++ maybe Dirichlet non-zero (i.e. positive) and scale to 1? +++

1 Green's theorem stuff

We wrote the terms in * and ** into vector form as (*, **), but it should have been * + **. I think it has to be a sum, since the term we start out with in the cost functional is the scalar quantity

$$\begin{aligned}\frac{1}{2}\int_\Omega \nabla\times\mathbf{w}^2 d\mathbf{x} &= \frac{1}{2}\int_\Omega\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)^2 d\mathbf{x}\\ &= \int_\Omega \frac{1}{2}\frac{\partial w_2}{\partial x_1}\frac{\partial w_2}{\partial x_1}-\frac{\partial w_2}{\partial x_1}\frac{\partial w_1}{\partial x_2}+\frac{1}{2}\frac{\partial w_1}{\partial x_2}\frac{\partial w_2}{\partial x_1}d\mathbf{x}.\end{aligned}$$

We take the derivative with respect to h_1, h_2

$$\begin{aligned}&\int_\Omega \frac{\partial h_2}{\partial x_1}\frac{\partial w_2}{\partial x_1}-\frac{\partial h_2}{\partial x_1}\frac{\partial w_1}{\partial x_2}-\frac{\partial w_2}{\partial x_1}\frac{\partial h_1}{\partial x_2}+\frac{\partial h_1}{\partial x_2}\frac{\partial w_1}{\partial x_2}d\mathbf{x}\\ &=\int_\Omega \frac{\partial h_2}{\partial x_1}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)+\frac{\partial h_1}{\partial x_2}\left(\frac{\partial w_1}{\partial x_2}-\frac{\partial w_2}{\partial x_1}\right)d\mathbf{x}\\ &=\int_\Omega \frac{\partial}{\partial x_1}\left(h_2\frac{\partial w_2}{\partial x_1}-h_2\frac{\partial w_1}{\partial x_2}\right)-\left(h_2\frac{\partial^2 w_2}{\partial x_1^2}-h_2\frac{\partial^2 w_1}{\partial x_1 x_2}\right)\\ &\quad+\frac{\partial}{\partial x_2}\left(h_1\frac{\partial w_1}{\partial x_2}-h_1\frac{\partial w_2}{\partial x_1}\right)-\left(h_1\frac{\partial^2 w_1}{\partial x_2^2}-h_1\frac{\partial^2 w_2}{\partial x_1 x_2}\right)d\mathbf{x}.\end{aligned}$$

Applying Green's theorem, we get

$$\begin{aligned}&\int_{\partial\Omega}\left(h_2\frac{\partial w_2}{\partial x_1}-h_2\frac{\partial w_1}{\partial x_2}\right)dx_1+\left(h_1\frac{\partial w_1}{\partial x_2}-h_1\frac{\partial w_2}{\partial x_1}\right)dx_2\\ &-\int_\Omega\left(h_2\frac{\partial^2 w_2}{\partial x_1^2}-h_2\frac{\partial^2 w_1}{\partial x_1 x_2}\right)+\left(h_1\frac{\partial^2 w_1}{\partial x_2^2}-h_1\frac{\partial^2 w_2}{\partial x_1 x_2}\right)d\mathbf{x}.\end{aligned}$$

Rewriting the boundary terms we get

$$\begin{aligned}&\int_{\partial\Omega}\nabla\times\mathbf{w}h_2 dx_1-\nabla\times\mathbf{w}h_1 dx_2\\ &-\int_\Omega\left(h_2\frac{\partial^2 w_2}{\partial x_1^2}-h_2\frac{\partial^2 w_1}{\partial x_1 x_2}\right)+\left(h_1\frac{\partial^2 w_1}{\partial x_2^2}-h_1\frac{\partial^2 w_2}{\partial x_1 x_2}\right)d\mathbf{x}.\end{aligned}$$

Green's Theorem can be written as

$$\int_{\partial\Omega}Ldx+Mdy=\int_{\partial\Omega}(M,-L)\cdot(dy,-dx)=\int_{\partial\Omega}(M,-L)\cdot\mathbf{n}ds,$$

where $ds = \sqrt{dx^2 + dy^2}$ and $\mathbf{n}$ chosen such that it is normalized and perpendicular to (dx, dy). Therefore, we have

$$\int_{\partial\Omega} \nabla \times \mathbf{w} h_2 dx_1 - \nabla \times \mathbf{w} h_1 dx_2 = \int_{\partial\Omega} \nabla \times \mathbf{w} \left(-h_1, -h_2\right) \cdot \left(dx_2, -dx_1\right)$$
$$= \int_{\partial\Omega} \nabla \times \mathbf{w} \left(-h_1, -h_2\right) \cdot \left(\frac{dx_2}{ds}, -\frac{dx_1}{ds}\right) ds,$$

Consider the boundary term from my derivation

$$\int_{\partial\Omega} \left(\nabla \times \mathbf{w}\right) \mathbf{h}_\perp \cdot \mathbf{n} ds$$
$$= \int_{\partial\Omega} \left(\nabla \times \mathbf{w}\right) \left(h_2, -h_1\right) \cdot (n_1, n_2) ds. \tag{1}$$

These two formulations agree if $n_1 = \frac{dx_1}{ds}$ and $n_2 = \frac{dx_2}{ds}$. However, this doesn't quite make sense to me, since these are not normal to dx_1 and dx_2.

