Chapter 7

Block Diagram Algebra and Transfer Functions of Systems

7.1 INTRODUCTION

It is pointed out in Chapters 1 and 2 that the block diagram is a shorthand, graphical representation of a physical system, illustrating the functional relationships among its components. This latter feature permits evaluation of the contributions of the individual elements to the overall performance of the system.

In this chapter we first investigate these relationships in more detail, utilizing the frequency domain and transfer function concepts developed in preceding chapters. Then we develop methods for reducing complicated block diagrams to manageable forms so that they may be used to predict the overall performance of a system.

7.2 REVIEW OF FUNDAMENTALS

In general, a block diagram consists of a specific configuration of four types of elements: blocks, summing points, takeoff points, and arrows representing unidirectional signal flow:

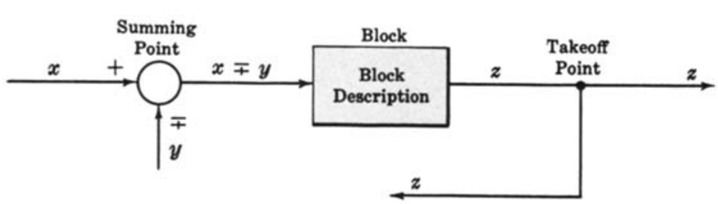


Fig. 7-1

The meaning of each element should be clear from Fig. 7-1.

Time-domain quantities are represented by lowercase letters.

EXAMPLE 7.1. $r = r(t)$ for continuous signals, and $r(t_k)$ or $r(k)$, $k = 1, 2, \ldots,$ for discrete-time signals.

Capital letters in this chapter are used for Laplace transforms, or z-transforms. The argument s or z is often suppressed, to simplify the notation, if the context is clear, or if the results presented are the same for both Laplace (continuous-time system) and z-(discrete-time system)transfer function domains.

EXAMPLE 7.2. $R = R(s)$ or $R = R(z)$.

The basic feedback control system configuration presented in Chapter 2 is reproduced in Fig. 7-2, with all quantities in abbreviated transform notation.

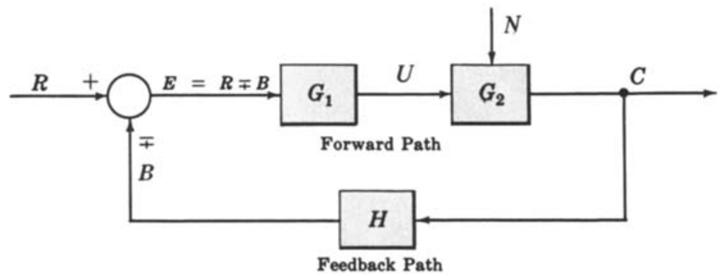


Fig. 7-2

The quantities G_1, G_2, and H are the transfer functions of the components in the blocks. They may be either Laplace or z-transform transfer functions.

EXAMPLE 7.3. $G_1 = U/E$ or $U = G_1 E$.

It is important to note that these results apply *either* to Laplace transform *or* to z-transform transfer functions, but not necessarily to *mixed* continuous/discrete block diagrams that include *samplers*. Samplers are linear devices, but they are not time-invariant. Therefore they cannot be characterized by an ordinary s-domain transfer function, as defined in Chapter 6. See Problem 7.38 for some exceptions, and Section 6.8 for a more extensive discussion of mixed continuous/discrete systems.

7.3 BLOCKS IN CASCADE

Any finite number of blocks in series may be algebraically combined by multiplication of transfer functions. That is, n components or blocks with transfer functions $G_1, G_2, \ldots, G_n$ connected in cascade are equivalent to a single element G with a transfer function given by

$$G = G_1 \cdot G_2 \cdot G_3 \cdots G_n = \prod_{i=1}^{n} G_i \tag{7.1}$$

The symbol for multiplication "·" is omitted when no confusion results.

EXAMPLE 7.4.

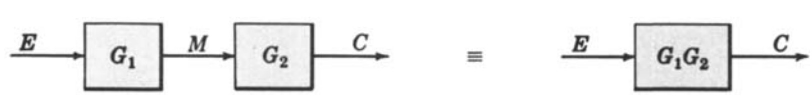


Fig. 7-3

Multiplication of transfer functions is *commutative*; that is,

$$G_i G_j = G_j G_i \tag{7.2}$$

for any i or j.

EXAMPLE 7.5.

Fig. 7-4

Loading effects (interaction of one transfer function upon its neighbor) must be accounted for in the derivation of the individual transfer functions before blocks can be cascaded. (See Problem 7.4.)

7.4 CANONICAL FORM OF A FEEDBACK CONTROL SYSTEM

The two blocks in the forward path of the feedback system of Fig. 7-2 may be combined. Letting $G \equiv G_1 G_2$, the resulting configuration is called the **canonical form** of a feedback control system. G and H are not necessarily unique for a particular system.

The following definitions refer to Fig. 7-5.

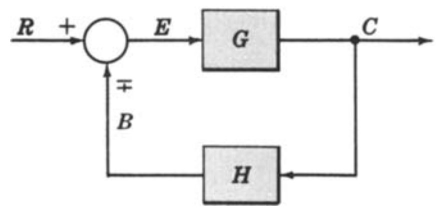


Fig. 7-5

Definition 7.1: $G \equiv$ direct transfer function $\equiv$ forward transfer function

Definition 7.2: $H \equiv$ feedback transfer function

Definition 7.3: $GH \equiv$ loop transfer function $\equiv$ open-loop transfer function

Definition 7.4: $C/R \equiv$ closed-loop transfer function $\equiv$ control ratio

Definition 7.5: $E/R \equiv$ actuating signal ratio $\equiv$ error ratio

Definition 7.6: $B/R \equiv$ primary feedback ratio

In the following equations, the $-$ sign refers to a *positive* feedback system, and the $+$ sign refers to a *negative* feedback system:

$$\frac{C}{R} = \frac{G}{1 \pm GH} \tag{7.3}$$

$$\frac{E}{R} = \frac{1}{1 \pm GH} \tag{7.4}$$

$$\frac{B}{R} = \frac{GH}{1 \pm GH} \tag{7.5}$$

The denominator of C/R determines the *characteristic equation* of the system, which is usually determined from $1 \pm GH = 0$ or, equivalently,

$$D_{GH} \pm N_{GH} = 0 \tag{7.6}$$

where D_{GH} is the denominator and N_{GH} is the numerator of GH, unless a pole of G cancels a zero of H (see Problem 7.9). Relations (*7.1*) through (*7.6*) are valid for both continuous (s-domain) and discrete (z-domain) systems.

7.5 BLOCK DIAGRAM TRANSFORMATION THEOREMS

Block diagrams of complicated control systems may be simplified using easily derivable transformations. The first important transformation, combining blocks in cascade, has already been presented in Section 7.3. It is repeated for completeness in the chart illustrating the transformation theorems (Fig. 7-6). The letter P is used to represent any transfer function, and W, X, Y, Z denote any transformed signals.

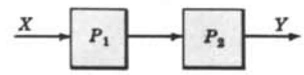
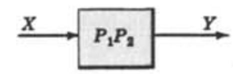
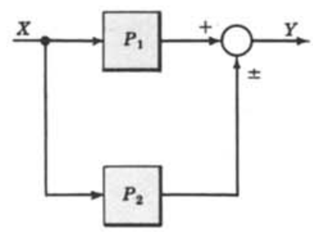
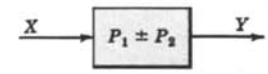
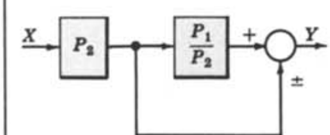
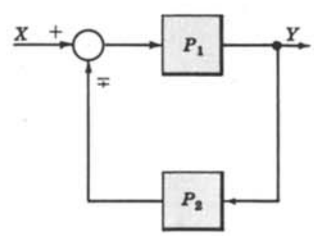
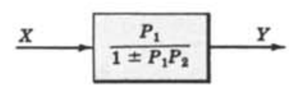
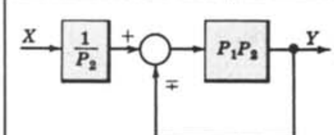
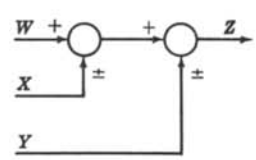
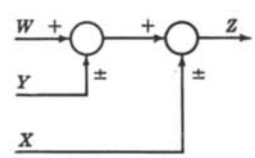
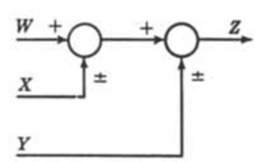
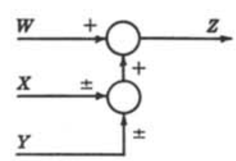
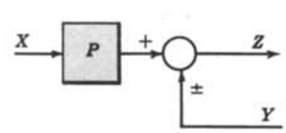
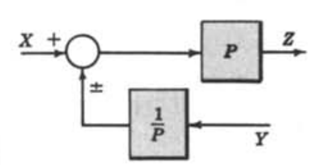
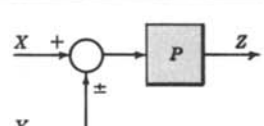
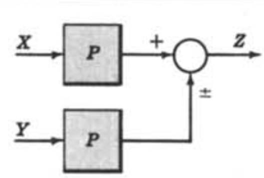
	Transformation	Equation	Block Diagram	Equivalent Block Diagram
1	Combining Blocks in Cascade	$Y = (P_1P_2)X$		
2	Combining Blocks in Parallel; or Eliminating a Forward Loop	$Y = P_1X \pm P_2X$		
3	Removing a Block from a Forward Path	$Y = P_1X \pm P_2X$		
4	Eliminating a Feedback Loop	$Y = P_1(X \mp P_2Y)$		
5	Removing a Block from a Feedback Loop	$Y = P_1(X \mp P_2Y)$		
6a	Rearranging Summing Points	$Z = W \pm X \pm Y$		
6b	Rearranging Summing Points	$Z = W \pm X \pm Y$		
7	Moving a Summing Point Ahead of a Block	$Z = PX \pm Y$		
8	Moving a Summing Point Beyond a Block	$Z = P[X \pm Y]$		

Fig. 7-6

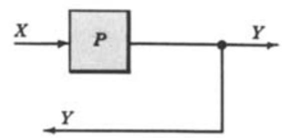
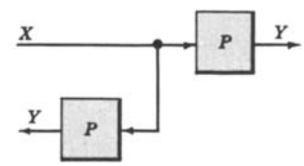
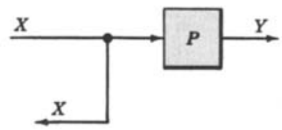
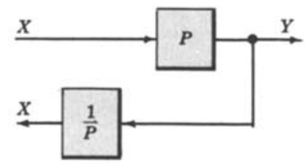
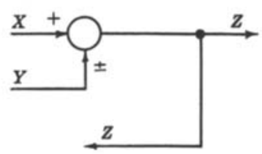
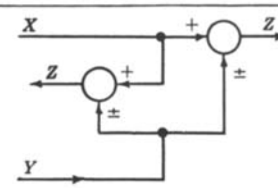
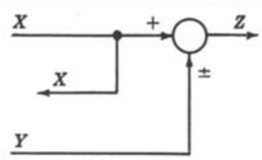
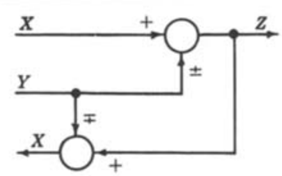
	Transformation	Equation	Block Diagram	Equivalent Block Diagram
9	Moving a Takeoff Point Ahead of a Block	$Y = PX$		
10	Moving a Takeoff Point Beyond a Block	$Y = PX$		
11	Moving a Takeoff Point Ahead of a Summing Point	$Z = X \pm Y$		
12	Moving a Takeoff Point Beyond a Summing Point	$Z = X \pm Y$		

Fig. 7-6 *Continued*

7.6 UNITY FEEDBACK SYSTEMS

Definition 7.7: A **unity feedback system** is one in which the primary feedback b is identically equal to the controlled output c.

EXAMPLE 7.6. $H = 1$ for a linear, unity feedback system (Fig. 7-7).

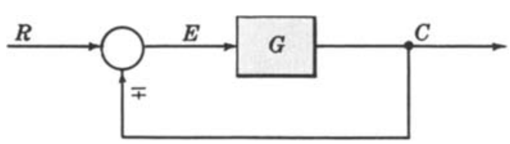


Fig. 7-7

Any feedback system with only linear time-invariant elements can be put into the form of a unity feedback system by using Transformation 5.

EXAMPLE 7.7.

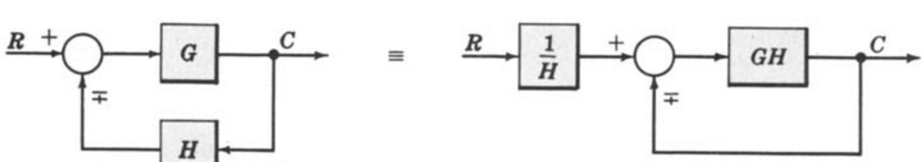


Fig. 7-8

The characteristic equation for the unity feedback system, determined from $1 \pm G = 0$, is

$$D_G \pm N_G = 0 \tag{7.7}$$

where D_G is the denominator and N_G the numerator of G.

7.7 SUPERPOSITION OF MULTIPLE INPUTS

Sometimes it is necessary to evaluate system performance when several inputs are simultaneously applied at different points of the system.

When multiple inputs are present in a *linear* system, each is treated independently of the others. The output due to all stimuli acting together is found in the following manner. We assume zero initial conditions, as we seek the system response only to inputs.

Step 1: Set all inputs except one equal to zero.

Step 2: Transform the block diagram to canonical form, using the transformations of Section 7.5.

Step 3: Calculate the response due to the chosen input acting alone.

Step 4: Repeat Steps 1 to 3 for each of the remaining inputs.

Step 5: Algebraically add all of the responses (outputs) determined in Steps 1 to 4. This sum is the total output of the system with all inputs acting simultaneously.

We reemphasize here that the above superposition process is dependent on the system being linear.

EXAMPLE 7.8. We determine the output C due to inputs U and R for Fig. 7-9.

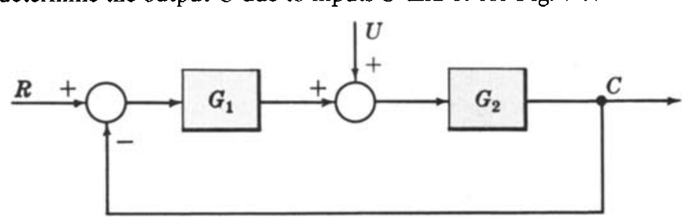
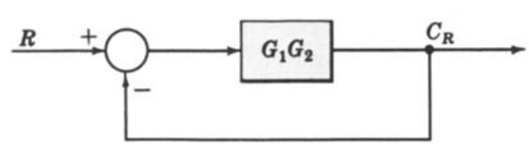


Fig. 7-9

Step 1: Put $U \equiv 0$.

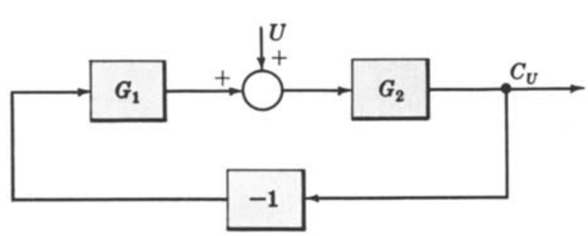
Step 2: The system reduces to



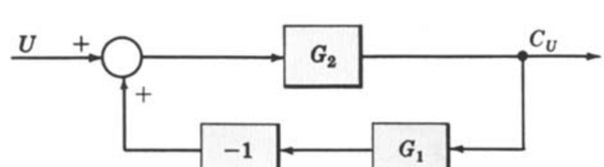
Step 3: By Equation (7.3), the output C_R due to input R is $C_R = [G_1G_2/(1 + G_1G_2)]R$.

Step 4a: Put $R = 0$.

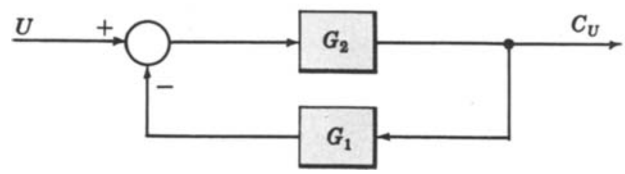
Step 4b: Put -1 into a block, representing the negative feedback effect:



Rearrange the block diagram:



Let the -1 block be absorbed into the summing point:



Step 4c: By Equation (*7.3*), the output C_U due to input U is $C_U = [G_2/(1 + G_1G_2)]U$.
Step 5: The total output is

$$C = C_R + C_U = \left[\frac{G_1G_2}{1 + G_1G_2}\right]R + \left[\frac{G_2}{1 + G_1G_2}\right]U = \left[\frac{G_2}{1 + G_1G_2}\right][G_1R + U]$$

7.8 REDUCTION OF COMPLICATED BLOCK DIAGRAMS

The block diagram of a practical feedback control system is often quite complicated. It may include several feedback or feedforward loops, and multiple inputs. By means of systematic block diagram reduction, every multiple loop linear feedback system may be reduced to canonical form. The techniques developed in the preceding paragraphs provide the necessary tools.

The following general steps may be used as a basic approach in the reduction of complicated block diagrams. Each step refers to specific transformations listed in Fig. 7-6.

Step 1: Combine all cascade blocks using Transformation 1.

Step 2: Combine all parallel blocks using Transformation 2.

Step 3: Eliminate all minor feedback loops using Transformation 4.

Step 4: Shift summing points to the left and takeoff points to the right of the major loop, using Transformations 7, 10, and 12.

Step 5: Repeat Steps 1 to 4 until the canonical form has been achieved for a particular input.

Step 6: Repeat Steps 1 to 5 for each input, as required.

Transformations 3, 5, 6, 8, 9, and 11 are sometimes useful, and experience with the reduction technique will determine their application.

EXAMPLE 7.9. Let us reduce the block diagram (Fig. 7-10) to canonical form.

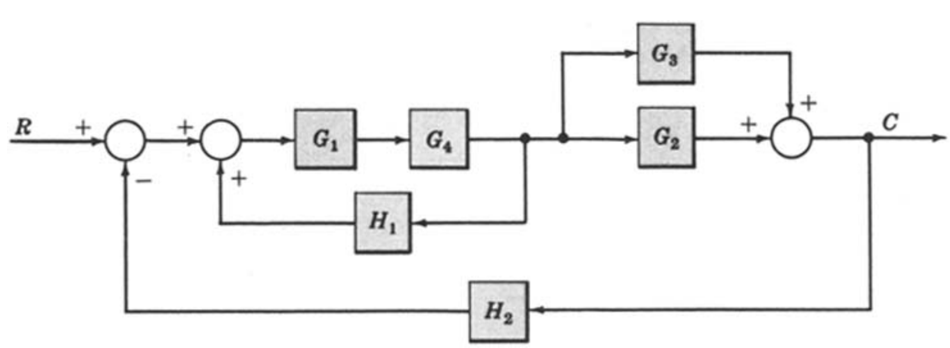


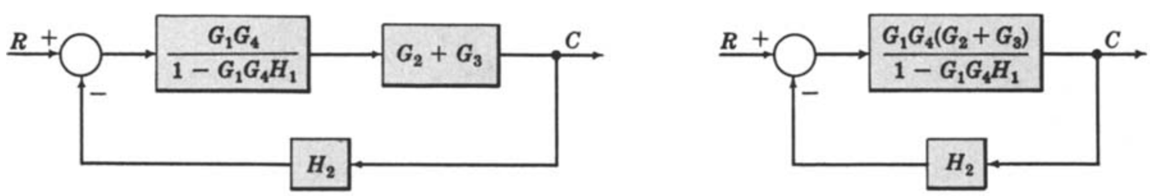
Fig. 7-10

Step 1:

Step 2:

Step 3:

Step 4: Does not apply.
Step 5:



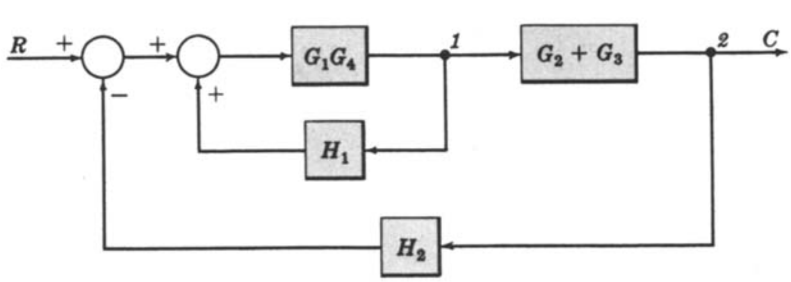
Step 6: Does not apply.

An occasional requirement of block diagram reduction is the isolation of a particular block in a feedback or feedforward loop. This may be desirable to more easily examine the effect of a particular block on the overall system.

