1 Optimality conditions for the sedimentation equations

The relevant part of the equation is:

$$\nabla \cdot \left(\rho \nabla \frac{\delta F[\rho]}{\delta \rho}\right) = \frac{1}{\beta}\left(\frac{\nabla^2 \rho}{1-\eta} + \nabla \rho \cdot \nabla \frac{(3-2\eta)}{(1-\eta)^2} - \rho \nabla^2 \frac{\eta - 2}{(\eta-1)^2}\right),$$

where $\eta = a\rho$ and $a = \pi\sigma^2/4$. Consider:

$$\begin{aligned} F_1(\rho) &= \nabla^2 \rho \frac{1}{1-a\rho} \\ F_2(\rho) &= \nabla \rho \cdot \nabla \left(\frac{3-2a\rho}{(1-a\rho)^2}\right) \\ F_3(\rho) &= \rho \nabla^2 \left(\frac{a\rho - 2}{(a\rho-1)^2}\right) \end{aligned}$$

Then

$$F_1(\rho+h) - F_1(\rho) = \nabla(\rho+h)\frac{1}{1-a(\rho+h)} - \nabla\rho \frac{1}{1-a\rho}$$

Using the expansion:

$$\frac{1}{c-x} = \frac{1}{c} + \frac{1}{c^2}x + O(x^2),$$

where $c = 1 - a\rho$, we get:

$$\begin{aligned} F_1(\rho+h) - F_1(\rho) &= \nabla^2(\rho+h)\left(\frac{1}{1-a\rho} + \frac{a}{(1-a\rho)^2}h\right) - \nabla^2\rho\frac{1}{1-a\rho} \\ &= \nabla^2 h\left(\frac{1}{1-a\rho}\right) + \nabla^2\rho\left(\frac{a}{(1-a\rho)^2}h\right) \end{aligned}$$

For F_2 we consider the expansion:

$$\frac{1}{(c-x)^2} = \frac{1}{c^2} + \frac{2}{c^3}x + O(x^2),$$

and get:

$$\begin{aligned} F_2(\rho+h) - F_2(\rho) &= \nabla(\rho+h)\cdot\nabla\left(\frac{3-2a(\rho+h)}{(1-a(\rho+h))^2}\right) - \nabla\rho\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) \\ &= \nabla(\rho+h)\cdot\nabla\left(\frac{3-2a(\rho+h)}{(1-a\rho)^2} + \frac{3-2a(\rho+h)}{(1-a\rho)^3}2ah\right) - \nabla\rho\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) \\ &= \nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + \nabla\rho\cdot\nabla\left(h\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right) \\ &= \nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + (\nabla h\cdot\nabla\rho)\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) \\ &+ h\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) \end{aligned}$$

Finally, we have:

$$\begin{aligned}
F_3(\rho+h) - F_3(\rho) &= (\rho+h)\nabla^2\left(\frac{a(\rho+h)-2}{(a(\rho+h)-1)^2}\right) - \rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
&= (\rho+h)\nabla^2\left(\frac{a(\rho+h)-2}{(1-a\rho)^2} + \frac{a(\rho+h)-2}{(1-a\rho)^3}2ah\right) - \rho\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
&= h\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) + \rho\nabla^2\left(h\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
&= h\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) + \rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\nabla^2 h \\
&\quad + \rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\cdot\nabla h + \rho h\nabla^2\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)
\end{aligned}$$

Combining these in the Lagrangian gives:

$$\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega & q\nabla^2 h\left(\frac{1}{1-a\rho}\right) + q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}h\right) \\
& + q\nabla h\cdot\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\left(\nabla h\cdot\nabla\rho\right)\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) \\
& + qh\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) \\
& - qh\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\nabla^2 h \\
& - q\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\cdot\nabla h - q\rho h\nabla^2\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right).
\end{aligned}$$

Rearranging gives:

$$\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega & h\Bigg(q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
& - q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) \\
& + \nabla h\cdot\Bigg(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) \\
& + \nabla^2 h\Bigg(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)
\end{aligned}$$

Integration by parts gives:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Bigg(& q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
& - q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) \\
& - h\nabla\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) - q\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
& + h\nabla^2\left(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)
\end{aligned}
$$

So we have:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_\Omega h\Bigg[& q\nabla^2\rho\left(\frac{a}{(1-a\rho)^2}\right) + q\nabla\rho\cdot\nabla\left(\frac{-2a}{(1-a\rho)^2}\right) - q\nabla^2\left(\frac{a\rho-2}{(a\rho-1)^2}\right) \\
& - q\rho\nabla^2\left(\frac{a}{(1-a\rho)^2}\right) - q\rho\nabla^2\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right) \\
& - \nabla\cdot\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}\right)\right) - \nabla\cdot\left(q\nabla\rho\left(\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right) \\
& + \nabla\cdot\left(q\rho\nabla\left(\frac{a}{(1-a\rho)^2}\right)\right) + \nabla\cdot\left(q\rho\nabla\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
& + \nabla^2\left(q\left(\frac{1}{1-a\rho}\right)\right) - \nabla^2\left(q\rho\left(\frac{a}{(1-a\rho)^2}\right)\right) - \nabla^2\left(q\rho\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)\Bigg]drdt
\end{aligned}
$$

1.1 Boundary Terms

We have the equation:

$$
\rho\nabla\frac{\delta F[\rho]}{\delta\rho} = \frac{1}{\beta}\left(\frac{\nabla\rho}{1-\eta} - \rho\nabla\frac{\eta-2}{(\eta-1)^2}\right)
$$

Then:

$$
\begin{aligned}
F_4(\rho) &= \frac{\nabla\rho}{1-a\rho} \\
F_5(\rho) &= \rho\nabla\frac{a\rho-2}{(a\rho-1)^2}
\end{aligned}
$$

