

Table 1: Errors compared to exact solution for different discretizations (into one, two and three shapes) of the box and a quadrilateral

	One Shape	Two Shapes	Three Shapes	Four Shapes	Quadrilateral
Abs. E., $N = 20$	2.2063×10^{-7}	3.2073×10^{-7}	3.9108×10^{-7}	3.902×10^{-7}	2.2844×10^{-7}
Rel. E., $N = 20$	1.1235×10^{-9}	1.1377×10^{-9}	1.15×10^{-9}	1.0026×10^{-9}	1.116×10^{-9}
Abs. E., $N = 30$	2.1913×10^{-7}	3.1878×10^{-7}	3.8859×10^{-7}	3.8767×10^{-7}	2.2505×10^{-7}
Rel. E., $N = 30$	1.1159×10^{-9}	1.1308×10^{-9}	1.1427×10^{-9}	9.9612×10^{-10}	1.0995×10^{-9}

1 Validation Tests

1.1 Exact Tests

Several examples are run, using exact solutions, to validate the multishape code. This is done using an exact solution to the advection diffusion equation on an infinite domain, so that Dirichlet boundary conditions, matching the value of the exact solution on the boundary of the multishape, can be applied. The exact solution is [1]

$$\rho = \exp(\alpha t + \beta_1 y_1 + \beta_2 y_2)$$

$$\mathbf{v} = \left(\beta_1 - \frac{\alpha}{2\beta_1} + p_1 \exp(-\beta_1 y_1), \beta_2 - \frac{\alpha}{2\beta_2} + p_2 \exp(-\beta_2 y_2) \right),$$

where $\beta_1 = 0.1$, $\beta_2 = 0.1$, $\alpha = -0.5$, $p_1 = -1$ and $p_2 = 1$. We compare the exact solution on a box of dimensions $[0, 2] \times [0, 2]$ with different discretizations of the box using multishape, see Figure 1. Each of the shapes are discretized with $N = 20$ and $N = 30$ points in each spatial direction, which means that the dissected box has more points in total than the original box. The ODE solver tolerances are 10^{-9} . The solution can be seen in Figure 2. The question is whether the results of the PDE on the box and the different discretizations of the box have a similar error when compared to the exact solution. The absolute and relative errors are measured in an L_2 norm in space and an L_∞ norm in time and are displayed in Table 1. The leftmost column contains errors for the non-rectangular quadrilateral, which is shown in Figure 3.

The same test can be done for a wedge. Here, a single wedge and discretized versions are considered, see Figure 4. Table 2 shows the errors measured against the exact solution for different discretizations of the wedge for $N = 20$ and $N = 30$. The solution can be seen in Figure 5.

Next the advection diffusion equation is solved on a multishape which is composed of four quadrilaterals, see Figure 6. The absolute error for $N = 20$ on each shape as compared to the exact solution is 6.1246×10^{-7} and the relative error is 1.3259×10^{-9} . The same is done for

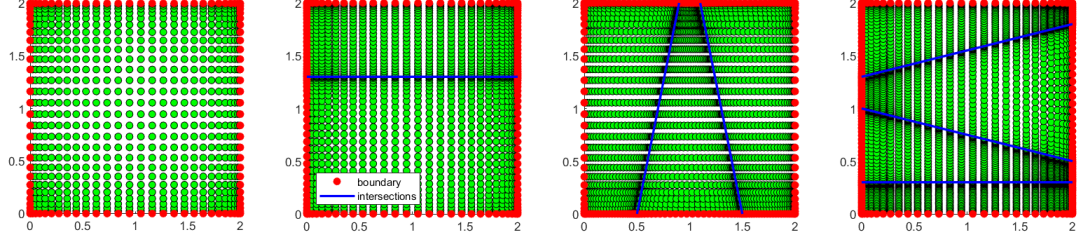


Figure 1: Different discretizations of the box.

