1 The Multiple Species Gradient Equation

We consider the derivative of the Lagrangian with respect to $\mathbf{w}$. However, we will need to consider the Frechét derivative of terms involving $F(\mathbf{w})$ first. If F is a function of $\mathbf{w}$ only and not of the position variable r, we can do the following. Otherwise, we will have to work with the definition of the Frechét derivative and derive the gradient equation like that. We consider the first order term of the Taylor expansion, so that we have:

$$F(\mathbf{w}+\mathbf{h}) - F(\mathbf{w}) = \left(\nabla_{\mathbf{w}} F(\mathbf{w})^T\right)\mathbf{h}$$

Then:

$$\begin{aligned}
\mathcal{L}_{\mathbf{w}}(\rho_a, \rho_b, \mathbf{w}, q_a, q_b)\mathbf{h} = &\int_0^T \int_\Omega \left(\beta \mathbf{w}\cdot\mathbf{h} - D_a \nabla\cdot(\rho_a \left(\nabla_{\mathbf{w}} F_a(\mathbf{w})^T\right)\mathbf{h})q_a - D_b\nabla\cdot(\rho_b\left(\nabla_{\mathbf{w}} F_b(\mathbf{w})^T\right)\mathbf{h})q_b\right) drdt \\
&+ \int_0^T \int_{\partial\Omega} \left(D_a\rho_a\left(\nabla_{\mathbf{w}} F_a(\mathbf{w})^T\right)\mathbf{h}q_{a,\partial\Omega} + D_b\rho_b\left(\nabla_{\mathbf{w}} F_b(\mathbf{w})^T\right)\mathbf{h}q_{b,\partial\Omega}\right)\cdot\mathbf{n}drdt \\
&= \int_0^T \int_\Omega \left(\beta\mathbf{w}\cdot\mathbf{h} + D_a\rho_a\left(\left(\nabla_{\mathbf{w}} F_a(\mathbf{w})^T\right)\mathbf{h}\right)\cdot\nabla q_a\right. \\
&\left. + D_b\rho_b\left(\left(\nabla_{\mathbf{w}} F_b(\mathbf{w})^T\right)\mathbf{h}\right)\cdot\nabla q_b\right)drdt \\
&- \int_0^T \int_{\partial\Omega} \left(D_a\rho_a\left(\nabla_{\mathbf{w}} F_a(\mathbf{w})^T\right)\mathbf{h}q_a + D_b\rho_b\left(\nabla_{\mathbf{w}} F_b(\mathbf{w})^T\right)\mathbf{h}q_b\right)\cdot\mathbf{n}drdt \\
&+ \int_0^T \int_{\partial\Omega} \left(D_a\rho_a\left(\nabla_{\mathbf{w}} F_a(\mathbf{w})^T\right)\mathbf{h}q_{a,\partial\Omega} + D_b\rho_b\left(\nabla_{\mathbf{w}} F_b(\mathbf{w})^T\right)\mathbf{h}q_{b,\partial\Omega}\right)\cdot\mathbf{n}drdt \\
&= \int_0^T \int_\Omega \left(\beta\mathbf{w}\cdot\mathbf{h} + D_a\rho_a\left(\left(\nabla_{\mathbf{w}} F_a(\mathbf{w})^T\right)\mathbf{h}\right)\cdot\nabla q_a\right. \\
&\left. + D_b\rho_b\left(\left(\nabla_{\mathbf{w}} F_b(\mathbf{w})^T\right)\mathbf{h}\right)\cdot\nabla q_b\right)drdt,
\end{aligned}$$

since $q_a = q_{a,\partial\Omega}$ and $q_b = q_{b,\partial\Omega}$ from the adjoint derivation.
Now we use the relation $((\nabla\mathbf{a})^T)\mathbf{b})\cdot\mathbf{c} = ((\mathbf{c}\cdot\nabla)\mathbf{a})\cdot\mathbf{b}$ (from year end review) to find that:

$$\begin{aligned}
\mathcal{L}_{\mathbf{w}}(\rho_a, \rho_b, \mathbf{w}, q_a, q_b)\mathbf{h} = \int_0^T \int_\Omega \Big(&\beta\mathbf{w}\cdot\mathbf{h} + D_a\rho_a\left(\left(\nabla_r q_a\cdot\nabla_{\mathbf{w}}\right)F_a(\mathbf{w})\right)\cdot\mathbf{h} \\
&+ D_b\rho_b\left(\left(\nabla_r q_b\cdot\nabla_{\mathbf{w}}\right)F_b(\mathbf{w})\right)\cdot\mathbf{h}\Big)drdt,
\end{aligned}$$

