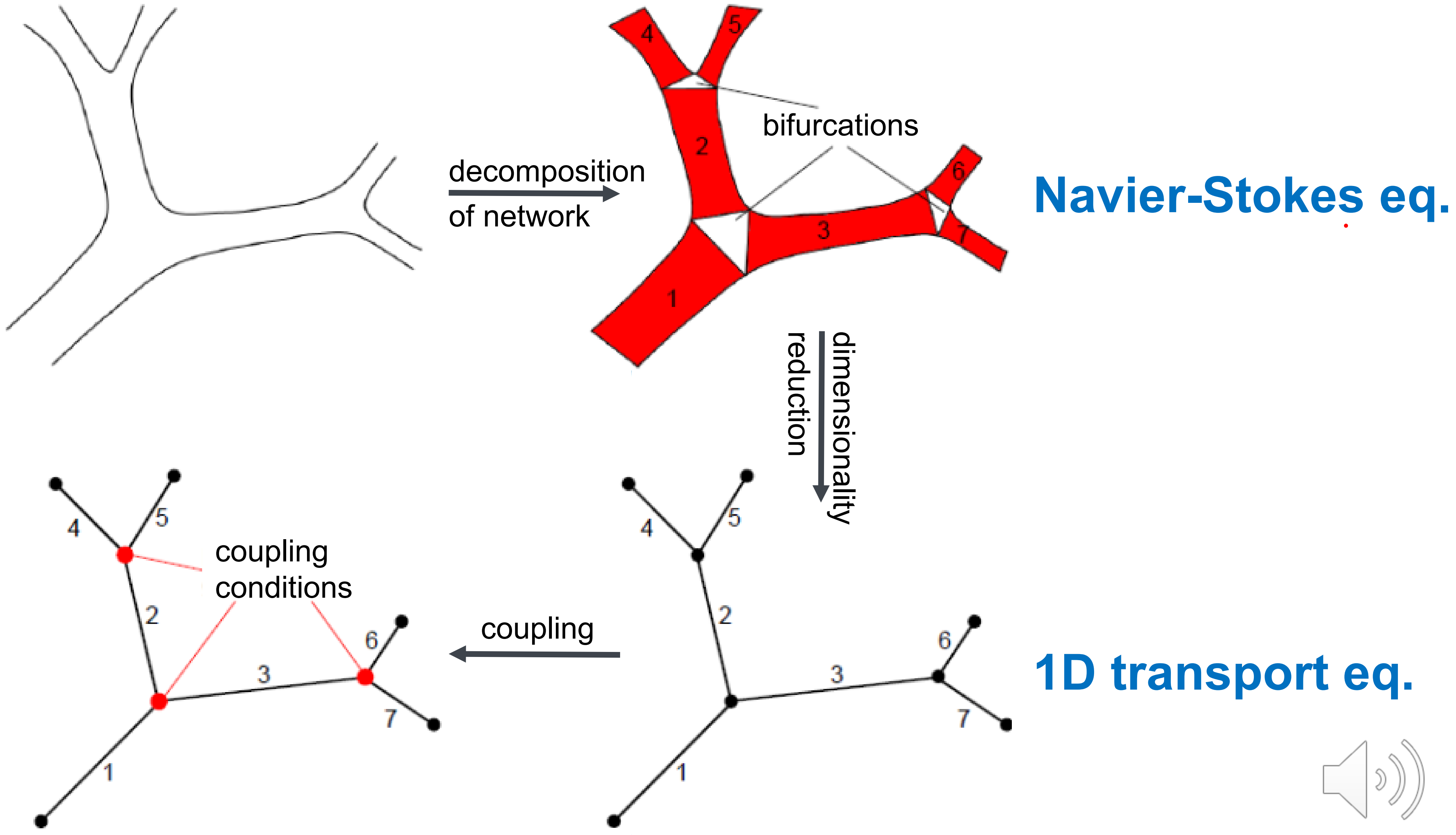
Repetition



decomposition of network
bifurcations
4
5
2
6
3
7
1
dimensionality reduction
coupling
coupling conditions
Navier-Stokes eq.
1D transport eq.

University of Stuttgart

1D Equations (summary)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0$$

mass balance

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial z}\left(\alpha \frac{Q^2}{A}\right) + \frac{A}{\rho}\frac{\partial p}{\partial z} + K_R \frac{Q}{A} = 0$$

momentum balance

$$p = P_{\text{ext}} + G_0\left(\sqrt{\frac{A}{A_0}} - 1\right) \qquad G_0 = \frac{\sqrt{\pi} E h_0}{(1-\nu^2)\sqrt{A_0}}$$

closure model

- A cross-sectional area
- Q flow rate
- p mean blood pressure
- α inertia correction coefficient (velocity profile)
- K_R friction coefficient
- ρ blood density
- P_{ext} external (tissue) pressure
- ν Poisson's ratio (vessel wall)
- E Young's modulus (vessel wall)
- h_0 Vessel wall thickness
- A_0 Unstressed cross-sectional area

1D Equations (transport form)

University of Stuttgart

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \quad z \in (0, l), \quad t > 0$$

$$\mathbf{U} = \begin{pmatrix} A \\ Q \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} Q \\ \frac{Q^2}{A} + \frac{G_0 A^{3/2}}{3\rho A_0^{1/2}} \end{pmatrix}, \quad \mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ -K_R \frac{Q}{A} \end{pmatrix}$$

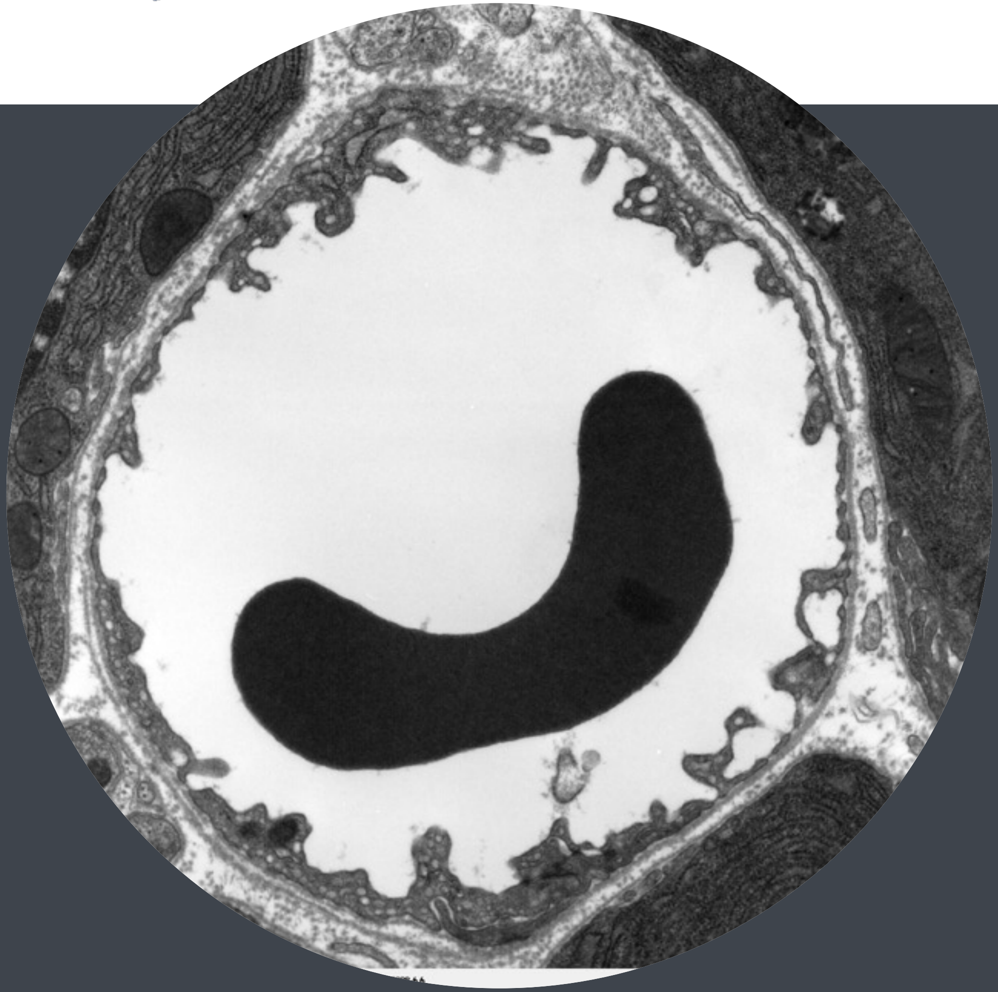
$$G_0 = \frac{\sqrt{\pi} E h_0}{(1 - \nu^2)\sqrt{A_0}}$$

(simplified for constant parameters, for varying parameters see lecture notes)

University of Stuttgart
Institute for Modelling Hydraulic and Environmental Systems

Chapter 3

Numerics of transport equations

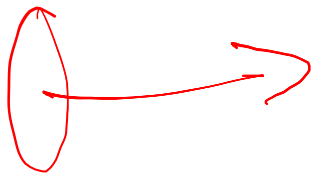


University of Stuttgart

Transport equations

General system of transport equations in 3D

$$\frac{\partial \mathbf{U}}{\partial t} + \cancel{\frac{\partial \mathbf{F}_1}{\partial x}(\mathbf{U})} + \cancel{\frac{\partial \mathbf{F}_2}{\partial y}(\mathbf{U})} + \frac{\partial \mathbf{F}_3}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U})$$



$$U = \begin{pmatrix} A \\ Q \end{pmatrix}$$

Primary variables: $\mathbf{U} \in \mathbb{R}^n,\ \mathbf{U} = (U_1, \ldots, U_n)^T$