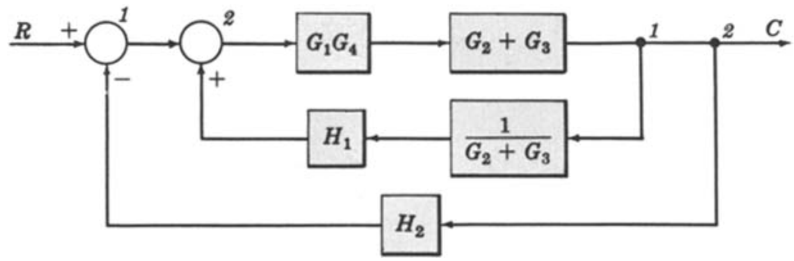
Isolation of a block generally may be accomplished by applying the same reduction steps to the system, but usually in a different order. Also, the block to be isolated cannot be combined with any others.

Rearranging Summing Points (Transformation 6) and Transformations 8, 9, and 11 are especially useful for isolating blocks.

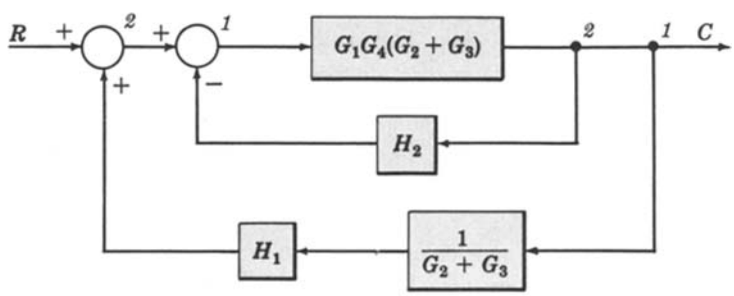
EXAMPLE 7.10. Let us reduce the block diagram of Example 7.9, isolating block H_1.
Steps 1 and 2:



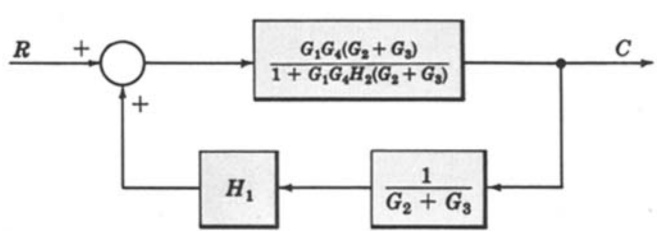
We do not apply Step 3 at this time, but go directly to Step 4, moving takeoff point *1* beyond block $G_2 + G_3$:



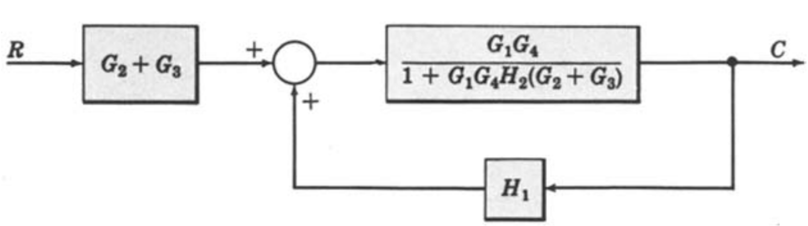
We may now rearrange summing points *1* and *2* and combine the cascade blocks in the forward loop using Transformation 6, then Transformation 1:



Step 3:



Finally, we apply Transformation 5 to remove $1/(G_2 + G_3)$ from the feedback loop:



Note that the same result could have been obtained after applying Step 2 by moving takeoff point *2 ahead* of $G_2 + G_3$, instead of takeoff point *1 beyond* $G_2 + G_3$. Block $G_2 + G_3$ has the same effect on the control ratio C/R whether it directly follows R or directly precedes C.

Solved Problems

BLOCKS IN CASCADE

7.1. Prove Equation (*7.1*) for blocks in cascade.

The block diagram for n transfer functions $G_1, G_2, \ldots, G_n$ in cascade is given in Fig. 7-11.

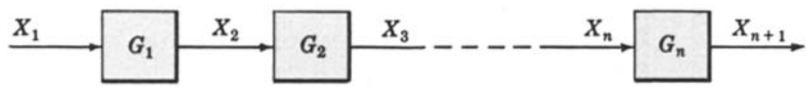


Fig. 7-11

The output transform for any block is equal to the input transform multiplied by the transfer function (see Section 6.1). Therefore $X_2 = X_1G_1$, $X_3 = X_2G_2, \ldots, X_n = X_{n-1}G_{n-1}$, $X_{n+1} = X_nG_n$. Combining these equations, we have

$$X_{n+1} = X_nG_n = X_{n-1}G_{n-1}G_n = \cdots = X_1G_1G_2 \cdots G_{n-1}G_n$$

Dividing both sides by X_1, we obtain $X_{n+1}/X_1 = G_1G_2 \cdots G_{n-1}G_n$.

7.2. Prove the commutativity of blocks in cascade, Equation (*7.2*).

Consider two blocks in cascade (Fig. 7-12):

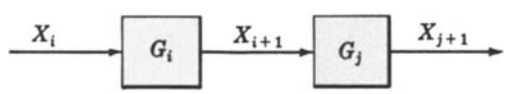


Fig. 7-12

From Equation (*6.1*) we have $X_{i+1} = X_iG_i = G_iX_i$ and $X_{j+1} = X_{i+1}G_j = G_jX_{i+1}$. Therefore $X_{j+1} = (X_iG_i)G_j = X_iG_iG_j$. Dividing both sides by X_i, $X_{j+1}/X_i = G_iG_j$.

Also, $X_{j+1} = G_j(G_iX_i) = G_jG_iX_i$. Dividing again by X_i, $X_{j+1}/X_i = G_jG_i$. Thus $G_iG_j = G_jG_i$.

This result is extended by mathematical induction to any finite number of transfer functions (blocks) in cascade.

7.3. Find X_n/X_1 for each of the systems in Fig. 7-13.

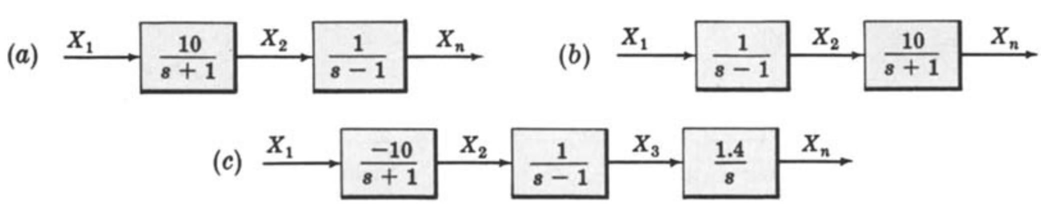


Fig. 7-13

(*a*) One way to work this problem is to first write X_2 in terms of X_1:

$$X_2 = \left(\frac{10}{s+1}\right)X_1$$

Then write X_n in terms of X_2:

$$X_n = \left(\frac{1}{s-1}\right)X_2 = \left(\frac{1}{s-1}\right)\left(\frac{10}{s+1}\right)X_1$$

Multiplying out and dividing both sides by X_1, we have $X_n/X_1 = 10/(s^2-1)$.

A shorter method is as follows. We know from Equation (*7.1*) that two blocks can be reduced to one by simply multiplying their transfer functions. Also, the transfer function of a single block is its output-to-input transform. Hence

$$\frac{X_n}{X_1}=\left(\frac{1}{s-1}\right)\left(\frac{10}{s+1}\right)=\frac{10}{s^2-1}$$

(*b*) This system has the same transfer function determined in part (*a*) because multiplication of transfer functions is commutative.

(*c*) By Equation (*7.1*), we have

$$\frac{X_n}{X_1}=\left(\frac{-10}{s+1}\right)\left(\frac{1}{s-1}\right)\left(\frac{1.4}{s}\right)=\frac{-14}{s(s^2-1)}$$

7.4. The transfer function of Fig. 7-14a is $\omega_0/(s+\omega_0)$, where $\omega_0=1/RC$. Is the transfer function of Fig. 7-14b equal to $\omega_0^2/(s+\omega_0)^2$? Why?

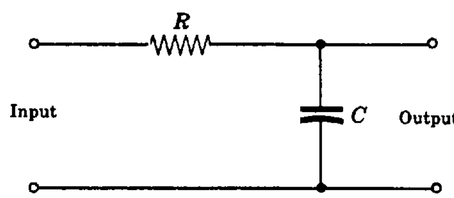


Fig. 7-14a

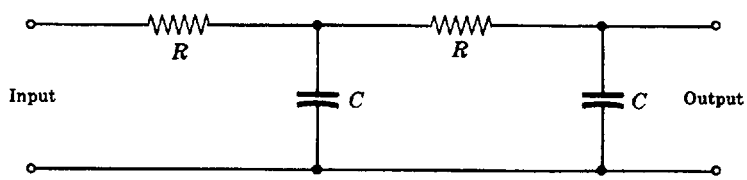


Fig. 7-14b

No. If two networks are connected in series (Fig. 7-15) the second loads the first by drawing current from it. Therefore Equation (*7.1*) cannot be directly applied to the combined system. The correct transfer function for the connected networks is $\omega_0^2/(s^2+3\omega_0 s+\omega_0^2)$ (see Problem 6.16), and this is *not* equal to $(\omega_0/(s+\omega_0))^2$.

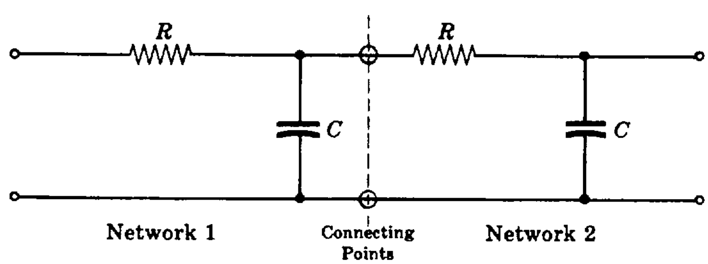


Fig. 7-15

CANONICAL FEEDBACK CONTROL SYSTEMS

7.5. Prove Equation (*7.3*), $C/R=G/(1\pm GH)$.

The equations describing the canonical feedback system are taken directly from Fig. 7-16. They are given by $E=R\mp B$, $B=HC$, and $C=GE$. Substituting one into the other, we have

$$\begin{aligned}C&=G(R\mp B)=G(R\mp HC)\\&=GR\mp GHC=GR+(\mp GHC)\end{aligned}$$

Subtracting $(\mp GHC)$ from both sides, we obtain $C\pm GHC=GR$ or $C/R=G/(1\pm GH)$.

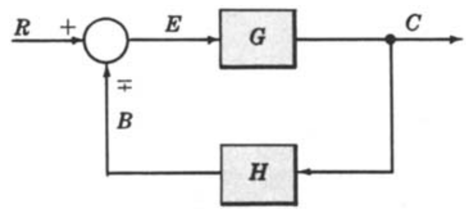


Fig. 7-16

7.6. Prove Equation (*7.4*), $E/R = 1/(1 \pm GH)$.

From the preceding problem, we have $E = R \mp B$, $B = HC$, and $C = GE$. Then $E = R \mp HC = R \mp HGE$, $E \pm GHE = R$, and $E/R = 1/(1 \pm GH)$.

7.7. Prove Equation (*7.5*), $B/R = GH/(1 \pm GH)$.

From $E = R \mp B$, $B = HC$, and $C = GE$, we obtain $B = HGE = HG(R \mp B) = GHR \mp GHB$. Then $B \pm GHB = GHR$, $B = GHR/(1 \pm GH)$, and $B/R = GH/(1 \pm GH)$.

7.8. Prove Equation (*7.6*), $D_{GH} \pm N_{GH} = 0$.

The characteristic equation is usually obtained by setting $1 \pm GH = 0$. (See Problem 7.9 for an exception.) Putting $GH \equiv N_{GH}/D_{GH}$, we obtain $D_{GH} \pm N_{GH} = 0$.

7.9. Determine (*a*) the loop transfer function, (*b*) the control ratio, (*c*) the error ratio, (*d*) the primary feedback ratio, (*e*) the characteristic equation, for the feedback control system in which K_1 and K_2 are constants (Fig. 7-17).

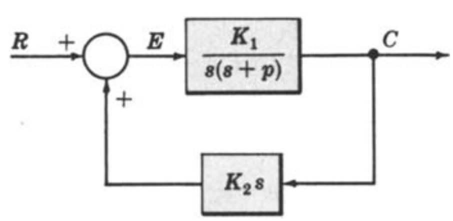


Fig. 7-17

(*a*) The loop transfer function is equal to *GH*.

Hence
$$GH = \left[\frac{K_1}{s(s+p)}\right]K_2 s = \frac{K_1K_2}{s+p}$$

(*b*) The control ratio, or closed-loop transfer function, is given by Equation (*7.3*) (with a minus sign for positive feedback):

$$\frac{C}{R} = \frac{G}{1-GH} = \frac{K_1}{s(s+p-K_1K_2)}$$

(*c*) The error ratio, or actuating signal ratio, is given by Equation (*7.4*):

$$\frac{E}{R} = \frac{1}{1-GH} = \frac{1}{1-K_1K_2/(s+p)} = \frac{s+p}{s+p-K_1K_2}$$

(*d*) The primary feedback ratio is given by Equation (*7.5*):

$$\frac{B}{R} = \frac{GH}{1-GH} = \frac{K_1K_2}{s+p-K_1K_2}$$

(*e*) The characteristic equation is given by the denominator of C/R above, $s(s+p-K_1K_2) = 0$. In this case, $1 - GH = s + p - K_1K_2 = 0$, which is *not* the characteristic equation, because the pole s of G cancels the zero s of H.

BLOCK DIAGRAM TRANSFORMATIONS

7.10. Prove the equivalence of the block diagrams for Transformation 2 (Section 7.5).

The equation in the second column, $Y = P_1 X \pm P_2 X$, governs the construction of the block diagram in the third column, as shown. Rewrite this equation as $Y = (P_1 \pm P_2)X$. The equivalent block diagram in the last column is clearly the representation of this form of the equation (Fig. 7-18)

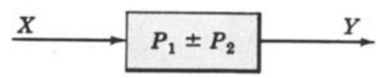


Fig. 7-18

7.11. Repeat Problem 7.10 for Transformation 3.

Rewrite $Y = P_1 X \pm P_2 X$ as $Y = (P_1/P_2)P_2 X \pm P_2 X$. The block diagram for this form of the equation is clearly given in Fig. 7-19.

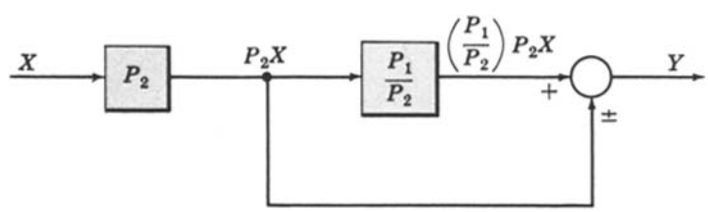


Fig. 7-19

7.12. Repeat Problem 7.10 for Transformation 5.

We have $Y = P_1[X \mp P_2 Y] = P_1 P_2[(1/P_2)X \mp Y]$. The block diagram for the latter form is given in Fig. 7-20.

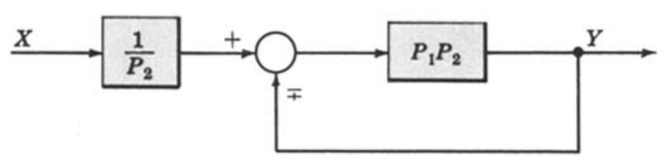


Fig. 7-20

7.13. Repeat Problem 7.10 for Transformation 7.

We have $Z = PX \pm Y = P[X \pm (1/P)Y]$, which yields the block diagram given in Fig. 7-21.

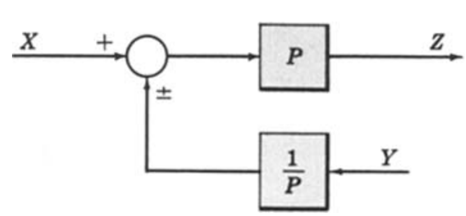


Fig. 7-21

7.14. Repeat Problem 7.10 for Transformation 8.

We have $Z = P(X \pm Y) = PX \pm PY$, whose block diagram is clearly given in Fig. 7-22.

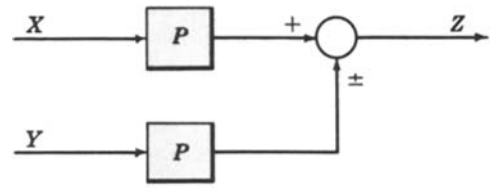


Fig. 7-22

UNITY FEEDBACK SYSTEMS

7.15. Reduce the block diagram given in Fig. 7-23 to unity feedback form and find the system characteristic equation.

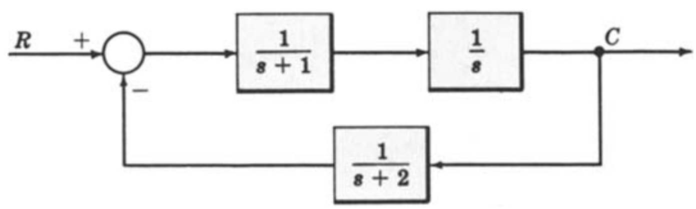


Fig. 7-23

Combining the blocks in the forward path, we obtain Fig. 7-24.

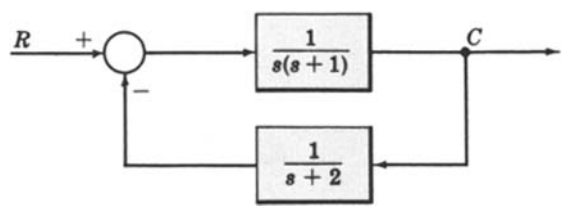


Fig. 7-24

Applying Transformation 5, we have Fig. 7-25.

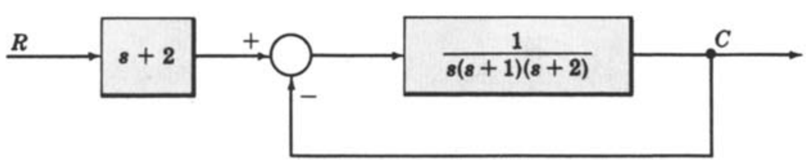


Fig. 7-25

By Equation (7.7), the characteristic equation for this system is $s(s+1)(s+2)+1=0$ or $s^3+3s^2+2s+1=0$.

MULTIPLE INPUTS AND OUTPUTS

7.16. Determine the output C due to U_1, U_2, and R for Fig. 7-26.

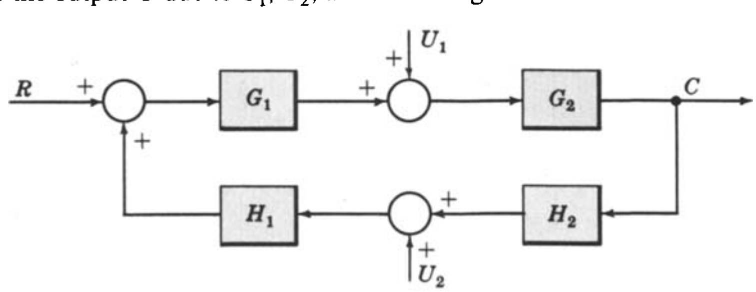


Fig. 7-26

Let $U_1 = U_2 = 0$. After combining the cascaded blocks, we obtain Fig. 7-27, where C_R is the output due to R acting alone. Applying Equation (*7.3*) to this system, $C_R = [G_1G_2/(1 - G_1G_2H_1H_2)]R$.

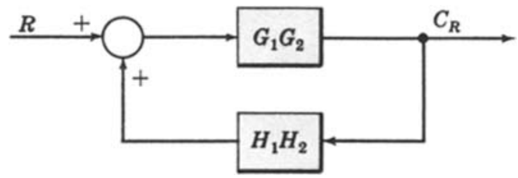


Fig. 7-27

Now let $R = U_2 = 0$. The block diagram is now given in Fig. 7-28, where C_1 is the response due to U_1 acting alone. Rearranging the blocks, we have Fig. 7-29. From Equation (*7.3*), we get $C_1 = [G_2/(1 - G_1G_2H_1H_2)]U_1$.

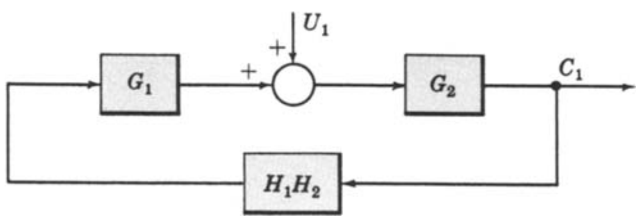


Fig. 7-28

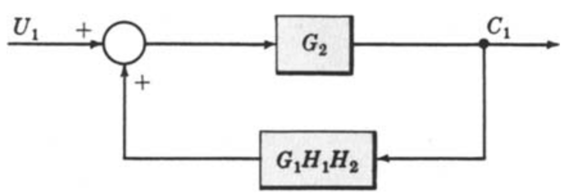


Fig. 7-29

Finally, let $R = U_1 = 0$. The block diagram is given in Fig. 7-30, where C_2 is the response due to U_2 acting alone. Rearranging the blocks, we get Fig. 7-31. Hence $C_2 = [G_1G_2H_1/(1 - G_1G_2H_1H_2)]U_2$.

