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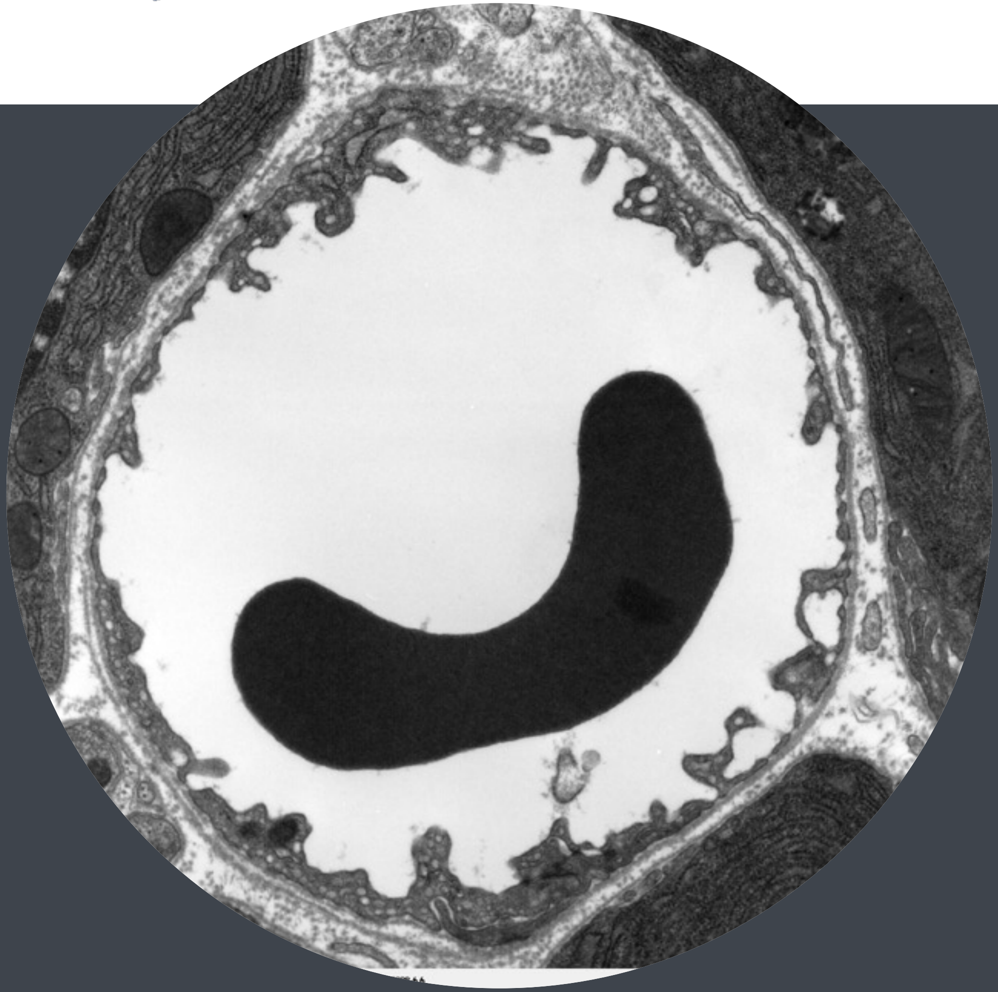

