Boundary Observation and Non-Constant Flux Boundary conditions

This set of notes is concerned with deriving non-zero (non-)constant flux boundary conditions, which are different for different parts of the boundary. Furthermore, the target distribution of particles $\hat{\rho}$ is prescribed on parts of the boundary only, instead of on all of Ω. Different parts of the boundary have different target distributions of particles.
+++ Boundary control correct here, since we don't directly apply the control on the boundary??+++

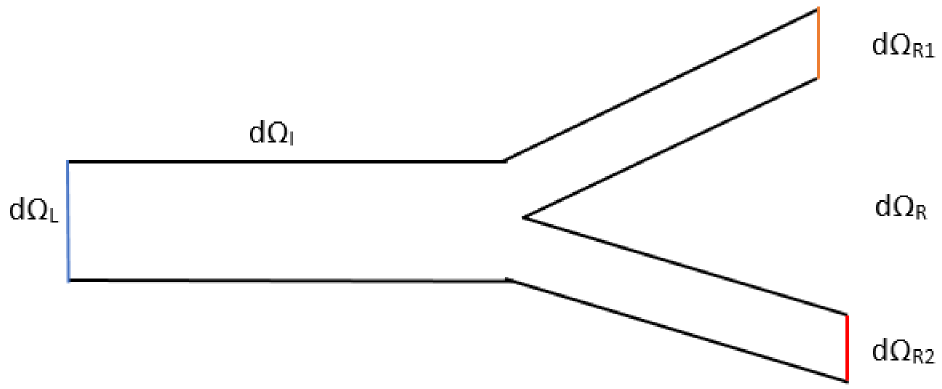

Figure 1: Boundary

The domain with the different parts of the boundary that are considered can be seen in Figure 1.

1 Distributed Observation and Constant Flux Boundary Conditions

The problem of interest is of the form:

$$\min_{\rho, f} \quad \frac{1}{2}||\rho - \hat{\rho}||^2_{L_2(Q)} + \frac{\beta}{2}||f||^2_{L_2(Q)}$$

subject to:

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \mathbf{w}_{Flow}) + \nabla \cdot (\rho \nabla V_{ext}) + \nabla \cdot \int_\Omega \rho(r)\rho(r')\nabla V_2(|r - r'|)dr' + f \qquad \text{in} \quad Q,$$

$$\rho = \rho_0 \quad \text{at} \quad t = 0$$

$$-\mathbf{j} \cdot \mathbf{n} = \mathbb{1}_{\partial\Omega_L} C_L + \mathbb{1}_{\partial\Omega_R} C_R + \mathbb{1}_{\partial\Omega_I} 0, \quad \text{on} \quad \partial Q,$$

where C_L, C_R are constants and $\mathbb{1}$ is the indicator function of the set (the parts of the boundary) of interest. Furthermore, $\mathbf{j}$ satisfies:

$$\mathbf{j} = \nabla\rho - (\rho\mathbf{w}_{Flow}) + (\rho\nabla V_{ext}) + \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr'.$$

The Lagrangian

The Lagrangian is of the form:

$$\begin{aligned}\mathcal{L}(\rho, f, p_Q, p_{\partial Q}) = &\int_0^T\int_\Omega \frac{1}{2}(\rho-\hat{\rho})^2 drdt + \frac{\beta}{2}\int_0^T\int_\Omega f^2 drdt \\ &+ \int_0^T\int_\Omega \Big(\partial_t\rho - \nabla^2\rho + \nabla\cdot(\rho\mathbf{w}_{Flow}) - \nabla\cdot(\rho\nabla V_{ext}) + \nabla\cdot\int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)\Big)p_Q drdt \\ &+ \int_0^T\int_{\partial\Omega}\Big(\Big(-\nabla\rho + (\rho\mathbf{w}_{Flow}) - (\rho\nabla V_{ext}) - \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr'\Big)\cdot\mathbf{n} \\ &- \mathbb{1}_{\partial\Omega_L}C_L - \mathbb{1}_{\partial\Omega_R}C_R - \mathbb{1}_{\partial\Omega_I}0\Big)p_{\partial Q}drdt.\end{aligned}$$