Flux function: $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$

Source term: $\mathbf{S} : \mathbb{R}^n \to \mathbb{R}^n$

Transport equations

General system of transport equations in 3D

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_1}{\partial x}(\mathbf{U}) + \frac{\partial \mathbf{F}_2}{\partial y}(\mathbf{U}) + \frac{\partial \mathbf{F}_3}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U})$$

Examples:

- Euler equations
- Shallow water equations (2D)
  → Saint-Venant equations (1D)
- Wave equation (after Reformulation)

Transport equations

General system of transport equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_1}{\partial x}(\mathbf{U}) + \frac{\partial \mathbf{F}_2}{\partial y}(\mathbf{U}) + \frac{\partial \mathbf{F}_3}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U})$$

- Euler equations
- Shallow water equations (2D)
  → Saint-Venant equations (1D)
- Wave equation (after Reformulation)

hyperbolic equations

DuMu^X

Transport equations

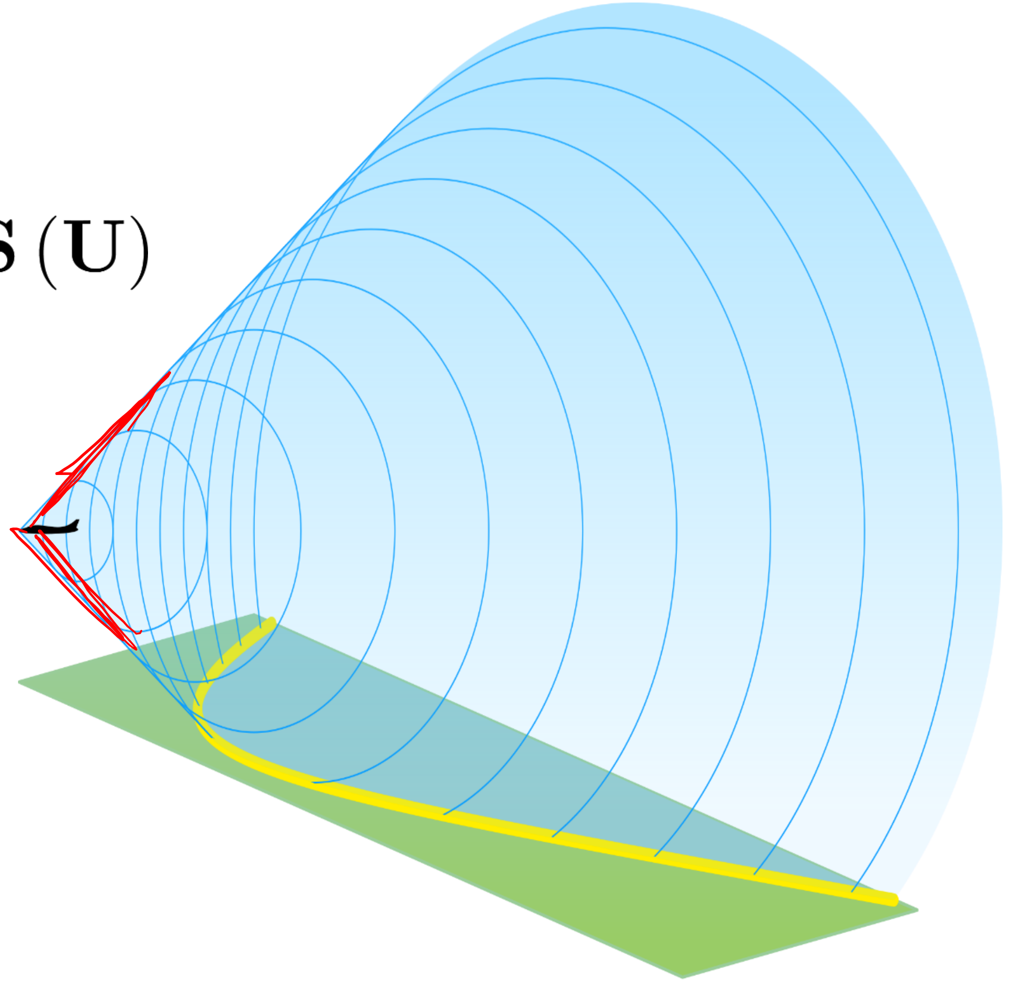
University of Stuttgart

General system of transport equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_1}{\partial x}(\mathbf{U}) + \frac{\partial \mathbf{F}_2}{\partial y}(\mathbf{U}) + \frac{\partial \mathbf{F}_3}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U})$$

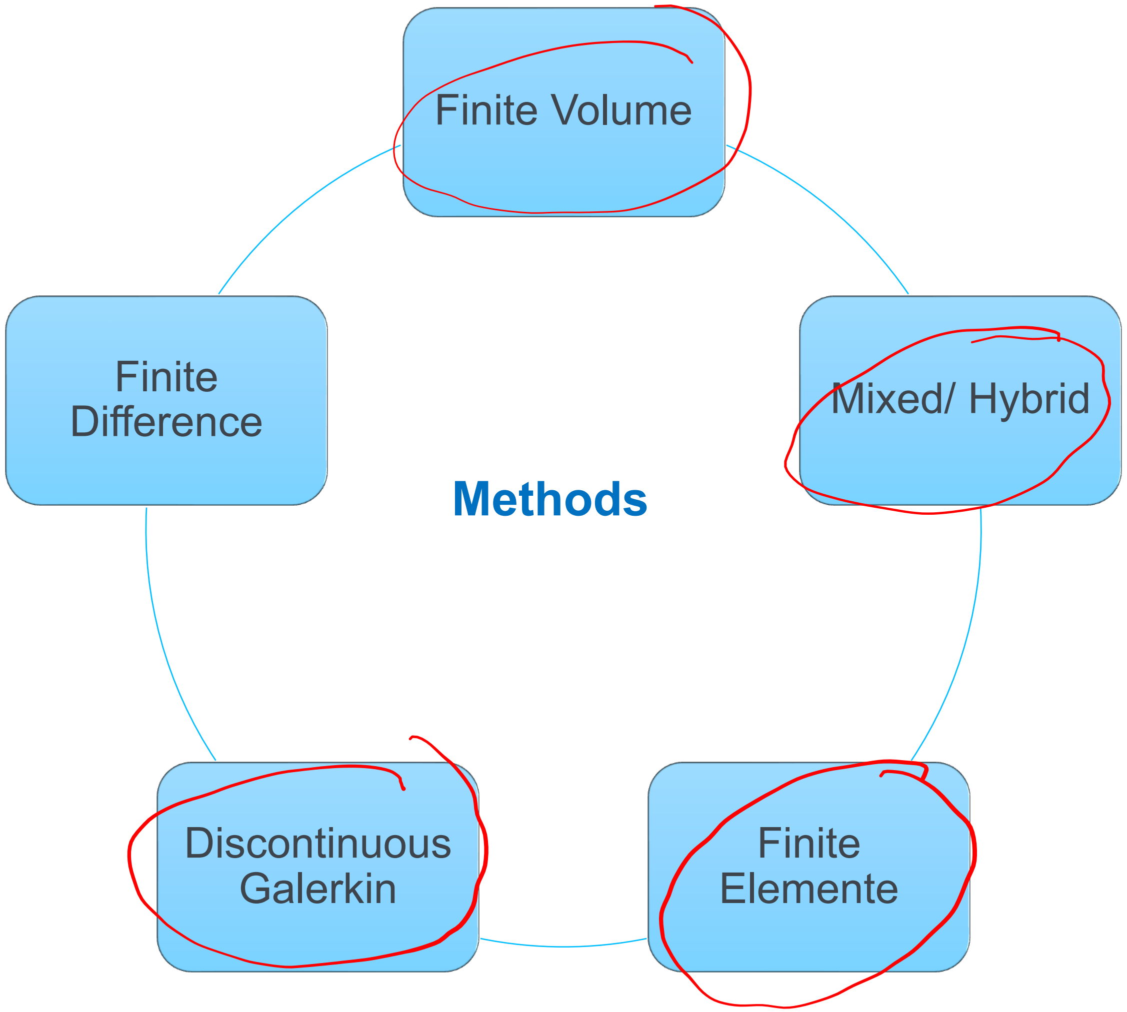
- Euler equations
- Shallow water equations (2D)
  → Saint-Venant equations (1D)
- Wave equation (after Reformulation)