1.1 Quick reminder on what I've done

We know that

$$\nabla \times \mathbf{w} = \nabla \cdot \mathbf{w}_\perp,$$

where $\mathbf{w}_\perp = (w_2, -w_1)$, the result of a rotation of $\mathbf{w}$ by $\pi/2$. The Lagrangian included a term of the form

$$\mathcal{L}(\rho, \mathbf{w}, q) = ... \frac{\eta}{2} \int_0^T \int_\Omega \left(\nabla \cdot \mathbf{w}_\perp\right)^2 d\mathbf{x} dt...$$

For the adjoint equation, we find the usual results. We take the derivative with respect to $\mathbf{w}$

$$\mathcal{L}_{\mathbf{w}}(\rho, \mathbf{w}, q)\mathbf{h} = \int_0^T \int_\Omega ... + \eta \left(\nabla \cdot \mathbf{h}_\perp\right) \left(\nabla \cdot \mathbf{w}_\perp\right) d\mathbf{x} dt.$$

Then we integrate by parts (or divergence theorem) to get

$$\mathcal{L}_{\mathbf{w}}(\rho, \mathbf{w}, q)\mathbf{h} = \int_0^T \int_\Omega .... - \eta \nabla \left(\nabla \cdot \mathbf{w}_\perp\right) \cdot \mathbf{h}_\perp d\mathbf{x} dt + \int_0^T \int_{\partial\Omega} \eta \left(\nabla \cdot \mathbf{w}_\perp\right) \mathbf{h}_\perp \cdot \mathbf{n} d\mathbf{x} dt.$$

2 Paper Examples

Source Control Neumann

We choose

$$\begin{aligned}
\rho_0 &= 0.25 \\
V_{ext} &= 1.5 \sin(\pi x_2/5) \cos(\pi x_1/5 + \pi/5) \\
\widehat{\rho} &= 0.25(1-t) + t(0.25 \sin(\pi(x_1 - 2)/2) \sin(\pi(x_2 - 2)/2) + 0.25)
\end{aligned}$$

We choose the domain $[-1, 1]^2$ with a time horizon $(0, 1)$. We have $N = 20$, $n = 11$. With $\beta = 10^{-3}$ we solve this problem in 333 seconds. When $\kappa = 1$, the cost functional is $J = 0.015$, $J_1 = 0.021$ and $J_2 = 0.9946$. (We compare this against the result with $\beta = 10^3$, where we get $J = 0.0153$. For $\beta = 10^{-5}$ we get $J = 8.5366 \times 10^{-4}$, see Figure 4.) Figure 2 displays the results, Figure 1 shows the external potential acting on the particles.

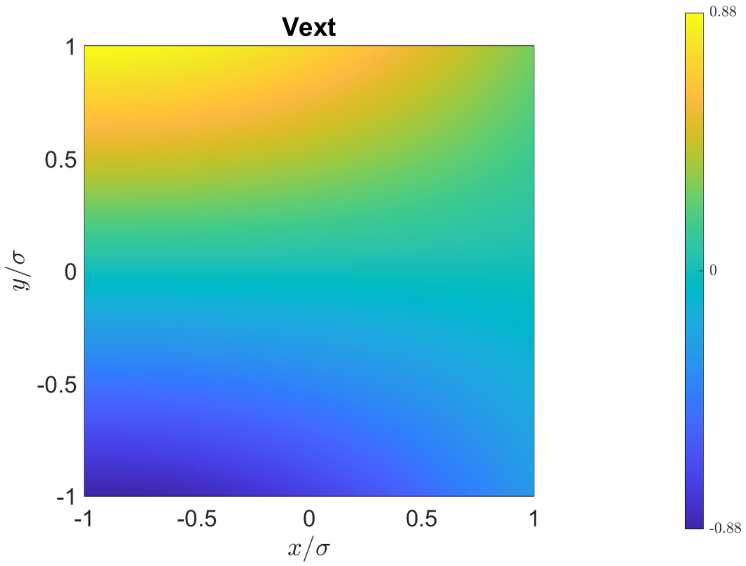

Figure 1: V_{ext} corresponding to results in Figure 2.

For $\kappa = -1$ we get $J = 0.013$, $J_1 = 0.017$ and $J_2 = 0.8846$. The result can be seen in Figure 3.(We compare this against the result with $\beta = 10^3$, where we get $J = 0.0171$. For $\beta = 10^{-5}$ we get $J = 8.0800 \times 10^{-4}$, see Figure 5.) We can see that the control behaviour for $\kappa = -1$ is very different to the one for $\kappa = 1$, since the repulsion supports the drive towards $\widehat{\rho}$, while the attraction counteracts it. For both cases we can observe that the control works harder in regions with steep external potential.