Then for F_4 we have:

$$\begin{aligned}
F_4(\rho+h)-F_4(\rho) &= \nabla(\rho+h)\frac{1}{1-a(\rho+h)}-\nabla\rho\frac{1}{1-a\rho} \\
&= \nabla(\rho+h)\left(\frac{1}{1-a\rho}+\frac{1}{(1-a\rho)^2}h\right) \\
&= \nabla h\left(\frac{1}{1-a\rho}\right)+\nabla\rho\left(\frac{1}{(1-a\rho)^2}h\right)
\end{aligned}$$

For F_5 we get:

$$\begin{aligned}
F_5(\rho+h)-F_5(\rho) &= (\rho+h)\nabla\frac{a(\rho+h)-2}{(a(\rho+h)-1)^2}-\rho\nabla\frac{a\rho-2}{(a\rho-1)^2} \\
&= (\rho+h)\nabla\left(\frac{a(\rho+h)-2}{(1-a\rho)^2}+\frac{a(\rho+h)-2}{(1-a\rho)^3}2ah\right)-\rho\nabla\frac{a\rho-2}{(a\rho-1)^2} \\
&= h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)+\rho\nabla\left(h\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
&= h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)+h\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)+\nabla h\left(\rho\frac{a}{(1-a\rho)^2}+\rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)
\end{aligned}$$

Then the boundary terms for the Lagrangian are:

$$\begin{aligned}
\mathcal{L}_{\rho,1}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}&\left(q_{\partial\Omega}\nabla h\left(\frac{1}{1-a\rho}\right)+q_{\partial\Omega}\nabla\rho\left(\frac{1}{(1-a\rho)^2}h\right)-q_{\partial\Omega}h\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right)\right. \\
&\left.-hq_{\partial\Omega}\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)-q_{\partial\Omega}\nabla h\left(\rho\frac{a}{(1-a\rho)^2}+\rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)\cdot\mathbf{n}drdt
\end{aligned}$$

From the integration by parts of the terms within the domain we get:

$$\begin{aligned}
\mathcal{L}_{\rho,2}(\rho,\mathbf{w},q)h = -\frac{1}{\beta}\int_0^T\int_{\partial\Omega}&\left(h\left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right)+q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right.\right. \\
&\left.-q\rho\nabla\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
&+\nabla h\left(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) \\
&\left.-h\nabla\left(q\left(\frac{1}{1-a\rho}\right)-q\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right)\right)\cdot\mathbf{n}drdt
\end{aligned}$$

Combining all of these give all boundary terms for the Lagrangian:

$$
\begin{aligned}
\mathcal{L}_\rho(\rho, \mathbf{w}, q)h = -\frac{1}{\beta}\int_0^T \int_{\partial\Omega} \Bigg(h\Bigg(& q_{\partial\Omega}\nabla\rho\left(\frac{1}{(1-a\rho)^2}\right) - q_{\partial\Omega}\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right) \\
& - q_{\partial\Omega}\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right) + \left(q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\right. \\
& \left. - q\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\right) - \nabla\Bigg(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg) \\
& + \nabla h\Bigg(q_{\partial\Omega}\left(\frac{1}{1-a\rho}\right) - q_{\partial\Omega}\left(\rho\frac{a}{(1-a\rho)^2} + \rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right) + q\left(\frac{1}{1-a\rho}\right) \\
& - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg) \cdot \mathbf{n}\, dr\, dt
\end{aligned}
$$

Comparing terms in ∇h (there should be $\cdot\mathbf{n}$ everywhere in the below):

$$
\begin{aligned}
& q_{\partial\Omega}\left(\frac{1}{1-a\rho}\right) - q_{\partial\Omega}\left(\rho\frac{a}{(1-a\rho)^2} + \rho\frac{2a^2\rho-4a}{(1-a\rho)^3}\right) \\
& + q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right) = 0.
\end{aligned}
$$

This holds when $q_{\partial\Omega} = -q$. Then for $h \neq 0$ we get:

$$
\begin{aligned}
& - q\nabla\rho\left(\frac{1}{(1-a\rho)^2}\right) + q\nabla\left(\frac{a\rho-2}{(1-a\rho)^2}\right) \\
& + q\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right) + q\nabla\left(\frac{3-2a\rho}{(1-a\rho)^2}\right) + q\nabla\rho\left(\frac{-2a}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right) \\
& - q\rho\nabla\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right) - \nabla\Bigg(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) = 0
\end{aligned}
$$

Then:

$$
\begin{aligned}
& q\Bigg(\nabla\left(\frac{a\rho-2}{(1-a\rho)^2} + \frac{3-2a\rho}{(1-a\rho)^2}\right) + \nabla\rho\left(\frac{-2a}{(1-a\rho)^2} - \frac{1}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\Bigg) \\
& - \nabla\Bigg(q\left(\frac{1}{1-a\rho}\right) - q\rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) = 0
\end{aligned}
$$

and so:

$$
\begin{aligned}
& q\Bigg(\nabla\left(\frac{a\rho-2}{(1-a\rho)^2} + \frac{3-2a\rho}{(1-a\rho)^2}\right) + \nabla\rho\left(\frac{-2a}{(1-a\rho)^2} - \frac{1}{(1-a\rho)^2} + \frac{6a-4a^2\rho}{(1-a\rho)^3}\right)\Bigg) \\
& + q\nabla\Bigg(\left(\frac{1}{1-a\rho}\right) - \rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) - \nabla q\Bigg(\left(\frac{1}{1-a\rho}\right) - \rho\left(\frac{a}{(1-a\rho)^2} + \frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg) = 0
\end{aligned}
$$

The terms with ∇q don't cancel nicely so there is a mistake somewhere.

$$q\Bigg(\nabla\left(\frac{-2a\rho+1}{(1-a\rho)^2}\right)+\nabla\rho\left(\frac{-2a}{(1-a\rho)^2}-\frac{1}{(1-a\rho)^2}+\frac{6a-4a^2\rho}{(1-a\rho)^3}\right)+\nabla\Bigg(\left(\frac{1}{1-a\rho}\right)-\rho\left(\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)\Bigg)$$
$$-\nabla q\Bigg(\left(\frac{1}{1-a\rho}\right)-\rho\left(\frac{a}{(1-a\rho)^2}+\frac{2a^2\rho-4a}{(1-a\rho)^3}\right)\Bigg)=0$$

2 Flow through constriction

First Question: Why are the stripes there. If we look at V_{ext} in one dimension, no oscillation is noticeable, see Figure 2 and Figure 3 and 1.

