Reinforcement Learning
Lecture 8: nstep bootstrapping and eligibility traces [1]

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June 22, 2023

[1] Many slides adapted from R. Sutton's course, D. Silver's course as well as previous RL courses given at U. of Stuttgart by J. Mainprice, D. Hennes, M. Toussaint, H. Ngo, and V. Ngo.

Outline

n-step bootstrapping

- Unifying Monte Carlo and TD
- n-step TD
- n-step Sarsa

Unifying Monte Carlo and TD

Monte Carlo Prediction (Estimation of v_π)

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

  $V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

  $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

  Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

  $G \leftarrow 0$

  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

    $G \leftarrow G + R_{t+1}$

    Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

      Append G to $Returns(S_t)$

      $V(S_t) \leftarrow$ average$(Returns(S_t))$

Backup diagram

- Entire episode included
- Only single choice considered at each state
- Thus, there will be an explore/exploit dilemma
- Value is estimated by mean return

Policy evaluation (prediction)

- We have some policy π which tells the agent which action a to choose in state s
- Find the value function $v_\pi(s)$ of this policy, i.e. **evaluate** the policy π

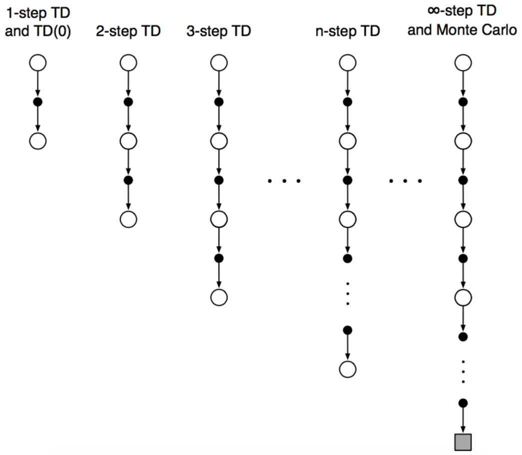
$$V(S_t) = V(S_t) + \alpha\Big[\underbrace{G_t}_{\text{MC target}} - V(S_t)\Big]$$

- Simplest temporal difference update TD(0):

$$V(S_t) = V(S_t) + \alpha\Big[\underbrace{\underbrace{R_{t+1} + \gamma V(S_{t+1})}_{\text{TD target}} - V(S_t)}_{\text{TD error}}\Big]$$

- TD error is error in the estimate made at a particular time step
- Reinforcement:
  - more reward than expected: $R_{t+1} + \gamma V(S_{t+1}) > V(S_t) \Rightarrow V(S_t) \uparrow$
  - less reward than expected: $R_{t+1} + \gamma V(S_{t+1}) < V(S_t) \Rightarrow V(S_t) \downarrow$

n-step TD prediction



n-step returns

- Monte Carlo:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots + \gamma^{T-t-1} R_T$$

- TD:

$$G_{t:t+1} = R_{t+1} + \gamma V_t(S_{t+1})$$

- 2-step return:

$$G_{t:t+2} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V_{t+1}(S_{t+2})$$

- n-step return:

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n V_{t+n-1}(S_{t+n})$$

Error-reduction property

- *Error reduction property* of n-step returns

$$\underbrace{\max_s \left| \mathbb{E}_\pi[G_{t:t+n} \mid S_t = s] - v_\pi(s) \right|}_{\text{Maximum error using } n\text{-step return}} \leq \underbrace{\gamma^n \max_s \left| V_{t+n-1}(s) - v_\pi(s) \right|}_{\text{Maximum error using } V}$$

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n V_{t+n-1}(S_{t+n})$$

- Using above, we can show that n-step methods converge
- Generalization of 1-step:

$$\max_s \left| \mathbb{E}_\pi[R_{t+1} + \gamma V(S_{t+1}) \mid S_t = s] - v_\pi(s) \right| \leq \gamma \max_s \left| V(s) - v_\pi(s) \right|$$

n-step TD

- n-step return

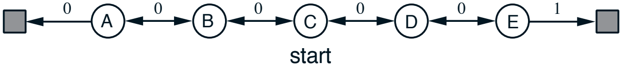
$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n V_{t+n-1}(S_{t+n})$$

- *Not available* until time $t+n$
- Natural algorithm is to wait until time $t+n$
- n-**step TD** update:

$$V_{\underbrace{t+n}_{\text{next step}}}(S_t) = V_{\underbrace{t+n-1}_{\text{previous step}}}(S_t) + \alpha\Big[G_{t:t+n} - V_{\underbrace{t+n-1}_{\text{previous step}}}(S_t)\Big]$$