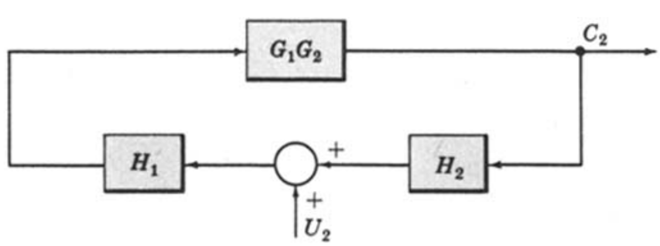


Fig. 7-30

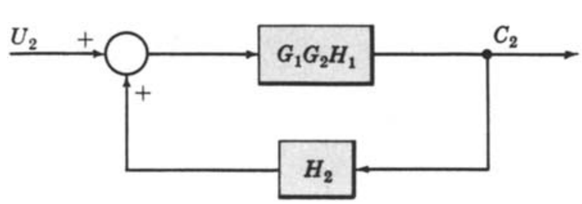


Fig. 7-31

By superposition, the total output is

$$C = C_R + C_1 + C_2 = \frac{G_1G_2R + G_2U_1 + G_1G_2H_1U_2}{1 - G_1G_2H_1H_2}$$

7.17. Figure 7-32 is an example of a multiinput-multioutput system. Determine C_1 and C_2 due to R_1 and R_2.

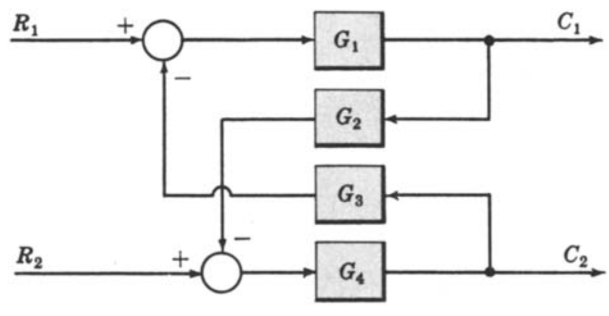


Fig. 7-32

First put the block diagram in the form of Fig. 7-33, ignoring the output C_2.

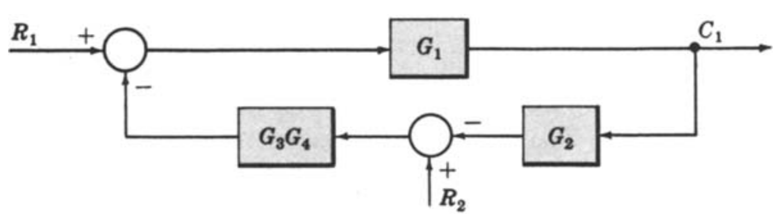


Fig. 7-33

Letting $R_2 = 0$ and combining the summing points, we get Fig. 7-34.

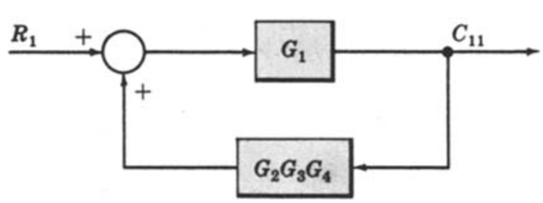


Fig. 7-34

Hence C_{11}, the output at C_1 due to R_1 alone, is $C_{11} = G_1R_1/(1 - G_1G_2G_3G_4)$. For $R_1 = 0$, we have Fig. 7-35.

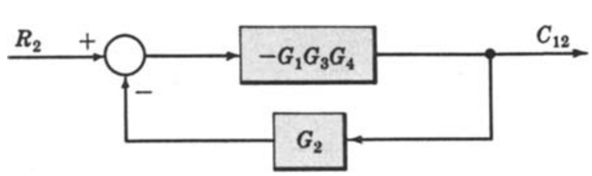


Fig. 7-35

Hence $C_{12} = -G_1G_3G_4R_2/(1 - G_1G_2G_3G_4)$ is the output at C_1 due to R_2 alone. Thus $C_1 = C_{11} + C_{12} = (G_1R_1 - G_1G_3G_4R_2)/(1 - G_1G_2G_3G_4)$.

Now we reduce the original block diagram, ignoring output C_1. First we obtain Fig. 7-36.

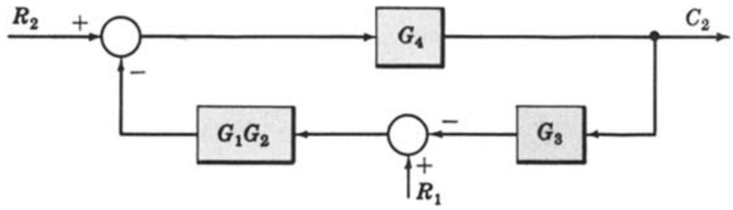


Fig. 7-36

Then we obtain the block diagram given in Fig. 7-37. Hence $C_{22} = G_4R_2/(1 - G_1G_2G_3G_4)$. Next, letting $R_2 = 0$, we obtain Fig. 7-38. Hence $C_{21} = -G_1G_2G_4R_1/(1 - G_1G_2G_3G_4)$. Finally, $C_2 = C_{22} + C_{21} = (G_4R_2 - G_1G_2G_4R_1)/(1 - G_1G_2G_3G_4)$.

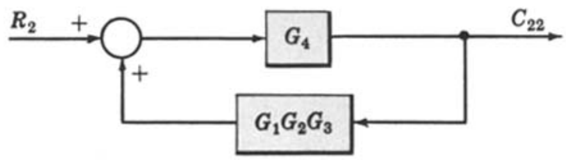


Fig. 7-37

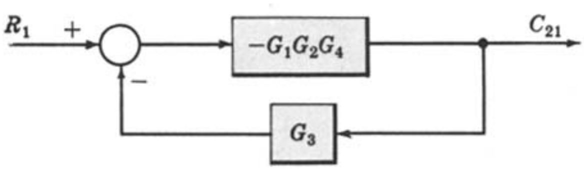


Fig. 7-38

BLOCK DIAGRAM REDUCTION

7.18. Reduce the block diagram given in Fig. 7-39 to canonical form, and find the output transform C. K is a constant.

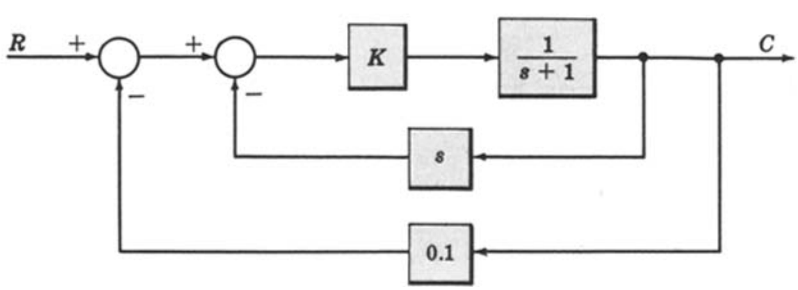


Fig. 7-39

First we combine the cascade blocks of the forward path and apply Transformation 4 to the innermost feedback loop to obtain Fig. 7-40.

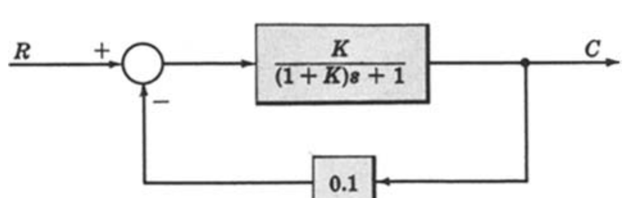


Fig. 7-40

Equation (7.3) or the reapplication of Transformation 4 yields $C = KR/[(1 + K)s + (1 + 0.1K)]$.

7.19. Reduce the block diagram of Fig. 7-39 to canonical form, isolating block K in the forward loop.

Mathcad

By Transformation 9 we can move the takeoff point ahead of the $1/(s+1)$ block (Fig. 7-41):

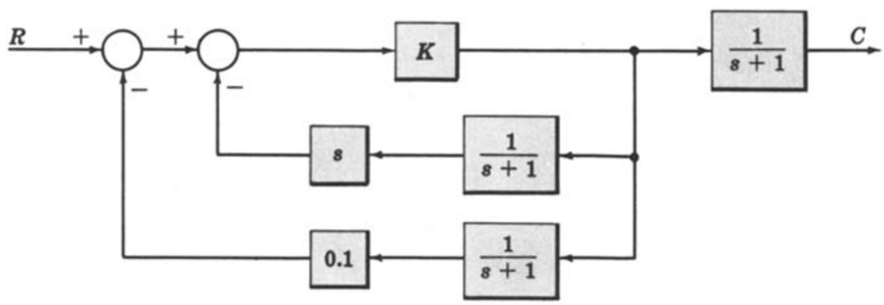


Fig. 7-41

Applying Transformations 1 and 6*b*, we get Fig. 7-42.

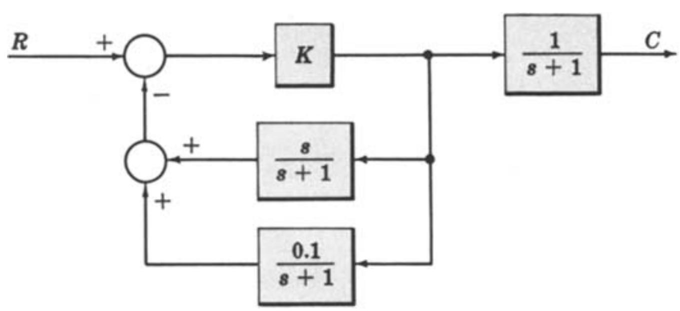


Fig. 7-42

Now we can apply Transformation 2 to the feedback loops, resulting in the final form given in Fig. 7-43.

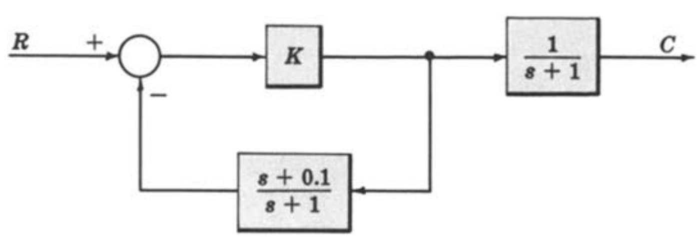


Fig. 7-43

7.20. Reduce the block diagram given in Fig. 7-44 to open-loop form.

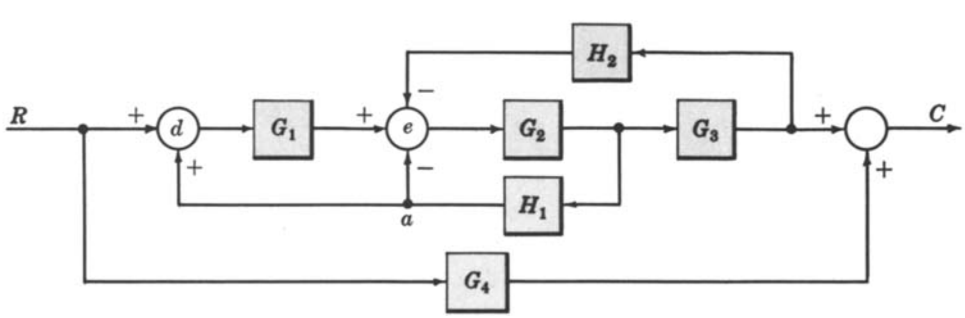


Fig. 7-44

First, moving the leftmost summing point beyond G_1 (Transformation 8), we obtain Fig. 7-45.

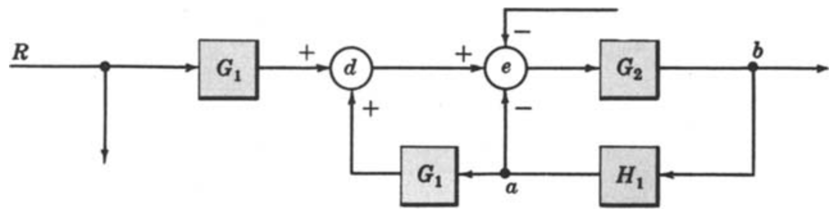


Fig. 7-45

Next, moving takeoff point a beyond G_1, we get Fig. 7-46.

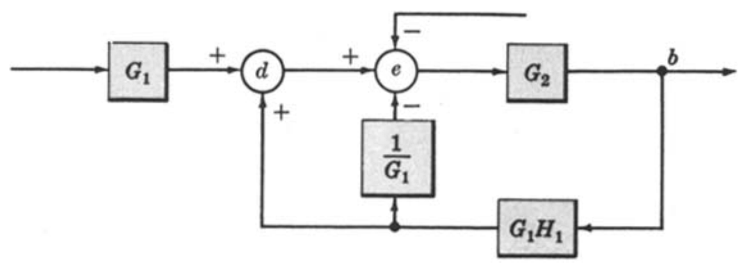


Fig. 7-46

Now, using Transformation 6b, and then Transformation 2, to combine the two lower feedback loops (from $G_1 H_1$) entering d and e, we obtain Fig. 7-47.

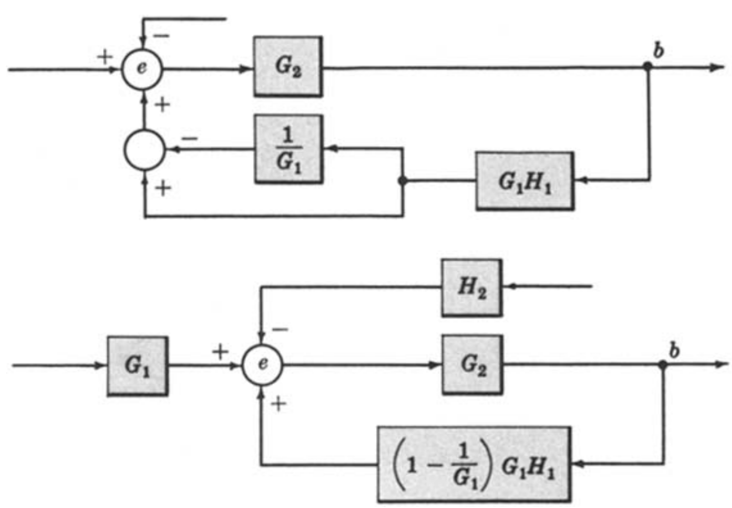
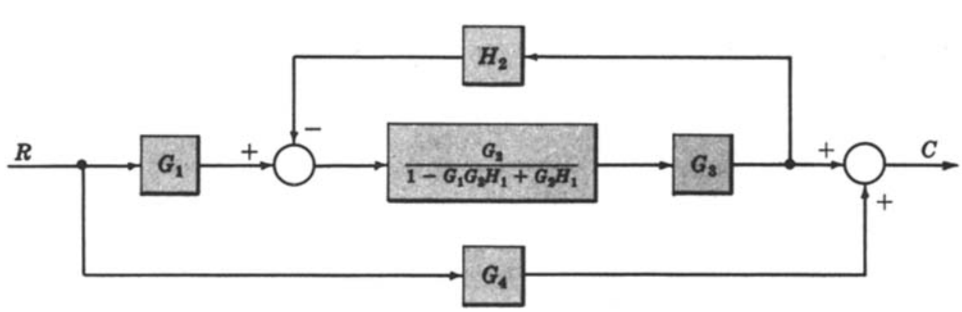
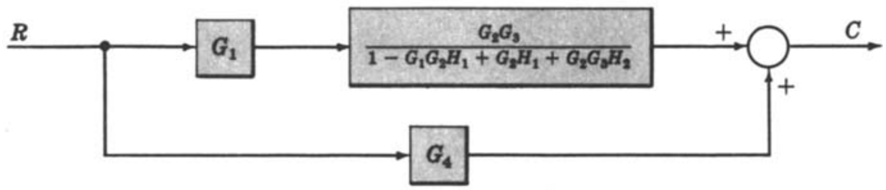


Fig. 7-47

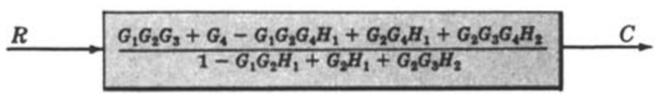
Applying Transformation 4 to this inner loop, the system becomes



Again, applying Transformation 4 to the remaining feedback loop yields



Finally, Transformation 1 and 2 give the open-loop block diagram:



MISCELLANEOUS PROBLEMS

7.21. Show that simple block diagram Transformation 1 of Section 7.5 (combining blocks in cascade) is not valid if the first block is (or includes) a *sampler*.

The output transform $U^*(s)$ of an ideal sampler was determined in Problem 4.39 as

$$U^*(s) = \sum_{k=0}^{\infty} e^{-skT}u(kT)$$

Taking $U^*(s)$ as the input of block P_2 of Transformation 1 of the table, the output transform $Y(s)$ of block P_2 is

$$Y(s) = P_2(s)U^*(s) = P_2(s)\sum_{k=0}^{\infty} e^{-skT}u(kT)$$

Clearly, the input transform $X(s) = U(s)$ cannot be factored from the right-hand side of $Y(s)$, that is, $Y(s) \neq F(s)U(s)$. The same problem occurs if P_1 includes other elements, as well as a sampler.

7.22. Why is the characteristic equation invariant under block diagram transformation?

Block diagram transformations are determined by *rearranging* the input-output equations of one or more of the subsystems that make up the total system. Therefore the final transformed system is governed by the same equations, probably arranged in a different manner than those for the original system.

Now, the characteristic equation is determined from the denominator of the overall system transfer function set equal to zero. Factoring or other rearrangement of the numerator and denominator of the system transfer function clearly does not change it, nor does it alter the denominator set equal to zero.

7.23. Prove that the transfer function represented by C/R in Equation (*7.3*) can be approximated by $\pm 1/H$ when $|G|$ or $|GH|$ are very large.

Dividing the numerator and denominator of $G/(1 \pm GH)$ by G, we get $1\bigg/\left(\dfrac{1}{G} \pm H\right)$. Then

$$\lim_{|G|\to\infty}\left[\frac{C}{R}\right] = \lim_{|G|\to\infty}\left[\frac{1}{\dfrac{1}{G}\pm H}\right] = \pm\frac{1}{H}$$

Dividing by GH and taking the limit, we obtain

$$\lim_{|GH|\to\infty}\left[\frac{C}{R}\right] = \lim_{|GH|\to\infty}\left[\frac{\dfrac{1}{H}}{\dfrac{1}{GH}\pm 1}\right] = \pm\frac{1}{H}$$

7.24. Assume that the characteristics of G change radically or unpredictably during system operation. Using the results of the previous problem, show how the system should be designed so that the output C can always be predicted reasonably well.

In problem 7.23 we found that

$$\lim_{|GH|\to\infty}\left[\frac{C}{R}\right] = \pm\frac{1}{H}$$

Thus $C \to \pm R/H$ as $|GH| \to \infty$, or C is independent of G for large $|GH|$. Hence the system should be designed so that $|GH| \gg 1$.

7.25. Determine the transfer function of the system in Fig. 7-48. Then let $H_1 = 1/G_1$ and $H_2 = 1/G_2$.

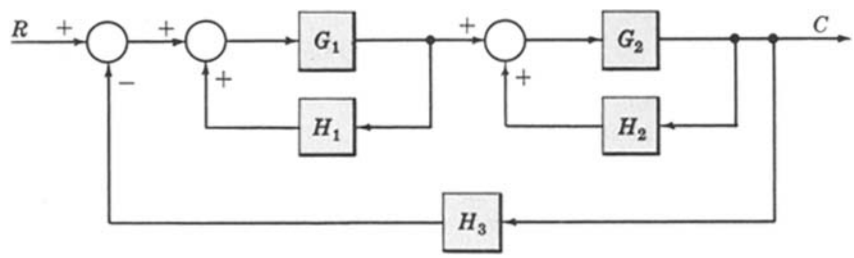


Fig. 7-48

Reducing the inner loops, we have Fig. 7-49.

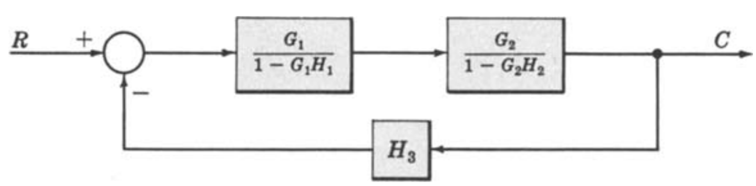


Fig. 7-49

Applying Transformation 4 again, we obtain Fig. 7-50.

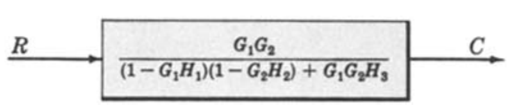


Fig. 7-50

Now put $H_1 = 1/G_1$ and $H_2 = 1/G_2$. This yields

$$\frac{C}{R} = \frac{G_1G_2}{(1-1)(1-1)+G_1G_2H_3} = \frac{1}{H_3}$$

7.26. Show that Fig. 7-51 is valid.

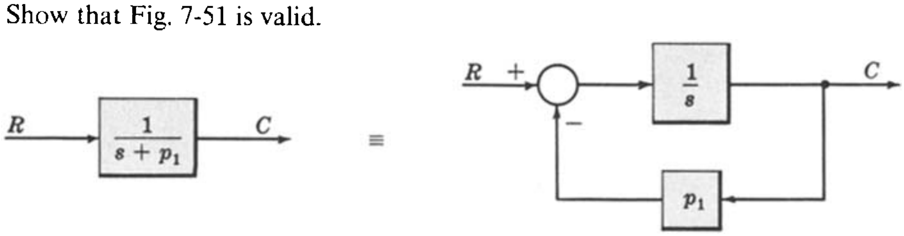


Fig. 7-51

From the open-loop diagram, we have $C = R/(s + p_1)$. Rearranging, $(s + p_1)C = R$ and $C = (1/s)(R - p_1 C)$. The closed-loop diagram follows from this equation.