Second Question: How to improve the performance with the transformation. We have $\rho = \exp(h - V_{ext})$. The issue was that ρ is zero at some point so $h = \ln \rho + V_{ext}$ was the point of trouble, since we get points at negative infinity. I added 10^{-5} to ρ_0 to circumvent that but it's still very slow and I am not sure if it's doing much better than the non-transformed problem. In fact, it runs to the end but gives a warning that the integration tolerances cannot be met, so it looks like it's worse than the original. And it even struggles with the case without the constriction. I use $n = 30$ and $N = 50$ and the initial condition is $\rho = \exp(-V_{ext})$.

3 Sedimentation Periodic

Now that the periodic case is working, I do not know why we do not get the same behaviour as in Archer's paper. We are choosing the two configurations in Archer's paper. $\bar{\rho} = 0.072/\sigma^2$ and $\bar{\rho} = 0.2/\sigma^2$. We set the domain to be 43×60, as in Archer's paper. We use $N = 50$ and $n = 30$, and $TMax = 60$.
At first we consider $\sigma = 1$ and so $\bar{\rho} = 0.072$, see Figure 4. Then we consider $\sigma = 0.5$ and so $\bar{\rho} = 0.072$, see Figure 5. We get a similar behaviour, but the times are not quite the same.
Then we consider $\sigma = 1$ and $\bar{\rho} = 0.2$, see Figure 6. Then we consider $\sigma = 0.5$ and $\bar{\rho} = 0.2$, see Figure 7.

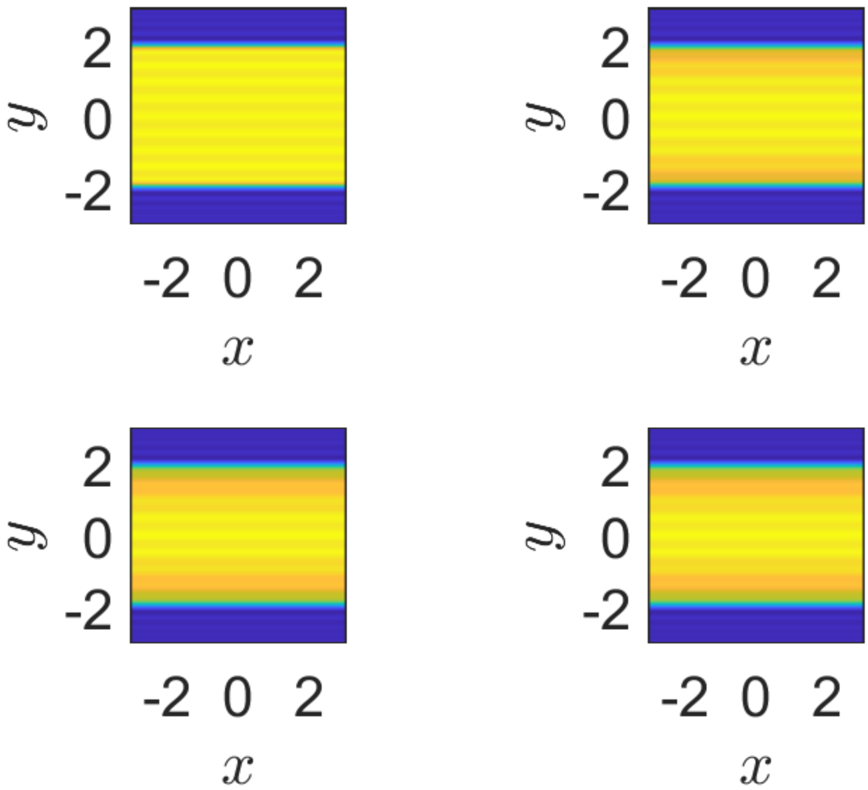

Figure 1: Stripes in ρ but no oscillation in V_{ext}

4 Sedimentation Multishape

One odd thing here is that using Archer's interaction potential two shapes are treated separately.This also happens when we use the Gaussian potential. The two potentials are:

$$V_A = \exp(-y/\sigma)$$
$$V_G = \exp(-y^2).$$

I am using two quadrilaterals and the external potential is acting downward as before, $n = 20$ and for each shape $N = 30$. Using V_A is displayed in Figure 8 and using V_9 is displayed in Figure 10. But looks like my problem is solved now... see Figure 9.
When using $\bar{\rho} = 0.2$, the code breaks due to lack of points I think. When $\bar{\rho} = 0.0001$, we get smooth behaviour in both. So if that is the only problem then we can make some pretty plots for the sake of it, see Figure 11.

