

## Literature Review on Related Work

While mean-field games were first introduced by Lasry and Lions, [1], [2],[3] and [4], and independently by Huang, Caines and Malhamé, [5], under the name Nash certainty equivalence, the optimal control of this class of problems is quite a new area of research. The main difficulty in the optimal control of mean-field equations is a non-linear, non-local particle interaction term. Therefore, standard results in optimal control theory cannot readily be applied, and new approaches have to be developed to address theoretical and numerical challenges.

There are two types of models that recent work has focussed on. The most popular model is a deterministic microscopic model, which is a generalization of the well-known Cucker-Smale model, see [6], [7]. In the mean-field limit, a Vlasov-type PDE arises. For control problems involving this class of models, the work by Fornasier et al. provides a range of theoretical results on the convergence of the microscopic optimal control problem to a corresponding macroscopic problem, using methods of optimal transport and a  $\Gamma$ -limit argument, proving existence of optimal controls in the mean-field setting, see [8], [9] and [10]. The work focusses on sparse control strategies, where one or more agents influence a larger crowd. Additional work on sparse control strategies can be found in [11], as well as in the review paper [12]. In [13], an alternative method, an  $L_2$  calculus, is developed, and convergence results are proved. The control in this work is applied through the interaction term.

Numerical advances have been made in [14] and [15], where sparse and other control strategies through the external agents are considered. In both papers a Strang-Splitting scheme, [16], is applied to solve the optimal control problem. The numerical results verify the convergence of the microscopic control problem to its mean-field limit. Furthermore, in [17], different selective control strategies are considered, and an iterative numerical method is chosen, where the interaction term is approximated stochastically.

Fewer work has been done on the optimal control of the Fokker-Planck PDE, which arises as the mean-field limit of a stochastic microscopic model. Some theoretical results on this model are published. In [18], the existence of optimal controls for microscopic and macroscopic versions of the problem are proved and a rigorous derivation of first-order optimality conditions is given. Following this, [19] discusses the existence and regularity of an optimal control problem of this type on periodic domains, including the well-posedness of the Fokker-Planck equation. In [20] and [21], the convergence of the microscopic optimal control problem to its mean-field limit is proved. Numerical results on the model include those presented in [20], where a Strang-Splitting scheme, [22], is applied, and in which convergence to the mean field optimal control problem is shown numerically. Furthermore, in [18], an optimal control hierarchy, including

instantaneous and Boltzmann-type controls, is proposed. The mean-field first-order optimality system in [18] is solved using a Chang-Cooper scheme for the forward equation, finite differences for the adjoint equation, while approximating the integrals using a Monte-Carlo scheme. This is coupled by a sweeping algorithm, where updates are made through the gradient equation. Some numerical results on a porous media version of the Fokker-Planck equation are presented in [21]. In [23] and [24], steady state solutions to a Fokker-Planck-type PDE are considered, however, the main focus are Boltzmann-type approaches to solving the optimal control problem.

The most common control types in the literature are flow control, e.g. [18], control through the interaction term, e.g. [20], as well as control through external agents, e.g. [9]. Most papers do not consider boundary conditions, because it is assumed that the particle distribution is of compact support, see [13], [10] or [15]. No-flux boundary conditions, which are of high relevance in applications, are not often found in the literature, but are considered in [18] and [21]. Our work considers the mean-field equation of Fokker-Planck type, flow-type control or control through a force term and no-flux or Dirichlet boundary conditions, in order to address a broad range of test problems and real world applications.

As described above, some numerical methods have been developed for solving optimal control problems involving non-local, non-linear PDEs. Most of these papers however focus on other methods and use the mean-field optimal control as verification tool, see [20], [18]. It takes large computational effort to solve these problems, which increases with dimensionality, see [14], [15]. We are proposing a new numerical framework for PDE-constrained optimization applied to multiscale particle dynamics, where a multiple shooting approach, solving the first-order optimality system, is combined with pseudospectral methods, discretizing space and time domains. This composition of methods offers an efficient and accurate solver for the class of problems discussed. To our knowledge, it is the first time that pseudospectral methods are used in the context of optimal control problems and that multiple shooting is applied to optimal control problems involving non-local PDEs.

The most relevant publications on the multiple shooting method, with regards to our approach, are by Carraro et al., see [25], [26], treating direct and indirect multiple shooting methods for parabolic optimal control problems. While direct multiple shooting is applied to the full statement of an optimal control problem, the indirect method solves the first-order optimality system instead. Our solver applies the indirect multiple shooting approach, by solving the first-order optimality system corresponding to the optimal control problem for multiscale particle dynamics.

