Paper Examples

1 Neumann Source Control

We choose

$$\begin{aligned}\rho_0 &= 0.25\\ V_{ext} &= 1.5\sin(\pi x_2/5)\cos(\pi x_1/5 - \pi/5)\\ \widehat{\rho} &= 0.25(1-t) + t(0.25\sin(\pi(x_1-2)/2)\sin(\pi(x_2-2)/2) + 0.25)\end{aligned}$$

We choose the domain $[-1,1]^2$ with a time horizon $(0,1)$ and $N = 20$, $n = 11$. For $\beta = 10^{-3}$, for $\kappa = -1$ we have $\mathcal{J}_c = 0.0018$, for $\kappa = 0$ (compared to $\mathcal{J}_{uc} = 0.0274$ from $\beta = 10^3$), $\mathcal{J}_c = 0.0017$ and for $\kappa = 1$, $\mathcal{J}_c = 0.0018$. Each of these computations takes around 200 seconds for 10 outer iterations. The results can be seen in Figures 2, 3 and 4 and the external potential acting on ρ is displayed in Figure 1.
The cost for these results is not that far from the cost with $\beta = 3$. Therefore we run the same example with $\beta = 10^{-5}$. This gives for $\kappa = -1$, $\mathcal{J}_c = 8.0673 \times 10^{-4}$, for $\kappa = 0$, $\mathcal{J}_c = 8.1989 \times 10^{-4}$, and for $\kappa = 1$, $\mathcal{J}_c = 8.4241 \times 10^{-4}$. Notably, these calculations only take around 20 seconds. The results are displayed in Figures 5, 6 and 7. The controls are larger, but the difference in dynamics is smaller for different interactions.

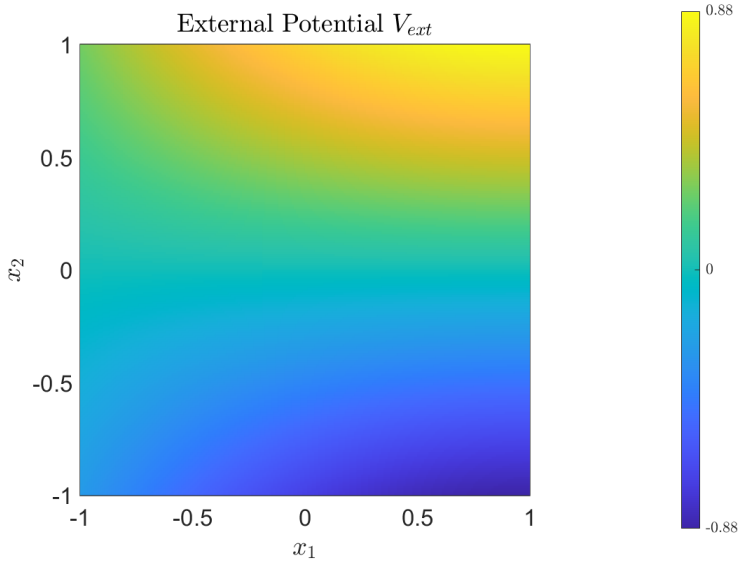

Figure 1: Neumann Source Control: External Potential V_{ext} acting on ρ.

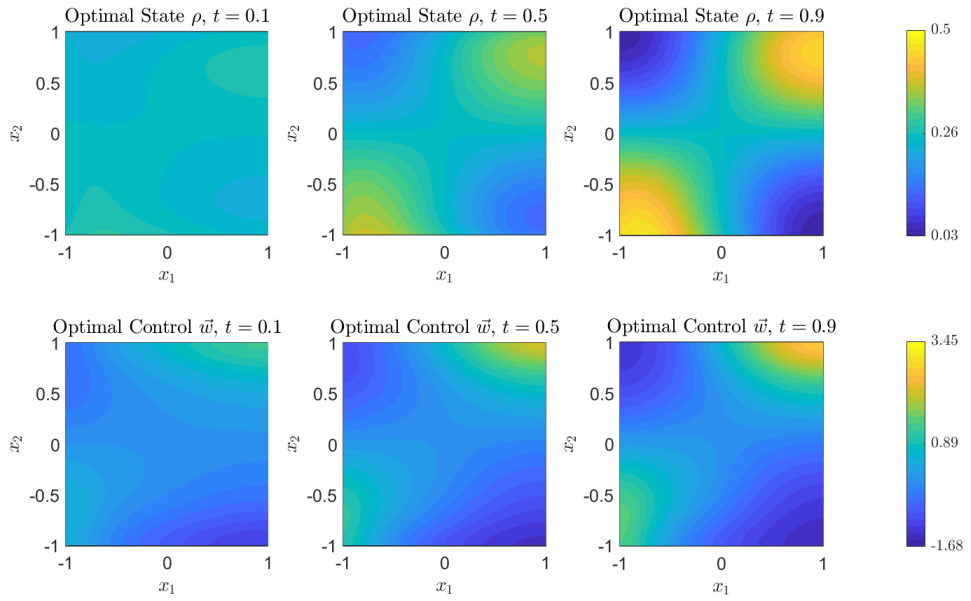

Figure 2: Neumann Source Control: Optimal ρ and optimal control for $\kappa = 0$ and $\beta = 10^{-3}$.

Dirichlet Source Control

We choose

$$\begin{aligned}
\rho_0 &= 0.25\cos(\pi x_1/2)\cos(\pi x_2/2) + 0.25 \\
V_{ext} &= (1-t)(-\cos(\pi x_1/2)\sin(\pi x_2/2) + 1) \\
\widehat{\rho} &= 0.25\cos(\pi x_1/2)\cos(\pi x_2/2)(1-t) - t(0.25\sin(\pi x_1)\sin(\pi x_2/2 - \pi/2) + 0.25)
\end{aligned}$$

so that the problem has Dirichlet boundary conditions at 0.25 ($\rho = 0.25$ on $\partial\Omega$). We choose the domain $[-1,1]^2$ with a time horizon $(0,1)$ and $N = 20$, $n = 11$. For $\beta = 10^{-3}$, for $\kappa = -1$ we have $\mathcal{J}_c = 0.0036$, for $\kappa = 0$ (compared to $\mathcal{J}_{uc} = 0.0219$ from $\beta = 10^3$), $\mathcal{J}_c = 0.0038$ and for $\kappa = 1$, $\mathcal{J}_c = 0.0043$. Each of these computations takes around 70 seconds for 10 outer iterations. The results can be seen in Figures 9, 10 and 11 and the external potential acting on ρ is displayed in Figure 8.