7.27. Prove Fig. 7-52.

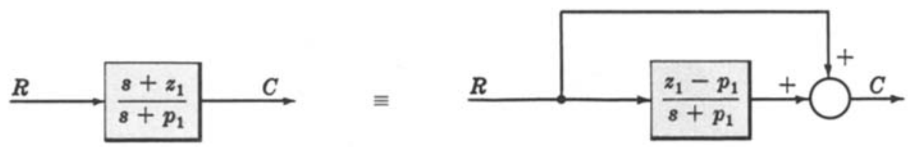


Fig. 7-52

This problem illustrates how a finite zero may be removed from a block.

From the forward-loop diagram, $C = R + (z_1 - p_1)R/(s + p_1)$. Rearranging,

$$C = \left(1 + \frac{z_1 - p_1}{s + p_1}\right)R = \left(\frac{s + p_1 + z_1 - p_1}{s + p_1}\right)R = \left(\frac{s + z_1}{s + p_1}\right)R$$

This mathematical equivalence clearly proves the equivalence of the block diagrams.

7.28. Assume that linear approximations in the form of transfer functions are available for each block of the Supply and Demand System of Problem 2.13, and that the system can be represented by Fig. 7-53.

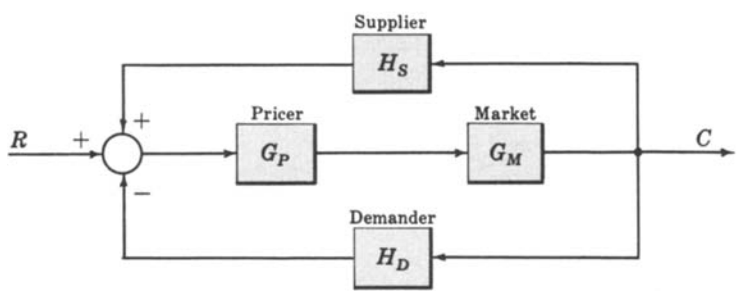


Fig. 7-53

Determine the overall transfer function of the system.

Block diagram Transformation 4, applied twice to this system, gives Fig. 7-54.

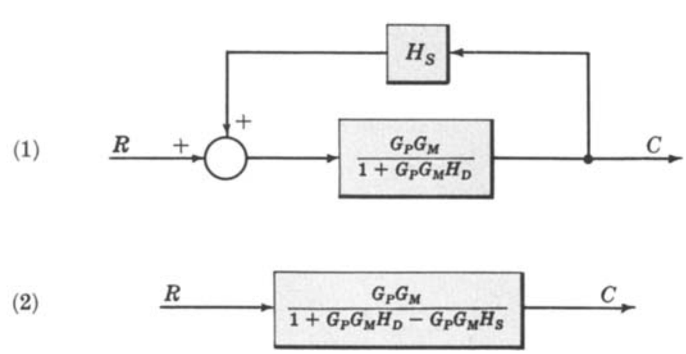


Fig. 7-54

Hence the transfer function for the linearized Supply and Demand model is: $\dfrac{G_P G_M}{1 + G_P G_M (H_D - H_S)}$.

Supplementary Problems

7.29. Determine C/R for each system in Fig. 7-55.

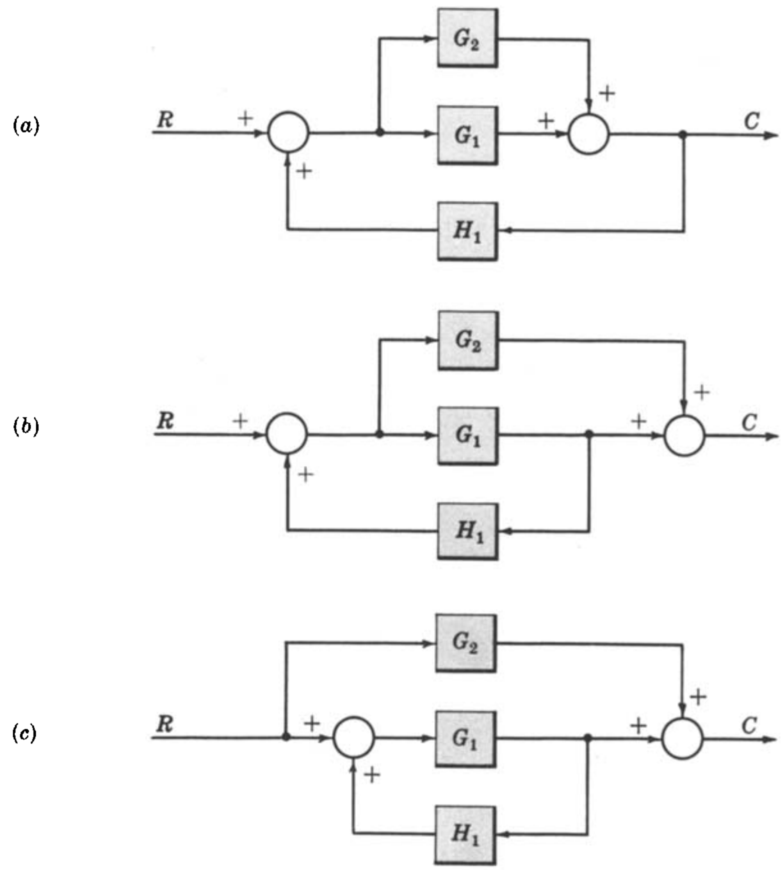


Fig. 7-55

7.30. Consider the blood pressure regulator described in Problem 2.14. Assume the vasomotor center (VMC) can be described by a linear transfer function $G_{11}(s)$, and the baroreceptors by the transfer function $k_1 s + k_2$ (see Problem 6.33). Transform the block diagram into its simplest, unity feedback form.

7.31. Reduce Fig. 7-56 to canonical form.

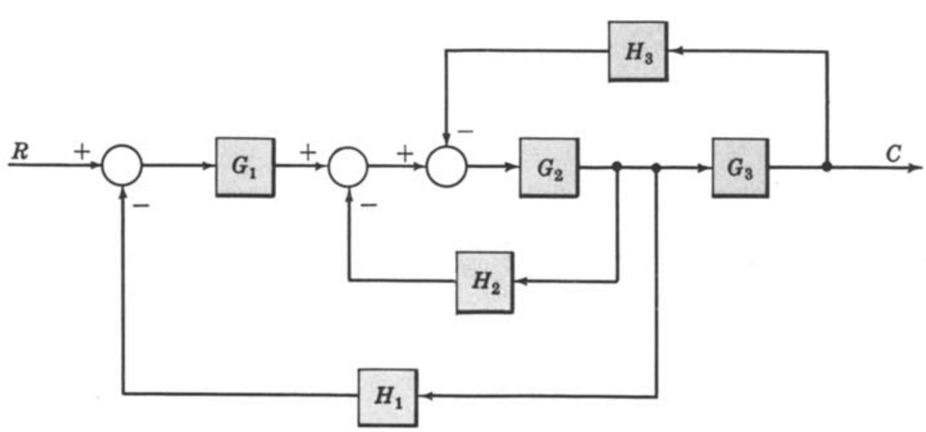


Fig. 7-56

7.32. Determine C for the system represented by Fig. 7-57.

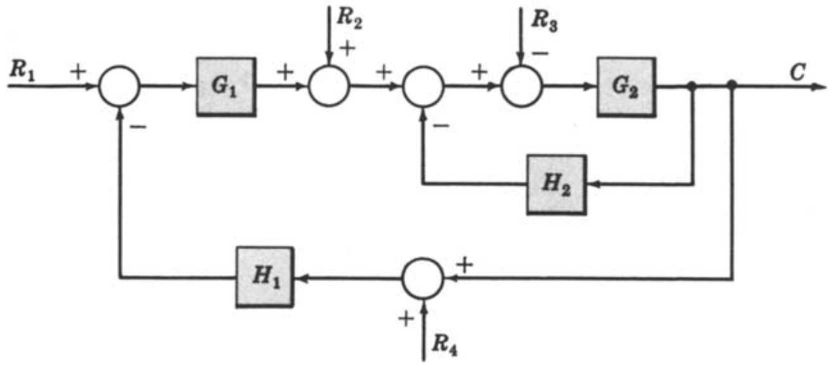


Fig. 7-57

7.33. Give an example of two feedback systems in canonical form having identical control ratios C/R but different G and H components.

7.34. Determine C/R_2 for the system given in Fig. 7-58.

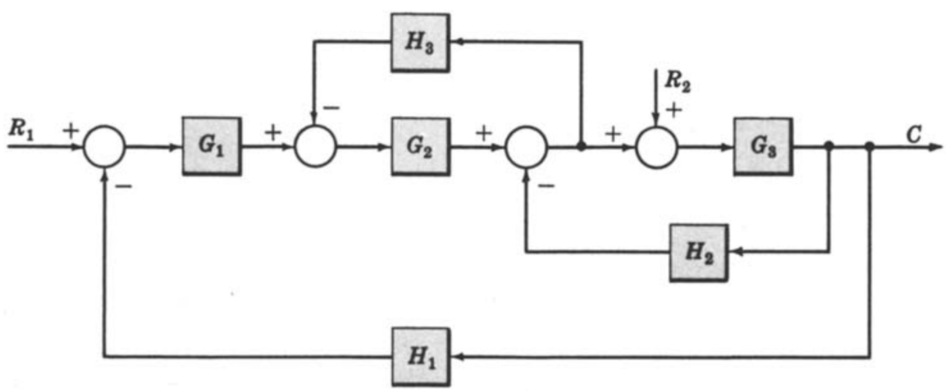


Fig. 7-58

7.35. Determine the complete output C, with both inputs R_1 and R_2 acting simultaneously, for the system given in the preceding problem.

7.36. Determine C/R for the system represented by Fig. 7-59.

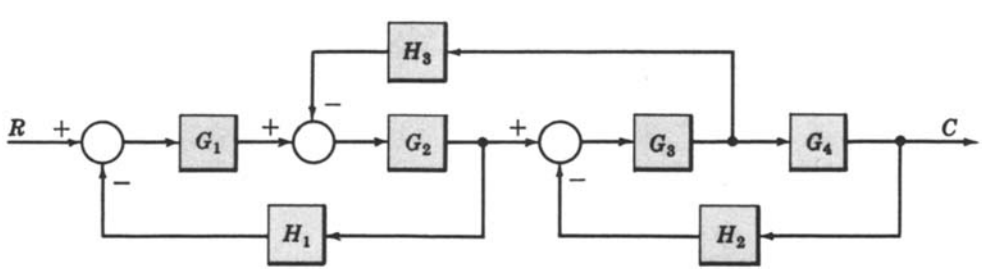


Fig. 7-59

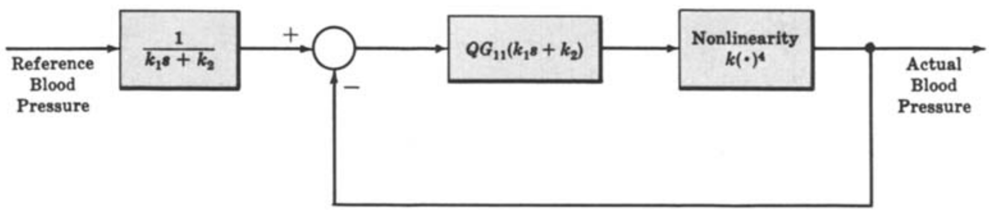
7.37. Determine the characteristic equation for each of the systems of Problems (a) 7.32, (b) 7.35, (c) 7.36.

7.38. What block diagram transformation rules in the table of Section 7.5 permit the inclusion of a sampler?

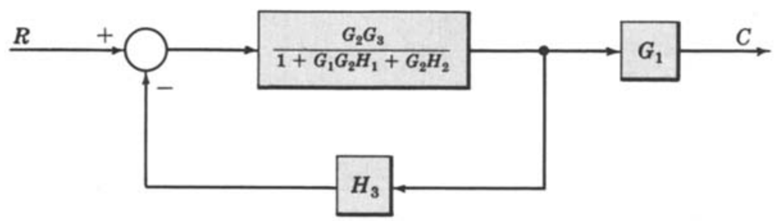
Answers to Supplementary Problems

7.29. See Problem 8.15.

7.30.



7.31.



7.32. $$C = \frac{G_1G_2R_1 + G_2R_2 - G_2R_3 - G_1G_2H_1R_4}{1 + G_2H_2 + G_1G_2H_1}$$

7.34. $$\frac{C}{R_2} = \frac{G_3(1 + G_2H_3)}{1 + G_3H_2 + G_2H_3 + G_1G_2G_3H_1}$$

7.35. $$C = \frac{G_1G_2G_3R_1 + G_3(1 + G_2H_3)R_2}{1 + G_3H_2 + G_2H_3 + G_1G_2G_3H_1}$$

7.36. $$\frac{C}{R} = \frac{G_1G_2G_3G_4}{(1 + G_1G_2H_1)(1 + G_3G_4H_2) + G_2G_3H_3}$$

7.37. (*a*) $1 + G_2H_2 + G_1G_2H_1 = 0$

(*b*) $1 + G_3H_2 + G_2H_3 + G_1G_2G_3H_1 = 0$

(*c*) $(1 + G_1G_2H_1)(1 + G_3G_4H_2) + G_2G_3H_3 = 0.$

7.38. The results of Problem 7.21 indicate that any transformation that involves any *product* of two or more transforms is not valid if a sampler is included. But all those that simply involve the sum or difference of signals are valid, that is, Transformations 6, 11, and 12. Each represents a simple rearrangement of signals as a linear-sum, and addition is a commutative operation, even for sampled signals, that is $Z = X \pm Y = Y \pm X$.

Chapter 8

Signal Flow Graphs

8.1 INTRODUCTION

The most extensively used graphical representation of a feedback control system is the block diagram, presented in Chapters 2 and 7. In this chapter we consider another model, the signal flow graph.

A **signal flow graph** is a pictorial representation of the simultaneous equations describing a system. It graphically displays the transmission of signals through the system, as does the block diagram. But it is easier to draw and therefore easier to manipulate than the block diagram.

The properties of signal flow graphs are presented in the next few sections. The remainder of the chapter treats applications.

8.2 FUNDAMENTALS OF SIGNAL FLOW GRAPHS

Let us first consider the simple equation

$$X_i = A_{ij} X_j \tag{8.1}$$

The variables X_i and X_j can be functions of time, complex frequency, or any other quantity. They may even be constants, which are "variables" in the mathematical sense.

For signal flow graphs, A_{ij} is a mathematical operator mapping X_j into X_i, and is called the **transmission function**. For example, A_{ij} may be a constant, in which case X_i is a constant times X_j in Equation (*8.1*); if X_i and X_j are functions of s or z, A_{ij} may be a transfer function $A_{ij}(s)$ or $A_{ij}(z)$.

The signal flow graph for Equation (*8.1*) is given in Fig. 8-1. This is the simplest form of a signal flow graph. Note that the variables X_i and X_j are represented by a small dot called a **node**, and the transmission function A_{ij} is represented by a line with an arrow, called a **branch**.

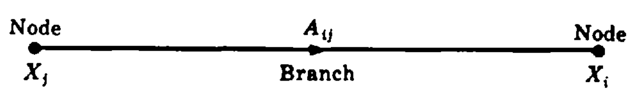


Fig. 8-1

Every variable in a signal flow graph is designated by a node, and every transmission function by a branch. Branches are always unidirectional. The arrow denotes the direction of signal flow.

EXAMPLE 8.1. Ohm's law states that $E = RI$, where E is a voltage, I a current, and R a resistance. The signal flow graph for this equation is given in Fig. 8-2.

Fig. 8-2

8.3 SIGNAL FLOW GRAPH ALGEBRA

1. The Addition Rule

The value of the variable designated by a node is equal to the sum of all signals entering the node. In other words, the equation

$$X_i = \sum_{j=1}^{n} A_{ij} X_j$$

is represented by Fig. 8-3.

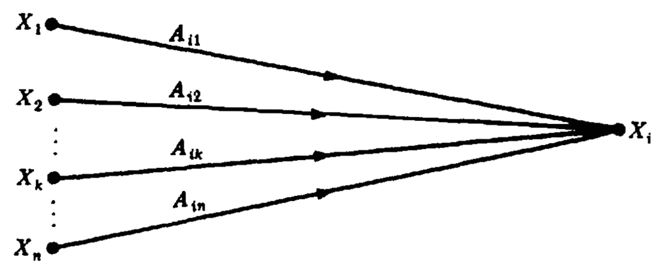


Fig. 8-3

EXAMPLE 8.2. The signal flow graph for the equation of a line in rectangular coordinates, $Y = mX + b$, is given in Fig. 8-4. Since b, the Y-axis intercept, is a constant it may represent a node (variable) or a transmission function.

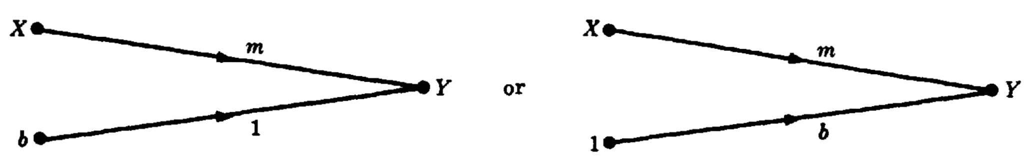


Fig. 8-4

2. The Transmission Rule

The value of the variable designated by a node is transmitted on every branch leaving that node. In other words, the equation

$$X_i = A_{ik} X_k \qquad i = 1, 2, \ldots, n, \ k \text{ fixed}$$

is represented by Fig. 8-5.

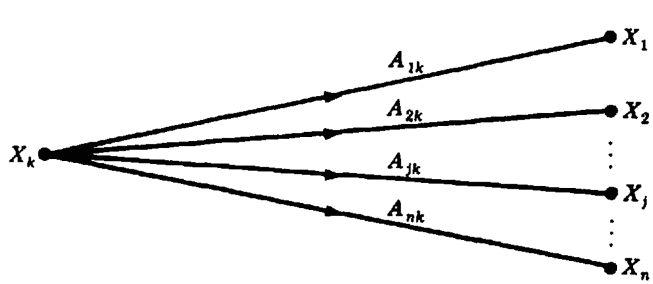


Fig. 8-5

EXAMPLE 8.3. The signal flow graph of the simultaneous equations $Y = 3X$, $Z = -4X$ is given in Fig. 8-6.

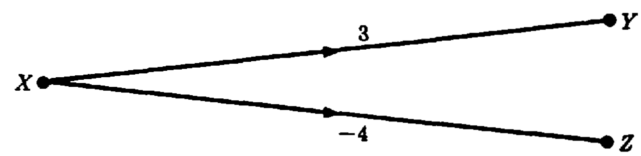


Fig. 8-6

3. The Multiplication Rule

A cascaded (series) connection of $n - 1$ branches with transmission functions $A_{21}, A_{32}, A_{43}, \ldots, A_{n(n-1)}$ can be replaced by a single branch with a new transmission function equal to the product of the old ones. That is,

$$X_n = A_{21} \cdot A_{32} \cdot A_{43} \cdots A_{n(n-1)} \cdot X_1$$

The signal flow graph equivalence is represented by Fig. 8-7.

Fig. 8-7

EXAMPLE 8.4. The signal flow graph of the simultaneous equations $Y = 10X$, $Z = -20Y$ is given in Fig. 8-8.

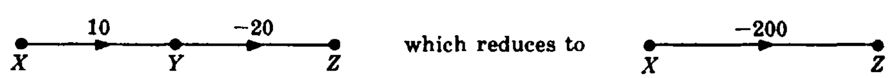


Fig. 8-8

8.4 DEFINITIONS

The following terminology is frequently used in signal flow graph theory. The examples associated with each definition refer to Fig. 8-9.

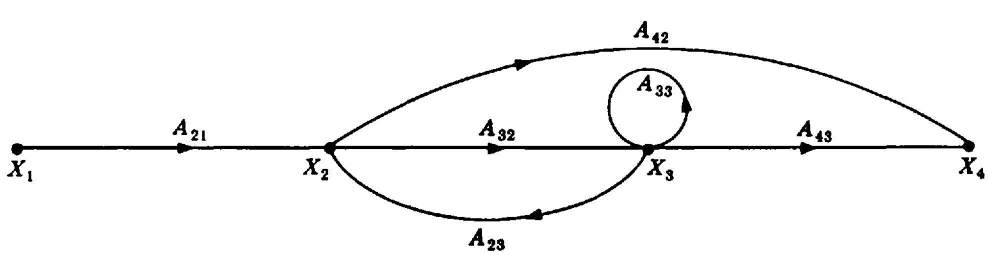


Fig. 8-9

Definition 8.1: A **path** is a continuous, unidirectional succession of branches along which no node is passed more than once. For example, X_1 to X_2 to X_3 to X_4, X_2 to X_3 and back to X_2, and X_1 to X_2 to X_4 are paths.

