1 Multishape OCP

(Note: MultiShapeFancyChannel3) Last week, there was an issue with the multishape OCP for small β in particular. I have found an alternative initial condition for the problem to work. Last week the initial condition was a Gaussian located in the first shape only. This causes the algorithm to converge to $wErr = 0.00$ in only a few iterations, while $J_{FW} < J_{Opt}$. I suspect this is happening because there is not enough mass in the system. When I ran the OCP on the last two shapes without changing the initial condition (by accident) the same mistake occurred, while it didn't occur when I ran it on the first shape. I now changed the initial condition to $\rho_0 = 0.5$. This works well with $\beta = 10^{-3}$ and $J_{FW} = 0.1218$, while $J_{Opt} = 0.0034$. The result can be seen in Figure 1.

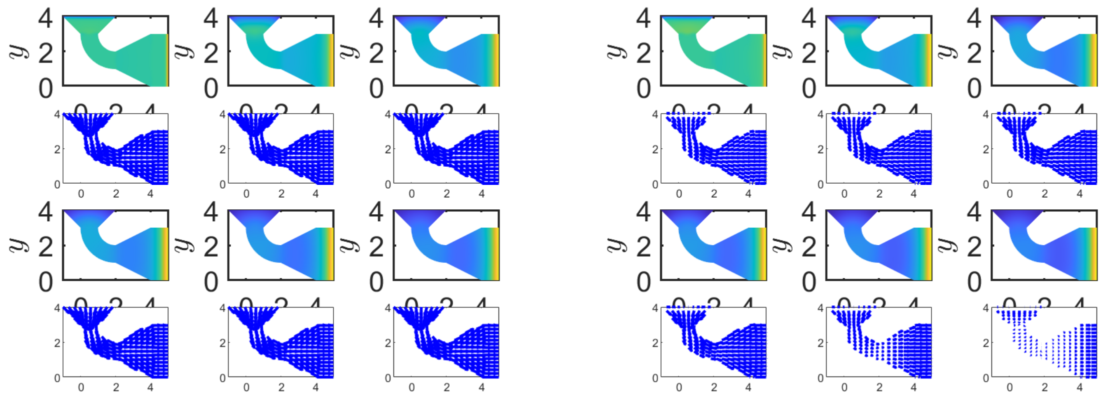

Figure 1: $\widehat{\rho}$ and optimal ρ with corresponding $\mathbf{w}$.

2 Periodic Boundary Conditions

We consider the advection diffusion equation with periodic boundary conditions and a corresponding OCP:

$$
\begin{aligned}
&\min \frac{1}{2}||\rho - \widehat{\rho}||^2 + \frac{\beta}{2}||\mathbf{w}||^2\\
&\text{subject to:}\\
&\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial \rho \mathbf{w}}{\partial x}\\
&\rho(a) = \rho(b)\\
&\frac{\partial \rho(a)}{\partial x} - \rho(a)\mathbf{w}(a) = \frac{\partial \rho(b)}{\partial x} - \rho(b)\mathbf{w}(b)
\end{aligned}
$$

The relevant part of the Lagrangian is then:

$$\begin{aligned}\mathcal{L} = ... &- \int_0^T \int_\Omega \left(\frac{\partial \rho}{\partial t} - \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial \rho \mathbf{w}}{\partial x}\right) q dr dt \\ &- \int_0^T \left(-\rho(b)q_1 + \rho(a)q_1 - \frac{\partial \rho(b)}{\partial x}q_2 + \rho(b)\mathbf{w}(b)q_2 + \frac{\partial \rho(a)}{\partial x}q_2 - \rho(a)\mathbf{w}(a)q_2\right) dt.\end{aligned}$$

Taking partial derivatives, the relevant part of the Lagrangian is:

$$\mathcal{L} = ... - \int_0^T \left[q\frac{\partial \rho}{\partial x} - \rho\frac{\partial q}{\partial x} - \rho \mathbf{w} q\right]_a^b - \left(-\rho(b)q_1 + \rho(a)q_1 - \frac{\partial \rho(b)}{\partial x}q_2 + \rho(b)\mathbf{w}(b)q_2 + \frac{\partial \rho(a)}{\partial x}q_2 - \rho(a)\mathbf{w}(a)q_2\right) dt.$$

Taking the derivative with respect to ρ gives:

$$\begin{aligned}\mathcal{L}_\rho h = ... - \int_0^T &\left[q\frac{\partial h}{\partial x} - h\frac{\partial q}{\partial x} - h\mathbf{w}q\right]_a^b \\ &- \left(-h(b)q_1 + h(a)q_1 - \frac{\partial h(b)}{\partial x}q_2 + h(b)\mathbf{w}(b)q_2 + \frac{\partial h(a)}{\partial x}q_2 - h(a)\mathbf{w}(a)q_2\right) dt\end{aligned}$$

Writing all terms explicitly:

$$\begin{aligned}\mathcal{L}_\rho h = ... + \int_0^T \Bigg(&- q(b)\frac{\partial h(b)}{\partial x} + h(b)\frac{\partial q(b)}{\partial x} + h(b)\mathbf{w}(b)q(b) + q(a)\frac{\partial h(a)}{\partial x} - h(a)\frac{\partial q(a)}{\partial x} - h(a)\mathbf{w}(a)q(a) \\ h(b)q_1 - h(a)q_1 &+ \frac{\partial h(b)}{\partial x}q_2 - h(b)\mathbf{w}(b)q_2 - \frac{\partial h(a)}{\partial x}q_2 + h(a)\mathbf{w}(a)q_2\Bigg) dt\end{aligned}$$

Then considering the terms that satisfy $\frac{\partial h}{\partial x} \neq 0$ at a and b separately we get:

$$\begin{aligned}\int_0^T -q(b)\frac{\partial h(b)}{\partial x} + \frac{\partial h(b)}{\partial x}q_2 dt = 0 \\ \int_0^T q(a)\frac{\partial h(a)}{\partial x} - \frac{\partial h(a)}{\partial x}q_2 dt = 0\end{aligned}$$

And therefore we find $q(b) = q_2$ and $q(a) = q_2$ and so:

$$q(a) = q(b).$$

Then considering the terms where $h \neq 0$, again separately for a and b we get:

$$\begin{aligned}\int_0^T h(b)\frac{\partial q(b)}{\partial x} + h(b)\mathbf{w}(b)q(b) + h(b)q_1 - h(b)\mathbf{w}(b)q_2 dt = 0 \\ \int_0^T -h(a)\frac{\partial q(a)}{\partial x} - h(a)\mathbf{w}(a)q(a) - h(a)q_1 + h(a)\mathbf{w}(a)q_2 dt = 0\end{aligned}$$

And using that $q(b) = q_2$ and $q(a) = q_2$ we get:

$$\begin{aligned}\frac{\partial q(b)}{\partial x} + \mathbf{w}(b)q(b) + q_1 - \mathbf{w}(b)q(b) = 0 \\ -\frac{\partial q(a)}{\partial x} - \mathbf{w}(a)q(a) - q_1 + \mathbf{w}(a)q(a) = 0\end{aligned}$$

and so:

$$\frac{\partial q(b)}{\partial x} = \frac{\partial q(a)}{\partial x}.$$

Therefore, the two boundary conditions for the adjoint equation are:

$$q(a) = q(b) \quad \frac{\partial q(b)}{\partial x} = \frac{\partial q(a)}{\partial x},$$

as expected.