Source Control Dirichlet

We choose

$$\rho_0 = 0.25 \cos(\pi x_1/2) \cos(\pi x_2/2)$$
$$V_{ext} = (1 - t)(-\cos(\pi x_1/2) \sin(\pi x_2/2) + 1)$$
$$\widehat{\rho} = 0.25 \cos(\pi x_1/2) \cos(\pi x_2/2)(1 - t) - t \sin(\pi x_1) \sin(\pi x_2/2 - \pi/2)$$

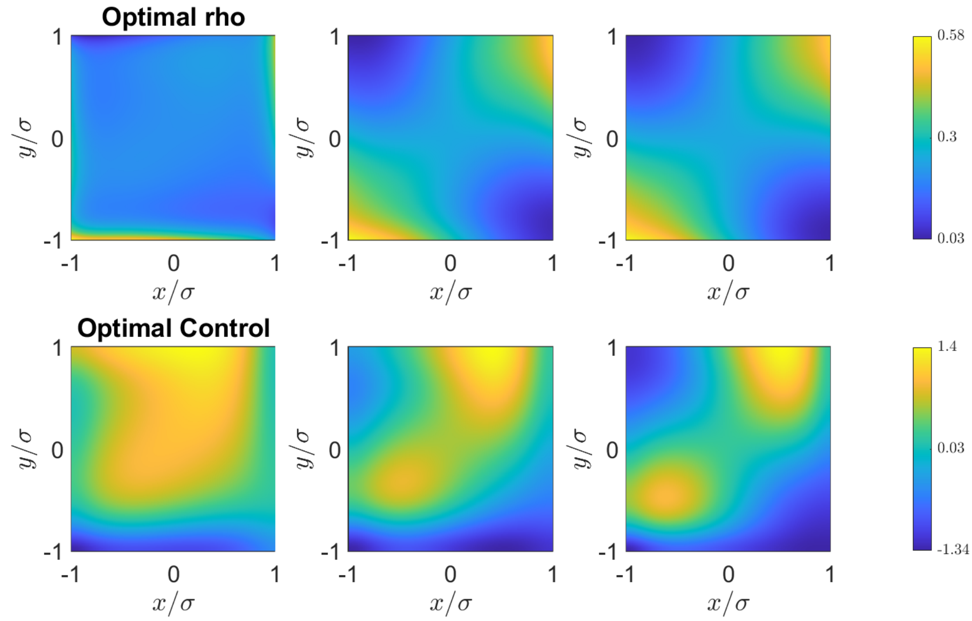

Figure 2: Optimal ρ and cost w, $\kappa = 1$, $\beta = 10^{-3}$

We choose the domain $[-1, 1]^2$ with a time horizon $(0, 1)$. We have $N = 20$, $n = 11$. For $\beta = 10^{-3}$, $\kappa = 1$ we get $J = 0.0268$, $J_1 = 0.0096$ and $J_2 = 43.9790$. (vs $\beta = 10^3$, where $J = 0.1709$.) For $\beta = 10^{-3}$, $\kappa = 1$ we get $J = 0.0257$, $J_1 = 0.0089$ and $J_2 = 42.4812$. (vs $\beta = 10^3$, where $J = 0.1707$.) Both example take around 60 to 80 seconds to solve.

Flow Control Neumann

We choose

$$
\begin{aligned}
\rho_0 &= 0.25 \\
V_{ext} &= ((x_1 + 0.3)^2 - 1)((x_1 - 0.4)^2 - 0.5)((x_2 + 0.3)^2 - 1)((x_2 - 0.4)^2 - 0.5) \\
\widehat{\rho} &= (1 - t)0.25 + t(1/1.3791)\exp\left(-2((x_1 + 0.2)^2 + (x_2 + 0.2)^2)\right)
\end{aligned}
$$

We choose the domain $[-1, 1]^2$ with a time horizon $(0, 1)$. We have $N = 20$, $n = 11$. For $\beta = 10^{-3}$, $\kappa = 1$ we get $J = 0.0059$, $J_1 = 0.0033$ and $J_2 = 8.4178$ (compare to $\beta = 10^3$ with $J = 0.0336$). For $\kappa = -1$ we get $J = 0.0030$, $J_1 = 0.0019$ and $J_2 = 4.1185$ (compare to $\beta = 10^3$ with $J = 0.0214$).