Chapter 3

Numerics of transport equations

Arteries

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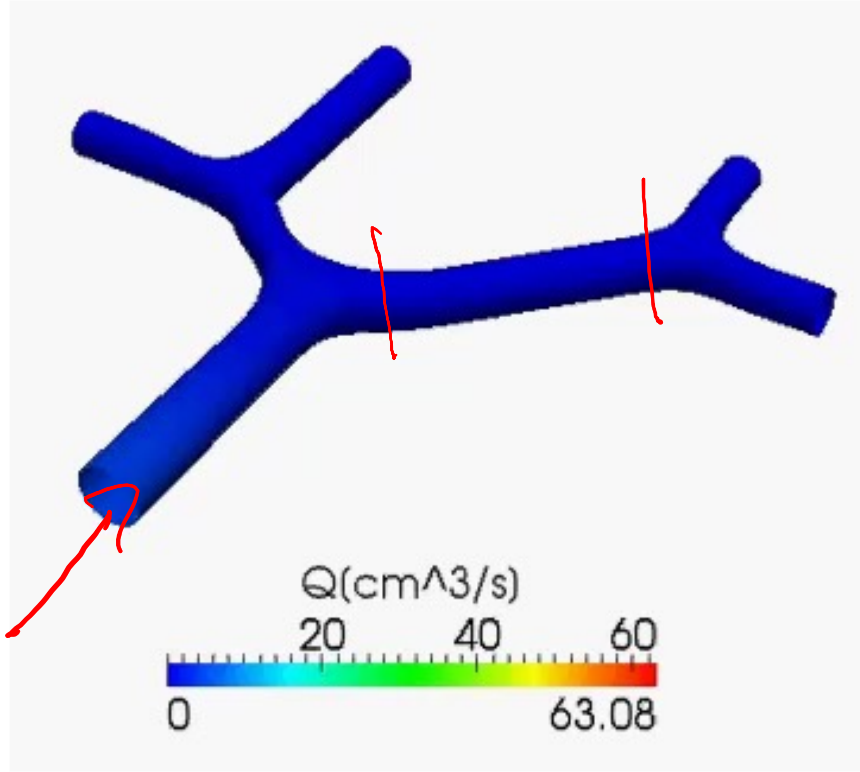
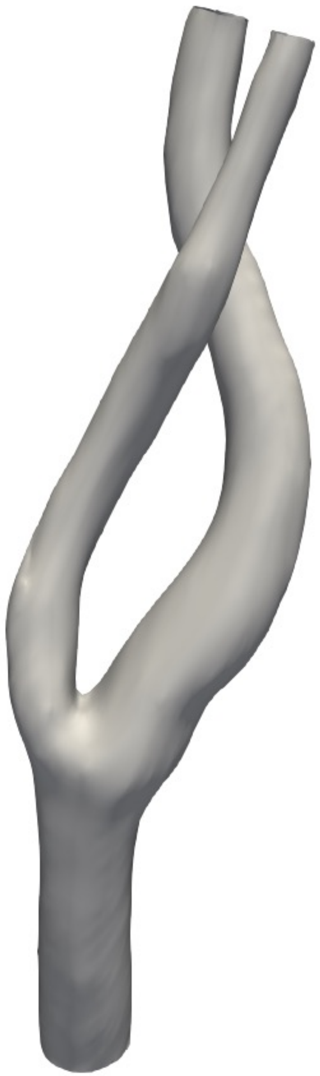
Blood flow through arteries





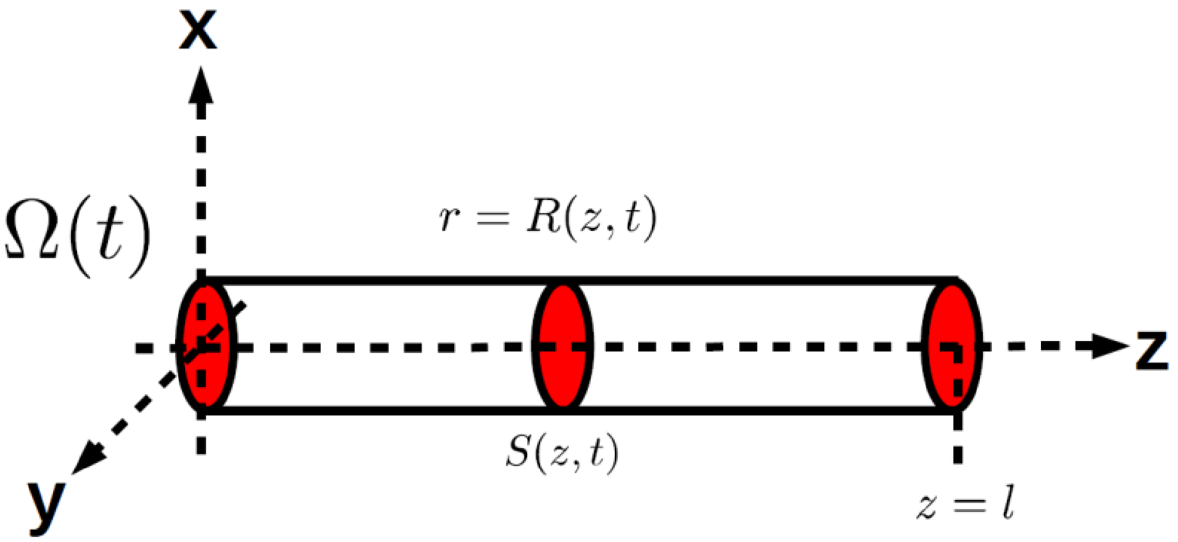
1D variables

- **Cross-sectional area**

$$A(z,t) = \int_{S(z,t)} dS$$

- **Flux:**

$$Q(z,t) = \int_{S(z,t)} u_z\left(\mathbf{x},t\right)\, dS$$



1D Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}) \, , \; z \in (0, l) \, , \; t > 0$$

$$\mathbf{U} = \begin{pmatrix} A \\ Q \end{pmatrix} , \; \mathbf{F}(\mathbf{U}) = \begin{pmatrix} Q \\ \frac{Q^2}{A} + \frac{G_0 A^{3/2}}{3\rho A_0^{1/2}} \end{pmatrix} , \; \mathbf{S}(\mathbf{U}) = \begin{pmatrix} 0 \\ -K_R \frac{Q}{A} \end{pmatrix}$$

