# 1 Optimality conditions for the sedimentation equations

The relevant part of the equation is:

$$\nabla\cdot\left(\rho\nabla\frac{\delta F[\rho]}{\delta\rho}\right)=\frac{1}{\beta}\left(\frac{\nabla^2\rho}{1-\eta}+\nabla\rho\cdot\nabla\frac{(3-2\eta)}{(1-\eta)^2}-\rho\nabla^2\frac{\eta-2}{(\eta-1)^2}\right),$$

where $\eta=a\rho$ and $a=\pi\sigma^2/4$. Consider:

$$\begin{aligned}F_1(\rho)&=\nabla^2\rho\frac{1}{1-a\rho}\\F_2(\rho)&=\nabla\rho\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\\F_3(\rho)&=\rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)\end{aligned}$$

Then

$$F_1(\rho+h)-F_1(\rho)=\nabla(\rho+h)\frac{1}{1-a(\rho+h)}-\nabla\rho\frac{1}{1-a\rho}$$

Using the expansion:

$$\frac{1}{c-x}=\frac{1}{c}+\frac{1}{c^2}x+O(x^2),$$

where $c=1-a\rho$, we get:

$$\begin{aligned}F_1(\rho+h)-F_1(\rho)&=\nabla^2(\rho+h)\left(\frac{1}{1-a\rho}+\frac{a}{(1-a\rho)^2}h\right)-\nabla^2\rho\frac{1}{1-a\rho}\\&=\nabla^2h\left(\frac{1}{1-a\rho}\right)+\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}h\right)\end{aligned}$$

For $F_2$ we consider the expansion:

$$\frac{1}{(c-x)^2}=\frac{1}{c^2}+\frac{2}{c^3}x+O(x^2),$$

and get:

$$\begin{aligned}F_2(\rho+h)-F_2(\rho)&=\nabla(\rho+h)\cdot\nabla\left(\frac{3-2a(\rho+h)}{(1-a(\rho+h))^2}\right)-\nabla\rho\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\\&=\nabla(\rho+h)\cdot\nabla\left(\frac{3-2a(\rho+h)}{(1-a\rho)^2}+\frac{3-2a(\rho+h)}{(1-a\rho)^3}2ah\right)-\nabla\rho\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\\&=\nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+\nabla\rho\cdot\nabla\left(h\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right)\\&=\nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+(\nabla h\cdot\nabla\rho)\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\&+h\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\end{aligned}$$

Finally, we have:

$$
\begin{aligned}
F_3(\rho+h) - F_3(\rho) &= (\rho+h)\nabla^2\left(\frac{a(\rho+h)-2}{(a(\rho+h)-1)^2}\right) - \rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
&= (\rho+h)\nabla^2\left(\frac{a(\rho+h)-2}{(1-a\rho)^2} + \frac{a(\rho+h)-2}{(1-a\rho)^3}2ah\right) - \rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
&= h\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) + \rho\nabla^2\left(h\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
&= h\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) + \rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\nabla^2 h \\
&+ 2\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\cdot\nabla h + \rho h\nabla^2\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)
\end{aligned}
$$

Combining these in the Lagrangian gives:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega & q\nabla^2 h\left(\frac{1}{1-a\rho}\right) + q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}h\right) \\
&+ q\nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\left(\nabla h\cdot\nabla\rho\right)\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) \\
&+ qh\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) \\
&- qh\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\nabla^2 h \\
&- q\rho\nabla\left(\frac{2a}{(1-a\rho)^2} + \frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\cdot\nabla h - q\rho h\nabla^2\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right).
\end{aligned}
$$

Rearranging gives:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega & h\Bigg(q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
&- q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) \\
&+ \nabla h\cdot\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\rho\nabla\left(\frac{2a}{(1-a\rho)^2} + \frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\right) \\
&+ \nabla^2 h\left(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)
\end{aligned}
$$

Integration by parts gives:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Bigg(q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)$$
$$- q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)$$
$$- h\nabla\Bigg(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\Bigg)$$
$$+ h\nabla^2\Bigg(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)$$

So we have:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Bigg[q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)$$
$$- q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2}\right) - q\rho\nabla^2\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)$$
$$- \nabla\cdot\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right)$$
$$+ \nabla\cdot\left(q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}\right)\right) + \nabla\cdot\left(q\rho\nabla\left(\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\right)$$
$$+ \nabla^2\left(q\left(\frac{1}{1-a\rho}\right)\right) - \nabla^2\left(q\rho\left(\frac{a}{(1-a\rho)^2}\right)\right) - \nabla^2\left(q\rho\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)\Bigg]drdt$$

