SURFACES ASSOCIATED WITH FIRST-ORDER ODES *

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ABSTRACT

A link between first-order ordinary differential equations (ODEs) and 2-dimensional Riemannian manifolds is explored. Given a first-order ODE, an associated Riemannian metric on the variable space is defined, and some properties of the resulting surface are studied, including a connection between Jacobi fields and Lie point symmetries. In particular, it is proven that if the associated surface is flat, then the ODE can be integrated by quadratures.

Next, deformations of the associated surfaces are considered. A relation between some Jacobi fields on the deformed surface and integrability of the ODE is established, showing that there is a class of vector fields, beyond Lie point symmetries, which are useful for solving first-order ODEs. As a result, it is concluded that the deformation into a constant curvature surface leads to the integrability of the given ODE.

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1 Introduction

First-order ordinary differential equations (ODEs) are foundational in many scientific fields, enabling the modeling and understanding of numerous phenomena including concentration and dilution problems, population models and even astrophysical research. Moreover, as the simplest form of ODEs, they provide an accessible gateway to more advanced mathematical investigations in this area.

The works of S. Lie and E. Cartan [1, 2, 3] provided a geometric perspective on differential equations. Their approach, which turned out to be particularly useful in understanding and solving nonlinear ODEs, gave rise to the concepts of symmetry and invariance. This interplay between differential equations and geometric objects has been extensively studied since then, and extends to recent decades [4, 5, 6, 7, 8, 9, 10, 11, 12].

Consider a first-order ODE in the form

$$\frac{du}{dx} = \phi(x, u), \tag{1}$$

where ϕ is a smooth function defined on an open subset $U \subseteq \mathbb{R}^2$. It is well known [13, 14] that the associated vector field

$$A = \partial_x + \phi(x, u)\partial_u, \tag{2}$$

encodes all the relevant information about the equation. The first integrals of A, in the sense of smooth functions $F \in \mathcal{C}^\infty(U)$ satisfying $A(F) = 0$, provide implicit descriptions of the solutions of (1).

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Usually, the standard volume form $\mathbf{\Omega} = dx \wedge du$ on $\mathbb{R}^2$ is considered, and then the 1-form ω, defined as

$$\omega := i_A \mathbf{\Omega} = -\phi(x,u)dx + du, \tag{3}$$

and where i_A denotes interior product, provides an alternative way of describing the ODE. Finding a function F satisfying $dF = \mu\omega$ – with μ a smooth function called integrating factor of (3) – is equivalent to solve equation (1). Indeed, such a function F would be a first integral of A, since $A(F) = dF(A) = \mu\omega(A) = 0$.

The computation of F, once an integrating factor is identified, is a straightforward process usually known as quadrature, so the finding of an integrating factor constitutes a key step in the integration of first-order ODEs.

But we could consider an alternative perspective, from which (x,u) are not coordinates of the Euclidean plane $\mathbb{R}^2$ with the standard metric – hence the choice of the standard volume form – but coordinates of a different 2-dimensional Riemannian manifold (or surface, for short), such that the corresponding volume form is $\mu\mathbf{\Omega}$. Determining which surface we are working on, therefore, implies solving equation (1).

Regarding the search for a suitable Riemannian metric, observe that the volume form $\mu\mathbf{\Omega}$ satisfies

$$\begin{aligned}\mathcal{L}_A \mu\Omega &= d(i_A(\mu\mathbf{\Omega})) + i_A(d(\mu\mathbf{\Omega}))\\ &= d(\mu i_A \mathbf{\Omega})\\ &= d(dF) = 0,\end{aligned}$$

so the flow of A conserve volumes. This fact suggests that the flow of A should play a role in the geometry of the surface. Moreover, note that the smooth function F allows us to define the change of coordinates $\varphi : U \to V \subseteq \mathbb{R}^2$, given by

$$\varphi(x,u) = (x,c) = (x, F(x,u)).$$

By pulling back the standard metric on V to U, we obtain a Riemannian metric on U such that the corresponding volume form is $\mu\mathbf{\Omega}$:

$$\begin{aligned}\varphi^*(dx \wedge dc) &= dx \wedge dF\\ &= dx \wedge \mu\omega\\ &= \mu\mathbf{\Omega}.\end{aligned}$$

Interestingly, the integral curves of A are geodesics with respect to this metric. This is because the change of coordinates φ transforms the integral curves of A into straight lines [13, 14, 15], which are geodesics with respect to the standard metric of $\mathbb{R}^2$.

This perspective led us to explore methods for associating a surface with a given first-order ODE (1), in such a way that the solutions of the equation are related to the geodesic curves of the surface. Our approach bears similarities to the one presented in [11, 12], but it stems from a different motivation.

In this paper, after introducing the notation and some basic results (Section 2), we present our approach in Section 3. We then explore the relationship between geometric notions such as geodesics, Jacobi fields, and curvature of the surface, and classical concepts in the theory of ODEs like Lie point symmetries and integrating factors. An important role of the curvature should be expected, due to its pivotal function in both mathematics and physics. And this is indeed the case: we show that within our framework, the curvature is intimately related to the integrability of the ODE. In particular, we will see that if the associated surface is flat (zero Gaussian curvature), then the ODE can be integrated by quadratures.

Posteriorly, in Section 4, we consider deformations of the associated surface, and study the implications of these deformations on the integrability of the original ODE. As we will see, the knowledge of a Jacobi field on any of the deformed surfaces (including the original surface) enables the integration of the equation. Thus, we are expanding the class of vector fields that can be used for integrating first-order ODEs beyond Lie point symmetries. Remarkably, we will show that deforming the surface into one with constant curvature leads to the integration of the ODE. In this context, the classical method of searching for an integrating factor will be interpreted as a deformation of the associated surface into a flat one.

2 Preliminaries

2.1 Differential equations and symmetries

During this work we will denote by U an open subset of $\mathbb{R}^2$, with coordinates (x, u). We will focus on real-valued smooth functions $f : I \to \mathbb{R}$, where I is an open interval containing 0, and we will denote by γ_f the curve on U given by:

$$\gamma_f : \begin{array}{ccc} I & \to & U \\ t & \mapsto & (t, f(t)) \end{array} .$$

If f is a solution to the equation (1), the curve γ_f is an integral curve of the associated vector field A defined in (2), i.e., $\dot{\gamma}_f(t) = A_{\gamma_f(t)}$. Conversely, any integral curve of A can be reparametrized to the form γ_f, for a certain solution f of (1) [13, 16, 17].

When looking for solutions to differential equations, Lie point symmetries constitute a key concept, and they play a fundamental role. Loosely speaking, they are local groups of transformations, which map every solution of the ODE to another solution. Within our setup, we will be focused on their infinitesimal generators, also referred to as Lie point symmetries by an abuse of language. It is a well-known fact that if $V = \xi(x, u)\partial_x + \eta(x, u)\partial_u$ is a Lie point symmetry, then $\mu = (\eta - \xi\phi)^{-1}$ is an integrating factor for (3). Therefore, the knowledge of a Lie point symmetry enables the integration of the given ODE by means of a quadrature [16, 17, 18].

