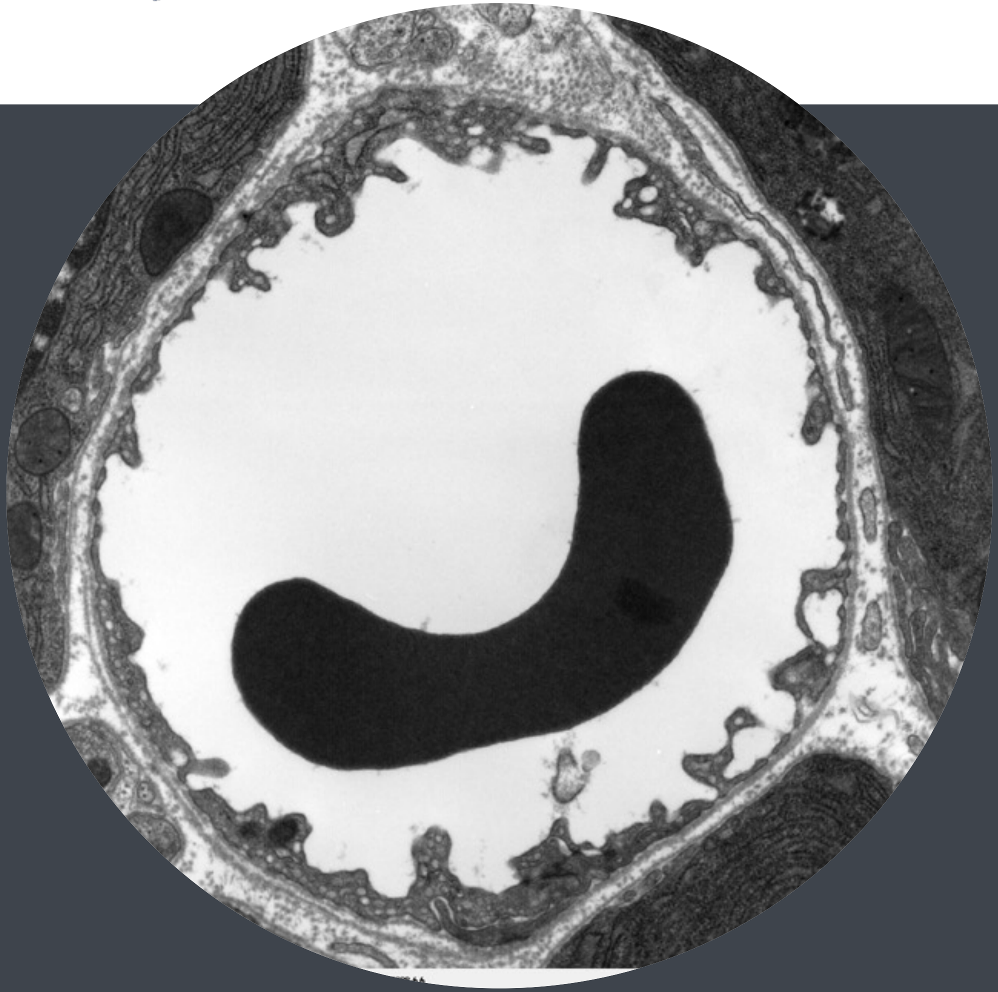
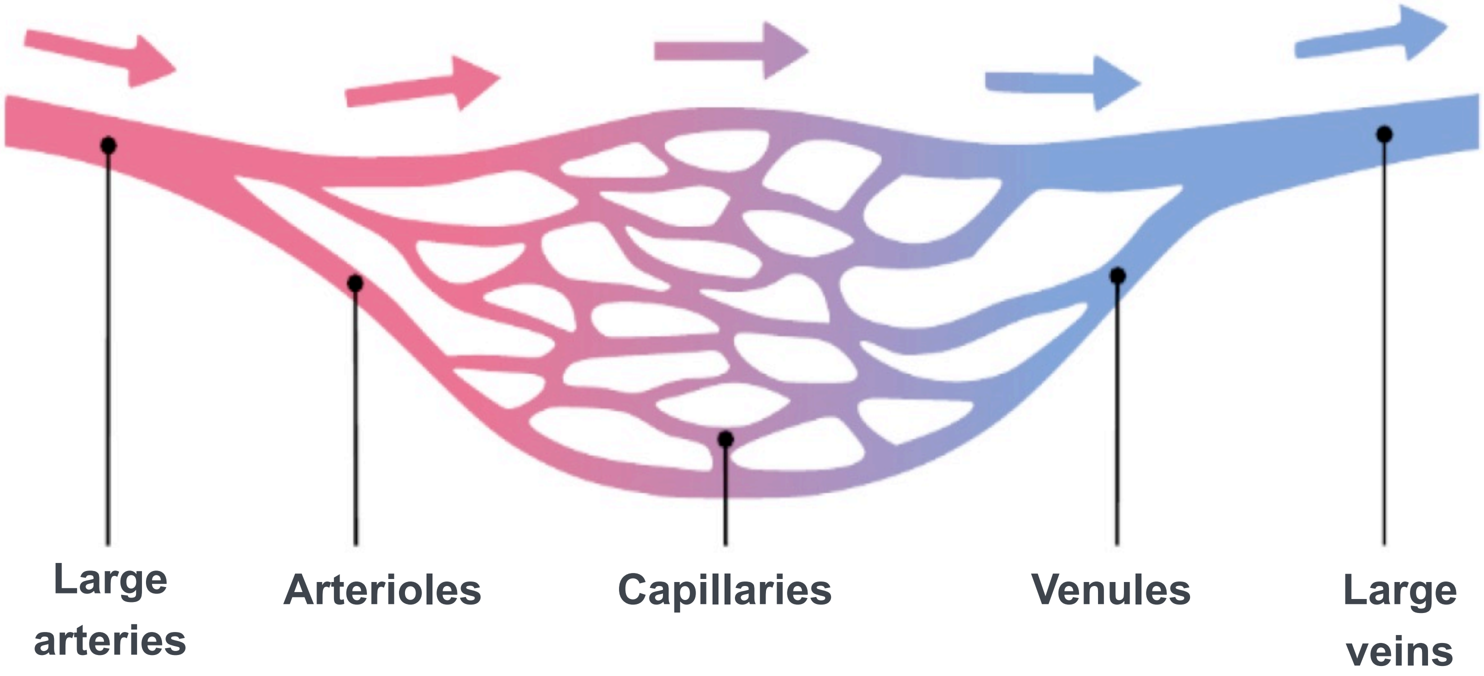
University of Stuttgart
Institute for Modelling Hydraulic and Environmental Systems

