Linear time exact solution trials

Perturbation Functions Considered

The one from last week:

$$f(t) = \frac{e^{-1/t}}{e^{-1/t} + e^{-1/(1-t)}}.$$

An new perturbation is given by:

$$g(t) = \frac{e^{-a/(t-t0)}}{e^{-a/(t-t0)} + e^{-a/(1-t-t0)}} \times \frac{e^{a/(t-t0)}}{e^{a/(t-t0)} + e^{a/(1-t-t0)}},$$

where $a = 0.7$ (mostly) and $t0 = -0.01$. The translation by $t0$ is due to the fact that otherwise you'd get NANs for $t = 0$. The factor a is flattening or sharpening the peak.

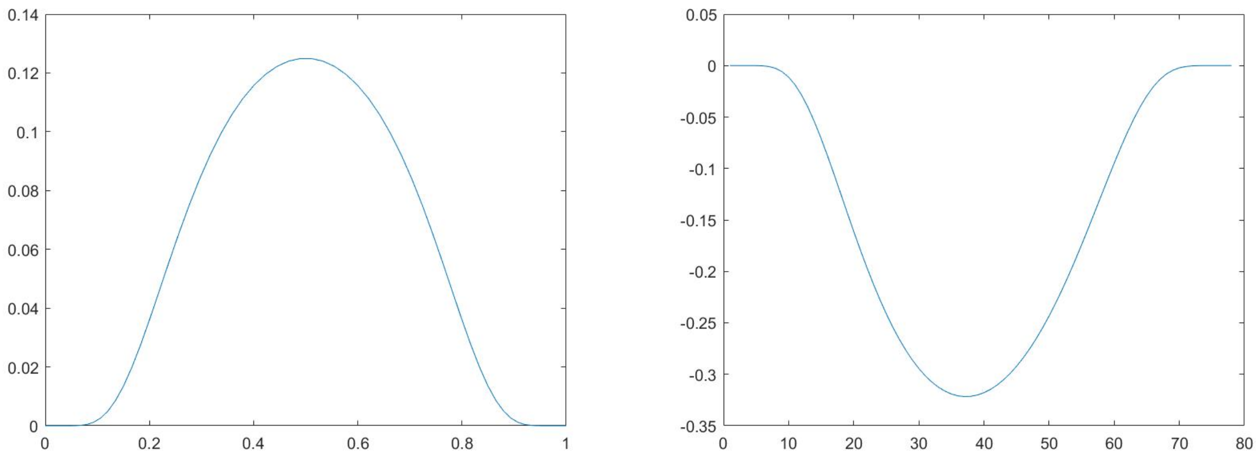

Figure 1: Perturbation $g(t)$ (left), w perturbed by $0.1g(t)$ (right).

Finally, we consider a similar perturbation in space.

$$h(x) = \frac{e^{-a/(1+x-x0)}}{e^{-a/(1+x-x0)} + e^{-a/(1-x-x0)}} \times \frac{e^{a/(1+x-x0)}}{e^{a/(1+x-x0)} + e^{a/(1-x-x0)}},$$

where an extra factor of 1 is added to the denominator to account for the space interval being $[-1, 1]$ and not $[0, 1]$. The factor a is the same as above and $x0 = t0$. The function and the effect on w can be seen in Figure 2.

Kalise

Mixed3

We are using the exact solution 'Mixed3' (second version of the Mixed BC exact solutions) with Dirichlet and Neumann BCs. All errors are measured in L2Linf, tolerances and n, N as

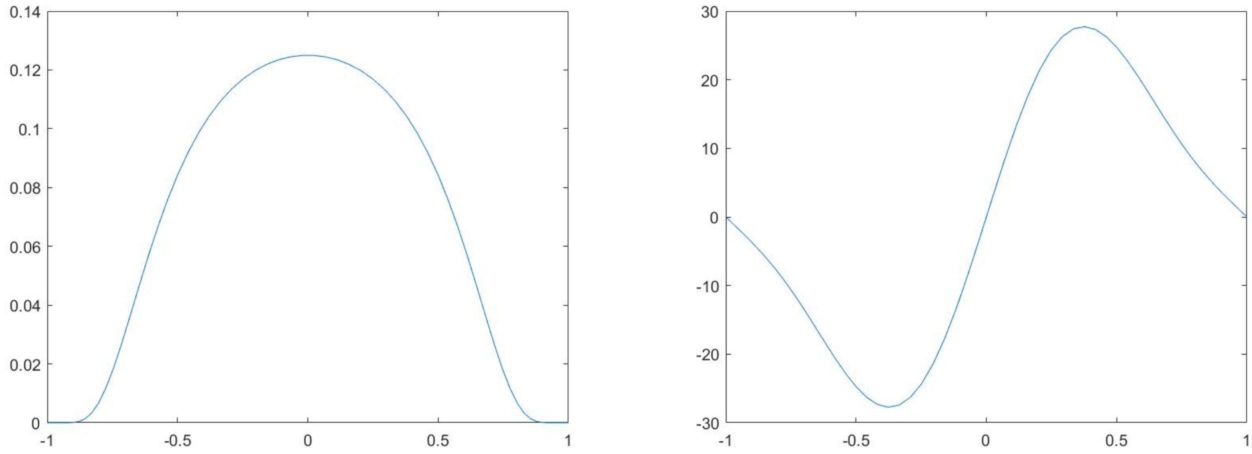

Figure 2: Perturbation $h(x)$ (left), w perturbed by $h(x)$ (right).