Table 2: Errors compared to exact solution for different discretization (into one, two and three shapes) of the wedge

	One Shape	Two Shapes (a)	Two Shapes (b)	Three Shapes
Abs. Error, $N = 20$	3.1141×10^{-7}	3.9526×10^{-7}	4.2335×10^{-7}	4.4145×10^{-7}
Rel. Error, $N = 20$	7.4762×10^{-10}	7.5984×10^{-10}	7.4942×10^{-10}	7.0897×10^{-10}
Abs. Error, $N = 30$	2.7613×10^{-7}	3.8484×10^{-7}	3.892×10^{-7}	4.3211×10^{-7}
Rel. Error, $N = 30$	7.5178×10^{-10}	7.4375×10^{-10}	7.5061×10^{-10}	6.9683×10^{-10}

a second example involving a wedge, see Figure 7. The absolute error to the exact solution is 3.7239×10^{-7} and the relative error is 8.9375×10^{-10} , when choosing $N = 20$ per shape.

2 Forward Problems on multishapes

We first consider a forward problem on the different discretizations of the box, see Figure 1. We compare the error of the discretizations to the first box without discretization. We choose the initial condition for ρ to be

$$\rho_0 = \exp(-2((y_1 - 0.7)^2 + (y_2 - 0.2)^2))$$

and impose a constant flow of strength 0.8 acting upward. We choose $N = 20$ and $N = 30$ as before. The errors are displayed in Table 3 and the result can be seen in Figure 8.

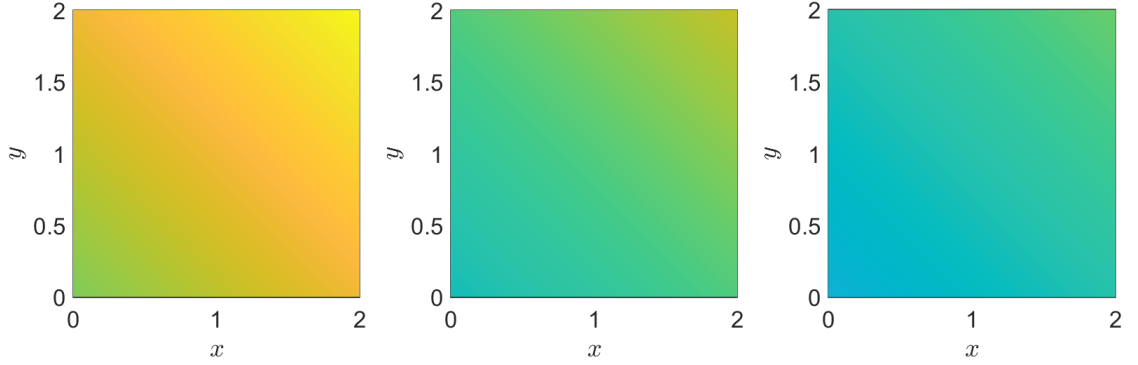


Figure 2: Exact solution on the box.

Table 3: Errors (compared to whole box) for different discretization (into two, three and four shapes) of the box

	Two Shapes	Three Shapes	Four Shapes
Abs. Error, $N = 20$	0.0197	0.0039	0.01
Rel. Error, $N = 20$	0.0015	0.0005	0.0007
Abs. Error, $N = 30$	0.0103	0.0021	0.0053
Rel. Error, $N = 30$	0.0005	0.0002	0.0002

We follow the same idea, but now consider the wedge discretizations, see Figure 4. We choose the initial condition for ρ to be

$$\rho_0 = \exp(-2((y_1 - 1.5)^2 + (y_2 - 4.5)^2))$$

and impose a constant flow of strength 3 acting from left to right, along the angular direction. We choose $N = 20$ and $N = 30$. The errors are displayed in Table 4 and the result can be seen in Figure 9.

