

## Observation on a Part of the Domain and Non-Constant Flux Boundary Conditions

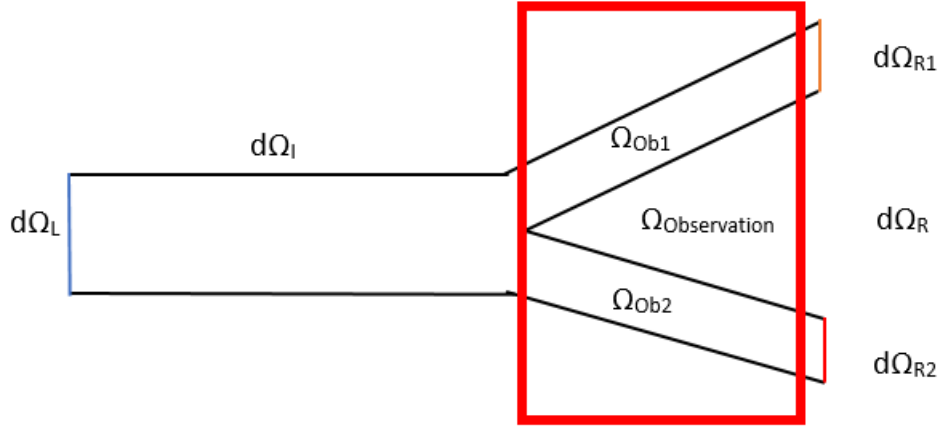


Figure 1: Domain of Interest

The problem of interest is of the form:

$$\min_{\rho, f} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(Q_{Ob})}^2 + \frac{\beta}{2} \|f\|_{L_2(Q)}^2$$

subject to:

$$\begin{aligned} \partial_t \rho &= \nabla^2 \rho - \nabla \cdot (\rho \mathbf{w}_{Flow}) + \nabla \cdot (\rho \nabla V_{ext}) + \nabla \cdot \int_{\Omega} \rho(r) \rho(r') \nabla V_2(|r - r'|) dr' + f \quad \text{in } Q, \\ \rho &= \rho_0 \quad \text{at } t = 0 \\ -\mathbf{j} \cdot \mathbf{n} &= \mathbb{1}_{\partial\Omega_L} (C_{L1} + C_{L2}\rho) + \mathbb{1}_{\partial\Omega_R} (C_{R1} + C_{R2}\rho) + \mathbb{1}_{\partial\Omega_I} 0, \quad \text{on } \partial Q, \end{aligned}$$

where  $C_{L1}, C_{L2}, C_{R1}, C_{R2}$  are constants and  $\mathbb{1}$  is the indicator function of the set (the parts of the boundary) of interest. Furthermore,  $\mathbf{j}$  satisfies:

$$\mathbf{j} = \nabla \rho - (\rho \mathbf{w}_{Flow}) + (\rho \nabla V_{ext}) + \int_{\Omega} \rho(r) \rho(r') \nabla V_2(|r - r'|) dr'.$$

Moreover, let  $\hat{\rho}$  be defined such that:

$$\hat{\rho} = \mathbb{1}_{\Omega_{Ob1}} \tilde{\rho} + \mathbb{1}_{\Omega_{Ob2}} 0.$$

## The Lagrangian

The Lagrangian is of the form:

$$\begin{aligned}\mathcal{L}(\rho, f, p_Q, p_{\partial Q}) &= \frac{1}{2} \int_0^T \int_{\Omega_{Ob}} (\rho - \hat{\rho})^2 dr dt + \frac{\beta}{2} \int_0^T \int_{\Omega} f^2 dr dt \\ &+ \int_0^T \int_{\Omega} \left( \partial_t \rho - \nabla^2 \rho + \nabla \cdot (\rho \mathbf{w}_{Flow}) - \nabla \cdot (\rho \nabla V_{ext}) + \nabla \cdot \int_{\Omega} \rho(r) \rho(r') \nabla V_2(|r - r'|) - f \right) p_Q dr dt \\ &+ \int_0^T \int_{\partial \Omega} \left( \left( -\nabla \rho + (\rho \mathbf{w}_{Flow}) - (\rho \nabla V_{ext}) - \int_{\Omega} \rho(r) \rho(r') \nabla V_2(|r - r'|) dr' \right) \cdot \mathbf{n} \right. \\ &\left. - \mathbb{1}_{\partial \Omega_L} (C_{L1} + C_{L2} \rho) - \mathbb{1}_{\partial \Omega_R} (C_{R1} + C_{R2} \rho) - \mathbb{1}_{\partial \Omega_I} 0 \right) p_{\partial Q} dr dt.\end{aligned}$$