above. In the Dirichlet Case both $\beta = 10^{-1}$ ($\lambda = 0.1$) and $\beta = 10^{-3}$ ($\lambda = 10$) converged. The perturbation was $10g(t)$ with $a = 0.7$, see Figure 3 and 4. For $\beta = 10^{-1}$ the initial exact error was $w_{errI} = 0.0324$, $p_{errI} = 0.0557$ and $\rho_{errI} = 0.0421$. Then after 1157 iterations it converged to $w_{err} = 2.2602 \times 10^{-7}$, $p_{err} = 4.2460 \times 10^{-7}$ and $\rho_{err} = 31175 \times 10^{-7}$. The errors for $\beta = 10^{-3}$ are similar.
The results for the Neumann case are very similar. The algorithm converges for both β values. For $\beta = 10^{-1}$ the initial exact errors are $w_{ErrI} = 0.0309$, $p_{errI} = 0.0755$ and $\rho_{ErrI} = 0.0466$. Then in 1175 Iterations, this converges to $w_{err} = 2.1936 \times 10^{-7}$, $p_{Err} = 6.5744 \times 10^{-7}$ and $\rho_{err} = 3.4265 \times 10^{-7}$. The results for $\beta = 10^{-3}$ are again similar.

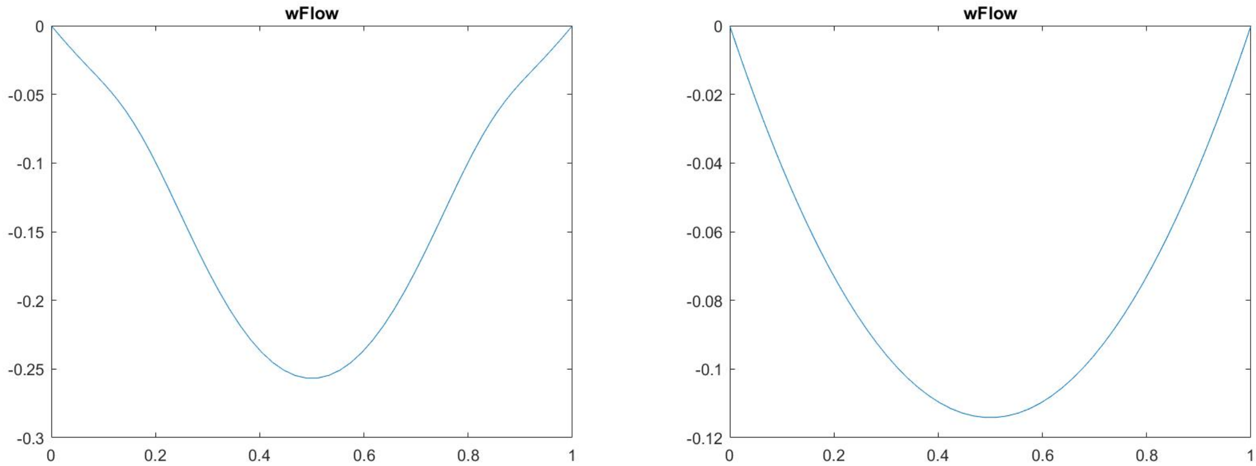

Figure 3: Perturbed w and exact w.

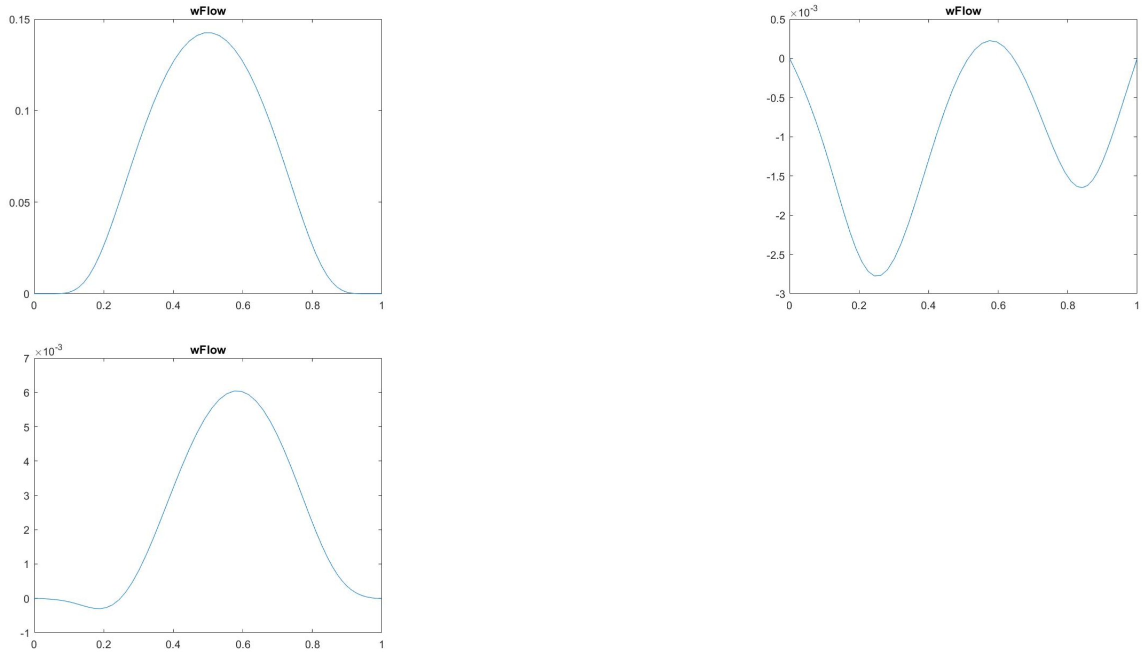

Figure 4: Perturbed w Error (first - left) (second: Dirichlet - Mid, Neumann - right).