Setting this to zero and since this holds for all permissible $\mathbf{h}$, we get:

$$\beta\mathbf{w} + D_a\rho_a\left(\left(\nabla_r q_a\cdot\nabla_{\mathbf{w}}\right)F_a(\mathbf{w})\right) + D_b\rho_b\left(\left(\nabla_r q_b\cdot\nabla_{\mathbf{w}}\right)F_b(\mathbf{w})\right) = 0.$$

Using that $\nabla\cdot(\mathbf{b}\mathbf{a}^T) = \mathbf{a}(\nabla\cdot\mathbf{b}) + (\mathbf{b}\cdot\nabla)\mathbf{a}$, and observing that $\nabla_{\mathbf{w}}\cdot(\nabla_r q) = 0$, we get:

$$\beta\mathbf{w} + D_a\rho_a\nabla_{\mathbf{w}}\cdot\left(\nabla q_a F_a(\mathbf{w})^T\right) + D_b\rho_b\nabla_{\mathbf{w}}\cdot\left(\nabla q_b F_b(\mathbf{w})^T\right) = 0.$$

Since $\nabla_r q$ does not depend on $\mathbf{w}$ we can rearrange this to get:

$$\beta \mathbf{w} + D_a \rho_a \left(\nabla_{\mathbf{w}} F_a(\mathbf{w})\right)^T \nabla q_a + D_b \rho_b \left(\nabla_{\mathbf{w}} F_b(\mathbf{w})\right)^T \nabla q_b = 0.$$

And finally we have:

$$\mathbf{w} = -\frac{1}{\beta} \left(D_a \rho_a \left(\nabla_{\mathbf{w}} F_a(\mathbf{w})\right)^T \nabla q_a + D_b \rho_b \left(\nabla_{\mathbf{w}} F_b(\mathbf{w})\right)^T \nabla q_b \right).$$

As an example, take $F_a(\mathbf{w}) = c_a \mathbf{w}$ and $F_b(\mathbf{w}) = c_b \mathbf{w}$. We get:

$$\mathbf{w} = -\frac{1}{\beta} \Bigg(D_a \rho_a c_a \mathbf{1} \nabla q_a + D_b \rho_b c_b \mathbf{1} \nabla q_b \Bigg).$$

2 Sedimentation

2.1 Free Energy Frechét Derivative

We have the general expression:

$$\begin{aligned}
\nabla \cdot \left(\rho \nabla \frac{\delta F[\rho]}{\delta \rho} \right) &= \frac{1}{\beta} \Bigg(\nabla \cdot \left(\frac{\nabla \rho}{1-\eta} \right) - \nabla \cdot \left(\rho \nabla \frac{\eta - 2}{(\eta-1)^2} \right) \Bigg) \\
&= \frac{1}{\beta} \Bigg(\frac{\nabla^2 \rho}{1-\eta} + \nabla \rho \cdot \nabla \frac{1}{1-\eta} - \nabla \rho \cdot \nabla \frac{\eta-2}{(\eta-1)^2} - \rho \nabla^2 \frac{\eta-2}{(\eta-1)^2} \Bigg) \\
&= \frac{1}{\beta} \Bigg(\frac{\nabla^2 \rho}{1-\eta} + \nabla \rho \cdot \nabla \frac{(3-2\eta)}{(1-\eta)^2} - \rho \nabla^2 \frac{\eta-2}{(\eta-1)^2} \Bigg)
\end{aligned}$$

We want to take the Frechét derivative of these terms. We set:

$$\begin{aligned}
F_1 &= \frac{\nabla^2 \rho}{1-\eta} = \frac{\nabla^2 \rho}{1 - a\rho} \\
F_2 &= \nabla \rho \cdot \nabla \frac{(3-2\eta)}{(1-\eta)^2} = \nabla \rho \cdot \nabla \frac{(3-2a\rho)}{(1-a\rho)^2} \\
F_3 &= \rho \nabla^2 \frac{\eta-2}{(\eta-1)^2} = \rho \nabla^2 \frac{a\rho - 2}{(a\rho-1)^2}.
\end{aligned}$$

We are looking at $F(\rho + h) - F(\rho)$. We use the expansions:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + O(x^2) \\
\frac{1}{(1-x)^2} &= 1 + 2x + O(x^2).
\end{aligned}$$

