

## Optimality System of the Averaged Advection Diffusion Equation

The optimality system is:

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{1}{r} \frac{\partial \rho}{\partial r} + \frac{\partial^2 \rho}{\partial r^2} + \frac{\partial \rho}{\partial z^2} - \frac{\rho \mathbf{w}_r}{r} - \nabla \cdot (\rho \mathbf{w}) + f \\ 0 &= \left( - \left( \frac{\partial \rho}{\partial r}, \frac{\partial \rho}{\partial z} \right) + \rho \mathbf{w} \right) \cdot \mathbf{n}\end{aligned}$$

$$\begin{aligned}\frac{\partial q}{\partial t} &= -\frac{1}{r} \frac{\partial q}{\partial r} - \frac{\partial^2 q}{\partial r^2} - \frac{\partial q}{\partial z^2} - \mathbf{w} \cdot \left( \frac{\partial q}{\partial r}, \frac{\partial q}{\partial z} \right) - \rho + \hat{\rho} \\ 0 &= \left( \frac{\partial q}{\partial r}, \frac{\partial q}{\partial z} \right) \cdot \mathbf{n}\end{aligned}$$

$$\mathbf{w} = -\frac{1}{\beta} \rho \left( \frac{\partial q}{\partial r}, \frac{\partial q}{\partial z} \right)$$

Log I used the expanded version of the term in the implementation, where  $\nabla$  is defined with respect to  $r$  and  $z$ :  $\nabla \cdot (\rho \mathbf{w}) = \mathbf{w}_r \frac{\partial \rho}{\partial r} + \mathbf{w}_z \frac{\partial \rho}{\partial z} + \rho \frac{\partial \mathbf{w}_r}{\partial r} + \rho \frac{\partial \mathbf{w}_z}{\partial z}$ .

### Exact Solution

We are choosing an exact solution which satisfies the boundary conditions, matches the final time condition for  $q$  and is invariant in  $\theta$ . We choose:

$$\begin{aligned}\rho &= \beta^{1/2} e^t \cos(\pi r) \cos(\pi z) \\ q &= \beta^{1/2} (e^T - e^t) \cos(\pi r) \cos(\pi z),\end{aligned}$$

and use these to determine the values of  $\mathbf{w}$ ,  $f$  and  $\hat{\rho}$ .