Observation on a Part of the Domain and Non-Constant Flux Boundary Conditions

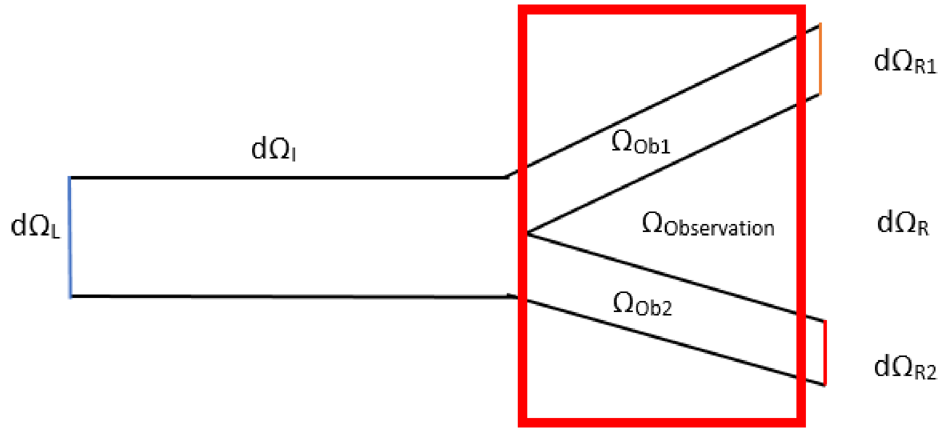

Figure 1: Domain of Interest

The problem of interest is of the form:

$$\min_{\rho, f} \quad \frac{1}{2}||\rho - \hat{\rho}||^2_{L_2(Q_{Ob})} + \frac{\beta}{2}||f||^2_{L_2(Q)}$$

subject to:

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \mathbf{w}_{Flow}) + \nabla \cdot (\rho \nabla V_{ext}) + \nabla \cdot \int_\Omega \rho(r)\rho(r')\nabla V_2(|r - r'|)dr' + f \qquad \text{in} \quad Q,$$

$$\rho = \rho_0 \quad \text{at} \quad t = 0$$

$$-\mathbf{j} \cdot \mathbf{n} = \mathbb{1}_{\partial\Omega_L}(C_{L1} + C_{L2}\rho) + \mathbb{1}_{\partial\Omega_R}(C_{R1} + C_{R2}\rho) + \mathbb{1}_{\partial\Omega_I} 0, \qquad \text{on} \quad \partial Q,$$

where C_{L1}, C_{L2}, C_{R1}, C_{R2} are constants and $\mathbb{1}$ is the indicator function of the set (the parts of the boundary) of interest. Furthermore, $\mathbf{j}$ satisfies:

$$\mathbf{j} = \nabla \rho - (\rho \mathbf{w}_{Flow}) + (\rho \nabla V_{ext}) + \int_\Omega \rho(r)\rho(r')\nabla V_2(|r - r'|)dr'.$$

Moreover, let $\hat{\rho}$ be defined such that:

$$\hat{\rho} = \mathbb{1}_{\Omega_{Ob1}}\tilde{\rho} + \mathbb{1}_{\Omega_{Ob2}} 0.$$

The Lagrangian

The Lagrangian is of the form:

$$\begin{aligned}\mathcal{L}(\rho, f, p_Q, p_{\partial Q}) = &\frac{1}{2}\int_0^T\int_{\Omega_{Ob}}(\rho-\hat{\rho})^2 drdt + \frac{\beta}{2}\int_0^T\int_\Omega f^2 drdt\\ &+\int_0^T\int_\Omega\Big(\partial_t\rho - \nabla^2\rho + \nabla\cdot(\rho\mathbf{w}_{Flow}) - \nabla\cdot(\rho\nabla V_{ext}) + \nabla\cdot\int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|) - f\Big)p_Q dr\\ &+\int_0^T\int_{\partial\Omega}\Big(\Big(-\nabla\rho + (\rho\mathbf{w}_{Flow}) - (\rho\nabla V_{ext}) - \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr'\Big)\cdot\mathbf{n}\\ &-\mathbb{1}_{\partial\Omega_L}(C_{L1}+C_{L2}\rho) - \mathbb{1}_{\partial\Omega_R}(C_{R1}+C_{R2}\rho) - \mathbb{1}_{\partial\Omega_I}0\Big)p_{\partial Q}drdt.\end{aligned}$$

The Adjoint Equation

The derivative of $\mathcal{L}$ with respect to ρ is, as taken from the extended project:

$$\begin{aligned}&\mathcal{L}_\rho(\rho, \mathbf{w}, p_Q, p_{\partial Q})h = \int_\Omega h(T)p_Q(T)dr\\ &+\int_0^T\int_\Omega\Big(\mathbb{1}_{\Omega_{Ob}}(\rho-\hat{\rho}) - \partial_t p_Q - \nabla p_Q\cdot\mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q\cdot\nabla V_{ext}\\ &+\int_\Omega(\nabla p_Q(r)+\nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' + \int_{\partial\Omega}(p_{\partial Q}(r') - p_Q(r'))\rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr'\Big)hdrdt\\ &+\int_0^T\int_{\partial\Omega}\Big(\Big(\frac{\partial p_Q}{\partial n} + p_Q\mathbf{w}\cdot\mathbf{n} - p_{\partial Q}\mathbf{w}_{Flow}\cdot\mathbf{n} + p_{\partial Q}\frac{\partial V_{ext}}{\partial n} - p_Q\frac{\partial V_{ext}}{\partial n} + (p_{\partial Q}-p_Q)\int_\Omega\rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr'\\ &-\mathbb{1}_{\partial\Omega_L}C_{L2}p_{\partial Q} - \mathbb{1}_{\partial\Omega_R}C_{R2}p_{\partial Q}\Big)h + \Big(p_{\partial Q} - p_Q\Big)\frac{\partial h}{\partial n}\Big)drdt = 0.\end{aligned}$$

Then, from appropriate analysis we find that:

$$p_{\partial Q} = p_Q,$$

and therefore we get:

$$\begin{aligned}\mathbb{1}_{\Omega_{Ob}}(\rho-\hat{\rho}) - \partial_t p_Q - \nabla p_Q\cdot\mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q\cdot\nabla V_{ext}&\\ +\int_\Omega(\nabla p_Q(r)+\nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' &= 0, \quad \text{in} \quad Q,\\ \frac{\partial p_Q}{\partial n} - \mathbb{1}_{\partial\Omega_L}C_{L2}p_Q - \mathbb{1}_{\partial\Omega_R}C_{R2}p_Q &= 0, \quad \text{on} \quad \partial Q.\end{aligned}$$

In particular, this is:

$$\begin{aligned}\mathbb{1}_{\Omega_{Ob1}}(\rho-\hat{\rho}) + \mathbb{1}_{\Omega_{Ob2}}\rho - \partial_t p_Q - \nabla p_Q\cdot\mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q\cdot\nabla V_{ext}&\\ +\int_\Omega(\nabla p_Q(r)+\nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' &= 0, \quad \text{in} \quad Q,\\ \frac{\partial p_Q}{\partial n} - \mathbb{1}_{\partial\Omega_L}C_{L2}p_Q - \mathbb{1}_{\partial\Omega_R}C_{R2}p_Q &= 0, \quad \text{on} \quad \partial Q.\end{aligned}$$