University of Stuttgart
Institute for Modelling Hydraulic and Environmental Systems

LH2

SimTech
Cluster of Excellence

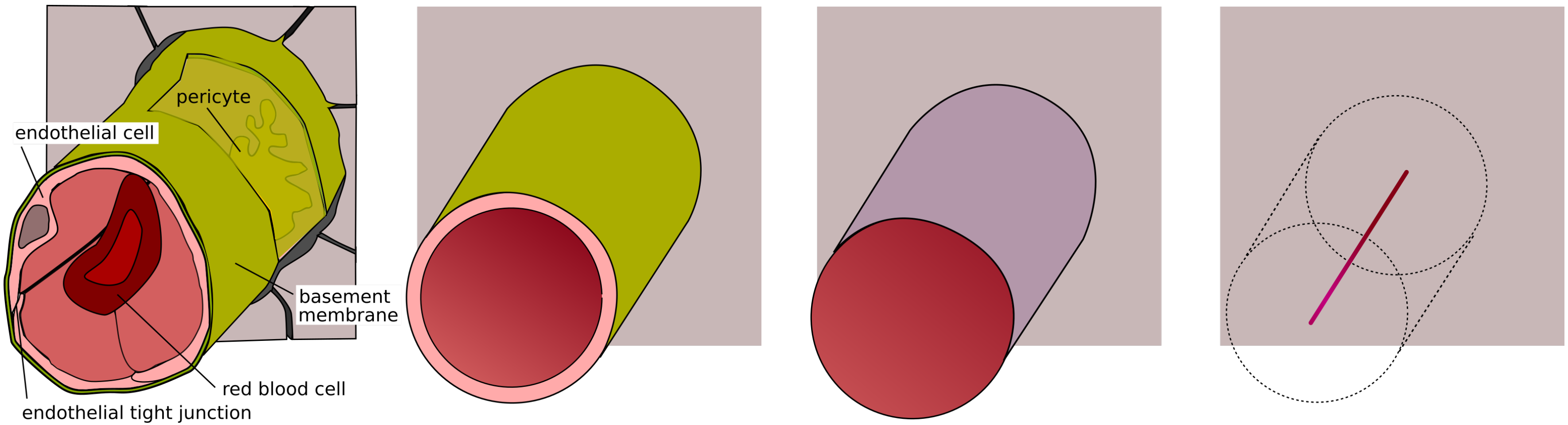
Chapter 4

Modeling flow in capillaries

Data-integrated Simulation Science A – Part D Biofluidmechanics

Modeling flow in capillaries

General idea



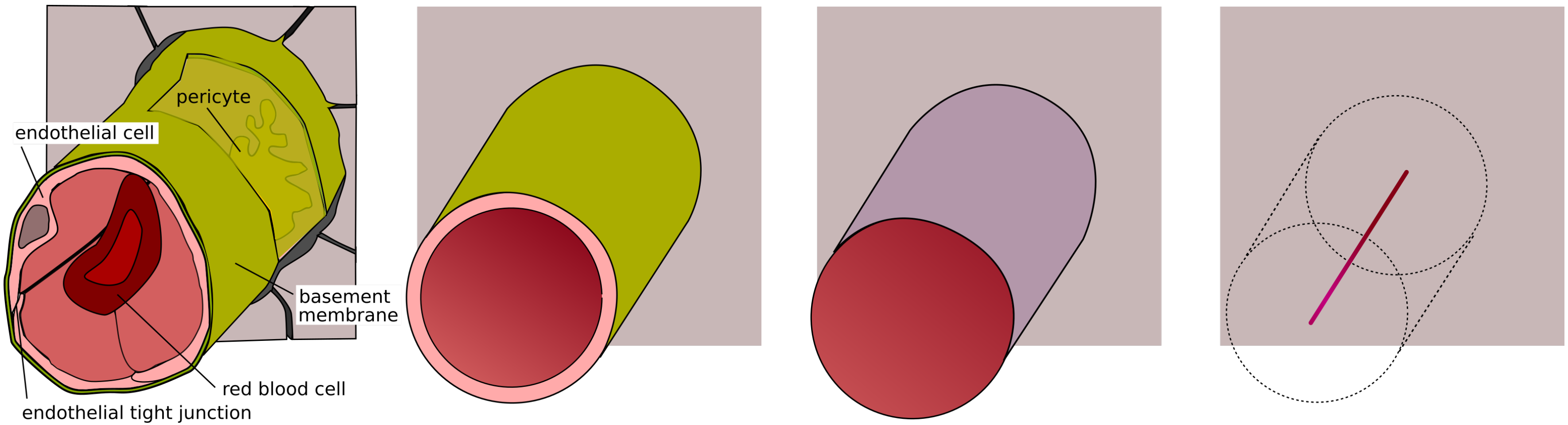
Abstraction

Reduce wall

Integrate over cross-section S

Modeling flow in capillaries

General idea



Abstraction → Reduce wall → **Integrate over cross-section S**

Modeling flow in capillaries

Assumptions

- Low Reynolds numbers (Re << 1)
  - Neglection of higher-order terms (quadratic in velocity)
  - Neglection of inertia terms (stationary flow)
- Rigid vessel wall

$$A(z,\cancel{t}) = |S(z,\cancel{t})| = 2\pi R(z,\cancel{t})$$

- Leaky vessel (fluid crosses the vessel wall)

$$v_r(z,R) := v_R(z)$$

Modeling flow in capillaries

Radially-symmetric Stokes equations

$$\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\hat{\mu}_B}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{\partial^2 v_r}{\partial z^2}\right) = 0$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\hat{\mu}_B}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right) = 0$$

$$\boxed{\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0}$$

mass balance equation

Modeling flow in capillaries

Mass balance equation – Integration over cross-section

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0,$$

$$A(z, \cancel{t}) = |S(z, \cancel{t})| = 2\pi R(z, \cancel{t})$$

$$\int_S \frac{1}{r}\frac{\partial(r v_r)}{\partial r}\,\mathrm{d}A + \int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0$$

Modeling flow in capillaries

Mass balance equation – Integration over cross-section

$$\boxed{\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A} + \int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0$$