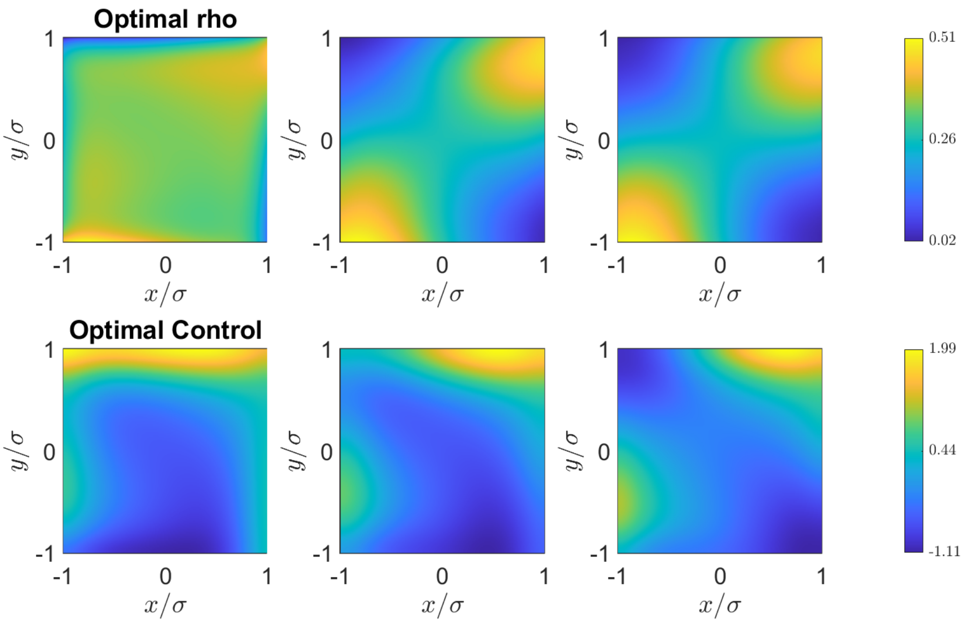

Figure 3: Optimal ρ and cost w, $\kappa = -1$, $\beta = 10^{-3}$.

Flow Control Dirichlet

For this example we need more points. In the following $N = 25$, but that's still not enough. We choose

$$\rho_0 = (0.25\pi)^2 \cos(\pi x_1/2)\cos(\pi x_2/2)$$

$$\widehat{\rho} = (1-t)(0.25\pi)^2 \cos(\pi x_1/2)\cos(\pi x_2/2) + t(0.5(0.25\pi)^2 \sin(\pi x_1)\sin(\pi x_2/2 - \pi/2))$$

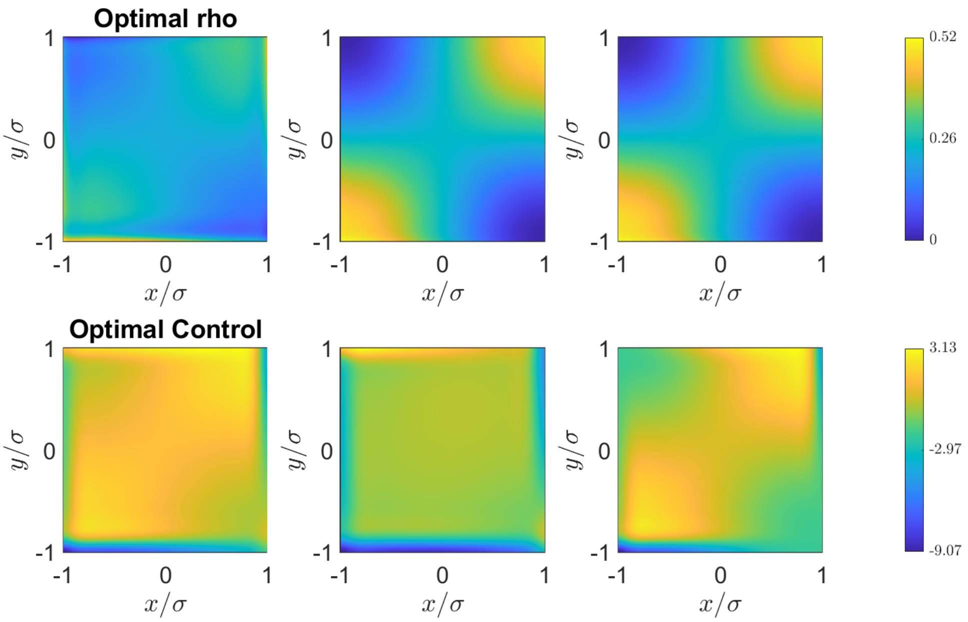

Figure 4: Optimal ρ and cost w, $\kappa = 1$, $\beta = 10^{-5}$.