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1D Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0,l), \; t > 0$$

$$\mathcal{U} = \begin{pmatrix} A \\ Q \end{pmatrix}$$

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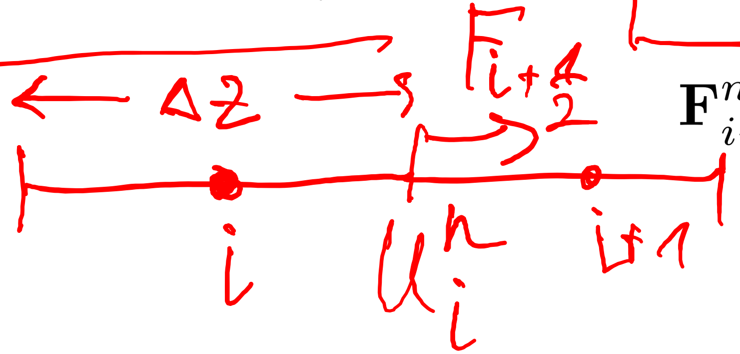
Finite Volume Method:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta z}\left(\mathbf{F}_{i+\frac{1}{2}}^n - \mathbf{F}_{i-\frac{1}{2}}^n\right),$$

$$\boxed{\mathbf{U}_i^n \approx \frac{1}{\Delta z}\int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} \mathbf{U}(z,t_n)\,dz}$$

$$\mathbf{F}_{i+\frac{1}{2}}^n \approx \mathbf{F}(\mathbf{U}(z_{i+\frac{1}{2}}, t_n))$$



1D Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0, l), \; t > 0$$

$$\frac{\partial c}{\partial t} + v \cdot \frac{\partial c}{\partial z} = 0$$

$v > 0$

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1D Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0, l), \; t > 0$$

$$\Leftrightarrow \frac{\partial \mathbf{U}}{\partial t} + \mathbf{H}(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial z} = \mathbf{S}(\mathbf{U}), \quad \underline{\mathbf{H}(\mathbf{U})} := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$$

Quasi-linear form

$$\frac{\partial F}{\cancel{\partial U}} \frac{\cancel{\partial U}}{\partial z} = \frac{\partial F}{\partial z}$$

Jacobian matrix

Hyperbolic Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0, l), \; t > 0$$

$$\Leftrightarrow \frac{\partial \mathbf{U}}{\partial t} + \mathbf{H}(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial z} = \mathbf{S}(\mathbf{U}), \quad \mathbf{H}(\mathbf{U}) := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$$

Mathematical definition of a hyperbolic system

- H has only real eigenvalues
- H is diagonalizable

Hyperbolic Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \ z \in (0,l), \ t > 0$$

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Mathematical definition of a hyperbolic system

- H has only real eigenvalues
- H is diagonalizable

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Transformation: $\mathbf{H} = \mathbf{L}^{-1}\boldsymbol{\Lambda}\mathbf{L}, \quad \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2)$

Hyperbolic Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0, l), \; t > 0$$

$$\Leftrightarrow \mathbf{L}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Lambda L}\frac{\partial \mathbf{U}}{\partial z} = \mathbf{LS}(\mathbf{U}), \quad \mathbf{H}(\mathbf{U}) := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}, \quad \mathbf{H} = \mathbf{L}^{-1}\mathbf{\Lambda L},$$

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Hyperbolic Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U})\,, \; z \in (0,l)\,, \; t > 0$$

$$\Leftrightarrow \mathbf{L}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Lambda}\mathbf{L}\frac{\partial \mathbf{U}}{\partial z} = \mathbf{L}\mathbf{S}(\mathbf{U})\,, \quad \mathbf{H}(\mathbf{U}) := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}, \quad \mathbf{H} = \mathbf{L}^{-1}\mathbf{\Lambda}\mathbf{L},$$

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad U = \begin{pmatrix} A \\ Q \end{pmatrix}$$

Variable transformation: $\dfrac{\partial \mathbf{W}}{\partial \mathbf{U}} := \mathbf{L}$

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Hyperbolic Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \; z \in (0, l), \; t > 0$$

$$\Leftrightarrow \mathbf{L}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Lambda}\mathbf{L}\frac{\partial \mathbf{U}}{\partial z} = \mathbf{L}\mathbf{S}(\mathbf{U}), \quad \mathbf{H}(\mathbf{U}) := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}, \quad \mathbf{H} = \mathbf{L}^{-1}\mathbf{\Lambda}\mathbf{L},$$

Variable transformation: $\dfrac{\partial \mathbf{W}}{\partial \mathbf{U}} := \mathbf{L}$

$$\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial z} = \mathbf{R}(\mathbf{W})$$

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Hyperbolic Equations

Transport system of equations with flux function $\mathbf{F}$ and source term $\mathbf{S}$:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial z}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \;\; z \in (0, l), \;\; t > 0$$

$$\Leftrightarrow \mathbf{L}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Lambda L}\frac{\partial \mathbf{U}}{\partial z} = \mathbf{LS}(\mathbf{U}), \quad \mathbf{H}(\mathbf{U}) := \frac{\partial \mathbf{F}}{\partial \mathbf{U}}, \quad \mathbf{H} = \mathbf{L}^{-1}\mathbf{\Lambda L},$$