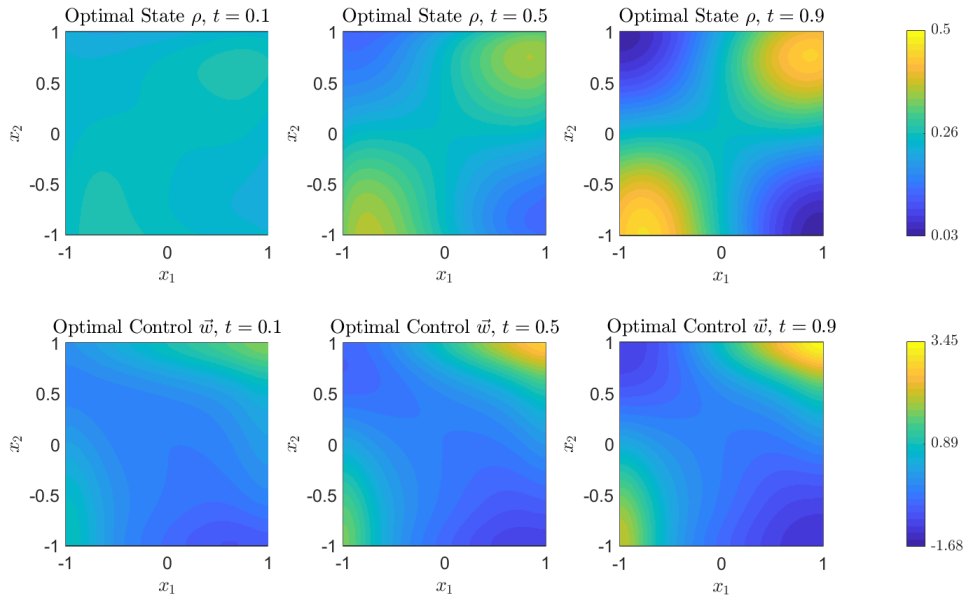

Figure 3: Neumann Source Control: Optimal ρ and optimal control for $\kappa = -1$ and $\beta = 10^{-3}$.

2 Neumann Flow Control

We choose

$$\begin{aligned}\rho_0 &= 0.25\\ V_{ext} &= ((x_1+0.3)^2-1)((x_1-0.4)^2-0.5)((x_2+0.3)^2-1)((x_2-0.4)^2-0.5)\\ \widehat{\rho} &= (1-t)0.25 + t(1/1.3791)\exp\left(-2((x_1+0.2)^2+(x_2+0.2)^2)\right)\end{aligned}$$

We choose the domain $[-1,1]^2$ with a time horizon $(0,1)$. We have $N = 20$, $n = 11$. For $\beta = 10^{-3}$, $\kappa = 1$ we get $\mathcal{J}_c = 0.0059$ (compare to $\beta = 10^3$ with $\mathcal{J}_{uc} = 0.0336$), for $\kappa = 0$, $\mathcal{J}_c = 0.0043$, and for $\kappa = -1$ we get $\mathcal{J}_c = 0.0030$, (compare to $\beta = 10^3$ with $\mathcal{J}_{uc} = 0.0214$). Each of the problems takes around 180 seconds to solve. The results can be seen in Figures 13, 14 and 15 and the external potential associated with it is shown in Figure 12.

3 Dirichlet Flow Control without V_{ext}

We choose

$$\rho_0 = (0.25\pi)^2\cos(\pi x_1/2)\cos(\pi x_2/2) + (0.25\pi)^2$$
$$\widehat{\rho} = (1-t)((0.25\pi)^2\cos(\pi x_1/2)\cos(\pi x_2/2) + (0.25\pi)^2) + t((0.25\pi)^2\cos(\pi x_1/2)\cos(3\pi x_2/2) + (0.25\pi)^2)$$

We choose the domain $[-1,1]^2$ with a time horizon $(0,1)$. We have $N = 20$, $n = 11$. For $\beta = 10^{-3}$, $\kappa = 1$ we get $\mathcal{J}_c = 0.0121$, for $\kappa = 0$, $\mathcal{J}_c = 0.0095$, and for $\kappa = -1$ we get

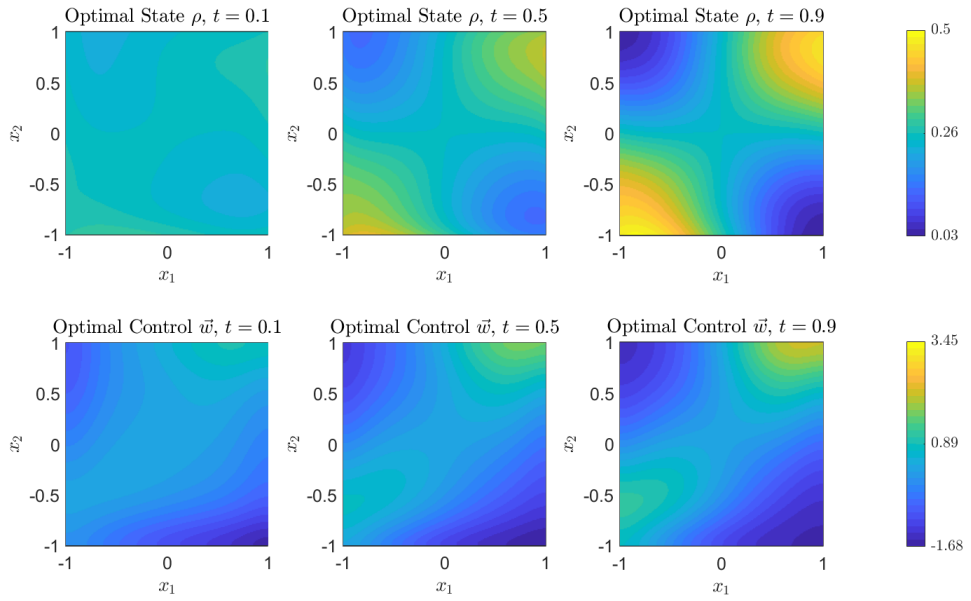

Figure 4: Optimal ρ and optimal control for $\kappa = 1$ and $\beta = 10^{-3}$.