Lie point symmetries of first-order ODEs can be characterized by the following property [16, 17]:

Proposition 2.1. A vector field V defined on U is a Lie point symmetry of (1) if $[V, A] = \rho A$, for a certain smooth function ρ.

Remark 2.2. To exclude trivial symmetry groups, mapping each solution of the ODE to itself, we will assume that the Lie point symmetries discussed throughout the text are vector fields pointwise linearly independent of A.

Consider now the following definition [19, 20]:

Definition 2.3. Given a vector field X pointwise linearly independent of A, a smooth function δ defined on U will be called a symmetrizing factor for X with respect to equation (1) if δX is a Lie point symmetry of (1).

The following result, which is a restatement of Lemma 2.1 in [20], shows that symmetrizing factors always exist, and that they are in close relationship with integrating factors:

Lemma 2.4. Let X be a vector field such that $i_X\omega = 1$, then a non-vanishing smooth function δ is a symmetrizing factor for X with respect to (1) if and only if $1/\delta$ is an integrating factor for ω.

If we consider, for instance, the vector field $X = \partial_u$, it turns out that it can be *converted* into a Lie point symmetry $\delta\partial_u$, by multiplication with a scalar function δ. Moreover, $1/\delta$ is an integrating factor for ω. Therefore, the knowledge of a symmetrizing factor for ∂_u is equivalent to the integration by quadratures of (1).

On the other hand, observe that the vector fields $\delta\partial_u$ and A commute, i.e., $[\delta\partial_u, A] = 0$. There is a well-known result (see, for instance, [15, Theorem 13.10]) concerning commuting vector fields that we state here for the sake of completeness:

Theorem 2.5. Two pointwise linearly independent vector fields Y_1, Y_2 defined on U satisfy $[Y_1, Y_2] = 0$ if and only if for every $p \in U$ there exists a local coordinate change

$$\varphi : \begin{array}{ccc} W & \to & V \\ (s, t) & \mapsto & (x, u) \end{array} ,$$

with $(0, 0) \in W \subseteq \mathbb{R}^2$ and $p \in V \subseteq U$ open sets, in such a way that $\varphi(0, 0) = p$ and

$$\varphi_* \left(\frac{\partial}{\partial s} \right) = Y_1, \quad \varphi_* \left(\frac{\partial}{\partial t} \right) = Y_2.$$

2.2 Riemannian geometry

In this subsection we recall some fundamental facts on Riemannian geometry. In this work the term *surface* will be used to indicate a pair $(\mathcal{S}, g)$ consisting of a 2-dimensional manifold $\mathcal{S}$ equipped with a Riemannian metric g, i.e., a two-times covariant symmetric tensor field, positive definite, and hence non-degenerate. We will refer to the surface simply by $\mathcal{S}$, whenever the context is clear.

Given an orthonormal frame $\{e_1, e_2\}$ in $\mathcal{S}$ with dual coframe $\{\omega_1, \omega_2\}$, the corresponding Riemannian metric can be written as [21]

$$g = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2. \tag{4}$$

It is a well-known fact that every surface $\mathcal{S}$ is endowed with a uniquely determined torsionless metric connection ∇, called the Levi-Civita connection. In the provided frame, this connection is described by the matrix of 1-forms

$$\Theta = \begin{pmatrix} 0 & -T_{12}^1\omega^1 - T_{12}^2\omega^2 \\ T_{12}^1\omega^1 + T_{12}^2\omega^2 & 0 \end{pmatrix}, \tag{5}$$

where T_{ij}^k are the structure coefficients of the coframe, i.e., smooth functions such that $d\omega^k = T_{ij}^k\omega^i \wedge \omega^j$. The details can be found, for instance, in [22, 23, 24].

Consider now two vector fields described by their components in the frame $\{e_1, e_2\}$:

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where x_i, y_i are smooth functions defined on $\mathcal{S}$. The covariant derivative $\nabla_X Y$ can be expressed in this frame as

$$\nabla_X Y = \begin{pmatrix} X(y_1) \\ X(y_2) \end{pmatrix} + i_X\Theta \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Here, $i_X\Theta$ is a matrix whose entries represent the interior product of X with each entry of Θ, and the dot denotes matrix multiplication. Therefore

$$\begin{aligned} \nabla_X Y &= \begin{pmatrix} X(y_1) \\ X(y_2) \end{pmatrix} + \begin{pmatrix} 0 & T_{12}^1 x_1 + T_{12}^2 x_2 \\ -T_{12}^1 x_1 - T_{12}^2 x_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} X(y_1) + T_{12}^1 x_1 y_2 + T_{12}^2 x_2 y_2 \\ X(y_2) - T_{12}^1 x_1 y_1 - T_{12}^2 x_2 y_1 \end{pmatrix}. \end{aligned} \tag{6}$$

Also, given a curve $\alpha : I \to \mathcal{S}$, where I is an open interval in $\mathbb{R}$, and a vector field Y along α, the covariant derivative $\nabla_{\dot\alpha(t)} Y(t)$ for $t \in I$ is computed as

$$\nabla_{\dot\alpha(t)} Y(t) = \begin{pmatrix} y_1'(t) + T_{12}^1 x_1(t) y_2(t) + T_{12}^2 x_2(t) y_2(t) \\ y_2'(t) - T_{12}^1 x_1(t) y_1(t) - T_{12}^2 x_2(t) y_1(t) \end{pmatrix}, \tag{7}$$

where $x_1(t), x_2(t)$ are the components of the tangent vector $\dot\alpha(t)$ in the frame $\{e_1, e_2\}$, and $y_1(t), y_2(t)$ are the components of $Y(t)$ in the same frame [21].

A fundamental feature of a surface is its Gaussian curvature $\mathcal{K}$. The connection form (5) allows us to obtain it by means of Gauss equation [22, 24]:

$$\mathcal{K} = d\Theta_2^1(e_1, e_2). \tag{8}$$

In curved spaces, the notion of a geodesic is used to extend the idea of a straight line in flat spaces. For the sake of completeness we recall the following definitions

Definition 2.6. A curve $\gamma : I \subseteq \mathbb{R} \to \mathcal{S}$ is called a pregeodesic if

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = \sigma(t)\dot\gamma(t)$$

for every $t \in I$, and being σ a smooth function. In case $\sigma(t) = 0$ for every t the curve is called a geodesic.

Definition 2.7. A vector field X will be called a pregeodesic vector field if

$$\nabla_X X = \sigma X,$$

being σ a smooth function. In case $\sigma = 0$ the vector field is called a geodesic vector field.

It turns out that a vector field is a geodesic (pregeodesic) vector field if and only if its integral curves are geodesics (pregeodesics) of the surface [21, 25, 26]. On the other hand, recall that a pregeodesic can be reparametrized to a geodesic [27], so a pregeodesic is a curve whose image is the same as a geodesic.

An intimately related concept is that of Jacobi field, which plays a crucial role in understanding how geodesics evolve, capturing the curvature of the surface and providing a local view of geodesic variations. Essentially, they offer insights into nearby geodesics' behavior.

Definition 2.8. A vector field J along a geodesic γ is called a Jacobi field if it satisfies the Jacobi equation:

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J + \mathcal{K}\left(J - g(J,\dot{\gamma})\dot{\gamma}\right) = 0, \tag{9}$$

for every point p in the image of γ.