Another weird thing, which doesn't have to do with the multishape is the inital condition

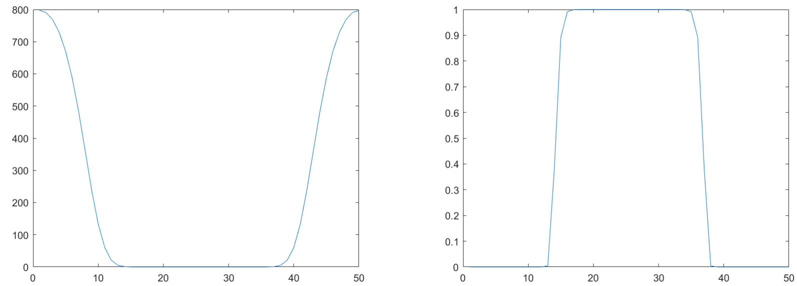

Figure 2: V_{ext} and ρ in 1D, $b = 1$

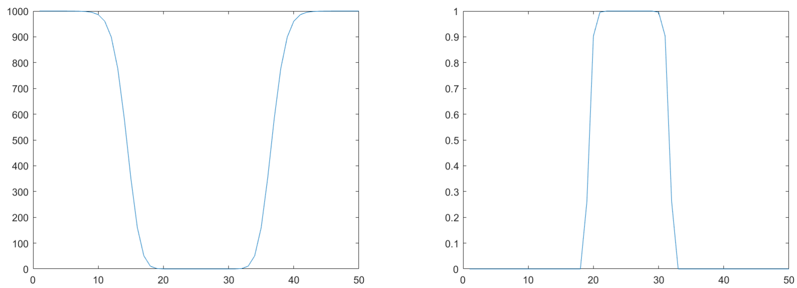

Figure 3: V_{ext} and ρ in 1D, $b = 0.7$

for ρ. When plotting the solution for ρ at time 0, this is not uniform but influenced by V_{ext} and V_2. We assumed that was due to the perturbation in ρ_0 but this is with uniform ρ_0. The different influences are displayed in Figure 13. I guess these effects are normal though?

