

PDE-Constrained Optimization for Multiscale Particle Dynamics

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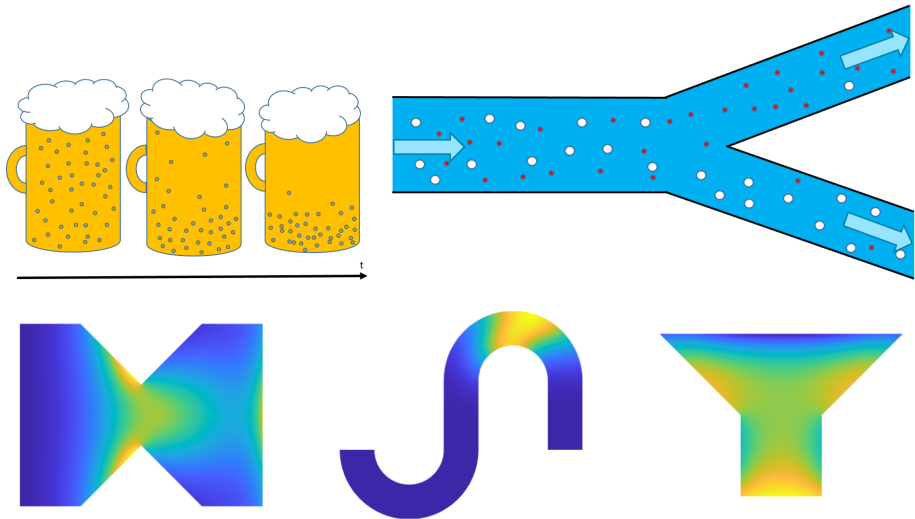
Joint work with Ben Goddard and John Pearson

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Structure of the Talk

- ▶ PDE-Constrained Optimization
- ▶ Optimization for DDFT
- ▶ Numerical Methods
- ▶ Results

Motivation



PDE-Constrained Optimization

A simple model

$$\min_{\rho, f} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|f\|_{L_2(\Sigma)}^2$$

subject to:

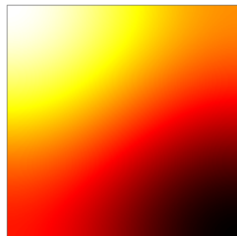
$$\partial_t \rho = \nabla^2 \rho + f \quad \text{in } \Sigma := (0, T) \times \Omega$$

BC and IC:

$$\frac{\partial \rho}{\partial n} = 0 \quad \text{on } \partial \Sigma := (0, T) \times \partial \Omega$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

Forward ρ



Target $\hat{\rho}$



PDE-Constrained Optimization

Deriving (first-order) optimality conditions

Define the Lagrangian \mathcal{L} :

$$\mathcal{L}(\rho, f, q) = \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|f\|_{L_2(\Sigma)}^2 - \int_{\Sigma} q \left(\partial_t \rho - \nabla^2 \rho - f \right) d\vec{x} dt - \int_{\partial\Sigma} q \frac{\partial \rho}{\partial n} d\vec{x} dt.$$

Compute directional derivatives and set equal to zero:

$$\mathcal{L}_q(\rho^*, f^*, q)h = 0, \quad \mathcal{L}_\rho(\rho^*, f^*, q)h = 0, \quad \mathcal{L}_f(\rho^*, f^*, q)h = 0.$$

PDE-Constrained Optimization

Deriving (first-order) optimality conditions

Define the Lagrangian \mathcal{L} :

$$\begin{aligned}\mathcal{L}(\rho, f, q) = & \frac{1}{2} \int_{\Sigma} (\rho - \hat{\rho})^2 d\vec{x} dt + \frac{\beta}{2} \int_{\Sigma} f^2 d\vec{x} dt - \int_{\Sigma} q \left(\partial_t \rho - \nabla^2 \rho - f \right) d\vec{x} dt \\ & - \int_{\partial \Sigma} q \frac{\partial \rho}{\partial n} d\vec{x} dt.\end{aligned}$$