$$\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A = \int_0^{2\pi}\int_0^{R} \frac{\partial(rv_r)}{\partial r}\mathrm{d}r\mathrm{d}\theta = 2\pi R v_r(z,R) := 2\pi R v_R(z).$$

„Leaky vessel“

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Mass balance equation – Integration over cross-section

$$\int_S \frac{1}{r}\frac{\partial (r v_r)}{\partial r}\,\mathrm{d}A + \boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0}$$

$$\int_S \frac{\partial u_z}{\partial z}\,\mathrm{d}A = \frac{\partial}{\partial z}\left(\int_S u_z\,\mathrm{d}A\right) - 2\pi R\,\underbrace{u_z(t,R)}_{=0}\frac{\partial R}{\partial z} = \frac{\partial Q}{\partial z}$$

Modeling flow in capillaries

Mass balance equation – 1D model

$$\int_S \frac{1}{r}\frac{\partial(r v_r)}{\partial r}\,\mathrm{d}A + \int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0$$

$$\Rightarrow \quad \frac{\partial Q(z)}{\partial z} = -2\pi R v_R(z)$$

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Momentum balance equation – 1D model

Same as for elastic tubes but without intertia / nonlinear term and constant A

$$\cancelto{0}{\frac{\partial Q}{\partial t}} + \cancelto{0}{\frac{\partial}{\partial z}\left(\alpha \frac{Q^2}{A}\right)} + \frac{\overset{\rightarrow A_0}{A}}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A} = 0.$$

$$\Longrightarrow \quad \frac{A_0}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A_0} = 0$$

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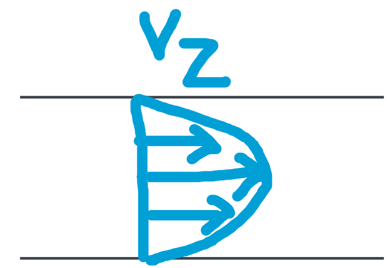
1D model

$$\frac{A_0}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A_0} = 0 \qquad\qquad \frac{\partial Q(z)}{\partial z} = -2\pi R v_R(z)$$

→ insert momenum balance into mass balance + power-law velocity profile

$$-\frac{\partial}{\partial z}\left(\frac{A_0^2}{\rho K_R}\frac{\partial p}{\partial z}\right) = -2\pi R v_R(z)$$

$$-\frac{\partial}{\partial z}\left(\frac{\rho}{\mu_B}\frac{\pi R^4}{2(\gamma+2)}\frac{\partial p}{\partial z}\right) = -\rho 2\pi R v_R(z)$$