Neumannplus2 linear time, trig space

Now, investigating Neumann (plus 2) with linear time terms (i.e. exact solution with trig functions in space, and linear time) gives the following: With $\beta = 10^{-1}$ the initial errors are $w_{errI} = 0.2353$, $p_{errI} = 0.8576$ and $\rho_{errI} = 0.3192$. This converges in 1167 Iterations to $w_{err} = 6.9449 \times 10^{-7}$, $p_{err} = 8.3209 \times 10^{-6}$ and $\rho_{err} = 2.1389 \times 10^{-6}$. The results are very similar for $\beta = 10^{-3}$ and can be seen in Figure 6. The perturbation in w can be seen in Figure 5. Note this also converges to a tolerance of 10^{-6} instead of 10^{-5}.

Dirichlet linear time, trig space (1)

Looking at Dirichlet Exact solutions with linear time terms (i.e. trig in space, linear in time), this also converges for the two β values. Accidentally the tolerance was set to 10^{-6} which converged as well ($\beta = 10^{-1}$. The perturbations in w can be seen in Figure 7. The results are $w_{err} = 8.6507 \times 10^{-7}$, $p_{err} = 2.9450 \times 10^{-7}$ and $\rho_{Err} = 10^{-7}$.
Considering $\beta + 10^{-3}$ seems to be a bit more difficult, simply because the scaling makes the perturbation much larger for this case. Therefore, different perturbations have to be tried, to see which one perturbs w a reasonable amount. It seems like even very small perturbations cannot reach convergence. This is most likely due to the fact that the Dirichlet exact solution

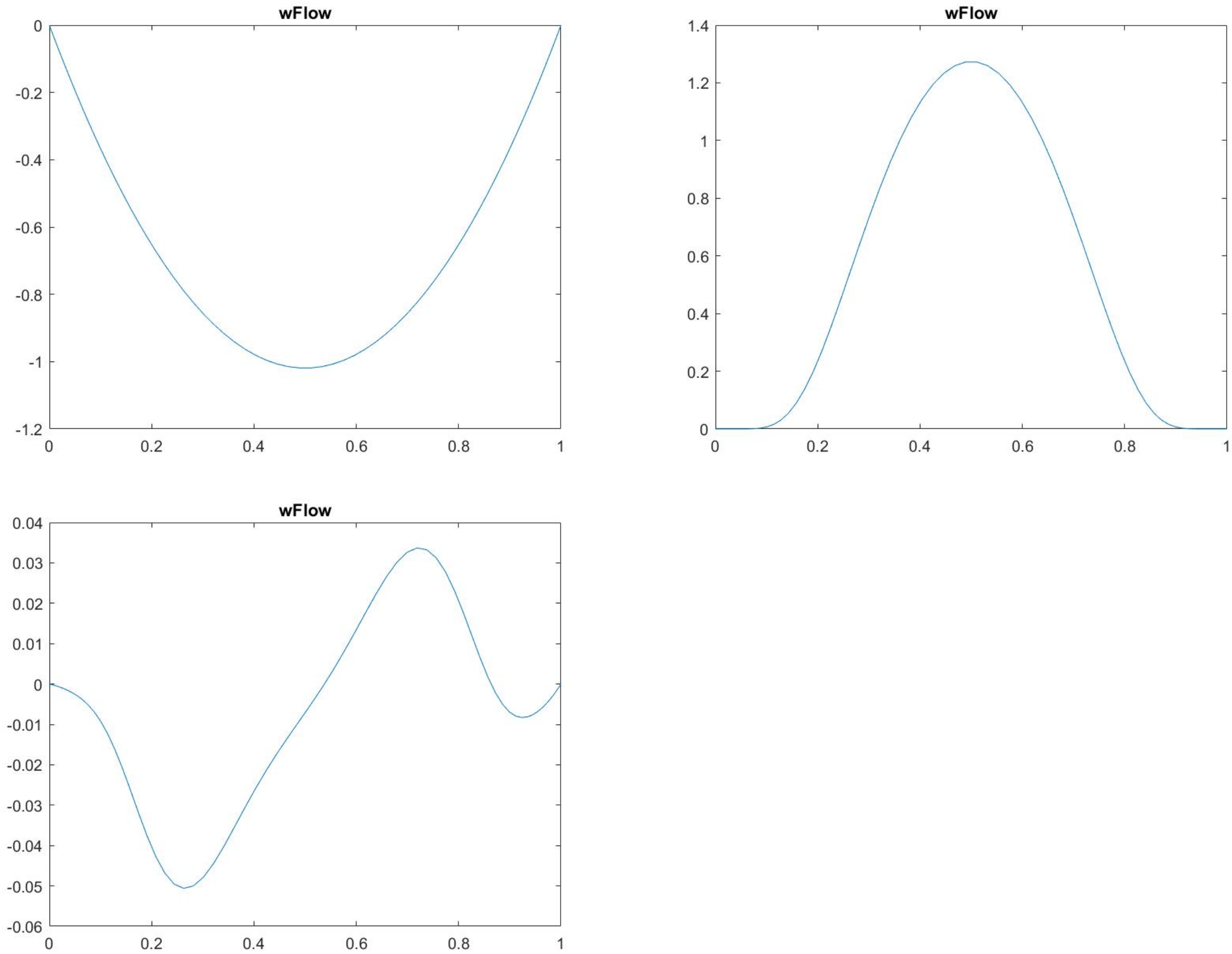

Figure 5: This is w exact (left), w first error (mid) and second error (right). Here with $\beta = 10^{-3}$ and perturbation $10g(t)$.

for w scales like $1/\beta$ (to be consistent with the original exact solutions). This is discussed next.