hyperbolic equations



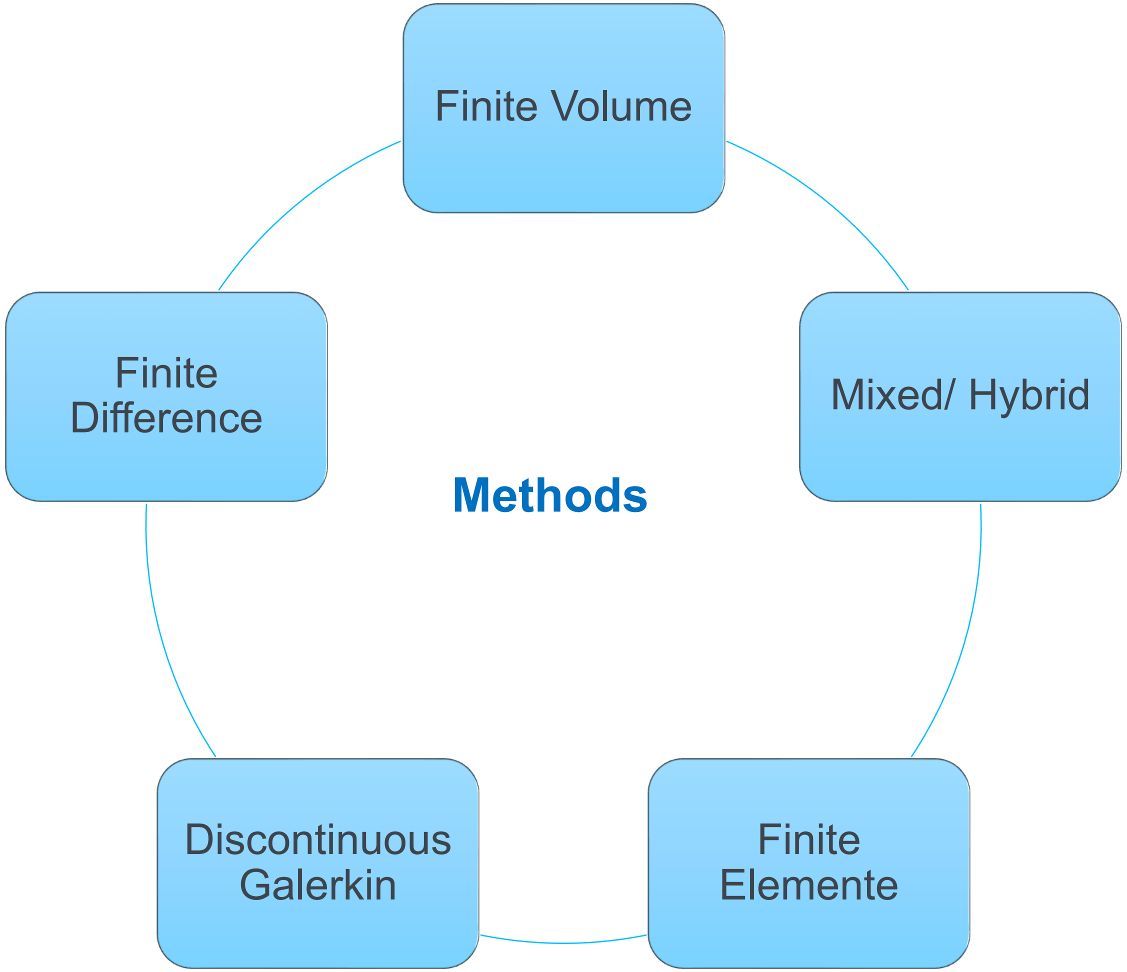
Discretization methods

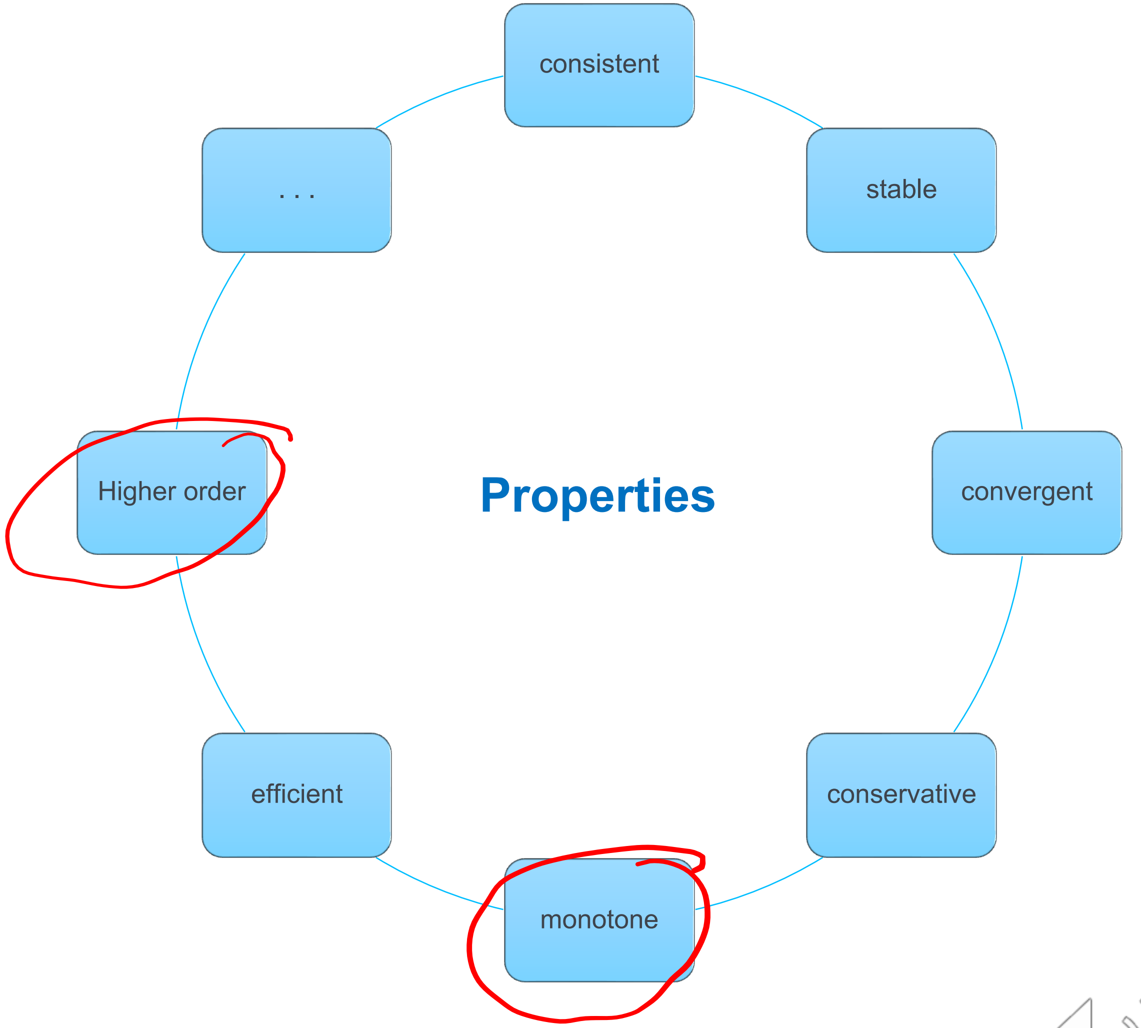
Discretisation schemes



University of Stuttgart

Discretisation schemes





Scalar transport equation

General system of transport equations in 1D

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}\left(\mathbf{U}\right) = \mathbf{S}\left(\mathbf{U}\right), \; z \in (0, l), \; t > 0$$

Scalar equation: $\mathbf{U} = u \in \mathbb{R}$

$$\mathbf{F}\left(\mathbf{U}\right) = f(u), \; \mathbf{S}\left(\mathbf{U}\right) \equiv 0$$

Scalar transport equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z} f(u) = 0, \; z \in (0, l), \; t > 0$$

Finite volume method

University of Stuttgart

Divergence theorem

Divergence theorem

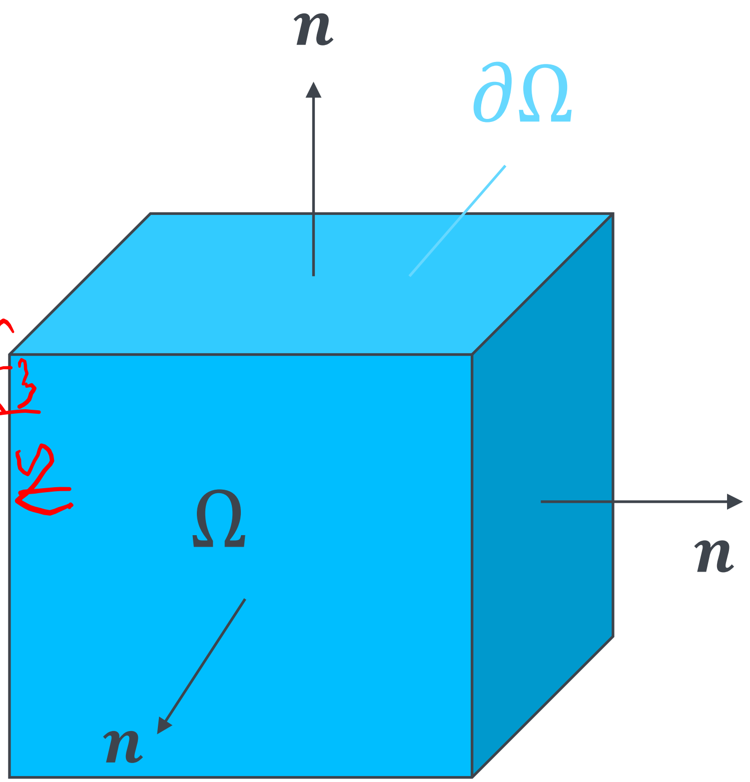
$$\int_\Omega \operatorname{div}(\boldsymbol{F})\,\mathrm{d}x = \int_{\partial\Omega} \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}s,$$