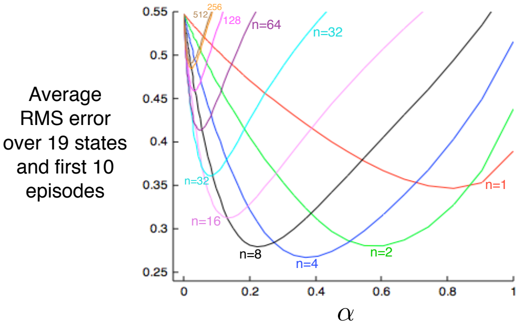
Initialize $V(s)$ arbitrarily, for all $s \in \mathcal{S}$
for all episode **do**
  Initialize and store $S_0 \neq$ terminal
  $T \leftarrow \infty$
  repeat for $t = 0, 1, 2, \ldots$
    if $t < T$ **then**
      Take an action according to $\pi(\cdot \mid S_t)$
      Observe and store next reward R_{t+1} and state S_{t+1}
      if S_{t+1} is terminal **then** $T \leftarrow t + 1$
    end if
    $\tau \leftarrow t - n + 1$ ▷ τ is the time whose state's estimate is updated
    if $\tau \geq 0$ **then**
      $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
      if $\tau + n < T$ **then** $G \leftarrow G + \gamma^n V(S_{\tau+n})$
      $V(S_\tau) \leftarrow V(S_\tau) + \alpha[G - V(S_\tau)]$
    end if
  until $\tau = T - 1$
end for

Random walk example



- Suppose the first episode progressed directly from C to the right, through D and E
- How does 2-step TD work here?
- How about 3-step TD?

19-state random walk



- An intermediate α is best
- An intermediate n is best
- Is there an optimal n? For every task?
- For larger n, smaller α seems best

n-step Sarsa

- Action–value of n-step return:

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n Q_{t+n-1}(S_{t+n}, A_{t+n})$$

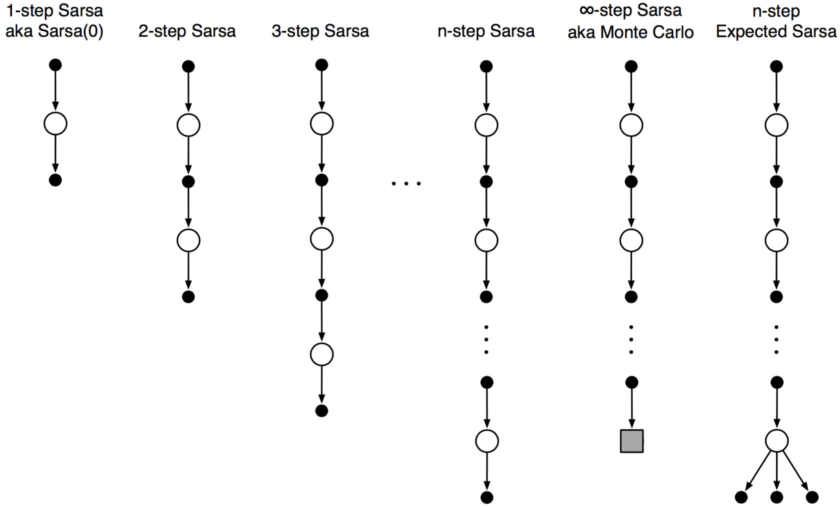
- n-step Sarsa:

$$Q_{t+n}(S_t, A_t) = Q_{t+n-1}(S_t, A_t) + \alpha \Big[G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \Big]$$

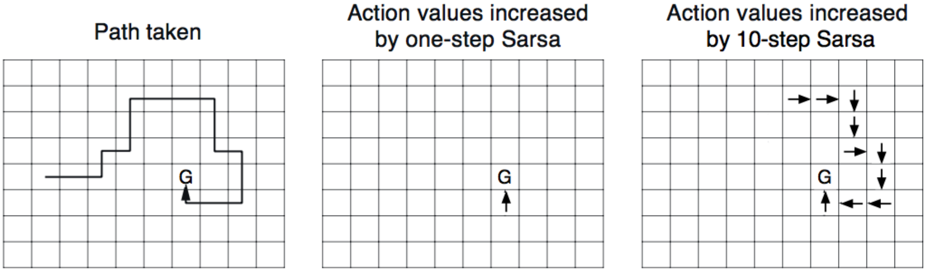
- n-step Expected Sarsa:
    - same update
    - *slightly* different n-step return