## References

- [1] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. i – le cas stationnaire. *Comptes Rendus Mathématique*, 343(9):619 – 625, 2006.
- [2] Jean-Michel Lasry and Pierre-Louis Lions. Jeux à champ moyen. ii – horizon fini et contrôle optimal. *Comptes Rendus Mathématique*, 343(10):679 – 684, 2006.
- [3] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Cahiers de la Chaire Finance et Développement Durable*, 2007.
- [4] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, Mar 2007.
- [5] Minyi Huang, Peter E. Caines, and Roland P. Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. In *42nd IEEE International Conference on Decision and Control (IEEE Cat. No.03CH37475)*, volume 1, pages 98–103, Dec 2003.
- [6] Felipe Cucker and Steve Smale. Emergent behavior in flocks. *Automatic Control, IEEE Transactions on*, 52:852 – 862, 06 2007.
- [7] Felipe Cucker and Steve Smale. On the mathematics of emergence. *Japan J Math*, 2:197–227, 03 2007.
- [8] Massimo Fornasier and Francesco Solombrino. Mean-field optimal control. *ESAIM: Control, Optimisation and Calculus of Variations*, 20(4):1123–1152, 2014.
- [9] Massimo Fornasier, Benedetto Piccoli, and Francesco Rossi. Mean-field sparse optimal control. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2028):20130400, 2014.
- [10] Massimo Fornasier, Stefano Lisini, Carlo Orrieri, and Guiseppe Savaré. Mean-field optimal control as gamma-limit of finite agent controls. *European Journal of Applied Mathematics*, 30(6):1153–1186, 2019.
- [11] Benedetto Piccoli, Francesco Rossi, and Emmanuel Trélat. Control to flocking of the kinetic Cucker–Smale model. *SIAM Journal on Mathematical Analysis*, 47(6):4685 – 4719, 2014.
- [12] Massimo Fornasier. Learning and sparse control of multiagent systems. *Proc. 7thECM*, 2016.
- [13] Martin Burger, René Pinnau, Claudia Totzeck, and Oliver Tse. Mean-field optimal control and optimality conditions in the space of probability measures (preprint?), 2019.

- [14] Martin Burger, Rene Pinnau, Claudia Totzeck, Oliver Tse, and Andreas Roth. Instantaneous control of interacting particle systems in the mean-field limit (preprint?), 2019.
- [15] Martin Burger, René Pinnau, Andreas Roth, Claudia Totzeck, and Oliver Tse. Controlling a self-organizing system of individuals guided by a few external agents – particle description and mean-field limit (preprint?), 2016.
- [16] C.Z Cheng and Georg Knorr. The integration of the Vlasov equation in configuration space. *Journal of Computational Physics*, 22(3):330–351, 1976.
- [17] Giacomo Albi and Lorenzo Pareschi. Selective model-predictive control for flocking systems (preprint). *Communications in Industrial and Applied Mathematics*, 2016.
- [18] Giacomo Albi, Young-Pil Choi, Massimo Fornasier, and Dante Kalise. Mean field control hierarchy. *Applied Mathematics and Optimization*, 76(1):93 – 135, 2016.
- [19] Jose A. Carrillo, Edgard A. Pimentel, and Vardan K. Voskanyan. On a mean field optimal control problem (preprint?), 2019.
- [20] René Pinnau, Claudia Totzeck, Oliver Tse, and Stephan Martin. A consensus-based model for global optimization and its mean-field limit. *Mathematical Models and Methods in Applied Sciences*, 27(01):183–204, 2017.
- [21] José Carrillo, Young-Pil Choi, Claudia Totzeck, and Oliver Tse. An analytical framework for consensus-based global optimization method. *Mathematical Models and Methods in Applied Sciences*, 28(6):1037–1066, 2018.
- [22] Gilbert Strang. On the construction and comparison of difference schemes. *SIAM Journal on Numerical Analysis*, 5(3):506–517, 1968.
- [23] Giacomo Albi, Lorenzo Pareschi, and Mattia Zanella. Boltzmann-type control of opinion consensus through leaders. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2028):20140138, 2014.
- [24] Giacomo Albi, Michael Herty, and Lorenzo Pareschi. Kinetic description of optimal control problems and applications to opinion consensus. 2014.
- [25] Thomas Carraro and Michael Geiger. Direct and indirect multiple shooting for parabolic optimal control problems. In P. Cinnella G. Zavarise and M. Campiti, editors, *Special Issue: 86th Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM)*, pages 35–67. 01 2015.

- [26] Thomas Carraro, Michael Geiger, and R Rannacher. Indirect multiple shooting for nonlinear parabolic optimal control problems with control constraints. *SIAM Journal on Scientific Computing*, 36, 01 2014.