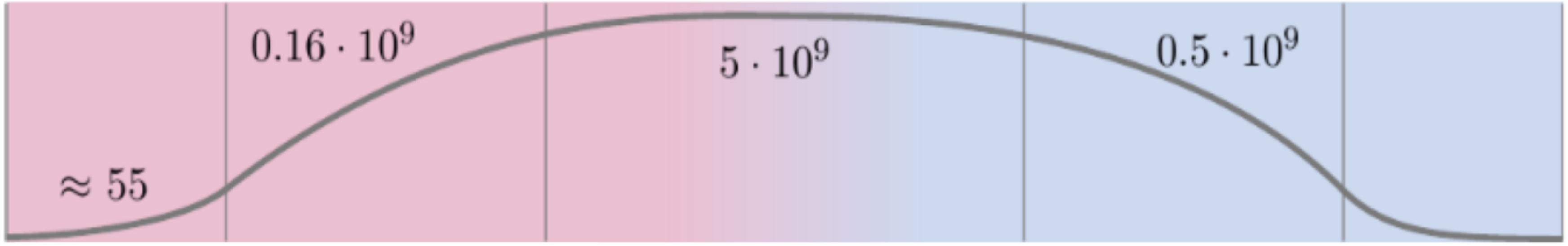
Chapter 2

# Mathematical modelling of (small) arteries by 1D models



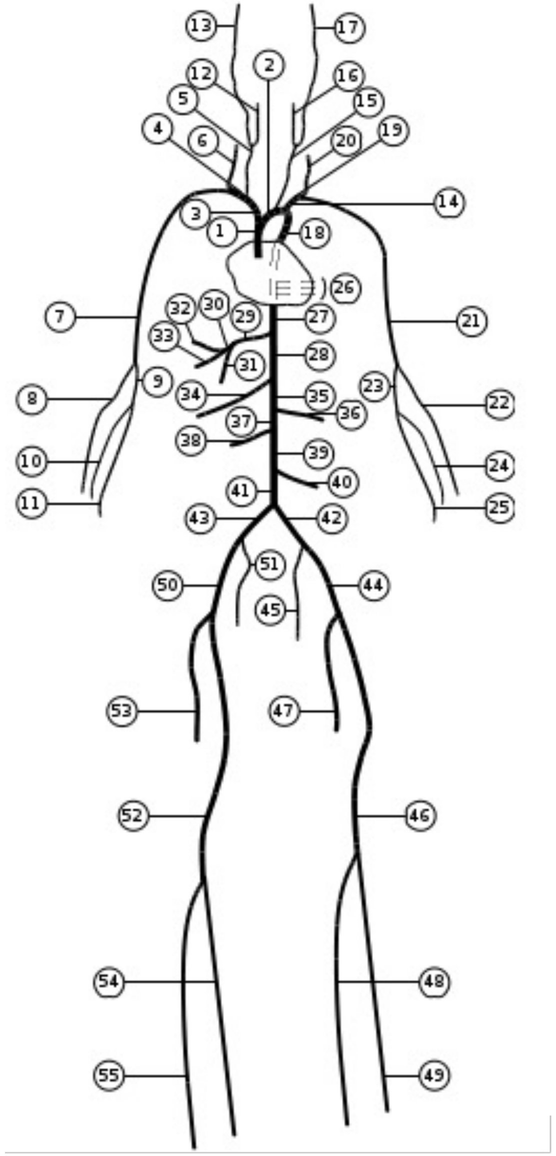
# Structure of the vascular system

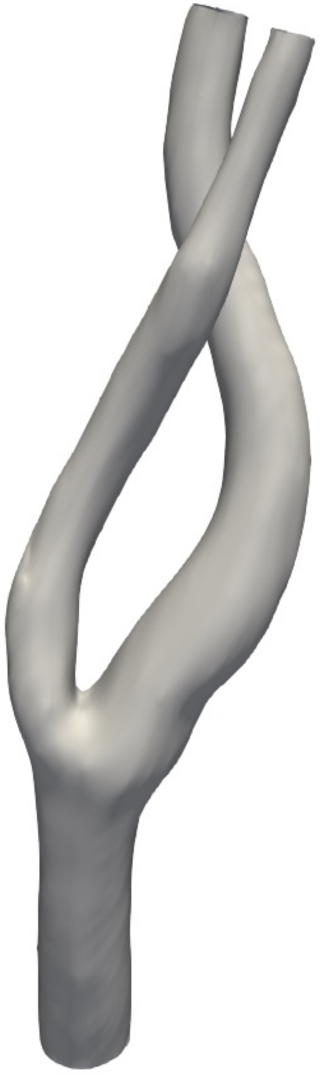


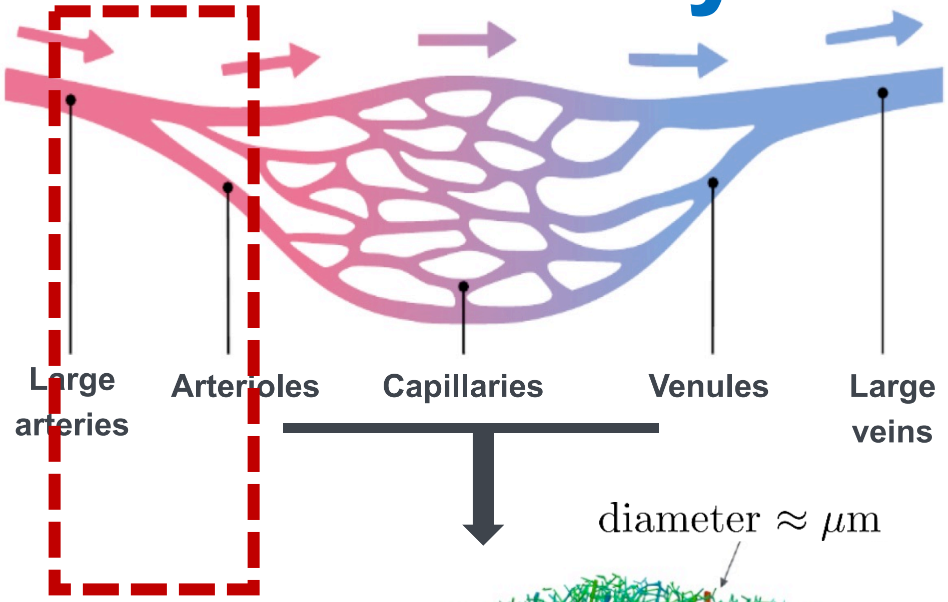

Number of vessels

# Structure of the vascular system

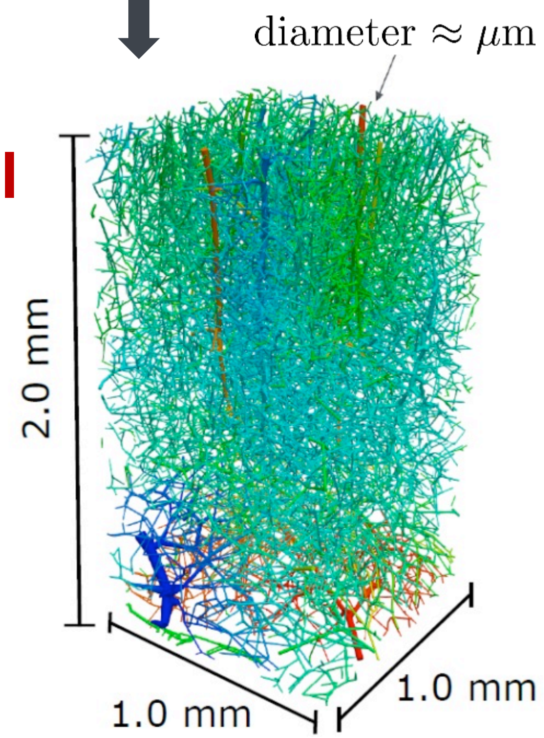


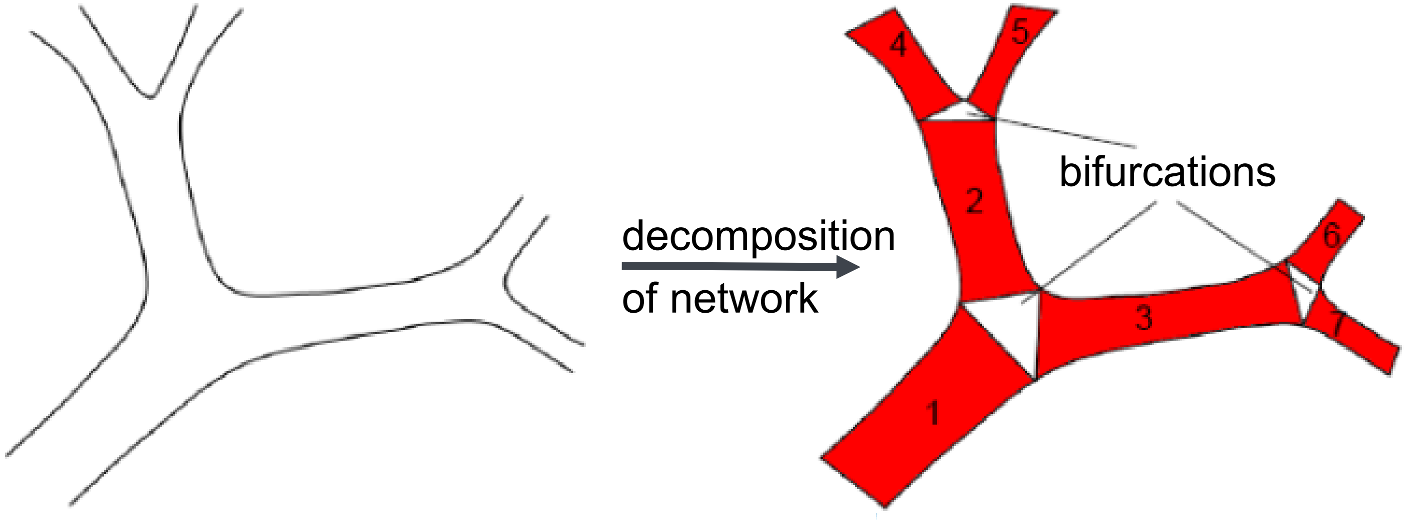
$d \approx mm$