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1D model

$$\frac{A_0}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A_0} = 0 \qquad \frac{\partial Q(z)}{\partial z} = -2\pi R v_R(z)$$

→ insert momenum balance into mass balance + power-law velocity profile

$$-\frac{\partial}{\partial z}\left(\frac{A_0^2}{\rho K_R}\frac{\partial p}{\partial z}\right) = -2\pi R v_R(z)$$

$$-\frac{\partial}{\partial z}\left(\frac{\rho}{\mu_B}\frac{\pi R^4}{2(\gamma+2)}\frac{\partial p}{\partial z}\right) = -\rho 2\pi R v_R(z)$$

Non-Newtonian behavior **Flow across vessel wall**

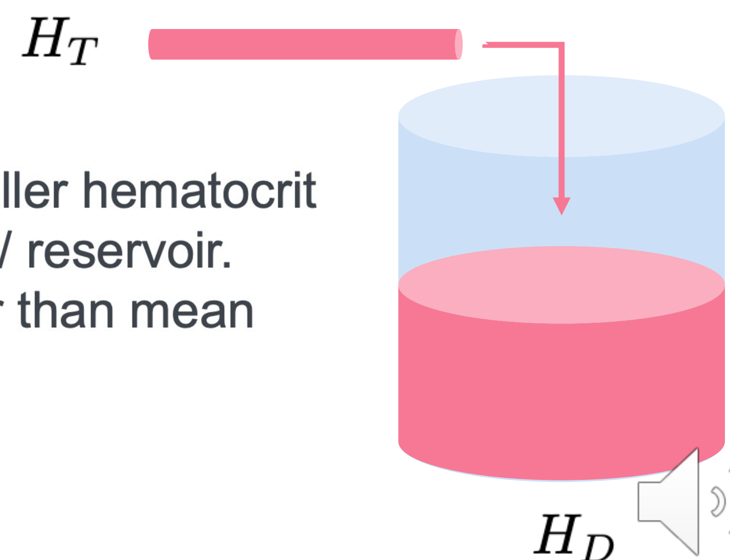
→ need for closure models

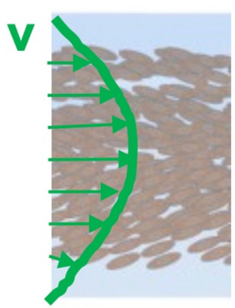
Modeling flow in capillaries

Non-Newtonian behavior

→ Fåhræus effect

$$\frac{H_T}{H_D} = H_D + (1 - H_D)\left(1 + 1.7e^{-0.415D} - 0.6e^{-0.011D}\right)$$





Axial migration leads to smaller hematocrit than in feeding large vessel / reservoir. Mean RBC velocity is higher than mean plasma velocity.

for references see script

Modeling flow in capillaries

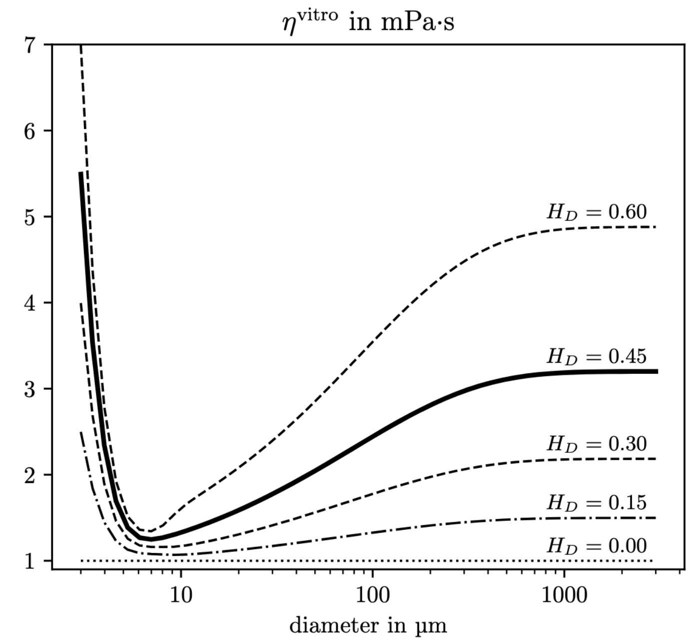
Non-Newtonian behavior

→ Empirical relationship for μ_B derived from experiments

$$\eta^{\text{vitro}} := \frac{\mu_B^{\text{vitro}}}{\mu_P} = 1 + (\eta_{0.45}^{\text{vitro}} - 1)\frac{(1-H_D)^C - 1}{(1-0.45)^C - 1},$$