And combining fractions gives:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Bigg[q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{2a(a\rho-2)}{(1-a\rho)^3}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right)$$
$$- q\rho\nabla^2\left(\frac{a(3-a\rho)}{(1-a\rho)^3}\right) - \nabla\cdot\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{2a(a\rho-2)}{(1-a\rho)^3}\right)\right)$$
$$+ \nabla\cdot\left(q\rho\nabla\left(\frac{-2a(a\rho-3)}{(1-a\rho)^3}\right)\right) + \nabla^2\left(q\left(\frac{1}{1-a\rho}\right)\right) - \nabla^2\left(q\rho\left(\frac{-a(a\rho-3)}{(1-a\rho)^3}\right)\right)\Bigg]drdt$$

According to Mathematica this is:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Bigg[\frac{1}{(a\rho-1)^3}\Bigg(4a\nabla\rho\cdot\nabla q + 2a(-1+a\rho)q\nabla^2\rho + (-1+5a\rho-2a^2\rho^2)\nabla^2 q\Bigg)\Bigg]drdt$$

And rewriting this is:

$$\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Bigg[\frac{4a\nabla\rho\cdot\nabla q}{(a\rho-1)^3} + \frac{2aq\nabla^2\rho}{(a\rho-1)^2} + \frac{(-1+5a\rho-2a^2\rho^2)\nabla^2 q}{(a\rho-1)^3}\Bigg]drdt$$

## 1.1 Boundary Terms

We have the equation:

$$\rho\nabla\frac{\delta F[\rho]}{\delta\rho} = \frac{1}{\beta}\left(\frac{\nabla\rho}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)$$

Then:

$$\begin{aligned}
F_4(\rho) &= \frac{\nabla\rho}{1-a\rho} \\
F_5(\rho) &= \rho\nabla\frac{a\rho-2}{(a\rho-1)^2}
\end{aligned}$$

Then for $F_4$ we have:

$$\begin{aligned}
F_4(\rho+h) - F_4(\rho) &= \nabla(\rho+h)\frac{1}{1-a(\rho+h)} - \nabla\rho\frac{1}{1-a\rho} \\
&= \nabla(\rho+h)\left(\frac{1}{1-a\rho} + \frac{a}{(1-a\rho)^2}h\right) \\
&= \nabla h\left(\frac{1}{1-a\rho}\right) + \nabla\rho\left(\frac{a}{(1-a\rho)^2}h\right)
\end{aligned}$$

For $F_5$ we get:

$$\begin{aligned}
F_5(\rho+h) - F_5(\rho) &= (\rho+h)\nabla\frac{a(\rho+h)-2}{(a(\rho+h)-1)^2} - \rho\nabla\frac{a\rho-2}{(a\rho-1)^2} \\
&= (\rho+h)\nabla\left(\frac{a(\rho+h)-2}{(1-a\rho)^2} + \frac{a(\rho+h)-2}{(1-a\rho)^3}2ah\right) - \rho\nabla\frac{a\rho-2}{(a\rho-1)^2} \\
&= h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right) + \rho\nabla\left(h\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
&= h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right) + h\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right) + \nabla h\left(\rho\frac{a}{(1-a\rho)^2} + \rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)
\end{aligned}$$

Then the boundary terms for the Lagrangian are:

$$\begin{aligned}
\mathcal{L}_{\rho,1}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Bigg(& q_{\partial\Omega}\nabla h\left(\frac{1}{1-a\rho}\right) + q_{\partial\Omega}\nabla\rho\left(\frac{1}{(1-a\rho)^2}h\right) - q_{\partial\Omega}h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right) \\
& - hq_{\partial\Omega}\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right) - q_{\partial\Omega}\nabla h\left(\rho\frac{a}{(1-a\rho)^2} + \rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\cdot\mathbf{n}drdt
\end{aligned}$$

From the integration by parts of the terms within the domain we get:

$$
\begin{aligned}
\mathcal{L}_{\rho,2}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Bigg(h\Bigg(&q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\
&-q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\Bigg)\\
&+\nabla h\Bigg(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\\
&-h\nabla\Bigg(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg)\cdot\mathbf{n}drdt
\end{aligned}
$$

Combining all of these give all boundary terms for the Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\rho}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}\Bigg(h\Bigg(&q_{\partial\Omega}\nabla\rho\left(\frac{1}{(1-a\rho)^2}\right)-q_{\partial\Omega}\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)\\
&-q_{\partial\Omega}\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+\Bigg(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\
&-q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)\Bigg)-\nabla\Bigg(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg)\\
&+\nabla h\Bigg(q_{\partial\Omega}\left(\frac{1}{1-a\rho}\right)-q_{\partial\Omega}\left(\rho\frac{a}{(1-a\rho)^2}+\rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+q\left(\frac{1}{1-a\rho}\right)\\
&-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg)\cdot\mathbf{n}drdt
\end{aligned}
$$