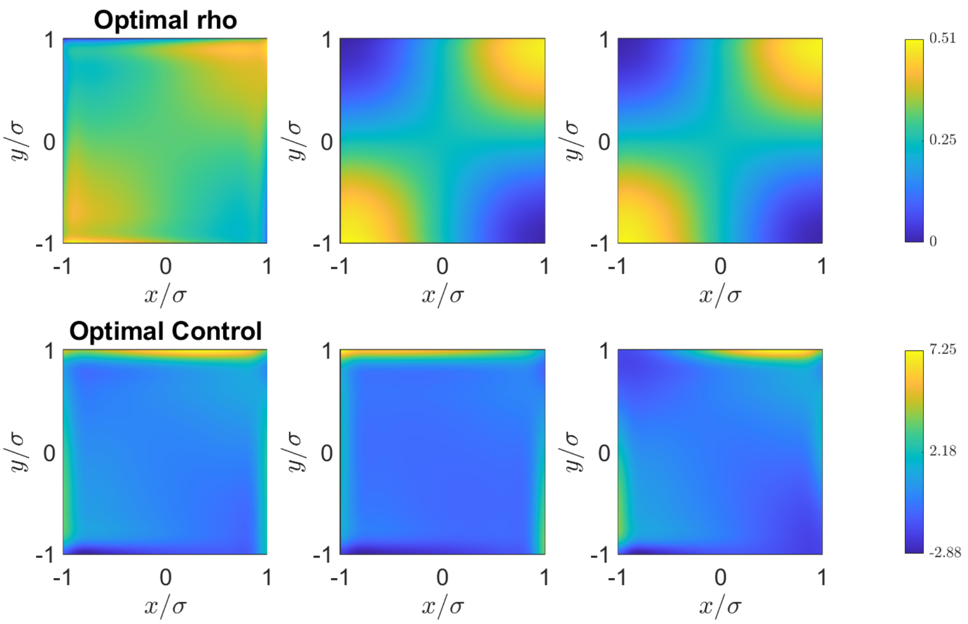

Figure 5: Optimal ρ and cost w, $\kappa = -1$, $\beta = 10^{-5}$.

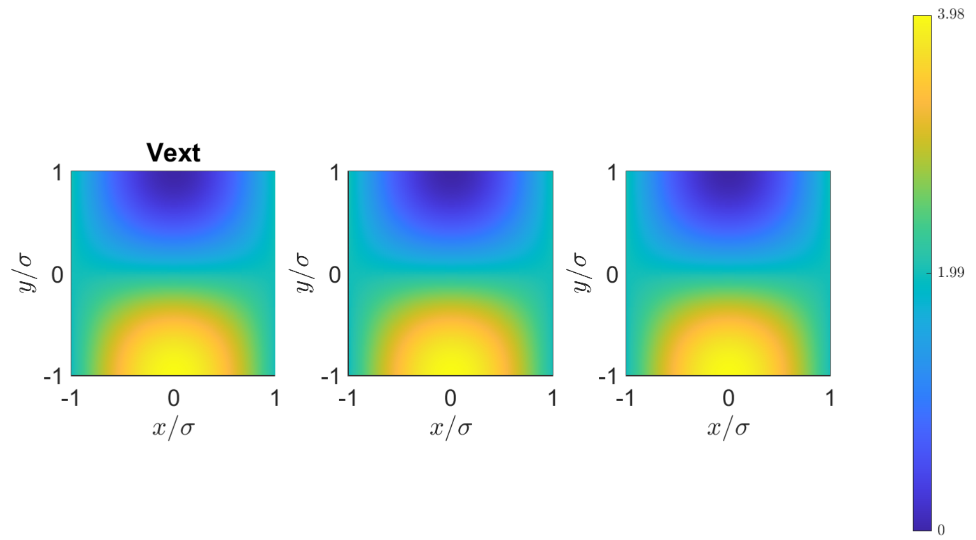

Figure 6: V_{ext} corresponding to results in Figure 7.

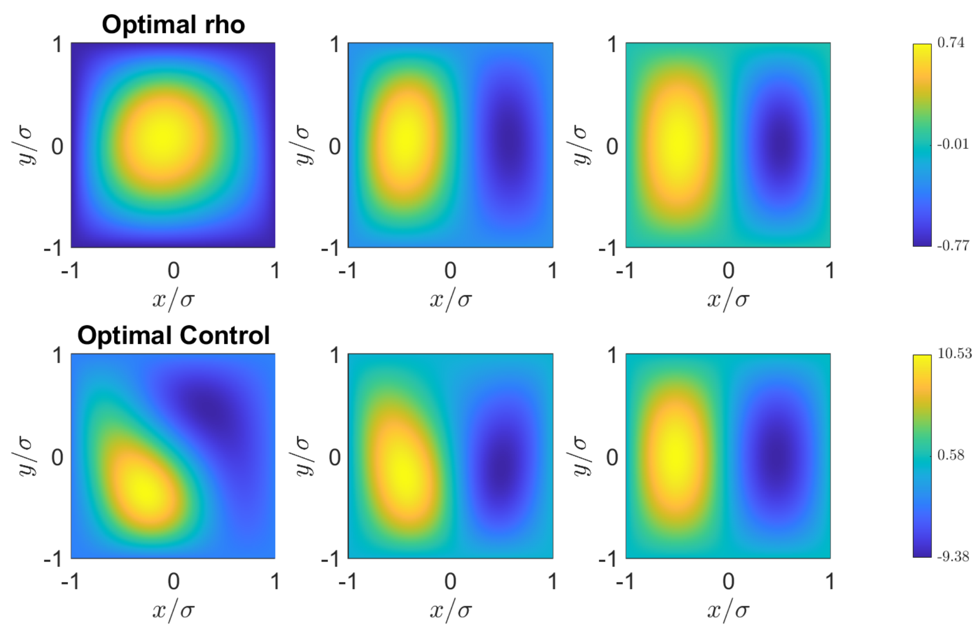

Figure 7: Optimal ρ and cost w, $\kappa = 1$.