Computing $\mathcal{L}_q(\rho^*, f^*, q)h = 0$ results in the forward problem:

$$\partial_t \rho = \nabla^2 \rho + f \quad \text{in } \Sigma$$

$$\frac{\partial \rho}{\partial n} = 0 \quad \text{on } \partial \Sigma$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

PDE-Constrained Optimization

Deriving (first-order) optimality conditions

Define the Lagrangian \mathcal{L} :

$$\begin{aligned}\mathcal{L}(\rho, f, q) = & \frac{1}{2} \int_{\Sigma} (\rho - \hat{\rho})^2 d\vec{x} dt + \frac{\beta}{2} \int_{\Sigma} f^2 d\vec{x} dt - \int_{\Sigma} q \left(\partial_t \rho - \nabla^2 \rho - f \right) d\vec{x} dt \\ & - \int_{\partial \Sigma} q \frac{\partial \rho}{\partial n} d\vec{x} dt.\end{aligned}$$

Computing $\mathcal{L}_{\rho}(\rho^*, f^*, q)h$:

$$\begin{aligned}\mathcal{L}_{\rho}(\rho^*, f^*, q)h = & \int_{\Omega} (q(T)h(T) - q(0)h(0)) d\vec{x} - \int_{\Sigma} \left(h(-\rho + \hat{\rho}) - h\partial_t q - h\nabla^2 q \right) d\vec{x} dt \\ & - \int_{\partial \Sigma} \left(q \frac{\partial h}{\partial n} - q \frac{\partial h}{\partial n} + h \frac{\partial q}{\partial n} \right) d\vec{x} dt.\end{aligned}$$

PDE-Constrained Optimization

Deriving (first-order) optimality conditions

Computing $\mathcal{L}_\rho(\rho^*, f^*, q)h = 0$:

$$\mathcal{L}_\rho(\rho^*, f^*, q)h = \int_{\Omega} q(T)h(T)d\vec{x} - \int_{\Sigma} h \left(-\rho + \hat{\rho} - \partial_t q - \nabla^2 q \right) d\vec{x}dt - \int_{\partial\Sigma} h \frac{\partial q}{\partial n} d\vec{x}dt = 0.$$

Adjoint equation:

$$\begin{aligned} \partial_t q &= -\nabla^2 q - \rho + \hat{\rho} && \text{in } \Sigma \\ \frac{\partial q}{\partial n} &= 0 && \text{on } \partial\Sigma \\ q(T, \vec{x}) &= 0 \end{aligned}$$

PDE-Constrained Optimization

Deriving (first-order) optimality conditions

Define the Lagrangian \mathcal{L} :

$$\begin{aligned}\mathcal{L}(\rho, f, q) = & \frac{1}{2} \int_{\Sigma} (\rho - \hat{\rho})^2 d\vec{x} dt + \frac{\beta}{2} \int_{\Sigma} f^2 d\vec{x} dt - \int_{\Sigma} q \left(\partial_t \rho - \nabla^2 \rho - f \right) d\vec{x} dt \\ & - \int_{\partial \Sigma} q \frac{\partial \rho}{\partial n} d\vec{x} dt.\end{aligned}$$

Computing $\mathcal{L}_f(\rho^*, f^*, q)h = 0$:

$$\mathcal{L}_f(\rho^*, f^*, q)h = \int_{\Sigma} h (\beta f + q) d\vec{x} dt = 0.$$

Gradient equation:

$$f = -\frac{1}{\beta} q.$$

PDE-Constrained Optimization

The (first-order) optimality system

$$\partial_t \rho = \nabla^2 \rho + f$$

$$\partial_t q = -\nabla^2 q - \rho + \hat{\rho}$$

$$f = -\frac{1}{\beta} q$$

$$\frac{\partial \rho}{\partial n} = 0, \quad \rho(0, \vec{x}) = \rho_0(\vec{x}),$$

$$\frac{\partial q}{\partial n} = 0, \quad q(T, \vec{x}) = 0.$$

PDE-Constrained Optimization

A simple model

$$\min_{\rho, f} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|f\|_{L_2(\Sigma)}^2$$

subject to:

$$\partial_t \rho = \nabla^2 \rho + f \quad \text{in } \Sigma$$

BC and IC:

$$\frac{\partial \rho}{\partial n} = 0 \quad \text{on } \partial \Sigma$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