For F_1 we get:

$$\begin{aligned}
F_1(\rho + h) - F_1(\rho) &= \frac{\nabla^2(\rho + h)}{1 - a(\rho + h)} - \frac{\nabla^2 \rho}{1 - a\rho} \\
&= \nabla^2(\rho + h)(1 + a(\rho + h)) - \nabla^2 \rho (1 + a\rho) \\
&= (\nabla^2 \rho)(1 + a\rho + ah - 1 - a\rho) + (\nabla^2 h)(1 + a\rho + ah) \\
&= (\nabla^2 \rho)(ah) + (\nabla^2 h)(1 + a\rho)
\end{aligned}$$

For F_2 we have:

$$
\begin{aligned}
F_2(\rho+h)-F_2(\rho) &= \nabla(\rho+h)\cdot\nabla\frac{(3-2a(\rho+h))}{(1-a(\rho+h))^2}-\nabla\rho\cdot\nabla\frac{(3-2a\rho)}{(1-a\rho)^2}\\
&= \nabla(\rho+h)\cdot\nabla\left((3-2a(\rho+h))(1+2a(\rho+h))\right)-\nabla\rho\cdot\nabla\left((3-2a\rho)(1+2a\rho)\right)\\
&= \nabla(\rho+h)\cdot\nabla\left(3+6a(\rho+h)-2a(\rho+h)-4a^2(\rho+h)^2\right)\\
&-\nabla\rho\cdot\nabla\left(3+6a\rho-2a\rho-4a^2\rho^2\right)\\
&= \nabla\rho\cdot\nabla\left(3+4a(\rho+h)-4a^2(\rho+h)^2-\left(3+4a\rho-4a^2\rho^2\right)\right)\\
&+\nabla h\cdot\nabla\left(3+6a(\rho+h)-2a(\rho+h)-4a^2(\rho+h)^2\right)\\
&= \nabla\rho\cdot\nabla\left(4ah-8a^2\rho h\right)+\nabla h\cdot\nabla\left(3+6a\rho-2a\rho-4a^2\rho^2\right)\\
&= \nabla\rho\cdot\nabla\left((4a-8a^2\rho)h\right)+\nabla h\cdot\nabla\left(4a\rho-4a^2\rho^2\right)\\
&= \nabla\rho\cdot h\nabla\left(4a-8a^2\rho\right)+\nabla\rho\cdot(4a-8a^2\rho)\nabla h+\nabla h\cdot\nabla\left(4a\rho-4a^2\rho^2\right)\\
&= -8a^2h\nabla\rho\cdot\nabla\rho+\nabla h\cdot\left(\nabla\rho(4a-8a^2\rho)+\nabla\left(4a\rho-4a^2\rho^2\right)\right)\\
&= -8a^2h\left(\nabla\rho\right)^2+\nabla h\cdot\left(8a\nabla\rho-16a^2\rho\nabla\rho\right)
\end{aligned}
$$

Finally F_3 is:

$$
\begin{aligned}
F_3(\rho+h)-F_3(\rho) &= (\rho+h)\nabla^2\left(\frac{a(\rho+h)-2}{(a(\rho+h)-1)^2}\right)-\rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)\\
&= (\rho+h)\nabla^2\left((a(\rho+h)-2)(1+2a(\rho+h))\right)-\rho\nabla^2\left((a\rho-2)(1+2a\rho)\right)\\
&= (\rho+h)\nabla^2\left(-2-3a(\rho+h)+2a^2(\rho+h)^2\right)-\rho\nabla^2\left(-2-3a\rho+2a^2\rho^2\right)\\
&= \rho\nabla^2\left(-3ah+4a^2\rho h\right)+h\nabla^2\left(-3a\rho+2a^2\rho^2\right)\\
&= -3a\rho\nabla^2h+4a^2\rho\nabla^2(\rho h)-3ah\nabla^2\rho+2a^2h\nabla^2\rho^2
\end{aligned}
$$

2.2 Lagrangian

We consider the part of the Lagrangian that is relevant:

$$
\mathcal{L}(\rho,\mathbf{w},q)=-\int_0^T\int_\Omega\left(\frac{1}{\beta}\left(\frac{\nabla^2\rho}{1-\eta}+\nabla\rho\cdot\nabla\frac{(3-2\eta)}{(1-\eta)^2}-\rho\nabla^2\frac{\eta-2}{(\eta-1)^2}\right)q\right)drdt
$$