Finally, two multishape examples are shown, which are of interesting shapes. The first of these examples is solving an advection diffusion problem on a multishape consisting of two quadrilaterals and two wedges, with constant velocity of strength ten. The initial condition for

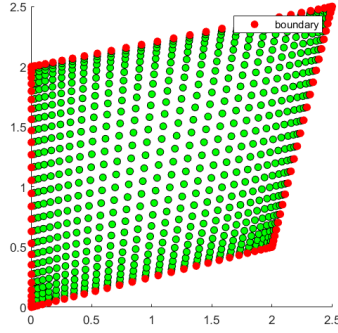


Figure 3: Quadrilateral domain.

Table 4: Errors (compared to whole wedge) for different discretization (into two and three shapes) of the wedge

	Two Shapes (a)	Two Shapes (b)	Three Shapes
Abs. Error, $N = 20$	0.0386	0.0076	0.0693
Rel. Error, $N = 20$	0.0044	0.0007	0.0047
Abs. Error, $N = 30$	0.0215	0.004	0.0383
Rel. Error, $N = 30$	0.0016	0.0002	0.0017

this problem is:

$$\rho_0 = \exp(-2(y_1 - 0.5)^2 - 2(y_2 + 1)^2).$$

The result, evaluated for $N = 20$ on each shape, can be seen in Figure 10.

In a second example, the velocity is of strength 5 and the initial condition is:

$$\rho_0 = \exp(-2(y_1 - 0.5)^2 - 2(y_2 - 1.5)^2).$$

The result, which is computed on a multishape made up of four quadrilaterals into a channel, can be seen in Figure 11.

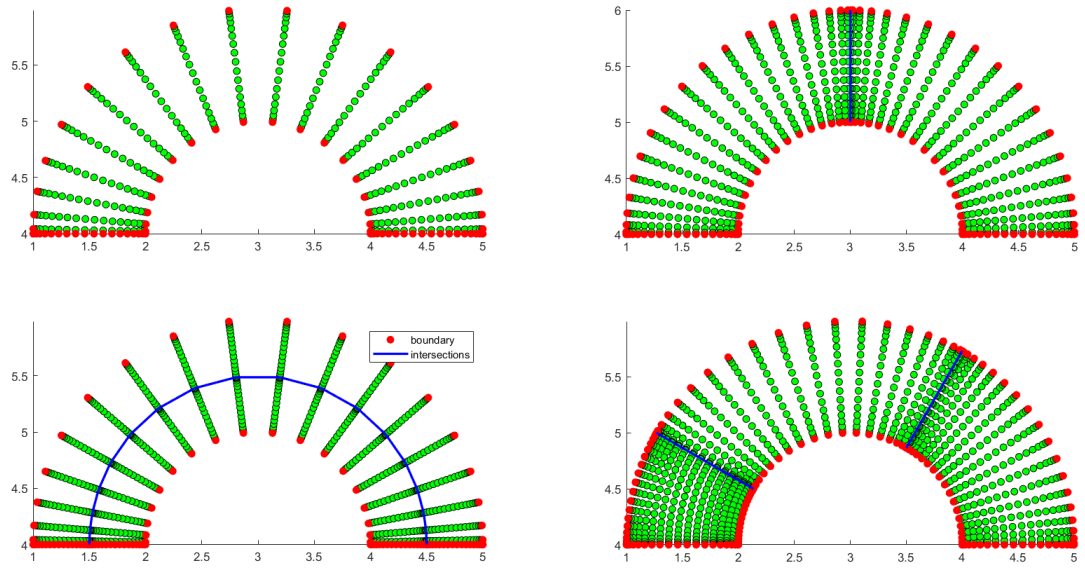


Figure 4: Different discretizations of the wedge.

References

- [1] G D Hutomo, J Kusuma, A Ribal, A G Mahie, and N Aris. Numerical solution of 2-d advection-diffusion equation with variable coefficient using du-fort frankel method. *Journal of Physics: Conference Series*, 1180:012009, feb 2019.

