1 Optimality conditions for the sedimentation equations

We consider the terms of the PDE and the boundary conditions separately here.

1.1 Terms that go into the PDE

The relevant part of the PDE is:

$$\nabla \cdot \left(\rho \nabla \frac{\delta F[\rho]}{\delta \rho}\right) = \frac{1}{\beta}\left(\frac{\nabla^2 \rho}{1-\eta} + \nabla \rho \cdot \nabla \frac{(3-2\eta)}{(1-\eta)^2} - \rho \nabla^2 \frac{\eta - 2}{(\eta-1)^2}\right),$$

where $\eta = a\rho$ and $a = \pi\sigma^2/4$. Consider:

$$\begin{aligned}
F_1(\rho) &= \nabla^2 \rho \frac{1}{1-a\rho} \\
F_2(\rho) &= \nabla \rho \cdot \nabla \left(\frac{3-2a\rho}{(1-a\rho)^2}\right) \\
F_3(\rho) &= \rho \nabla^2 \left(\frac{a\rho - 2}{(a\rho-1)^2}\right)
\end{aligned}$$

Then

$$F_1(\rho + h) - F_1(\rho) = \nabla(\rho + h)\frac{1}{1-a(\rho+h)} - \nabla \rho \frac{1}{1-a\rho}$$

Using the expansion:

$$\frac{1}{c-x} = \frac{1}{c} + \frac{1}{c^2}x + O(x^2),$$

where $c = 1 - a\rho$, we get:

$$\begin{aligned}
F_1(\rho + h) - F_1(\rho) &= \nabla^2(\rho + h)\left(\frac{1}{1-a\rho} + \frac{a}{(1-a\rho)^2}h\right) - \nabla^2 \rho \frac{1}{1-a\rho} \\
&= \nabla^2 h \left(\frac{1}{1-a\rho}\right) + \nabla^2 \rho \left(\frac{a}{(1-a\rho)^2}h\right)
\end{aligned}$$

For F_2 we consider the expansion:

$$\frac{1}{(c-x)^2} = \frac{1}{c^2} + \frac{2}{c^3}x + O(x^2),$$

and get:

$$
\begin{aligned}
F_2(\rho+h)-F_2(\rho) &= \nabla(\rho+h)\cdot\nabla\left(\frac{3-2a(\rho+h)}{(1-a(\rho+h))^2}\right)-\nabla\rho\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\\
&= \nabla(\rho+h)\cdot\nabla\left(\frac{3-2a(\rho+h)}{(1-a\rho)^2}+\frac{3-2a(\rho+h)}{(1-a\rho)^3}2ah\right)-\nabla\rho\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\\
&= \nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+\nabla\rho\cdot\nabla\left(h\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right)\\
&= \nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+(\nabla h\cdot\nabla\rho)\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\
&+ h\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)
\end{aligned}
$$

Finally, we have:

$$
\begin{aligned}
F_3(\rho+h)-F_3(\rho) &= (\rho+h)\nabla^2\left(\frac{a(\rho+h)-2}{(a(\rho+h)-1)^2}\right)-\rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)\\
&= (\rho+h)\nabla^2\left(\frac{a(\rho+h)-2}{(1-a\rho)^2}+\frac{a(\rho+h)-2}{(1-a\rho)^3}2ah\right)-\rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)\\
&= h\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)+\rho\nabla^2\left(h\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)\\
&= h\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)+\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\nabla^2 h\\
&+ 2\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\cdot\nabla h+\rho h\nabla^2\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)
\end{aligned}
$$

Combining these in the Lagrangian gives:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h = &-\frac{1}{\beta}\int_0^T\int_\Omega q\nabla^2 h\left(\frac{1}{1-a\rho}\right)+q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}h\right)\\
&+ q\nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+q\left(\nabla h\cdot\nabla\rho\right)\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\
&+ qh\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\
&- qh\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\nabla^2 h\\
&- q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\cdot\nabla h-q\rho h\nabla^2\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right).
\end{aligned}
$$