Taking the derivatives with respect to ρ gives:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h=-\frac{1}{\beta}\int_0^T\int_\Omega\Big(&(\nabla^2\rho)(ah)+(\nabla^2h)(1+a\rho)-8a^2h\left(\nabla\rho\right)^2+\nabla h\cdot\left(8a\nabla\rho-16a^2\rho\nabla\rho\right)\\
&+3a\rho\nabla^2h-4a^2\rho\nabla^2(\rho h)+3ah\nabla^2\rho-2a^2h\nabla^2\rho^2\Big)qdrdt
\end{aligned}
$$

Integrate by parts the term involving $\nabla^2(\rho h)$:

$$\begin{aligned}
\int_0^T \int_\Omega q\rho \nabla^2(\rho h) drdt &= \int_0^T \int_{\partial\Omega} q\rho \nabla(\rho h) \cdot \mathbf{n} drdt - \int_0^T \int_\Omega \nabla(q\rho) \cdot \nabla(\rho h) drdt \\
&= \int_0^T \int_{\partial\Omega} q\rho \left(\rho \nabla h + h \nabla \rho\right) \cdot \mathbf{n} drdt - \int_0^T \int_{\partial\Omega} \rho h \nabla(q\rho) \cdot \mathbf{n} drdt + \int_0^T \int_\Omega \rho h \nabla^2(q\rho) drdt \\
&= \int_0^T \int_{\partial\Omega} \Big(q\rho^2 \nabla h + q\rho h \nabla \rho - \rho^2 h \nabla q - q\rho h \nabla \rho\Big) \cdot \mathbf{n} drdt + \int_0^T \int_\Omega \rho h \nabla^2(q\rho) drdt \\
&= \int_0^T \int_{\partial\Omega} \Big(q\rho^2 \nabla h - \rho^2 h \nabla q\Big) \cdot \mathbf{n} drdt + \int_0^T \int_\Omega \rho h \nabla^2(q\rho) drdt
\end{aligned}$$

Then we have the terms involving $\nabla^2 h$:

$$\begin{aligned}
\int_0^T \int_\Omega (\nabla^2 h)(q + 4aq\rho) drdt &= \int_0^T \int_{\partial\Omega} (\nabla h)(q + 4aq\rho) \cdot \mathbf{n} drdt - \int_0^T \int_\Omega (\nabla h) \cdot \nabla(q + 4aq\rho) drdt \\
&= \int_0^T \int_{\partial\Omega} \big((\nabla h)(q + 4aq\rho) - h\nabla q - 4ah\nabla(q\rho)\big) \cdot \mathbf{n} drdt \\
&+ \int_0^T \int_\Omega h\nabla^2 q + 4ah\nabla^2(q\rho) drdt \\
&= \int_0^T \int_{\partial\Omega} \big((\nabla h)(q + 4aq\rho) - h\nabla q - 4ah\nabla(q\rho)\big) \cdot \mathbf{n} drdt \\
&+ \int_0^T \int_\Omega h\nabla^2 q + 4ahq\nabla^2 \rho + 4ah\rho\nabla^2 q + 4ah\nabla\rho \cdot \nabla q drdt
\end{aligned}$$

Finally, the terms involving ∇h:

$$\begin{aligned}
\int_0^T \int_\Omega \nabla h \cdot \Big(8aq\nabla\rho - 16a^2 q\rho\nabla\rho\Big) drdt &= \int_0^T \int_{\partial\Omega} h \Big(8aq\nabla\rho - 16a^2 q\rho\nabla\rho\Big) \cdot \mathbf{n} drdt \\
&- \int_0^T \int_\Omega h\nabla \cdot \Big(8aq\nabla\rho - 16a^2 q\rho\nabla\rho\Big) drdt \\
&= \int_0^T \int_{\partial\Omega} h \Big(8aq\nabla\rho - 16a^2 q\rho\nabla\rho\Big) \cdot \mathbf{n} drdt \\
&- \int_0^T \int_\Omega h \Big(8a\nabla \cdot (q\nabla\rho) - 16a^2 \nabla \cdot (q\rho\nabla\rho)\Big) drdt \\
&= \int_0^T \int_{\partial\Omega} h \Big(8aq\nabla\rho - 16a^2 q\rho\nabla\rho\Big) \cdot \mathbf{n} drdt \\
&- \int_0^T \int_\Omega h \Big(8a\nabla q \cdot \nabla\rho + 8aq\nabla^2\rho \\
&- 16a^2 q(\nabla\rho)^2 - 16a^2 \rho\nabla\rho \cdot \nabla q - 16a^2 q\rho\nabla^2\rho \Big) drdt
\end{aligned}$$