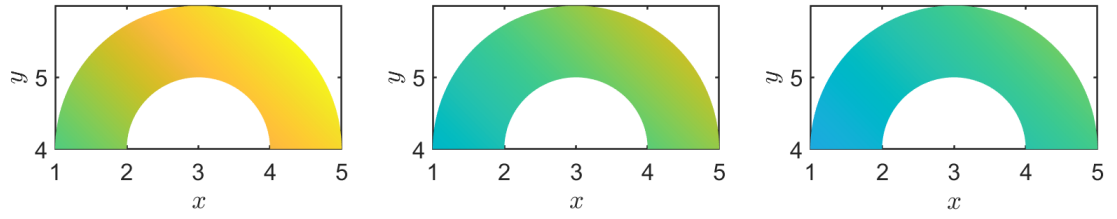


Figure 5: Exact solution on the wedge.

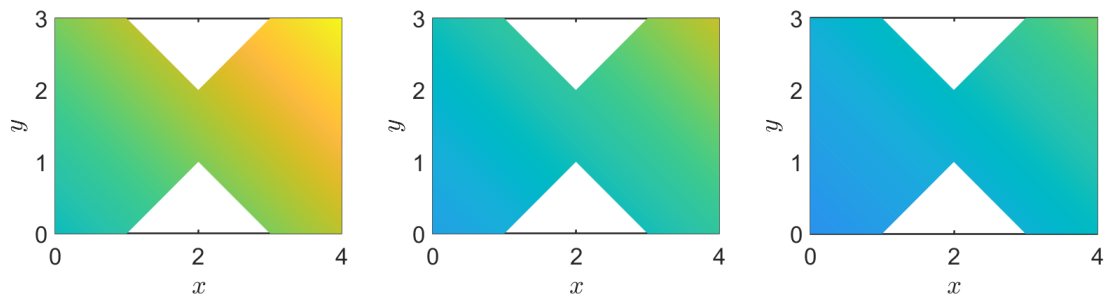


Figure 6: Example 1 multishape

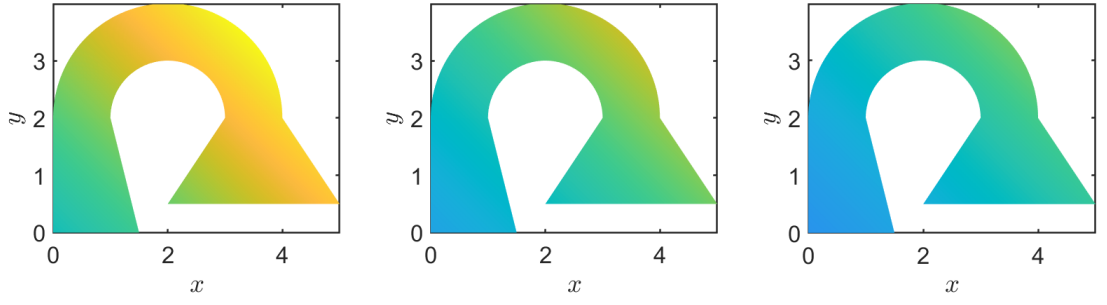


Figure 7: Example 2 multishape

