Bio-fluid mechanics Exercise Part I

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1 Characteristic variables

The 1D model for elastic vessels can be written in quasi-linear form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}}\frac{\partial \boldsymbol{U}}{\partial z} + \boldsymbol{S}(\boldsymbol{U}) = 0, \tag{1}$$

with the entries of the Jacobian matrix of the flux function $\boldsymbol{J}(\boldsymbol{U})$ given by $J_{ij} = \frac{\partial F_i}{\partial U_j}$ and

$$\boldsymbol{U} = \begin{bmatrix} A \\ Q \end{bmatrix}, \quad \boldsymbol{F}(\boldsymbol{U}) = \begin{bmatrix} Q \\ \alpha\frac{Q^2}{A} + \frac{G_0}{3\rho}A_0\left(\frac{A}{A_0}\right)^{\frac{3}{2}} \end{bmatrix}, \quad \boldsymbol{S}(\boldsymbol{U}) = \begin{bmatrix} 0 \\ K_R\frac{Q}{A} \end{bmatrix}. \tag{2}$$

(1.1) Let's assume $\alpha = 1$. Show that the Jacobian is given by

$$\boldsymbol{J}(\boldsymbol{U}) = \begin{pmatrix} 0 & 1 \\ -\bar{v}^2 + c_1^2(A) & 2\bar{v} \end{pmatrix}, \quad \bar{v} := \frac{Q}{A}, \quad c_1(A) := \sqrt{\sqrt{\frac{A}{A_0}}\frac{G_0}{2\rho}}. \tag{3}$$

If $\boldsymbol{J}$ has only real eigenvalues and is diagonizable, the system of transport equations is hyperbolic.

(1.2) Show that the eigenvalues of the Jacobian are given by

$$\lambda_1 = \bar{v} - c_1, \quad \lambda_2 = \bar{v} + c_1. \tag{4}$$

The corresponding (scaled) left eigenvectors are given by

$$\boldsymbol{l}_1^T(\boldsymbol{J} - \lambda_1\boldsymbol{I}) = 0 \;\Rightarrow\; \boldsymbol{l}_1 = \frac{1}{A}(\bar{v} + c_1, -1)^T, \quad \boldsymbol{l}_2^T(\boldsymbol{J} - \lambda_2\boldsymbol{I}) = 0 \;\Rightarrow\; \boldsymbol{l}_2 = \frac{1}{A}(c_1 - \bar{v}, 1)^T. \tag{5}$$

With this eigenbasis the Jacobian matrix can be diagonalized, i.e. $\boldsymbol{J} = \boldsymbol{L}^{-1}\boldsymbol{\Lambda}\boldsymbol{L}$, where the rows of $\boldsymbol{L}$ correspond to the left eigenvectors and $\boldsymbol{\Lambda} := \operatorname{diag}(\lambda_i)$ is the diagonal matrix of eigenvalues. Using the eigendecomposition and multiplying Eq. (1) with $\boldsymbol{L}$ gives

$$\boldsymbol{L}\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{\Lambda}\boldsymbol{L}\frac{\partial \boldsymbol{U}}{\partial z} + \boldsymbol{L}\boldsymbol{S}(\boldsymbol{U}) = 0. \tag{6}$$

If we find new variables such that $\frac{\partial W_i}{\partial \boldsymbol{U}} = \boldsymbol{l}_i$ or in vector-notation $\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{U}} = \boldsymbol{L}$, then we deduce a system of coupled scalar transport equations for the characteristic variables:

$$\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{U}}\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{\Lambda}\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{U}}\frac{\partial \boldsymbol{U}}{\partial z} + \boldsymbol{L}\boldsymbol{S}(\boldsymbol{U}) = 0 \;\Leftrightarrow\; \frac{\partial \boldsymbol{W}}{\partial t} + \boldsymbol{\Lambda}\frac{\partial \boldsymbol{W}}{\partial z} + \boldsymbol{L}\boldsymbol{S}(\boldsymbol{U}) = 0. \tag{7}$$

(1.3) Compute the characteristic variables W_1 and W_2 that are the entries of $\boldsymbol{W}$. (Use integration.)

(1.4) Show that the physical variables A and Q can be expressed in terms of W_1 and W_2 as

$$A(W_1, W_2) = A_0\left(\frac{W_1 + W_2}{8c_0}\right)^4, \quad Q(W_1, W_2) = A\left(\frac{W_2 - W_1}{2}\right), \tag{8}$$

where $c_0 := c_1(A_0) = \sqrt{G_0/(2\rho)}$.