Revised (linear in time) Exact Solution for the Flow Control Problem with Dirichlet and Neumann BCs

Optimality System

$$
\begin{aligned}
\partial_t \rho &= \Delta \rho - \nabla \cdot (w \rho) + f \\
\partial_t p &= -\Delta p - \rho + \hat{\rho} - w \cdot \nabla p \\
w &= -\frac{1}{\beta} \rho \nabla p
\end{aligned}
$$

Neumann plus 2 with linear time

Choose

$$
\begin{aligned}
\rho &= \beta^{1/2} t (\cos(\pi x) + 2) \\
p &= \beta^{1/2} (T - t) \cos(\pi x) \\
w &= \pi t (T - t) \sin(\pi x)(\cos(\pi x) + 2),
\end{aligned}
$$

then

$$
\begin{aligned}
f &= \beta^{1/2}(\cos(\pi x) + 2) + \pi^2 \beta^{1/2} t \cos(\pi x) + \pi^2 \beta^{1/2} t^2 (T - t)(\cos(\pi x) + 2)\Big((\cos(\pi x) + 2)\cos(\pi x) - 2\sin(\pi x)^2\Big) \\
\hat{\rho} &= \beta^{1/2} t (\cos(\pi x) + 2) - \beta^{1/2} \cos(\pi x) - \pi^2 \beta^{1/2} (T - t) \cos(\pi x) - \pi^2 \beta^{1/2} t (T - t)^2 \sin(\pi x)^2 (\cos(\pi x) + 2).
\end{aligned}
$$

Dirichlet with linear time (1) - not ideal but close to original

Choose (similar to the original choices)

$$
\begin{aligned}
\rho &= 2t \cos(\frac{\pi x}{2}) \\
p &= (T - t) \cos(\frac{\pi x}{2}) \\
w &= \frac{\pi}{\beta} t (T - t) \sin(\frac{\pi x}{2}) \cos(\frac{\pi x}{2}),
\end{aligned}
$$

then

$$
\begin{aligned}
f &= 2\cos(\frac{\pi x}{2}) + \frac{\pi^2}{2} t \cos(\frac{\pi x}{2}) + \frac{2\pi^2}{\beta} t^2 (T - t)\left(\frac{\pi}{2} \cos(\frac{\pi x}{2})^3 - \pi \sin(\frac{\pi x}{2})^2 \cos(\frac{\pi x}{2})\right) \\
\hat{\rho} &= 2t \cos(\frac{\pi x}{2}) - \cos(\frac{\pi x}{2}) - \frac{\pi^2}{4} (T - t) \cos(\frac{\pi x}{2}) - \frac{\pi^2}{2\beta} t (T - t)^2 \sin(\frac{\pi x}{2})^2 \cos(\frac{\pi x}{2}).
\end{aligned}
$$

Dirichlet with linear time (2) - better numerically

Choose (for better numerical results)

$$
\begin{aligned}
\rho &= 2\beta^{1/2} t \cos(\frac{\pi x}{2}) \\
p &= (T-t)\beta^{1/2} \cos(\frac{\pi x}{2}) \\
w &= \pi t (T-t) \sin(\frac{\pi x}{2}) \cos(\frac{\pi x}{2}),
\end{aligned}
$$

then

$$
\begin{aligned}
f &= 2\beta^{1/2} \cos(\frac{\pi x}{2}) + \frac{\pi^2}{2}\beta^{1/2} t \cos(\frac{\pi x}{2}) + \beta^{1/2} 2\pi^2 t^2 (T-t) \left(\frac{\pi}{2} \cos(\frac{\pi x}{2})^3 - \pi \sin(\frac{\pi x}{2})^2 \cos(\frac{\pi x}{2}) \right) \\
\hat{\rho} &= 2\beta^{1/2} t \cos(\frac{\pi x}{2}) - \beta^{1/2} \cos(\frac{\pi x}{2}) - \beta^{1/2} \frac{\pi^2}{4} (T-t) \cos(\frac{\pi x}{2}) - \frac{\pi^2}{2} \beta^{1/2} t (T-t)^2 \sin(\frac{\pi x}{2})^2 \cos(\frac{\pi x}{2}).
\end{aligned}
$$