Variable transformation: $\dfrac{\partial \mathbf{W}}{\partial \mathbf{U}} := \mathbf{L}$

$$\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{W}}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial z} = \mathbf{R}(\mathbf{W})$$

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda}\frac{\partial \mathbf{W}}{\partial z} = \mathbf{R}(\mathbf{W})$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

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Characteristic Equations

Coupled scalar transport equations:

$$\frac{\partial W_1}{\partial t} + \lambda_1 (W_1, W_2) \frac{\partial W_1}{\partial z} = R_1(W_1, W_2)$$
$$\frac{\partial W_2}{\partial t} + \lambda_2 (W_1, W_2) \frac{\partial W_2}{\partial z} = R_2 (W_1, W_2)$$

$H = \frac{\partial F}{\partial u}$

$\frac{\partial w}{\partial u} : L$

Eigenvalues:

$$\lambda_1 (W_1, W_2) = -\frac{5}{8} W_1 + \frac{3}{8} W_2$$
$$\lambda_2 (W_1, W_2) = -\frac{3}{8} W_1 + \frac{5}{8} W_2$$

Characteristic Equations

Transformation:

$W_1 = W_1(A, Q)$

U

$$W_1 = -\frac{Q}{A} + 4c(A) \qquad W_2 = \frac{Q}{A} + 4c(A)$$

$$\boxed{\begin{array}{c}\text{Physical Variables}\\ Q, A\end{array}} \quad \longrightarrow \text{Transformation} \longrightarrow \quad \boxed{\begin{array}{c}\text{Characteristic Variables}\\ W_1, W_2\end{array}}$$

$H = \frac{\partial F}{\partial U}$

$$A = A_0 \left(\frac{W_1 + W_2}{8c_0} \right)^4 \qquad Q = A \left(\frac{W_2 - W_1}{2} \right)$$

$$\lambda_1 (Q, A) = \frac{Q}{A} - c$$

$$\lambda_2 (Q, A) = \frac{Q}{A} + c$$

$$\boxed{c(A) = \sqrt{\frac{G_0}{2\rho} \left(\frac{A}{A_0} \right)^{\frac{1}{4}}}}, \quad c_0 = \sqrt{\frac{G_0}{2\rho}}$$

$\frac{Q}{A} = \overline{V} < C$

$$\lambda_1 (W_1, W_2) = -\frac{5}{8} W_1 + \frac{3}{8} W_2$$

$$\lambda_2 (W_1, W_2) = -\frac{3}{8} W_1 + \frac{5}{8} W_2$$

Characteristic Equations

Transformation:

$$W_1 = -\frac{Q}{A} + 4c(A) \qquad W_2 = \frac{Q}{A} + 4c(A)$$

Physical Variables

$$Q, A$$

Transformation

Characteristic Variables

$$W_1, W_2$$

$$A = A_0 \left(\frac{W_1 + W_2}{8c_0} \right)^4 \qquad Q = A \left(\frac{W_2 - W_1}{2} \right)$$

$$\boxed{\begin{aligned} \lambda_1 (Q, A) &= \frac{Q}{A} - c < 0 \\ \lambda_2 (Q, A) &= \frac{Q}{A} + c > 0 \end{aligned}}$$

$$c(A) = \sqrt{\frac{G_0}{2\rho}} \left(\frac{A}{A_0} \right)^{\frac{1}{4}}, \quad c_0 = \sqrt{\frac{G_0}{2\rho}}$$

$$\lambda_1 (W_1, W_2) = -\frac{5}{8} W_1 + \frac{3}{8} W_2$$

$$\lambda_2 (W_1, W_2) = -\frac{3}{8} W_1 + \frac{5}{8} W_2$$

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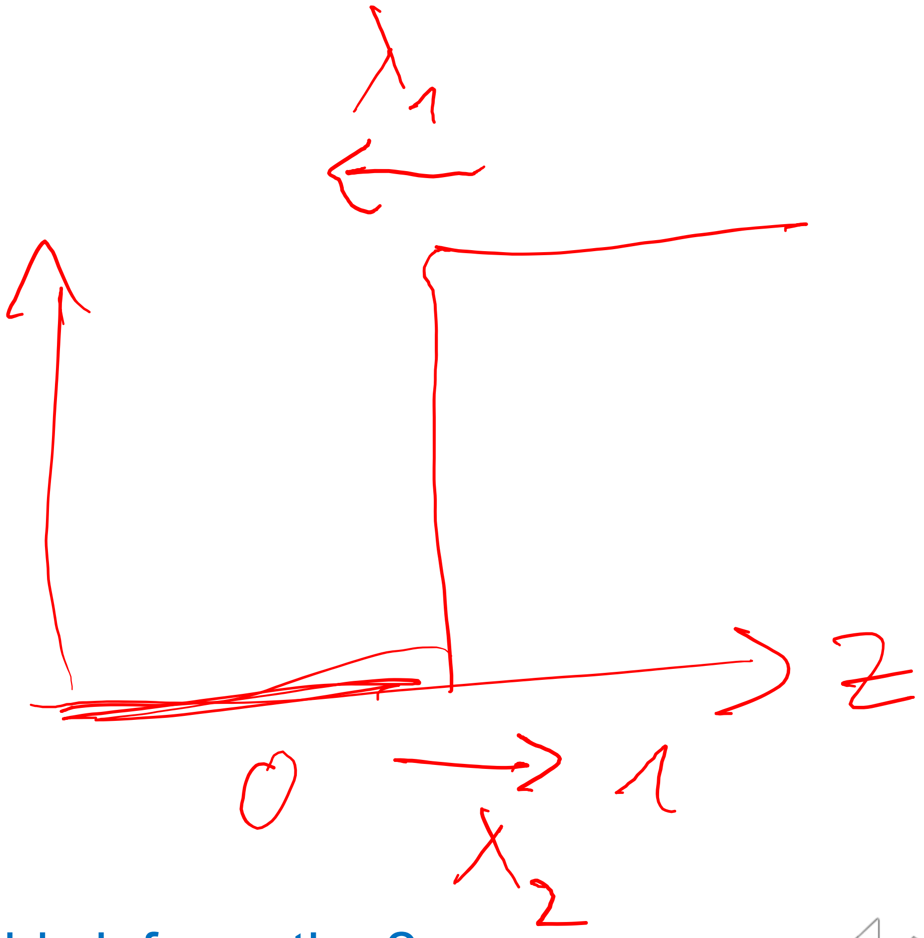
Characteristic Equations