Comparing terms in $\nabla h$ (there should be $\cdot\mathbf{n}$ everywhere in the below):

$$
\begin{aligned}
\Bigg[&q_{\partial\Omega}\left(\frac{1}{1-a\rho}\right)-q_{\partial\Omega}\left(\rho\frac{a}{(1-a\rho)^2}+\rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\\
&+q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg]\cdot\mathbf{n}=0.
\end{aligned}
$$

This holds when $q_{\partial\Omega}=-q$. Then for $h\neq 0$ we get:

$$
\begin{aligned}
\Bigg[&-q\nabla\rho\left(\frac{1}{(1-a\rho)^2}\right)+q\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)\\
&+q\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\\
&-q\rho\nabla\left(\frac{2a}{(1-a\rho)^2}+\frac{4a^2\rho-8a}{(1-a\rho)^3}\right)-\nabla\Bigg(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg]\cdot\mathbf{n}=0
\end{aligned}
$$

According to Mathematica this is:

$$
\frac{1}{(a\rho-1)^3}\Big[(-1+a)(-1+a\rho)q\nabla\rho+(1+a\rho)\nabla q\Big]\cdot\mathbf{n}=0
$$

Rearranging in terms of $q$ gives:

$$\left[q\frac{(a-1)\nabla\rho}{(a\rho-1)^2} + \nabla q\frac{(1+a\rho)}{(a\rho-1)^3}\right] \cdot \mathbf{n} = 0$$

# 2 Constriction Flow

## 2.1 Equilibrium

We choose $N = 50$, $n = 30$ and a box from $-3$ to $3$ in both directions. We choose the strength of $V_{ext}$ to be 10 instead of 1000 as in the paper. We vary constriction width $b$ and interaction strength $\kappa$ in equilibrium.
For $b = 0.6$ and varying interaction strengths, the results are shown in Figures 1, 2 and 3.

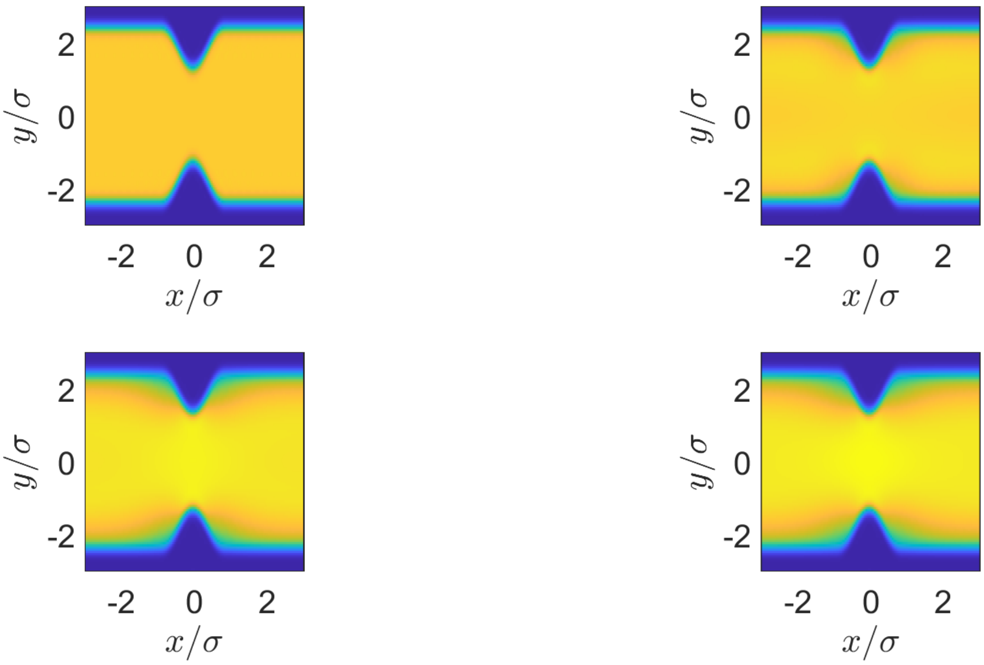


Figure 1: $b = 0.6$, $\kappa = -0.2$

In Figure 4 we see how the dynamics changes for a wider constriction (compare to Figure 2).

## 2.2 Non-Equilibrium

We impose a flow of size 1 on the equilibrium setup.
For $b = 0.6$ the results are displayed in Figures 5, 6, 7 and 8.