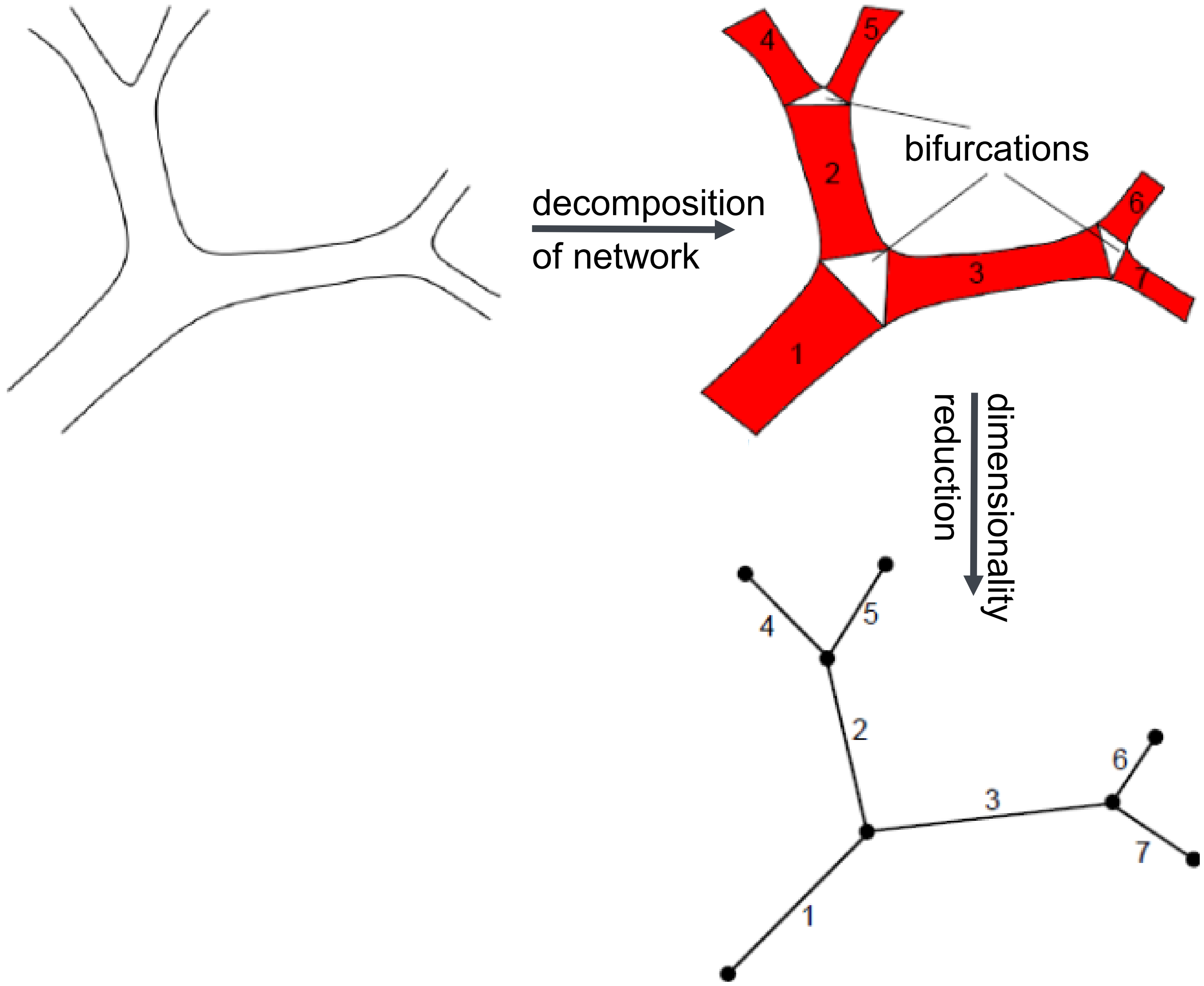

**Target of the 1D model we will derive**



# Domain decomposition

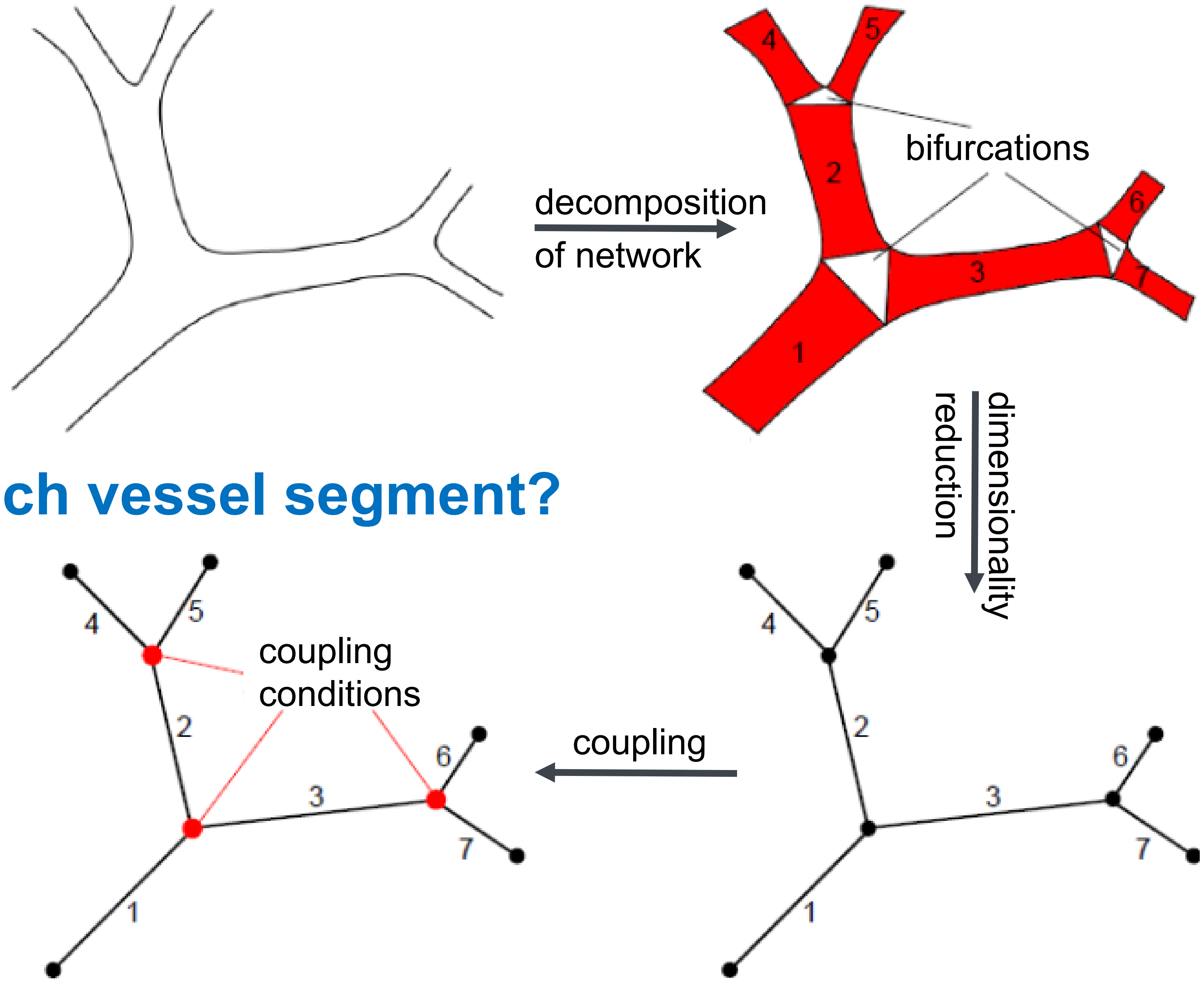


# Domain decomposition



University of Stuttgart

# Domain decomposition




## Modelling each vessel segment?

# Cylindrical geometry

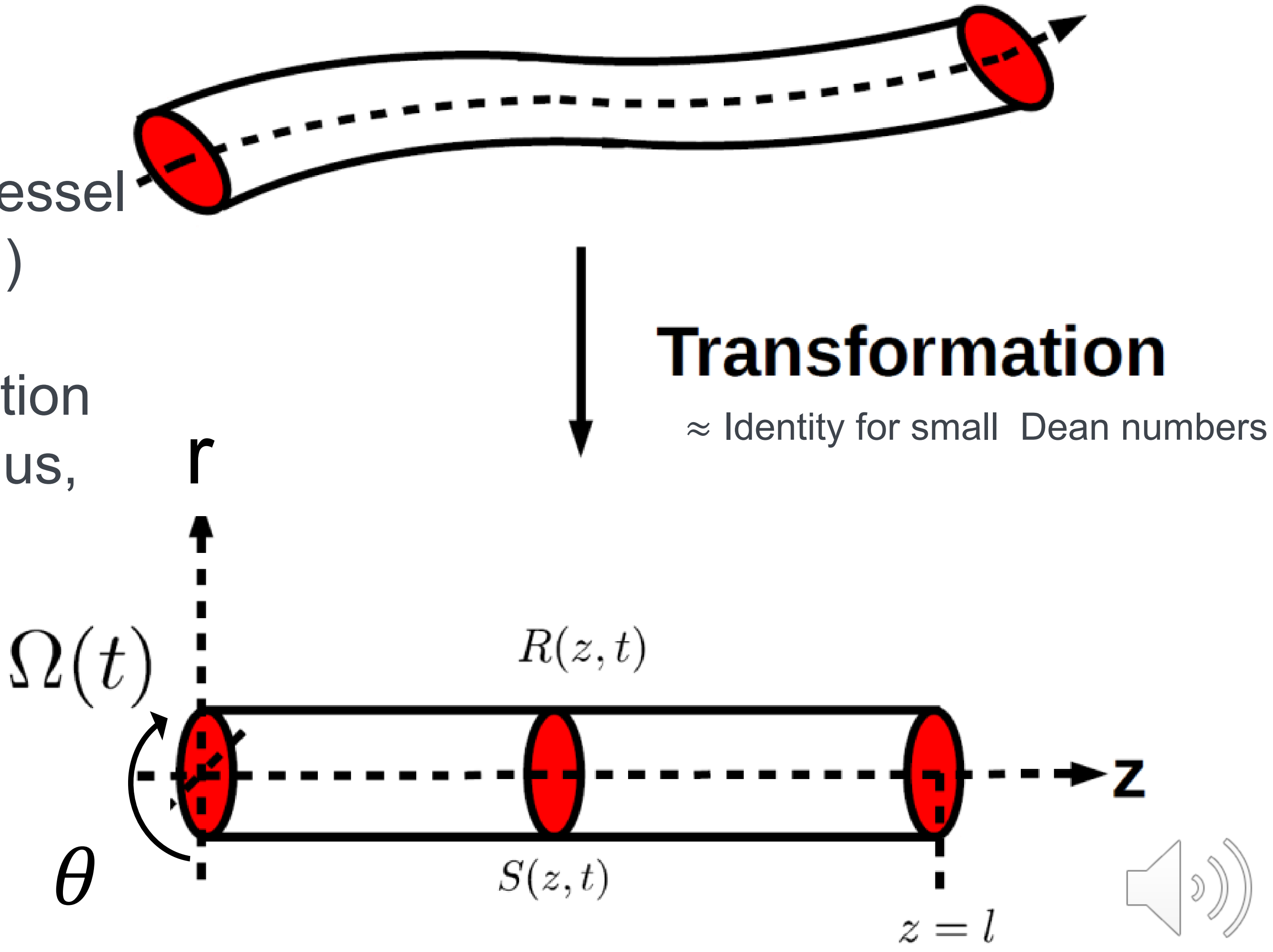
curved vessel

Assumption: circular cross section

z axis coincides with main axis of vessel (can be achieved by transformation)

