1 Periodic Boundary Conditions

We consider the advection diffusion equation with periodic boundary conditions and a corresponding OCP:

$$
\begin{aligned}
&\min \frac{1}{2}||\rho - \widehat{\rho}||^2 + \frac{\beta}{2}||\mathbf{w}||^2 \\
&\text{subject to:} \\
&\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial \rho \mathbf{w}}{\partial x} \\
&\rho(a) = \rho(b) \\
&\frac{\partial \rho(a)}{\partial x} - \rho(a)\mathbf{w}(a) = -\frac{\partial \rho(b)}{\partial x} + \rho(b)\mathbf{w}(b)
\end{aligned}
$$

The relevant part of the Lagrangian is then:

$$
\begin{aligned}
\mathcal{L} = ... - &\int_0^T \int_\Omega \left(\frac{\partial \rho}{\partial t} - \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial \rho \mathbf{w}}{\partial x} \right) q dr dt \\
- &\int_0^T \left(\rho(b) q_1 - \rho(a) q_1 + \frac{\partial \rho(b)}{\partial x} q_2 - \rho(b)\mathbf{w}(b) q_2 + \frac{\partial \rho(a)}{\partial x} q_2 - \rho(a)\mathbf{w}(a) q_2 \right) dt.
\end{aligned}
$$

Taking partial derivatives, the relevant part of the Lagrangian is:

$$
\mathcal{L} = ... - \int_0^T \left[q\frac{\partial \rho}{\partial x} - \rho \frac{\partial q}{\partial x} - \rho \mathbf{w} q \right]_a^b - \left(\rho(b) q_1 - \rho(a) q_1 + \frac{\partial \rho(b)}{\partial x} q_2 - \rho(b)\mathbf{w}(b) q_2 + \frac{\partial \rho(a)}{\partial x} q_2 - \rho(a)\mathbf{w}(a) q_2 \right) dt.
$$

Taking the derivative with respect to ρ gives:

$$
\begin{aligned}
\mathcal{L}_\rho h = ... - &\int_0^T \left[q\frac{\partial h}{\partial x} - h\frac{\partial q}{\partial x} - h\mathbf{w} q \right]_a^b \\
&- \left(h(b) q_1 - h(a) q_1 + \frac{\partial h(b)}{\partial x} q_2 - h(b)\mathbf{w}(b) q_2 + \frac{\partial h(a)}{\partial x} q_2 - h(a)\mathbf{w}(a) q_2 \right) dt
\end{aligned}
$$

Writing all terms explicitly:

$$
\begin{aligned}
\mathcal{L}_\rho h = ... + \int_0^T \Bigg(&- q(b)\frac{\partial h(b)}{\partial x} + h(b)\frac{\partial q}{\partial x} + h(b)\mathbf{w}(b)q(b) + q(a)\frac{\partial h(a)}{\partial x} - h(a)\frac{\partial q(a)}{\partial x} - h(a)\mathbf{w}(a)q(a) \\
&- h(b) q_1 + h(a) q_1 - \frac{\partial h(b)}{\partial x} q_2 + h(b)\mathbf{w}(b) q_2 - \frac{\partial h(a)}{\partial x} q_2 + h(a)\mathbf{w}(a) q_2 \Bigg) dt
\end{aligned}
$$

Then considering the terms that satisfy $\frac{\partial h}{\partial x} \neq 0$ we get:

$$
-q(b) + q(a) - q_2(b) - q_2(a) = 0,
$$

so that $q(b) = -q_2(b)$ and $q(a) = q_2(a)$. For $h \neq 0$ we have:

$$
\frac{\partial q(b)}{\partial x} + \mathbf{w}(b)q(b) - \frac{\partial q(a)}{\partial x} - \mathbf{w}(a)q(a) - q_1(b) + q_1(a) + \mathbf{w}(b)q_2(b) + \mathbf{w}(a)q_2(a) = 0
$$

