow Y(s)$$

a) Función de transferencia original

$$U(s) \longrightarrow \boxed{\frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_2 s^2 + a_1 s + a_0}} \xrightarrow{Z(s)} \boxed{c_m s^m + c_{m-1}s^{m-1} + \cdots + c_1 s + c_0} \longrightarrow Y(s)$$

b) Diagrama de bloques equivalente

variable auxiliar Z(s).

Espacio de Estados Canónico

$$x_1(t) = z(t)$$

$$x_2(t) = \frac{dz(t)}{dt} = \dot{z}(t)$$

$$x_3(t) = \frac{d^2y(t)}{dt^2} = \ddot{z}(t)$$

$$\vdots$$

$$x_m(t) = \frac{d^{m-1}z(t)}{dt^{m-1}} = \overset{m-1}{z}(t)$$

$$x_{m+1}(t) = \frac{d^{m}z(t)}{dt^{m}} = \overset{m}{z}(t)$$

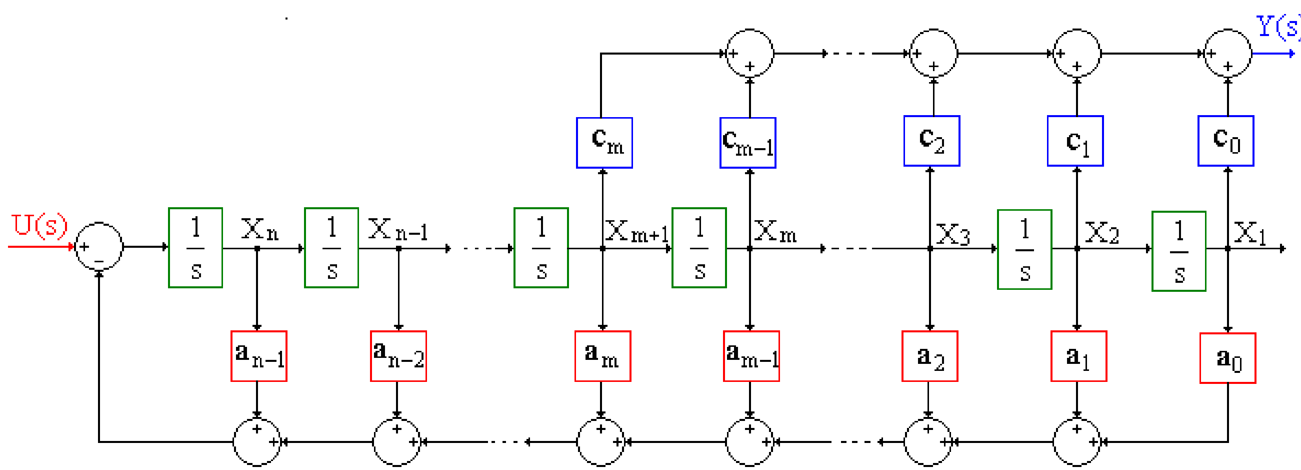
$$\vdots$$

$$x_{n-1}(t) = \frac{d^{n-2}z(t)}{dt^{n-2}} = \overset{n-2}{z}(t)$$

$$x_n(t) = \frac{d^{n-1}z(t)}{dt^{n-1}} = \overset{n-1}{z}(t)$$

$$\frac{Y(s)}{Z(s)} = c_m s^m + c_{m-1}s^{m-1} + \cdots + c_1 s + c_0$$

$$y(t) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{m-1} & c_m & \cdots & 0 & 0 \end{bmatrix} \mathbf{x(t)}$$



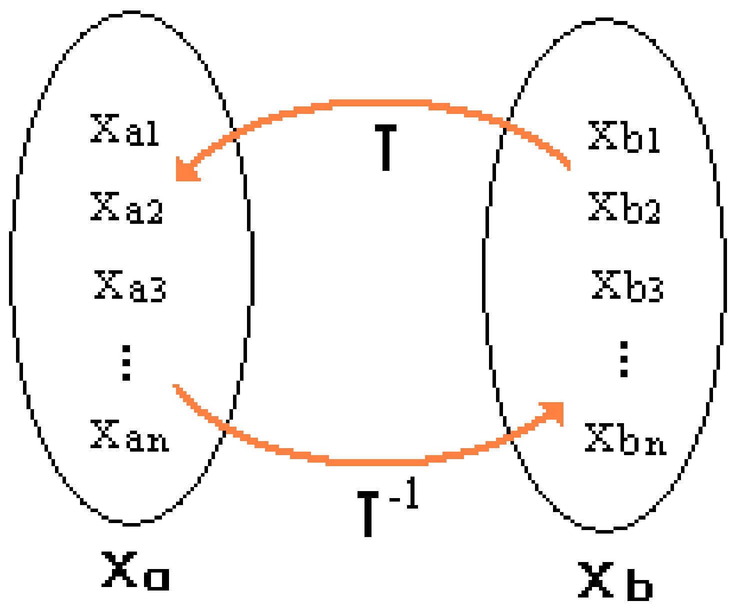
Espacio de Estados con variables Físicas
Motor D.C
Tanques en serie