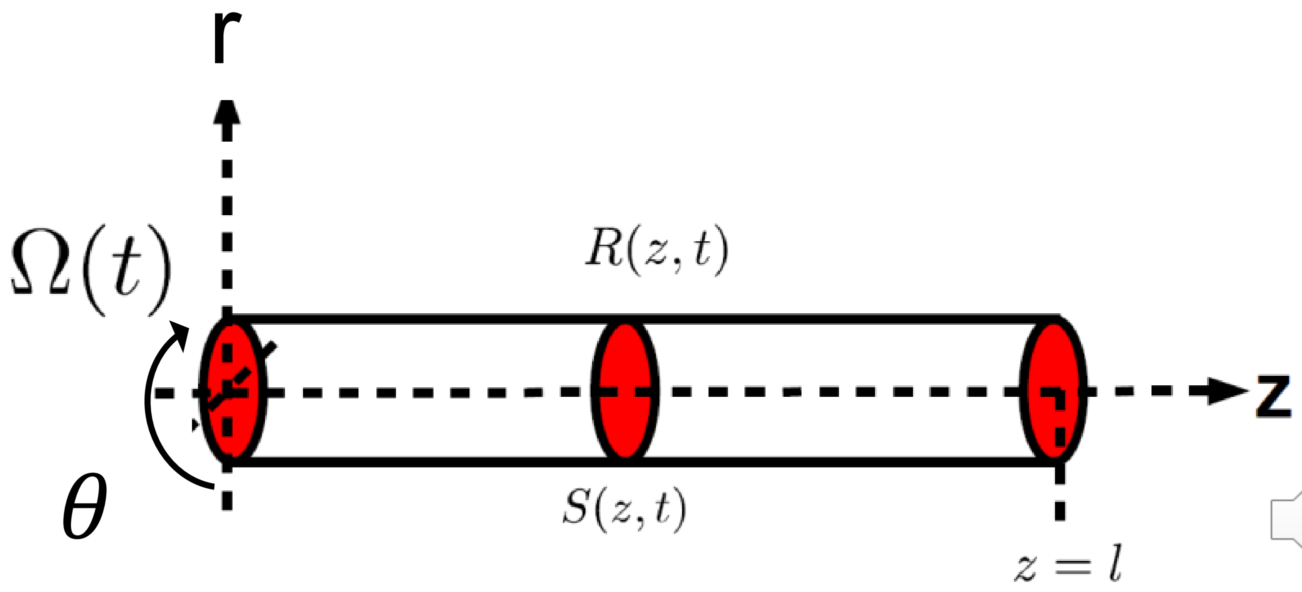
S(z,t) denotes cross section at position z and time t, R(z,t) denotes the radius, accordingly

l: length of vessel



# Navier-Stokes equations (3D)

$$\Omega(t) = \{(r, \theta, z) \in \mathbb{R}^3 \mid 0 < z < l,\ 0 \le r < R(z,t),\ 0 \le \theta < 2\pi\} \subset \mathbb{R}^3$$



# Navier-Stokes equations (3D)

Momentum and continuity equations:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} + \nabla P - \mathbf{div} \left( 2\mu \mathbf{D} \right) = \mathbf{f}, \text{ in } \Omega(t), \; t > 0,$$

$$\operatorname{div} \left( \mathbf{u} \right) = 0, \text{ in } \Omega(t), \; t > 0.$$

$\rho$ : density, blood is assumed to be incompressible

University of Stuttgart

# Navier-Stokes equations (3D)

Momentum and continuity equations:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} + \nabla P - \mathbf{div} \left( 2\mu \mathbf{D} \right) = \mathbf{f}, \text{ in } \Omega(t), \; t > 0,$$

$$\mathrm{div} \left( \mathbf{u} \right) = 0, \text{ in } \Omega(t), \; t > 0.$$

$\rho :$ density, blood is assumed to be incompressible

$\mathbf{f} \equiv \mathbf{0} :$ no external forces, as for example gravity

# Navier-Stokes equations (3D)



Momentum and continuity equations:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} + \nabla P - \mathbf{div} \left( 2\mu \mathbf{D} \right) = \mathbf{f}, \text{ in } \Omega(t), \; t > 0,$$

$$\mathrm{div} \left( \mathbf{u} \right) = 0, \text{ in } \Omega(t), \; t > 0.$$

$\rho$ : density, blood is assumed to be incompressible

$\mathbf{f} \equiv \mathbf{0}$ : no external forces, as for example gravity

$$\mathbf{D} \left( \mathbf{u} \right) = \tfrac{1}{2} \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T \right), \; \nabla \mathbf{u} = \begin{pmatrix} \nabla u_1^T \\ \nabla u_2^T \\ \nabla u_3^T \end{pmatrix}$$

Remark:

$$\mathrm{div}(p\mathbf{I}) = \nabla p$$

University of Stuttgart

# Cylinder coordinates
# Navier-Stokes equations (3D)

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) - b_r = 0$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} - \frac{v_r v_\theta}{r} + \frac{1}{r\rho}\frac{\partial p}{\partial \theta} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right) - b_\theta = 0$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) - b_z = 0$$

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

velocity vector field $\boldsymbol{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}$

radial component

circumferential component

axial component

University of Stuttgart

# Cylinder coordinates
# Navier-Stokes equations (3D)

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}+\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)-\cancel{b_r}=0$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}-\frac{v_r v_\theta}{r}+\frac{1}{r\rho}\frac{\partial p}{\partial \theta}-\frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)-\cancel{b_\theta}=0$$

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)-\cancel{b_z}=0$$

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Neglect gravity / body forces: **b** = 0

University of Stuttgart

# Cylinder coordinates

# Navier-Stokes equations (3D)

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + v_z\frac{\partial v_r}{\partial z} - \cancel{\frac{v_\theta^2}{r}} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right) - \cancel{b_r} = 0$$

$$\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z\frac{\partial v_\theta}{\partial z}} - \cancel{\frac{v_r v_\theta}{r}} + \cancel{\frac{1}{r\rho}\frac{\partial p}{\partial \theta}} - \frac{\mu}{\rho}\left(\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} - \cancel{\frac{v_\theta}{r^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}\right) - \cancel{b_\theta} = 0$$

$$\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + v_z\frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \frac{\partial^2 v_z}{\partial z^2}\right) - \cancel{b_z} = 0$$

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \frac{\partial v_z}{\partial z} = 0$$

Neglect gravity / body forces: **b** = 0

Irrotational flow: $v_\theta = 0$ and $\frac{\partial}{\partial \theta} = 0$

University of Stuttgart

# Simplified (irrotational, no gravity) Navier-Stokes equations

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right) = 0$$

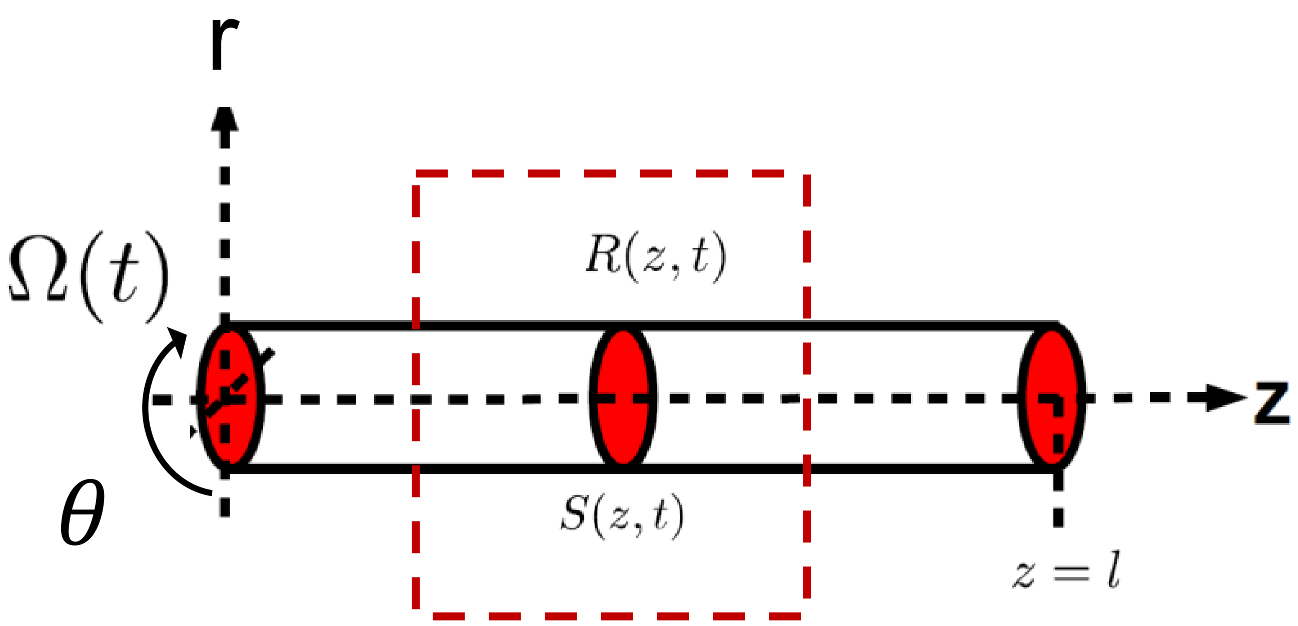
$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right) = 0$$

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0$$

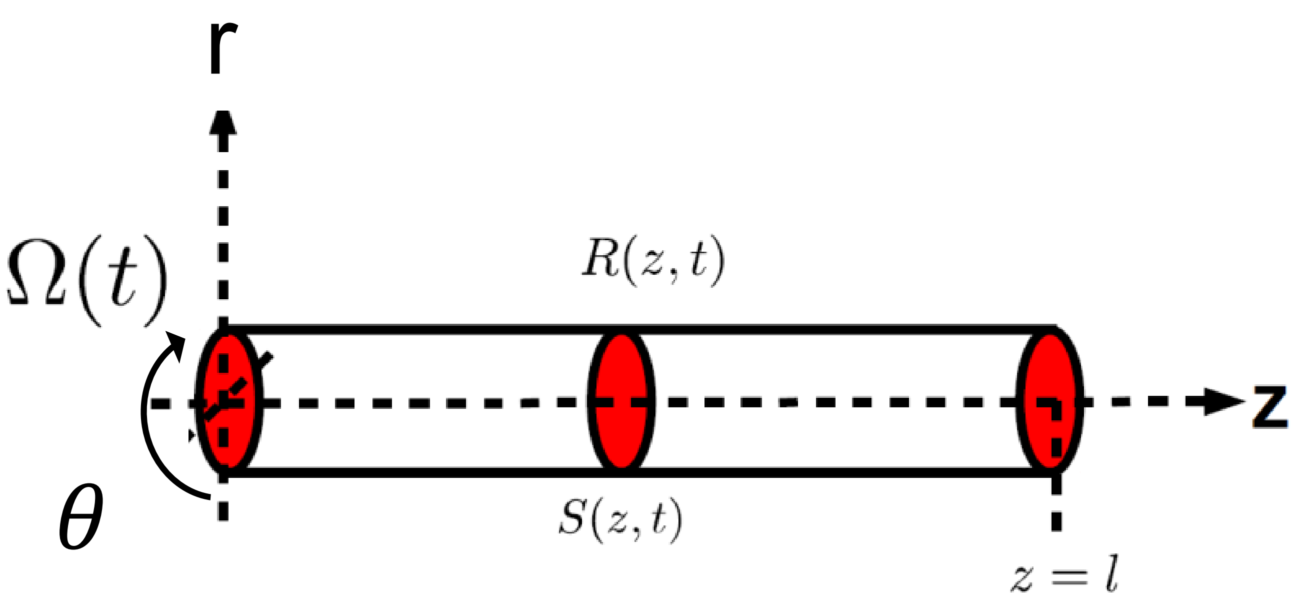
Neglect gravity / body forces: **b** = 0