$\mathcal{J}_c = 0.0104$, (compare to $\beta = 10^3$ with $\mathcal{J}_{uc} = 0.5272$). Each of the problems takes around 50 seconds to solve. The results can be seen in Figures 16, 17 and 18.

4 Dirichlet Flow Control with V_{ext}

We add the following external potential to the above problem, see Figure 19

$$V_{ext} = 10 \sin(\pi x_2/3 - \pi/2) \sin(\pi x_1/2)$$

For $\beta = 10^{-3}$, $\kappa = 1$ we get $\mathcal{J}_c = 0.0130$, for $\kappa = 0$, $\mathcal{J}_c = 0.0106$, and for $\kappa = -1$ we get $\mathcal{J}_c = 0.0113$. (Compare these to $\beta = 10^3$ with $\mathcal{J}_{uc} = 0.0898$) Each of the problems takes around 50 seconds to solve. The results can be seen in Figures 20, 21 and 22.

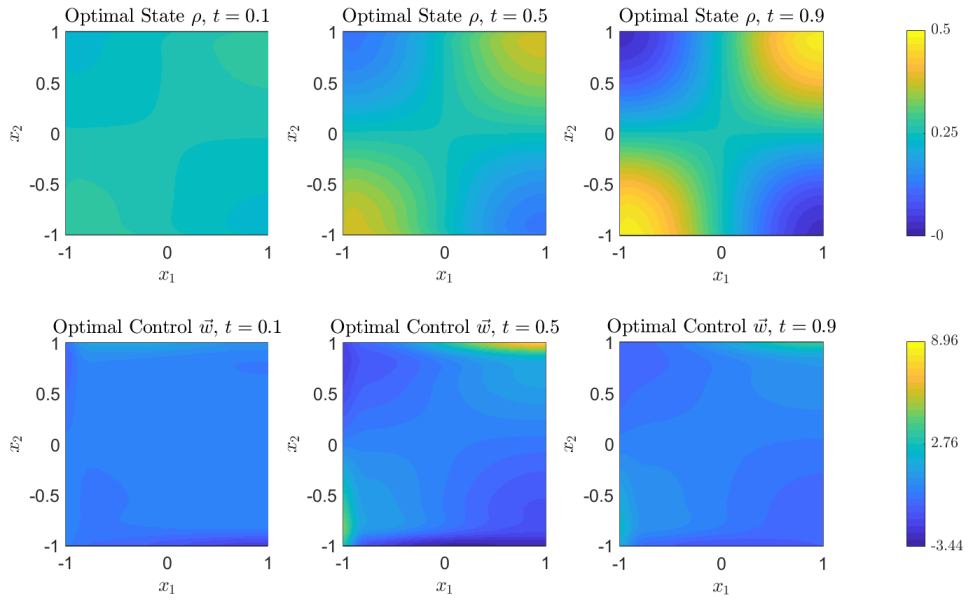

Figure 5: Neumann Source Control: Optimal ρ and optimal control for $\kappa = 0$ and $\beta = 10^{-5}$.

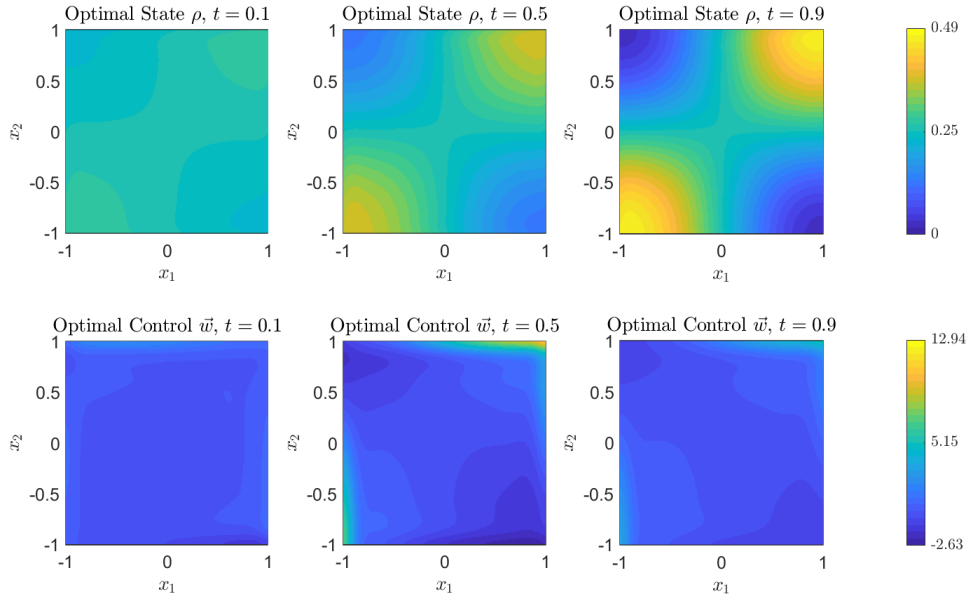

Figure 6: Neumann Source Control: Optimal ρ and optimal control for $\kappa = -1$ and $\beta = 10^{-5}$.