$$\eta_{0.45}^{\text{vitro}}(D) = 220e^{-1.3D} - 2.44e^{-0.06D^{0.645}} + 3.2,$$

$$C(D) = (0.8 + e^{-0.075D})\left(\frac{1}{1+10^{-11}D^{12}} - 1\right) + \frac{1}{1+10^{-11}D^{12}},$$



In-vitro: experiments with blood in glass tubes

for references see script

Modeling flow in capillaries

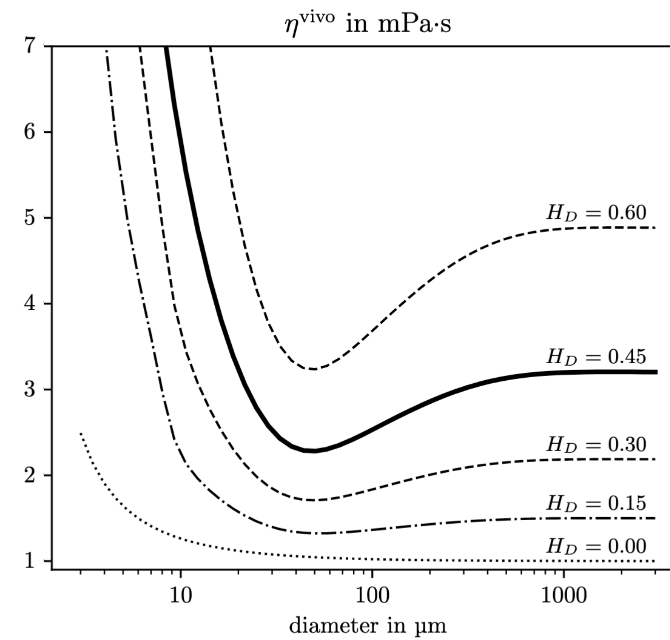
Non-Newtonian behavior

→ Empirical relationship for μ_B derived from experiments

$$\eta^{\text{vivo}} := \frac{\mu_B^{\text{vivo}}}{\mu_P} = \left[1 + (\eta_{0.45}^{\text{vivo}} - 1)\frac{(1-H_D)^C - 1}{(1-0.45)^C - 1}\left(\frac{D}{D-1.1}\right)^2\right]\left(\frac{D}{D-1.1}\right)^2,$$

$$\eta_{0.45}^{\text{vivo}}(D) = 6e^{-0.085D} - 2.44e^{-0.06D^{0.645}} + 3.2,$$

$$C(D) = (0.8 + e^{-0.075D})\left(\frac{1}{1+10^{-11}D^{12}} - 1\right) + \frac{1}{1+10^{-11}D^{12}},$$