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$\operatorname{div}(F) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Divergence theorem (simplified for 1D)

$$\int_a^b \frac{\partial f}{\partial z}\,\mathrm{d}z = \int_\Omega \frac{\partial f}{\partial z}\,\mathrm{d}z = \int_{\partial\Omega} f n_z\,\mathrm{d}s = f(b) - f(a)$$



Applied to incompressible fluid flow (no source)

$$\int_\Omega \operatorname{div}(\boldsymbol{v})\,\mathrm{d}x = \underbrace{\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}s}_{Q} = 0,$$

$$\operatorname{div}(v) = 0$$

Finite Volume Method

Scalar transport equation

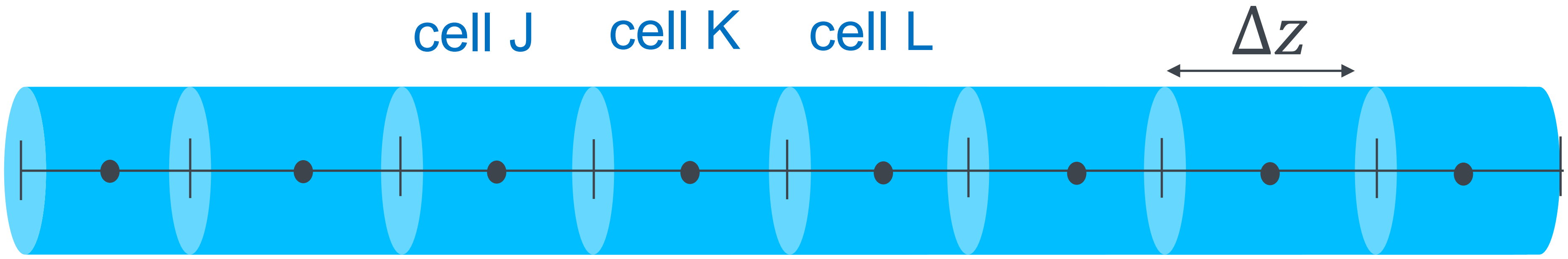
$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial z}f(u)=0,\ z\in(0,l)\,,\ t>0$$

University of Stuttgart

Finite Volume Method

Scalar transport equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z} f(u) = 0, \;\; z \in (0, l)\,, \;\; t > 0$$



- cell-center, degree of freedom, e.g. $u_K := \frac{1}{|K|} \int_K u(z,t)dz$

| face, flux integration point

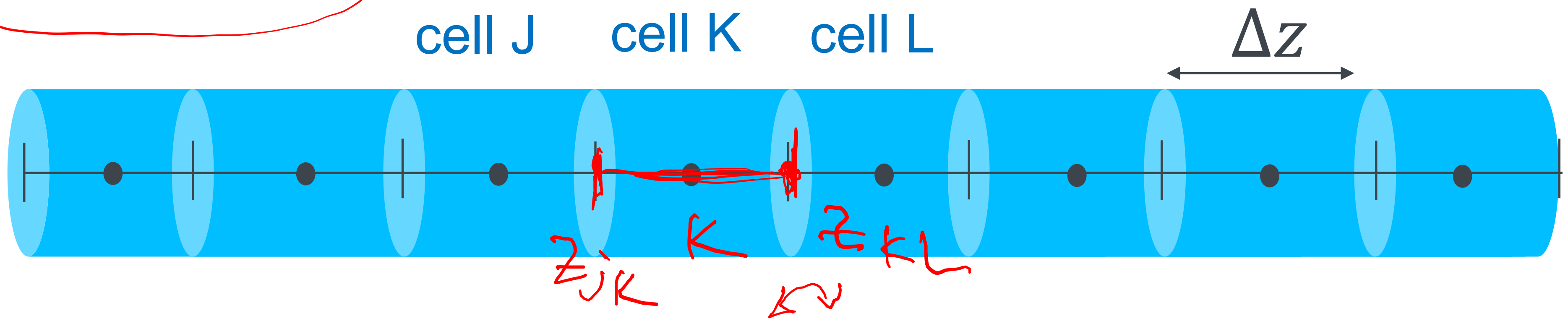
Finite Volume Method

cell average

$$u_K := \frac{1}{\Delta z}\int_K u(z,t)\,dz$$

Scalar transport equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z} f(u) = 0,\ z \in (0,l)\,,\ t > 0$$



$\Delta z = |K|$

$$\int_{t_n}^{t_{n+1}}\int_K \frac{\partial u}{\partial t}\,dz\,dt + \int_{t_n}^{t_{n+1}}\int_K \frac{\partial f(u)}{\partial z}\,dz\,dt = \int_K\int_{t_n}^{t_{n+1}} \frac{\partial u}{\partial t}\,dt\,dz + \int_{t_n}^{t_{n+1}}\int_K \frac{\partial f(u)}{\partial z}\,dz\,dt$$

$$= \int_K u(z,t_{n+1}) - u(z,t_n)\ dz + \int_{t_n}^{t_{n+1}}\int_K \frac{\partial f(u)}{\partial z}\,dz\,dt$$

$$= |K|u_K^{n+1} - |K|u_K^n + \int_{t_n}^{t_{n+1}} \boxed{f(u(z_{KL}))} - f(u(z_{JK}))\ dt$$

Integrate in time and space

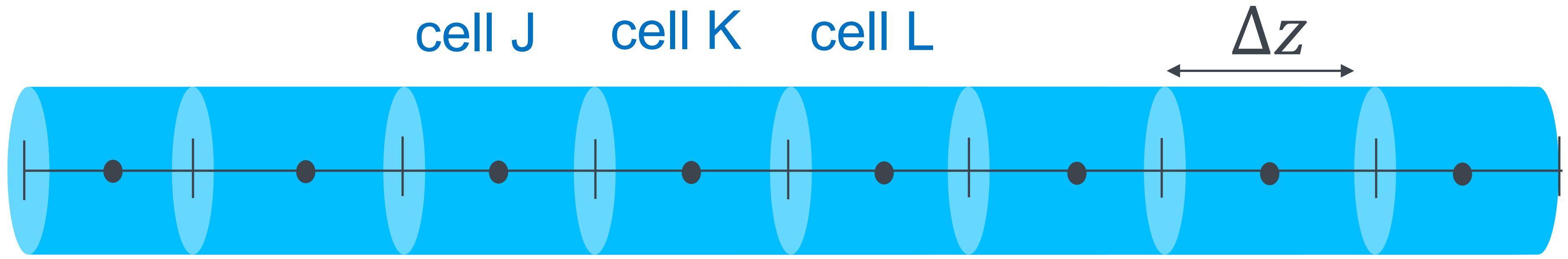
University of Stuttgart

Finite Volume Method

Scalar transport equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z} f(u) = 0,\ \ z \in (0, l)\ ,\ \ t > 0$$

$$u_K := \frac{1}{\Delta z} \int_K u(z,t)\,dz$$



$$\int_{t_n}^{t_n+1} \int_K \frac{\partial u}{\partial t}\,dz\,dt + \int_{t_n}^{t_n+1} \int_K \frac{\partial f(u)}{\partial z}\,dz\,dt = \Delta z\, u_K^{n+1} - \Delta z\, u_K^{n} + \int_{t_n}^{t_n+1} f(u(z_{KL})) - f(u(z_{JK}))\ dt$$

Flux approximation: $f(u(z_{KL})) \approx F_{KL}$

$$= |K| u_K^{n+1} - |K| u_K^{n} + \int_{t_n}^{t_n+1} F_{KL} - F_{JK}\ dt$$

Explicit Euler:

$$= |K| u_K^{n+1} - |K| u_K^{n} + \Delta t (F_{KL}^{n} - F_{JK}^{n}) = 0$$

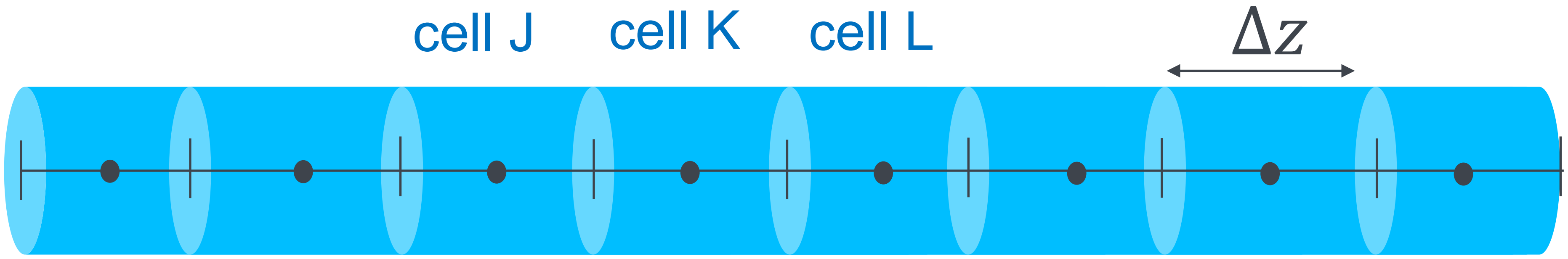
$\Delta t = t_{n+1} - t_n$

Finite Volume Method

Scalar transport equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z} f(u) = 0, \;\; z \in (0, l)\;, \;\; t > 0$$

$$\boxed{u_K := \frac{1}{\Delta z} \int_K u(z,t)\,dz}$$



$$u_K^{n+1} = u_k^n - \frac{\Delta t}{\Delta z}\left(F_{KL}^n - F_{JK}^n\right)$$