Coupled scalar transport equations:

$$\frac{\partial W_1}{\partial t} + \lambda_1 \left(W_1, W_2\right) \frac{\partial W_1}{\partial z} = R_1(W_1, W_2)$$

$$\frac{\partial W_2}{\partial t} + \lambda_2 \left(W_1, W_2\right) \frac{\partial W_2}{\partial z} = R_2 \left(W_1, W_2\right)$$



Eigenvalues:

$\lambda_1 \left(W_1, W_2\right) < 0$

$\lambda_2 \left(W_1, W_2\right) > 0$

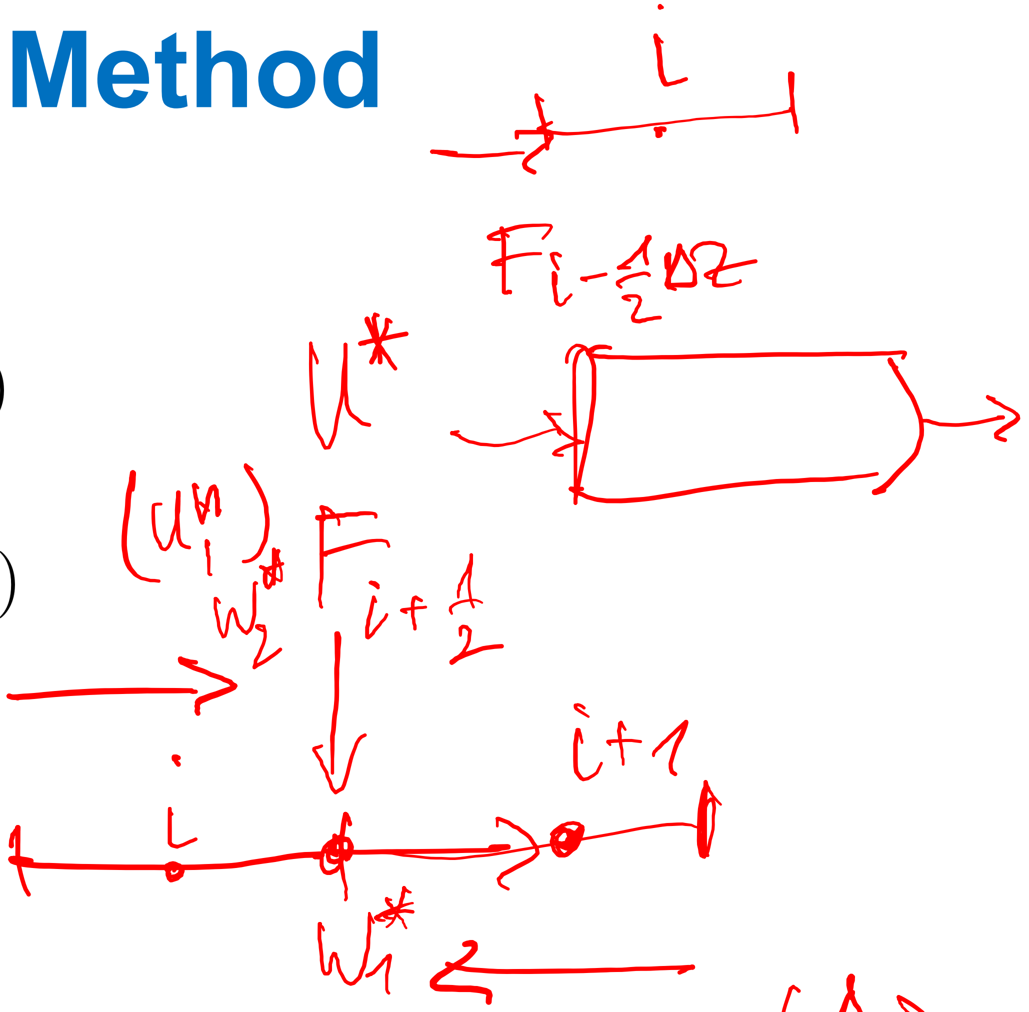
Is it possible to construct a Riemann solver by using this information?

