We wrote the terms in * and ** into vector form as (*, **), but it should have been * + **. I think it has to be a sum, since the term we start out with in the cost functional is the scalar quantity

$$
\begin{aligned}
\frac{1}{2}\int_\Omega \nabla\times\mathbf{w}^2 d\mathbf{x} &= \frac{1}{2}\int_\Omega \left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)^2 d\mathbf{x}\\
&= \int_\Omega \frac{1}{2}\frac{\partial w_2}{\partial x_1}\frac{\partial w_2}{\partial x_1}-\frac{\partial w_2}{\partial x_1}\frac{\partial w_1}{\partial x_2}+\frac{1}{2}\frac{\partial w_1}{\partial x_2}\frac{\partial w_2}{\partial x_1}d\mathbf{x}.
\end{aligned}
$$

We take the derivative with respect to h_1, h_2

$$
\begin{aligned}
&\int_\Omega \frac{\partial h_2}{\partial x_1}\frac{\partial w_2}{\partial x_1}-\frac{\partial h_2}{\partial x_1}\frac{\partial w_1}{\partial x_2}-\frac{\partial w_2}{\partial x_1}\frac{\partial h_1}{\partial x_2}+\frac{\partial h_1}{\partial x_2}\frac{\partial w_1}{\partial x_2}d\mathbf{x}\\
&=\int_\Omega \frac{\partial h_2}{\partial x_1}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)+\frac{\partial h_1}{\partial x_2}\left(\frac{\partial w_1}{\partial x_2}-\frac{\partial w_2}{\partial x_1}\right)d\mathbf{x}\\
&=\int_\Omega \frac{\partial}{\partial x_1}\left(h_2\frac{\partial w_2}{\partial x_1}-h_2\frac{\partial w_1}{\partial x_2}\right)-\left(h_2\frac{\partial^2 w_2}{\partial x_1^2}-h_2\frac{\partial^2 w_1}{\partial x_1 x_2}\right)\\
&\quad+\frac{\partial}{\partial x_2}\left(h_1\frac{\partial w_1}{\partial x_2}-h_1\frac{\partial w_2}{\partial x_1}\right)-\left(h_1\frac{\partial^2 w_1}{\partial x_2^2}-h_1\frac{\partial^2 w_2}{\partial x_1 x_2}\right)d\mathbf{x}.
\end{aligned}
$$

Applying Green's theorem, we get

$$
\begin{aligned}
&\int_{\partial\Omega}\left(h_2\frac{\partial w_2}{\partial x_1}-h_2\frac{\partial w_1}{\partial x_2}\right)dx_1+\left(h_1\frac{\partial w_1}{\partial x_2}-h_1\frac{\partial w_2}{\partial x_1}\right)dx_2\\
&-\int_\Omega\left(h_2\frac{\partial^2 w_2}{\partial x_1^2}-h_2\frac{\partial^2 w_1}{\partial x_1 x_2}\right)+\left(h_1\frac{\partial^2 w_1}{\partial x_2^2}-h_1\frac{\partial^2 w_2}{\partial x_1 x_2}\right)d\mathbf{x}.
\end{aligned}
$$

Rewriting the boundary terms we get

$$
\begin{aligned}
&\int_{\partial\Omega}\nabla\times\mathbf{w}h_2dx_1-\nabla\times\mathbf{w}h_1dx_2\\
&-\int_\Omega\left(h_2\frac{\partial^2 w_2}{\partial x_1^2}-h_2\frac{\partial^2 w_1}{\partial x_1 x_2}\right)+\left(h_1\frac{\partial^2 w_1}{\partial x_2^2}-h_1\frac{\partial^2 w_2}{\partial x_1 x_2}\right)d\mathbf{x}.
\end{aligned}
$$

Green's Theorem can be written as

$$
\int_{\partial\Omega}Ldx+Mdy=\int_{\partial\Omega}(M,-L)\cdot(dy,-dx)=\int_{\partial\Omega}(M,-L)\cdot\mathbf{n}ds,
$$

where $ds=\sqrt{dx^2+dy^2}$ and $\mathbf{n}$ chosen such that it is normalized and perpendicular to (dx,dy). Therefore, we have

$$
\begin{aligned}
\int_{\partial\Omega}\nabla\times\mathbf{w}h_2dx_1-\nabla\times\mathbf{w}h_1dx_2&=\int_{\partial\Omega}\nabla\times\mathbf{w}\,(-h_1,-h_2)\cdot(dx_2,-dx_1)\\
&=\int_{\partial\Omega}\nabla\times\mathbf{w}\,(-h_1,-h_2)\cdot\left(\frac{dx_2}{ds},-\frac{dx_1}{ds}\right)ds,
\end{aligned}
$$

Consider the boundary term from my derivation

$$\begin{aligned}
&\int_{\partial\Omega} (\nabla \times \mathbf{w})\, \mathbf{h}_{\perp} \cdot \mathbf{n} ds \\
=&\int_{\partial\Omega} (\nabla \times \mathbf{w})\, (h_2, -h_1) \cdot (n_1, n_2) ds.
\end{aligned} \tag{1}$$

These two formulations agree if $n_1 = \frac{dx_1}{ds}$ and $n_2 = \frac{dx_2}{ds}$. However, this doesn't quite make sense to me, since these are not normal to dx_1 and dx_2.