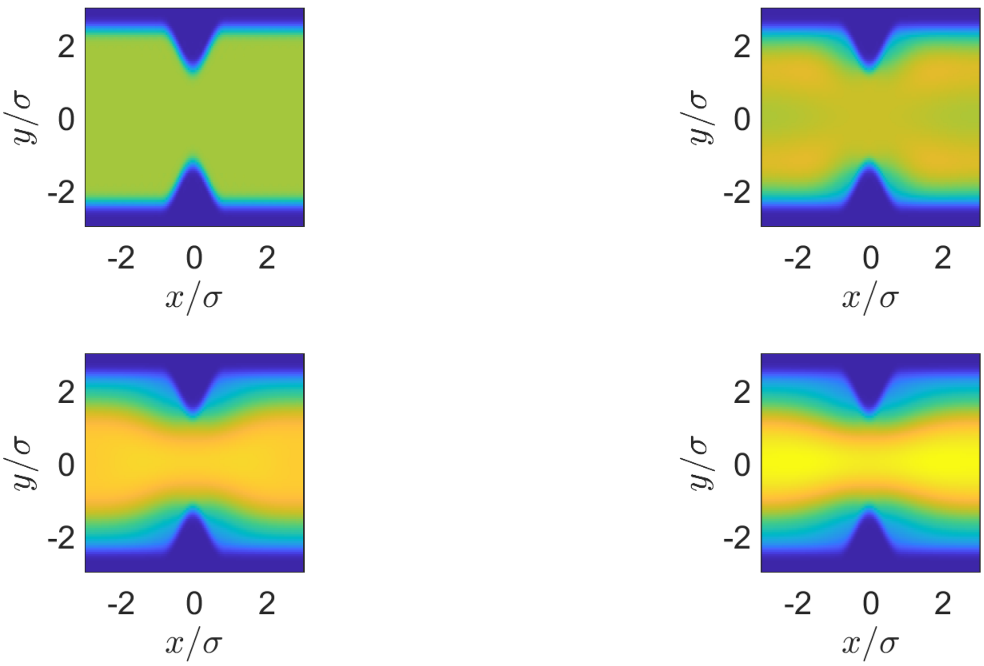


Figure 2: $b = 0.6$, $\kappa = -0.5$

## 3 Sedimentation

The result for the configurations as in Figure 8 in Archer's paper can be seen in Figure 9 (needs more points).

The result for the configurations as in Figure 10 in Archer's paper can be seen in Figures 10 and 11 (needs more points).