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n \sum_a \pi(a \mid S_{t+n}) Q_{t+n-1}(S_{t+n}, a)$$

n-step Sarsa

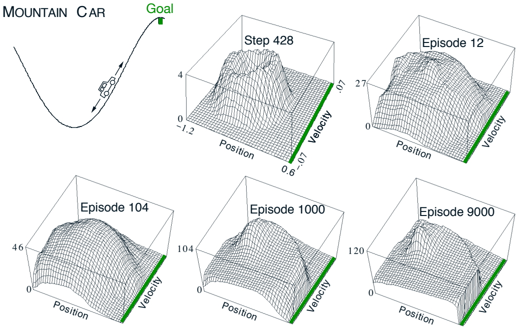

1-step Sarsa
aka Sarsa(0)
2-step Sarsa
3-step Sarsa
n-step Sarsa
∞-step Sarsa
aka Monte Carlo
n-step
Expected Sarsa

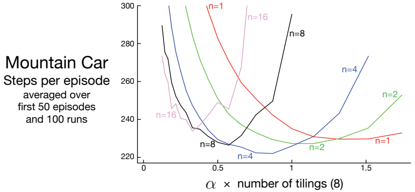
n-step Sarsa example



Initialize action-value function parameterization $\hat{q}$
for all episode **do**
  Initialize and store $S_0 \neq$ terminal
  Select and store an action $A_0 \sim \pi(\cdot \mid S_0)$ or ϵ-greedy wrt $\hat{q}$
  $T \leftarrow \infty$
  repeat for $t = 0, 1, 2, \ldots$
    if $t < T$ **then**
      Take an action A_t
      Observe and store next reward R_{t+1} and state S_{t+1}
      if S_{t+1} is terminal **then** $T \leftarrow t + 1$
      else Select and store an action $A_{t+1} \sim \pi(\cdot \mid S_{t+1})$ or ϵ-greedy wrt $\hat{q}$
    end if
    $\tau \leftarrow t - n + 1$ ▷ τ is the time whose state's estimate is updated
    if $\tau \geq 0$ **then**
      $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
      if $\tau + n < T$ **then** $G \leftarrow G + \gamma^n \hat{q}(S_{\tau+n}, A_{\tau+n})$
      $\boldsymbol{w} \leftarrow \boldsymbol{w} + \alpha[G - \hat{q}(S_\tau, A_\tau, \boldsymbol{w})] \boldsymbol{\nabla} \hat{q}(S_\tau, A_\tau, \boldsymbol{w})$
    end if
  until $\tau = T - 1$
end for

n-step Sarsa with function approximation





n-step off–policy learning

- Recall the importance sampling ratio:

$$\rho_{t:h} = \prod_{k=t}^{\min(h,T-1)} \frac{\pi(A_k \mid S_k)}{\mu(A_k \mid S_k)}$$

- Off–policy methods weight updates by this ratio
- Off–policy n-step TD:

$$V_{t+n}(S_t) = V_{t+n-1}(S_t) + \alpha\rho_{t:t+n-1}\Big[G_{t:t+n} - V_{t+n-1}(S_t)\Big]$$

n-step off–policy learning (part 2)

- Recall the importance sampling ratio:

$$\rho_{t:h} = \prod_{k=t}^{\min(h,T-1)} \frac{\pi(A_k \mid S_k)}{\mu(A_k \mid S_k)}$$

- Off–policy n-step Sarsa:

$$Q_{t+n}(S_t, A_t) = Q_{t+n-1}(S_t, A_t) + \alpha \rho_{t+1:t+n} \Big[G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \Big]$$

- Off–policy n-step Expected Sarsa:

$$Q_{t+n}(S_t, A_t) = Q_{t+n-1}(S_t, A_t) + \alpha \rho_{t+1:t+n-1} \Big[G_{t:t+n} - Q_{t+n-1}(S_t, A_t) \Big] \text{ , with}$$

$$G_{t:t+n} = R_{t+1} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n \sum_a \pi(a \mid s) Q_{t+n-1}(s, a)$$

Summary

- n-step bootstrapping generalizes TD and MC learning methods
  - $n = 1$ is TD
  - $n = \infty$ is MC
  - intermediate n is often better than either extreme
  - applies to both continuing and episodic domains
- Additional cost in computation
  - we need to remember the last n states
  - learning is delayed by n steps
  - per-step computation is small (like TD)
- Everything generalizes nicely:
  - error-reduction theory
  - Sarsa, off–policy by importance sampling, Expected Sarsa Backup

Eligibility Traces

Scaling-up reinforcement learning

- Sparse rewards
  - e.g. gridworld
- Large *state* spaces:
  - Go: $\log_{10}|S| = \log_{10}(3^{19\times 19}) \approx 170 > 82$
  - Camera images , e.g. $\log_{10}|S| = \log_{10}((256^3)^{1280\times 720}) \gg 82$ (3 color channels, 8 bits each)
  - Continuous spaces: e.g. inverted pendulum, mobile robot, etc.