Definition 8.2: An **input node** or **source** is a node with only outgoing branches. For example, X_1 is an input node.

Definition 8.3: An **output node** or **sink** is a node with only incoming branches. For example, X_4 is an output node.

Definition 8.4: A **forward path** is a path from the input node to the output node. For example, X_1 to X_2 to X_3 to X_4, and X_1 to X_2 to X_4 are forward paths.

Definition 8.5: A **feedback path** or **feedback loop** is a path which originates and terminates on the same node. For example, X_2 to X_3 and back to X_2 is a feedback path.

Definition 8.6: A **self-loop** is a feedback loop consisting of a single branch. For example, A_{33} is a self-loop.

Definition 8.7: The **gain** of a branch is the transmission function of that branch when the transmission function is a multiplicative operator. For example, A_{33} is the gain of the self-loop if A_{33} is a constant or transfer function.

Definition 8.8: The **path gain** is the product of the branch gains encountered in traversing a path. For example, the path gain of the forward path from X_1 to X_2 to X_3 to X_4 is $A_{21}A_{32}A_{43}$.

Definition 8.9: The **loop gain** is the product of the branch gains of the loop. For example, the loop gain of the feedback loop from X_2 to X_3 and back to X_2 is $A_{32}A_{23}$.

Very often, a variable in a system is a function of the output variable. The canonical feedback system is an obvious example. In this case, if the signal flow graph were to be drawn directly from the equations, the "output node" would require an outgoing branch, contrary to the definition. This problem may be remedied by adding a branch with a transmission function of unity entering a "dummy" node. For example, the two graphs in Fig. 8-10 are equivalent, and Y_4 is an output node. Note that $Y_4 = Y_3$.

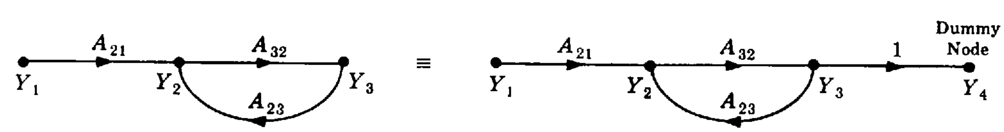


Fig. 8-10

8.5 CONSTRUCTION OF SIGNAL FLOW GRAPHS

The signal flow graph of a linear feedback control system whose components are specified by noninteracting transfer functions can be constructed by direct reference to the block diagram of the system. Each variable of the block diagram becomes a node and each block becomes a branch.

EXAMPLE 8.5. The block diagram of the canonical feedback control system is given in Fig. 8-11.

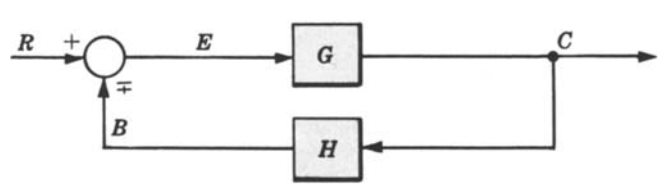


Fig. 8-11

The signal flow graph is easily constructed from Fig. 8-12. Note that the $-$ or $+$ sign of the summing point is associated with H.

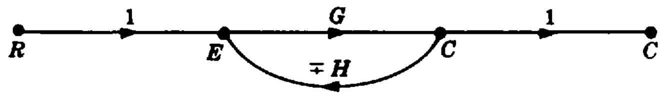


Fig. 8-12

The signal flow graph of a system described by a set of simultaneous equations can be constructed in the following general manner.

1. Write the system equations in the form

$$X_1 = A_{11}X_1 + A_{12}X_2 + \cdots + A_{1n}X_n$$

$$X_2 = A_{21}X_1 + A_{22}X_2 + \cdots + A_{2n}X_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$X_m = A_{m1}X_1 + A_{m2}X_2 + \cdots + A_{mn}X_n$$

   An equation for X_1 is not required if X_1 is an input node.
2. Arrange the m or n (whichever is larger) nodes from left to right. The nodes may be rearranged if the required loops later appear too cumbersome.
3. Connect the nodes by the appropriate branches A_{11}, A_{12}, etc.
4. If the desired output node has outgoing branches, add a dummy node and a unity gain branch.
5. Rearrange the nodes and/or loops in the graph to achieve maximum pictorial clarity.

EXAMPLE 8.6. Let us construct a signal flow graph for the simple resistance network given in Fig. 8-13. There are five variables, v_1, v_2, v_3, i_1, and i_2. v_1 is known. We can write four independent equations from Kirchhoff's voltage and current laws. Proceeding from left to right in the schematic, we have

$$i_1 = \left(\frac{1}{R_1}\right)v_1 - \left(\frac{1}{R_1}\right)v_2 \qquad v_2 = R_3 i_1 - R_3 i_2 \qquad i_2 = \left(\frac{1}{R_2}\right)v_2 - \left(\frac{1}{R_2}\right)v_3 \qquad v_3 = R_4 i_2$$

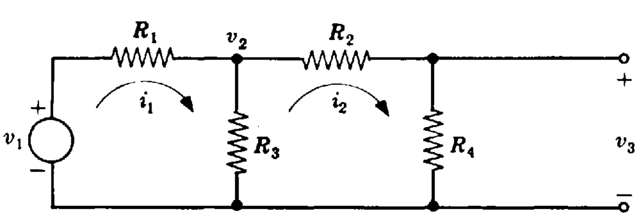


Fig. 8-13

Laying out the five nodes in the same order with v_1 as an input node, and connecting the nodes with the appropriate branches, we get Fig. 8-14. If we wish to consider v_3 as an output node, we must add a unity gain

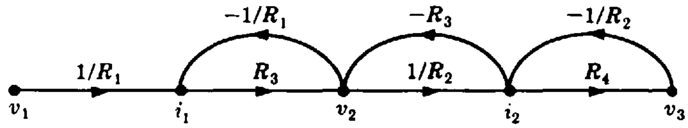


Fig. 8-14

branch and another node, yielding Fig. 8-15. No rearrangement of the nodes is necessary. We have one forward path and three feedback loops clearly in evidence.

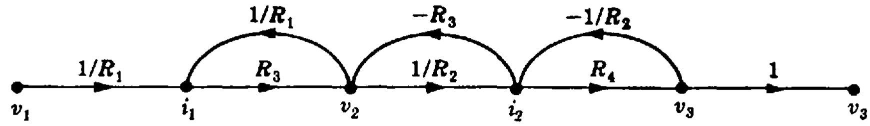


Fig. 8-15

Note that signal flow graph representations of equations are not unique. For example, the addition of a unity gain branch followed by a dummy node changes the graph, but not the equations it represents.

8.6 THE GENERAL INPUT-OUTPUT GAIN FORMULA

We found in Chapter 7 that we can reduce complicated block diagrams to canonical form, from which the control ratio is easily written as

$$\frac{C}{R}=\frac{G}{1\pm GH}$$

It is possible to simplify signal flow graphs in a manner similar to that of block diagram reduction. But it is also possible, and much less time-consuming, to write down the input-output relationship *by inspection* from the original signal flow graph. This can be accomplished using the formula presented below. This formula can also be applied directly to block diagrams, but the signal flow graph representation is easier to read—especially when the block diagram is very complicated.

Let us denote the ratio of the input variable to the output variable by T. For linear feedback control systems, $T=C/R$. For the general signal flow graph presented in preceding paragraphs $T=X_n/X_1$, where X_n is the output and X_1 is the input.

The general formula for any signal flow graph is

$$T=\frac{\Sigma_i P_i\Delta_i}{\Delta} \tag{8.2}$$

where P_i = the ith forward path gain

P_{jk} = jth possible product of k nontouching loop gains

$$\Delta = 1-(-1)^{k+1}\sum_k\sum_j P_{jk}$$

$$=1-\sum_j P_{j1}+\sum_j P_{j2}-\sum_j P_{j3}+\cdots$$

= 1 − (sum of all loop gains) + (sum of all gain products of two nontouching loops) − (sum of all gain products of three nontouching loops) + ⋯

Δ_i = Δ evaluated with all loops touching P_i eliminated

Two loops, paths, or a loop and a path are said to be **nontouching** if they have no nodes in common.

Δ is called the **signal flow graph determinant** or **characteristic function**, since $\Delta=0$ is the system characteristic equation.

The application of Equation (*8.2*) is considerably more straightforward than it appears. The following examples illustrate this point.

EXAMPLE 8.7. Let us first apply Equation (*8.2*) to the signal flow graph of the canonical feedback system (Fig. 8-16).

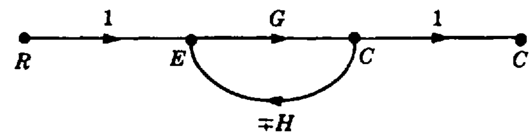


Fig. 8-16

There is only one forward path; hence

$$P_1 = G$$
$$P_2 = P_3 = \cdots = 0$$

There is only one (feedback) loop. Hence

$$P_{11} = \mp GH$$
$$P_{jk} = 0 \qquad j \neq 1 \qquad k \neq 1$$

Then

$$\Delta = 1 - P_{11} = 1 \pm GH \qquad \text{and} \qquad \Delta_1 = 1 - 0 = 1$$

Finally,

$$T = \frac{C}{R} = \frac{P_1\Delta_1}{\Delta} = \frac{G}{1 \pm GH}$$

EXAMPLE 8.8. The signal flow graph of the resistance network of Example 8.6 is shown in Fig. 8-17. Let us apply Equation (*8.2*) to this graph and determine the voltage gain $T = v_3/v_1$ of the resistance network.

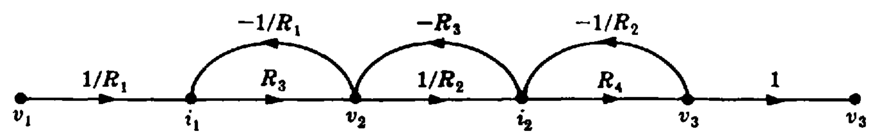


Fig. 8-17

There is one forward path (Fig. 8-18). Hence the forward path gain is

$$P_1 = \frac{R_3 R_4}{R_1 R_2}$$

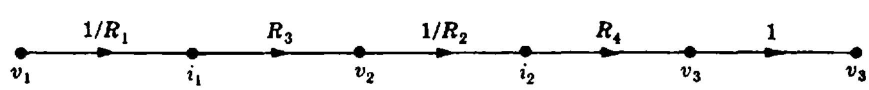


Fig. 8-18

There are three feedback loops (Fig. 8-19). Hence the loop gains are

$$P_{11} = -\frac{R_3}{R_1} \qquad P_{21} = -\frac{R_3}{R_2} \qquad P_{31} = -\frac{R_4}{R_2}$$

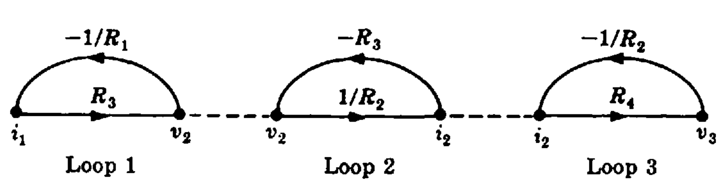


Fig. 8-19

There are two nontouching loops, loops 1 and 3. Hence

$$P_{12} = \text{gain product of the only two nontouching loops} = P_{11} \cdot P_{31} = \frac{R_3 R_4}{R_1 R_2}$$

There are no three loops that do not touch. Therefore

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 + \frac{R_3}{R_1} + \frac{R_3}{R_2} + \frac{R_4}{R_2} + \frac{R_3 R_4}{R_1 R_2}$$

$$= \frac{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_3 + R_3 R_4}{R_1 R_2}$$

Since all loops touch the forward path, $\Delta_1 = 1$. Finally,

$$\frac{v_3}{v_1} = \frac{P_1 \Delta_1}{\Delta} = \frac{R_3 R_4}{R_1 R_2 + R_1 R_3 + R_1 R_4 + R_2 R_3 + R_3 R_4}$$

8.7 TRANSFER FUNCTION COMPUTATION OF CASCADED COMPONENTS

Loading effects of interacting components require little special attention using signal flow graphs. Simply combine the graphs of the components at their normal joining points (output node of one to the input node of another), account for loading by adding new loops at the joined nodes, and compute the overall gain using Equation (*8.2*). This procedure is best illustrated by example.

EXAMPLE 8.9. Assume that two identical resistance networks are to be cascaded and used as the control elements in the forward loop of a control system. The networks are simple voltage dividers of the form given in Fig. 8-20.

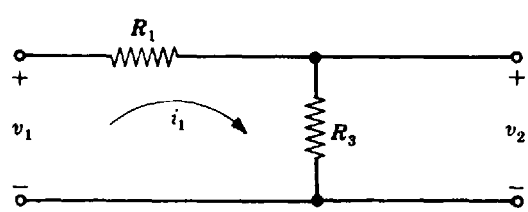


Fig. 8-20

Two independent equations for this network are

$$i_1 = \left(\frac{1}{R_1}\right) v_1 - \left(\frac{1}{R_1}\right) v_2 \qquad \text{and} \qquad v_2 = R_3 i_1$$

The signal flow graph is easily drawn (Fig. 8-21). The gain of this network is, by inspection, equal to

$$\frac{v_2}{v_1} = \frac{R_3}{R_1 + R_3}$$

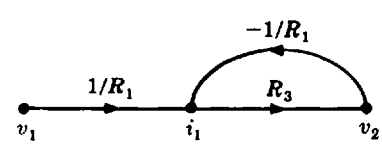


Fig. 8-21

If we were to ignore loading, the overall gain of two cascaded networks would simply be determined by multiplying the individual gains:

$$\left(\frac{v_2}{v_1}\right)^2 = \frac{R_3^2}{R_1^2 + R_3^2 + 2R_1 R_3}$$

This answer is incorrect. We prove this in the following manner. When the two identical networks are cascaded, we note that the result is equivalent to the network of Example 8.6, with $R_2 = R_1$ and $R_4 = R_3$ (Fig. 8-22).

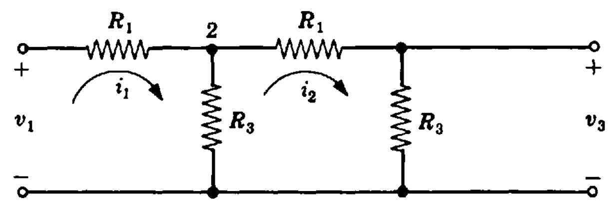


Fig. 8-22

The signal flow graph of this network was also determined in Example 8.6 (Fig. 8-23).

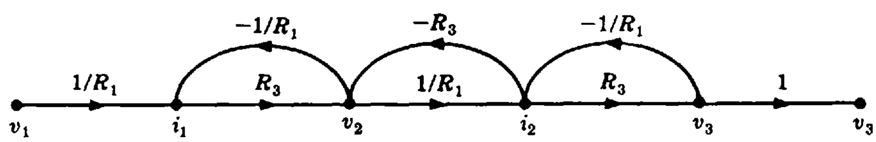


Fig. 8-23

We observe that the feedback branch $-R_3$ in Fig. 8-23 does not appear in the signal flow graph of the cascaded signal flow graphs of the individual networks connected from node v_2 to v_1' (Fig. 8-24). This means that, as a result of connecting the two networks, the second one loads the first, changing the equation for v_2 from

$$v_2 = R_3 i_1 \qquad \text{to} \qquad v_2 = R_3 i_1 - R_3 i_2$$

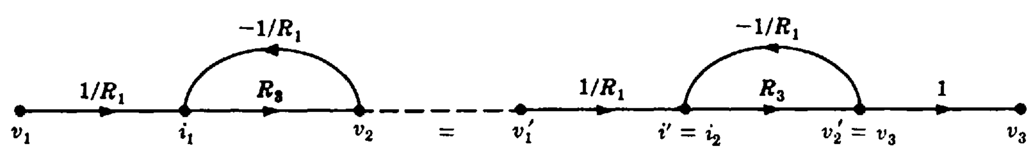


Fig. 8-24

This result could also have been obtained by directly writing the equations for the combined networks. In this case, only the equation for v_2 would have changed form.

The gain of the combined networks was determined in Example 8.8 as

$$\frac{v_3}{v_1} = \frac{R_3^2}{R_1^2 + R_3^2 + 3R_1R_3}$$

when R_2 is set equal to R_1 and R_4 is set equal to R_3. We observe that

$$\left(\frac{v_2}{v_1}\right)^2 = \frac{R_3^2}{R_1^2 + R_3^2 + 2R_1R_3} \neq \frac{v_3}{v_1}$$

It is good general practice to calculate the gain of cascaded networks directly from the *combined* signal flow graph. Most practical control system components load each other when connected in series.

8.8 BLOCK DIAGRAM REDUCTION USING SIGNAL FLOW GRAPHS AND THE GENERAL INPUT-OUTPUT GAIN FORMULA

Often, the easiest way to determine the control ratio of a complicated block diagram is to translate the block diagram into a signal flow graph and apply Equation (*8.2*). Takeoff points and summing points must be separated by a unity gain branch in the signal flow graph when using Equation (*8.2*).

If the elements G and H of a canonical feedback representation are desired, Equation (8.2) also provides this information. The direct transfer function is

$$G = \sum_i P_i \Delta_i \tag{8.3}$$

The loop transfer function is

$$GH = \Delta - 1 \tag{8.4}$$

Equations (8.3) and (8.4) are solved simultaneously for G and H, and the canonical feedback control system is drawn from the result.

EXAMPLE 8.10. Let us determine the control ratio C/R and the canonical block diagram of the feedback control system of Example 7.9 (Fig. 8-25).

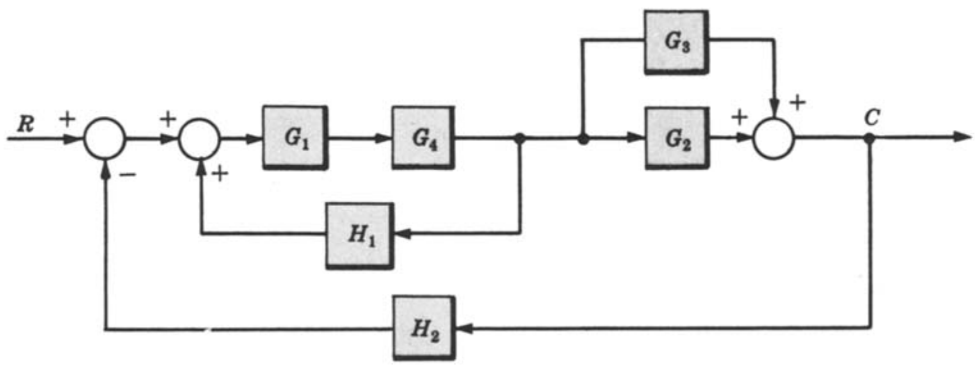


Fig. 8-25

The signal flow graph is given in Fig. 8-26. There are two forward paths:

$$P_1 = G_1G_2G_4 \qquad P_2 = G_1G_3G_4$$

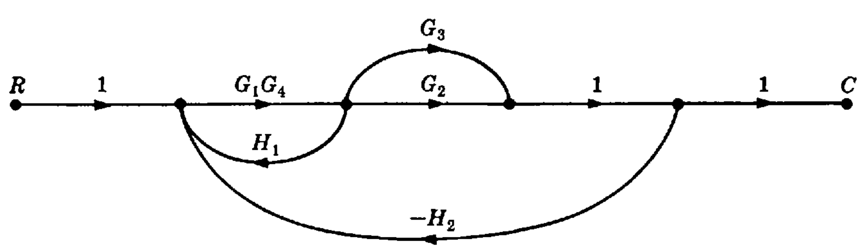


Fig. 8-26

There are three feedback loops:

$$P_{11} = G_1G_4H_1 \qquad P_{21} = -G_1G_2G_4H_2 \qquad P_{31} = -G_1G_3G_4H_2$$

There are no nontouching loops, and all loops touch both forward paths; then

$$\Delta_1 = 1 \qquad \Delta_2 = 1$$

Therefore the control ratio is

$$T = \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

$$= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

From Equations (*8.3*) and (*8.4*), we have

$$G = G_1G_4(G_2 + G_3) \qquad \text{and} \qquad GH = G_1G_4(G_3H_2 + G_2H_2 - H_1)$$

Therefore
$$H = \frac{GH}{G} = \frac{(G_2 + G_3)H_2 - H_1}{G_2 + G_3}$$

The canonical block diagram is therefore given in Fig. 8-27.

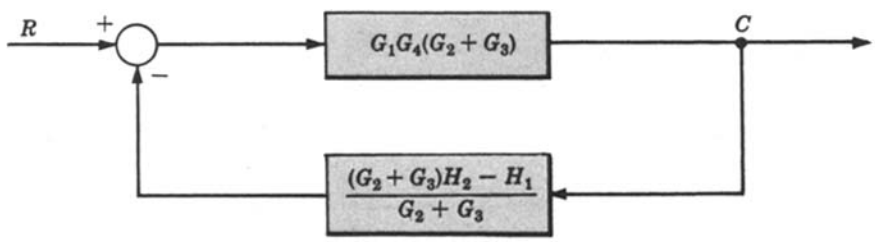


Fig. 8-27

The negative summing point sign for the feedback loop is a result of using a positive sign in the *GH* formula above. If this is not obvious, refer to Equation (*7.3*) and its explanation in Section 7.4.