Dirichlet linear time, trig space (2)

Scaling $\rho Exact$ and p_{Exact} with $\beta^{1/2}$ makes w_{Exact} independent of β. Trying again $10 * g(t)$ and it can be observed that the magnitude of perturbation looks more reasonable now, see Figures 8 and 9. With $\beta = 10^{-3}$ and $\lambda = 10$, we get final errors at Iteration 1132 of $w_{err} = 4.4521 \times 10^{-7}$, $p_{err} = 5.9025 \times 10^{-7}$ and $\rho_{err} = 4.8031 \times 10^{-7}$, see Figure 10. The initial errors for this were $w_{errI} = 0.1037$, $p_{errI} = 0.0885$ and $\rho_{errI} = 0.0737$. The results for $\beta = 10^{-1}$ look basically the same.

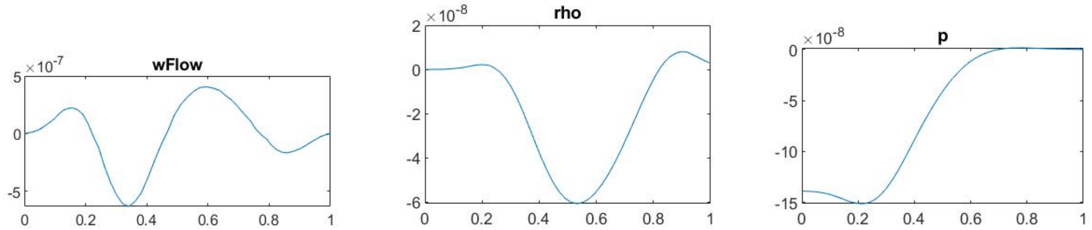

Figure 6: Final error in variables when Neumann with linear time converged. Here with $\beta = 10^{-3}$ and perturbation $10g(t)$.

Multiple Shooting

Mixed3

Using 'Mixed3' again with Dirichlet BCs, and the perturbation $10g(t)$,with $\beta = 10^{-1}$ and measured in L2Linf norm, Multiple Shooting converges within 724 Iterations. The exact errors are then $\rho_{err} = 0.00000791$ and $p_{err} = 0.00015429$, see Figure 11. The initial errors were $cons_{err} = 0.02820414$, $\rho_{ErrI} = 0.03215880$ and $p_{err} = 0.01302147$.

The algorithm also converges for Neumann BCs using $10g(t)$. The initial error is 0.01835403, $\rho_{errI} = 0.03490495$ and $p_{errI} = 0.00321550$. Then, after 481 iterations, the algorithm converges and gives $\rho_{err} = 0.00000738$ and $p_{err} = 0.00009099$.

Neumann plus2 linear t

Using $10g(t)$ and $\beta = 10^{-1}$, $\lambda = 0.1$ converges. The initial errors were 0.19989140, $\rho_{errI} = 0.25206626$ and $p_{errI} = 0.09993510$. It converges in 982 Iterations to $\rho_{err} = 0.00000003$ and $p_{err} = 0.00001598$, see Figure 12. With $\beta = 10^{-3}$ the result is basically the same, only takes double the iterations at the same λ.

Dirichlet linear t (2)

This also converges for $10g(t)$, $\beta = 10^{-1}$ and $\lambda = 10^{-1}$. The initial errors are $conc_{err} = 0.08618302$, $\rho_{errI} = 0.06506113$ and $p_{errI} = 0.06437616$. After 948 Iterations it converges and the errors are $\rho_{err} = 0.00000700$ and $p_{err} = 0.00005981$. This is similar for $\beta = 10^{-3}$.

Trying original perturbation $f(t)$ - Kalise

Just a check that the original perturbation $f(t)$ also works on this version of the Neumann problem (as it had with the exponential exact solutions). Furthermore, does the Dirichlet problem now converge with this? Yes to both. Tried Neumann with $\beta = 10^{-3}$ and it converges