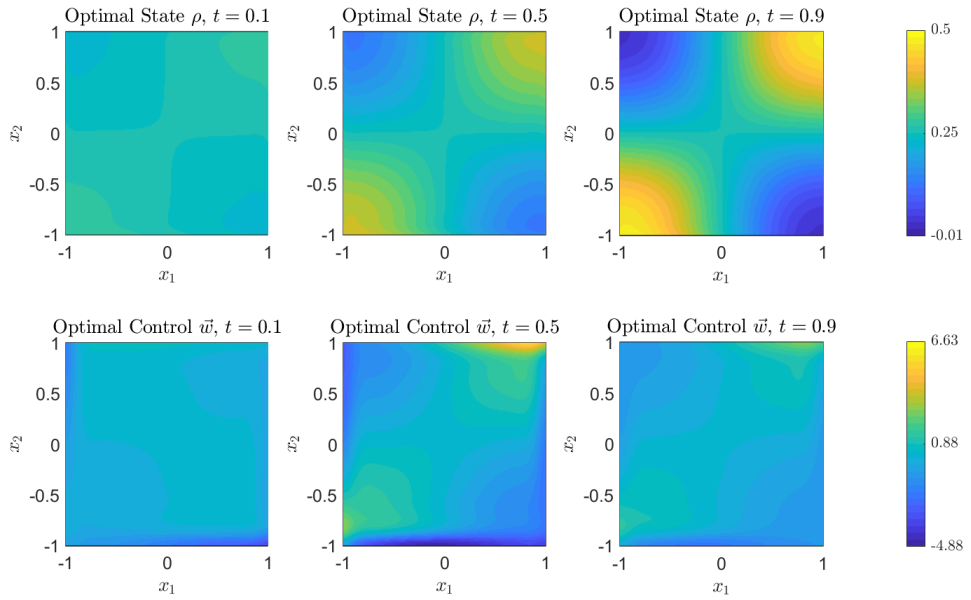

Figure 7: Neumann Source Control: Optimal ρ and optimal control for $\kappa = 1$ and $\beta = 10^{-5}$.

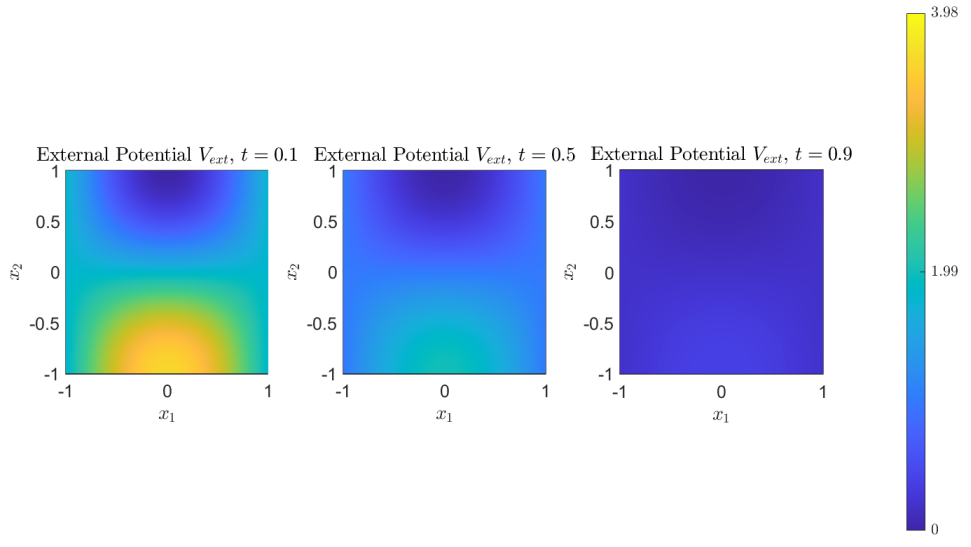

Figure 8: Dirichlet Source Control: External Potential V_{ext} acting on ρ.

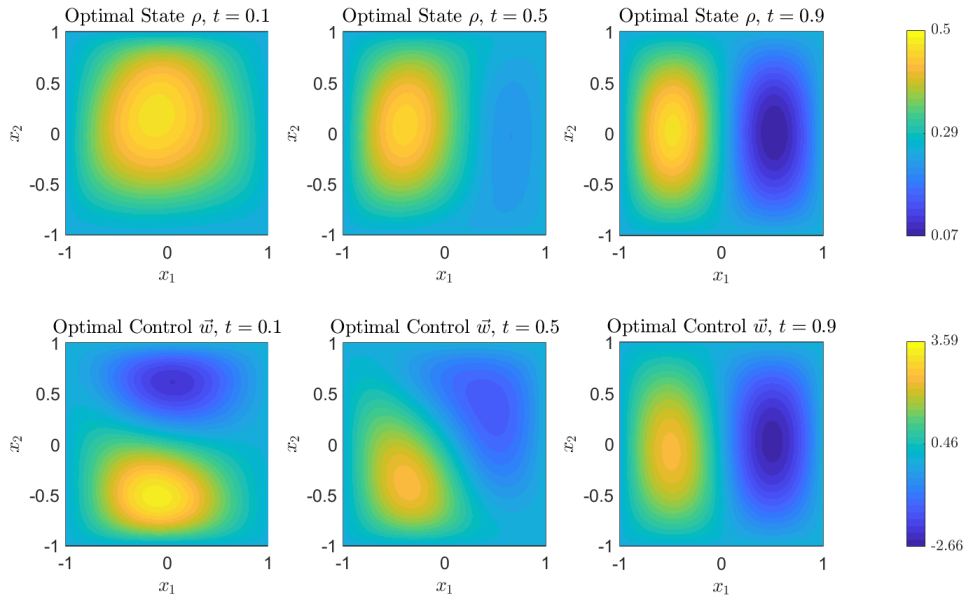

Figure 9: Dirichlet Source Control: Optimal ρ and optimal control for $\kappa = 0$ and $\beta = 10^{-3}$.