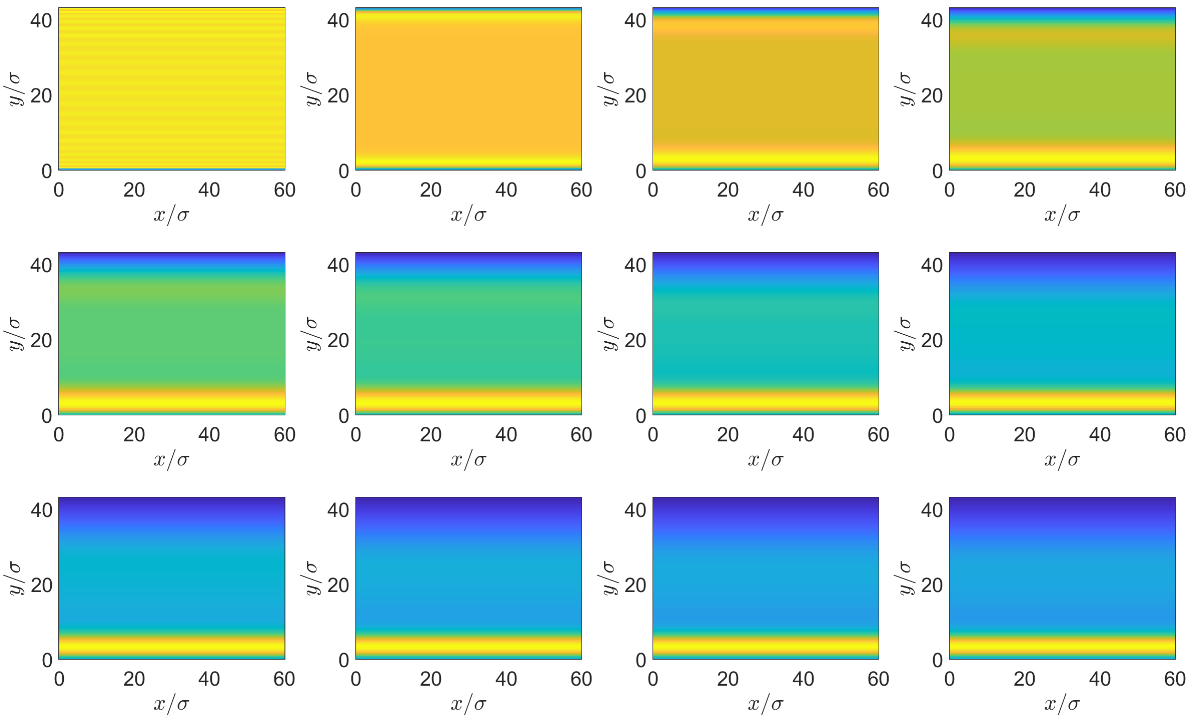

Figure 4: $\sigma = 1$, $\bar{\rho} = 0.072$

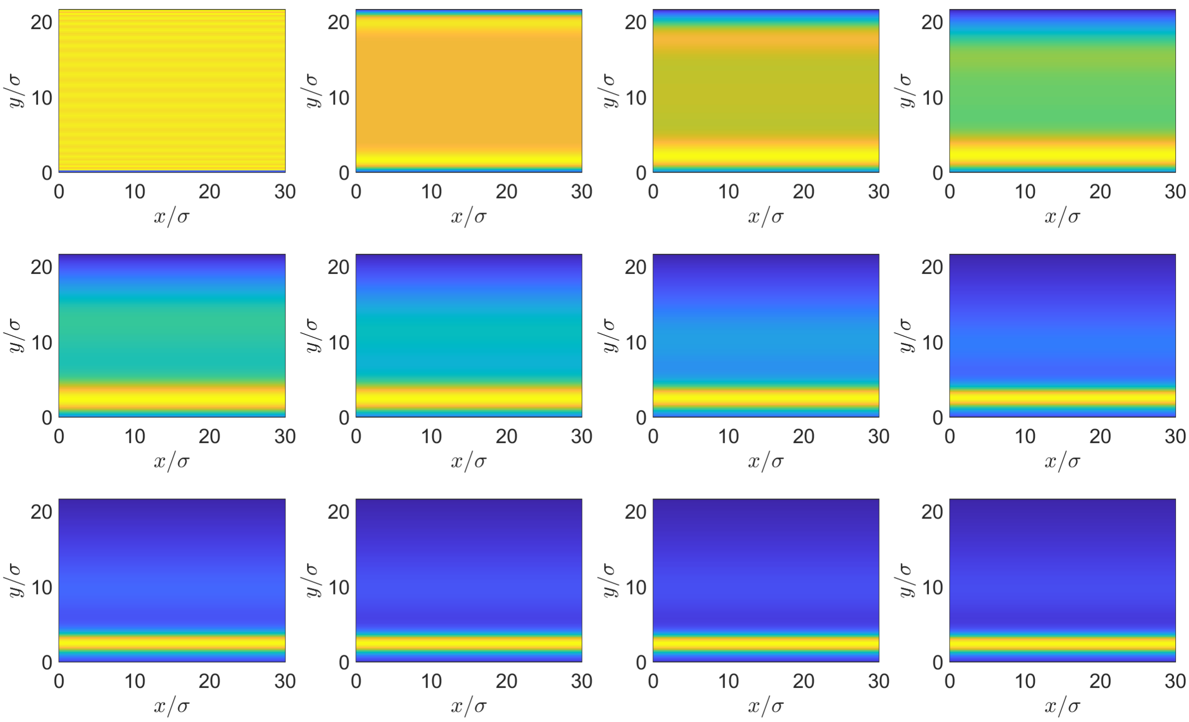

Figure 5: $\sigma = 0.5$, $\bar{\rho} = 0.072$

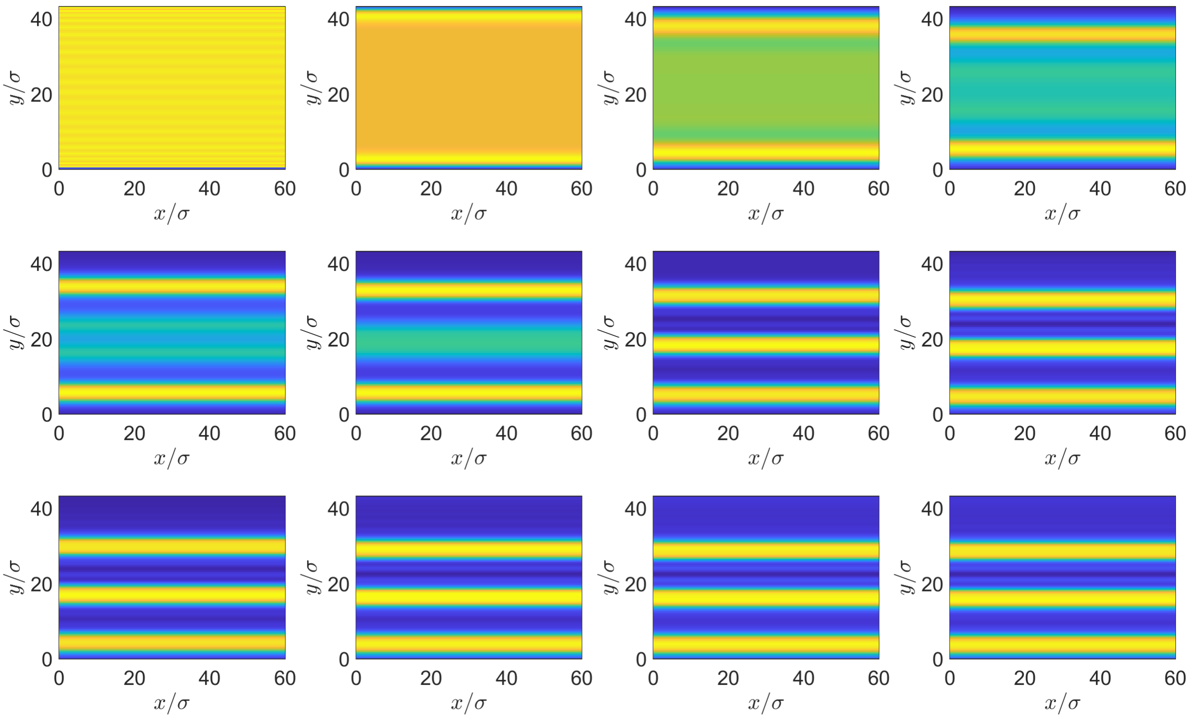

Figure 6: $\sigma = 1$, $\bar{\rho} = 0.2$