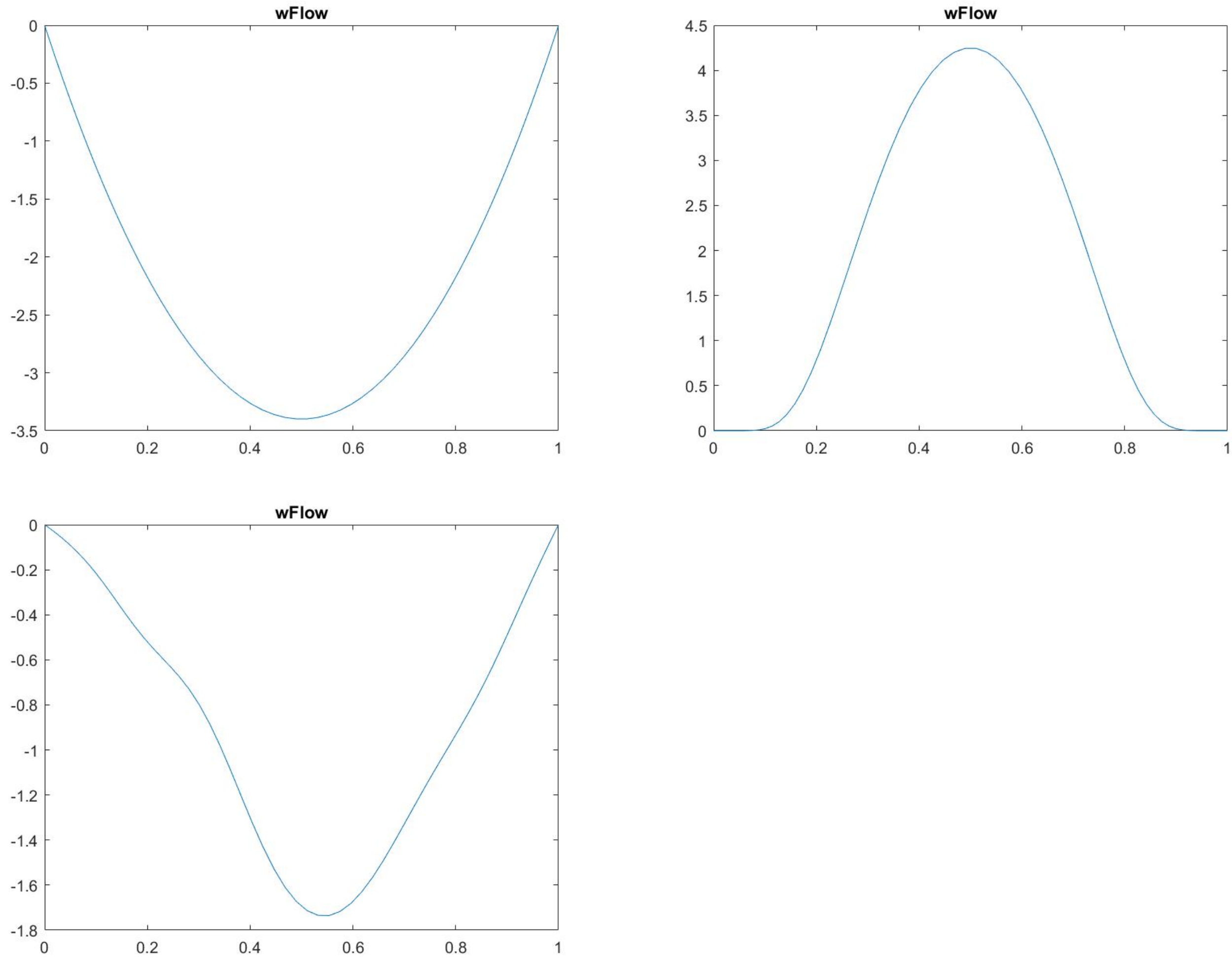

Figure 7: This is w exact (left), w first error (mid) and second error (right). Here with $\beta = 10^{-1}$ and perturbation $10g(t)$.

in 1078 Iterations from O(10^{-1}) to O(10^{-7}). Similarly, chose Dirichlet with $\beta = 10^{-1}$ and got similar convergence results for this.

Trying $g(t)h(x)$ perturbation

Kalise

Starting with Kalise and Neumann(plus2) Flow control, we choose the perturbation $100g(t)h(x)$, see first section in document. The effect of w can be seen in Figure 13. Choosing $\beta = 10^{-1}$, $\lambda = 0.1$, the initial error is $w_{errI} = 0.2627$, $p_{ErrI} = 0.7837$ and $\rho_{errI} = 0.3350$, with consistency error of $w_{cons} = 0.6782$. After 1142 Iterations this converges to $w_{err} = 7.1695 \times 10^{-7}$, $p_{err} = 8.3103 \times 10^{-6}$ and $\rho_{err} = 2.2422 \times 10^{-6}$. Choosing $\beta = 10^{-3}$ works as well. THe dirichlet case

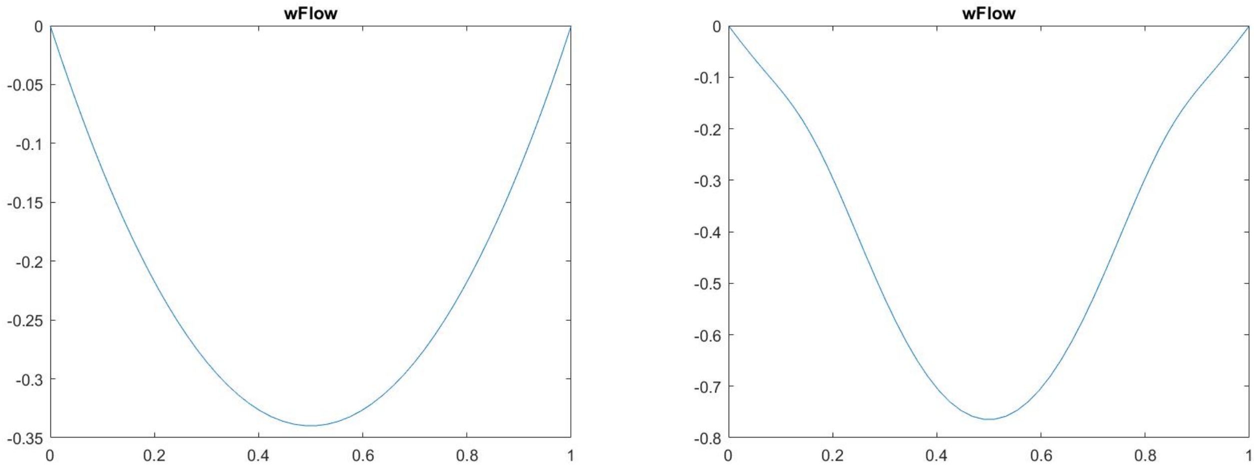

Figure 8: Exact w and perturbed w. Here with $\beta = 10^{-3}$ and perturbation $10g(t)$.