Combining all of these gives:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega \Big(q(\nabla^2\rho)(ah) - q8a^2h\,(\nabla\rho)^2$$
$$+ 8ah\nabla q\cdot\nabla\rho + 8aqh\nabla^2\rho - 16a^2qh(\nabla\rho)^2 - 16ha^2\rho\nabla\rho\cdot\nabla q - 16ha^2q\rho\nabla^2\rho$$
$$- 4a^2\rho hq\nabla^2(\rho) - 8a^2\rho h\nabla q\cdot\nabla\rho - 4a^2h\rho^2\nabla^2 q + 3ahq\nabla^2\rho - 2a^2qh\nabla^2\rho^2$$
$$h\nabla^2 q + 4ahq\nabla^2\rho + 4ah\rho\nabla^2 q + 4ah\nabla\rho\cdot\nabla q\Big)drdt$$

Rearranging and cancelling results in:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega hq\Big(a\nabla^2\rho - 8a^2\,(\nabla\rho)^2 + 8a\nabla^2\rho - 16a^2(\nabla\rho)^2$$
$$- 16a^2\rho\nabla^2\rho - 4a^2\rho\nabla^2(\rho) + 3a\nabla^2\rho - 2a^2\nabla^2\rho^2 + 4a\nabla^2\rho\Big)$$
$$+ h\nabla q\cdot\Big(8a\nabla\rho - 16a^2\rho\nabla\rho - 8a^2\rho\nabla\rho + 4a\nabla\rho\Big)$$
$$+ h\nabla^2 q\Big(1 - 4a^2\rho^2 + 4a\rho\Big)drdt$$

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega hq\Big(16a\nabla^2\rho - 28a^2\,(\nabla\rho)^2 - 24a^2\rho\nabla^2\rho\Big)$$
$$+ h\nabla q\cdot\Big(12a\nabla\rho - 24a^2\rho\nabla\rho\Big) + h\nabla^2 q\Big(1 - 4a^2\rho^2 + 4a\rho\Big)drdt$$

Adding the other terms to it we get the adjoint equation:

$$\frac{\partial q}{\partial t} = -\frac{1}{\beta}q\Big(16a\nabla^2\rho - 28a^2\,(\nabla\rho)^2 - 24a^2\rho\nabla^2\rho\Big)$$
$$-\frac{1}{\beta}\nabla q\cdot\Big(12a\nabla\rho - 24a^2\rho\nabla\rho\Big) - \frac{1}{\beta}\nabla^2 q\Big(1 - 4a^2\rho^2 + 4a\rho\Big)$$
$$+ \nabla V_{ext}\cdot\nabla q - \mathbf{w}\cdot\nabla q - \rho + \widehat{\rho} + \int_\Omega(\nabla_r q(r) - \nabla_{r'}q(r'))\rho(r')\mathbf{K}(r,r')dr'$$

2.3 Boundary Terms

We had the flux term:

$$\begin{aligned}
-\rho\nabla\frac{\delta F[\rho]}{\delta\rho} &= -\frac{1}{\beta}\left(\nabla\rho + \frac{\rho\nabla\eta}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\\
&= -\frac{1}{\beta}\left(\nabla\rho + \frac{\eta\nabla\rho}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\\
&= -\frac{1}{\beta}\left(\nabla\rho + \frac{\nabla\rho}{1-\eta} - \nabla\rho - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\\
&= -\frac{1}{\beta}\left(\frac{\nabla\rho}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)
\end{aligned}$$

Taking Frechét derivatives gives:

$$\begin{aligned}F_4(\rho+h)-F(\rho) &= \nabla(\rho+h)(1+a\rho+ah)-\nabla\rho(1+a\rho)\\ &= (\nabla\rho)(ah)+\nabla h(1+a\rho)\\ &= ah\nabla\rho+\nabla h+a\rho\nabla h,\end{aligned}$$