Optimization for DDFT

A (simple) DDFT model

$$\min_{\rho, \vec{w}} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|\vec{w}\|_{L_2(\Sigma)}^2$$

subject to:

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \vec{w}) \quad \text{in } \Sigma$$

BC and IC:

$$\frac{\partial \rho}{\partial n} - \rho \vec{w} \cdot \vec{n} = 0 \quad \text{on } \partial \Sigma$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

Optimization for DDFT

A (simple) DDFT model

$$\min_{\rho, \vec{w}} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|\vec{w}\|_{L_2(\Sigma)}^2$$

subject to:

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \vec{w}) + \nabla \cdot (\rho \nabla V_{\text{ext}}) \quad \text{in } \Sigma$$

BC and IC:

$$\frac{\partial \rho}{\partial n} - \rho \vec{w} \cdot \vec{n} + \rho \frac{\partial V_{\text{ext}}}{\partial n} = 0 \quad \text{on } \partial \Sigma$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

Optimization for DDFT

A (simple) DDFT model

$$\min_{\rho, \vec{w}} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|\vec{w}\|_{L_2(\Sigma)}^2$$

subject to:

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \vec{w}) + \nabla \cdot (\rho \nabla V_{\text{ext}}) + \nabla \cdot \int_{\Omega} \rho(\vec{x}) \rho(\vec{x}') \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}' \quad \text{in } \Sigma$$

BC and IC:

$$\frac{\partial \rho}{\partial n} - \rho \vec{w} \cdot \vec{n} + \rho \frac{\partial V_{\text{ext}}}{\partial n} + \int_{\Omega} \rho(\vec{x}) \rho(\vec{x}') \frac{\partial V_2}{\partial n}(|\vec{x} - \vec{x}'|) d\vec{x}' = 0 \quad \text{on } \partial \Sigma$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

Optimization for DDFT

The (first-order) optimality system

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \vec{w}) + \nabla \cdot (\rho \nabla V_{\text{ext}}) + \nabla \cdot \int_{\Omega} \rho(\vec{x}) \rho(\vec{x}') \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}'$$

$$\begin{aligned} \partial_t q = & -\nabla^2 q - \nabla q \cdot \vec{w} + \nabla q \cdot \nabla V_{\text{ext}} - \rho + \hat{\rho} \\ & + \int_{\Omega} \rho(\vec{x}') \left(\nabla_{\vec{x}} q(\vec{x}) - \nabla_{\vec{x}'} q(\vec{x}') \right) \cdot \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}' \end{aligned}$$

$$\vec{w} = -\frac{1}{\beta} \rho \nabla q$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x}), \quad q(T, \vec{x}) = 0, \quad + \text{BCs}$$

Optimization for DDFT

Problem: Negative diffusion term in q causes numerical instability.

Solution: Change of time variable for this PDE: $\tau = T - t$.

$$\begin{aligned}\partial_t \rho(t, \vec{x}) = & \nabla^2 \rho(t, \vec{x}) - \nabla \cdot (\rho(t, \vec{x}) \vec{w}(t, \vec{x})) + \nabla \cdot (\rho(t, \vec{x}) \nabla V_{\text{ext}}(t, \vec{x})) \\ & + \nabla \cdot \int_{\Omega} \rho(t, \vec{x}) \rho(t, \vec{x}') \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}'\end{aligned}$$