Finite Volume Method

Coupled scalar transport equations:

$$\frac{\partial W_1}{\partial t} + \lambda_1 (W_1, W_2) \frac{\partial W_1}{\partial z} = R_1(W_1, W_2)$$

$$\frac{\partial W_2}{\partial t} + \lambda_2 (W_1, W_2) \frac{\partial W_2}{\partial z} = R_2 (W_1, W_2)$$



Finite Volume Method:

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta z} \left(\underline{\mathbf{F}_{i+\frac{1}{2}}^n} - \underline{\mathbf{F}_{i-\frac{1}{2}}^n} \right),$$

$$\mathbf{F}_{i+\frac{1}{2}}^n := \mathbf{F}\left(\mathbf{U}_{i+\frac{1}{2}}^*(W_1, W_2)\right), \; W_1 = W_1(\mathbf{U}_{i+1}^n), W_2 = W_2(\mathbf{U}_i^n)$$

$$\boxed{A_{i+\frac{1}{2}}^* = A_0 \left(\frac{W_1 + W_2}{8c_0} \right)^4, \quad Q_{i+\frac{1}{2}}^* = A \left(\frac{W_2 - W_1}{2} \right)}$$

$\begin{pmatrix} A \\ Q \end{pmatrix}$

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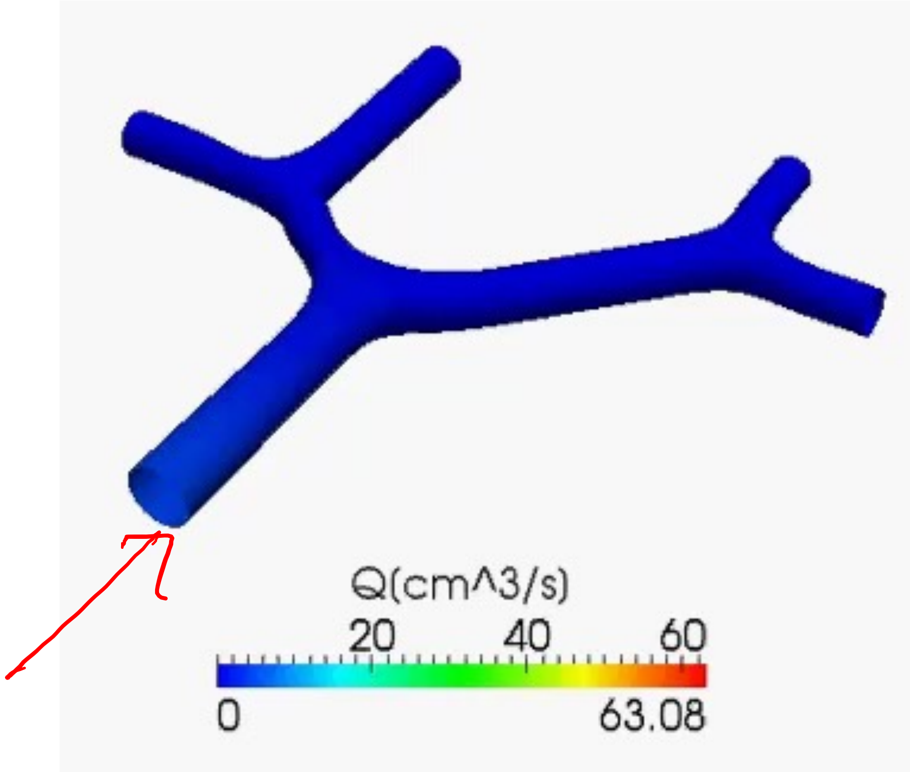
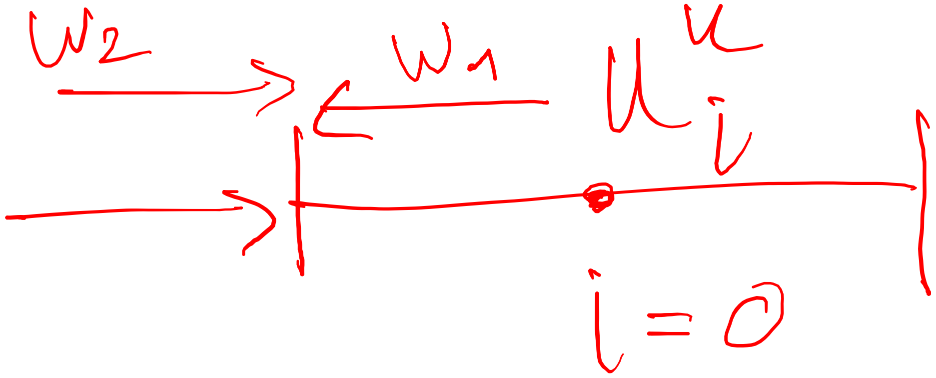
Inflow Boundary





$$w_1 = w_1(u_i^n) = w_1\left(\begin{pmatrix} A \\ Q \end{pmatrix}\right)$$

$$\boxed{Q_{in}(t) = \begin{cases} Q_{\max} \sin\left(\frac{\pi}{T} t\right) \ \frac{cm^3}{s} \\ 0 \ \frac{cm^3}{s} \end{cases}}$$ für $0.0 \leq t \leq T$, für $T < t \leq T_d$.