Using the first condition, we get:

$$\frac{\partial q(b)}{\partial x} + \mathbf{w}(b)q(b) - \frac{\partial q(a)}{\partial x} - \mathbf{w}(a)q(a) - q_1(b) + q_1(a) - \mathbf{w}(b)q(b) + \mathbf{w}(a)q(a) = 0,$$

so that terms involving $\mathbf{w}$ cancel to give:

$$\frac{\partial q(b)}{\partial x} - \frac{\partial q(a)}{\partial x} - q_1(b) + q_1(a) = 0.$$

If $q_1 = q_2$ we get:

$$\frac{\partial q(b)}{\partial x} - \frac{\partial q(a)}{\partial x} + q(b) + q(a) = 0.$$

2 Time independent control

We wanted to see whether the time independent flow control is similar to the ∇V_{ext} of the target. The target state was influenced by $V_{ext} = 0.1y_2$. The forward state for the OCP was influenced by $V_{ext} = 0.01y_2$. Figure 1 shows the control and ∇V_{ext} of the target. Why is one positive and one negative? Is it because they are opposite signs in PDE?

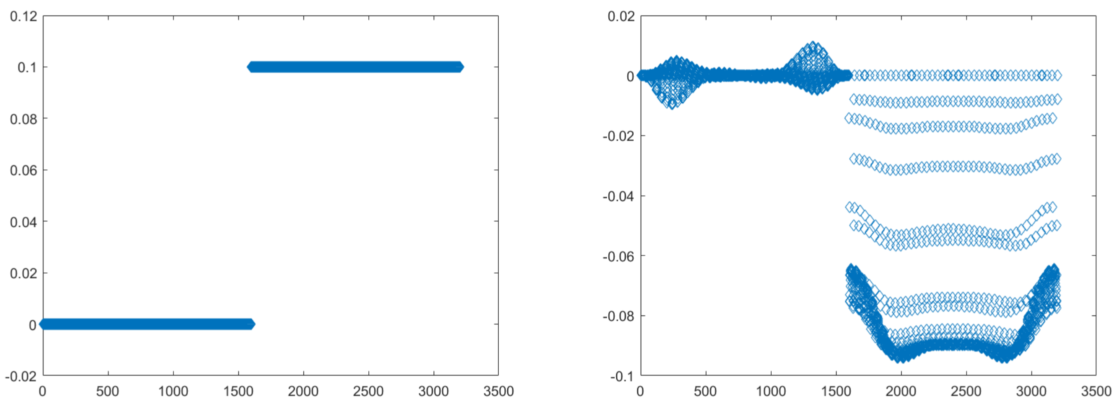

Figure 1: ∇V_{ext} of target and optimal control

3 Multishape Channel

The target flow profile works now and can be seen in Figure 2. We chose $N = 20$, $n = 20$ for each shape and $T = 5$. The optimal control problem is still not quite working. For $\beta = 10^{-1}$, it seems to work or almost, but for 10^{-3} it converges in three iterations, where the last error is zero. However, $J_{FW} < J_{Opt}$. When decreasing λ from 0.01 to 0.001, the convergence steps are smaller and hopefully therefore it will converge to a minimum. The target and optimal ρ for $\beta = 10^{-1}$ are displayed in Figure 3. We have $J_{FW} = 0.2092$ and $J_{Opt} = 0.1563$.