and

$$\begin{aligned}F_5(\rho+h)-F_5(\rho) &= (\rho+h)\nabla\left((a\rho+ah-2)(1+2a\rho+2ah)\right)-\rho\nabla\left((a\rho-2)(1+2a\rho)\right)\\ &= \rho\nabla(a\rho+2a^2\rho+2a^2\rho h+ah+2a^2\rho h+2a^2h^2-2-4a\rho-4ah)\\ &-\rho\nabla(a\rho+2a^2\rho^2-2-4a\rho)\\ &+h\nabla(a\rho+2a^2\rho+2a^2\rho h+ah+2a^2\rho h+2a^2h^2-2-4a\rho-4ah)\\ &= \rho\nabla\left(4a^2\rho h-3ah\right)\\ &+h\nabla\left(2a^2\rho-2-3a\rho\right)\\ &= 4a^2\rho^2\nabla h+4a^2h\rho\nabla\rho-3a\rho\nabla h+2a^2h\nabla\rho-3ah\nabla\rho\end{aligned}$$

Then the contribution to the Lagrangian is:

$$\mathcal{L}(\rho,\mathbf{w},q)=\int_0^T\int_{\partial\Omega}-\frac{1}{\beta}\left(\frac{\nabla\rho}{1-\eta}-\rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)\cdot\mathbf{n}drdt.$$

Then the contribution to the derivative with respect to ρ is:

$$\begin{aligned}\mathcal{L}_{\rho,1}(\rho,\mathbf{w},q)h &= \frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Big(ah\nabla\rho+\nabla h+a\rho\nabla h\\ &-4a^2\rho^2\nabla h-4a^2h\rho\nabla\rho+3a\rho\nabla h-2a^2h\nabla\rho+3ah\nabla\rho\Big)q_{\partial\Omega}\cdot\mathbf{n}drdt\\ &= \frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Big(\nabla h(1+a\rho-4a^2\rho^2+3a\rho)+h(4a\nabla\rho-4a^2\rho\nabla\rho-2a^2\nabla\rho)\Big)q_{\partial\Omega}\cdot\mathbf{n}drdt\end{aligned}$$

Now, collecting the boundary terms from integrating by parts in the main body we get:

$$\begin{aligned}\mathcal{L}_{\rho,2}(\rho,\mathbf{w},q)h &= -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Big((-4a^2q\rho^2\nabla h-\rho^2h\nabla q)\\ &+(\nabla h(q+4aq\rho)-h\nabla q-4ah\nabla(q\rho))\\ &+h(8aq\nabla\rho-16a^2q\rho\nabla\rho)\Big)\cdot\mathbf{n}dr\\ &= -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Big(\nabla h(-4a^2q\rho^2+q+4aq\rho)\\ &+h(-\rho^2\nabla q-\nabla q-4a\rho\nabla q+4aq\nabla\rho-16a^2q\rho\nabla\rho)\Big)\cdot\mathbf{n}drdt\end{aligned}$$

Combining the two gives:

$$\mathcal{L}_\rho(\rho, \mathbf{w}, q)h = \frac{1}{\beta}\int_0^T \int_{\partial\Omega} \nabla h\Big(q_{\partial\Omega} + aq_{\partial\Omega}\rho - 4a^2 q_{\partial\Omega}\rho^2 + 3aq_{\partial\Omega}\rho + 4a^2 q\rho^2 - q - 4aq\rho\Big)\cdot\mathbf{n}$$
$$+ h\Big(4aq_{\partial\Omega}\nabla\rho - 4a^2 q_{\partial\Omega}\rho\nabla\rho - 2a^2 q_{\partial\Omega}\nabla\rho + \rho^2\nabla q + \nabla q + 4a\rho\nabla q - 4aq\nabla\rho + 16a^2 q\rho\nabla\rho\Big)\cdot\mathbf{n}drdt$$

Considering $h = 0$ and $\nabla h \neq 0$ on $\partial\Omega$ we get that:

$$\Big(q_{\partial\Omega} + aq_{\partial\Omega}\rho - 4a^2 q_{\partial\Omega}\rho^2 + 3aq_{\partial\Omega}\rho + 4a^2 q\rho^2 - q - 4aq\rho\Big)\cdot\mathbf{n} = 0$$

Equating this for powers of ρ (allowed?) we conclude that $q_{\partial\Omega} = q$. Then if $h \neq 0$ we get:

$$\Big(-2a^2 q\nabla\rho + \rho^2\nabla q + \nabla q + 4a\rho\nabla q + 12a^2 q\rho\nabla\rho\Big)\cdot\mathbf{n} = 0$$

Since this doesn't cancel, there may be a mistake in the calculations.