The block diagram above may be put into the final form of Examples 7.9 or 7.10 by using the transformation theorems of Section 7.5.

Solved Problems

SIGNAL FLOW GRAPH ALGEBRA AND DEFINITIONS

8.1. Simplify the signal flow graphs given in Fig. 8-28.

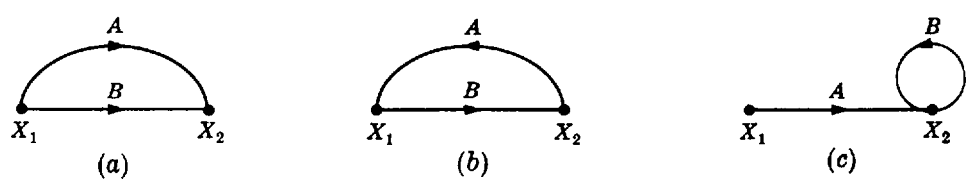
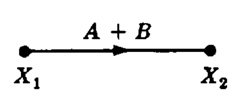
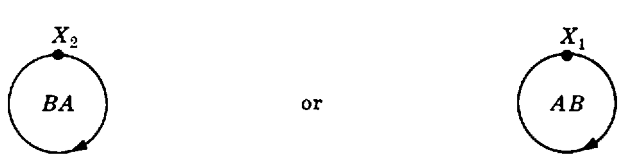


Fig. 8-28

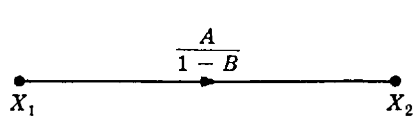
(*a*) Clearly, $X_2 = AX_1 + BX_1 = (A + B)X_1$. Therefore we have



(*b*) We have $X_2 = BX_1$ and $X_1 = AX_2$. Hence $X_2 = BAX_2$, or $X_1 = ABX_1$, yielding



(*c*) If A and B are multiplicative operators (e.g., constants or transfer functions), we have $X_2 = AX_1 + BX_2 = (A/(1 - B))X_1$. Hence the signal flow graph becomes



8.2. Draw signal flow graphs for the block diagrams in Problem 7.3 and reduce them by the multiplication rule (Fig. 8-29).

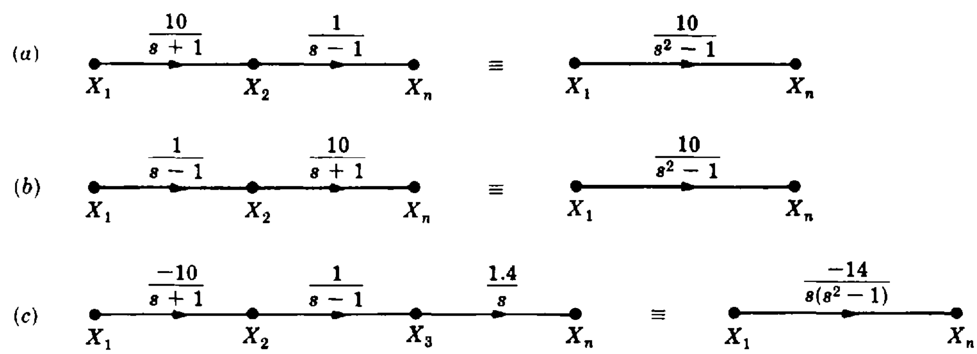


Fig. 8-29

8.3. Consider the signal flow graph in Fig. 8-30.

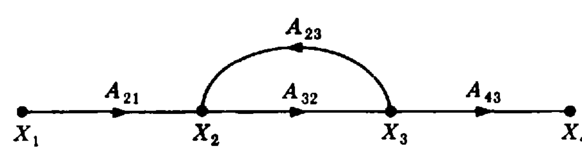


Fig. 8-30

(*a*) Draw the signal flow graph for the system equivalent to that graphed in Fig. 8-30, but in which X_3 becomes kX_3 (k constant) and X_1, X_2, and X_4 remain the same.

(*b*) Repeat part (*a*) for the case in which X_2 and X_3 become k_2X_2 and k_3X_3, and X_1 and X_4 remain the same (k_2 and k_3 are constants).

This problem illustrates the fundamentals of a technique that can be used for *scaling* variables.

(*a*) For the system to remain the same when a node variable is multiplied by a constant, all signals entering the node must be multiplied by the same constant, and all signals leaving the node divided by that constant. Since X_1, X_2, and X_4 must remain the same, the *branches* are modified (Fig. 8-31).

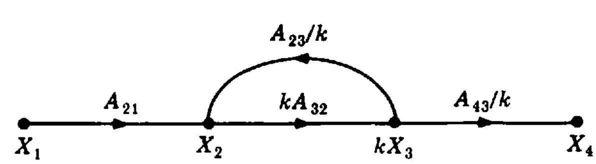


Fig. 8-31

(*b*) Substitute k_2X_2 for X_2, and k_3X_3 for X_3 (Fig. 8-32)

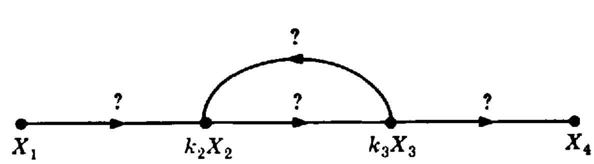


Fig. 8-32

It is clear from the graph that A_{21} becomes $k_2 A_{21}$, A_{32} becomes $(k_3/k_2)A_{32}$, A_{23} becomes $(k_2/k_3)A_{23}$, and A_{43} becomes $(1/k_3)A_{43}$ (Fig. 8-33).

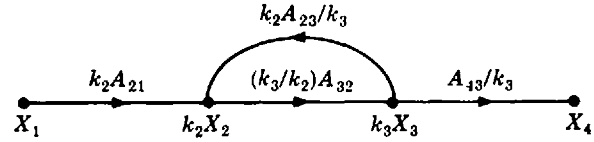


Fig. 8-33

8.4. Consider the signal flow graph given in Fig. 8-34.

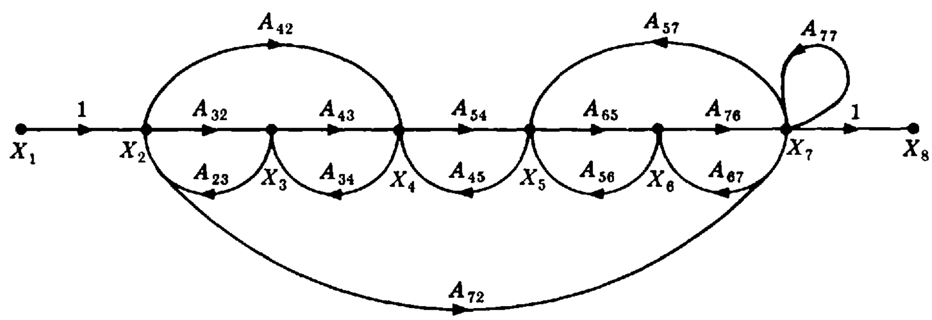


Fig. 8-34

Identify the (a) input node, (b) output node, (c) forward paths, (d) feedback paths, (e) self-loop. Determine the (f) loop gains of the feedback loops, (g) path gains of the forward paths.

(a) X_1

(b) X_8

(c) X_1 to X_2 to X_3 to X_4 to X_5 to X_6 to X_7 to X_8
X_1 to X_2 to X_7 to X_8
X_1 to X_2 to X_4 to X_5 to X_6 to X_7 to X_8

(d) X_2 to X_3 to X_2; X_3 to X_4 to X_3; X_4 to X_5 to X_4; X_2 to X_4 to X_3 to X_2; X_2 to X_7 to X_5 to X_4 to X_3 to X_2; X_5 to X_6 to X_5; X_6 to X_7 to X_6; X_5 to X_6 to X_7 to X_5; X_7 to X_7; X_2 to X_7 to X_6 to X_5 to X_4 to X_3 to X_2

(e) X_7 to X_7

(f) $A_{32}A_{23}$; $A_{43}A_{34}$; $A_{54}A_{45}$; $A_{65}A_{56}$; $A_{76}A_{67}$; $A_{65}A_{76}A_{57}$; A_{77}; $A_{42}A_{34}A_{23}$; $A_{72}A_{57}A_{45}A_{34}A_{23}$; $A_{72}A_{67}A_{56}A_{45}A_{34}A_{23}$

(g) $A_{32}A_{43}A_{54}A_{65}A_{76}$; A_{72}; $A_{42}A_{54}A_{65}A_{76}$

SIGNAL FLOW GRAPH CONSTRUCTION

8.5. Consider the following equations in which $x_1, x_2, \dots, x_n$ are variables and $a_1, a_2, \dots, a_n$ are coefficients or mathematical operators:

(a) $x_3 = a_1x_1 + a_2x_2 \mp 5$ $\qquad$ (b) $x_n = \sum_{k=1}^{n-1} a_k x_k + 5$

What are the minimum number of nodes and the minimum number of branches required to construct the signal flow graphs of these equations? Draw the graphs.

(a) There are four variables in this equation: x_1 x_2, x_3, and ± 5. Therefore a minimum of four nodes are required. There are three coefficients or transmission functions on the right-hand side of the equation:

a_1, a_2, and ∓ 1. Hence a minimum of three branches are required. A minimal signal flow graph is shown in Fig. 8-35(a).

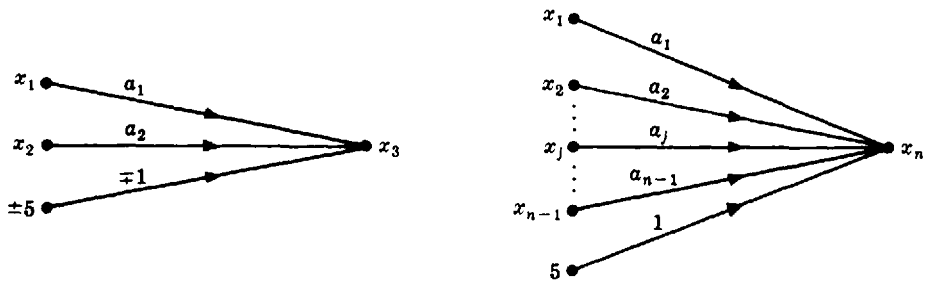


Fig. 8-35

(b) There are $n+1$ variables: $x_1, x_2, \ldots, x_n$, and 5; and there are n coefficients: $a_1, a_2, \ldots, a_{n-1}$, and 1. Therefore a minimal signal flow graph is shown in Fig. 8-35(b).

8.6. Draw signal flow graphs for

$$(a)\quad x_2 = a_1\left(\frac{dx_1}{dt}\right) \qquad (b)\quad x_3 = \frac{d^2x_2}{dt^2} + \frac{dx_1}{dt} - x_1 \qquad (c)\quad x_4 = \int x_3\,dt$$

(a) The operations called for in this equation are a_1 and d/dt. Let the equation be written as $x_2 = a_1 \cdot (d/dt)(x_1)$. Since there are two operations, we may define a new variable dx_1/dt and use it as an intermediate node. The signal flow graph is given in Fig. 8-36.

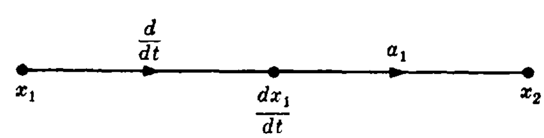


Fig. 8-36

(b) Similarly, $x_3 = (d^2/dt^2)(x_2) + (d/dt)(x_1) - x_1$. Therefore we obtain Fig. 8-37

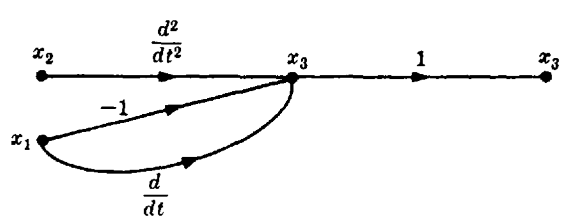


Fig. 8-37

(c) The operation is integration. Let the operator be denoted by $\int dt$. The signal flow graph is given in Fig. 8-38.

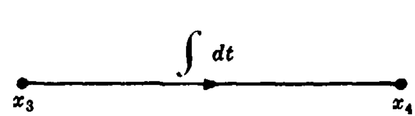


Fig. 8-38

8.7. Construct the signal flow graph for the following set of simultaneous equations:

$$x_2 = A_{21}x_1 + A_{23}x_3 \qquad x_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \qquad x_4 = A_{42}x_2 + A_{43}x_3$$

There are four variables: $x_1, \ldots, x_4$. Hence four nodes are required. Arranging them from left to right and connecting them with the appropriate branches, we obtain Fig. 8-39.

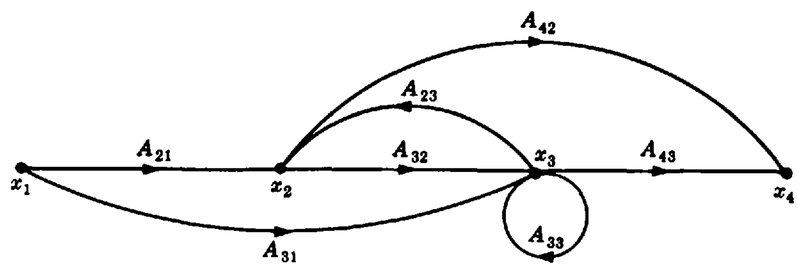


Fig. 8-39

A neater way to arrange this graph is shown in Fig. 8-40.

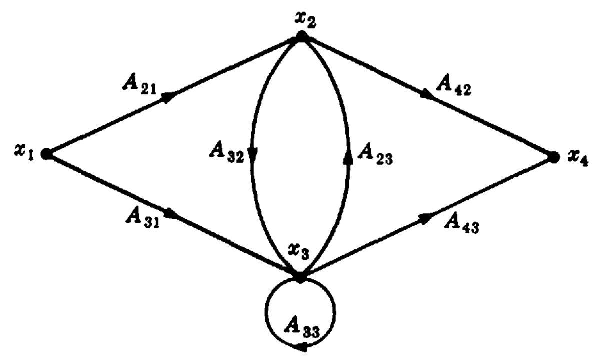


Fig. 8-40

8.8. Draw a signal flow graph for the resistance network shown in Fig. 8-41 in which $v_2(0) = v_3(0) = 0$. v_2 is the voltage across C_1.

Mathcad

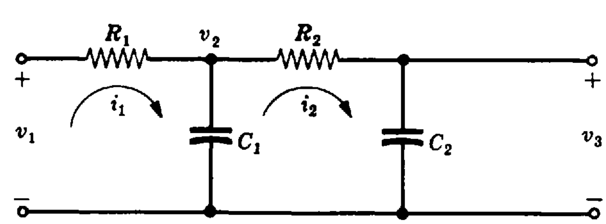


Fig. 8-41

The five variables are v_1, v_2, v_3, i_1, and i_2; and v_1 is the input. The four independent equations derived from Kirchhoff's voltage and current laws are

$$i_1 = \left(\frac{1}{R_1}\right)v_1 - \left(\frac{1}{R_1}\right)v_2 \qquad v_2 = \frac{1}{C_1}\int_0^t i_1\,dt - \frac{1}{C_1}\int_0^t i_2\,dt$$

$$i_2 = \left(\frac{1}{R_2}\right)v_2 - \left(\frac{1}{R_2}\right)v_3 \qquad v_3 = \frac{1}{C_2}\int_0^t i_2\,dt$$

The signal flow graph can be drawn directly from these equations (Fig. 8-42).

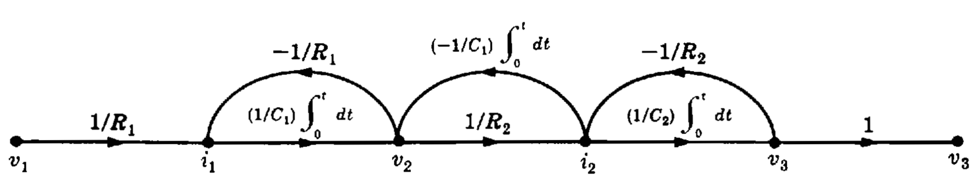


Fig. 8-42

In Laplace transform notation, the signal flow graph is given in Fig. 8-43.

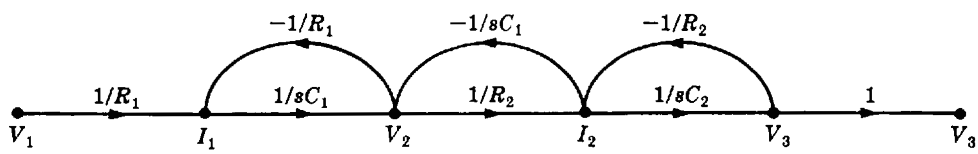


Fig. 8-43

THE GENERAL INPUT-OUTPUT GAIN FORMULA

8.9. The transformed equations for the mechanical system given in Fig. 8-44 are

$$\text{(i)} \qquad F + k_1 X_2 = \left(M_1 s^2 + f_1 s + k_1\right) X_1$$

$$\text{(ii)} \qquad k_1 X_1 = \left(M_2 s^2 + f_2 s + k_1 + k_2\right) X_2$$

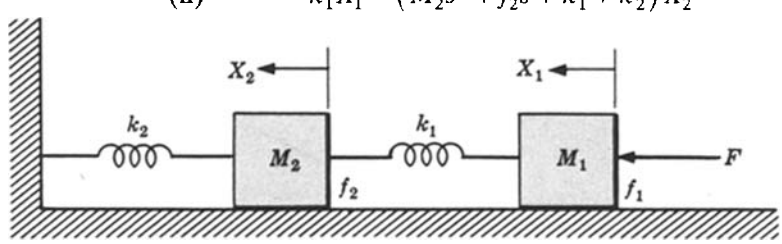


Fig. 8-44

where F is force, M is mass, k is spring constant, f is friction, and X is displacement. Determine X_2/F using Equation (*8.2*).

There are three variables: X_1, X_2, and F. Therefore we need three nodes. In order to draw the signal flow graph, divide Equation (i) by A and Equation (ii) by B, where $A \equiv M_1 s^2 + f_1 s + k_1$, and $B \equiv M_2 s^2 + f_2 s + k_1 + k_2$:

$$\text{(iii)} \qquad \left(\frac{1}{A}\right) F + \left(\frac{k_1}{A}\right) X_2 = X_1$$

$$\text{(iv)} \qquad \left(\frac{k_1}{B}\right) X_1 = X_2$$

Therefore the signal flow graph is given in Fig. 8-45.