$$\begin{aligned}\partial_{\tau} q(\tau, \vec{x}) = & \nabla^2 q(\tau, \vec{x}) + \nabla q(\tau, \vec{x}) \cdot \vec{w}(\tau, \vec{x}) - \nabla q(\tau, \vec{x}) \cdot \nabla V_{\text{ext}}(\tau, \vec{x}) + \rho(\tau, \vec{x}) - \hat{\rho}(\tau, \vec{x}) \\ & - \int_{\Omega} \rho(\tau, \vec{x}') \left(\nabla_{\vec{x}} q(\tau, \vec{x}) - \nabla_{\vec{x}'} q(\tau, \vec{x}') \right) \cdot \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}'\end{aligned}$$

$$\vec{w}(t, \vec{x}) = -\frac{1}{\beta} \rho(t, \vec{x}) \nabla q(t, \vec{x})$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x}), \quad q(0, \vec{x}) = 0, \quad + \text{BCs}$$

Numerical Methods

- ▶ Challenge 1: Particle interaction term is nonlinear and nonlocal (+ nonlocal BCs). How to avoid shortcomings of standard methods (FEM/FDM)?

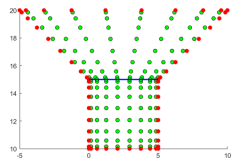
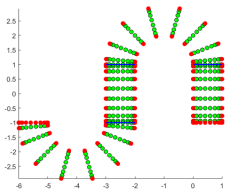
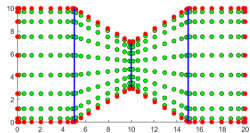
⇒ **Pseudospectral methods**

- ▶ Challenge 2: One PDE is forward in time, the other backward. How to do time stepping?

⇒ **Fixed point algorithm**

Numerical Methods

Pseudospectral Methods



- ▶ Reduce both PDEs to systems of ODEs.
- ▶ Discretize time (accurate interpolation).
- ▶ Equations can now be solved using a DAE solver (when given all necessary inputs).
- ▶ For more 'complex' domains, pseudospectral methods are extended to spectral element methods.



Numerical Methods

Fixed point algorithm

Initialize with guess $\vec{w}^{(0)}$.

1. Solve

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \vec{w}^{(i)}) + \nabla \cdot (\rho \nabla V_{\text{ext}}) + \nabla \cdot \int_{\Omega} \rho(\vec{x}) \rho(\vec{x}') \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}'.$$

2. Solve $\partial_{\tau} q = \nabla^2 q + \nabla q \cdot \vec{w}^{(i)} - \nabla q \cdot \nabla V_{\text{ext}} + \rho^{(i)} - \hat{\rho}$

$$- \int_{\Omega} \rho^{(i)}(\vec{x}') \left(\nabla q(\vec{x}) - \nabla q(\vec{x}') \right) \cdot \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}'.$$

3. Solve $\vec{w}_g^{(i)} = -\frac{1}{\beta} \rho^{(i)} \nabla q^{(i)}.$

4. Measure the error: $\mathcal{E} = \|\vec{w}^{(i)} - \vec{w}_g^{(i)}\|.$

5. Update control, with $\lambda \in [0, 1]$: $\vec{w}^{(i+1)} = (1 - \lambda) \vec{w}^{(i)} + \lambda \vec{w}_g^{(i)}.$

Iterate until $\mathcal{E} < TOL$.

Optimization for DDFT

Reminder: (Simple) DDFT model

$$\min_{\rho, \vec{w}} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|\vec{w}\|_{L_2(\Sigma)}^2$$

subject to:

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \vec{w}) + \nabla \cdot (\rho \nabla V_{\text{ext}}) + \nabla \cdot \int_{\Omega} \rho(\vec{x}) \rho(\vec{x}') \nabla V_2(|\vec{x} - \vec{x}'|) d\vec{x}' \quad \text{in } \Sigma$$