Irrotational flow: $v_\theta = 0$ and $\frac{\partial}{\partial \theta} = 0$

# Main idea: Integration over cross-section



University of Stuttgart




## Circular cross-section

$$A(z,t) = |S(z,t)| = \pi R^2(z,t),$$

## Mean (axial) velocity and velocity profile

$$v_z(r,z,t) = \bar{v}(z,t)s(r^*), \quad \bar{v} := \frac{1}{A}\int_S v_z \mathrm{d}A \qquad r^* = r/R(z,t)$$

## Flow rate

$$Q(z,t) := \int_S v_z \mathrm{d}A = A\bar{v}.$$

University of Stuttgart

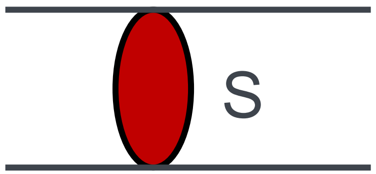
# Simplified (irrotational, no gravity) Navier-Stokes equations

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right) = 0$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right) = 0$$

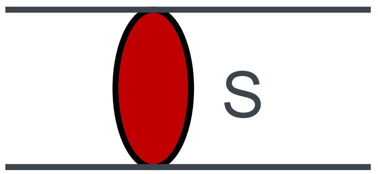
$$\boxed{\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0}$$

University of Stuttgart




# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \longrightarrow \int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A + \int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0.$$



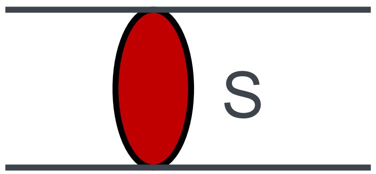

# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0 \longrightarrow \boxed{\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A}+\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A=0$$

$$\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A=\int_0^{2\pi}\int_0^{R}\frac{\partial(rv_r)}{\partial r}\mathrm{d}r\mathrm{d}\theta$$

**area element**

$$\mathrm{d}A = r\,\mathrm{d}r\mathrm{d}\theta$$

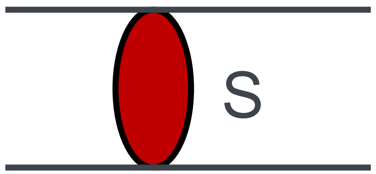



# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad \longrightarrow \quad \boxed{\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A} + \int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0$$

$$\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A = \int_0^{2\pi}\int_0^{R} \frac{\partial(rv_r)}{\partial r}\mathrm{d}r\mathrm{d}\theta = 2\pi R\, v_r(R,t)$$

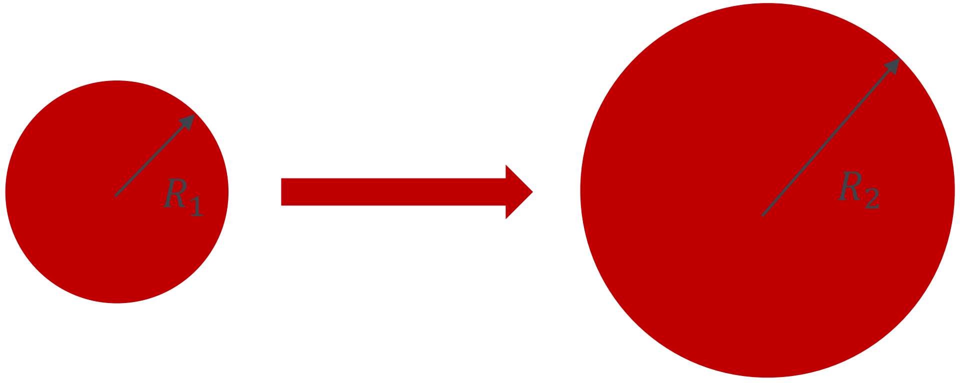
$v_r$ independent of $\theta$




# Example: mass/volume balance equation

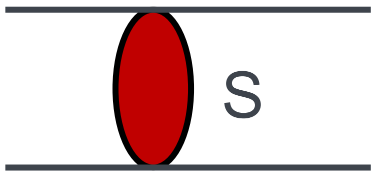
$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \longrightarrow \boxed{\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A} + \int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0$$

$$\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A = \int_0^{2\pi}\int_0^{R} \frac{\partial(rv_r)}{\partial r}\,\mathrm{d}r\mathrm{d}\theta = 2\pi R\, v_r(R,t) = 2\pi R\frac{\partial R}{\partial t}$$




boundary condition:
$v_r$ at r=R is wall velocity $\frac{\partial R}{\partial t}$

University of Stuttgart



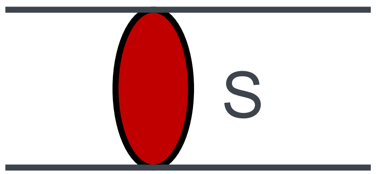

# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0 \quad \longrightarrow \quad \boxed{\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A}+\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A=0$$

$$\int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A=\int\limits_0^{2\pi}\int\limits_0^{R}\frac{\partial(rv_r)}{\partial r}\mathrm{d}r\mathrm{d}\theta=2\pi R\,\underbrace{v_r(R,t)}_{=\partial R/\partial t}=2\pi R\frac{\partial R}{\partial t}=\frac{\partial(\pi R^2)}{\partial t}\boxed{=\frac{\partial A}{\partial t}}$$

use chain rule or product rule

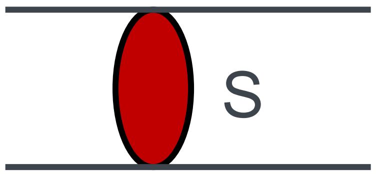
University of Stuttgart




# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \longrightarrow \int_S \frac{1}{r}\frac{\partial (r v_r)}{\partial r}\,\mathrm{d}A + \boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A} = 0.$$

University of Stuttgart

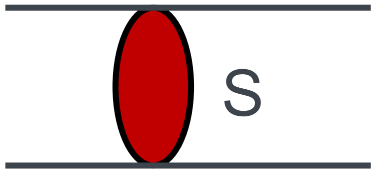



# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0 \quad\longrightarrow\quad \int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A+\boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A}=0$$

## Leibniz' rule for parameter integral

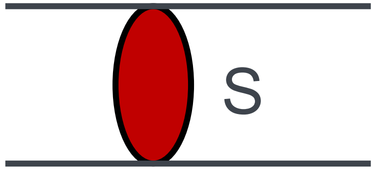
$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_{a(z)}^{b(z)} f(z,r,t)\mathrm{d}r\right)=\underbrace{\int_{a(z)}^{b(z)}\frac{\partial}{\partial z}f(z,r,t)\mathrm{d}r}_{\text{bounds constant}}+\underbrace{f(z,b(z),t)\frac{\mathrm{d}}{\mathrm{d}z}b(z)-f(z,a(z),t)\frac{\mathrm{d}}{\mathrm{d}z}a(z)}_{\text{change due to changes in bounds}}$$




# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0 \quad\longrightarrow\quad \int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A+\boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A}=0$$

$$\int_S \frac{\partial u_z}{\partial z}\,\mathrm{d}A=\int_0^{R(z)}\int_0^{2\pi}\frac{\partial u_z}{\partial z}r\,\mathrm{d}\theta\mathrm{d}r=\int_0^{R(z)}\frac{\partial}{\partial z}\left(\int_0^{2\pi}u_z r\mathrm{d}\theta\right)\mathrm{d}r$$



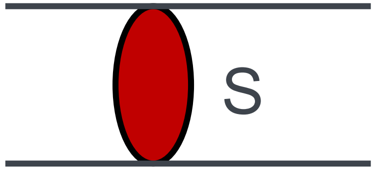

# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad \longrightarrow \quad \int_S \frac{1}{r}\frac{\partial (r v_r)}{\partial r}\,\mathrm{d}A + \boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A} = 0$$

$$\int_S \frac{\partial u_z}{\partial z}\,\mathrm{d}A = \int_0^{R(z)}\int_0^{2\pi} \frac{\partial u_z}{\partial z} r\,\mathrm{d}\theta \mathrm{d}r = \int_0^{R(z)} \frac{\partial}{\partial z}\left(\int_0^{2\pi} u_z r \mathrm{d}\theta\right)\mathrm{d}r \quad = \int_0^{R(z)} \frac{\partial g}{\partial z}\mathrm{d}r$$

$$g(z,r,t) := \int_0^{2\pi} v_z r \mathrm{d}\theta = 2\pi v_z r$$

University of Stuttgart




# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \longrightarrow \int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A + \boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A} = 0$$

$$\int_S \frac{\partial u_z}{\partial z}\,\mathrm{d}A = \int_0^{R(z)}\int_0^{2\pi} \frac{\partial u_z}{\partial z} r\,\mathrm{d}\theta\mathrm{d}r = \int_0^{R(z)} \frac{\partial}{\partial z}\left(\int_0^{2\pi} u_z r\mathrm{d}\theta\right)\mathrm{d}r = \int_0^{R(z)} \frac{\partial g}{\partial z}\mathrm{d}r$$