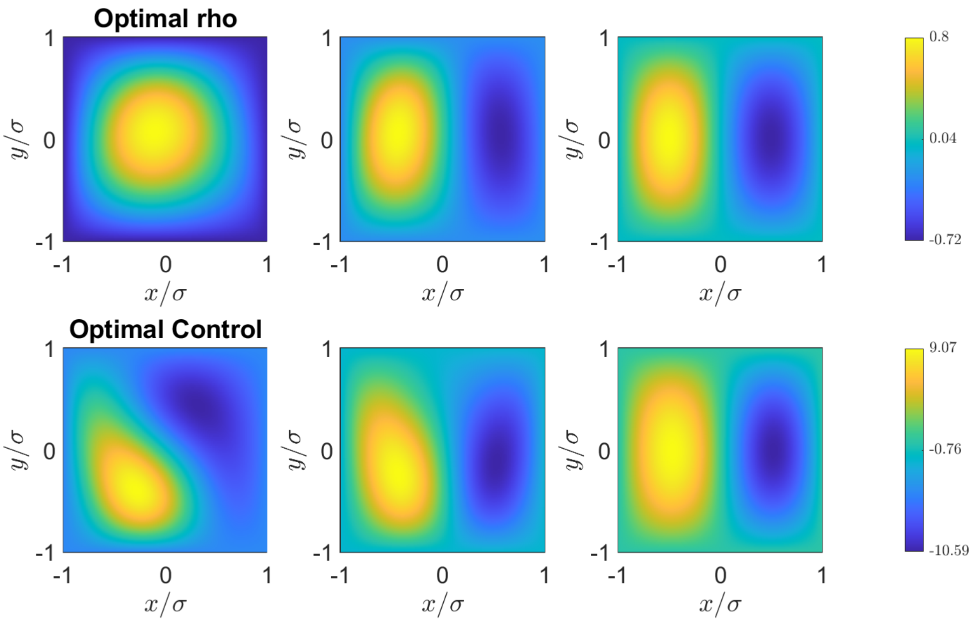

Figure 8: Optimal ρ and cost w, $\kappa = -1$.

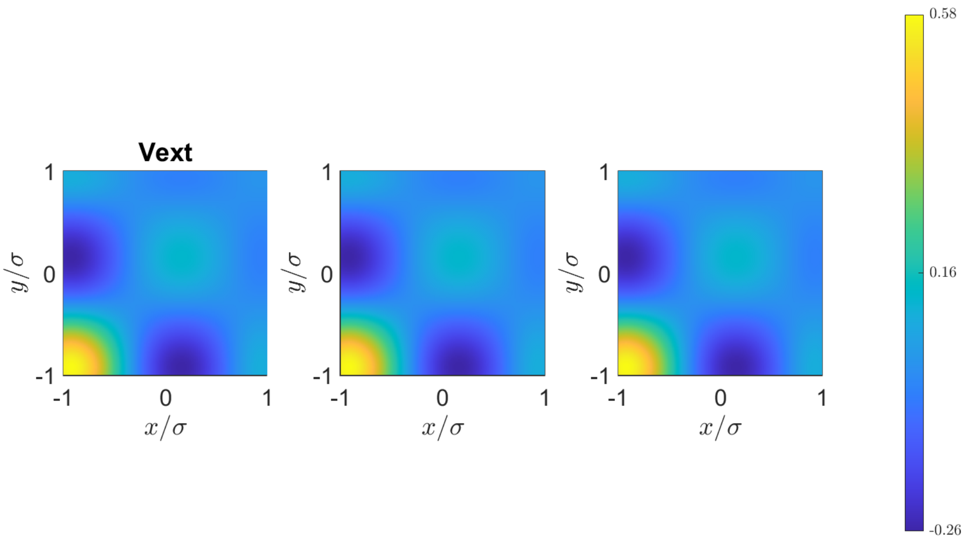

Figure 9: V_{ext} corresponding to results in Figure 10.