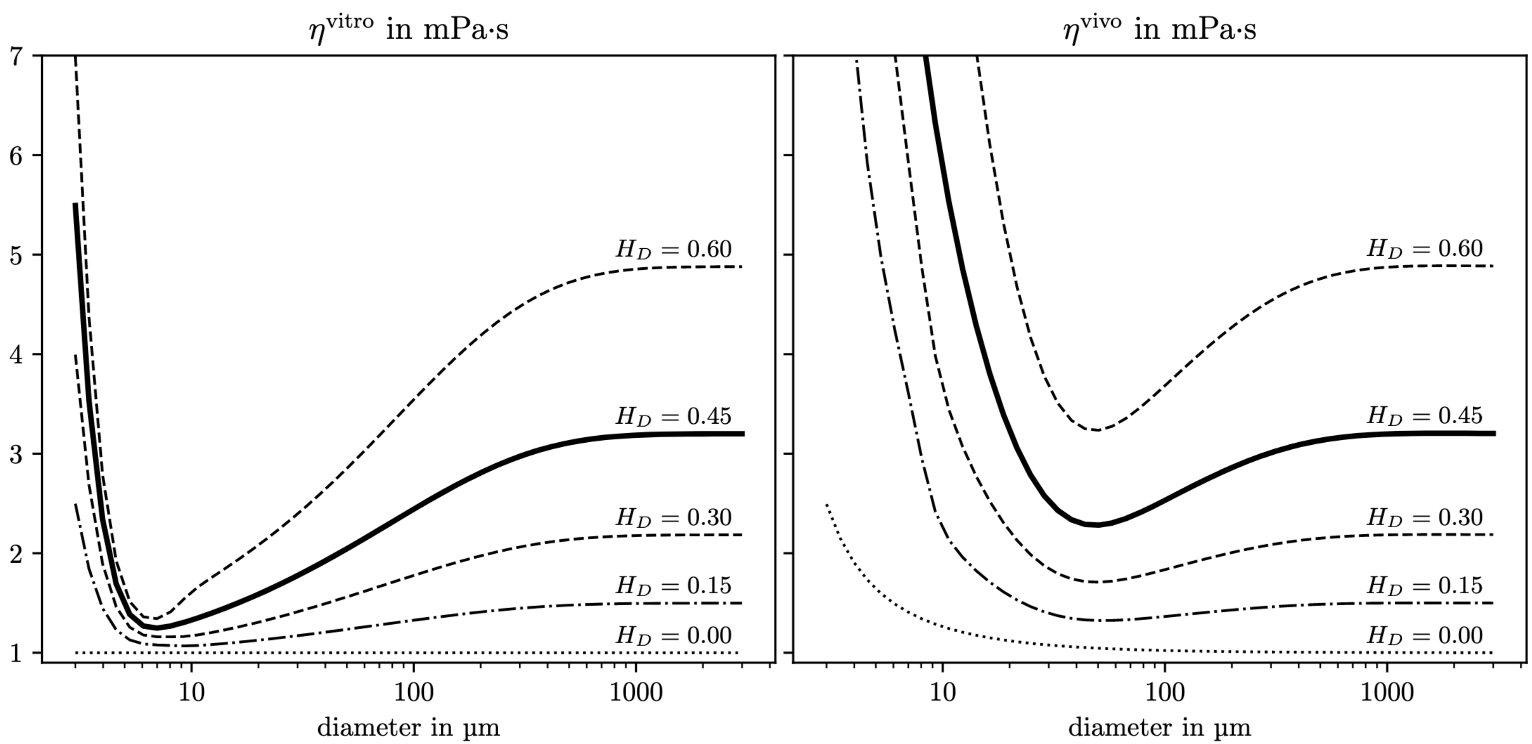
In-vivo: experiments with alive rats

for references see script

Modeling flow in capillaries

Non-Newtonian behavior

→ Empirical relationship for μ_B derived from experiments



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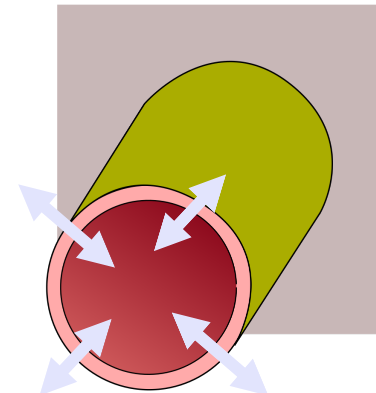
Modeling flow in capillaries

Transport across capillary wall



- Assumption: wall is porous
- Driving forces: hydrostatic pressure, osmotic pressure
  - Starling's law:

$$v_R(z) = L_p(p - P_{\text{ext}} - \sigma_w \Delta\Pi)$$

L_p	Filtration coefficient (measuerement data)
p	Pressure in capillary (1d model)
P_{ext}	Pressure outside wall (model?)
$\Delta\Pi$	Osmotic pressure difference (measurement data)
σ_w	Osmotic reflection coefficient (measurement data)

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Thank you!

Timo Koch (Oslo)

e-mail timokoch@uio.no

SFB 1313
sfb1313.uni-stuttgart.de
Interface-Driven Multi-Field Processes in Porous Media

https://www.iws.uni-stuttgart.de/lh2/

University of Stuttgart
Pfaffenwaldring 61, 70569 Stuttgart

Institute for Modelling Hydraulic and Environmental Systems,

Department of Hydromechanics and Modelling of Hydrosystems

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