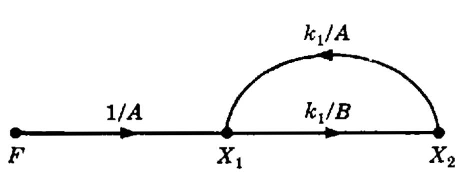


Fig. 8-45

The forward path gain is $P_1 = k_1/AB$. The feedback loop gain is $P_{11} = k_1^2/AB$. then $\Delta = 1 - P_{11} = (AB - k_1^2)/AB$ and $\Delta_1 = 1$. Finally,

$$\frac{X_2}{F} = \frac{P_1 \Delta_1}{\Delta} = \frac{k_1}{AB - k_1^2} = \frac{k_1}{\left(M_1 s^2 + f_1 s + k_1\right)\left(M_2 s^2 + f_2 s + k_1 + k_2\right) - k_1^2}$$

8.10. Determine the transfer function for the block diagram in Problem 7.20 by signal flow graph techniques.

The signal flow graph, Fig. 8-46, is drawn directly from Fig. 7-44. There are two forward paths. The path gains are $P_1 = G_1G_2G_3$ and $P_2 = G_4$. The three feedback loop gains are $P_{11} = -G_2H_1$, $P_{21} = G_1G_2H_1$, and $P_{31} = -G_2G_3H_2$. No loops are nontouching. Hence $\Delta = 1 - (P_{11} + P_{21} + P_{31})$. Also, $\Delta_1 = 1$; and since no loops touch the nodes of P_2, $\Delta_2 = \Delta$. Thus

$$T = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_4 + G_2G_4H_1 - G_1G_2G_4H_1 + G_2G_3G_4H_2}{1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2}$$

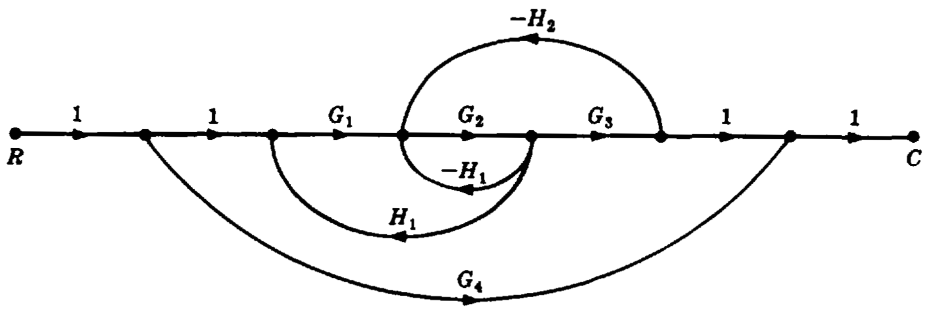


Fig. 8-46

8.11. Determine the transfer function V_3/V_1 from the signal flow graph of Problem 8.8.

The single forward path gain is $1/(s^2R_1R_2C_1C_2)$. The loop gains of the three feedback loops are $P_{11} = -1/(sR_1C_1)$, $P_{21} = -1/(sR_2C_1)$, and $P_{31} = -1/(sR_2C_2)$. The gain product of the only two nontouching loops is $P_{12} = P_{11} \cdot P_{31} = 1/(s^2R_1R_2C_1C_2)$. Hence

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = \frac{s^2R_1R_2^2C_1^2C_2 + s(R_2^2C_1C_2 + R_1R_2C_1C_2 + R_1R_2C_1^2) + R_2C_1}{s^2R_1R_2^2C_1^2C_2}$$

Since all loops touch the forward path, $\Delta_1 = 1$. Finally,

$$\frac{V_3}{V_1} = \frac{P_1\Delta_1}{\Delta} = \frac{1}{s^2R_1R_2C_1C_2 + s(R_2C_2 + R_1C_2 + R_1C_1) + 1}$$

8.12. Solve Problem 7.16 with signal flow graph techniques.

The signal flow graph is drawn directly from Fig. 7-26, as shown in Fig. 8-47:

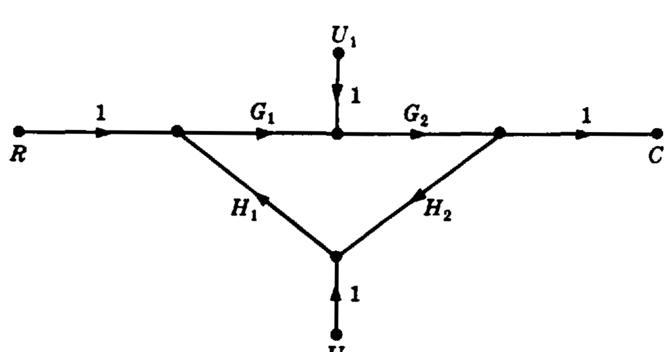


Fig. 8-47

With $U_1 = U_2 = 0$, we have Fig. 8-48. Then $P_1 = G_1G_2$ and $P_{11} = G_1G_2H_1H_2$. Hence $\Delta = 1 - P_{11} = 1 - G_1G_2H_1H_2$, $\Delta_1 = 1$, and

$$C_R = TR = \frac{P_1\Delta_1 R}{\Delta} = \frac{G_1G_2R}{1 - G_1G_2H_1H_2}$$

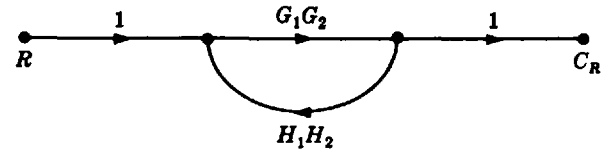


Fig. 8-48

Now put $U_2 = R = 0$ (Fig. 8-49).

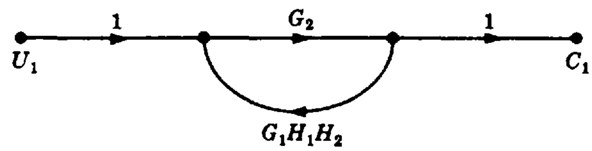


Fig. 8-49

Then $P_1 = G_2$, $P_{11} = G_1G_2H_1H_2$, $\Delta = 1 - G_1G_2H_1H_2$, $\Delta_1 = 1$, and

$$C_1 = TU_1 = \frac{G_2U_1}{1 - G_1G_2H_1H_2}$$

Now put $R = U_1 = 0$ (Fig. 8-50).

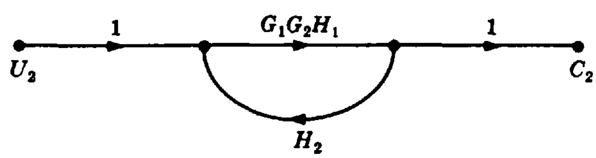


Fig. 8-50

Then $P_1 = G_1G_2H_1$, $P_{11} = G_1G_2H_1H_2$, $\Delta = 1 - G_1G_2H_1H_2$, $\Delta_1 = 1$, and

$$C_2 = TU_2 = \frac{P_1\Delta_1 U_2}{\Delta} = \frac{G_1G_2H_1U_2}{1 - G_1G_2H_1H_2}$$

Finally, we have

$$C = C_R + C_1 + C_2 = \frac{G_1G_2R + G_2U_1 + G_1G_2H_1U_2}{1 - G_1G_2H_1H_2}$$

TRANSFER FUNCTION COMPUTATION OF CASCADED COMPONENTS

8.13. Determine the transfer function for two of the networks in cascade shown in Fig. 8-51.

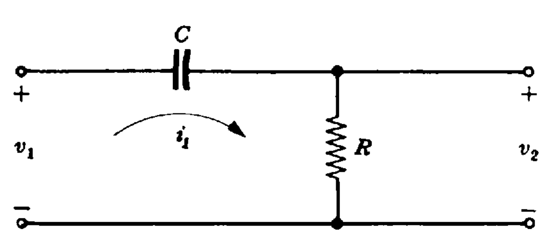


Fig. 8-51

In Laplace transform notation the network becomes Fig. 8-52.

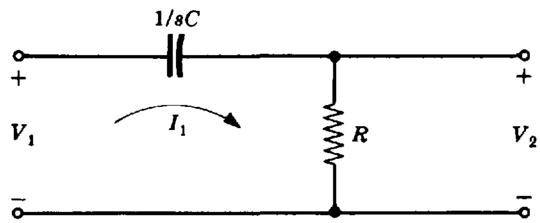


Fig. 8-52

By Kirchhoff's laws, we have $I_1 = sCV_1 - sCV_2$ and $V_2 = RI_1$. The signal flow graph is given in Fig. 8.53.

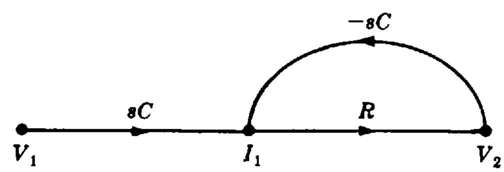


Fig. 8-53

For two networks in cascade (Fig. 8-54) the V_2 equation is also dependent on I_2: $V_2 = RI_1 - RI_2$. Hence two networks are joined at node 2 (Fig. 8-55) and a feedback loop $(-RI_2)$ is added between I_2 and V_2 (Fig. 8-56).

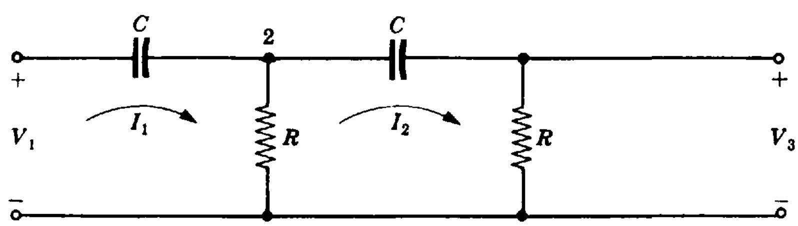


Fig. 8-54

Fig. 8-55

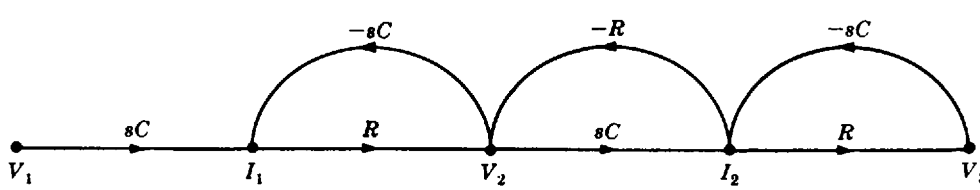


Fig. 8-56

Then $P_1 = s^2R^2C^2$, $P_{11} = P_{31} = -sRC$, $P_{12} = P_{11} \cdot P_{31} = s^2R^2C^2$, $\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 + 3sRC + s^2R^2C^2$, $\Delta_1 = 1$, and

$$T = \frac{P_1\Delta_1}{\Delta} = \frac{s^2}{s^2 + (3/RC)s + 1/(RC)^2}$$

8.14. Two resistance networks in the form of that in Example 8.6 are to be used for control elements in the forward path of a control system. They are to be cascaded and shall have identical respective component values as shown in Fig. 8-57. Find v_5/v_1 using Equation (*8.2*).

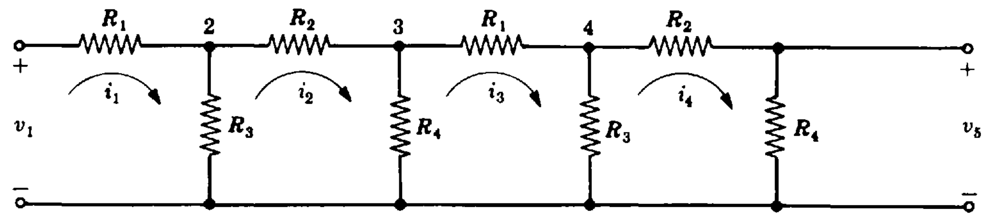


Fig. 8-57

There are nine variables: v_1, v_2, v_3, v_4, v_5, i_1, i_2, i_3, and i_4. Eight independent equations are

$$i_1 = \left(\frac{1}{R_1}\right)v_1 - \left(\frac{1}{R_1}\right)v_2 \qquad i_3 = \left(\frac{1}{R_1}\right)v_3 - \left(\frac{1}{R_1}\right)v_4$$

$$v_2 = R_3 i_1 - R_3 i_2 \qquad v_4 = R_3 i_3 - R_3 i_4$$

$$i_2 = \left(\frac{1}{R_2}\right)v_2 - \left(\frac{1}{R_2}\right)v_3 \qquad i_4 = \left(\frac{1}{R_2}\right)v_4 - \left(\frac{1}{R_2}\right)v_5$$

$$v_3 = R_4 i_2 - R_4 i_3 \qquad v_5 = R_4 i_4$$

Only the equation for v_3 is different from those of the single network of Example 8.6; it has an extra term, $(-R_4 i_3)$. Therefore the signal flow diagram for each network alone (Example 8.6) may be joined at node v_3, and an extra branch of gain $-R_4$ drawn from i_3 to v_3. The resulting signal flow graph for the double network is given in Fig. 8-58.

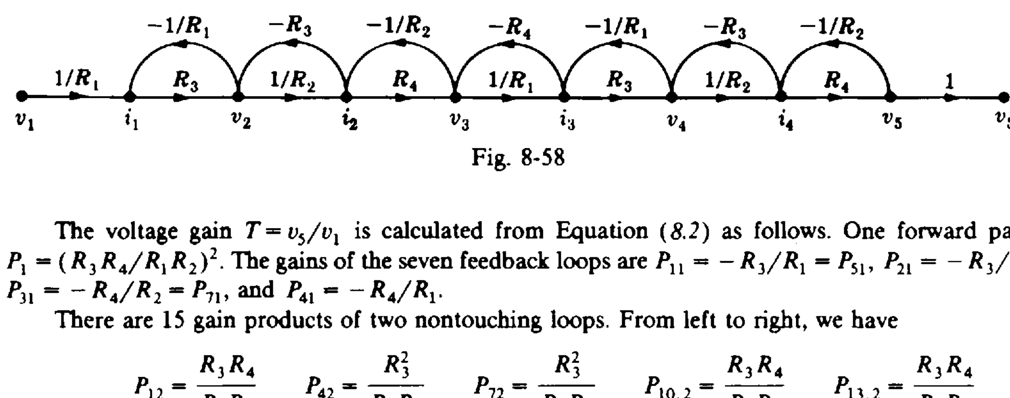


Fig. 8-58

The voltage gain $T = v_5/v_1$ is calculated from Equation (*8.2*) as follows. One forward path yields $P_1 = (R_3 R_4/R_1 R_2)^2$. The gains of the seven feedback loops are $P_{11} = -R_3/R_1 = P_{51}$, $P_{21} = -R_3/R_2 = P_{61}$, $P_{31} = -R_4/R_2 = P_{71}$, and $P_{41} = -R_4/R_1$.

There are 15 gain products of two nontouching loops. From left to right, we have

$$P_{12} = \frac{R_3 R_4}{R_1 R_2} \qquad P_{42} = \frac{R_3^2}{R_1 R_2} \qquad P_{72} = \frac{R_3^2}{R_1 R_2} \qquad P_{10,2} = \frac{R_3 R_4}{R_1 R_2} \qquad P_{13,2} = \frac{R_3 R_4}{R_1 R_2}$$

$$P_{22} = \frac{R_3 R_4}{R_1^2} \qquad P_{52} = \frac{R_3 R_4}{R_1 R_2} \qquad P_{82} = \left(\frac{R_3}{R_2}\right)^2 \qquad P_{11,2} = \frac{R_3 R_4}{R_2^2} \qquad P_{14,2} = \frac{R_4^2}{R_1 R_2}$$

$$P_{32} = \left(\frac{R_3}{R_1}\right)^2 \qquad P_{62} = \frac{R_3 R_4}{R_1 R_2} \qquad P_{92} = \frac{R_3 R_4}{R_2^2} \qquad P_{12,2} = \left(\frac{R_4}{R_2}\right)^2 \qquad P_{15,2} = \frac{R_3 R_4}{R_1 R_2}$$

There are 10 gain products of three nontouching loops. From left to right, we have

$$P_{13} = \frac{R_3^2 R_4}{R_1^2 R_2} \qquad P_{33} = -\frac{R_3 R_4^2}{R_1 R_2^2} \qquad P_{63} = -\frac{R_3^2 R_4}{R_1^2 R_2} \qquad P_{83} = -\frac{R_3 R_4^2}{R_1 R_2^2} \qquad P_{53} = -\frac{R_3 R_4^2}{R_1^2 R_2}$$

$$P_{23} = -\frac{R_3^2 R_4}{R_1 R_2^2} \qquad P_{43} = -\frac{R_3^2 R_4}{R_1^2 R_2} \qquad P_{73} = -\frac{R_3^2 R_4}{R_1 R_2^2} \qquad P_{93} = -\frac{R_3^2 R_4}{R_1 R_2^2} \qquad P_{10,3} = -\frac{R_3 R_4^2}{R_1 R_2^2}$$

There is one gain product of four nontouching loops: $P_{14} = P_{11}P_{31}P_{51}P_{71} = (R_3R_4/R_1R_2)^2$. Therefore the determinant is

$$\Delta = 1 - \sum_{j=1}^{7} P_{j1} + \sum_{j=1}^{15} P_{j2} - \sum_{j=1}^{10} P_{j3} + P_{14}$$

$$= 1 + \frac{R_1R_3 + R_1R_4 + R_2R_3 + R_2R_4 + 6R_3R_4 + 2R_3^2 + R_4^2}{R_1R_2} + \frac{R_3R_4 + R_3^2}{R_1^2} + \frac{R_3^2 + R_4^2 + R_3R_4}{R_2^2}$$

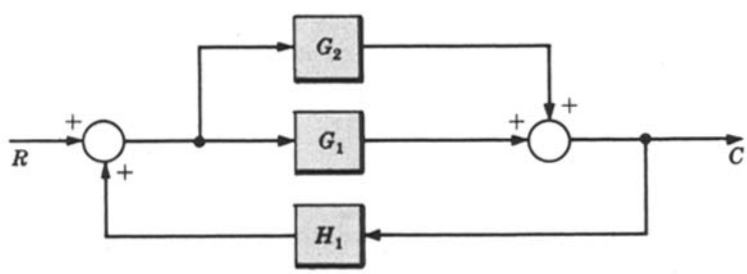
Since all loops touch the forward path, $\Delta_1 = 1$ and

$$T = \frac{P_1\Delta_1}{\Delta} = \frac{(R_3R_4)^2}{\begin{array}{c}(R_1R_2)^2 + R_1^2(R_2R_3 + R_2R_4 + R_3R_4 + R_3^2 + R_4^2) + R_2^2(R_3^2 + R_1R_3 + R_1R_4 + R_3R_4)\\ +2R_1R_2R_3^2 + R_1R_2R_4^2 + 6R_1R_2R_3R_4\end{array}}$$

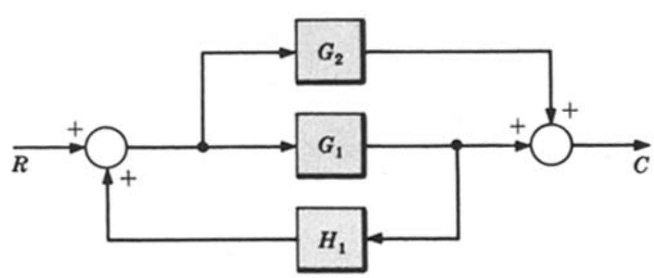
BLOCK DIAGRAM REDUCTION

8.15. Determine C/R for each system shown in Fig. 8-59 using Equation (8.2).

(a)



(b)



(c)

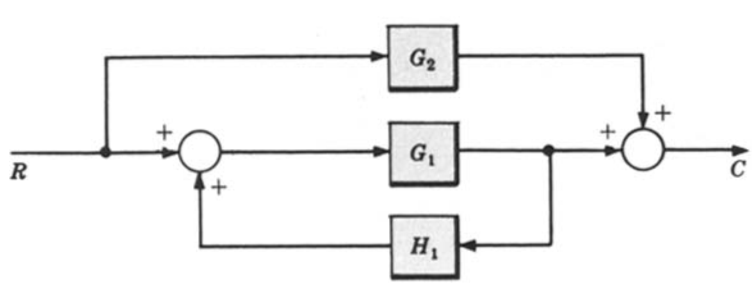


Fig. 8-59

(a) The signal flow graph is given in Fig. 8-60. The two forward path gains are $P_1 = G_1$, $P_2 = G_2$. The two feedback loop gains are $P_{11} = G_1H_1$, $P_{21} = G_2H_1$. Then

$$\Delta = 1 - (P_{11} + P_{21}) = 1 - G_1H_1 - G_2H_1$$

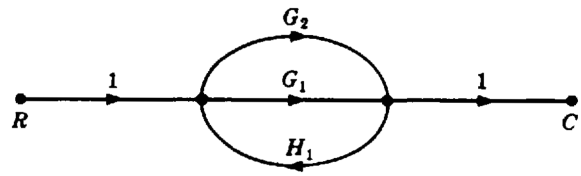


Fig. 8-60

Now, $\Delta_1 = 1$ and $\Delta_2 = 1$ because both paths touch the feedback loops at both interior nodes. Hence

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1 + G_2}{1 - G_1H_1 - G_2H_1}$$

(*b*) The signal flow graph is given in Fig. 8-61. Again, we have $P_1 = G_1$ and $P_2 = G_2$. But now there is only one feedback loop, and $P_{11} = G_1H_1$; then $\Delta = 1 - G_1H_1$. The forward path through G_1 clearly touches the feedback loop at nodes a and b; thus $\Delta_1 = 1$. The forward path through G_2 touches the feedback loop at node a; then $\Delta_2 = 1$. Hence

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1 + G_2}{1 - G_1H_1}$$

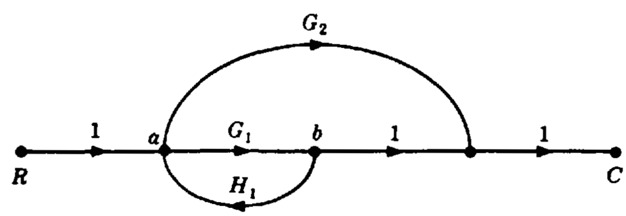


Fig. 8-61

(*c*) The signal flow graph is given in Fig. 8-62. Again, we have $P_1 = G_1$, $P_2 = G_2$, $P_{11} = G_1H_1$, $\Delta = 1 - G_1H_1$, and $\Delta_1 = 1$. But the feedback path *does not* touch the forward path through G_2 at *any* node. Therefore $\Delta_2 = \Delta = 1 - G_1H_1$ and

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1 + G_2(1 - G_1H_1)}{1 - G_1H_1}$$

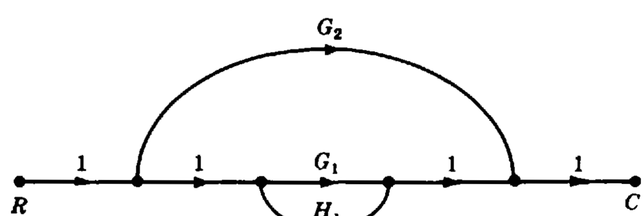


Fig. 8-62

This problem illustrates the importance of separating summing points and takeoff points with a branch of unity gain when applying Equation (*8.2*).