Discrete transport equation

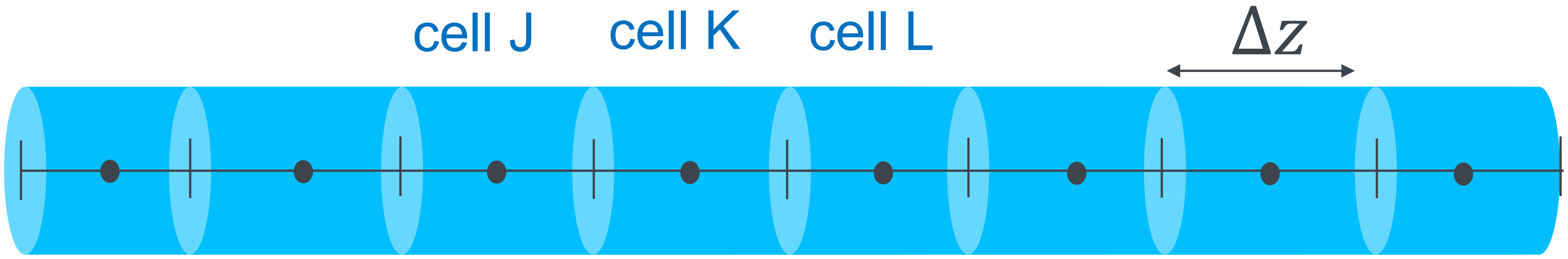
University of Stuttgart

Finite Volume Method

Scalar transport equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z} f(u) = 0,\ z \in (0, l)\,,\ t > 0$$

$$u_K := \frac{1}{\Delta z} \int_K u(z,t)\,dz$$



$$\Rightarrow \quad \boxed{u_K^{n+1} = u_K^n - \frac{\Delta t}{\Delta z}\left(F_{KL}^n - F_{JK}^n\right)}$$

Discrete transport equation

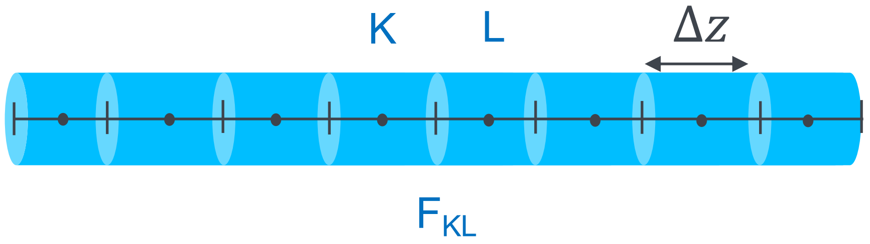
numerical flux: $\underline{F_{KL}^n := F(u_K^n, u_L^n)}$ **(„two-point flux“)**

Finite Volume Method

Derivation in higher dimensions is very similar, see lecture notes

University of Stuttgart

Finite Volume Method



Discrete scalar transport equation

$$u_K^{n+1} = u_K^n - \frac{\Delta t}{\Delta z}\left(F_{KL}^n - F_{JK}^n\right)$$

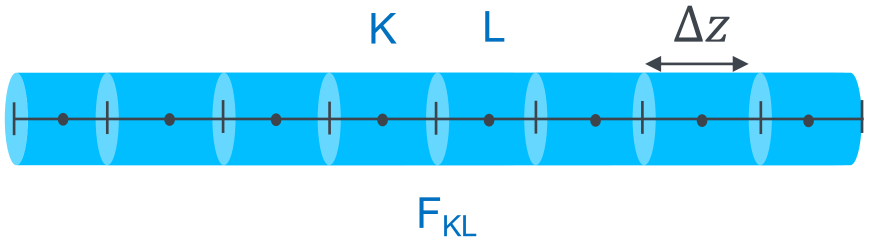
Numerical flux

$$F_{KL}^n := F(u_K^n, u_L^n)$$

Finite Volume Method

Discrete scalar transport equation

$$\boxed{u_K^{n+1} = u_K^n - \frac{\Delta t}{\Delta z}(F_{KL}^n - F_{JK}^n)}$$



Numerical flux

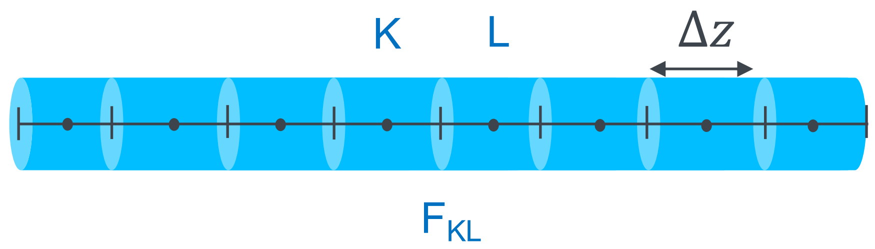
$$F_{KL}^n := F(u_K^n, u_L^n)$$

Properties:

- conservative $\quad F_{KL} = -F_{LK}$
- consistent $\quad F(a,a) = f(a), \quad \forall a \in \mathbb{R}$
- monotone (stability) $\quad c_K^n \leq u_K^n \quad \forall K \quad \Rightarrow \quad c_K^{n+1} \leq u_K^{n+1} \quad \forall K$

University of Stuttgart

Finite Volume Method



Discrete equation

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta z}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right)$$

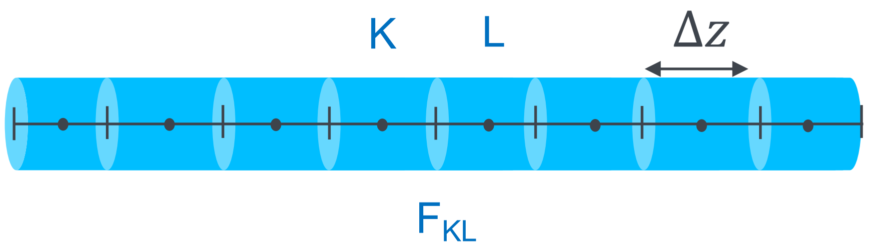
Numerical flux

$$F_{i+\frac{1}{2}}^n := F(u_i^n, u_{i+1}^n)$$

Examples:

- Central Flux: $F(a,b) = \dfrac{f(a) + f(b)}{2}$
- Lax-Friedrichs Flux: $F(a,b) = \dfrac{f(a) + f(b)}{2} - \dfrac{\Delta x}{2\Delta t}(b-a)$
- Upwind Flux: $F(a,b) = f(a)$ (reasonable for $f' > 0$)

Finite Volume Method



Discrete equation

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta z}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right)$$

Numerical flux

$$F^n_{i+\frac{1}{2}} := F(u_i^n, u_{i+1}^n)$$

Examples:

- Central Flux: $$F(a,b) = \frac{f(a)+f(b)}{2}$$
- Lax-Friedrichs Flux: $$F(a,b) = \frac{f(a)+f(b)}{2} - \frac{\Delta x}{2\Delta t}(b-a)$$
- Upwind Flux: $$F(a,b) = f(a)$$ (Generalisation: **Riemann solver (Godunov)**)

University of Stuttgart

Simple Example 1

Linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(vu) = 0,\ z \in (0, l)\,,\ t > 0,\ v = 0.5$$

$$u(z,0) = u_0(z), \text{(initial condition)}$$

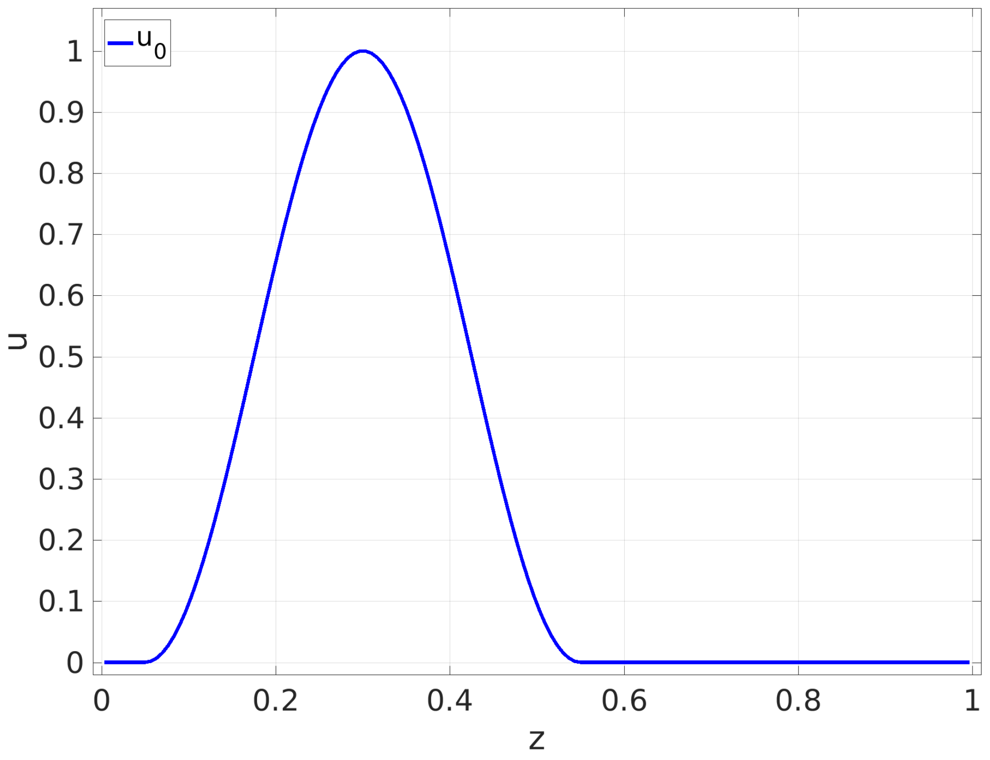
Exact solution (weak solution):

$$u(z,t) = u_0\left(z - vt\right)$$

Simple Example 1

Linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(vu) = 0, \; z \in (0, l), \; t > 0, \; v = 0.5$$