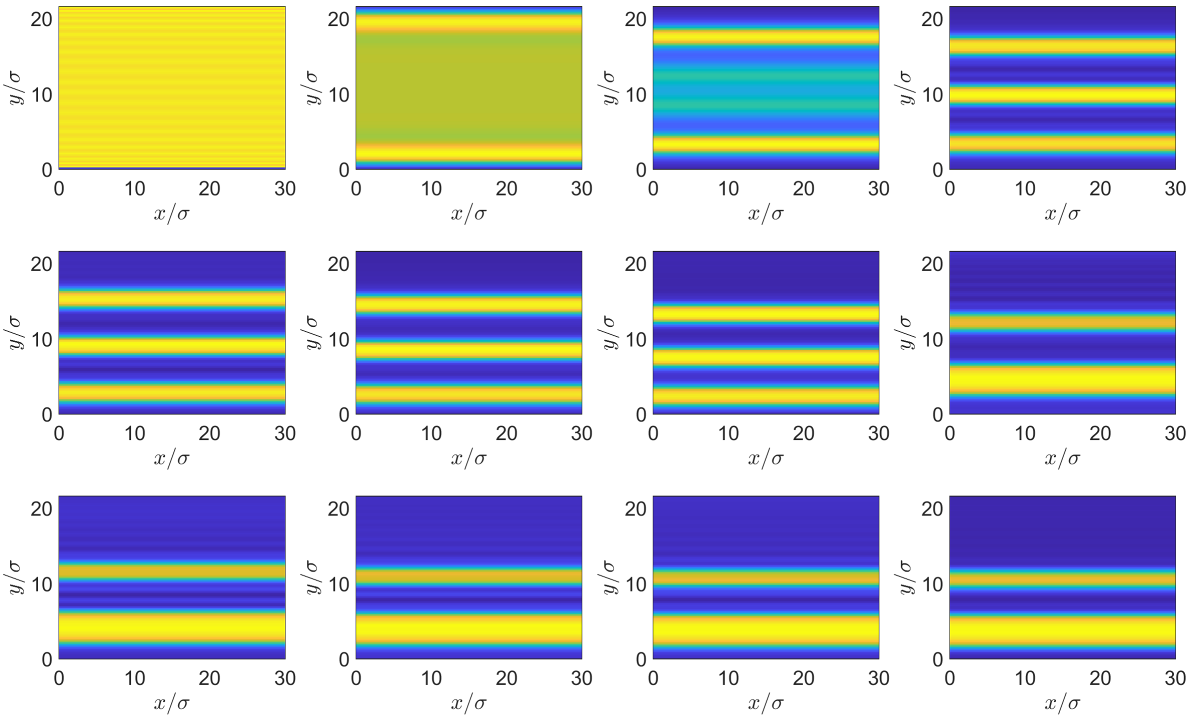

Figure 7: $\sigma = 0.5$, $\bar{\rho} = 0.2$

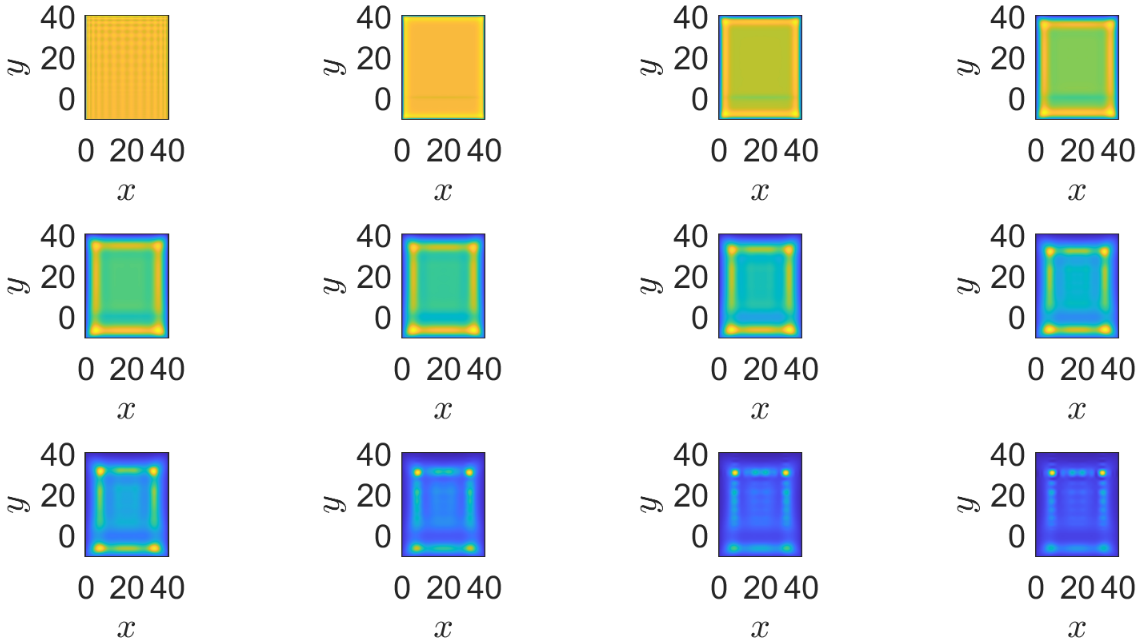

Figure 8: $\sigma = 1$, $\bar{\rho} = 0.072$, $TMax = 20$, V_A

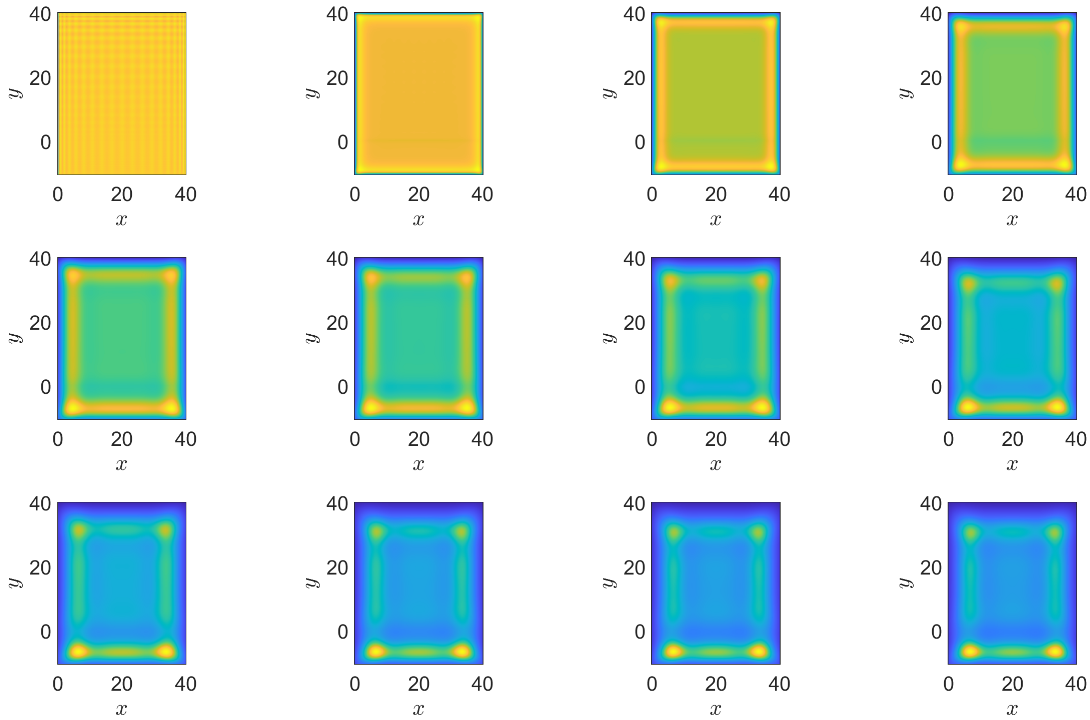

Figure 9: $\sigma = 1$, $\bar{\rho} = 0.072$, $TMax = 20$, V_A more points