## The Adjoint Equation

The derivative of  $\mathcal{L}$  with respect to  $\rho$  is, as taken from the extended project:

$$\begin{aligned}\mathcal{L}_{\rho}(\rho, \mathbf{w}, p_Q, p_{\partial Q})h &= \int_{\Omega} h(T) p_Q(T) dr \\ &+ \int_0^T \int_{\Omega} \left( \mathbb{1}_{\Omega_{Ob}} (\rho - \hat{\rho}) - \partial_t p_Q - \nabla p_Q \cdot \mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q \cdot \nabla V_{ext} \right. \\ &+ \int_{\Omega} (\nabla p_Q(r) + \nabla p_Q(r')) \rho(r') \nabla V_2(|r - r'|) dr' + \int_{\partial \Omega} (p_{\partial Q}(r') - p_Q(r')) \rho(r') \frac{\partial V_2(|r - r'|)}{\partial n} dr' \left. \right) h dr dt \\ &+ \int_0^T \int_{\partial \Omega} \left( \left( \frac{\partial p_Q}{\partial n} + p_Q \mathbf{w} \cdot \mathbf{n} - p_{\partial Q} \mathbf{w}_{Flow} \cdot \mathbf{n} + p_{\partial Q} \frac{\partial V_{ext}}{\partial n} - p_Q \frac{\partial V_{ext}}{\partial n} + (p_{\partial Q} - p_Q) \int_{\Omega} \rho(r') \frac{\partial V_2(|r - r'|)}{\partial n} dr' \right. \right. \\ &\left. \left. - \mathbb{1}_{\partial \Omega_L} C_{L2} p_{\partial Q} - \mathbb{1}_{\partial \Omega_R} C_{R2} p_{\partial Q} \right) h + \left( p_{\partial Q} - p_Q \right) \frac{\partial h}{\partial n} \right) dr dt = 0.\end{aligned}$$

Then, from appropriate analysis we find that:

$$p_{\partial Q} = p_Q,$$

and therefore we get:

$$\begin{aligned}\mathbb{1}_{\Omega_{Ob}} (\rho - \hat{\rho}) - \partial_t p_Q - \nabla p_Q \cdot \mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q \cdot \nabla V_{ext} \\ + \int_{\Omega} (\nabla p_Q(r) + \nabla p_Q(r')) \rho(r') \nabla V_2(|r - r'|) dr' &= 0, \quad \text{in } Q, \\ \frac{\partial p_Q}{\partial n} - \mathbb{1}_{\partial \Omega_L} C_{L2} p_Q - \mathbb{1}_{\partial \Omega_R} C_{R2} p_Q &= 0, \quad \text{on } \partial Q.\end{aligned}$$

In particular, this is:

$$\begin{aligned}\mathbb{1}_{\Omega_{Ob1}} (\rho - \hat{\rho}) + \mathbb{1}_{\Omega_{Ob2}} \rho - \partial_t p_Q - \nabla p_Q \cdot \mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q \cdot \nabla V_{ext} \\ + \int_{\Omega} (\nabla p_Q(r) + \nabla p_Q(r')) \rho(r') \nabla V_2(|r - r'|) dr' &= 0, \quad \text{in } Q, \\ \frac{\partial p_Q}{\partial n} - \mathbb{1}_{\partial \Omega_L} C_{L2} p_Q - \mathbb{1}_{\partial \Omega_R} C_{R2} p_Q &= 0, \quad \text{on } \partial Q.\end{aligned}$$