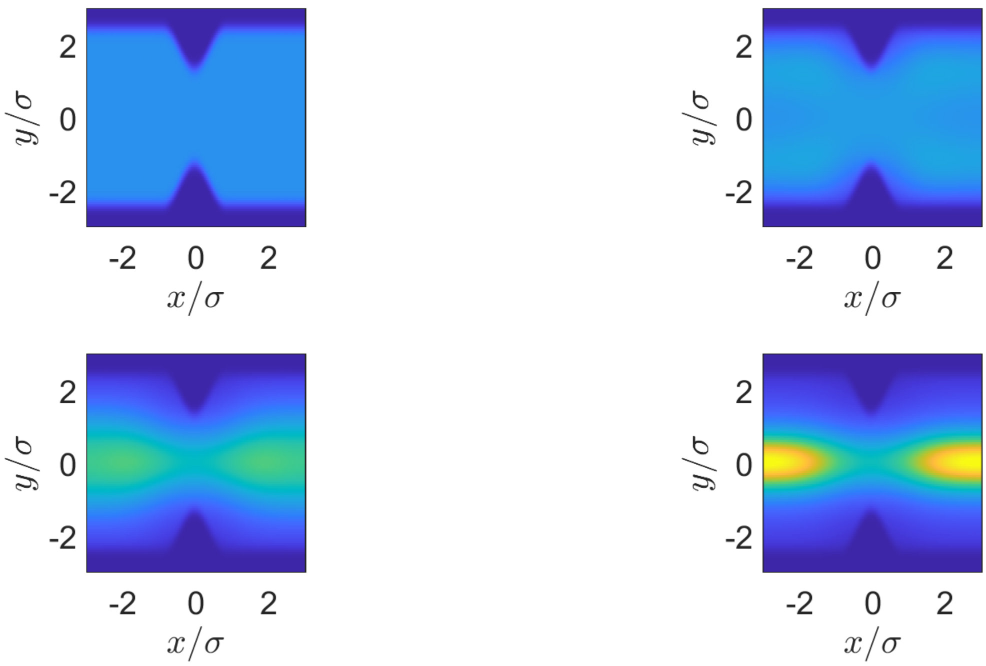


Figure 3: $b = 0.6$, $\kappa = -0.8$

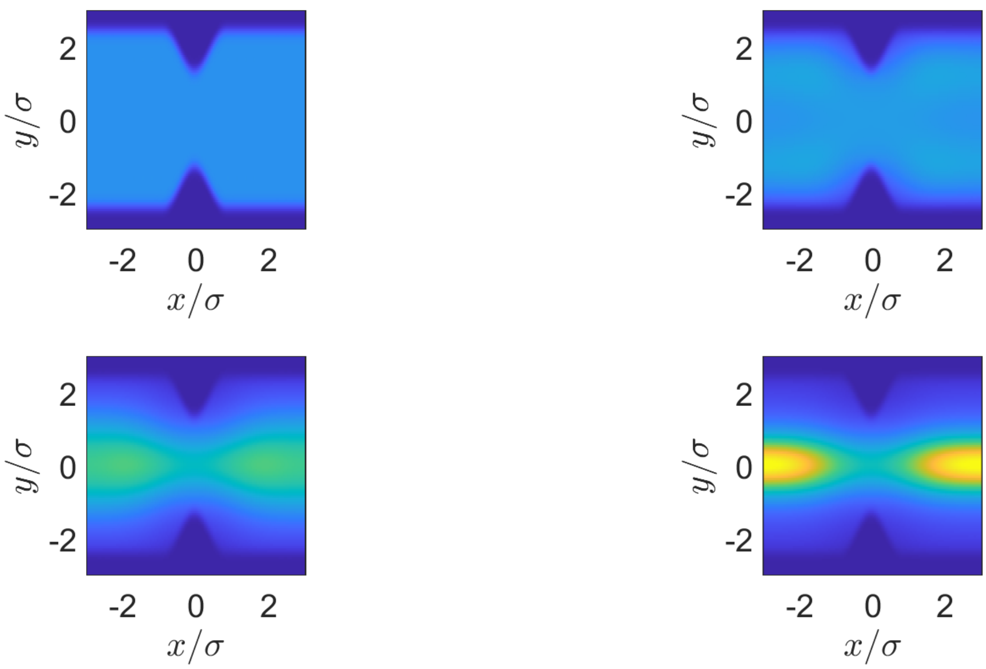


Figure 4: $b = 0.8$, $\kappa = -0.5$

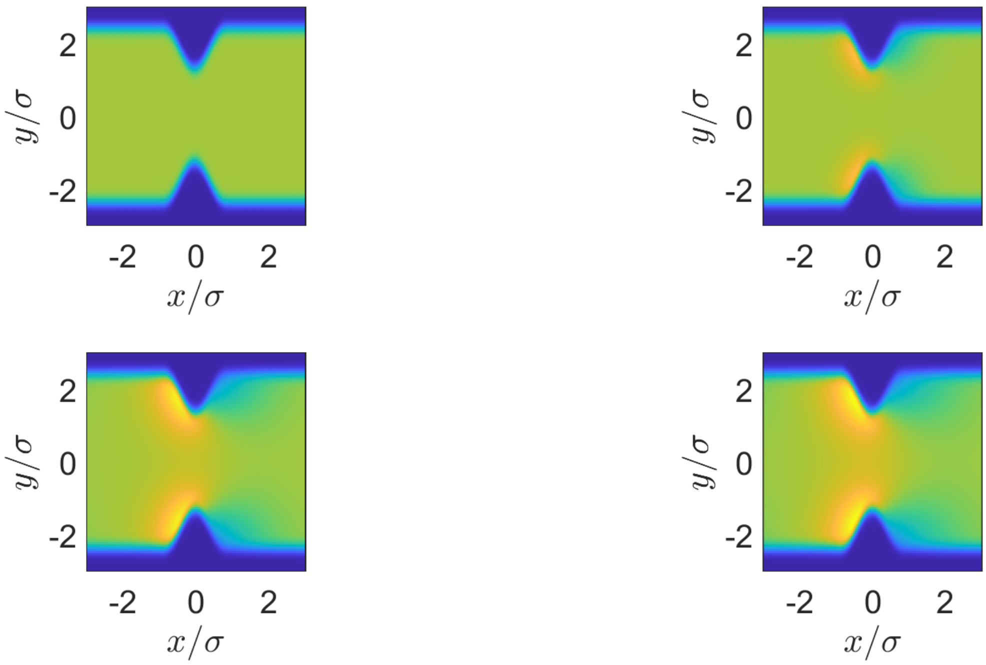