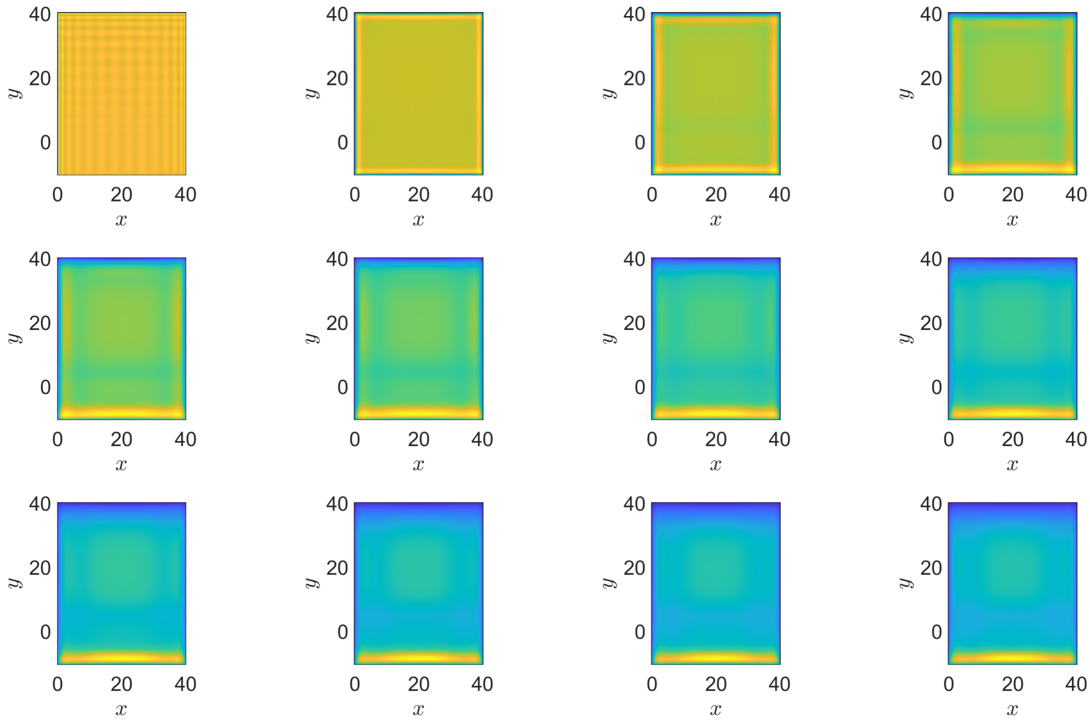

Figure 10: $\sigma = 1$, $\bar{\rho} = 0.072$, $TMax = 20$, V_G

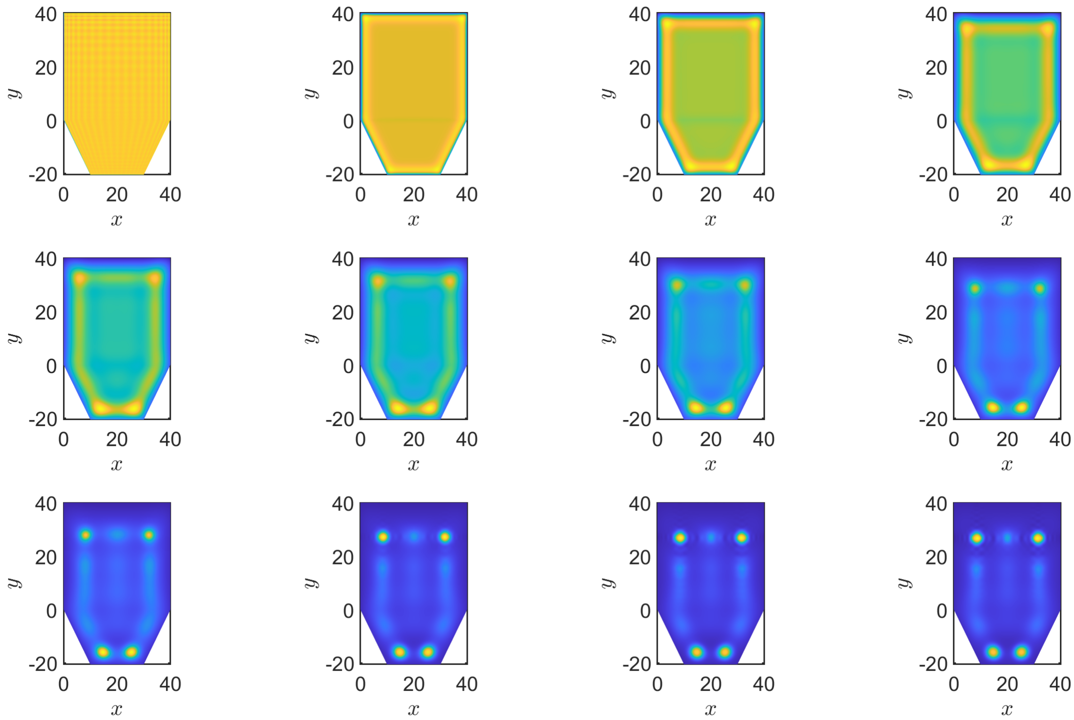

Figure 11: $\sigma = 1$, $\bar{\rho} = 0.072$, $TMax = 40$, V_A

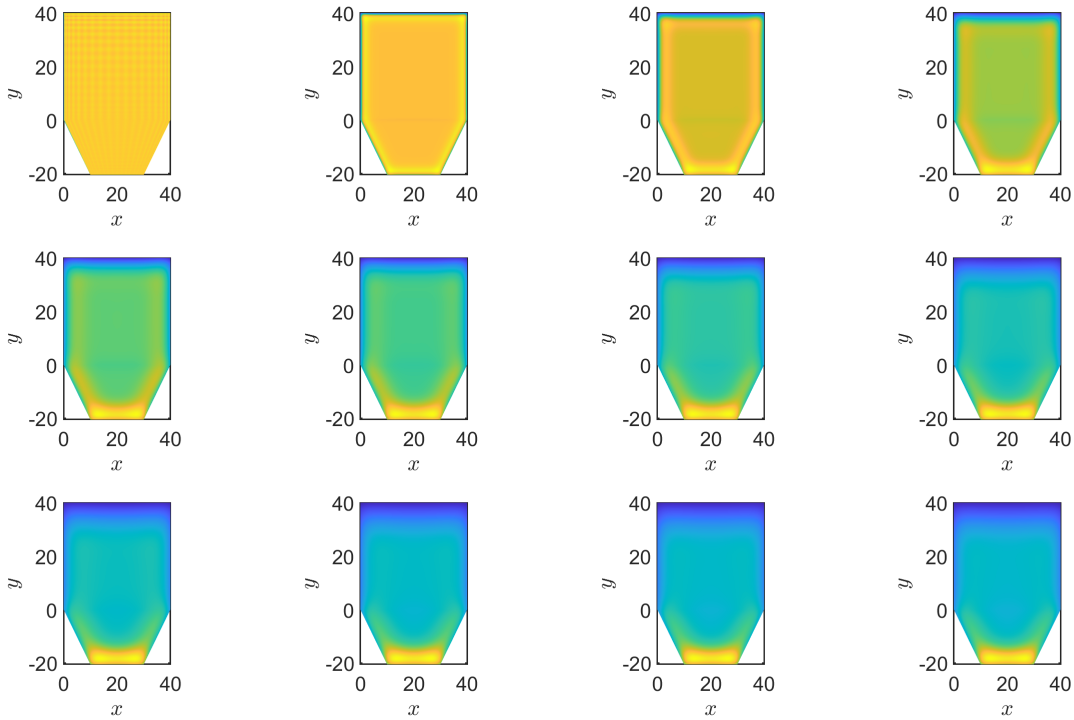