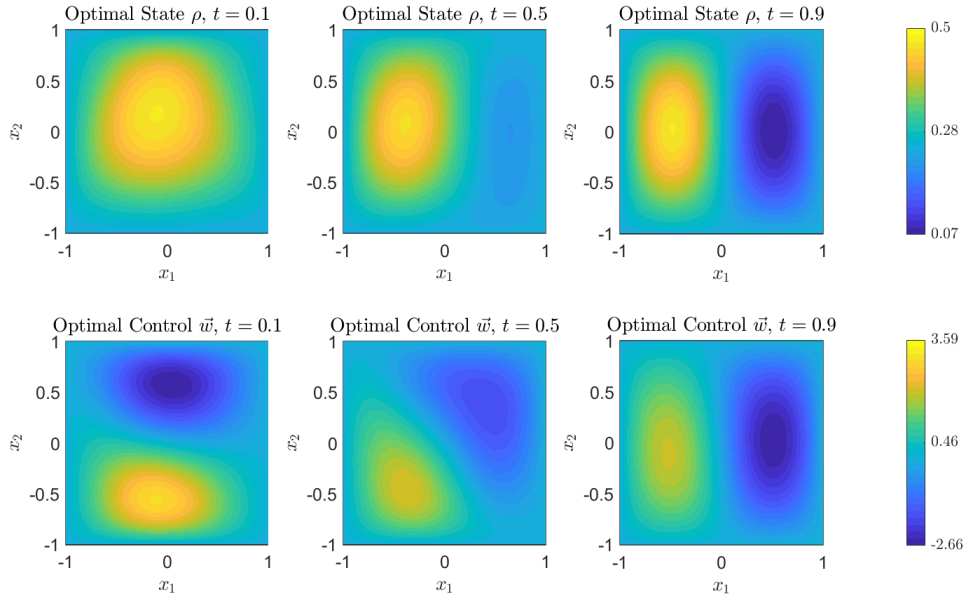

Figure 10: Dirichlet Source Control: Optimal ρ and optimal control for $\kappa = -1$ and $\beta = 10^{-3}$.

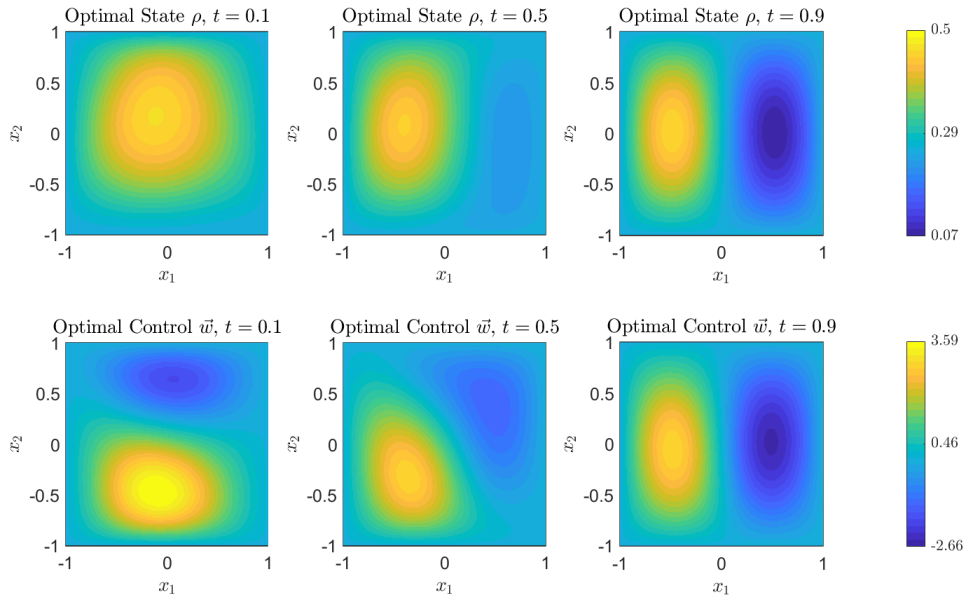

Figure 11: Dirichlet Source Control: Optimal ρ and optimal control for $\kappa = 1$ and $\beta = 10^{-3}$.

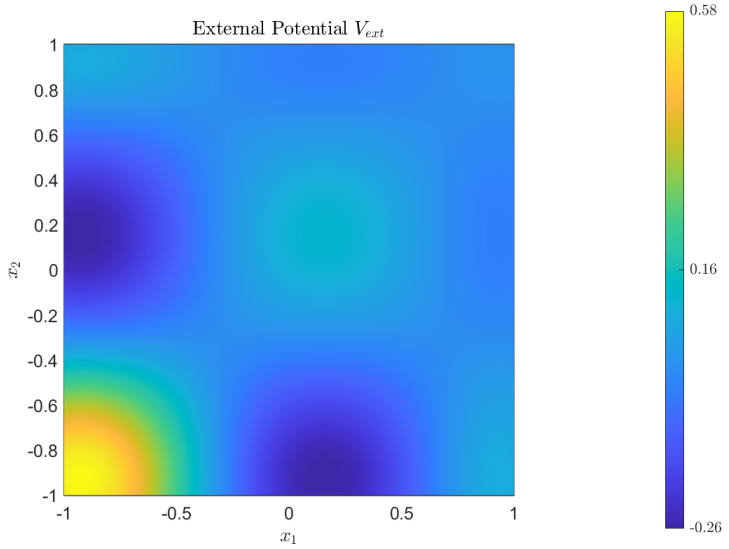

Figure 12: Neumann Flow Control: External Potential V_{ext} acting on ρ.

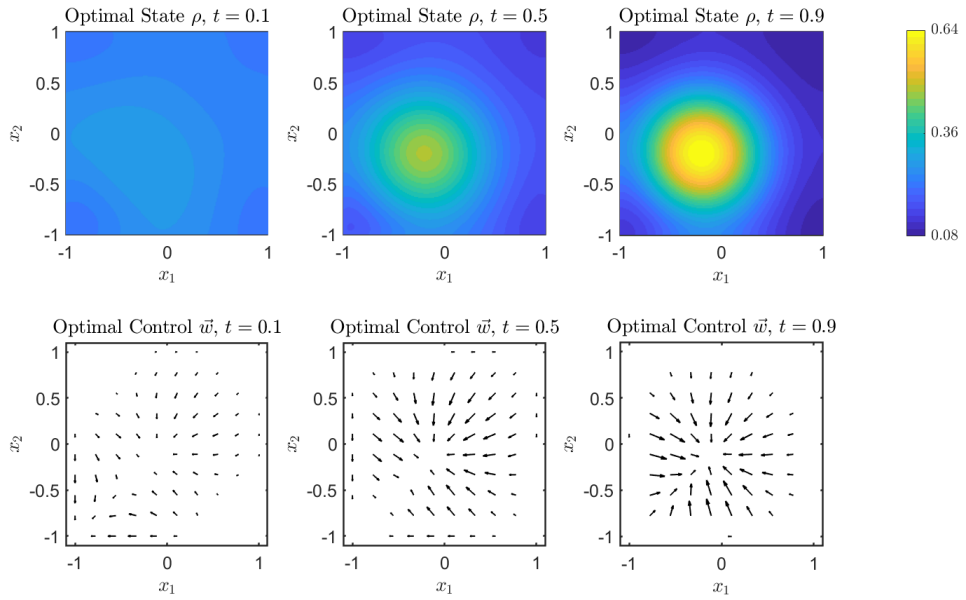

Figure 13: Neumann Flow Control: Optimal ρ and optimal control for $\kappa = 0$ and $\beta = 10^{-3}$.