Figure 5: $b = 0.6$, $\kappa = 0$

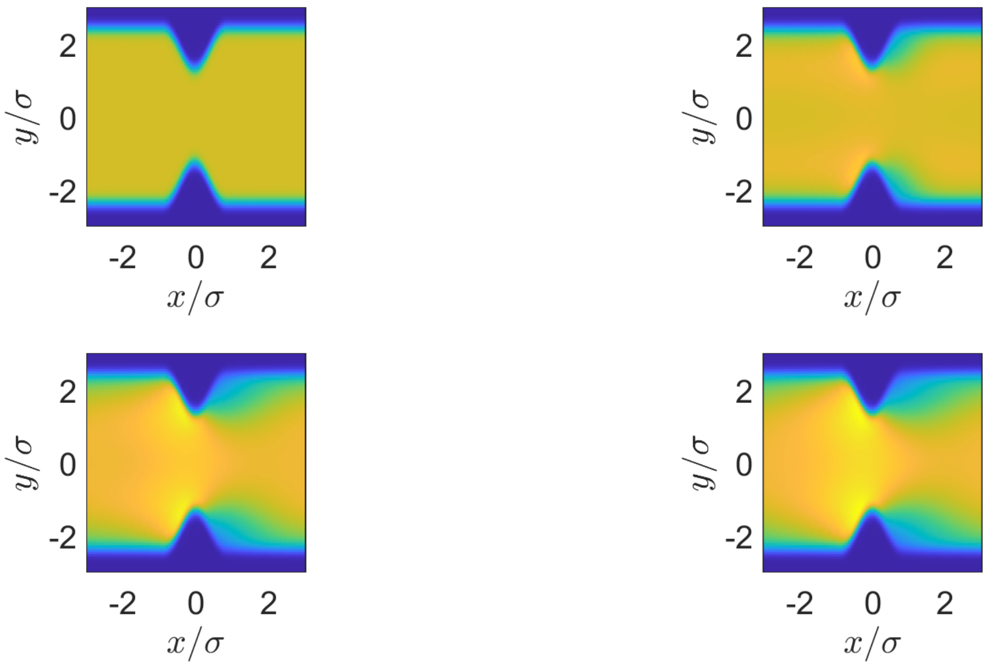


Figure 6: $b = 0.6$, $\kappa = -0.2$

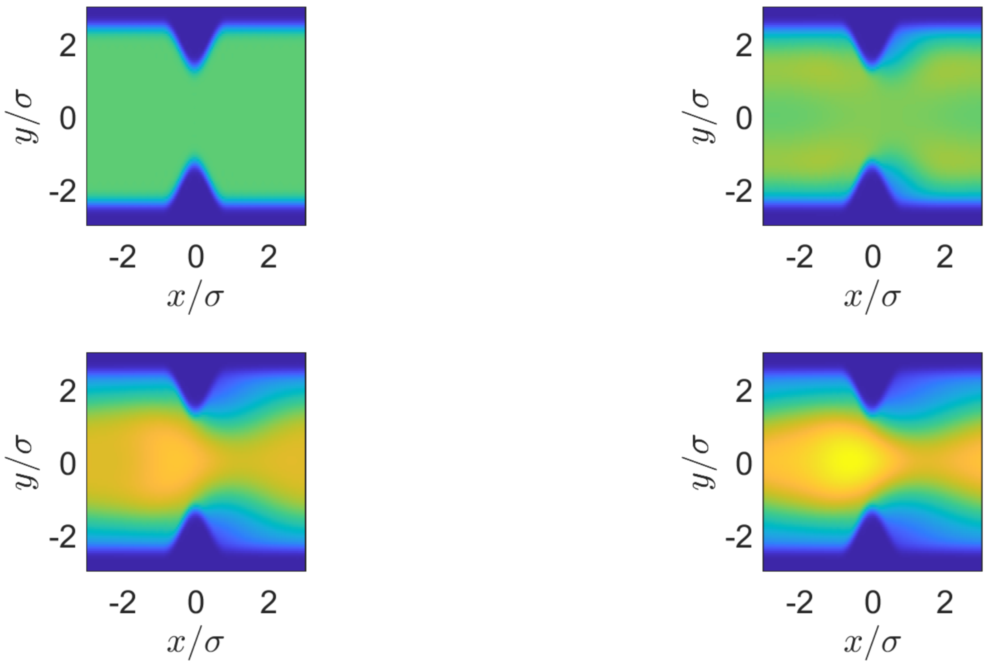


Figure 7: $b = 0.6$, $\kappa = -0.5$

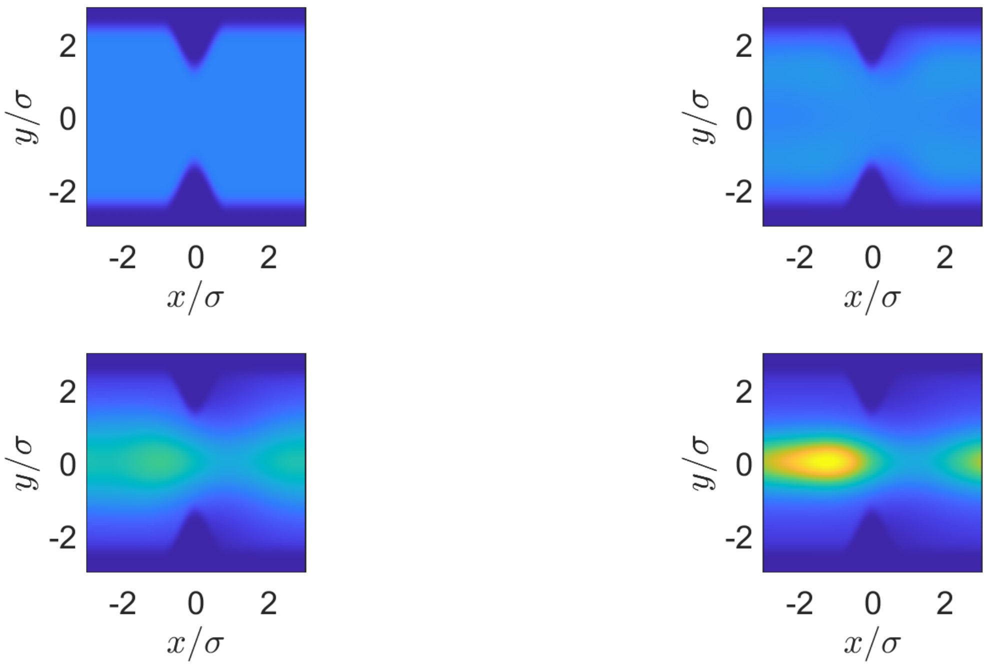