Rearranging gives:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho, \mathbf{w}, q)h = -\frac{1}{\beta}\int_0^T\int_\Omega & h\Bigg(q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
& - q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) \\
& + \nabla h\cdot\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\right) \\
& + \nabla^2 h\left(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)
\end{aligned}
$$

Integration by parts gives:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho, \mathbf{w}, q)h = -\frac{1}{\beta}\int_0^T\int_\Omega & h\Bigg(q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
& - q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) \\
& - h\nabla\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\right) \\
& + h\nabla^2\left(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)
\end{aligned}
$$

So we have:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho, \mathbf{w}, q)h = -\frac{1}{\beta}\int_0^T\int_\Omega & h\Bigg[q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
& - q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2}\right) - q\rho\nabla^2\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right) \\
& - \nabla\cdot\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right) \\
& + \nabla\cdot\left(q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}\right)\right) + \nabla\cdot\left(q\rho\nabla\left(\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\right) \\
& + \nabla^2\left(q\left(\frac{1}{1-a\rho}\right)\right) - \nabla^2\left(q\rho\left(\frac{a}{(1-a\rho)^2}\right)\right) - \nabla^2\left(q\rho\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)\Bigg]drdt
\end{aligned}
$$

And combining fractions gives:

$$\begin{aligned}\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Big[& q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{2a(a\rho-2)}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\ & - q\rho\nabla^2\left(\frac{a(3-a\rho)}{(1-a\rho)^3}\right) - \nabla\cdot\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{2a(a\rho-2)}{(1-a\rho)^3}\right)\right) \\ & + \nabla\cdot\left(q\rho\nabla\left(\frac{-2a(a\rho-3)}{(1-a\rho)^3}\right)\right) + \nabla^2\left(q\left(\frac{1}{1-a\rho}\right)\right) - \nabla^2\left(q\rho\left(\frac{-a(a\rho-3)}{(1-a\rho)^3}\right)\right)\Big]drdt\end{aligned}$$

According to Mathematica this is:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\left[\frac{1}{(a\rho-1)^3}\Big(4a\nabla\rho\cdot\nabla q + 2a(-1+a\rho)q\nabla^2\rho + (-1+5a\rho-2a^2\rho^2)\nabla^2 q\Big)\right]drdt$$

And rewriting this is:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\left[\frac{4a\nabla\rho\cdot\nabla q}{(a\rho-1)^3} + \frac{2aq\nabla^2\rho}{(a\rho-1)^2} + \frac{(-1+5a\rho-2a^2\rho^2)\nabla^2 q}{(a\rho-1)^3}\right]drdt$$

1.2 Boundary Terms

We have the equation:

$$-\mathbf{j}\cdot\mathbf{n} = -\rho\nabla\frac{\delta F[\rho]}{\delta\rho}\cdot\mathbf{n} = -\frac{1}{\beta}\left(\frac{\nabla\rho}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\cdot\mathbf{n}$$

Then:

$$\begin{aligned}F_4(\rho) &= \frac{\nabla\rho}{1-a\rho} \\ F_5(\rho) &= \rho\nabla\frac{a\rho-2}{(a\rho-1)^2}\end{aligned}$$

Then for F_4 we have:

$$\begin{aligned}F_4(\rho+h) - F_4(\rho) &= \nabla(\rho+h)\frac{1}{1-a(\rho+h)} - \nabla\rho\frac{1}{1-a\rho} \\ &= \nabla(\rho+h)\left(\frac{1}{1-a\rho} + \frac{a}{(1-a\rho)^2}h\right) \\ &= \nabla h\left(\frac{1}{1-a\rho}\right) + \nabla\rho\left(\frac{a}{(1-a\rho)^2}h\right)\end{aligned}$$