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nb. of atoms in the universe: $N = 10^{82}$ $(\log_{10} N = 82)$

Recall: n-step return

- n-step returns for $n = 1, 2, \ldots \infty$:

$$
\begin{array}{lll}
n = 1 & TD & R_{t+1} + \gamma V(S_{t+1}) \\
n = 2 & & R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2}) \\
n = 3 & & R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3}) \\
\vdots & & \vdots \\
n = \infty & MC & R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots + \gamma^{T-1} R_T
\end{array}
$$

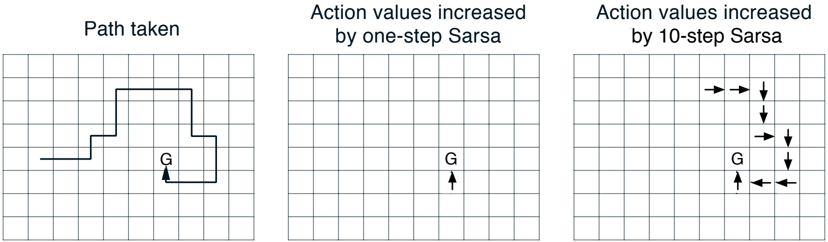
- n-step return:

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- n-step temporal difference update:

$$V(S_t) \leftarrow V(S_t) + \alpha \left[G_{t:t+n} - V(S_t) \right]$$

Example: n-step Sarsa



– Reward 0 except for G

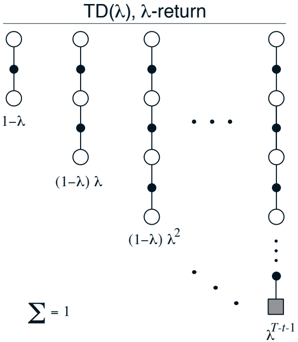
– Which action values would be updated upon reaching the goal?

– How to choose n?

λ-return

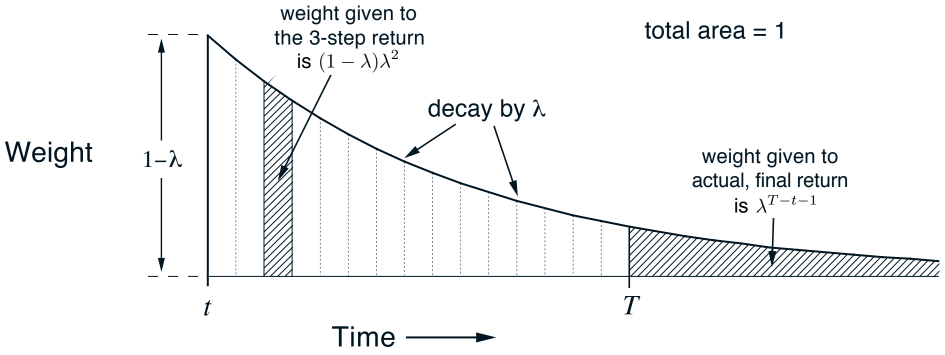
- The λ-return G_t^λ combines all n-step returns (weighted averaging):

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n}$$



λ-return weighting function

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}G_{t:t+n} = (1-\lambda)\sum_{n=1}^{T-t-1}\lambda^{n-1}G_{t:t+n} + \lambda^{T-t-1}G_t$$



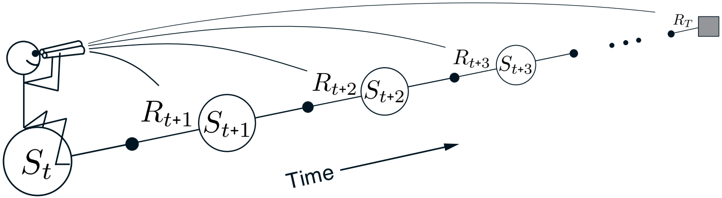
λ-return weighting function (part 2)