Figure 12: $\sigma = 1$, $\bar{\rho} = 0.05$, $TMax = 40$, V_A, less attraction ($\kappa = -2$)

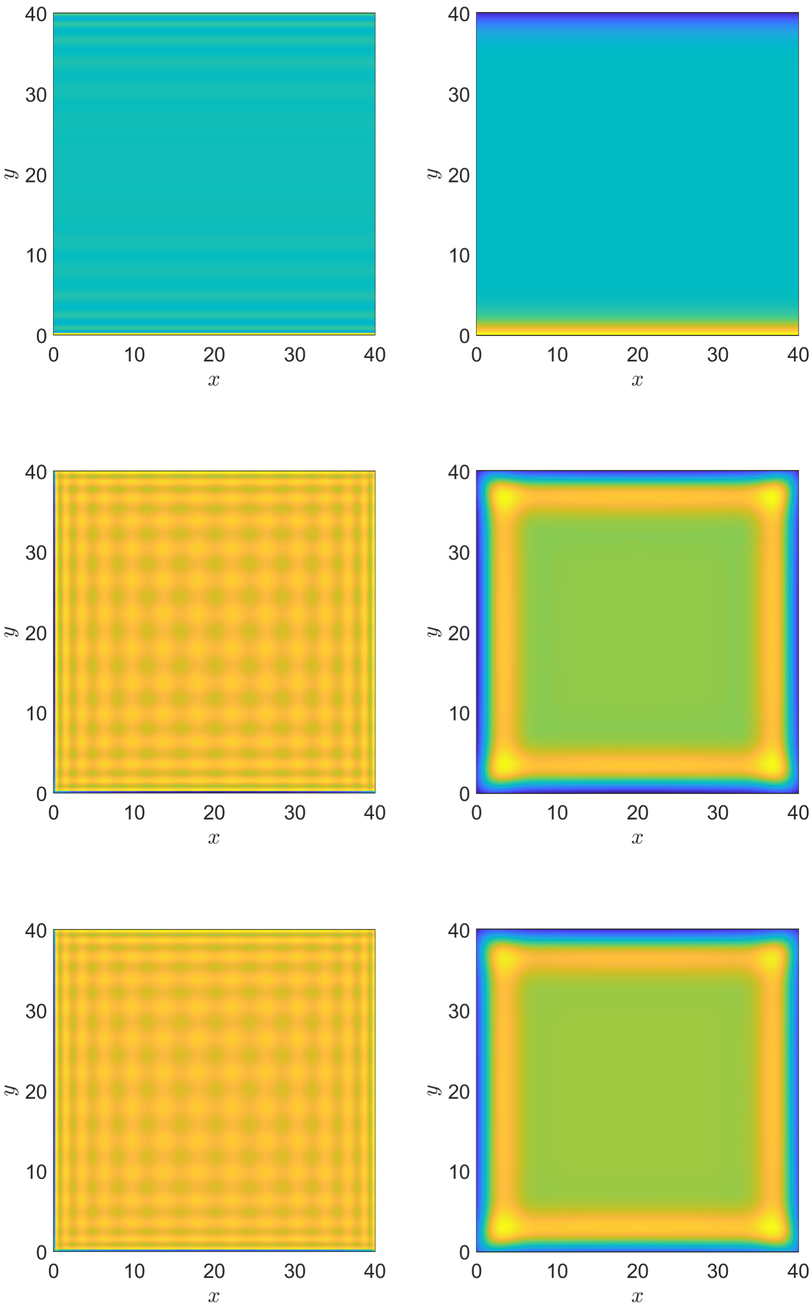

Figure 13: Plot1 shows V_{ext}only, Plot2 shows V_2 only and Plot3 shows both. At time 1 and time 10 out of 20.