For F_5 we get:

$$
\begin{aligned}
F_5(\rho+h)-F_5(\rho) &= (\rho+h)\nabla\frac{a(\rho+h)-2}{(a(\rho+h)-1)^2}-\rho\nabla\frac{a\rho-2}{(a\rho-1)^2}\\
&= (\rho+h)\nabla\left(\frac{a(\rho+h)-2}{(1-a\rho)^2}+\frac{a(\rho+h)-2}{(1-a\rho)^3}2ah\right)-\rho\nabla\frac{a\rho-2}{(a\rho-1)^2}\\
&= h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)+\rho\nabla\left(h\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)\\
&= h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)+h\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+\nabla h\left(\rho\frac{a}{(1-a\rho)^2}+\rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)
\end{aligned}
$$

Then the boundary terms for the Lagrangian are:

$$
\begin{aligned}
\mathcal{L}_{\rho,1}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Bigg(&-q_{\partial\Omega}\nabla h\left(\frac{1}{1-a\rho}\right)-q_{\partial\Omega}\nabla\rho\left(\frac{a}{(1-a\rho)^2}h\right)+q_{\partial\Omega}h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)\\
&+hq_{\partial\Omega}\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+q_{\partial\Omega}\nabla h\left(\rho\frac{a}{(1-a\rho)^2}+\rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\cdot\mathbf{n}drdt
\end{aligned}
$$

From the integration by parts of the terms within the domain (in the previous section) we get:

$$
\begin{aligned}
\mathcal{L}_{\rho,2}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Bigg(&h\Bigg(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\
&-q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\Bigg)\\
&+\nabla h\Bigg(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\\
&-h\nabla\Bigg(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg)\cdot\mathbf{n}drdt
\end{aligned}
$$

Combining all of these give all boundary terms for the Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\rho}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Bigg(&h\Bigg(-q_{\partial\Omega}\nabla\rho\left(\frac{a}{(1-a\rho)^2}\right)+q_{\partial\Omega}\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)\\
&+q_{\partial\Omega}\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right.\\
&\left.-q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\right)-\nabla\Bigg(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg)\\
&+\nabla h\Bigg(-q_{\partial\Omega}\left(\frac{1}{1-a\rho}\right)+q_{\partial\Omega}\left(\rho\frac{a}{(1-a\rho)^2}+\rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+q\left(\frac{1}{1-a\rho}\right)\\
&-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg)\cdot\mathbf{n}drdt
\end{aligned}
$$

Comparing terms in ∇h:

$$\begin{aligned}\Bigg[&- q_{\partial\Omega}\left(\frac{1}{1-a\rho}\right) + q_{\partial\Omega}\left(\rho\frac{a}{(1-a\rho)^2} + \rho\frac{2a^2\rho - 4a}{(1-a\rho)^3}\right) \\ &+ q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho - 4a}{(1-a\rho)^3}\right)\Bigg] \cdot \mathbf{n} = 0.\end{aligned}$$

This holds when $q_{\partial\Omega} = q$. Then for $h \neq 0$ we get:

$$\begin{aligned}&\Bigg[- q\nabla\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\left(\frac{a\rho - 2}{(1-a\rho)^2}\right) \\ &+ q\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho - 4a}{(1-a\rho)^3}\right) + q\nabla\left(\frac{3 - 2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a - 4a^2\rho}{(1-a\rho)^3}\right) \\ &- q\rho\nabla\left(\frac{2a}{(1-a\rho)^2} + \frac{4a^2\rho - 8a}{(1-a\rho)^3}\right) - \nabla\left(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho - 4a}{(1-a\rho)^3}\right)\right)\Bigg] \cdot \mathbf{n} = 0\end{aligned}$$

According to Mathematica this is:

$$\frac{(1 + a\rho)\nabla q}{(a\rho - 1)^3} \cdot \mathbf{n} = 0$$

Since $a\rho > 0$ by definition, this is:

$$\frac{\partial q}{\partial n} = 0.$$