Remark 2.9. It is a well-known result [21] that J is a Jacobi field along a geodesic γ defined on an open interval I containing 0 if and only if it is the variation field of some variation of γ through geodesics, i.e., if and only if there exists a smooth map $\Gamma : (-\varepsilon,\varepsilon)\times I \to \mathcal{S}$, with $\varepsilon > 0$, such that:

(a) The curve $\alpha_s(t) := \Gamma(s,t)$ is a geodesic for each $s \in (-\varepsilon,\varepsilon)$.

(b) $\alpha_0 = \gamma$.

(c) $J(t) = \Gamma_*\left(\dfrac{\partial}{\partial s}\right)_{\Gamma(0,t)}$, for $t \in I$.

The following definition is a natural extension of the concept of Jacobi field to vector fields defined on the surface, not necessarily along a geodesic:

Definition 2.10. Let X be a geodesic vector field of constant length 1 defined on a surface $\mathcal{S}$. A vector field J on $\mathcal{S}$ such that the restriction of J to each integral curve γ of X is a Jacobi vector field along γ will be called a Jacobi vector field relative to X.

Remark 2.11. As it can be deduced from equation (9), J is a Jacobi vector field relative to X if and only if

$$\nabla_X\nabla_X J + \mathcal{K}\left(J - g(J,X)X\right) = 0 \tag{10}$$

for every point $p \in \mathcal{S}$. This last expression allows us to conclude that any constant multiple of X is a Jacobi vector field relative to X. We will call them *trivial* Jacobi vector fields relative to X.

3 Surface associated to a first-order ODE

In this section, we present a systematic way to assign a surface $\mathcal{S}$ to a given first-order ODE, and we analyse some properties of this surface.

Definition 3.1. Given a first-order ODE like (1), we define the associated surface $\mathcal{S}$ to be the open set U together with the Riemannian metric given by

$$g = (1+\phi^2)dx\otimes dx - \phi dx\otimes du - \phi du\otimes dx + du\otimes du, \tag{11}$$

or in matrix form

$$g = \begin{pmatrix} 1+\phi^2 & -\phi \\ -\phi & 1 \end{pmatrix}. \tag{12}$$

Remark 3.2. The vector fields A and ∂_u on $\mathcal{S}$ satisfies the following properties, as can easily be checked:

(a) $\|A\| = 1$,

(b) A and ∂_u are orthogonal

In addition, $\|\partial_u\| = 1$, so the pair of vector fields $\{A, \partial_u\}$ constitutes an orthonormal frame for the surface $\mathcal{S}$. The corresponding dual coframe is given by the 1-forms

$$\begin{aligned} \omega^1 &= dx, \\ \omega^2 &= -\phi dx + du. \end{aligned}$$

To calculate the connection 1-form of the corresponding Levi-Civita connection, observe that $d\omega^1 = 0$ and $d\omega^2 = \phi_u dx \wedge du = \phi_u \omega^1 \wedge \omega^2$, so according to equation (5) we have:

$$\Theta = \begin{pmatrix} 0 & -\phi_u\omega^2 \\ \phi_u\omega^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \phi\phi_u dx - \phi_u du \\ -\phi\phi_u dx + \phi_u du & 0 \end{pmatrix}.$$

Now, we turn our attention to exploring several interesting properties regarding the geodesics of the surface $\mathcal{S}$. To avoid clutter, we will omit the arguments of the function ϕ and its partial derivatives ϕ_x, ϕ_u when they are clear from the context. We start with the following result:

Proposition 3.3. Consider an ODE like (1) such that $\phi_u \neq 0$, and let f be a smooth function. Then the curve γ_f is a geodesic of $\mathcal{S}$ if and only if f is a solution of (1).

Proof. First, observe that the components of the tangent vector $\dot{\gamma}_f(t)$ in the frame $\{A, \partial_u\}$ are

$$\dot{\gamma}_f(t) = \begin{pmatrix} 1 \\ f'(t) - \phi \end{pmatrix}, \tag{13}$$

and then, using equation (7), we obtain

$$\nabla_{\dot{\gamma}_f(t)}\dot{\gamma}_f(t) = \begin{pmatrix} -\phi_u(f'(t) - \phi)^2 \\ -\phi_u\phi - \phi_x + f''(t) \end{pmatrix}. \tag{14}$$

If f is a solution of (1) then $f'(t) = \phi(t, f(t))$. Differentiating both sides with respect to t gives

$$f''(t) = \phi_x + \phi_u f'(t) = \phi_x + \phi_u\phi.$$

Hence, by substituting in (14) we obtain

$$\nabla_{\dot{\gamma}_f(t)}\dot{\gamma}_f(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so γ_f is a geodesic.

Conversely, provided that γ_f is a geodesic of $\mathcal{S}$ we have that $\nabla_{\dot{\gamma}_f(t)}\dot{\gamma}_f(t) = 0$, and then by (14)

$$\phi_u\left(f'(t) - \phi\right)^2 = 0, \tag{15a}$$
$$f''(t) - \phi\phi_u - \phi_x = 0. \tag{15b}$$

Under the assumption $\phi_u \neq 0$, equation (15a) reduces to

$$f'(t) - \phi = 0, \tag{16}$$

and therefore f is a solution of (1). □

Remark 3.4. Observe that, in particular, this proposition shows that the vector field A is a geodesic vector field (this result appears, also, in [11]). Moreover, it is the only geodesic vector field V such that $i_{\partial_x}V = 1$.

Remark 3.5. For degenerated cases where $\phi_u = 0$, it is easy to see that there are geodesics which are not solutions of (1). Consider, for example,

$$\frac{du}{dx} = \sin(x).$$

Then, for $f(t) = t - \cos(t)$ the curve γ_f is a geodesic, since system (15) is satisfied. But $f(t)$ is clearly not a solution of the ODE.

At this point, it is appropriate to wonder about the nature of the rest of the geodesics of this surface. In this direction we state the following proposition regarding pregeodesic curves

Proposition 3.6. Given the first-order ODE (1), a curve γ_f is a pregeodesic of the associated surface $\mathcal{S}$ if and only if f satisfies the second-order ODE

$$\frac{d^2u}{dx^2} = A(\phi) - \phi_u\left(\frac{du}{dx} - \phi\right)^3. \tag{17}$$

Proof. First, observe that every solution of (1), which by Proposition 3.3 yields a geodesic γ_f and therefore a pregeodesic, is also a solution of (17). Indeed, differentiating $f'(t) = \phi(t, f(t))$ we obtain

$$f''(t) = \phi_x + \phi_u f'(t) = \phi_x + \phi\phi_u = A(\phi)$$

so (17) is satisfied.