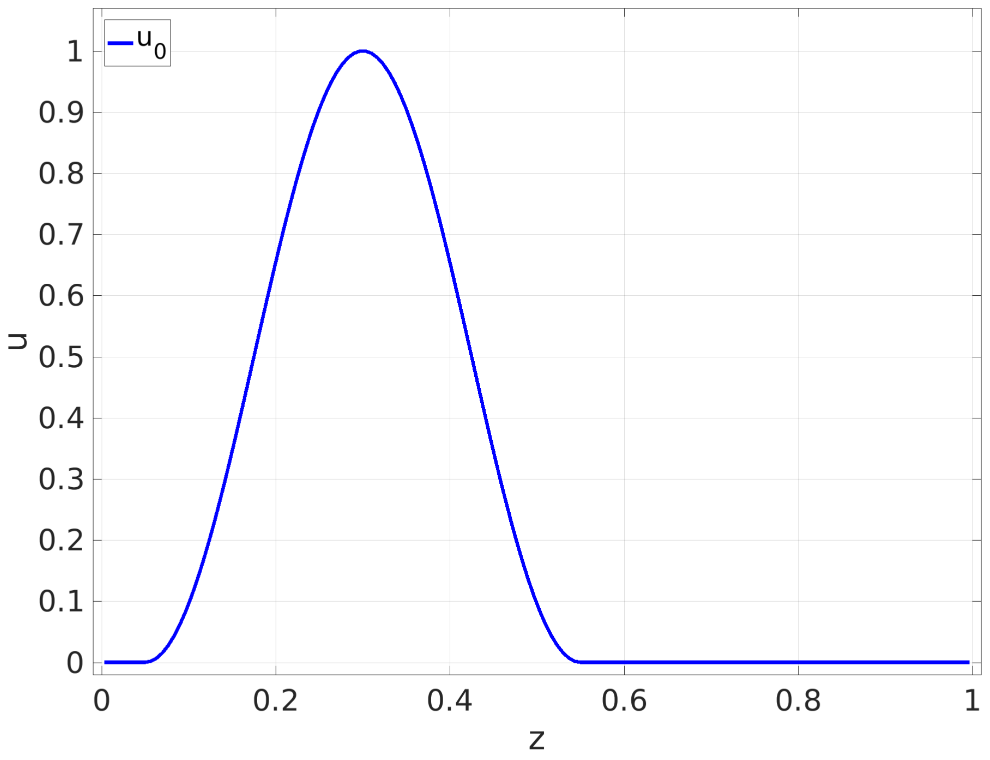


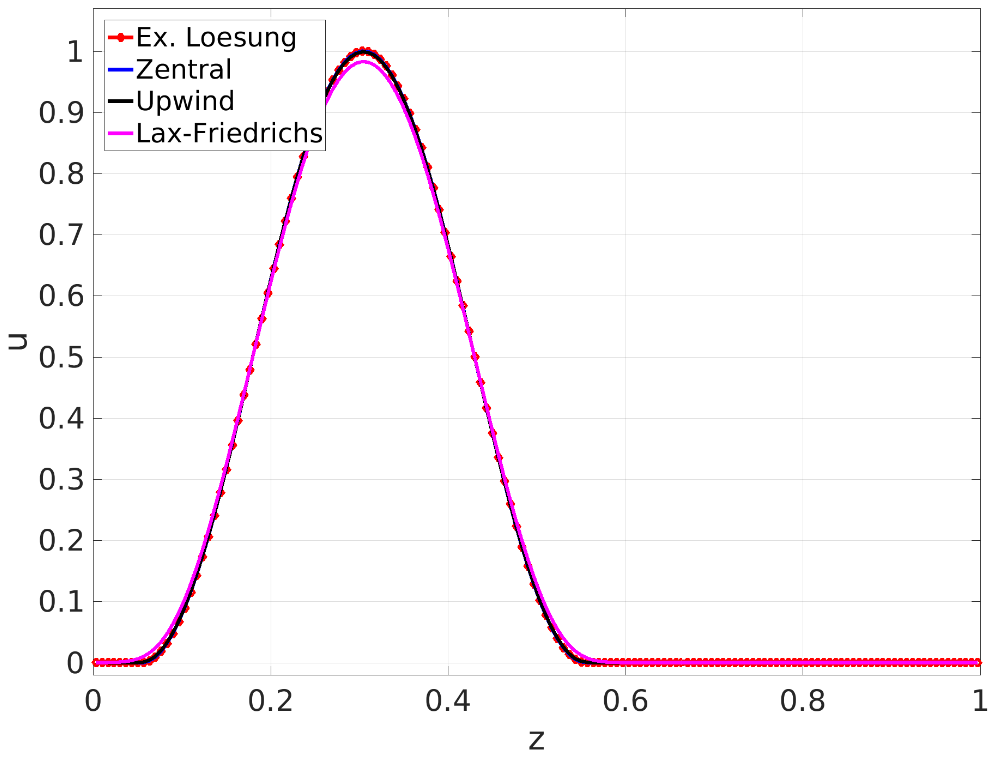
Simple Example 1

University of Stuttgart

Linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(vu) = 0, \ z \in (0, l), \ t > 0, \ v = 0.5$$