2.4 Implementation

While there is probably a mistake in the above, I implemented this as flow control problem. I set the target to be the forward problem with a certain V_{ext} and the initial configuration has a bit of a weaker V_{ext}, so that we would need some more help from the flow term to get to the first one. However, the difference is $a0 = 0.1$, $a1 = 0.099$, so that it's hard to see what the difference actually is, see Figures 1, 3 and 2. It may be better to actually do V_{ext} control?

3 Constriction Example Equation

We have the equation

$$\frac{\partial\rho}{\partial t} = \nabla^2\rho + \nabla(\rho\nabla V_{ext}).$$

We then make the substitution $\rho = e^{h - V_{ext}}$. Then

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}e^{h-V_{ext}} = e^{h-V_{ext}}\frac{\partial h}{\partial t}\\
\nabla\rho &= \nabla e^{h-V_{ext}} = e^{h-V_{ext}}\nabla(h - V_{ext})\\
\nabla^2\rho &= \nabla\Big(e^{h-V_{ext}}\nabla(h - V_{ext})\Big)\\
&= e^{h-V_{ext}}\Big(\nabla^2 h - \nabla^2 V_{ext}\Big) + e^{h-V_{ext}}\Big((\nabla h)^2 - 2\nabla h\cdot\nabla V_{ext} + (\nabla V_{ext})^2\Big)\\
\nabla\Big(\rho\nabla V_{ext}\Big) &= \nabla(e^{h-V_{ext}}\nabla V_{ext}) = e^{h-V_{ext}}\nabla^2 V_{ext} + e^{h-V_{ext}}\Big(\nabla h\cdot\nabla V_{ext} - (\nabla V_{ext})^2\Big)
\end{aligned}$$

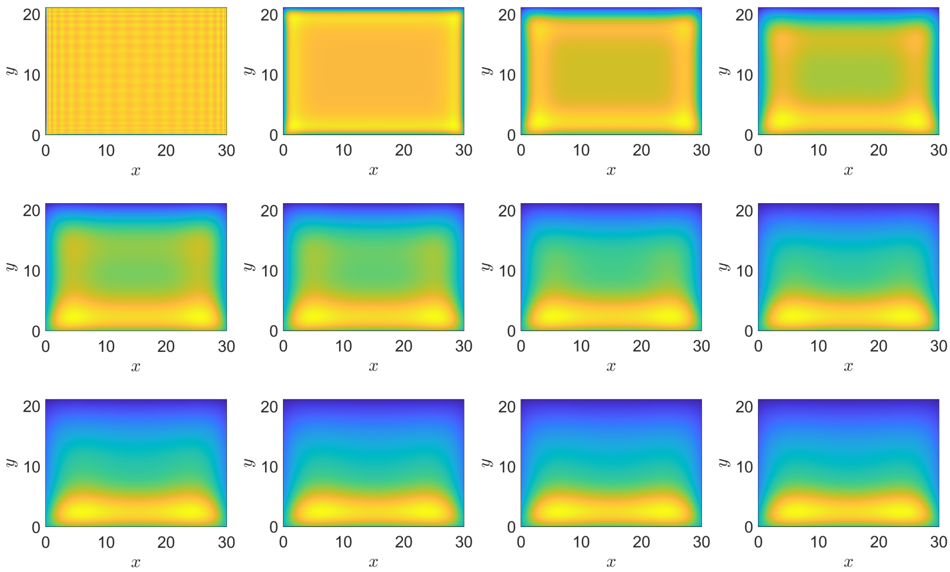

Figure 1: Sedimentation Forward Problem, $a = 0.099$

Then we get:

$$\begin{aligned}\frac{\partial h}{\partial t} &= \nabla^2 h - \nabla^2 V_{ext} + (\nabla h)^2 - 2\nabla h \cdot \nabla V_{ext} + (\nabla V_{ext})^2 + \nabla^2 V_{ext} + \nabla h \cdot \nabla V_{ext} - (\nabla V_{ext})^2 \\ &= \nabla^2 h + (\nabla h)^2 - \nabla h \cdot \nabla V_{ext}\end{aligned}$$

Problem: This crashes in the implementation, so either it's the wrong equation or the implementation is wrong.

4 Periodic Box

I am not sure yet what this is doing (the scaling of the box and the periodicity). Gravity is acting from left to right, top and bottom are periodic. Can we change that? See Figure 4.

5 CV

Questions:
Cover letter: How much detail of what I am doing/ how technical? What else should I include about myself?
CV: Should I include the preprint? Is the order of sections correct? What level of Matlab skills do I have?