(1 heartbeat / systole + diastole)

5-6l blood per minute

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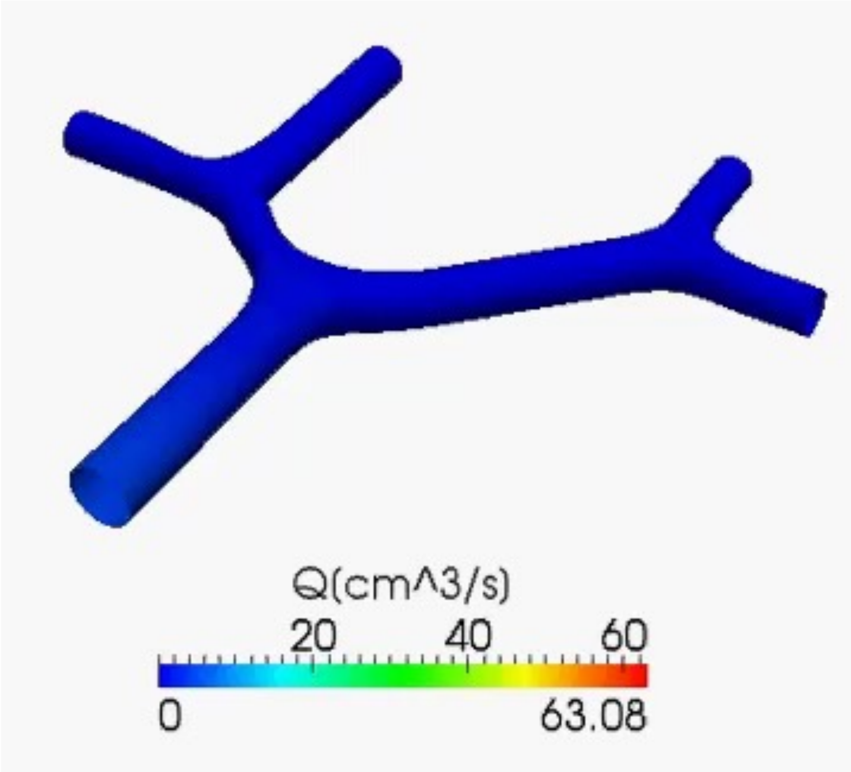
Inflow Boundary



$W_2 \rightarrow$

$\leftarrow W_1(U_K^n)$

$$Q_{in} = A_0 \left(\frac{W_1 + W_2}{8c_0} \right)^4 \left(\frac{W_2 - W_1}{2} \right)$$

$\rightarrow$ Newton's method

$$Q_{in}(t) = \begin{cases} Q_{\max} \sin\left(\frac{\pi}{T} t\right) \ \frac{cm^3}{s} & \text{für } 0.0 \leq t \leq T, \\ 0 \ \frac{cm^3}{s} & \text{für } T < t \leq T_d. \end{cases}$$

(1 heartbeat / systole + diastole)