8.16. Find the transfer function C/R for the system shown in Fig. 8-63 in which K is a constant.

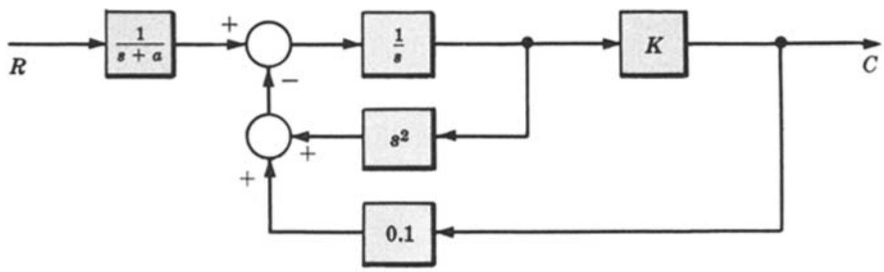


Fig. 8-63

The signal flow graph is given in Fig. 8-64. The only forward path gain is

$$P_1 = \left(\frac{1}{s+a}\right) \cdot \left(\frac{1}{s}\right) K = \frac{K}{s(s+a)}$$

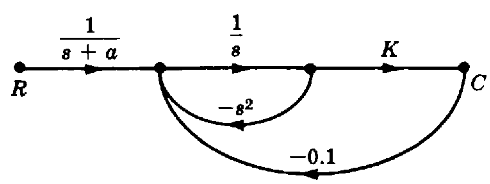


Fig. 8-64

The two feedback loop gains are $P_{11} = (1/s) \cdot (-s^2) = -s$ and $P_{21} = -0.1K/s$. There are no nontouching loops. Hence

$$\Delta = 1 - (P_{11} + P_{21}) = \frac{s^2 + s - 0.1K}{s} \qquad \Delta_1 = 1 \qquad \frac{C}{R} = \frac{P_1\Delta_1}{\Delta} = \frac{K}{(s+a)(s^2+s+0.1K)}$$

8.17. Solve Problem 7.18 using signal flow graph techniques.

The signal flow graph is given in Fig. 8-65.

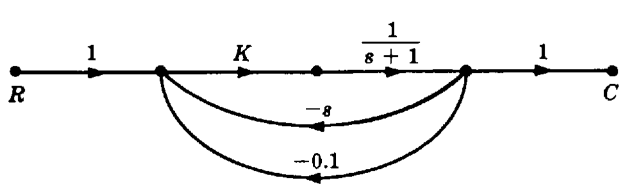


Fig. 8-65

Applying the multiplication and addition rules, we obtain Fig. 8-66. Now

$$P_1 = \frac{K}{s+1} \qquad P_{11} = -\frac{K(s+0.1)}{s+1} \qquad \Delta = 1 + \frac{K(s+0.1)}{s+1} \qquad \Delta_1 = 1,$$

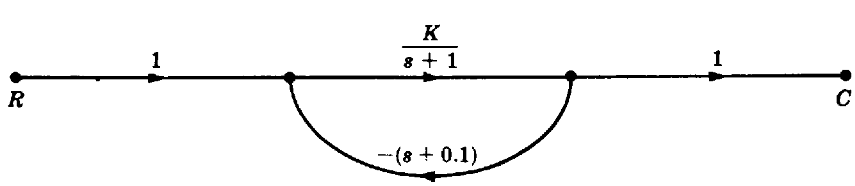


Fig. 8-66

and

$$C = TR = \frac{P_1 \Delta_1 R}{\Delta} = \frac{KR}{(1+K)s + 1 + 0.1K}$$

8.18. Find C/R for the control system given in Fig. 8-67.

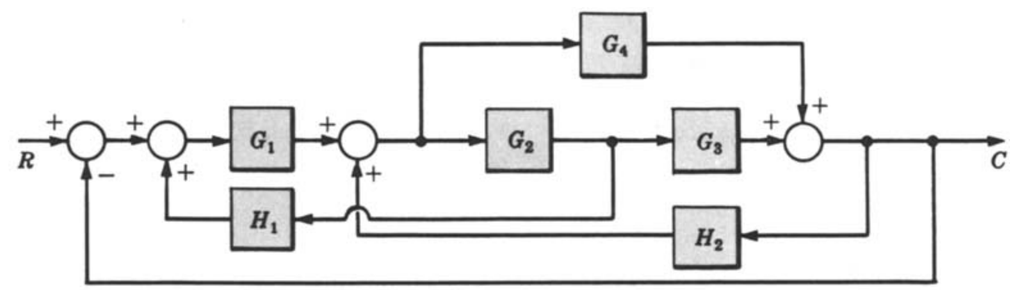


Fig. 8-67

The signal flow graph is given in Fig. 8.68. The two forward path gains are $P_1 = G_1G_2G_3$ and $P_2 = G_1G_4$. The five feedback loop gains are $P_{11} = G_1G_2H_1$, $P_{21} = G_2G_3H_2$, $P_{31} = -G_1G_2G_3$, $P_{41} = G_4H_2$, and $P_{51} = -G_1G_4$. Hence

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51}) = 1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4$$

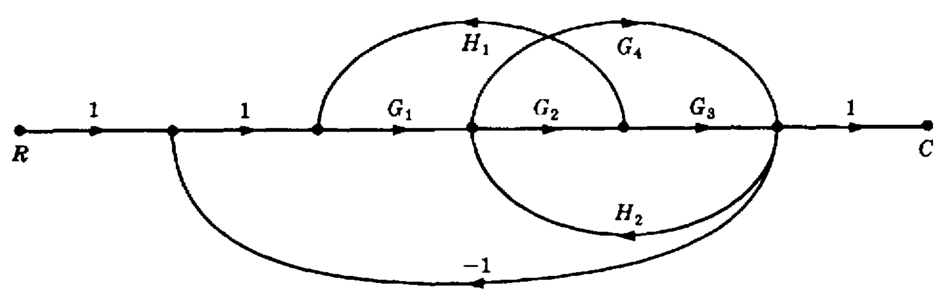


Fig. 8-68

and $\Delta_1 = \Delta_2 = 1$. Finally,

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4}$$

8.19. Determine C/R for the system given in Fig. 8-69. *Then* put $G_3 = G_1G_2H_2$.

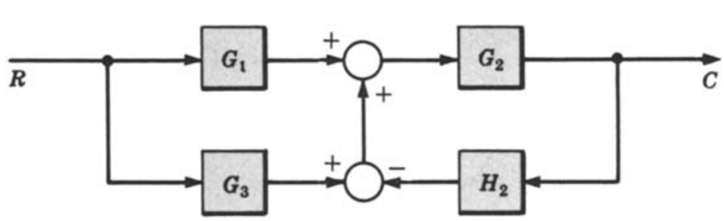


Fig. 8-69

The signal flow graph is given in Fig. 8-70. We have $P_1 = G_1G_2$, $P_2 = G_2G_3$, $P_{11} = -G_2H_2$, $\Delta = 1 + G_2H_2$, $\Delta_1 = \Delta_2 = 1$, and

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_2(G_1 + G_3)}{1 + G_2H_2}$$

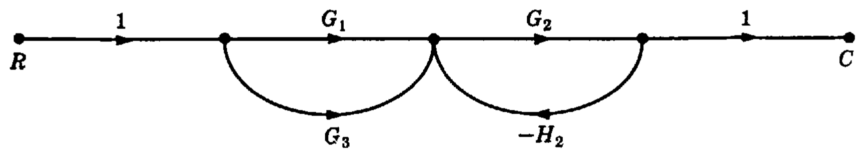


Fig. 8-70

Putting $G_3 = G_1G_2H_2$, we obtain $C/R = G_1G_2$ and the system transfer function becomes open-loop.

8.20. Determine the elements for a canonical feedback system for the system of Problem 8.10.

From Problem 8.10, $P_1 = G_1G_2G_3$, $P_2 = G_4$, $\Delta = 1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2$, $\Delta_1 = 1$, and $\Delta_2 = \Delta$. From Equation (*8.3*) we have

$$G = \sum_{i=1}^{2} P_i\Delta_i = G_1G_2G_s + G_4 + G_2G_4H_1 - G_1G_2G_4H_1 + G_2G_3G_4H_2$$

and from Equation (*8.4*) we obtain

$$H = \frac{\Delta - 1}{G} = \frac{G_2H_1 - G_1G_2H_1 + G_2G_3H_2}{G_1G_2G_3 + G_4 + G_2G_4H_1 - G_1G_2G_4H_1 + G_2G_3G_4H_2}$$

Supplementary Problems

8.21. Find C/R for Fig. 8-71, using Equation (*8.2*).

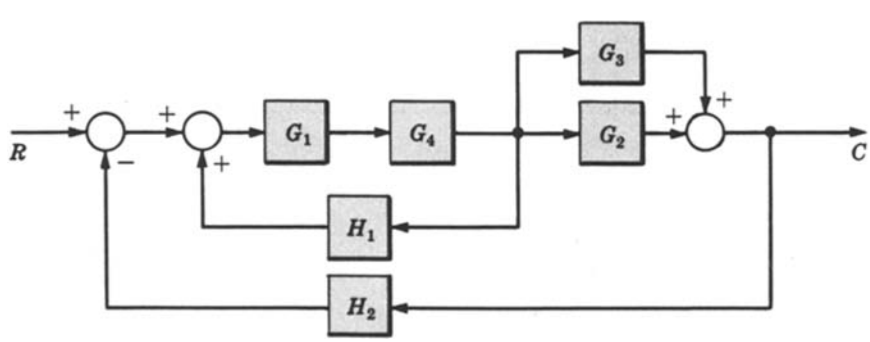


Fig. 8-71

8.22. Determine a set of canonical feedback system transfer functions for the preceding problem, using Equations (*8.3*) and (*8.4*).

8.23. Scale the signal flow graph in Fig. 8-72 so that X_3 becomes $X_3/2$ (see Problem 8.3).

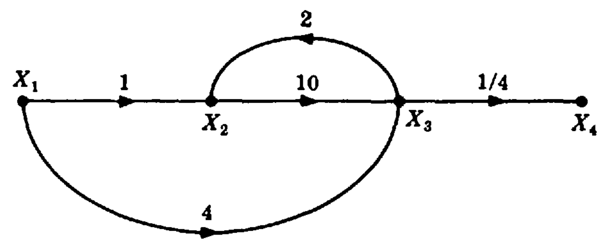


Fig. 8-72

8.24. Draw a signal flow graph for several nodes of the lateral inhibition system described in Problem 3.4 by the equation

$$c_k = r_k - \sum_{i=1}^{n} a_{k-i} c_i$$

8.25. Draw a signal flow graph for the system presented in Problem 7.31.

8.26. Draw a signal flow graph for the system presented in Problem 7.32.

8.27. Determine C/R_4 from Equation (*8.2*) for the signal flow graph drawn in Problem 8.26.

8.28. Draw a signal flow graph for the electrical network in Fig. 8-73.

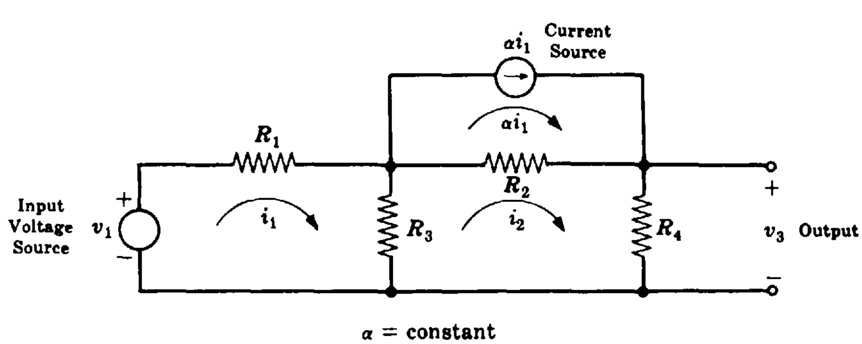


Fig. 8-73

8.29. Determine V_3/V_1 from Equation (*8.2*) for the network of Problem 8.28.

8.30. Determine the elements for a canonical feedback system for the network of Problem 8.28, using Equations (*8.3*) and (*8.4*).

8.31. Draw the signal flow graph for the analog computer circuit in Fig 8-74.

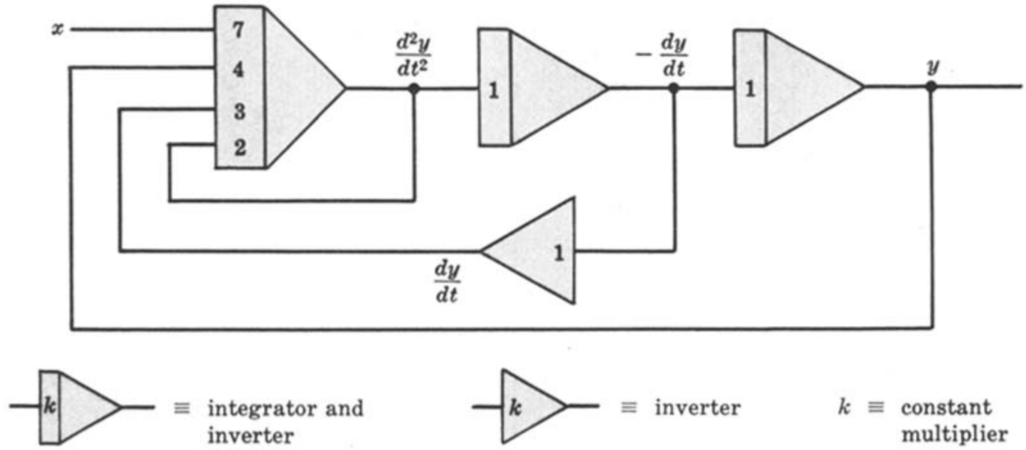


Fig. 8-74

8.32. Scale the analog computer circuit of Problem 8.31 so that y becomes $10y$, dy/dt becomes $20(dy/dt)$, and d^2y/dt^2 becomes $5(d^2y/dt^2)$.

Answers to Supplementary Problems

8.21. $P_1 = G_1G_2G_4$; $P_2 = G_1G_3G_4$, $P_{11} = G_1G_4H_1$, $P_{21} = -G_1G_2G_4H_2$, $P_{31} = -G_1G_3G_4H_2$, $\Delta = 1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2$, and $\Delta_1 = \Delta_2 = 1$. Therefore

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_4(G_2+G_3)}{1 - G_1G_4[H_1 - H_2(G_2+G_3)]}$$

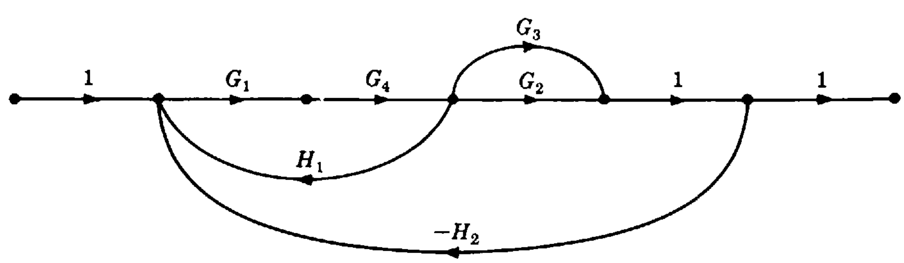


Fig. 8-75

8.22. $G = P_1\Delta_1 + P_2\Delta_2 = G_1G_4(G_2+G_3)$ $\qquad H = \dfrac{\Delta - 1}{G} = H_2 - \dfrac{H_1}{G_2+G_3}$

8.23.

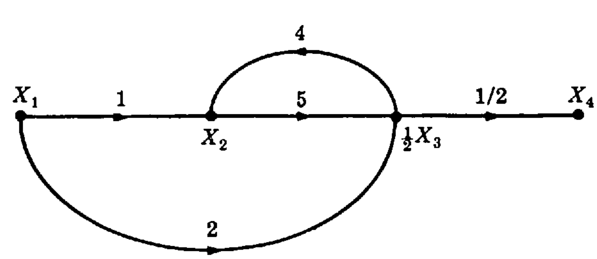


Fig. 8-76

8.24.

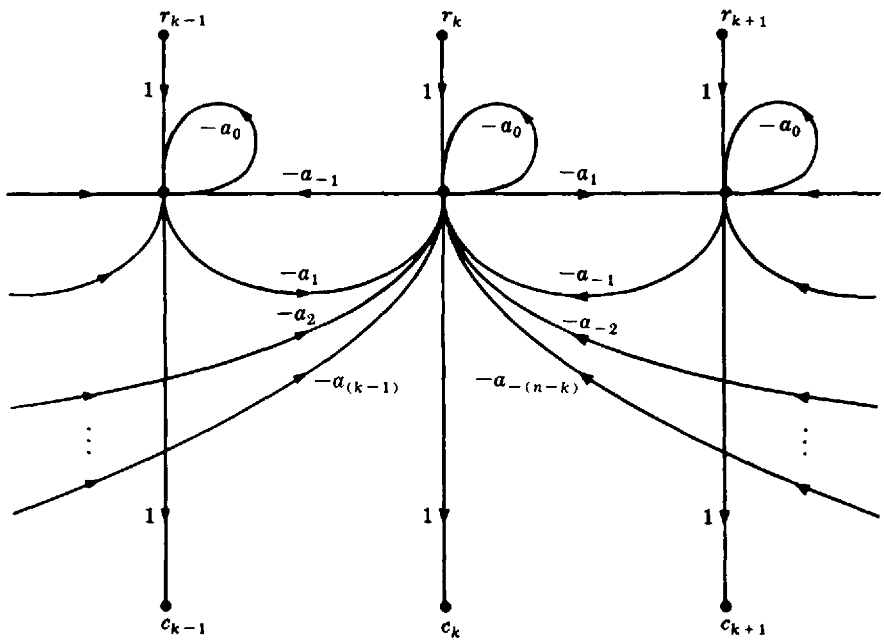


Fig. 8-77

8.25.

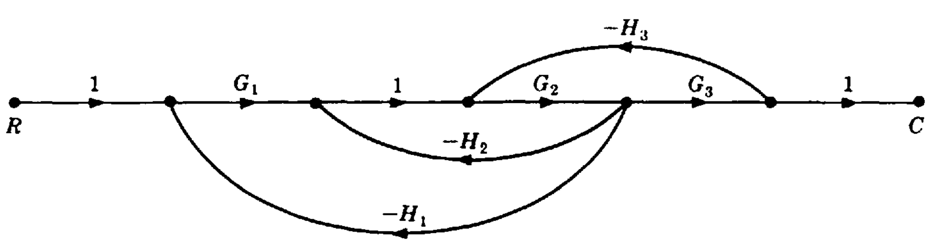


Fig. 8-78

8.26.

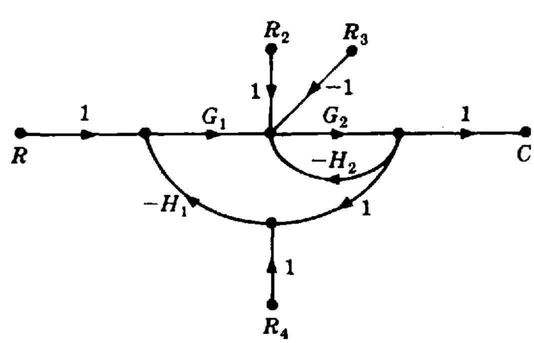


Fig. 8-79

8.27. $\dfrac{C}{R_4} = \dfrac{-G_1G_2H_1}{1 + G_2H_2 + G_1G_2H_1}$

8.28.

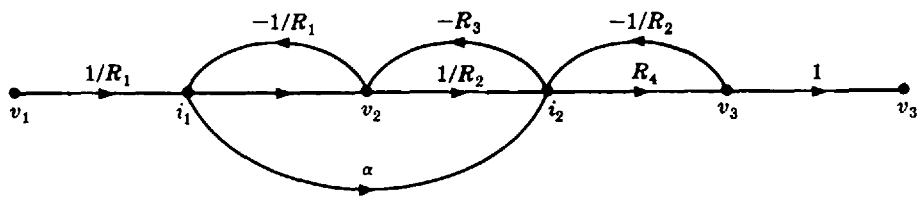


Fig. 8-80

8.29. $$\frac{V_3}{V_1} = \frac{R_3R_4 + \alpha R_2R_4}{R_1R_2 + R_1R_3 + R_1R_4 + R_2R_3 + R_3R_4 - \alpha R_2R_3}$$

8.30. $G = R_4(R_3 + \alpha R_2)$

$$H = \frac{R_1(R_2 + R_3 + R_4) + R_3R_4 + R_2R_3(1-\alpha)}{R_4(R_3 + \alpha R_2)}$$

8.31.

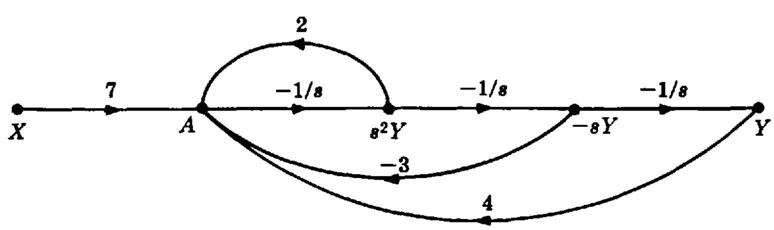


Fig. 8-81

8.32.

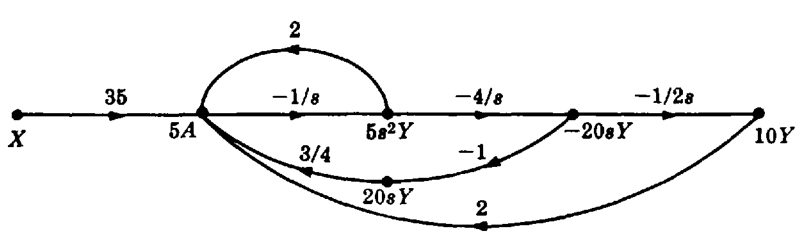


Fig. 8-82