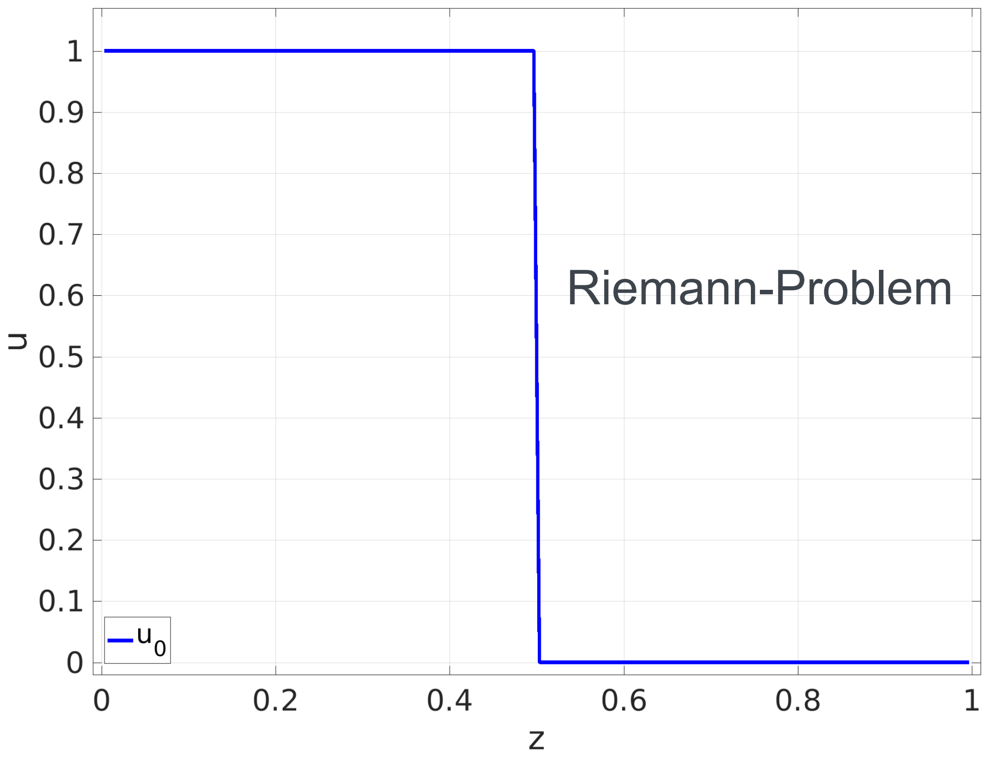




Simple Example 1

Linear advection equation

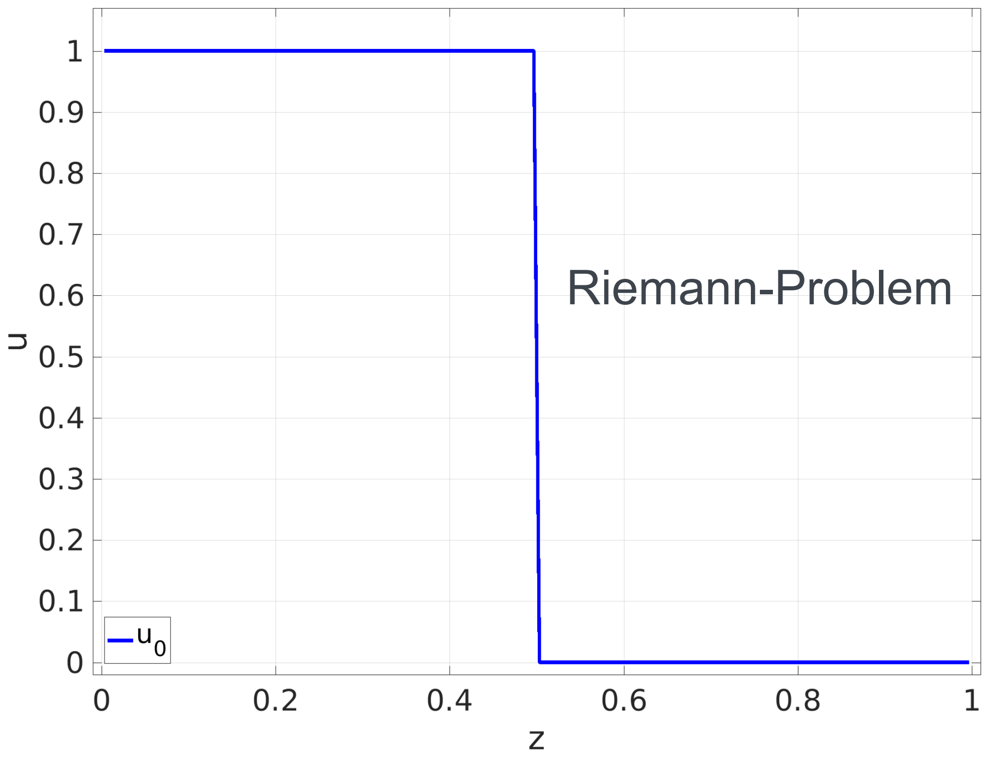
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(vu) = 0,\ z \in (0, l)\,,\ t > 0,\ v = 0.5$$

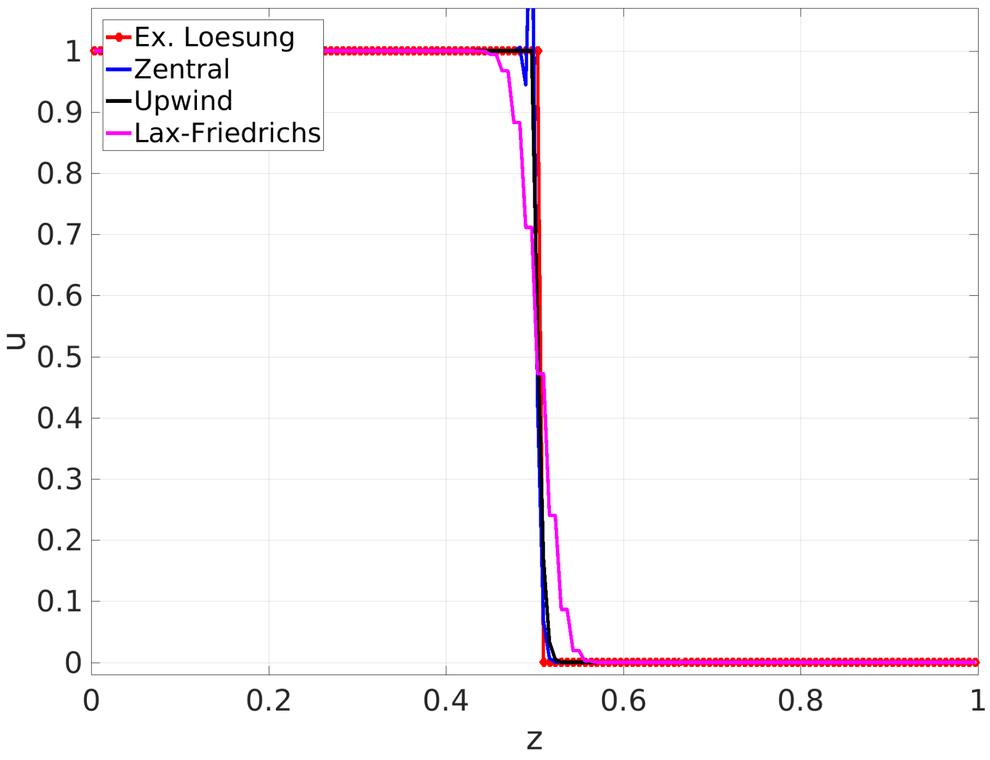


Simple Example 2

Linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(vu) = 0,\ z \in (0, l)\,,\ t > 0,\ v = 0.5$$

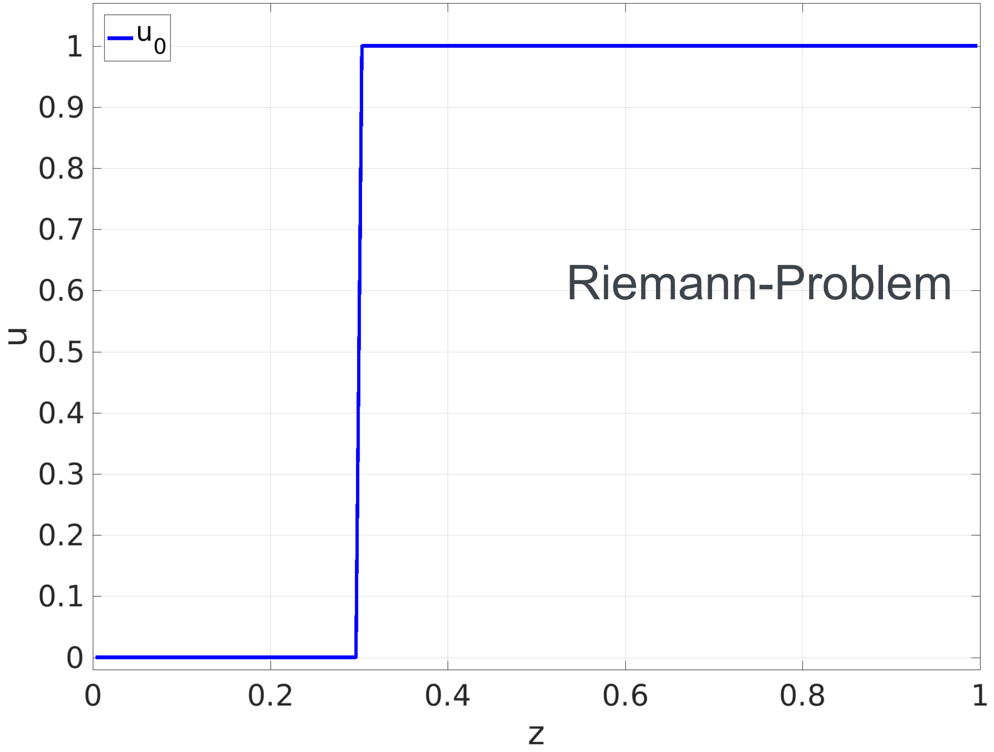




Simple Example 3

Burgers equation

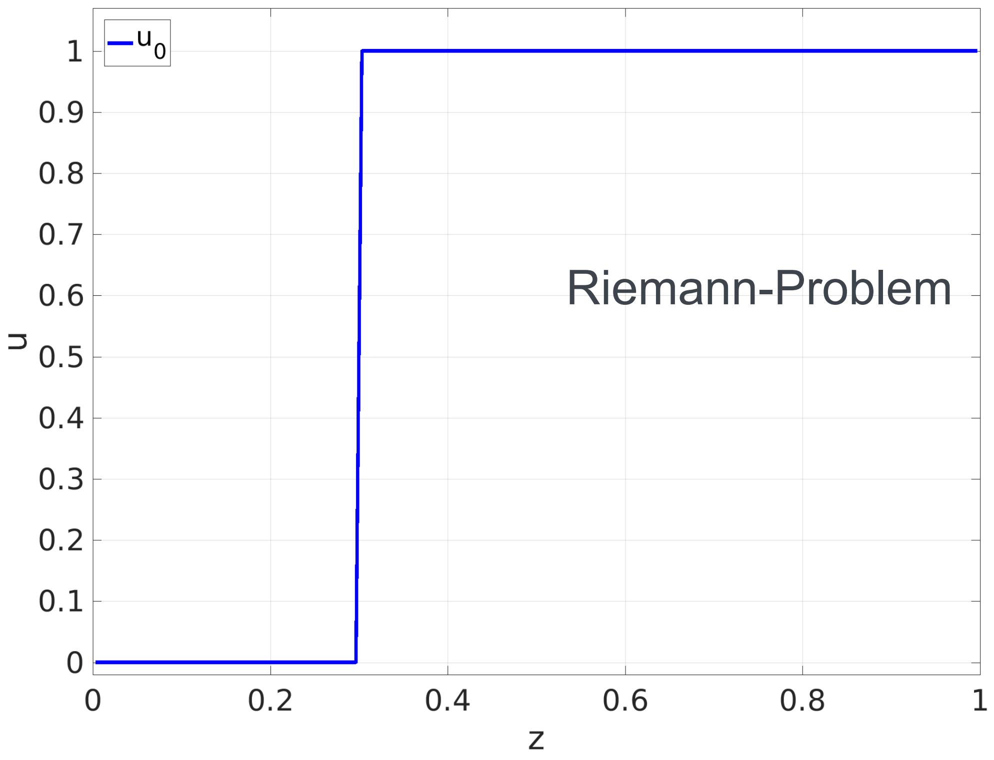
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}\left(\frac{1}{2}u^2\right) = 0,\ \ z \in (0, l)\ ,\ \ t > 0$$