Therefore, we will focus on the case when f is not a solution of (1), i.e. $f'(t) - \phi \neq 0$. The curve γ_f will be a pregeodesic of $\mathcal{S}$ if and only if

$$\nabla_{\dot{\gamma}_f(t)}\dot{\gamma}_f(t) = \sigma(t)\dot{\gamma}_f(t),$$

or, equivalently, if the vectors (13) and (14) appearing in the proof of Proposition 3.3 are proportional. So we have that γ_f is a pregeodesic curve if and only if

$$\frac{-\phi_u(f'(t)-\phi)^2}{1} = \frac{-\phi_u\phi - \phi_x + f''(t)}{f'(t)-\phi},$$

and then

$$-\phi_u(f'(t)-\phi)^3 = -\phi_u\phi - \phi_x + f''(t).$$

Rearranging terms we obtain the equivalent condition

$$f''(t) = A(\phi) - \phi_u(f'(t)-\phi)^3,$$

and the result is proven. □

Remark 3.7. If we consider an arbitrary geodesic $\gamma(t) = (a(t), b(t))$ of $\mathcal{S}$ such that $a'(t) \neq 0$, then it can be reparametrized as a curve $\gamma_f(t) = (t, f(t))$ for a certain smooth function f. Upon this reparametrization, the curve γ_f qualifies as a pregeodesic, and it retains the same image as the original geodesic γ. Hence, Proposition 3.6 essentially characterizes the images of a majority of geodesics on $\mathcal{S}$. In simpler terms, the smooth functions f which are solutions to the equation (17) yield curves γ_f which, upon arclength reparametrization, give rise to geodesics.

Following our analysis of geodesics on the surface $\mathcal{S}$, it is natural to proceed with the study of Jacobi fields. We will show a relation between these fields and Lie point symmetries. Given a vector field V we will denote by $V^\perp$ the component of V orthogonal to A:

$$V^\perp = V - g(V, A)A.$$

We state the following lemma:

Lemma 3.8. Let V be a Lie point symmetry of equation (1), with $[V, A] = \rho A$. Then, the following statements hold:

(a) $\rho = -A(g(V, A))$.

(b) $V^\perp$ is also a Lie point symmetry of (1), satisfying $[V^\perp, A] = 0$.

Proof. First, observe that $V^\perp = \delta\partial_u$ with $\delta = g(V, \partial_u)$. Then

$$\begin{aligned}[V^\perp, A] &= -A(\delta)\partial_u + \delta[\partial_u, A]\\ &= -A(\delta)\partial_u + \delta\phi_u\partial_u\\ &= (\delta\phi_u - A(\delta))\partial_u.\end{aligned}$$

Taking into account that $V = g(V, A)A + V^\perp$, we have

$$\begin{aligned}[V, A] &= [g(V, A)A + V^\perp, A]\\ &= -A(g(V, A))A + [V^\perp, A]\\ &= -A(g(V, A))A + (\delta\phi_u - A(\delta))\partial_u.\end{aligned}$$

Since $[V, A] = \rho A$, we conclude that $\rho = -A(g(V, A))$ and $[V^\perp, A] = 0$, and the result is proven. □

Remark 3.9. The orthogonal component $V^\perp$ corresponds to what is called in [16] the *evolutionary representative* of V, being $g(V, \partial_u)$ its *characteristic*. Moreover, the term $g(V, \partial_u)$ is a symmetrizing factor for ∂_u respect to the ODE (1), in the sense of Definition 2.3.

Now, we establish a link between Jacobi fields and Lie point symmetries.

Theorem 3.10. Consider a Lie point symmetry V of (1) with $[V, A] = \rho A$. Then V is a Jacobi field relative to A if and only if ρ is a first integral of (2).

Proof. First, we will show that if V is a Lie point symmetry of (1) then $V^\perp$ is a Jacobi field relative to A. Consider a point $p = (x_0, u_0) \in U$ and let f be a solution to (1) such that $f(x_0) = u_0$. We consider the geodesic $\gamma(t) := \gamma_f(t + x_0)$, defined on an open interval I containing 0, which satisfies $\gamma(0) = p$ and $\dot\gamma(0) = A_p$.

On the other hand, by Lemma 3.8 part (b) we can assert that $[V^\perp, A] = 0$ so, according to Theorem 2.5, there exists a local coordinate change φ around p satisfying

$$\varphi_*\left(\frac{\partial}{\partial s}\right) = V^\perp, \tag{18a}$$

$$\varphi_*\left(\frac{\partial}{\partial t}\right) = A. \tag{18b}$$

We can adjust the domain I of γ such that the map φ is defined on $(-\varepsilon, \varepsilon) \times I$ for some $\epsilon > 0$, and it satisfies $\varphi(0, 0) = p$. Then, it holds that φ is a variation through geodesics for the geodesic γ (see Remark 2.9), since:

(a) For every s, the curve defined by $\alpha_s(t) = \varphi(s, t)$ is an integral curve of A, according to (18b). Since A is a geodesic vector field, α_s is a geodesic for every s.

(b) In particular, $\alpha_0 = \gamma$, given that both α_0 and γ are geodesics, and they satisfy $\alpha_0(0) = \gamma(0)$ and $\dot\alpha_0(0) = \dot\gamma(0)$.

(c) Finally, $V^\perp_{\gamma(t)} = \varphi_*\left(\frac{\partial}{\partial s}\right)_{\varphi(0,t)}$ for $t \in I$, according to (18a).

Therefore, the restriction of $V^\perp$ to γ is a Jacobi field for γ, and it satisfies Jacobi equation (9) at p. Since this is true for every p and the corresponding integral curve of A through p, we conclude that $V^\perp$ is a Jacobi field relative to A, and then

$$\nabla_A \nabla_A V^\perp + \mathcal{K} V^\perp = 0. \tag{19}$$

Now, to prove the theorem, consider a Lie point symmetry V. We can write the left-hand side of equation (10) for $J = V$ and $X = A$, and using (19) we obtain:

$$\begin{aligned}
\nabla_A \nabla_A V + \mathcal{K}\left(V - g(V, A)A\right) &= \nabla_A \nabla_A (V^\perp + g(V, A)A) + \mathcal{K} V^\perp \\
&= \nabla_A \nabla_A V^\perp + \mathcal{K} V^\perp + \nabla_A \nabla_A (g(V, A)A) \\
&= \nabla_A \nabla_A (g(V, A)A) \\
&= A^2(g(V, A))A.
\end{aligned} \tag{20}$$

From Lemma 3.8 part (a), we have that $\rho = -A(g(V, A))$. Substituting this into equation (20), we obtain

$$\nabla_A \nabla_A V + \mathcal{K}\left(V - g(V, A)A\right) = -A(\rho)A.$$

This implies that the condition for V to be a Jacobi field relative to A reduces to ensuring $A(\rho) = 0$, and the result is proven. □

An immediate consequence of this theorem is the following result

Corollary 3.11. Any Lie point symmetry of (1) orthogonal to A is a Jacobi field relative to A.

And from here, we can conclude the following formula

Corollary 3.12. Any symmetrizing factor δ for ∂_u satisfies

$$A^2(\delta) + \mathcal{K}\delta = 0. \tag{21}$$

Proof. If δ is a symmetrizing factor then $\delta\partial_u$ is a Lie point symmetry orthogonal to A, so it is a Jacobi field relative to A. Writing equation (10) for $\delta\partial_u$ we obtain

$$\nabla_A \nabla_A \delta\partial_u + \mathcal{K}\delta\partial_u = 0. \tag{22}$$

Now, since

$$\nabla_A \partial_u = i_A \Theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

equation (22) gives rise to (21), and the result is proven. □

Given that symmetrizing factors are closely related to the integrability of the ODE (1) (see Lemma 2.4 and the subsequent discussion), it is noteworthy that equation (21) reveals the existence of a relation between solving the ODE and a pure geometric concept, namely curvature. In the rest of this section we will delve into this relation.