with $\beta = 10^{-1}$ converges as well in 1124 itrations and the solutions are of order $10^{-6}/10^{-7}$.
Trying a larger perturbation $100g(t)h(x)$ for the Neumann case with $\beta = 10^{-1}$ gives the two initial errors in w as seen in Figure 14. The initial errors are $w_{errI} = 0.9313$, $p_{errI} = 2.3058$ and $\rho_{ErrI} = 1.2690$. After 1355 Iterations this converges to $w_{err} = 7.0739 \times 10^{-7}$, $p_{err} = 8.3938 \times 10^{-6}$ and $\rho_{err} = 2.2085 \times 10^{-6}$.
This larger perturbation also converges for Dirichlet with $\beta = 10^{-1}$, however, the perturbations seems a bit smaller, see Figure 15 Initial errors are $w_{errI} = 0.6337$, $p_{errI} = 0.5188$ and $\rho_{errI} = 0.5428$. Then after 1345 Iterations, the errors are $w_{err} = 4.4080 \times 10^{-7}$, $p_{err} = 5.4380 \times 10^{-7}$ and $\rho_{err} = 5.2775 \times 10^{-7}$. Note that Figure 15 shows the perturbations in time, while Figure 16 shows the perturbations in space.

Multiple Shooting

Both Dirichlet and Neumann converge for $\beta = 10^{-1}$ and $100g(t)h(x)$. Need between 900 and 1000 Iterations.

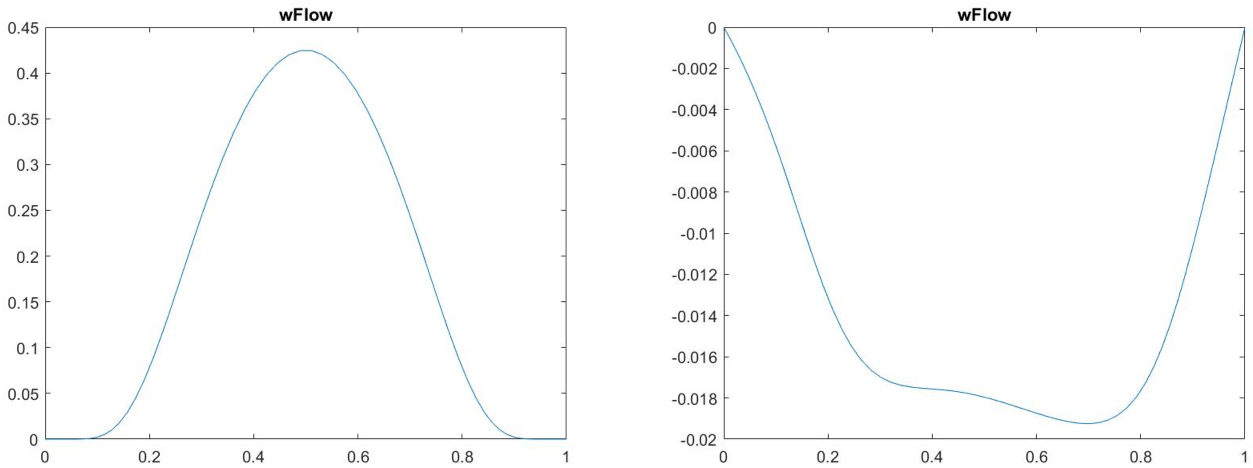

Figure 9: w error 1 (left) and 2 (right). Here with $\beta = 10^{-3}$ and perturbation $10g(t)$.

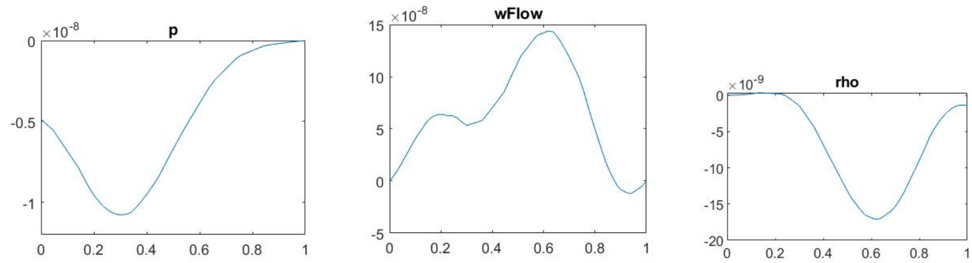

Figure 10: Exact final errors. Here with $\beta = 10^{-3}$ and perturbation $10g(t)$.