Figure 8: $b = 0.6$, $\kappa = -0.8$

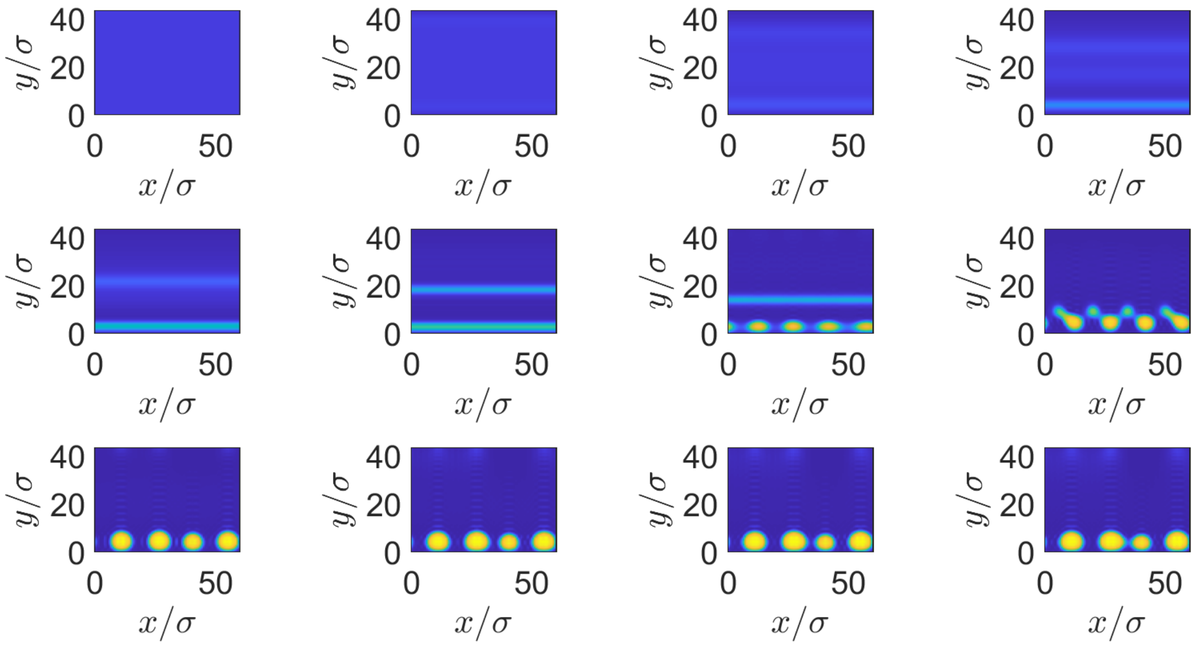


Figure 9: Result corresponding to Figure 8 in Archer's paper

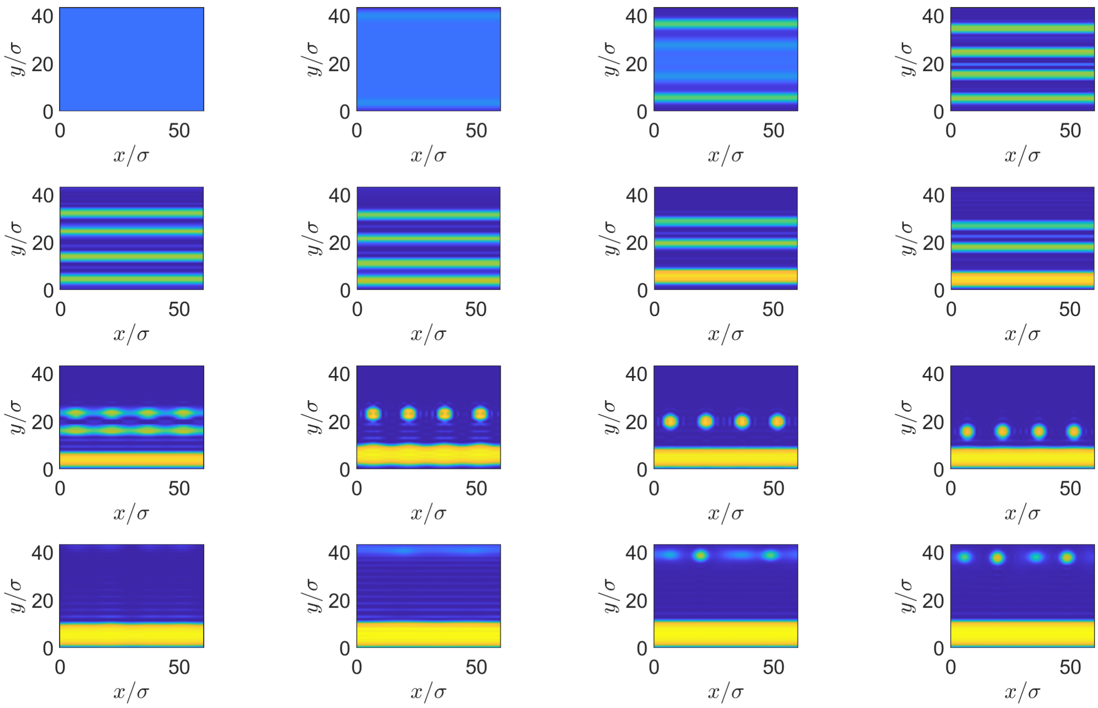