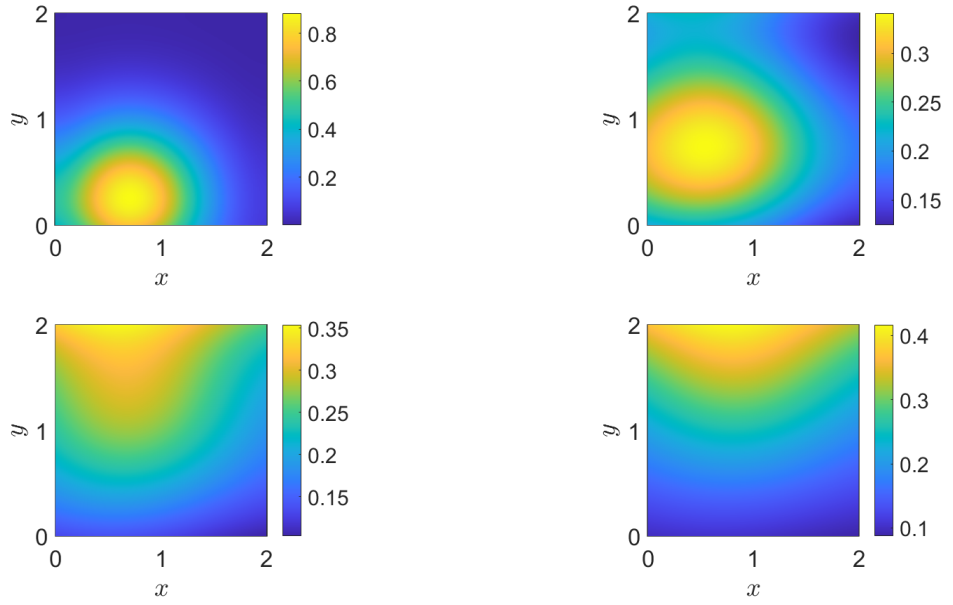


Figure 8: Forward Example on box discretizations

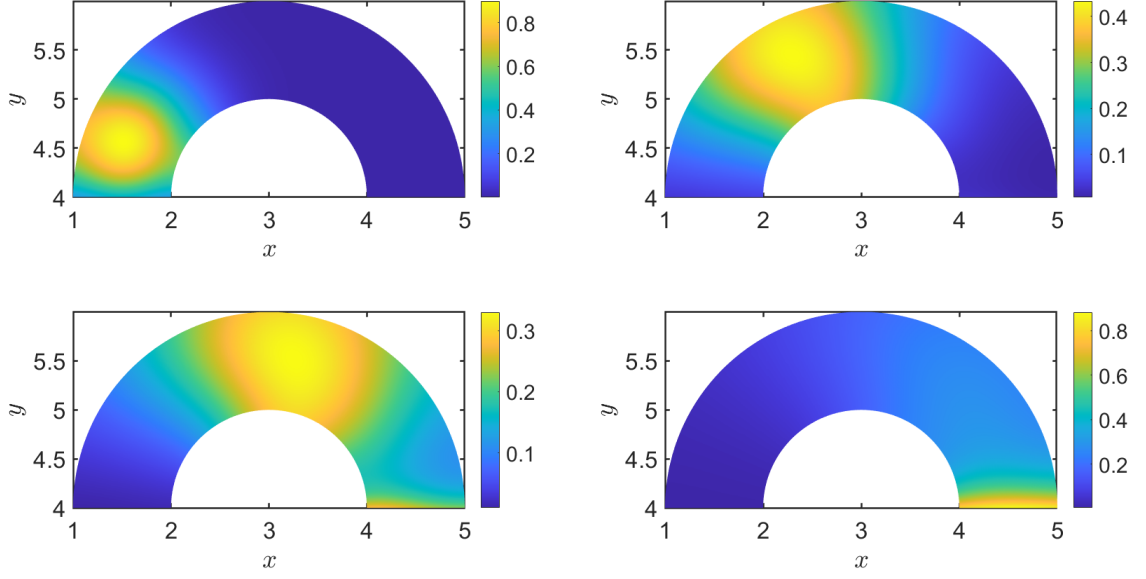


Figure 9: Forward Example on wedge discretizations

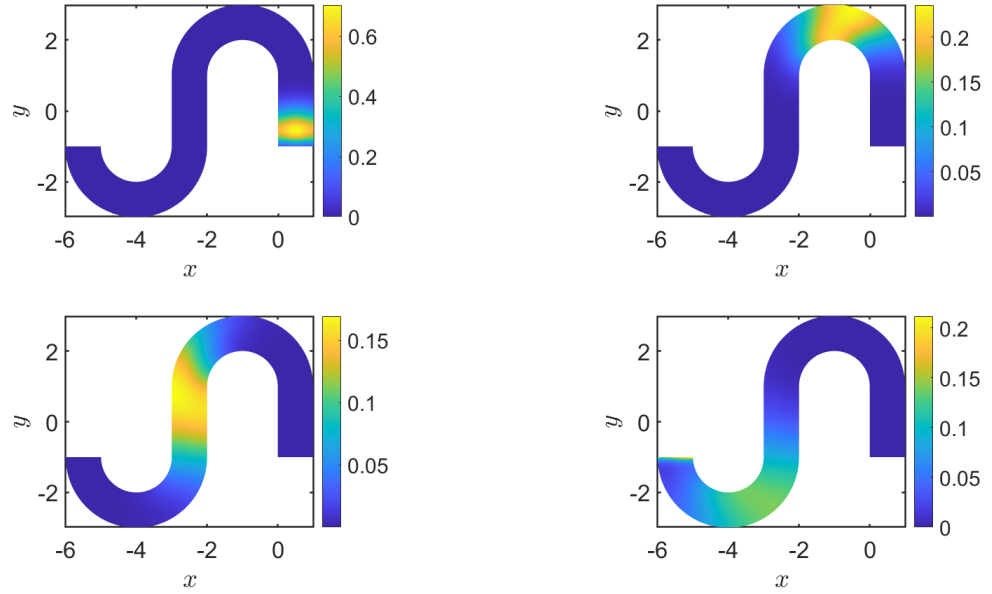