5-6l blood per minute

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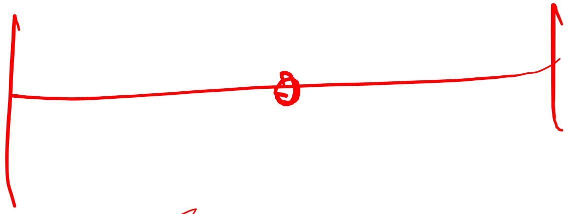
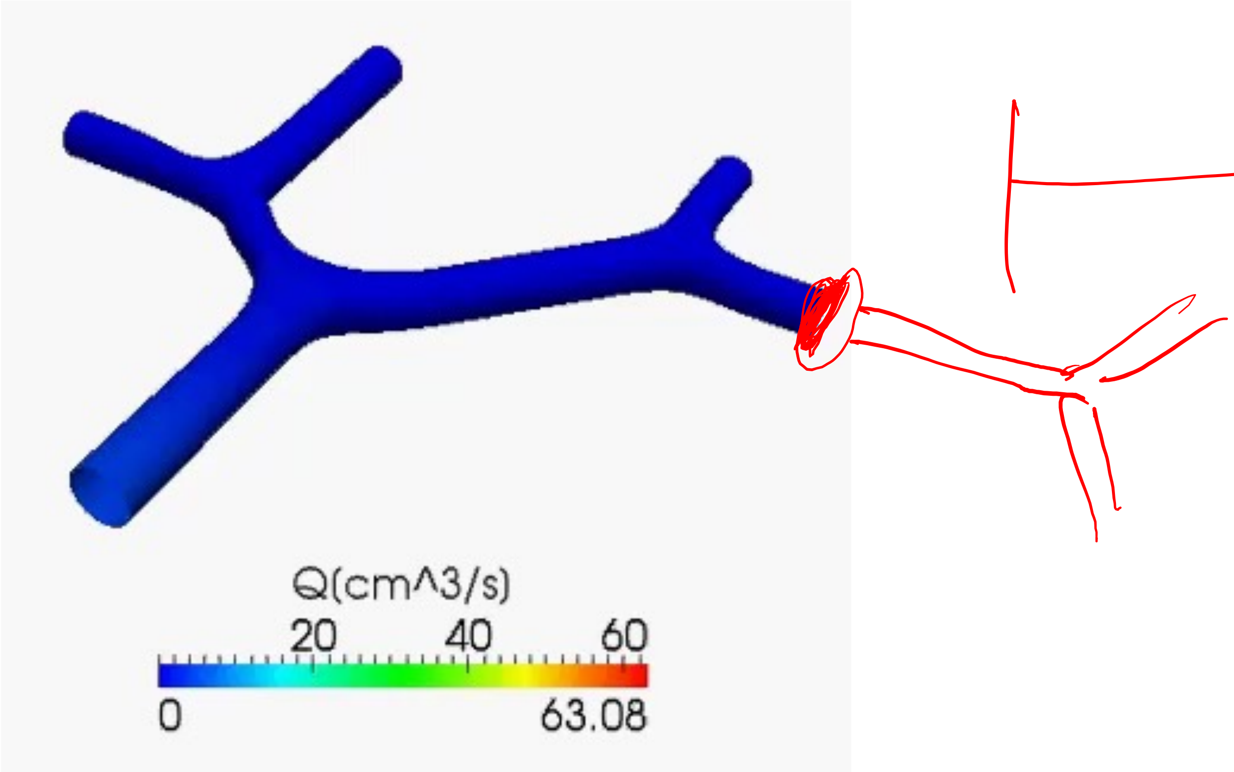
Outflow Boundary

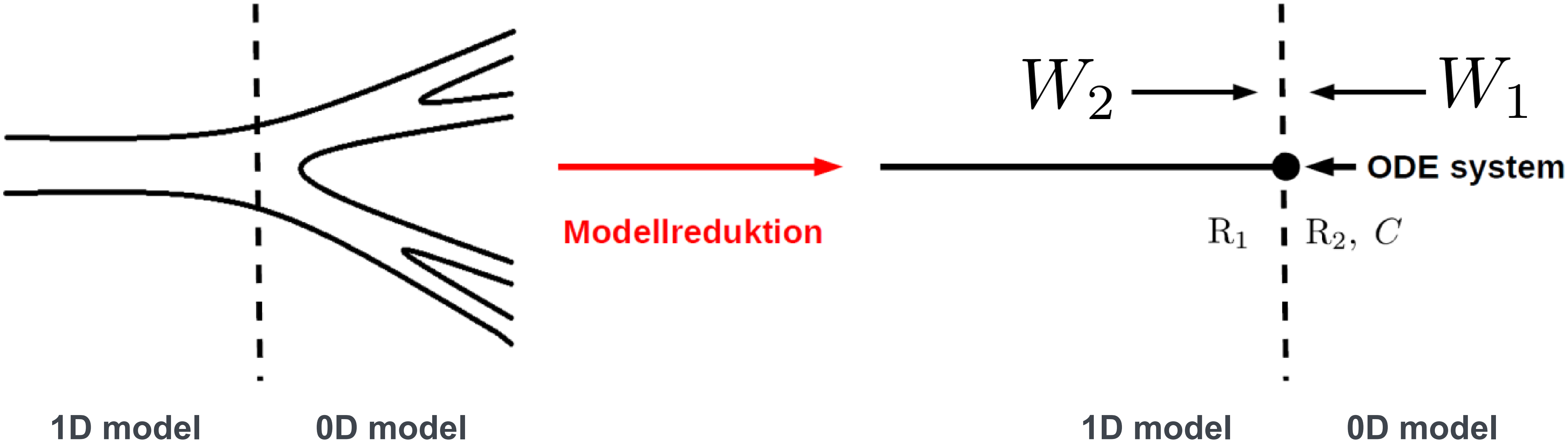
$$w_2 \rightarrow \quad \leftarrow w_1 = c_0 = \sqrt{\frac{G_0}{2\rho}}$$





$$w_2 = w_1$$

Outflow Boundary



Lumped model (ODE system, 0D model) for a single point

Summary

- Transformation to characteristic variables
- Characteristic variables yield information of the system
- Use characteristics for incorporating boundary conditions as well
- Riemann solver
- What we know so far allows to model single arterial segments

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Thank you!

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