Proposition 3.13. The Gaussian curvature of the surface $\mathcal{S}$ is given by the expression

$$\mathcal{K} = -\partial_u(A(\phi)). \tag{23}$$

Proof. Considering equation (8) we have that

$$\begin{aligned}
\mathcal{K} &= d\Theta_2^1(A, \partial_u) \\
&= d(\phi\phi_u dx - \phi_u du)(A, \partial_u) \\
&= \left(-\phi_u^2 - \phi\phi_{uu} - \phi_{xu}\right) dx \wedge du(A, \partial_u) \\
&= (-\phi\phi_u - \phi_x)_u \\
&= -\partial_u(A(\phi)).
\end{aligned}$$

□

The expression (23) for the Gaussian curvature was used by Z. Bayrakdar and T. Bayrakdar in [11] to establish that the first-order ODEs giving rise to associated surfaces with curvature $\mathcal{K} = \mathcal{K}(x, u)$ satisfy the inhomogeneous inviscid Burgers' equations:

$$\phi_x + \phi\phi_u = \int^u \mathcal{K}(x, t)dt.$$

Additionally, they gave conditions for a linear first-order ODE to have constant curvature. Remarkably, nonlinear first-order ODEs can also have constant curvature. The following example illustrates this for curvatures 1 and -1.

Example 3.14. Consider the first-order ODEs

$$\frac{du}{dx} = -1 + \sqrt{1 - (x+u)^2}$$

and

$$\frac{du}{dx} = -1 + \sqrt{1 + (x+u)^2}.$$

The associated surfaces have constant curvature $\mathcal{K} = 1$ and $\mathcal{K} = -1$, so they are examples of spherical and hyperbolic geometries, respectively.

Interestingly, we can perform the change of variables $Y = \frac{du}{dx} + 1$, $Z = x + u$, these ODEs correspond to the following algebraic equations

$$\begin{aligned}
Y^2 + Z^2 &= 1, \\
Y^2 - Z^2 &= 1,
\end{aligned}$$

which are the equations of a circle and a hyperbola, respectively.

The expression obtained in Proposition 3.13 allows us to establish a link between the flatness of the surface and the integrability of the corresponding ODE:

Theorem 3.15. Given the first-order ODE (1), if the Gaussian curvature $\mathcal{K}$ of the associated surface $\mathcal{S}$ is zero, then the ODE can be fully integrated.

Proof. Since $\mathcal{S}$ is flat, it follows that $\mathcal{K} = -\partial_u(A(\phi)) = 0$, so $A(\phi)$ is a smooth function which does not depend on the variable u. We now consider Ψ to be any smooth function on x satisfying

$$\frac{d\Psi}{dx} = A(\phi).$$

Then, the expression $F(x,u) = \phi(x,u) - \Psi(x)$ is a first integral of (2). In fact, we have that

$$A(F) = A(\phi - \Psi) = A(\phi) - \frac{d\Psi}{dx} = 0.$$

So the general solution of (1) is given by

$$\phi(x,u) - \Psi(x) = C,$$

with $C \in \mathbb{R}$. □

Example 3.16. Consider the first-order ODE

$$(1-x)\frac{du}{dx} = e^{(x-1)\frac{du}{dx} - u} + 1. \tag{24}$$

We can put this equation in normal form

$$\frac{du}{dx} = \frac{\mathbf{W}(e^{-u-1}) + 1}{1-x},$$

defined on $U = \{(x,u) \in \mathbb{R}^2 : x \neq 1\}$. Here, $\mathbf{W}$ is the smooth function that satisfies

$$\mathbf{W}(x)e^{\mathbf{W}(x)} = x,$$

known as the Lambert function [28]. Recall that

$$\frac{d\mathbf{W}(x)}{dx} = \frac{1}{e^{\mathbf{W}(x)}\left(1 + \mathbf{W}(x)\right)}.$$

The associated vector field is

$$A = \partial_x + \frac{\mathbf{W}(e^{-u-1}) + 1}{1-x}\partial_u,$$

and the curvature $\mathcal{K}$ of the associated surface is:

$$\begin{aligned}
\mathcal{K} &= -\partial_u\left(A\left(\frac{\mathbf{W}\left(e^{-u-1}\right) + 1}{1-x}\right)\right) \\
&= -\partial_u\left(\frac{1}{(x-1)^2}\right) \\
&= 0.
\end{aligned}$$

Since this surface is flat, equation (24) can be integrated following the proof of Theorem 3.15. Consider the function $\Psi(x) = \frac{1}{1-x}$, which satisfies

$$\frac{d\Psi}{dx} = \frac{1}{(x-1)^2} = A\left(\phi\right).$$

Then, a first integral of A is

$$\begin{aligned}
F(x,u) &= \phi(x,u) - \Psi(x) \\
&= \frac{\mathbf{W}(e^{-u-1}) + 1}{1-x} - \frac{1}{1-x} \\
&= \frac{\mathbf{W}(e^{-u-1})}{1-x}.
\end{aligned}$$

Finally, we can isolate u in

$$\frac{\mathbf{W}(e^{-u-1})}{1-x} = C,$$

where C is a nonzero constant, obtaining the general solution

$$u(x) = -\ln(C(1-x)) - C(1-x) - 1,$$

defined for $C \neq 0$ and $x \in \mathbb{R}$ such that $C(1-x) > 0$.

In the next section, we will see that the condition $\mathcal{K} = 0$ is not the only one that guarantees the integrability of (1). In fact, we will show that any constant curvature is a sufficient condition for the integrability of (1).

4 Deformation of the associated surface

In this section, we explore the idea of deforming the surface defined in the previous section while preserving its essential features in relation to equation (1), and we study the role of this deformation in the integrability of the equation. As we will show, the discovery of an integrating factor for (1) corresponds to deforming the surface $\mathcal{S}$ into a zero curvature surface, but it is not the only suitable deformation in order to achieve integrability.

We start by considering the family $\mathcal{G}$ of all Riemannian metrics on U that satisfy conditions (a) and (b) in Remark 3.2:

(a) $\|A\| = 1$,

(b) A and ∂_u are orthogonal in this metric.

This family of metrics can be *indexed* by the set $\mathcal{C}^\infty(U)$ of smooth functions defined on U:

Proposition 4.1. Every Riemannian metric in the family $\mathcal{G}$ can be expressed in coordinates (x, u) as the matrix

$$g_\epsilon = \begin{pmatrix} 1+\phi^2 e^{2\epsilon} & -\phi e^{2\epsilon} \\ -\phi e^{2\epsilon} & e^{2\epsilon} \end{pmatrix}, \tag{25}$$

with $\epsilon = \epsilon(x, u) \in \mathcal{C}^\infty(U)$.