The Adjoint Equation

The derivative of $\mathcal{L}$ with respect to ρ is, as taken from the extended project:

$$\begin{aligned}&\mathcal{L}_\rho(\rho, \mathbf{w}, p_\Omega, p_{\partial\Omega})h = \int_\Omega h(T)p_Q(T)dr \\ &+ \int_0^T\int_\Omega \Big((\rho-\hat{\rho}) - \partial_t p_Q - \nabla p_Q\cdot\mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q\cdot\nabla V_{ext} \\ &+ \int_\Omega(\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' + \int_{\partial\Omega}(p_{\partial Q}(r') - p_Q(r'))\rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr'\Big)hdrdt \\ &+ \int_0^T\int_{\partial\Omega}\Big(\Big(\frac{\partial p_Q}{\partial n} + p_Q\mathbf{w}\cdot\mathbf{n} - p_{\partial Q}\mathbf{w}_{Flow}\cdot\mathbf{n} + p_{\partial Q}\frac{\partial V_{ext}}{\partial n} - p_Q\frac{\partial V_{ext}}{\partial n} + (p_{\partial Q} - p_Q)\int_\Omega \rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr'\Big)h \\ &+ \Big(p_{\partial Q} - p_Q\Big)\frac{\partial h}{\partial n}\Big)drdt = 0.\end{aligned}$$

Then, from appropriate analysis we find that:

$$p_{\partial Q} = p_Q,$$

and therefore we get:

$$\begin{aligned}(\rho-\hat{\rho}) - \partial_t p_Q - \nabla p_Q\cdot\mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q\cdot\nabla V_{ext}& \\ + \int_\Omega(\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' &= 0, \quad \text{in} \quad Q, \\ \frac{\partial p_Q}{\partial n} &= 0, \quad \text{on} \quad \partial Q.\end{aligned}$$

2 Distributed Observation and Non-Constant Flux Boundary Conditions

The problem of interest is of the form:

$$\min_{\rho, f} \quad \frac{1}{2}||\rho - \hat{\rho}||^2_{L_2(Q)} + \frac{\beta}{2}||f||^2_{L_2(Q)}$$

subject to:

$$\begin{aligned}
&\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \mathbf{w}_{Flow}) + \nabla \cdot (\rho \nabla V_{ext}) + \nabla \cdot \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr' + f \qquad \text{in} \quad Q,\\
&\rho = \rho_0 \quad \text{at} \quad t = 0\\
&-\mathbf{j}\cdot\mathbf{n} = \mathbb{1}_{\partial\Omega_L}(C_{L1} + C_{L2}\rho) + \mathbb{1}_{\partial\Omega_R}(C_{R1} + C_{R2}\rho) + \mathbb{1}_{\partial\Omega_I}0, \qquad \text{on} \quad \partial Q
\end{aligned}$$

where C_{L1}, C_{L2}, C_{R1}, C_{R2} are constants and $\mathbb{1}$ is the indicator function of the set (the parts of the boundary) of interest. Furthermore, $\mathbf{j}$ satisfies:

$$\mathbf{j} = \nabla\rho - (\rho \mathbf{w}_{Flow}) + (\rho\nabla V_{ext}) + \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr'$$

The Lagrangian

The Lagrangian is of the form:

$$\begin{aligned}
\mathcal{L}(\rho, f, p_Q, p_{\partial Q}) = &\int_0^T \int_\Omega \frac{1}{2}(\rho - \hat{\rho})^2 dr dt + \frac{\beta}{2}\int_0^T\int_\Omega f^2 dr dt\\
&+ \int_0^T\int_\Omega \Big(\partial_t\rho - \nabla^2\rho + \nabla\cdot(\rho\mathbf{w}_{Flow}) - \nabla\cdot(\rho\nabla V_{ext}) + \nabla\cdot\int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)\Big)p_Q dr dt\\
&+ \int_0^T\int_{\partial\Omega}\Bigg(\Big(-\nabla\rho + (\rho\mathbf{w}_{Flow}) - (\rho\nabla V_{ext}) - \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr'\Big)\cdot\mathbf{n}\\
&- \mathbb{1}_{\partial\Omega_L}(C_{L1} + C_{L2}\rho) - \mathbb{1}_{\partial\Omega_R}(C_{R1} + C_{R2}\rho) - \mathbb{1}_{\partial\Omega_I}0\Bigg)p_{\partial Q} dr dt.
\end{aligned}$$