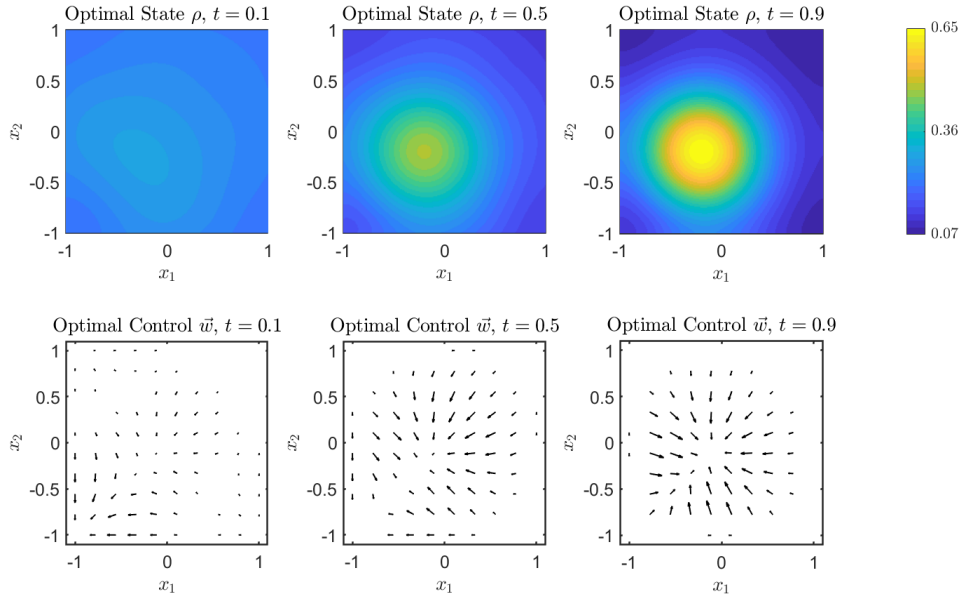

Figure 14: Neumann Flow Control: Optimal ρ and optimal control for $\kappa = -1$ and $\beta = 10^{-3}$.

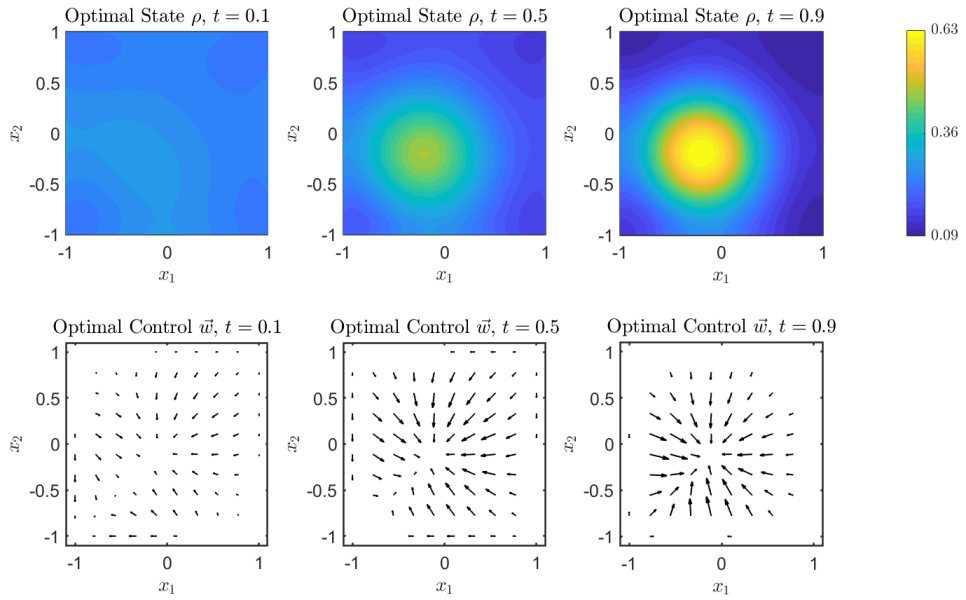

Figure 15: Neumann Flow Control: Optimal ρ and optimal control for $\kappa = 1$ and $\beta = 10^{-3}$.

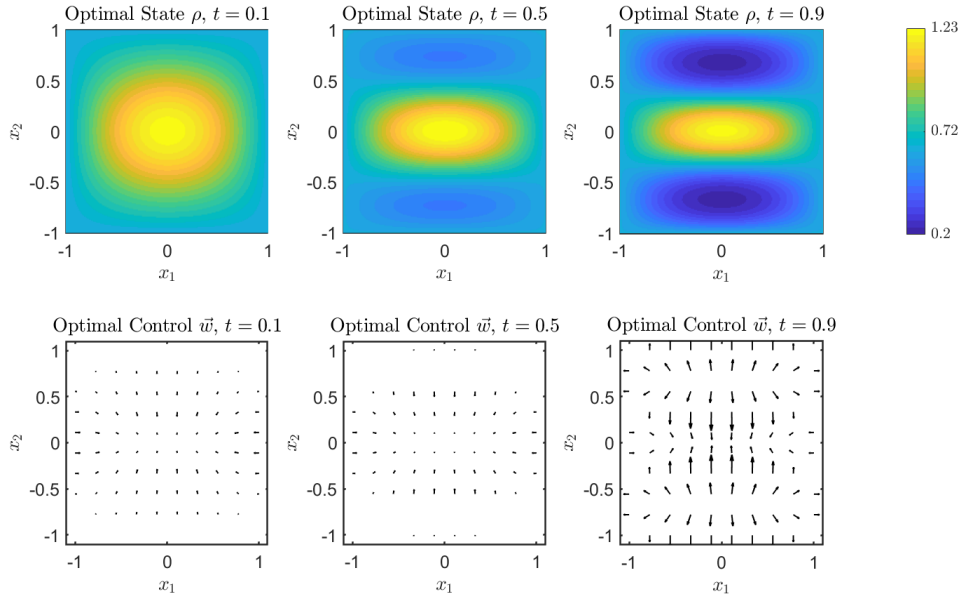

Figure 16: Dirichlet Flow Control: Optimal ρ and optimal control for $\kappa = 0$ and $\beta = 10^{-3}$.