Proof. Consider any Riemannian metric $\tilde{g}$ in $\mathcal{G}$. The vector field $\frac{1}{\sqrt{\tilde{g}(\partial_u, \partial_u)}}\partial_u$ is of unit length. This fact, combined with conditions (a) and (b), implies that the pair of vector fields

$$\left\{A, \frac{1}{\sqrt{\tilde{g}(\partial_u, \partial_u)}}\partial_u\right\}$$

forms an orthonormal frame with respect to $\tilde{g}$. Defining $\epsilon(x, u) := \ln\left(\sqrt{\tilde{g}(\partial_u, \partial_u)}\right)$, this frame can be rewritten as

$$\{A, e^{-\epsilon}\partial_u\}, \tag{26}$$

and the corresponding orthonormal dual coframe, denoted by $\{\omega_\epsilon^1, \omega_\epsilon^2\}$, is given by

$$\begin{aligned} \omega_\epsilon^1 &= dx, \\ \omega_\epsilon^2 &= -e^\epsilon \phi dx + e^\epsilon du. \end{aligned} \tag{27}$$

Then, the Riemannian metric $\tilde{g}$ can be expressed as in equation (4):

$$\begin{aligned} \tilde{g} &= \omega_\epsilon^1 \otimes \omega_\epsilon^1 + \omega_\epsilon^2 \otimes \omega_\epsilon^2 \\ &= dx \otimes dx + (-e^\epsilon \phi dx + e^\epsilon du) \otimes (-e^\epsilon \phi dx + e^\epsilon du) \\ &= dx \otimes dx + \phi^2 e^{2\epsilon} dx \otimes dx - \phi e^{2\epsilon} dx \otimes du - \phi e^{2\epsilon} du \otimes dx + e^{2\epsilon} du \otimes du \\ &= (1+\phi^2 e^{2\epsilon}) dx \otimes dx - \phi e^{2\epsilon} dx \otimes du - \phi e^{2\epsilon} du \otimes dx + e^{2\epsilon} du \otimes du. \end{aligned}$$

In matrix form, this becomes:

$$\tilde{g} = \begin{pmatrix} 1+\phi^2 e^{2\epsilon} & -\phi e^{2\epsilon} \\ -\phi e^{2\epsilon} & e^{2\epsilon} \end{pmatrix}. \tag{28}$$

□

Remark 4.2. Let us denote by g_ϵ the metric given by the matrix (28), and by $\mathcal{S}_\epsilon$ the surface given by the pair (U, g_ϵ). Observe that the surface $\mathcal{S}$ from Definition 3.1 is, obviously, the member of the family $\mathcal{G}$ corresponding to the choice $\epsilon = 0$, that is, $\mathcal{S} = \mathcal{S}_0$. Hence, the set

$$\{\mathcal{S}_\epsilon : \epsilon \in \mathcal{C}^\infty(U)\}$$

can be regarded as a *deformation* of $\mathcal{S}$.

For every surface $\mathcal{S}_\epsilon$, we will consider the function

$$\Delta_\epsilon := A(\epsilon) + \phi_u$$

to define two operators which will facilitate the computations.

Definition 4.3. Given a first-order ODE (1) we define two operators, $\mathfrak{T}_\epsilon$ and $\mathfrak{S}_\epsilon$, as follows:

$$\begin{aligned} \mathfrak{T}_\epsilon(h) &:= A(h) + \Delta_\epsilon h, \\ \mathfrak{S}_\epsilon(h) &:= A(h) - \Delta_\epsilon h, \end{aligned} \tag{29}$$

where $h \in \mathcal{C}^\infty(U)$.

These operators satisfy the following proposition

Proposition 4.4. For every $h \in \mathcal{C}^\infty(U)$ we have that

(a) $\mathfrak{T}_\epsilon(h) = 0$ if and only if $e^\epsilon h$ is an integrating factor for (3).

(b) If h is non-vanishing in U, then $\mathfrak{S}_\epsilon(h) = 0$ if and only if $e^{-\epsilon}h$ is a symmetrizing factor of the vector field ∂_u with respect to (1).

Proof. For the proof of part (a), observe that

$$\begin{aligned} d\left(e^\epsilon h(-\phi dx + du)\right) &= (-e^\epsilon h\phi)_u du \wedge dx + (e^\epsilon h)_x dx \wedge du \\ &= \left((e^\epsilon h)_u \phi + e^\epsilon h \phi_u + (e^\epsilon h)_x\right) dx \wedge du \\ &= \left(A(e^\epsilon h) + \phi_u e^\epsilon h\right) dx \wedge du \\ &= \left(e^\epsilon A(\epsilon) h + e^\epsilon A(h) + \phi_u e^\epsilon h\right) dx \wedge du \\ &= e^\epsilon \left(A(\epsilon)h + A(h) + \phi_u h\right) dx \wedge du \\ &= e^\epsilon \mathfrak{T}_\epsilon(h) dx \wedge du. \end{aligned} \tag{30}$$

Then, $e^\epsilon h$ is an integrating factor of (3) if and only if $\mathfrak{T}_\epsilon(h) = 0$.

To prove part (b), according to Lemma 2.4 we only have to show that $\frac{1}{e^{-\epsilon}h} = \frac{e^\epsilon}{h}$ is an integrating factor for (3). Therefore, proceeding in a similar way as in the previous case, we have

$$\begin{aligned} d\left(\frac{e^\epsilon}{h}(-\phi dx + du)\right) &= \left(-\frac{e^\epsilon}{h}\phi\right)_u du \wedge dx + \left(\frac{e^\epsilon}{h}\right)_x dx \wedge du \\ &= \left(\left(\frac{e^\epsilon}{h}\right)_u \phi + \frac{e^\epsilon}{h}\phi_u + \left(\frac{e^\epsilon}{h}\right)_x\right) dx \wedge du \\ &= \left(A\left(\frac{e^\epsilon}{h}\right) + \phi_u \frac{e^\epsilon}{h}\right) dx \wedge du \\ &= \frac{e^\epsilon A(\epsilon)h - e^\epsilon A(h) + e^\epsilon h \phi_u}{h^2} dx \wedge du \\ &= \frac{e^\epsilon}{h^2}\left(A(\epsilon)h - A(h) + h\phi_u\right) dx \wedge du \\ &= -\frac{e^\epsilon}{h^2}\mathfrak{S}_\epsilon(h) dx \wedge du. \end{aligned} \tag{31}$$

Therefore, $e^{-\epsilon}h$ is a symmetrizing factor of ∂_u with respect to (1) if and only if $\mathfrak{S}_\epsilon(h) = 0$. $\square$

Now we proceed to explore the geometry of the deformed surfaces $\mathcal{S}_\epsilon$. Recall that Proposition 3.3 establishes that the solutions to equation (1) correspond to geodesics of $\mathcal{S}_0$. Notably, this set of geodesics remains as geodesics through the deformation:

Proposition 4.5. The vector field A is a geodesic vector field for every surface $\mathcal{S}_\epsilon$.