The Adjoint Equation

The derivative of $\mathcal{L}$ with respect to ρ is, as taken from the extended project:

$$\mathcal{L}_\rho(\rho, \mathbf{w}, p_\Omega, p_{\partial\Omega})h = \int_\Omega h(T)p_Q(T)dr$$
$$+ \int_0^T \int_\Omega \Bigg((\rho - \hat{\rho}) - \partial_t p_Q - \nabla p_Q \cdot \mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q \cdot \nabla V_{ext}$$
$$+ \int_\Omega (\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r - r'|)dr' + \int_{\partial\Omega} (p_{\partial Q}(r') - p_Q(r'))\rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr' \Bigg) hdrdt$$
$$+ \int_0^T \int_{\partial\Omega} \Bigg(\Bigg(\frac{\partial p_Q}{\partial n} + p_Q \mathbf{w}\cdot\mathbf{n} - p_{\partial Q}\mathbf{w}_{Flow}\cdot\mathbf{n} + p_{\partial Q}\frac{\partial V_{ext}}{\partial n} - p_Q \frac{\partial V_{ext}}{\partial n} + (p_{\partial Q} - p_Q)\int_\Omega \rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr'$$
$$- \mathbb{1}_{\partial\Omega_L} C_{L2} p_{\partial Q} - \mathbb{1}_{\partial\Omega_R} C_{R2} p_{\partial Q} \Bigg) h$$
$$+ \left(p_{\partial Q} - p_Q \right)\frac{\partial h}{\partial n} \Bigg) drdt = 0.$$

Then, from appropriate analysis we find that:

$$p_{\partial Q} = p_Q,$$

and therefore we get:

$$\begin{aligned}
(\rho - \hat{\rho}) - \partial_t p_Q - \nabla p_Q \cdot \mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q \cdot \nabla V_{ext} \\
+ \int_\Omega (\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' = 0, \quad \text{in} \quad Q, \\
\frac{\partial p_Q}{\partial n} - \mathbb{1}_{\partial\Omega_L} C_{L2} p_Q - \mathbb{1}_{\partial\Omega_R} C_{R2} p_Q = 0, \quad \text{on} \quad \partial Q.
\end{aligned}$$

3 Boundary Observation and Constant Flux Boundary Conditions

The problem of interest is of the form:

$$\min_{\rho, f} \quad \frac{1}{2}||\rho - \hat{\rho}||^2_{L_2(\partial Q_R)} + \frac{\beta}{2}||f||^2_{L_2(Q)}$$

subject to:

$$\begin{aligned}
&\partial_t \rho = \nabla^2\rho - \nabla\cdot(\rho \mathbf{w}_{Flow}) + \nabla\cdot(\rho\nabla V_{ext}) + \nabla\cdot\int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr' + f \qquad \text{in} \quad Q, \\
&\rho = \rho_0 \quad \text{at} \quad t = 0 \\
&-\mathbf{j}\cdot\mathbf{n} = \mathbb{1}_{\partial\Omega_L} C_L + \mathbb{1}_{\partial\Omega_R} C_R + \mathbb{1}_{\partial\Omega_I} 0, \qquad \text{on} \quad \partial Q,
\end{aligned}$$

where C_L, C_R are constants and $\mathbb{1}$ is the indicator function of the set (the parts of the boundary) of interest. Furthermore, $\mathbf{j}$ satisfies:

$$\mathbf{j} = \nabla\rho - (\rho\mathbf{w}_{Flow}) + (\rho\nabla V_{ext}) + \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr'.$$

Moreover, let $\hat{\rho}$ be defined such that:

$$\hat{\rho} = \mathbb{1}_{\partial\Omega_{R1}}\tilde{\rho} + \mathbb{1}_{\partial\Omega_{R2}}0.$$