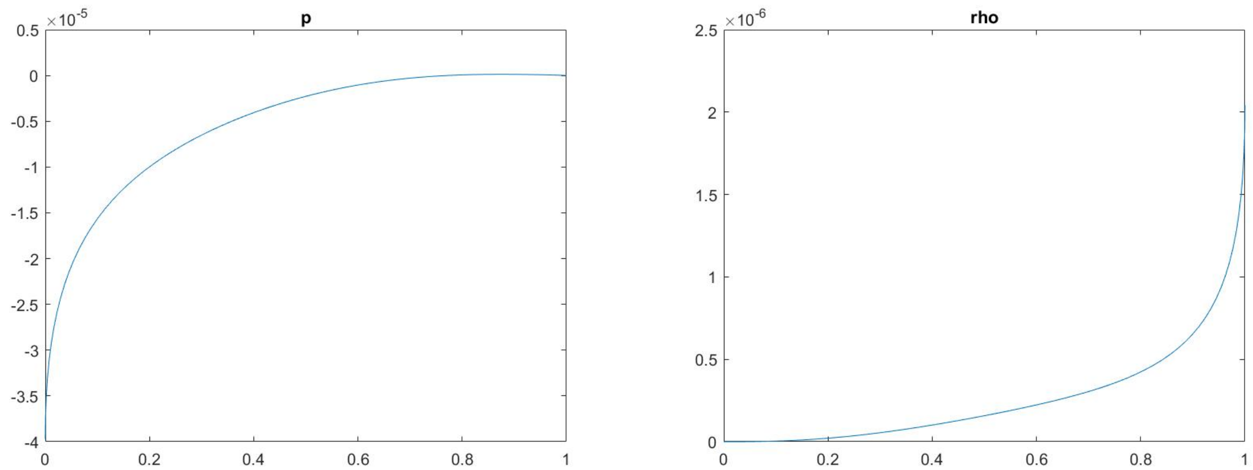

Figure 11: Final exact error in ρ and p. Here with $\beta = 10^{-1}$ and perturbation $10g(t)$.

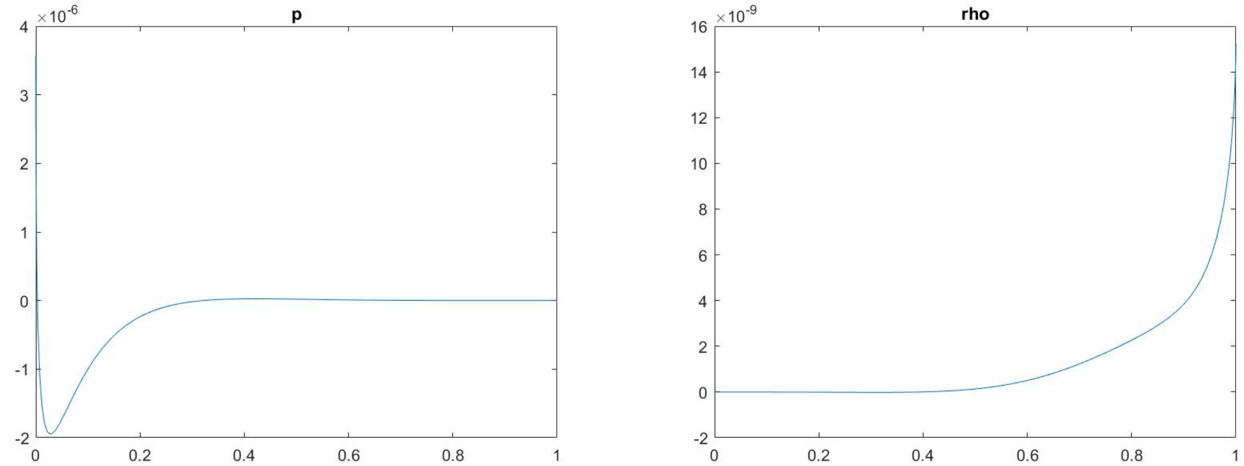

Figure 12: Final exact error in ρ and p. Here with $\beta = 10^{-1}$ and perturbation $10g(t)$.

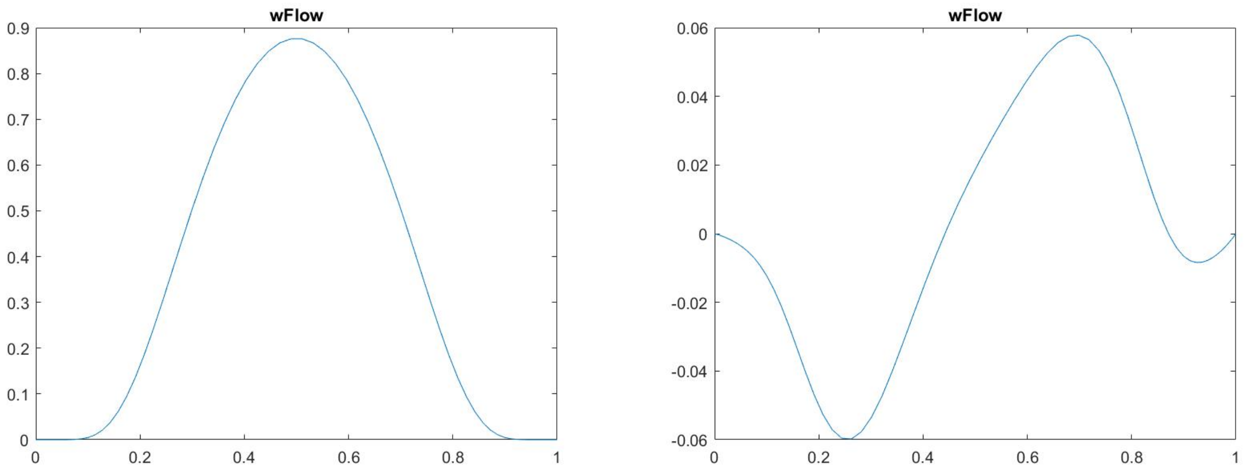

Figure 13: First and second error in w. Here with $\beta = 10^{-1}$ and perturbation $10g(t)h(x)$.