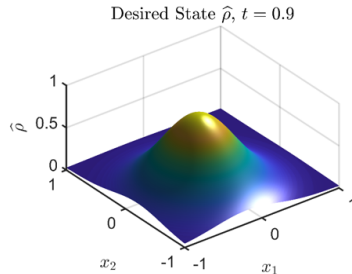
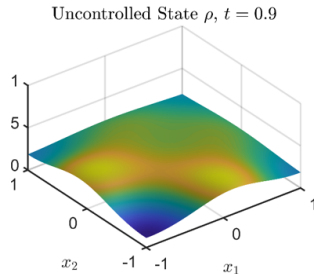
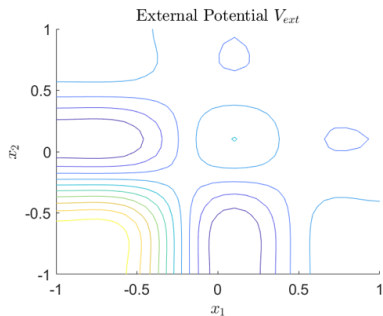
BC and IC:

$$\frac{\partial \rho}{\partial n} - \rho \vec{w} \cdot \vec{n} + \rho \frac{\partial V_{\text{ext}}}{\partial n} + \int_{\Omega} \rho(\vec{x}) \rho(\vec{x}') \frac{\partial V_2}{\partial n} (|\vec{x} - \vec{x}'|) d\vec{x}' = 0 \quad \text{on } \partial \Sigma$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

Results - Example 1

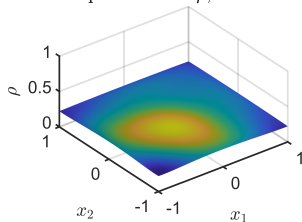
Overall Cost: $\mathcal{J} = \frac{1}{2}\|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2}\|\vec{w}\|_{L_2(\Sigma)}^2$, $\mathcal{J}_{\vec{w}=\vec{0}} = 0.0400$.



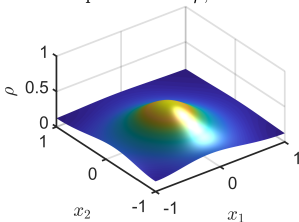
Results - Example 1

Overall Cost: $\mathcal{J} = \frac{1}{2}\|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2}\|\vec{w}\|_{L_2(\Sigma)}^2$, $\mathcal{J}_{\vec{w}=\vec{0}} = 0.0400$, $\mathcal{J}_{\text{opt}} = 0.0046$.

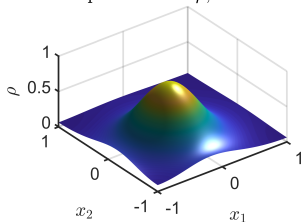
Optimal State ρ , $t = 0.1$



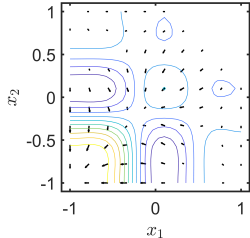
Optimal State ρ , $t = 0.5$



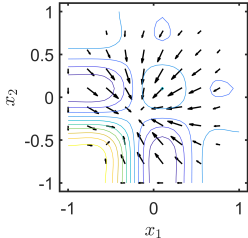
Optimal State ρ , $t = 0.9$



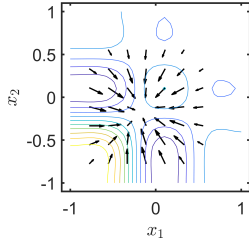
Optimal Control \vec{w} , $t = 0.1$



Optimal Control \vec{w} , $t = 0.5$



Optimal Control \vec{w} , $t = 0.9$



Optimization for DDFT

A more general DDFT model

$$\min_{\rho, \vec{w}} \quad \frac{1}{2} \|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2} \|\vec{w}\|_{L_2(\Sigma)}^2$$

subject to:

$$\partial_t \rho = \nabla \cdot \left(\rho \nabla \frac{\delta \mathcal{F}[\rho]}{\delta \rho} - \rho \vec{w} \right) := -\nabla \cdot \vec{j} \quad \text{in } \Sigma$$

$$\mathcal{F}[\rho] = \mathcal{F}_{id}[\rho] + \mathcal{F}_{ext}[\rho] + \mathcal{F}_{int}[\rho] + \int_{\Omega} \rho \left(-1 - \ln(1 - a\rho) + \frac{1}{1 - a\rho} \right) d\vec{x}$$