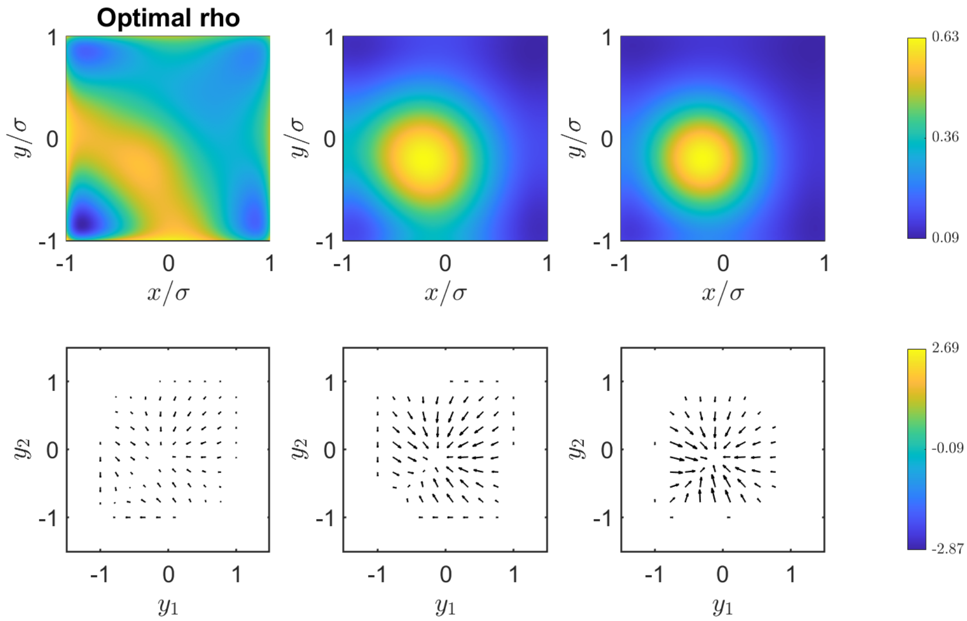

Figure 10: Optimal ρ and cost w, $\kappa = 1$.

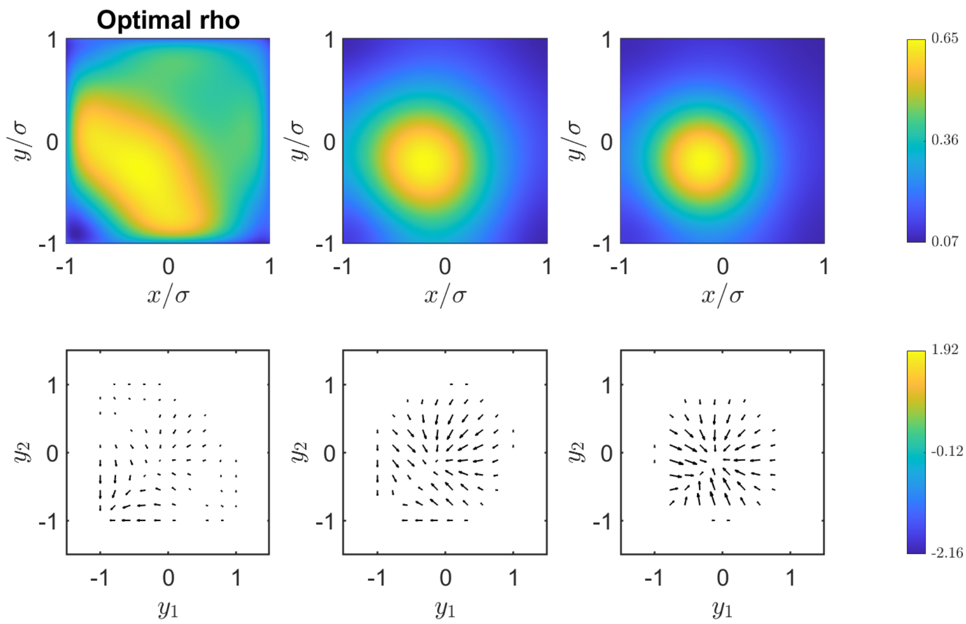

Figure 11: Optimal ρ and cost w, $\kappa = -1$.