- exact problem not exact yet - multishape JFW ¡ JOpt

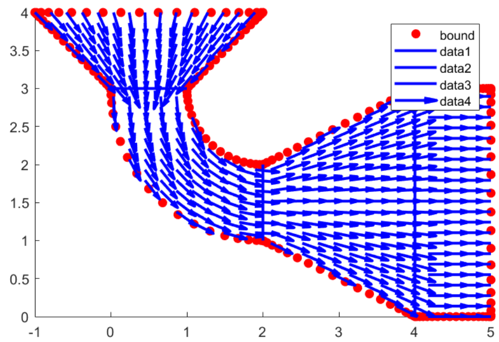

Figure 2: Target flow setup

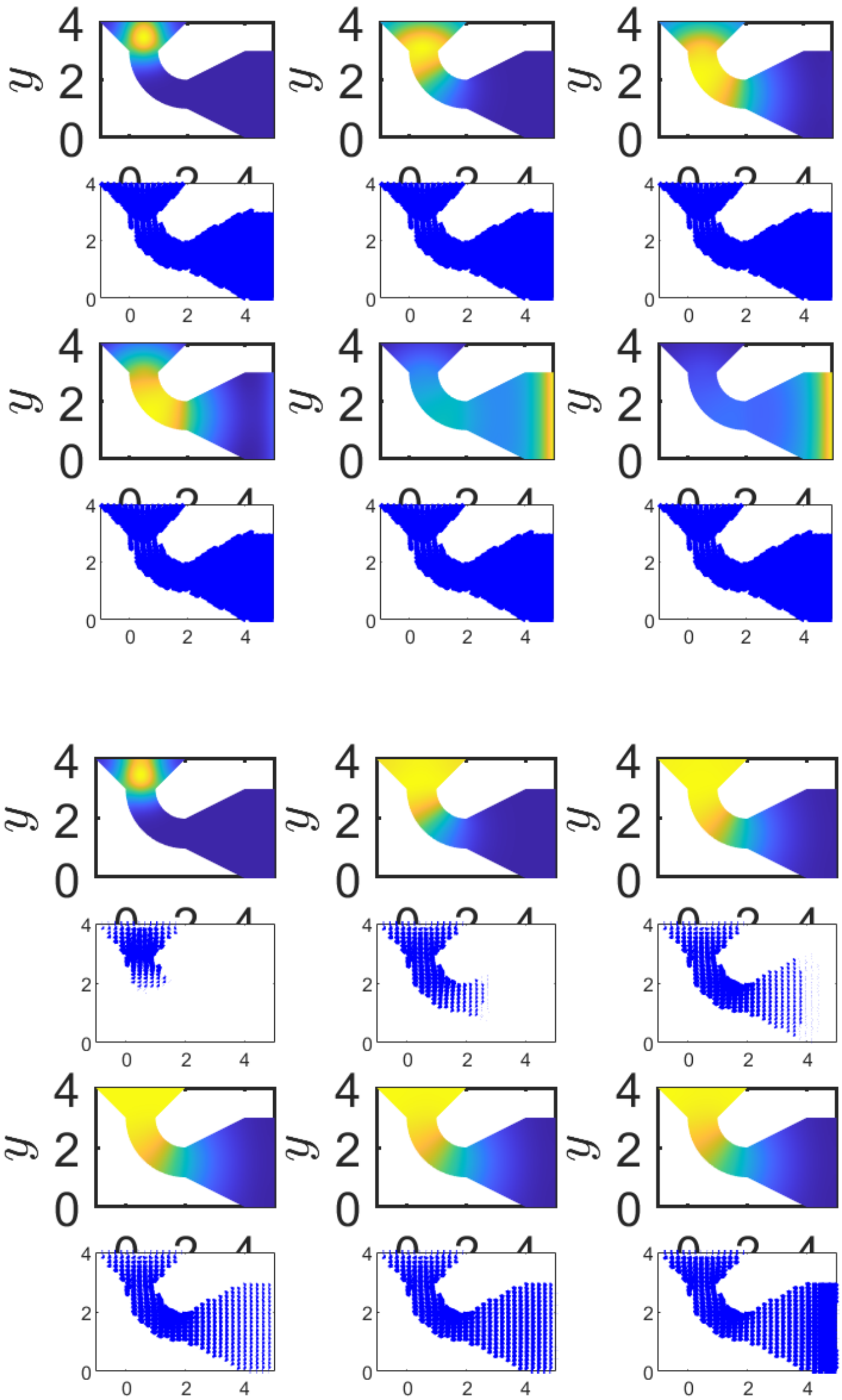

Figure 3: $\widehat{\rho}$ and optimal ρ, $\beta = 10^{-1}$.