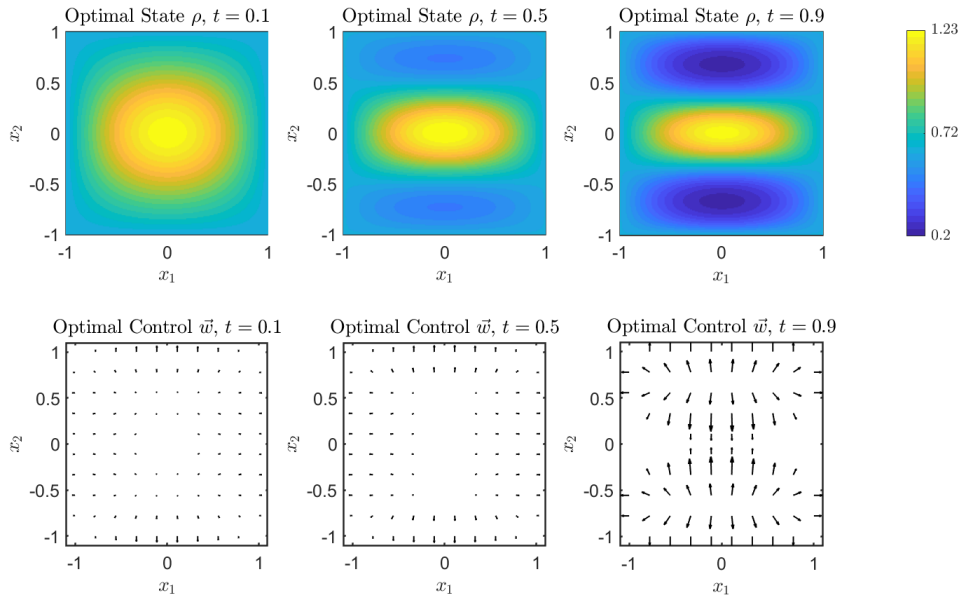

Figure 17: Dirichlet Flow Control: Optimal ρ and optimal control for $\kappa = -1$ and $\beta = 10^{-3}$.

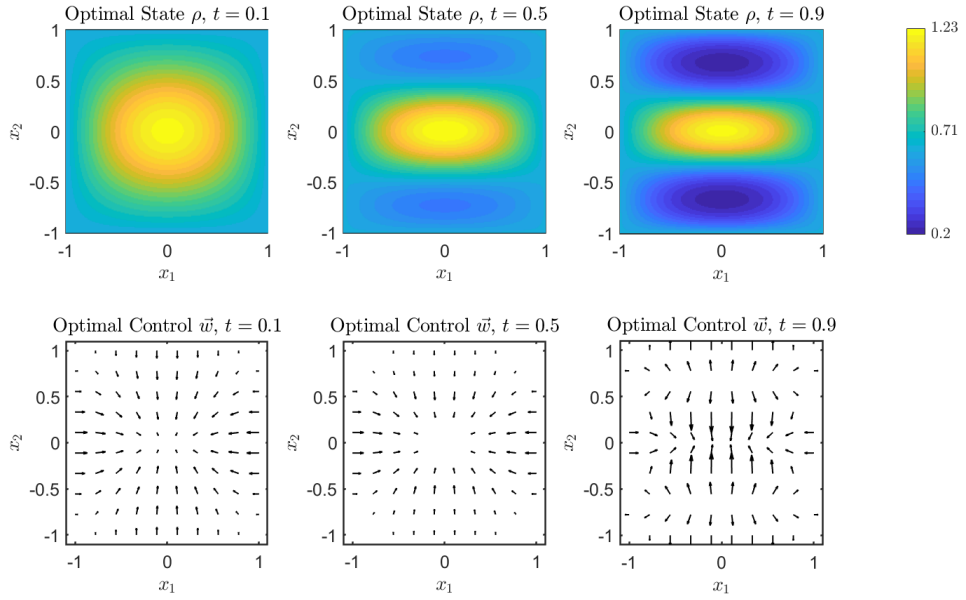

Figure 18: Dirichlet Flow Control: Optimal ρ and optimal control for $\kappa = 1$ and $\beta = 10^{-3}$.

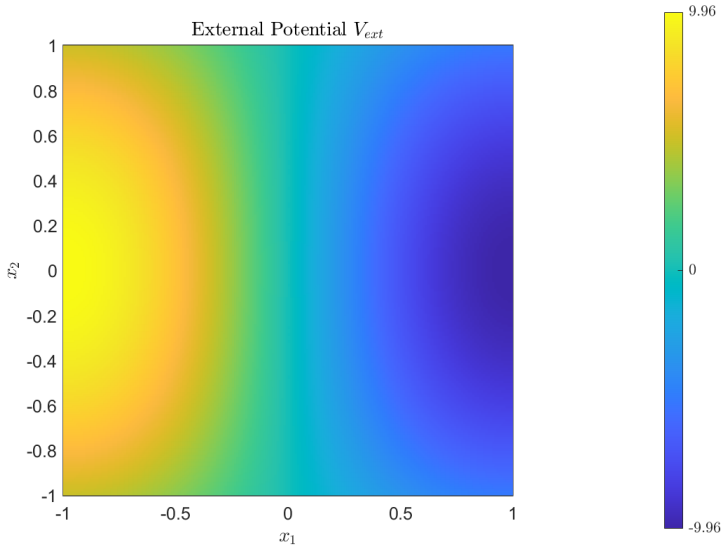

Figure 19: Dirichlet Flow Control 2: External Potential V_{ext} acting on ρ.