Proof. Taking into account that

$$\begin{aligned} d\omega_\epsilon^1 &= 0, \\ d\omega_\epsilon^2 &= (\epsilon_x + \epsilon_u \phi + \phi_u)\omega_\epsilon^1 \wedge \omega_\epsilon^2 = \Delta_\epsilon \omega_\epsilon^1 \wedge \omega_\epsilon^2, \end{aligned}$$

and according to (5), we can write the connection form for the Levi-Civita connection of S_ϵ with respect to the orthonormal frame (26) as

$$\Theta = \begin{pmatrix} 0 & -\Delta_\epsilon \omega_\epsilon^2 \\ \Delta_\epsilon \omega_\epsilon^2 & 0 \end{pmatrix}. \tag{32}$$

Hence, $\nabla_A A = 0$ and A is a geodesic vector field. □

We will show how the Gaussian curvature of the deformed surfaces, denoted as $\mathcal{K}_\epsilon$, relates to the integrability of (1). First we need an explicit expression for $\mathcal{K}_\epsilon$:

Proposition 4.6. The Gaussian curvature of $\mathcal{S}_\epsilon$ is given by the expression

$$\mathcal{K}_\epsilon = -A(\Delta_\epsilon) - \Delta_\epsilon^2. \tag{33}$$

Proof. Note that

$$\begin{aligned} d\Theta_2^1 &= -d\Delta_\epsilon \wedge w^2 - \Delta_\epsilon \wedge dw^2 \\ &= -\left((\Delta_\epsilon)_x + (\Delta_\epsilon)_u \phi + \Delta_\epsilon^2\right) w^1 \wedge w^2 \\ &= -\left(A(\Delta_\epsilon) + \Delta_\epsilon^2\right) w^1 \wedge w^2, \end{aligned}$$

and according to equation (8)

$$\mathcal{K}_\epsilon = d\Theta_2^1(A, e^{-\epsilon}\partial_u) = -A(\Delta_\epsilon) - \Delta_\epsilon^2.$$

□

Now, we prove that there is a relationship between the curvature $\mathcal{K}_\epsilon$ and the operators $\mathfrak{T}_\epsilon, \mathfrak{S}_\epsilon$:

Lemma 4.7. It is satisfied that

$$\mathfrak{T}_\epsilon \circ \mathfrak{S}_\epsilon = A^2 + \mathcal{K}_\epsilon. \tag{34}$$

Proof. Consider a smooth function h defined on U, then

$$\begin{aligned} \mathfrak{T}_\epsilon \circ \mathfrak{S}_\epsilon(h) &= (A + \Delta_\epsilon) \circ (A - \Delta_\epsilon)(h) \\ &= A^2(h) - A(\Delta_\epsilon h) + \Delta_\epsilon A(h) - \Delta_\epsilon^2 h \\ &= A^2(h) - A(\Delta_\epsilon)h - \Delta_\epsilon A(h) + \Delta_\epsilon A(h) - \Delta_\epsilon^2 h \\ &= A^2(h) + \mathcal{K}_\epsilon h. \end{aligned}$$

□

We are in conditions to prove the following result:

Theorem 4.8. The knowledge of a non-trivial Jacobi vector field relative to A on any of the surfaces $\mathcal{S}_\epsilon$ leads to the integration of the first-order ODE (1).

Proof. Let J be a non-trivial Jacobi field relative to A on $\mathcal{S}_\epsilon$, for certain $\epsilon \in \mathcal{C}^\infty(U)$. We can decompose J into its components with respect to the frame $\{A, e^{-\epsilon}\partial_u\}$ as follows:

$$J = \sigma A + \delta e^{-\epsilon}\partial_u,$$

where $\sigma = g_\epsilon(J, A)$ and $\delta = g_\epsilon(J, e^{-\epsilon}\partial_u)$.

Consider, first, the case where $\delta \neq 0$. We assume, without loss of generality, that δ is non-vanishing in U (if necessary, U can be appropriately reduced). Since J is a Jacobi field relative to A, equation (10) holds:

$$\nabla_A \nabla_A \left(\sigma A + \delta e^{-\epsilon}\partial_u\right) + \mathcal{K}_\epsilon \delta e^{-\epsilon}\partial_u = 0. \tag{35}$$

Now, since $\nabla_A A = \nabla_A \left(e^{-\epsilon}\partial_u\right) = 0$, as can be deduced from the connection 1-form (32), we obtain

$$A^2(\sigma)A + A^2(\delta)e^{-\epsilon}\partial_u + \mathcal{K}_\epsilon \delta e^{-\epsilon}\partial_u = 0. \tag{36}$$

Therefore, δ must satisfy

$$A^2(\delta) + \mathcal{K}_\epsilon \delta = 0. \tag{37}$$

According to Lemma 4.7, we have that

$$\left(\mathfrak{T}_\epsilon \circ \mathfrak{S}_\epsilon\right)(\delta) = 0,$$

and we can use Proposition 4.4 to conclude that

- either $\mathfrak{S}_\epsilon(\delta) = 0$ and then $e^{-\epsilon}\delta$ is a symmetrizing factor for ∂_u;
- or $\mathfrak{S}_\epsilon(\delta) \neq 0$ and $\mathfrak{T}_\epsilon(\mathfrak{S}_\epsilon(\delta)) = 0$, so $e^{\epsilon}\mathfrak{S}_\epsilon(\delta)$ is an integrating factor for (1).

In both cases, we obtain an integrating factor for (1), and therefore we can integrate it by quadratures.

Suppose now that $\delta = 0$. Observe that σ is not a constant function, given that J is non-trivial (see Remark 2.11). If we write equation (10) in this particular case we obtain:

$$\nabla_A \nabla_A (\sigma A) = A^2(\sigma) A = 0, \tag{38}$$

and $A^2(\sigma) = 0$. Consequently, if $A(\sigma)$ is not constant, it is a first integral of A. On the other hand, if $A(\sigma) = k \in \mathbb{R}$, then $A(\sigma - kx) = 0$, and $\sigma - kx$ is a first integral of A. In both cases, equation (1) is solved, and the result is proven. □

In conclusion, deforming $\mathcal{S}_0$ into any surface that admits a non-trivial Jacobi field with respect to A enables the integration of equation (1). More specifically, we have the following result:

Theorem 4.9. Given the first-order ODE (1), the deformation of the associated surface $\mathcal{S}_0$ into a surface of constant curvature leads to its integration by quadratures.

Proof. Consider a first-order ODE (1) and suppose that ϵ is such that $\mathcal{S}_\epsilon$ has constant curvature $\mathcal{K}_\epsilon = k \in \mathbb{R}$. A non-vanishing smooth function δ satisfying

$$A^2(\delta) + k\delta = 0$$

gives rise to the Jacobi field $J = \delta e^{-\epsilon}\partial_u$ relative to A. We can assume $\delta = \delta(x)$ and solve the second-order ODE

$$\delta'' + k\delta = 0 \tag{39}$$

to obtain δ.

It is well known that the solutions to this ODE are

$$\delta(x) = \begin{cases} A\cos\sqrt{k}x + B\sin\sqrt{k}x, & \text{if } k > 0,\\ A + Bx, & \text{if } k = 0,\\ A\cosh\sqrt{-k}x + B\sinh\sqrt{-k}x, & \text{if } k < 0, \end{cases} \tag{40}$$

with $A, B \in \mathbb{R}$.