Simple Example 3

Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}\left(\frac{1}{2}u^2\right) = 0,\ z \in (0, l)\,,\ t > 0$$

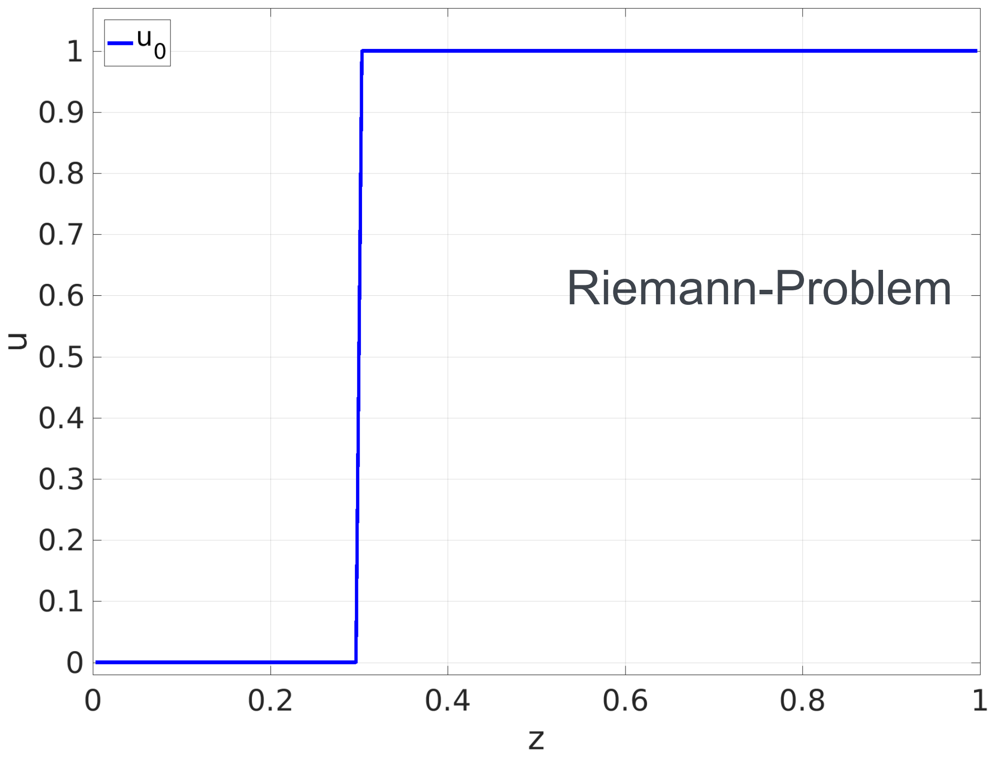


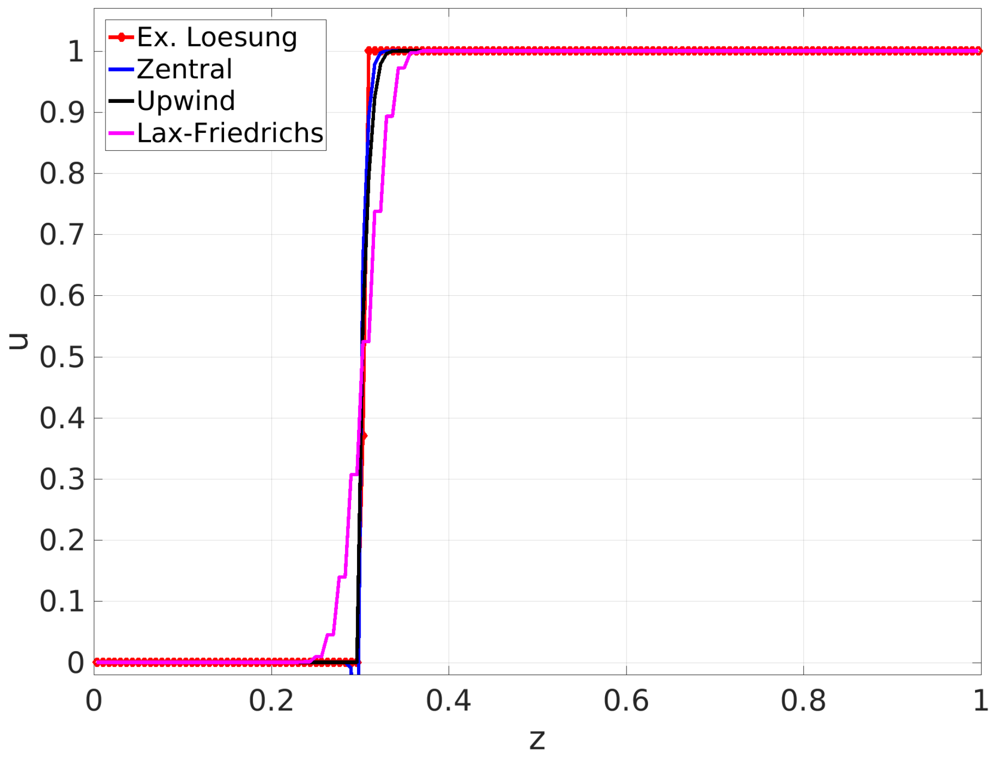
- Physical solution: **rarefaction wave**
- **There are also unphysical solutions (e.g. shock)**

Simple Example 3

Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial z}\left(\frac{1}{2}u^2\right) = 0,\ \ z \in (0, l)\ ,\ \ t > 0$$

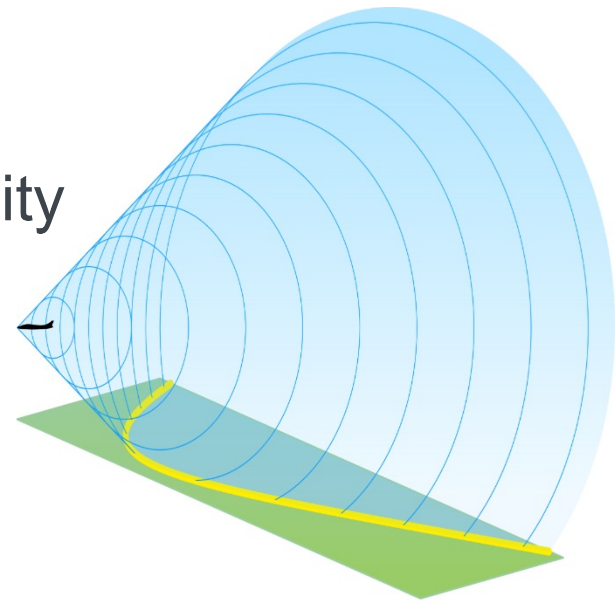




University of Stuttgart

Summary

- Shock waves and rarefaction waves can be physical solutions of hyperbolic equations
- FV methods are conservative and consistent, allow discontinuity
- Monotone schemes are stable (but in general only first order)
  - Godunov's order barrier theorem
- (Unphysical) oscillations occur for schemes of higher order
- (Approximate) Riemann solvers (State-of-the-Art in CFD)
  → HLLC-like solvers (Harten-Lax-van Leer-Contact)



Source:https://en.wikipedia.org/wiki/Supersonic_speed#/media/File:Supersonic_shockwave_cone.svg (CC-BY-SA 4.0)

Monotonicity (stability) + consistency →convergence to physical solution

Literature

- R. Leveque, “Finite Volume Methods for Hyperbolic Problems”

University of Stuttgart

Systems of transport equations

General system of transport equations in 1D

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0, l), \; t > 0$$

$$u = \begin{pmatrix} A \\ Q \end{pmatrix}$$

Scalar equation: $\mathbf{U} = u \in \mathbb{R}$

$$\mathbf{F}(\mathbf{U}) = f(u), \; \mathbf{S}(\mathbf{U}) \equiv 0$$

Quasi-linear form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \underline{\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} \frac{\partial \boldsymbol{U}}{\partial z}} + \boldsymbol{S}(\boldsymbol{U}) = 0$$

University of Stuttgart

Systems of transport equations

Quasi-linear form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}} \frac{\partial \boldsymbol{U}}{\partial z} + \boldsymbol{S}(\boldsymbol{U}) = 0$$

$$\underline{\mathbf{H}(\mathbf{U})} := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$$

Mathematical definition of a hyperbolic system

$$H = L^{-1} \Lambda L$$

- H has only real eigenvalues
- H is diagonalizable

University of Stuttgart
Institute for Modelling Hydraulic and Environmental Systems

SimTech
Cluster of Excellence

Thank you!

Timo Koch (Oslo)

e-mail timokoch@uio.no

https://www.iws.uni-stuttgart.de/lh2/

University of Stuttgart

Pfaffenwaldring 61, 70569 Stuttgart

Institute for Modelling Hydraulic and Environmental Systems,

Department of Hydromechanics and Modelling of Hydrosystems

SFB 1313
sfb1313.uni-stuttgart.de
Interface-Driven Multi-Field Processes in Porous Media

DuMu^x **dumux.org**

Dune
http://dune-project.org/