Transformación entre Espacio de Estados

$$\dot{\mathbf{x}}_a = \mathbf{A}_a \mathbf{x}_a + \mathbf{B}_a u(t)$$
$$y(t) = \mathbf{C}_a \mathbf{x}_a$$

$$\dot{\mathbf{x}}_b = \mathbf{A}_b \mathbf{x}_b + \mathbf{B}_b u(t)$$
$$y(t) = \mathbf{C}_b \mathbf{x}_b$$

$$\mathbf{x}_a = \mathbf{T}\mathbf{x}_b$$



$$\mathbf{x}_b = \mathbf{T}^{-1}\mathbf{x}_a$$

Transformación entre Espacio de Estados

y

$$\dot{\mathbf{x}}_a = \mathbf{A}_a \mathbf{x}_a + \mathbf{B}_a u(t)$$
$$y(t) = \mathbf{C}_a \mathbf{x}_a$$

$$\dot{\mathbf{x}}_b = \mathbf{A}_b \mathbf{x}_b + \mathbf{B}_b u(t)$$
$$y(t) = \mathbf{C}_b \mathbf{x}_b$$

$$\boxed{\mathbf{x}_a = \mathbf{T}\mathbf{x}_b}$$

$$\mathbf{T}\dot{\mathbf{x}}_b = \mathbf{A}_a \mathbf{T}\mathbf{x}_b + \mathbf{B}_a u(t)$$
$$y(t) = \mathbf{C}_a \mathbf{T}\mathbf{x}_b$$

$$\dot{\mathbf{x}}_b = \boxed{\mathbf{T}^{-1}\mathbf{A}_a \mathbf{T}}\mathbf{x}_b + \boxed{\mathbf{T}^{-1}\mathbf{B}_a}u(t)$$
$$y(t) = \boxed{\mathbf{C}_a \mathbf{T}}\mathbf{x}_b$$

$$\dot{\mathbf{x}}_b = \boxed{\mathbf{A}_b}\mathbf{x}_b + \boxed{\mathbf{B}_b}u(t)$$
$$y(t) = \boxed{\mathbf{C}_b}\mathbf{x}_b$$

Transformación entre Espacio de Estados

y

$$\dot{\mathbf{x}}_b = \boxed{\mathbf{T}^{-1}\mathbf{A}_a\mathbf{T}}\mathbf{x}_b + \boxed{\mathbf{T}^{-1}\mathbf{B}_a}u(t)$$

$$y(t) = \boxed{\mathbf{C}_a\mathbf{T}}\mathbf{x}_b$$

$$\dot{\mathbf{x}}_b = \boxed{\mathbf{A}_b}\mathbf{x}_b + \boxed{\mathbf{B}_b}u(t)$$

$$y(t) = \boxed{\mathbf{C}_b}\mathbf{x}_b$$

$$\boxed{\begin{aligned}\mathbf{A}_a\mathbf{T} &= \mathbf{T}\mathbf{A}_b\\ \mathbf{B}_a &= \mathbf{T}\mathbf{B}_b\\ \mathbf{C}_a\mathbf{T} &= \mathbf{C}_b\end{aligned}}$$

Ejemplo 1

y

$$\dot{\mathbf{x}}_c = \begin{bmatrix} 0 & 1 \\ 15 & 2 \end{bmatrix} \mathbf{x}_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad \dot{\mathbf{x}}_f = \begin{bmatrix} -3 & 0 \\ 1 & 5 \end{bmatrix} \mathbf{x}_f + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 4 & 0 \end{bmatrix} \mathbf{x}_c \qquad y(t) = \begin{bmatrix} 0 & 4 \end{bmatrix} \mathbf{x}_f$$

$$\mathbf{A_c T} = \mathbf{T A_f} \qquad \mathbf{B_c} = \mathbf{T B_f} \qquad \mathbf{C_c T} = \mathbf{C_f} \qquad \mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

$$\mathbf{A_c T} = \mathbf{T A_f} \therefore \begin{bmatrix} 0 & 1 \\ 15 & 2 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 1 & 5 \end{bmatrix}$$

$$\mathbf{B_c} = \mathbf{T B_f} \therefore \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{C_c T} = \mathbf{C_f} \therefore \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$$

Ejemplo 1

y

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \qquad \mathbf{x_c} = \mathbf{T}\mathbf{x_f}$$

$$\mathbf{A_c T} = \mathbf{T A_f}$$

$$\mathbf{B_c} = \mathbf{T B_f}$$

$$\mathbf{C_c T} = \mathbf{C_f}$$

$$\begin{bmatrix} 4t_{11} & 4t_{12} \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & t_{22} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 15 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & t_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & t_{22} \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t_{22} \\ 2 & 15+2t_{22} \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -3+t_{22} & 5t_{22} \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$$

Del E.E. a la Función de Transferencia

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\mathcal{L}[\dot{x}(t)] = \mathcal{L}[Ax(t) + Bu(t)]$$
$$sI\,X(s) = AX(s) + BU(s)$$
$$sI\,X(s) - AX(s) = BU(s)$$
$$[sI - A]X(s) = BU(s)$$

$$[sI - A]^{-1}[sI - A]X(s) = [sI - A]^{-1}BU(s)$$

$$X(s) = [sI - A]^{-1}BU(s)$$

$$\mathcal{L}[y(t)] = \mathcal{L}[Cx(t) + Du(t)]$$

$$Y(s) = CX(s) + DU(s)$$

$$Y(s) = C[sI - A]^{-1}BU(s) + DU(s)$$

$$Y(s) = \left[C[sI - A]^{-1}B + D\right]U(s)$$

$$\boxed{M(s) = C[sI - A]^{-1}B + D}$$