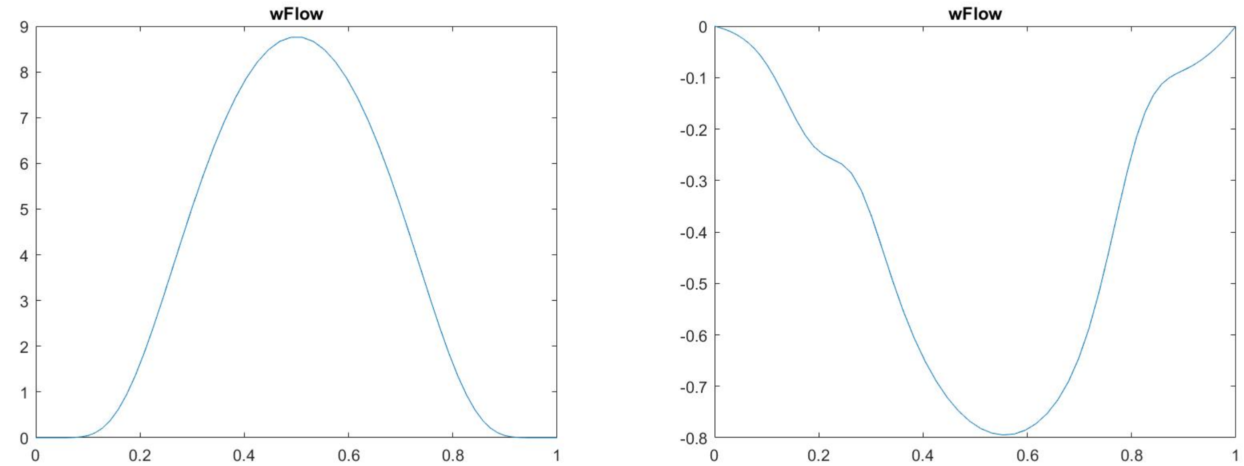

Figure 14: First and second error in w. Here with $\beta = 10^{-1}$ and perturbation $100g(t)h(x)$.

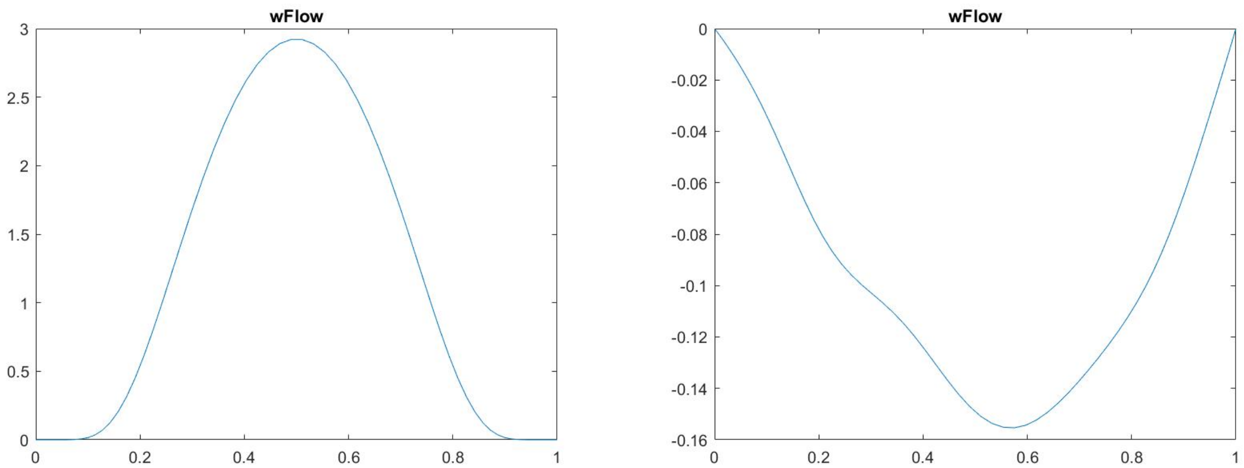

Figure 15: First and second error in w. Here with $\beta = 10^{-1}$ and perturbation $100g(t)h(x)$.

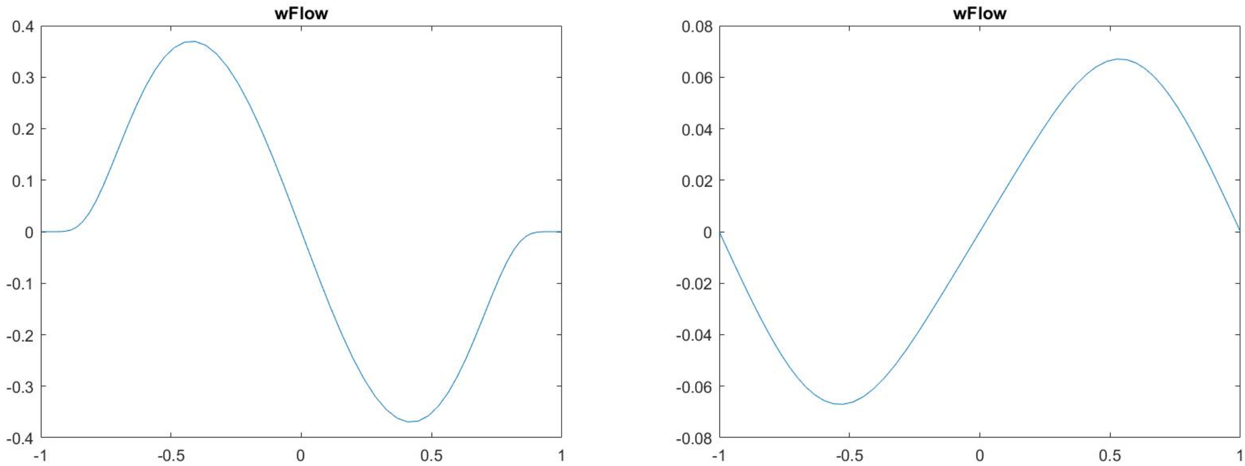

Figure 16: First and second error in w in space. Here with $\beta = 10^{-1}$ and perturbation $100g(t)h(x)$.