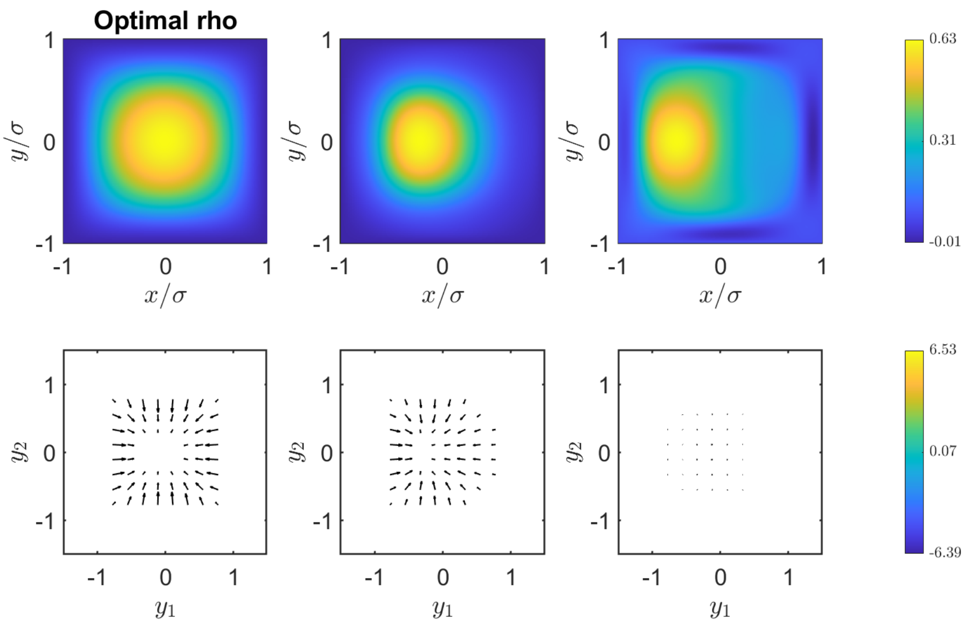

Figure 12: Optimal ρ and cost w, $\kappa = 1$.

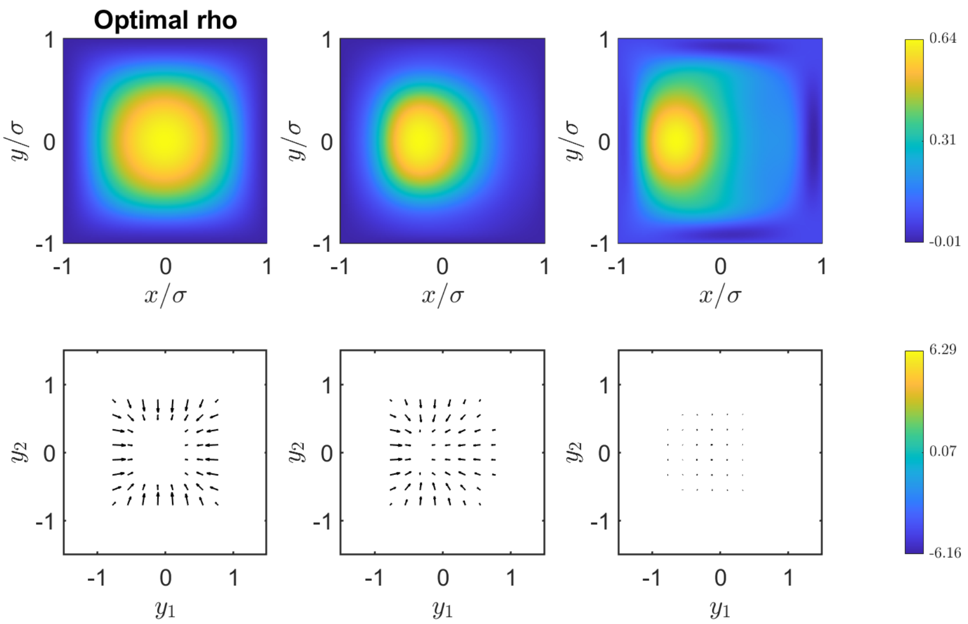

Figure 13: Optimal ρ and cost w, $\kappa = -1$.

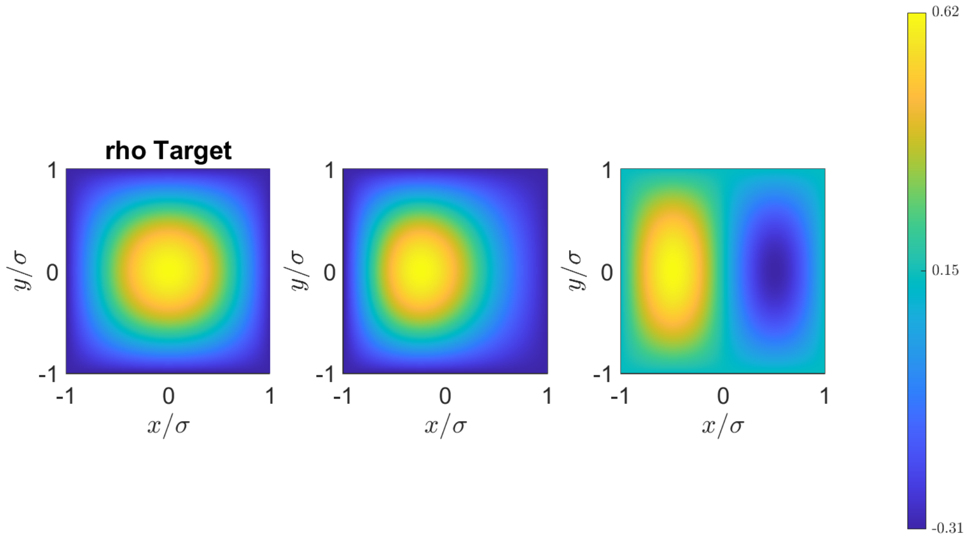

Figure 14: Target ρ.