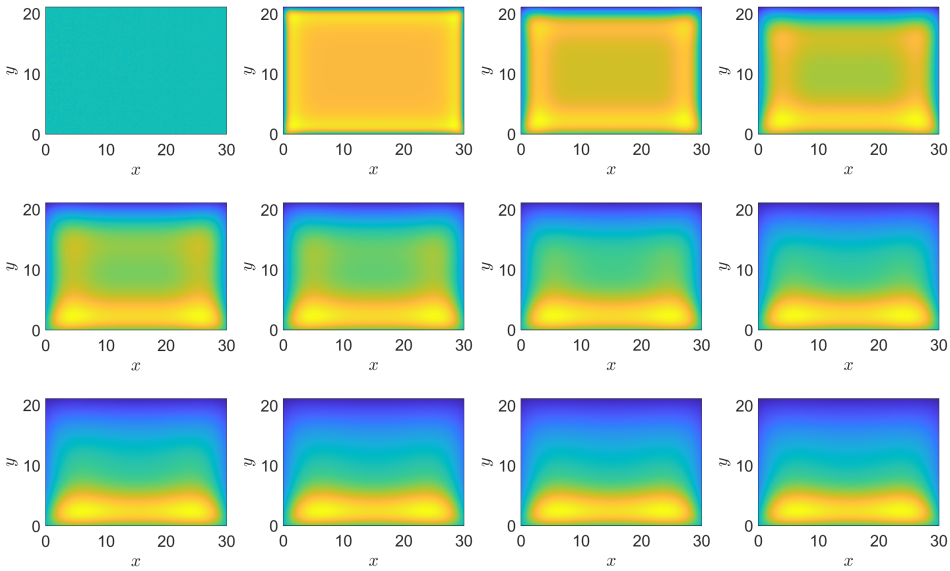

Figure 2: Sedimentation Optimal ρ, target $a = 0.1$

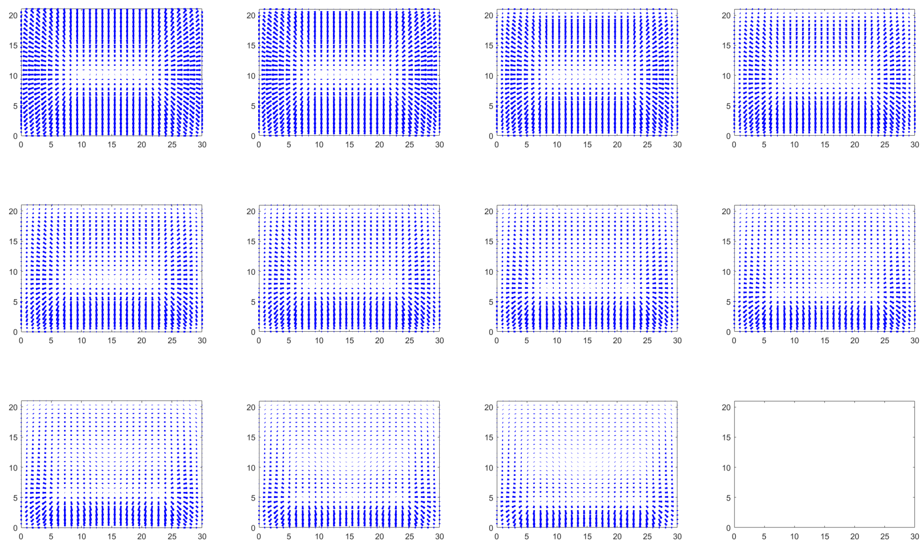

Figure 3: Sedimentation Control

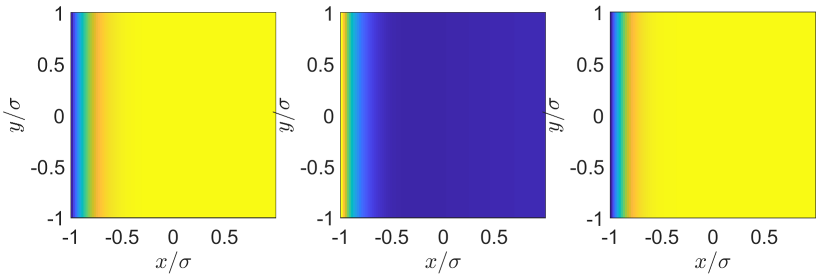

Figure 4: Periodic Box 1