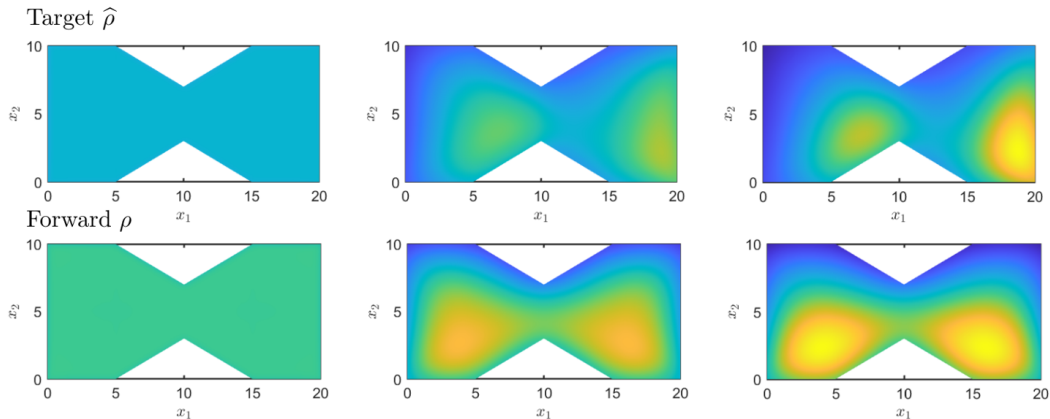
BC and IC:

$$\vec{j} \cdot \vec{n} = 0 \quad \text{on } \partial\Sigma$$

$$\rho(0, \vec{x}) = \rho_0(\vec{x})$$

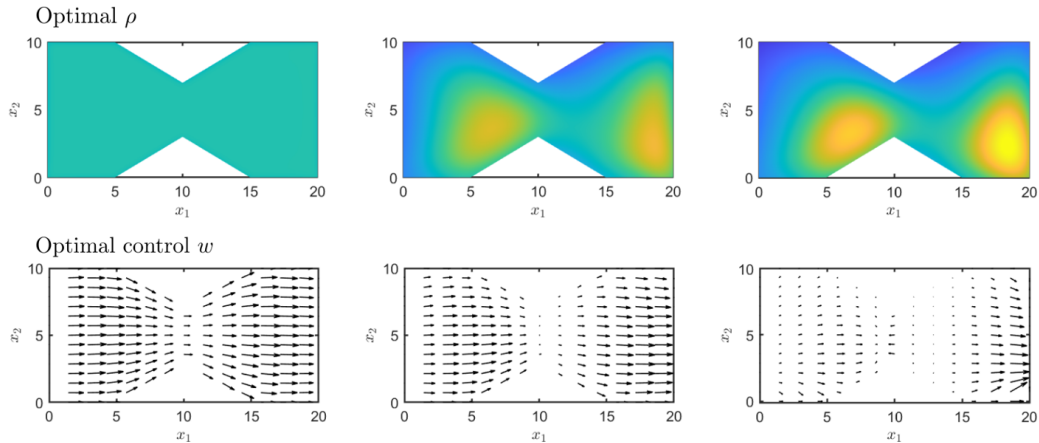
Results - Example 2

Overall Cost: $\mathcal{J} = \frac{1}{2}\|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2}\|\vec{w}\|_{L_2(\Sigma)}^2$, $\mathcal{J}_{\vec{w}=\vec{0}} = 0.0484$.

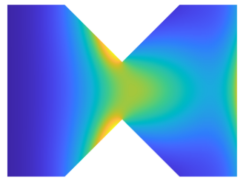


Results - Example 2

Overall Cost: $\mathcal{J} = \frac{1}{2}\|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2}\|\vec{w}\|_{L_2(\Sigma)}^2$, $\mathcal{J}_{\vec{w}=\vec{0}} = 0.0484$, $\mathcal{J}_{\text{opt}} = 0.0146$.