- General weighting function:

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{T-t-1} \lambda^{n-1} G_{t:t+n} + \lambda^{T-t-1} G_t$$

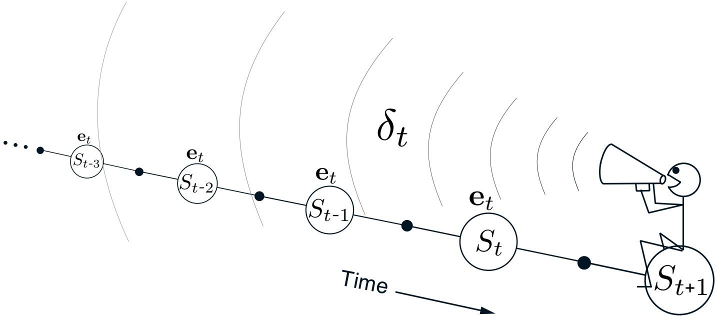
- For $\lambda = 1$: $G_t^\lambda = G_t$ (Monte Carlo)
- For $\lambda = 0$: $G_t^\lambda = G_{t:t+1}$ (1-step TD)

Forward view



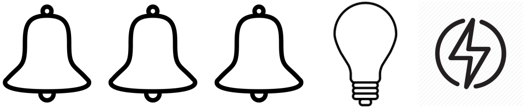
- Update values by looking forward to future rewards and states
- Update values towards λ-return
- Can only be computed for *terminated* sequences

Backward view



- Forward view provides theory
- Backward view provides a mechanism how to perform updates
- Update every step; works for *incomplete* sequences

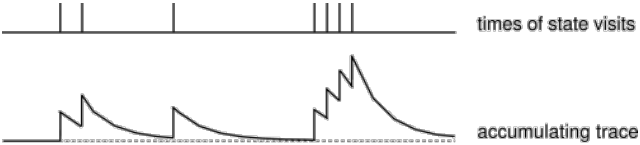
Eligibility traces



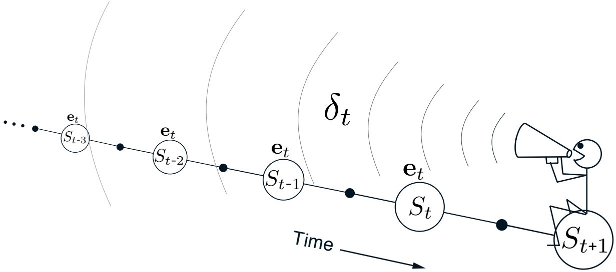
Credit assignment problem:

- **Frequency:** assign credit to most frequent states
- **Recency:** assign credit to recent states

$$\forall s : e(s) \leftarrow \gamma\lambda e(s)$$
$$e(S_t) \leftarrow e(S_t) + 1$$



Backward view



- Keep an eligibility trace for *every* state s
- Update value $V(s)$ for *every* state s:

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$\forall s : V(s) \leftarrow V(s) + \alpha \delta_t e_t(s)$$

TD(λ) and TD(0)

- When $\lambda = 0$:

$$e(s) = \begin{cases} 1 & \text{for} \quad s = S_t \\ 0 & \text{else} \end{cases}$$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$\forall s : V(s) \leftarrow V(s) + \alpha \delta_t e_t(s)$$

- Same as TD(0):

$$V(S_t) \leftarrow V(S_t) + \alpha \left[R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \right]$$

- What if $\lambda = 1$? Monte-Carlo

TD(λ) with function approximation

- Eligibility trace vector $\boldsymbol{e}$ keeps track which components have contributed to recent state evaluations
- Indicate the *eligibility* of each component for undergoing learning

$$\delta = R_{t+1} + \gamma \hat{v}(S_{t+1}, \boldsymbol{w}) - \hat{v}(S_t, \boldsymbol{w})$$
$$\boldsymbol{e} \leftarrow \gamma \lambda \boldsymbol{e} + \nabla \hat{v}(S_t, \boldsymbol{w})$$
$$\boldsymbol{w} \leftarrow \boldsymbol{w} + \alpha \delta \boldsymbol{e}$$

- Update weight vector proportional to scalar TD error and eligibility trace vector

Summary

- We saw a way to **unify TD and MC**
- n-step returns interpolate between TD(0) and MC ($n = \infty$)
- We can get a combination of different n-step returns by using λ returns
- **Eligibility traces** are a computationally efficient way to implement them
- In the high-dimensional function approximation case this becomes tricky to use and is an active area of research