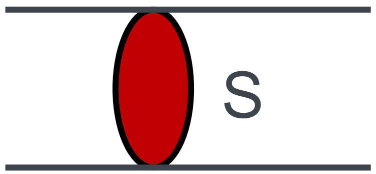
(Leibniz' rule)

$$= \frac{\mathrm{d}}{\mathrm{d}z}\left(\int_0^{R(z)} g\mathrm{d}r\right) - g(z,R(z),t)\frac{\mathrm{d}R(z)}{\mathrm{d}z} + g(z,0,t)\frac{\mathrm{d}0}{\mathrm{d}z}$$

$g(z,r,t) := \int_0^{2\pi} v_z r \mathrm{d}\theta = 2\pi v_z r$

$$= \frac{\mathrm{d}}{\mathrm{d}z}\left(\int_S u_z\,\mathrm{d}A\right) - 2\pi R\,u_z(t,R)\,\frac{\partial R}{\partial z}$$

University of Stuttgart




# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0 \quad\longrightarrow\quad \int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A+\boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A}=0$$

$$\int_S \frac{\partial u_z}{\partial z}\,\mathrm{d}A=\int_0^{R(z)}\int_0^{2\pi}\frac{\partial u_z}{\partial z}r\,\mathrm{d}\theta\mathrm{d}r=\int_0^{R(z)}\frac{\partial}{\partial z}\left(\int_0^{2\pi}u_z r\,\mathrm{d}\theta\right)\mathrm{d}r\;=\int_0^{R(z)}\frac{\partial g}{\partial z}\mathrm{d}r$$

(Leibniz' rule)

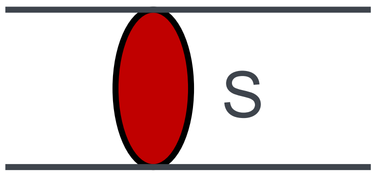
$$=\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_0^{R(z)}g\,\mathrm{d}r\right)-g(z,R(z),t)\frac{\mathrm{d}R(z)}{\mathrm{d}z}+g(z,0,t)\frac{\mathrm{d}0}{\mathrm{d}z}$$

$g(z,r,t):=\int_0^{2\pi}v_z r\,\mathrm{d}\theta=2\pi v_z r$

$$=\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_S u_z\,\mathrm{d}A\right)-2\pi R\,u_z(t,R)\,\frac{\partial R}{\partial z}$$

**what is $u_z(R,t)$?**

University of Stuttgart




# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z}=0 \longrightarrow \int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A+\boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A}=0$$

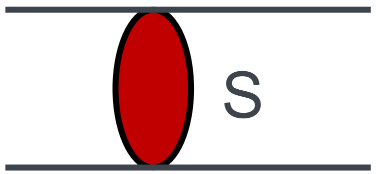
$$\int_S \frac{\partial u_z}{\partial z}\,\mathrm{d}A=\int_0^{R(z)}\int_0^{2\pi}\frac{\partial u_z}{\partial z}r\,\mathrm{d}\theta\mathrm{d}r=\int_0^{R(z)}\frac{\partial}{\partial z}\left(\int_0^{2\pi}u_z r\mathrm{d}\theta\right)\mathrm{d}r \quad =\int_0^{R(z)}\frac{\partial g}{\partial z}\mathrm{d}r$$

(Leibniz' rule)
$$=\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_0^{R(z)}g\mathrm{d}r\right)-g(z,R(z),t)\frac{\mathrm{d}R(z)}{\mathrm{d}z}+g(z,0,t)\frac{\mathrm{d}0}{\mathrm{d}z}$$

$g(z,r,t):=\int_0^{2\pi}v_z r\mathrm{d}\theta=2\pi v_z r$
$$=\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_S u_z\,\mathrm{d}A\right)-2\pi R\,u_z(t,R)\,\frac{\partial R}{\partial z}$$

boundary condition:
$v_z$ at r=R is 0 (no-slip

University of Stuttgart



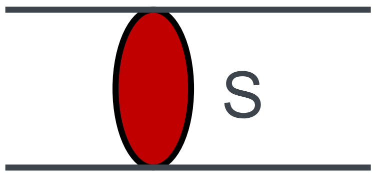

# Example: mass/volume balance equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad \longrightarrow \quad \int_S \frac{1}{r}\frac{\partial(rv_r)}{\partial r}\,\mathrm{d}A + \boxed{\int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A} = 0$$

$$\int_S \frac{\partial u_z}{\partial z}\,\mathrm{d}A = \frac{\mathrm{d}}{\mathrm{d}z}\left(\int_S u_z\,\mathrm{d}A\right) - 2\pi R\,\underbrace{u_z(t,R)}_{=0}\,\frac{\partial R}{\partial z}$$

$$= \frac{\partial Q}{\partial z};$$

boundary condition:
$v_z$ at r=R is 0 (no-slip)




University of Stuttgart

# Mass balance (summary)

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad \longrightarrow \quad \int_S \frac{1}{r}\frac{\partial (r v_r)}{\partial r}\,\mathrm{d}A + \int_S \frac{\partial v_z}{\partial z}\,\mathrm{d}A = 0$$

$$\longrightarrow \quad \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0$$

1. boundary condition:
$v_r$ at r=R is wall velocity $\frac{\partial R}{\partial t}$

2. boundary condition:
$v_z$ at r=R is 0 (no-slip)

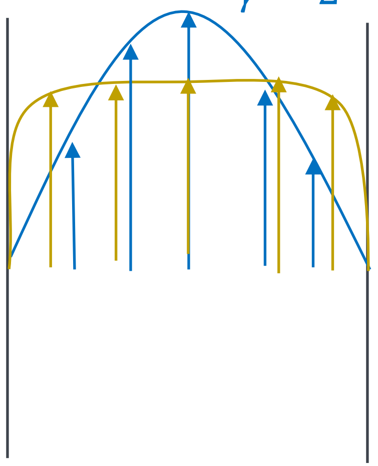
# Momentum balance

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial z}\left(\alpha \frac{Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A} = 0,$$

$$v_z(r,z,t) = \bar{v}(z,t)s(r^*)$$

$$\alpha := \frac{\int_S v_z^2 \mathrm{d}A}{A\bar{v}^2} = \frac{1}{A}\int_S s^2 \mathrm{d}A,$$

$$K_R = -2\pi \frac{\mu}{\rho} s'(1)$$




## typical choice: power law profile

$$v_z(r,z,t) = \bar{v}(z,t)s(r^*) = \bar{v}(z,t)\frac{\gamma+2}{\gamma}\left[1-(r^*)^\gamma\right] = \bar{v}(z,t)\frac{\gamma+2}{\gamma}\left[1-\left(\frac{r}{R}\right)^\gamma\right]$$

$$K_R = -2\pi\frac{\mu}{\rho}s'(1) = 2\pi\frac{\mu}{\rho}(\gamma+2), \quad \gamma = \frac{2-\alpha}{\alpha-1}$$

(no boundary layer effects, small Womersley numbers)


**Čanić, Sunčica, and Eun Heui Kim.** „Mathematical analysis of the quasilinear effects in a hyperbolic model blood flow through compliant axi-symmetric vessels." *Mathematical Methods in the Applied Sciences* 26.14 (2003): 1161-1186

# 1D Equations (summary)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0,$$

**mass balance**

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial z}\left(\alpha \frac{Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A} = 0.$$