The Lagrangian

The Lagrangian is of the form:

$$\begin{aligned}\mathcal{L}(\rho, f, p_Q, p_{\partial Q}) = &\int_0^T \int_{\partial\Omega_R} \frac{1}{2}(\rho-\hat{\rho})^2 drdt + \frac{\beta}{2}\int_0^T \int_\Omega f^2 drdt \\ &+ \int_0^T \int_\Omega \Big(\partial_t\rho - \nabla^2\rho + \nabla\cdot(\rho\mathbf{w}_{Flow}) - \nabla\cdot(\rho\nabla V_{ext}) + \nabla\cdot\int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)\Big)p_Q drdt \\ &+ \int_0^T \int_{\partial\Omega} \Big(\Big(-\nabla\rho + (\rho\mathbf{w}_{Flow}) - (\rho\nabla V_{ext}) - \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr'\Big)\cdot\mathbf{n} \\ &- \mathbb{1}_{\partial\Omega_L}C_L - \mathbb{1}_{\partial\Omega_R}C_R - \mathbb{1}_{\partial\Omega_I}0\Big)p_{\partial Q} drdt.\end{aligned}$$

The Adjoint Equation

The derivative of $\mathcal{L}$ with respect to ρ is, as taken from the extended project:

$$\begin{aligned}&\mathcal{L}_\rho(\rho, \mathbf{w}, p_\Omega, p_{\partial\Omega})h = \int_\Omega h(T)p_Q(T)dr \\ &+ \int_0^T \int_\Omega \Big(-\partial_t p_Q - \nabla p_Q\cdot\mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q\cdot\nabla V_{ext} \\ &+ \int_\Omega (\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' + \int_{\partial\Omega}(p_{\partial Q}(r') - p_Q(r'))\rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr'\Big)hdrdt \\ &+ \int_0^T \int_{\partial\Omega}\Big(\Big(\frac{\partial p_Q}{\partial n} + p_Q\mathbf{w}\cdot\mathbf{n} - p_{\partial Q}\mathbf{w}_{Flow}\cdot\mathbf{n} + p_{\partial Q}\frac{\partial V_{ext}}{\partial n} - p_Q\frac{\partial V_{ext}}{\partial n} + (p_{\partial Q} - p_Q)\int_\Omega \rho(r')\frac{\partial V_2(|r-r'|)}{\partial n}dr' \\ &(\rho-\hat{\rho})\mathbb{1}_{\partial\Omega_R}\Big)h + \Big(p_{\partial Q} - p_Q\Big)\frac{\partial h}{\partial n}\Big)drdt = 0.\end{aligned}$$

Then, from appropriate analysis we find that:

$$p_{\partial Q} = p_Q,$$

and therefore we get:

$$\begin{aligned}&-\partial_t p_Q - \nabla p_Q\cdot\mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q\cdot\nabla V_{ext} \\ &+ \int_\Omega (\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' = 0, \quad \text{in} \quad Q, \\ &\qquad\qquad\qquad \frac{\partial p_Q}{\partial n} + \mathbb{1}_{\partial\Omega_R}(\rho-\hat{\rho}) = 0, \quad \text{on} \quad \partial Q.\end{aligned}$$

In particular, the boundary condition is:

$$\frac{\partial p_Q}{\partial n} + \mathbb{1}_{\partial\Omega_{R1}}(\rho - \tilde{\rho}) + \mathbb{1}_{\partial\Omega_{R2}}\rho = 0, \quad \text{on} \quad \partial Q.$$

4 Boundary Observation and Non-Constant Flux Boundary Conditions

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$$\rho = \rho_0 \quad \text{at} \quad t = 0$$

$$-\mathbf{j}\cdot\mathbf{n} = \mathbb{1}_{\partial\Omega_L}(C_{L1} + C_{L2}\rho) + \mathbb{1}_{\partial\Omega_R}(C_{R1} + C_{R2}\rho) + \mathbb{1}_{\partial\Omega_I}0, \qquad \text{on} \quad \partial Q,$$

where C_{L1}, C_{L2}, C_{R1}, C_{R2} are constants and $\mathbb{1}$ is the indicator function of the set (the parts of the boundary) of interest. Furthermore, $\mathbf{j}$ satisfies:

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Moreover, let $\hat{\rho}$ be defined such that:

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The Lagrangian

The Lagrangian is of the form:

$$\begin{aligned}
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& + \int_0^T \int_\Omega \Big(\partial_t\rho - \nabla^2\rho + \nabla\cdot(\rho\mathbf{w}_{Flow}) - \nabla\cdot(\rho\nabla V_{ext}) + \nabla\cdot\int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)\Big) p_Q dr dt \\
& + \int_0^T \int_{\partial\Omega} \bigg(\Big(-\nabla\rho + (\rho\mathbf{w}_{Flow}) - (\rho\nabla V_{ext}) - \int_\Omega \rho(r)\rho(r')\nabla V_2(|r-r'|)dr' \Big)\cdot\mathbf{n} \\
& - \mathbb{1}_{\partial\Omega_L}(C_{L1} + C_{L2}\rho) - \mathbb{1}_{\partial\Omega_R}(C_{R1} + C_{R2}\rho) - \mathbb{1}_{\partial\Omega_I}0 \bigg) p_{\partial Q} dr dt.
\end{aligned}$$

The Adjoint Equation

The derivative of $\mathcal{L}$ with respect to ρ is, as taken from the extended project:

$$\mathcal{L}_\rho(\rho, \mathbf{w}, p_\Omega, p_{\partial\Omega})h = \int_\Omega h(T)p_Q(T)dr$$
$$+ \int_0^T \int_\Omega \Big(- \partial_t p_Q - \nabla p_Q \cdot \mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q \cdot \nabla V_{ext}$$
$$+ \int_\Omega (\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' + \int_{\partial\Omega} (p_{\partial Q}(r') - p_Q(r'))\rho(r') \frac{\partial V_2(|r-r'|)}{\partial n} dr' \Big) h dr dt$$
$$+ \int_0^T \int_{\partial\Omega} \Big(\Big(\frac{\partial p_Q}{\partial n} + p_Q \mathbf{w} \cdot \mathbf{n} - p_{\partial Q} \mathbf{w}_{Flow} \cdot \mathbf{n} + p_{\partial Q} \frac{\partial V_{ext}}{\partial n} - p_Q \frac{\partial V_{ext}}{\partial n} + (p_{\partial Q} - p_Q) \int_\Omega \rho(r') \frac{\partial V_2(|r-r'|)}{\partial n} dr'$$
$$\mathbb{1}_{\partial\Omega_R}(\rho - \hat{\rho}) - \mathbb{1}_{\partial\Omega_L} C_{L2} p_{\partial Q} - \mathbb{1}_{\partial\Omega_R} C_{R2} p_{\partial Q} \Big) h + \Big(p_{\partial Q} - p_Q \Big) \frac{\partial h}{\partial n} \Big) dr dt = 0.$$

Then, from appropriate analysis we find that:

$$p_{\partial Q} = p_Q,$$

and therefore we get:

$$\begin{aligned} -\partial_t p_Q - \nabla p_Q \cdot \mathbf{w}_{Flow} - \nabla^2 p_Q + \nabla p_Q \cdot \nabla V_{ext} & \\ + \int_\Omega (\nabla p_Q(r) + \nabla p_Q(r'))\rho(r')\nabla V_2(|r-r'|)dr' &= 0, \quad \text{in} \quad Q, \\ \frac{\partial p_Q}{\partial n} + \mathbb{1}_{\partial\Omega_R}(\rho - \hat{\rho}) - \mathbb{1}_{\partial\Omega_L} C_{L2} p_Q - \mathbb{1}_{\partial\Omega_R} C_{R2} p_Q &= 0, \quad \text{on} \quad \partial Q. \end{aligned}$$

Again, in particular the boundary condition is:

$$\frac{\partial p_Q}{\partial n} + \mathbb{1}_{\partial\Omega_{R1}}(\rho - \tilde{\rho} - C_{R2}p_Q) + \mathbb{1}_{\partial\Omega_{R2}}(\rho - C_{R2}p_Q) - \mathbb{1}_{\partial\Omega_L} C_{L2} p_Q = 0, \quad \text{on} \quad \partial Q.$$