Figure 10: Forward Problem 1, different colour scale for each plot to highlight particle mass location

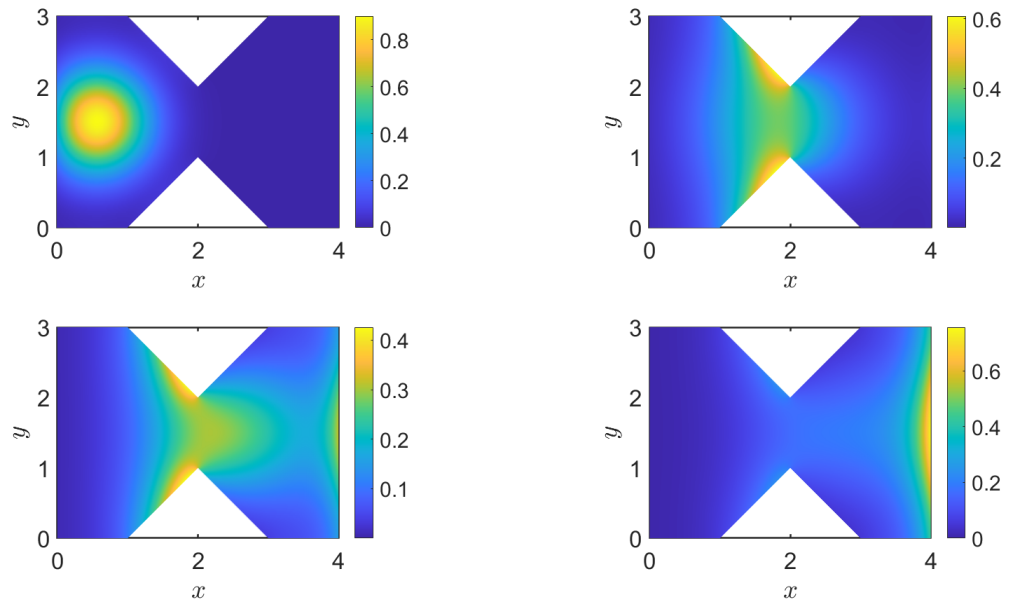


Figure 11: Forward Problem 2, different colour scale for each plot to highlight particle mass location