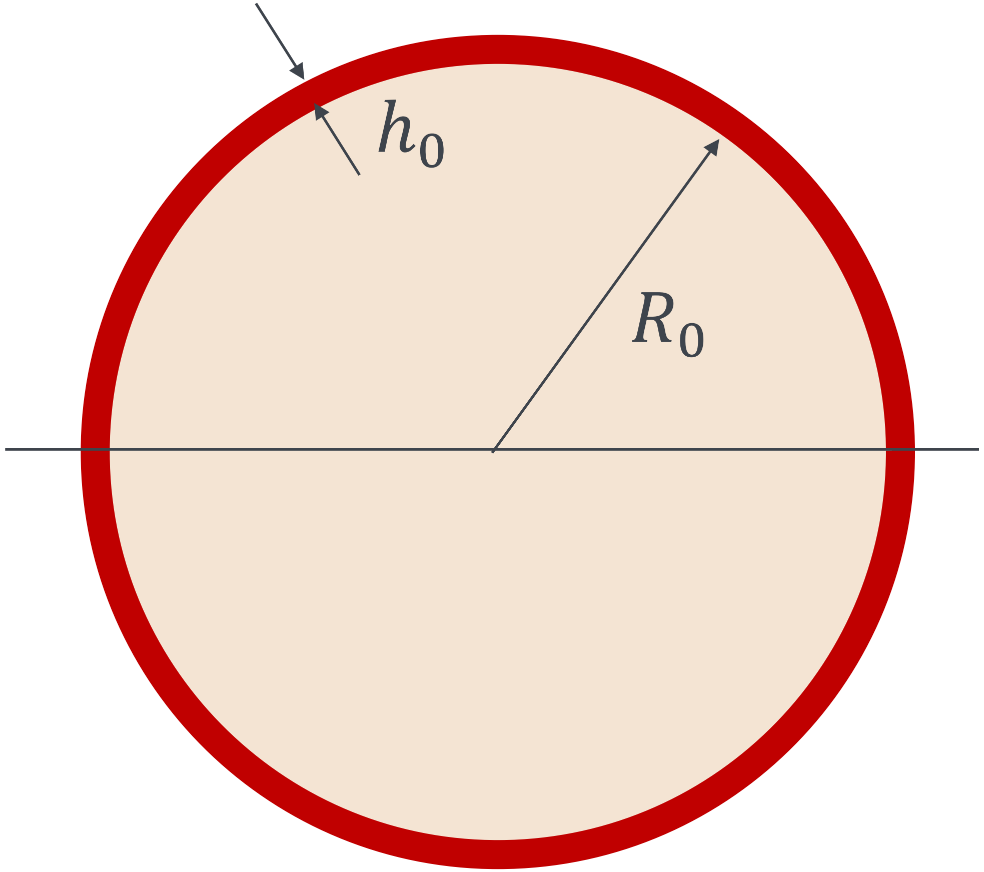
**momentum balance**

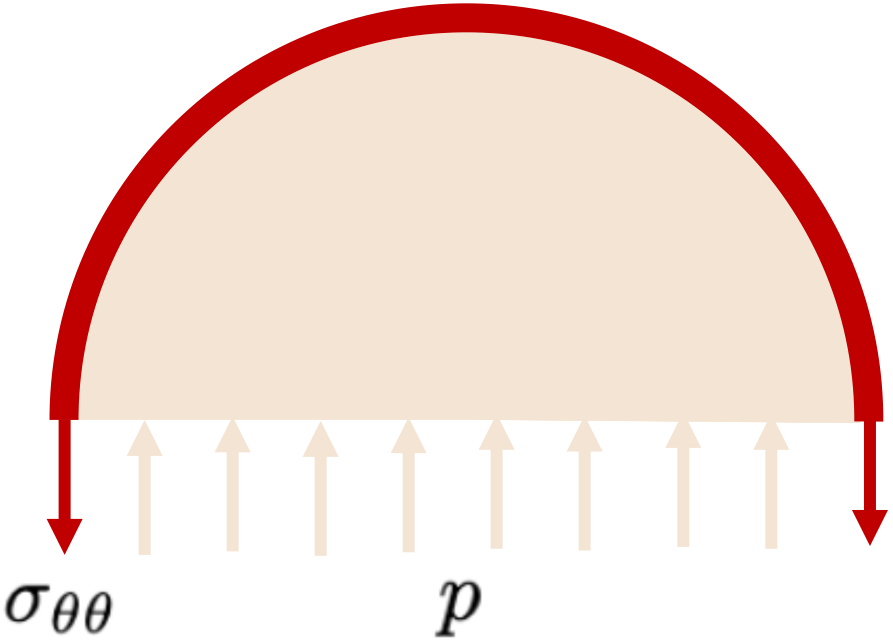
**3 unknowns (A, Q, p), 2 equations**

**→ Find relationship between A and p (closure model)**

University of Stuttgart

# Closure model



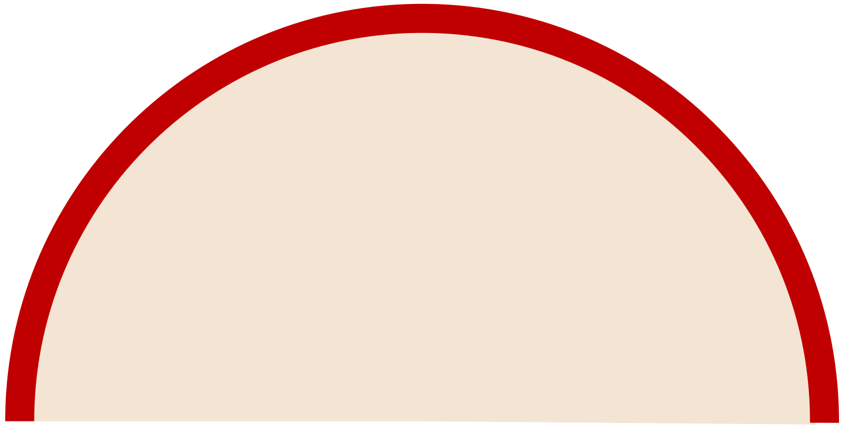



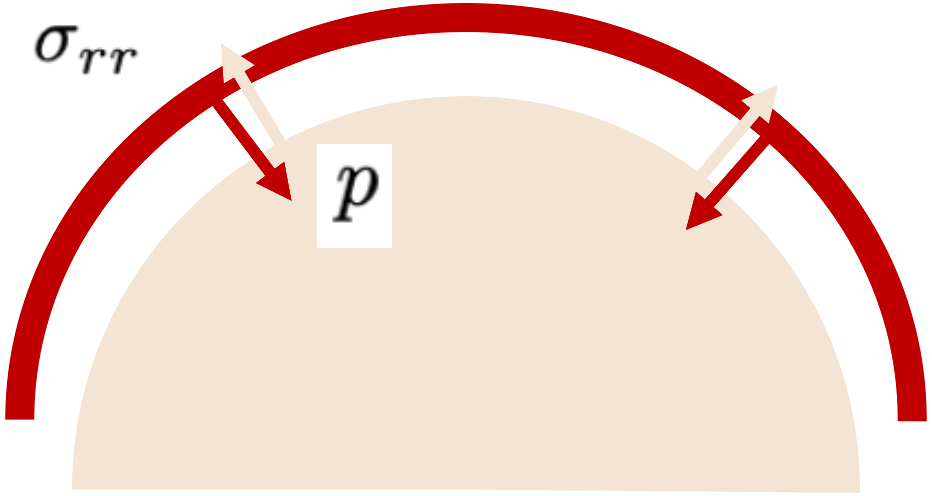

$$F_{p\theta} = 2R_0 p = 2h_0\sigma_{\theta\theta} = F_{\sigma\theta}$$

$$\Rightarrow \sigma_{\theta\theta} = \frac{pR_0}{h_0}$$

University of Stuttgart

# Closure model



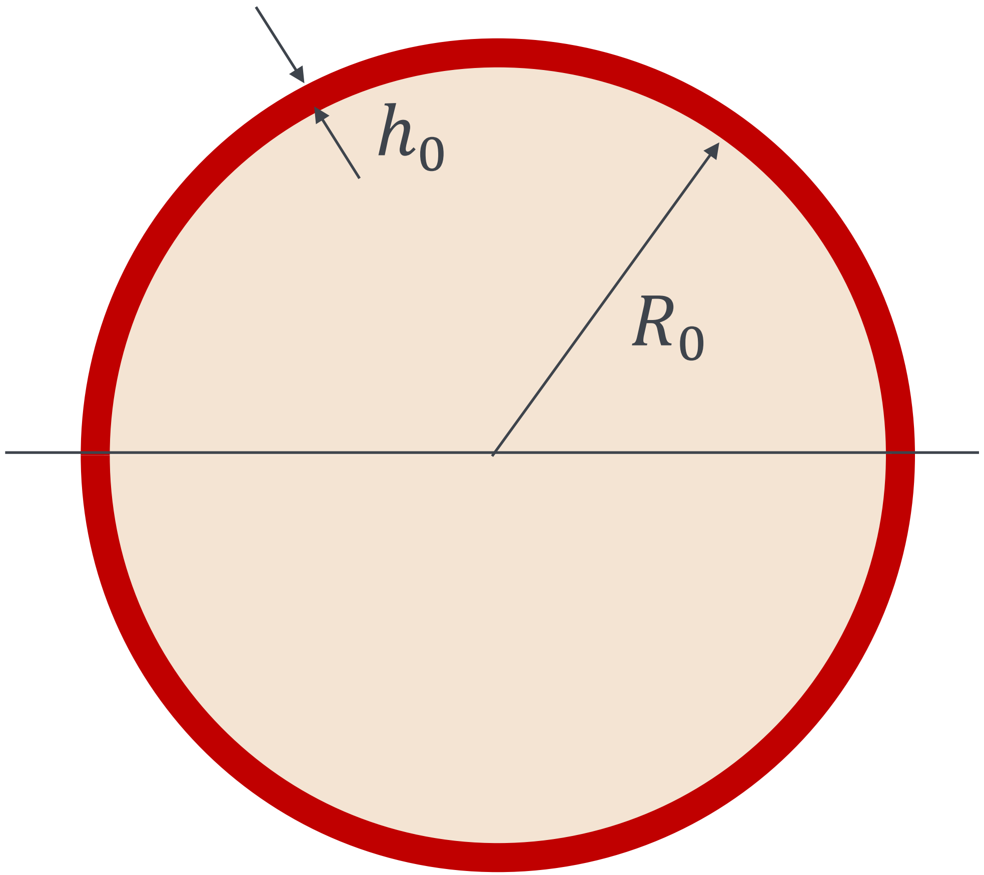



$$F_{pr} = \pi R_0 p = -\pi R_0 \sigma_{rr} = F_{\sigma r}$$

$$\Rightarrow \sigma_{rr} = -p$$

$h_0 \ll R_0;$

# Closure model




$$F_{pr} = \pi R_0 p = -\pi R_0 \sigma_{rr} = F_{\sigma r}$$

$$\Rightarrow \sigma_{rr} = -p$$

$$F_{p\theta} = 2R_0 p = 2h_0 \sigma_{\theta\theta} = F_{\sigma\theta}$$

$$\Rightarrow \sigma_{\theta\theta} = \frac{pR_0}{h_0}$$

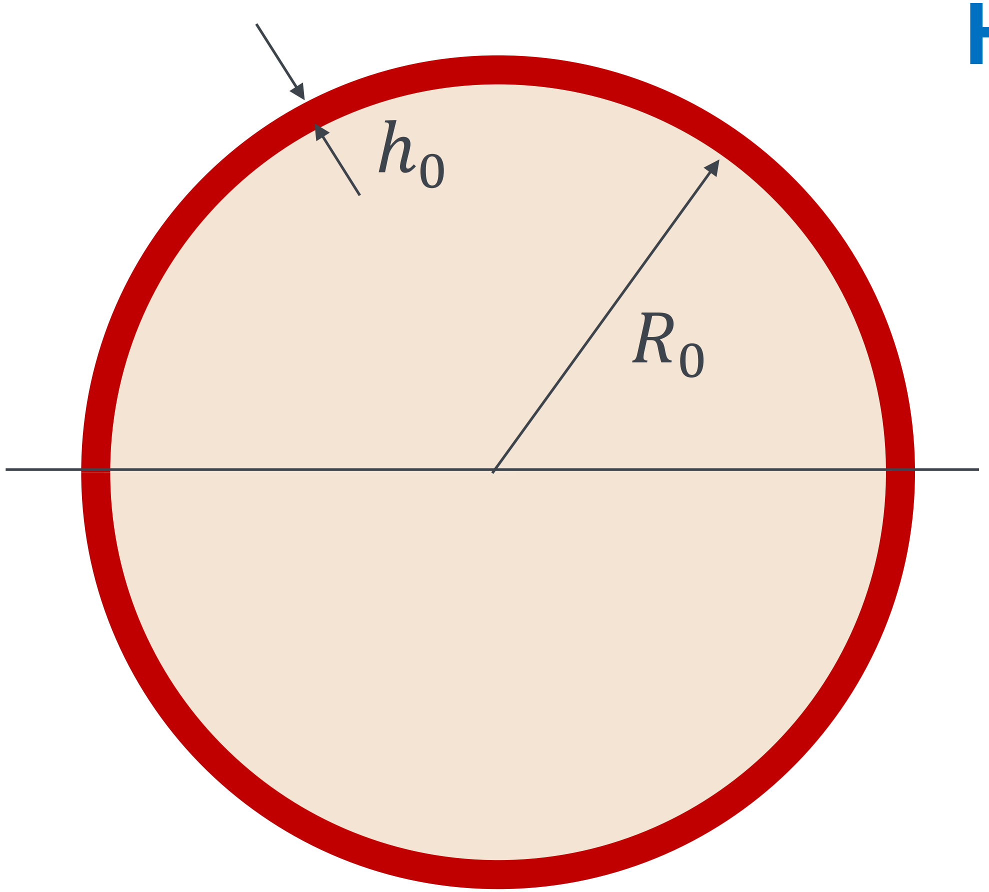
$$h_0 \ll R_0 \longrightarrow \sigma_{rr} \ll \sigma_{\theta\theta}$$

**assumption on axial strain** $\epsilon_{zz} \approx 0$

University of Stuttgart

# Closure model

## Hooke's law




$$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right]$$

$$\epsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\left(\sigma_{rr} + \sigma_{zz}\right)\right] \approx \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{zz}\right],$$

$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{\theta\theta} + \sigma_{rr}\right)\right] \approx \frac{1}{E}\left[\sigma_{zz} - \nu\sigma_{\theta\theta}\right] \approx 0.$$

$$\epsilon_{\theta\theta} \approx \frac{1}{E}\left[\sigma_{\theta\theta} - \nu^2\sigma_{\theta\theta}\right] = \frac{(1-\nu^2)}{E}\sigma_{\theta\theta} = \frac{(1-\nu^2)}{E}\frac{pR_0}{h_0}$$

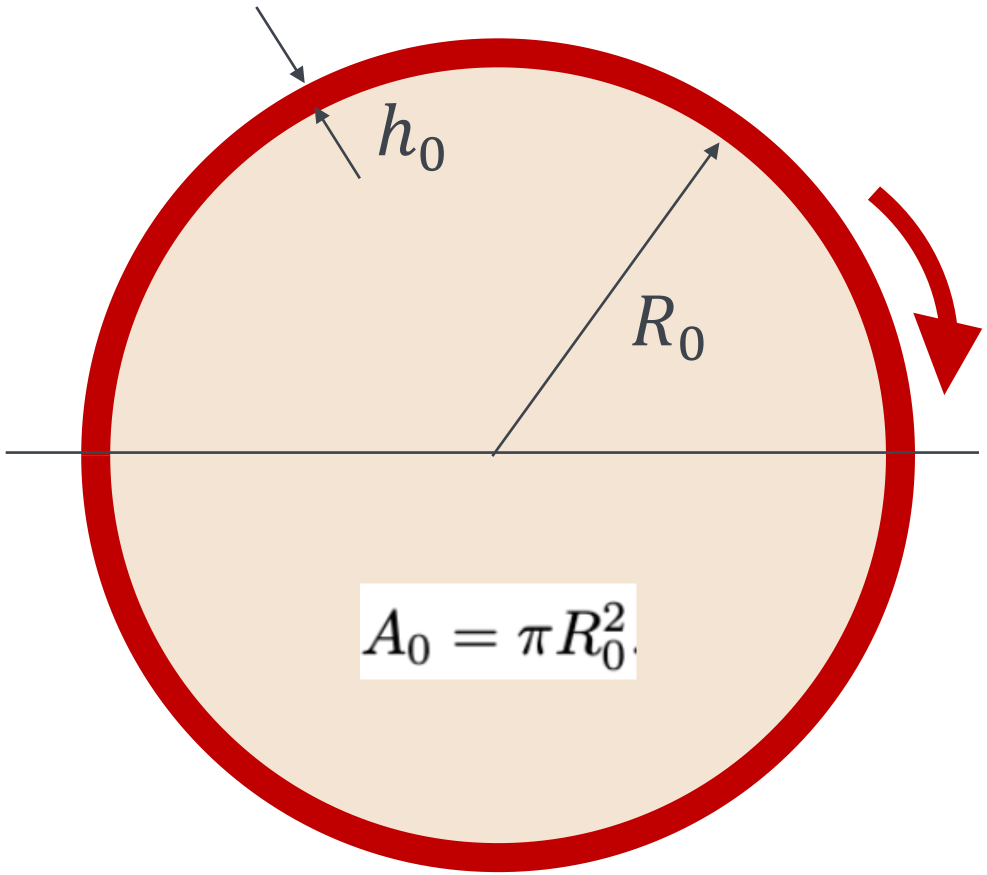
$$= \frac{(1-\nu^2)}{E}\frac{\Delta pR_0}{h_0}$$

$$\Delta p = p - P_{ext}$$

$$\sigma_{rr} \ll \sigma_{\theta\theta} \qquad \epsilon_{zz} \approx 0 \qquad \sigma_{\theta\theta} = \frac{pR_0}{h_0}$$

**relative pressure, generalize**

# Closure model




$$\epsilon_{\theta\theta} = \frac{R - R_0}{R_0} \frac{2\pi}{2\pi}$$

$$\epsilon_{\theta\theta} \approx \frac{(1-\nu^2)}{E} \frac{pR_0}{h_0}$$

$$p = P_{ext} + \frac{Eh_0}{(1-\nu^2)R_0}\epsilon_{\theta\theta} = P_{ext} + \frac{Eh_0}{(1-\nu^2)R_0} \frac{R - R_0}{R_0} = P_{ext} + \frac{\sqrt{\pi}Eh_0}{(1-\nu^2)\sqrt{A_0}}\left(\sqrt{\frac{A}{A_0}} - 1\right)$$

# 1D Equations (summary again)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0$$

**mass balance**

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial z}\left(\alpha \frac{Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A} = 0$$