Then, in any case, we have a non-trivial Jacobi field $J = \delta e^{-\epsilon}\partial_u$ relative to A, and according to Theorem 4.8 we can integrate equation (1).

□

In the following example we illustrate the case of a first-order ODE whose associated surface can be deformed into a surface of constant positive curvature.

Example 4.10. Consider the first-order ODE

$$\frac{du}{dx} = u^2, \tag{41}$$

whose associated surface $\mathcal{S}_0$ has Gaussian curvature

$$\mathcal{K}_0 = -\partial_u(A(u^2)) = -6u^2.$$

To deform this surface into a constant positive curvature surface we take

$$\epsilon = \ln\left(\frac{1}{u^2}\sin\left(\frac{1}{u}\right)\right).$$

In this case, we have $\Delta_\epsilon = -\cot\left(\frac{1}{u}\right)$, and then the surface $\mathcal{S}_\epsilon$ has curvature

$$\mathcal{K}_\epsilon = -A(\Delta_\epsilon) - \Delta_\epsilon^2 = 1,$$

where $A = \partial_x + u^2\partial_u$.

Therefore, we can take the particular solution of (39) given by $\delta(x) = \sin(x)$, which is hence a solution to

$$A^2(\delta) + \delta = 0.$$

By Lemma 4.7, we have that

$$(\mathfrak{T}_\epsilon \circ \mathfrak{S}_\epsilon)(\delta) = (\mathfrak{T}_\epsilon \circ \mathfrak{S}_\epsilon)(\sin(x)) = 0.$$

Finally, since

$$\mathfrak{S}_\epsilon(\sin(x)) = A(\sin(x)) - \Delta_\epsilon \sin(x) = \cos(x) + \cot\left(\frac{1}{u}\right)\sin(x) \neq 0,$$

we conclude that, necessarily

$$\mathfrak{T}_\epsilon(\mathfrak{S}_\epsilon(\sin(x))) = 0,$$

and then, by part (a) in Proposition 4.4, we have that

$$\begin{aligned} e^\epsilon \mathfrak{S}_\epsilon(\sin(x)) &= \frac{1}{u^2}\sin\left(\frac{1}{u}\right)\left(\cos(x) + \cot\left(\frac{1}{u}\right)\sin(x)\right) \\ &= \frac{1}{u^2}\sin\left(x + \frac{1}{u}\right) \end{aligned}$$

is an integrating factor for (41).

Now we show an example of a first-order ODE whose associated surface can be deformed into a surface of constant negative curvature.

Example 4.11. Consider the first-order ODE

$$\frac{du}{dx} = \frac{1-3xu}{x^2}. \tag{42}$$

The associated vector field is $A = \partial_x + \frac{1-3xu}{x^2}\partial_u$, and the Gaussian curvature of the associated surface $\mathcal{S}_0$ is

$$\mathcal{K}_0 = -\partial_u\left(A\left(\frac{1-3xu}{x^2}\right)\right) = -\frac{12}{x^2}.$$

We can deform it by using $\epsilon = x + 3\ln(x)$, in such a way that now $\Delta_\epsilon = 1$. Then

$$\mathcal{K}_\epsilon = -A(\Delta_\epsilon) - \Delta_\epsilon^2 = -1.$$

In this case a suitable δ giving rise to a Jacobi field relative to A is

$$\delta(x) = \sinh(x),$$

so by Lemma 4.7 we have that

$$(\mathfrak{T}_\epsilon \circ \mathfrak{S}_\epsilon)(\sinh(x)) = 0.$$

Observe that $\mathfrak{S}_\epsilon(\sinh(x)) = A(\sinh(x)) - \Delta_\epsilon \sinh(x) = \cosh(x) - \sinh(x) = e^{-x}$, which is not constant. Since

$$\mathfrak{T}_\epsilon(e^{-x}) = 0,$$

we have by Proposition 4.4 part (a) that

$$e^\epsilon e^{-x} = e^{x+3\ln(x)}e^{-x} = x^3$$

is an integrating factor for (42).

Recall that Theorem 3.15 established the integrability of (1) when the associated surface has zero curvature. This result can be extended to equations whose associated surface has any constant curvature, as shown by the following immediate corollary of Theorem 4.9:

Corollary 4.12. If the associated surface $\mathcal{S}_0$ has constant curvature, then the ODE can be integrated by quadratures.

Remark 4.13. Within our approach, the integration of equation (1) through the identification of an integrating factor for the 1-form (3) can be seen as the particular case of Theorem 4.9 in which we deform the surface $\mathcal{S}_0$ into a surface of zero curvature. Indeed, observe that such an integrating factor μ satisfies

$$d(\mu\phi dx - \mu du) = (-\mu_u\phi - \mu\phi_u - \mu_x)dx \wedge du = 0. \tag{43}$$

So, if we take $\epsilon = \ln\mu$ (where we possibly have to restrict the open set U of μ to ensure that ϵ is well defined), then

$$\Delta_\epsilon = A(\epsilon) + \phi_u = \frac{A(\mu)}{\mu} + \phi_u = \frac{\mu_x + \mu_u\phi + \mu\phi_u}{\mu} = 0,$$

because of (43). Thus, for this choice of ϵ we have

$$\mathcal{K}_\epsilon = -A(\Delta_\epsilon) - \Delta_\epsilon^2 = 0,$$

so it corresponds to the deformation into a surface of zero curvature.

In conclusion, we can think of Theorem 4.9 as a generalization of the integrating factor approach to solving first-order ODEs.

Remark 4.14. In the situation described in the previous remark, the volume form $\mathbf{\Omega}_\epsilon$ of the surface $\mathcal{S}_\epsilon$ conforms to the situation described in the introduction, that is, we can interpret the variables space (x, u) as a surface whose volume form is invariant under the flow of A. In fact, we have that the corresponding volume form is

$$\begin{aligned}\mathbf{\Omega}_\epsilon &= \omega_\epsilon^1 \wedge \omega_\epsilon^2\\ &= e^\epsilon dx \wedge du\\ &= \mu dx \wedge du\\ &= \mu\mathbf{\Omega}.\end{aligned}$$

5 Final remarks

In this paper, we have studied a specific approach for associating a surface with a first-order ODE. We have demonstrated that notions such as solutions, Lie point symmetries, and quadratures possess geometric counterparts within this framework.

Furthermore, we introduced a specific way of deforming the associated surface. We have showed that knowing a Jacobi field relative to A on any deformed surface leads to the integration of the ODE. In this sense it is interesting to note that the Jacobi fields relative to A of the associated surface itself allow us to integrate the ODE. Therefore, since there are Jacobi fields relative to A that are not Lie point symmetries of the ODE, this finding reveals the existence of additional vector fields, distinct from Lie point symmetries, that can also be utilized for ODE integration.

In particular, we have shown that the deformation of the associated surface into a constant curvature surface implies the integrability by quadratures of the ODE, and the particular case of the deformation to a zero curvature surface is the geometric counterpart of the knowledge of an integrating factor for the equation.

We find it interesting that a classical geometric notion such as the curvature of the associated surface, or its deformation, is directly related to the integrability of the equation. This new relationship offers a fresh perspective on the understanding of first-order ODEs.

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