Optimality System of the Averaged Advection Diffusion Equation

The optimality system is:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{1}{r}\frac{\partial \rho}{\partial r} + \frac{\partial^2 \rho}{\partial r^2} + \frac{\partial \rho}{\partial z^2} - \frac{\rho \mathbf{w}_r}{r} - \nabla \cdot (\rho \mathbf{w}) + f \\ 0 &= \left(- \left(\frac{\partial \rho}{\partial r}, \frac{\partial \rho}{\partial z} \right) + \rho \mathbf{w} \right) \cdot \mathbf{n}\end{aligned}$$

$$\begin{aligned}\frac{\partial q}{\partial t} &= -\frac{1}{r}\frac{\partial q}{\partial r} - \frac{\partial^2 q}{\partial r^2} - \frac{\partial q}{\partial z^2} - \mathbf{w} \cdot \left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z} \right) - \rho + \widehat{\rho} \\ 0 &= \left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z} \right) \cdot \mathbf{n}\end{aligned}$$

$$\mathbf{w} = -\frac{1}{\beta}\rho \left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z} \right)$$

I used the expanded version of the term in the implementation, where ∇ is defined with respect to r and z: $\nabla \cdot (\rho \mathbf{w}) = \mathbf{w}_r \frac{\partial \rho}{\partial r} + \mathbf{w}_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial \mathbf{w}_r}{\partial r} + \rho \frac{\partial \mathbf{w}_z}{\partial z}$.

Exact Solution

We are choosing an exact solution which satisfies the boundary conditions, matches the final time condition for q and is invariant in θ. We choose:

$$\begin{aligned}\rho &= \beta^{1/2} e^t \cos(\pi r) \cos(\pi z) \\ q &= \beta^{1/2} (e^T - e^t) \cos(\pi r) \cos(\pi z),\end{aligned}$$

and use these to determine the values of $\mathbf{w}$, f and $\widehat{\rho}$.