**momentum balance**

$$p = P_{\text{ext}} + G_0\left(\sqrt{\frac{A}{A_0}} - 1\right) \qquad G_0 = \frac{\sqrt{\pi} E h_0}{(1-\nu^2)\sqrt{A_0}}$$

**closure model**

# 1D Equations (transport form)

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$ :

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0, l), \; t > 0$$

$$\mathbf{U} = \begin{pmatrix} A \\ Q \end{pmatrix}, \; \mathbf{F}(\mathbf{U}) = \begin{pmatrix} Q \\ \frac{Q^2}{A} + \frac{G_0 A^{3/2}}{3\rho A_0^{1/2}} \end{pmatrix}, \; \mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ -K_R \frac{Q}{A} \end{pmatrix}$$

$$G_0 = \frac{\sqrt{\pi} E h_0}{(1-\nu^2)\sqrt{A_0}}$$

(simplified for constant parameters, for varying parameters see lecture notes)

University of Stuttgart

# Literature


**John, L., P. Pustějovská, and O. Steinbach.** "On the influence of the wall shear stress vector form on hemodynamic indicators." *Computing and Visualization in Science* 18.4-5 (2017): 113-122

**D'Angelo, Carlo.** "Multiscale modelling of metabolism and transport phenomena in living tissues." *Bibliotheque de l'EPFL, Lausanne* (2007)

**Čanić, Sunčica, and Eun Heui Kim.** "Mathematical analysis of the quasilinear effects in a hyperbolic model blood flow through compliant axi-symmetric vessels." *Mathematical Methods in the Applied Sciences* 26.14 (2003): 1161-1186

**Westerhof, Nicolaas, et al.** "Analog studies of the human systemic arterial tree." *Journal of biomechanics* 2.2 (1969): 121IN1135IN3137IN5139-134136138143

**Olufsen, Mette S.** "Structured tree outflow condition for blood flow in larger systemic arteries." *American journal of physiology-Heart and circulatory physiology* 276.1 (1999): H257-H268

**Fernández, Miguel Ángel, Vuk Milisic, and Alfio Quarteroni**. "Analysis of a geometrical multiscale blood flow model based on the coupling of ODEs and hyperbolic PDEs." *Multiscale Modeling & Simulation* 4.1 (2005): 215-236

# Literature

**Köppl, T., Schneider, M., Pohl, U., & Wohlmuth, B.** (2014). The influence of an unilateral carotid artery stenosis on brain oxygenation. *Medical engineering & physics*, *36*(7), 905-914

**Galdi, G., Rannacher, R., Robertson, A., Turek, S**.: *Hemodynamical Flows: Modeling, Analysis and Simulation*, vol. 37. Birkhäuser,Basel (2008)

# Thank you!

**Timo Koch (Oslo)**

e-mail timokoch@uio.no

University of Stuttgart
Pfaffenwaldring 61, 70569 Stuttgart

**Institute for Modelling Hydraulic and Environmental Systems,**

**Department of Hydromechanics and Modelling of Hydrosystems**

**https://www.iws.uni-stuttgart.de/lh2/**

SimTech
Cluster of Excellence

**SFB 1313**
**sfb1313.uni-stuttgart.de**
Interface-Driven Multi-Field Processes in Porous Media

DuMu$^X$ **dumux.org**

Dune
http://dune-project.org/