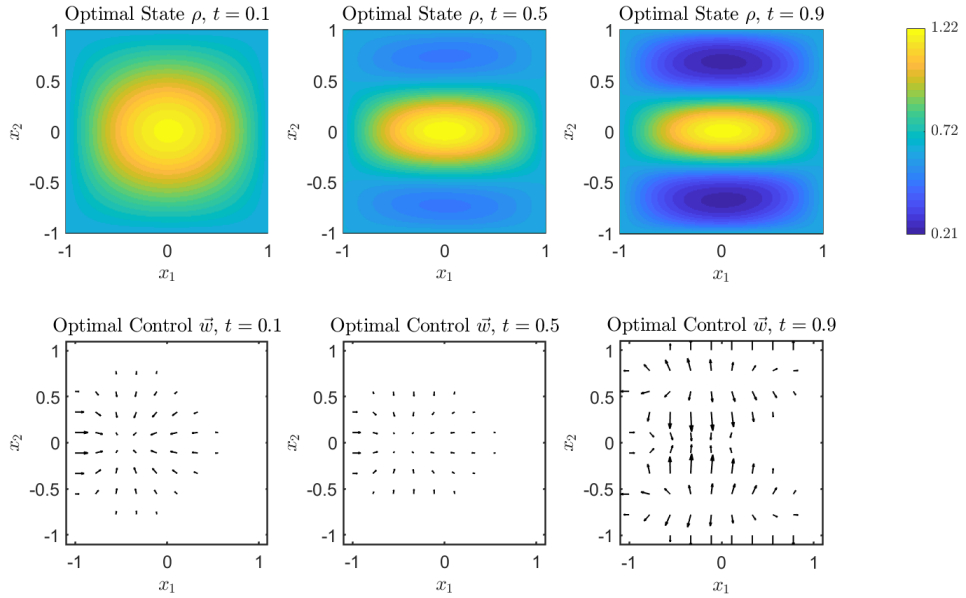

Figure 20: Dirichlet Flow Control 2: Optimal ρ and optimal control for $\kappa = 0$ and $\beta = 10^{-3}$.

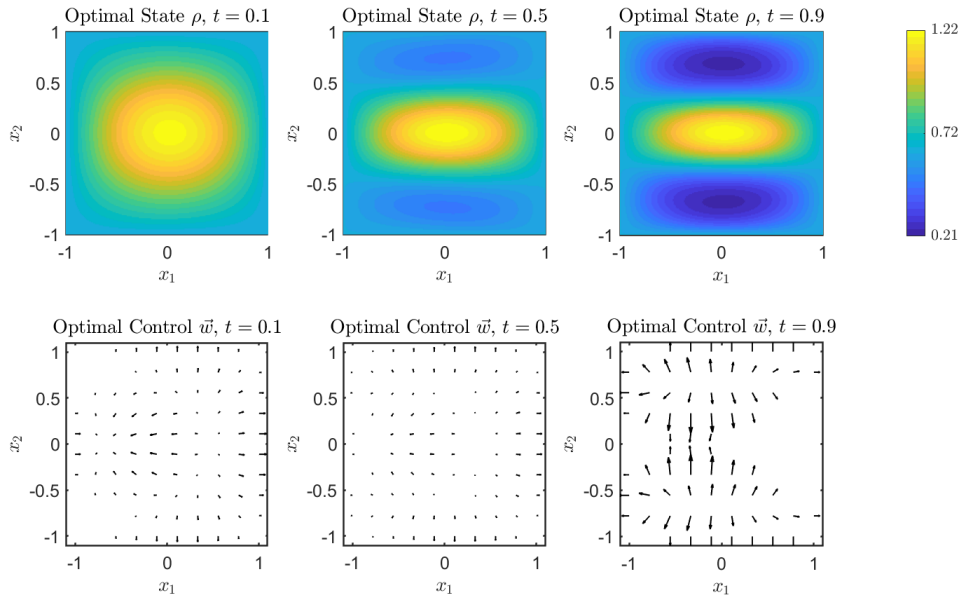

Figure 21: Dirichlet Flow Control 2: Optimal ρ and optimal control for $\kappa = -1$ and $\beta = 10^{-3}$.

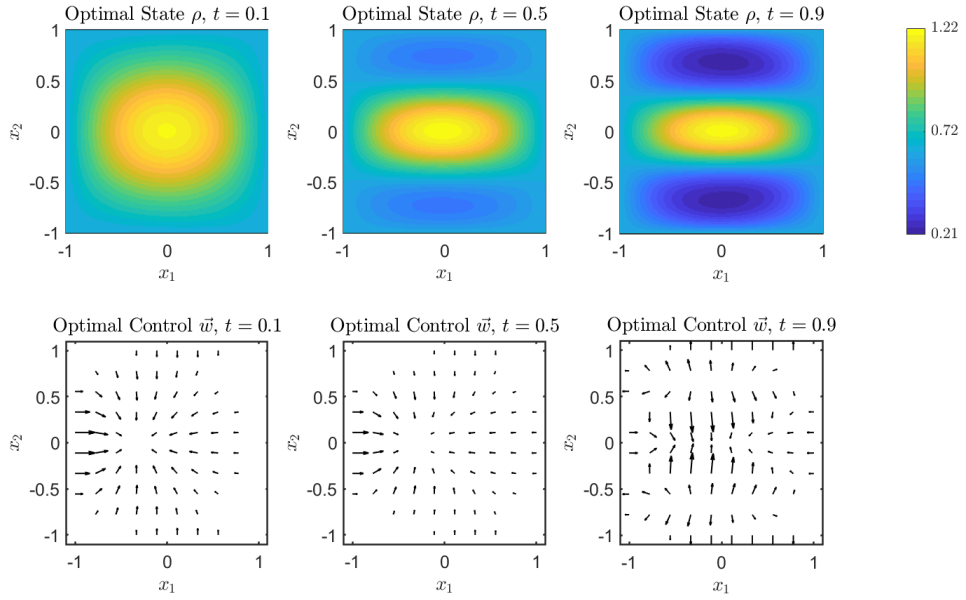

Figure 22: Dirichlet Flow Control 2: Optimal ρ and optimal control for $\kappa = 1$ and $\beta = 10^{-3}$.