Summary



Up to now:

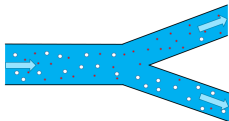
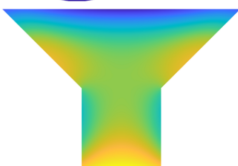
- ▶ Deriving PDE-constrained optimization models.
- ▶ Developing a suitable numerical method to solve them.

Current:




- ▶ Complex domains.
- ▶ Extended PDE models.
- ▶ Different boundary conditions.

Up next:


- ▶ More extended models.
- ▶ Different controls.
- ▶ Application to industrial processes.



Some References

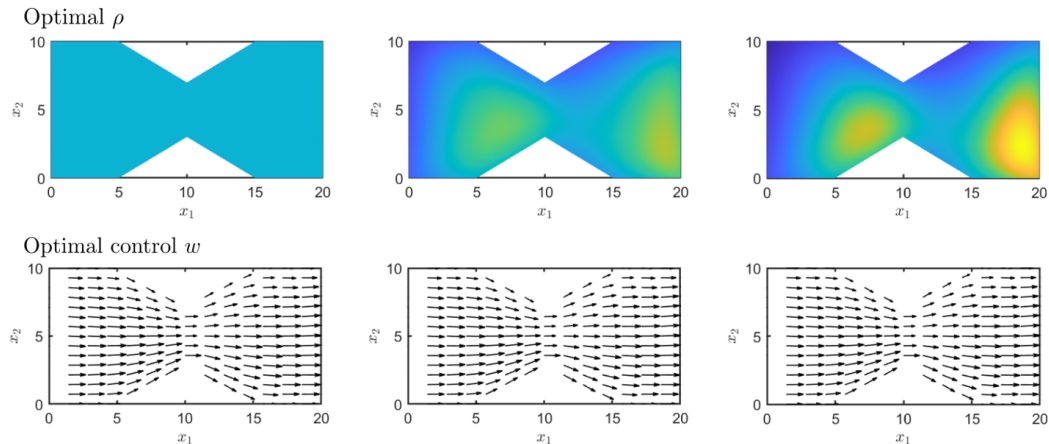
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Some More References

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Discrete and Continuous Dynamical Systems - Series B, 19(5), 1311-1333, 2014.

Results - Example 2, with Time-Independent Control

Overall Cost: $\mathcal{J} = \frac{1}{2}\|\rho - \hat{\rho}\|_{L_2(\Sigma)}^2 + \frac{\beta}{2}\|\vec{w}\|_{L_2(\Omega)}^2$, $\mathcal{J}_{\vec{w}=\vec{0}} = 0.0484$, $\mathcal{J}_{\text{opt}} = 0.0258$.



Data for Example 1

		$\beta = 10^{-3}$	$\beta = 10^{-1}$	$\beta = 10^1$	$\beta = 10^3$
$\kappa = -1$	$\mathcal{J}_{\vec{w}=\vec{0}}$	0.0400	0.0400	0.0400	0.0400
	\mathcal{J}_{opt}	0.0046	0.0370	0.0400	0.0400
	Iter	717	778	347	1
$\kappa = 0$	$\mathcal{J}_{\vec{w}=\vec{0}}$	0.0478	0.0478	0.0478	0.0478
	\mathcal{J}_{opt}	0.0064	0.0450	0.0478	0.0478
	Iter	718	784	343	1
$\kappa = 1$	$\mathcal{J}_{\vec{w}=\vec{0}}$	0.0556	0.0556	0.0556	0.0556
	\mathcal{J}_{opt}	0.0085	0.0530	0.0556	0.0556
	Iter	720	787	339	1

Table: Cost when $\vec{w} = \vec{0}$, optimal control cost, and iterations required, for a range of values for interaction strength κ and regularization parameter β .

References: Figures



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WEST Brewery *www.westbeer.com*