We have:

$$F[\rho] = \frac{1}{\beta}\int \rho(\ln \Lambda^2 \rho) - 2\rho - \rho\ln(1-\eta) + \frac{\rho}{1-\eta}dr$$
$$\eta = a\rho = \frac{\pi\sigma^2}{4}\rho$$

Taking the functional derivative of F gives:

$$\begin{aligned}\frac{\delta F[\rho]}{\delta\rho} &= \frac{1}{\beta}\left(1 + \ln\rho + \Lambda^2 - 2 - \ln(1-\eta) + a\frac{\rho}{1-\eta} + \frac{1}{1-\eta} + a\frac{\rho}{(1-\eta)^2}\right)\\ &= \frac{1}{\beta}\left(1 + \ln\rho + \Lambda^2 - 2 - \ln(1-\eta) + \frac{1}{(\eta-1)^2} - \frac{1}{\eta-1} - 1\right)\\ &= \frac{1}{\beta}\left(\ln\rho + \Lambda^2 - 2 - \ln(1-\eta) - \frac{\eta-2}{(\eta-1)^2}\right),\end{aligned}$$

using partial fractions.

$$\begin{aligned}\nabla\frac{\delta F[\rho]}{\delta\rho} &= \frac{1}{\beta}\left(\nabla\ln\rho + \nabla(\Lambda^2 - 2) - \nabla\ln(1-\eta) - \nabla\frac{\eta-2}{(\eta-1)^2}\right)\\ &= \frac{1}{\beta}\left(\frac{\nabla\rho}{\rho} - \frac{\nabla(1-\eta)}{1-\eta} - \nabla\frac{\eta-2}{(\eta-1)^2}\right)\\ &= \frac{1}{\beta}\left(\frac{\nabla\rho}{\rho} + \frac{\nabla\eta}{1-\eta} - \nabla\frac{\eta-2}{(\eta-1)^2}\right)\end{aligned}$$

Then multiplying by ρ gives:

$$\begin{aligned}\rho\nabla\frac{\delta F[\rho]}{\delta\rho} &= \frac{1}{\beta}\left(\nabla\rho + \frac{\rho\nabla\eta}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\\ &= \frac{1}{\beta}\left(\nabla\rho + \frac{\eta\nabla\rho}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\\ &= \frac{1}{\beta}\left(\nabla\rho + \frac{\nabla\rho}{1-\eta} - \nabla\rho - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\\ &= \frac{1}{\beta}\left(\frac{\nabla\rho}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\end{aligned}$$

Finally we take the divergence:

$$\begin{aligned}\nabla\cdot\left(\rho\nabla\frac{\delta F[\rho]}{\delta\rho}\right) &= \frac{1}{\beta}\left(\nabla\cdot\left(\frac{\nabla\rho}{1-\eta}\right) - \nabla\cdot\left(\rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\right)\\ &= \frac{1}{\beta}\left(\frac{\nabla^2\rho}{1-\eta} + \nabla\rho\cdot\nabla\frac{1}{1-\eta} - \nabla\rho\cdot\nabla\frac{\eta-2}{(\eta-1)^2} - \rho\nabla^2\frac{\eta-2}{(\eta-1)^2}\right)\\ &= \frac{1}{\beta}\left(\frac{\nabla^2\rho}{1-\eta} + \nabla\rho\cdot\nabla\frac{(3-2\eta)}{(1-\eta)^2} - \rho\nabla^2\frac{\eta-2}{(\eta-1)^2}\right)\end{aligned}$$