Figure 10: Result corresponding to Figure 10 in Archer's paper, at equispaced times

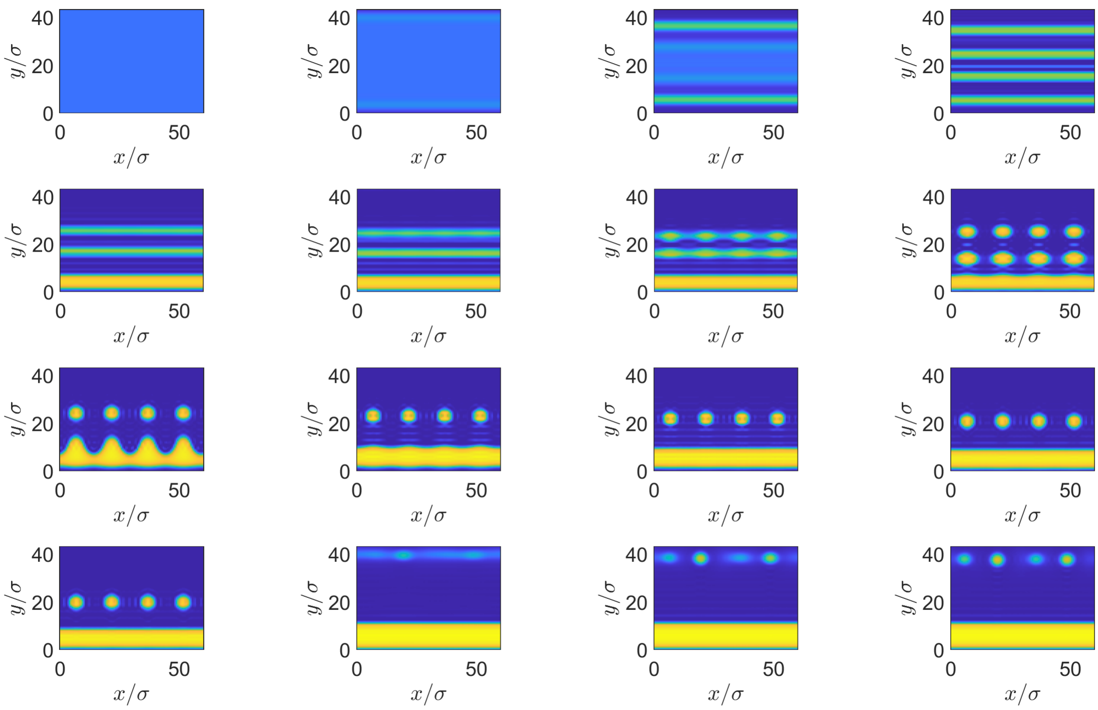


Figure 11: Result corresponding to Figure 10 in Archer's paper, at times 1,3,6,9,22,23,24,25,26,27,28,29,30, 43,46,49