

We have:

$$F[\rho] = \frac{1}{\beta} \int \rho (\ln \Lambda^2 \rho) - 2\rho - \rho \ln(1 - \eta) + \frac{\rho}{1 - \eta} dr$$

$$\eta = a\rho = \frac{\pi\sigma^2}{4}\rho$$

Taking the functional derivative of F gives:

$$\begin{aligned} \frac{\delta F[\rho]}{\delta \rho} &= \frac{1}{\beta} \left(1 + \ln \rho + \Lambda^2 - 2 - \ln(1 - \eta) + a \frac{\rho}{1 - \eta} + \frac{1}{1 - \eta} + a \frac{\rho}{(1 - \eta)^2} \right) \\ &= \frac{1}{\beta} \left(1 + \ln \rho + \Lambda^2 - 2 - \ln(1 - \eta) + \frac{1}{(\eta - 1)^2} - \frac{1}{\eta - 1} - 1 \right) \\ &= \frac{1}{\beta} \left(\ln \rho + \Lambda^2 - 2 - \ln(1 - \eta) - \frac{\eta - 2}{(\eta - 1)^2} \right), \end{aligned}$$

using partial fractions.

$$\begin{aligned} \nabla \frac{\delta F[\rho]}{\delta \rho} &= \frac{1}{\beta} \left(\nabla \ln \rho + \nabla(\Lambda^2 - 2) - \nabla \ln(1 - \eta) - \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \\ &= \frac{1}{\beta} \left(\frac{\nabla \rho}{\rho} - \frac{\nabla(1 - \eta)}{1 - \eta} - \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \\ &= \frac{1}{\beta} \left(\frac{\nabla \rho}{\rho} + \frac{\nabla \eta}{1 - \eta} - \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \end{aligned}$$

Then multiplying by ρ gives:

$$\begin{aligned} \rho \nabla \frac{\delta F[\rho]}{\delta \rho} &= \frac{1}{\beta} \left(\nabla \rho + \frac{\rho \nabla \eta}{1 - \eta} - \rho \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \\ &= \frac{1}{\beta} \left(\nabla \rho + \frac{\eta \nabla \rho}{1 - \eta} - \rho \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \\ &= \frac{1}{\beta} \left(\nabla \rho + \frac{\nabla \rho}{1 - \eta} - \nabla \rho - \rho \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \\ &= \frac{1}{\beta} \left(\frac{\nabla \rho}{1 - \eta} - \rho \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \end{aligned}$$

Finally we take the divergence:

$$\begin{aligned} \nabla \cdot \left(\rho \nabla \frac{\delta F[\rho]}{\delta \rho} \right) &= \frac{1}{\beta} \left(\nabla \cdot \left(\frac{\nabla \rho}{1 - \eta} \right) - \nabla \cdot \left(\rho \nabla \frac{\eta - 2}{(\eta - 1)^2} \right) \right) \\ &= \frac{1}{\beta} \left(\frac{\nabla^2 \rho}{1 - \eta} + \nabla \rho \cdot \nabla \frac{1}{1 - \eta} - \nabla \rho \cdot \nabla \frac{\eta - 2}{(\eta - 1)^2} - \rho \nabla^2 \frac{\eta - 2}{(\eta - 1)^2} \right) \\ &= \frac{1}{\beta} \left(\frac{\nabla^2 \rho}{1 - \eta} + \nabla \rho \cdot \nabla \frac{(3 - 2\eta)}{(1 - \eta)^2} - \rho \nabla^2 \frac{\eta - 2}{(\eta - 1)^2} \right) \end{aligned}$$