

1 Newton-Krylov

One question I have is why the solver needs more than one iteration, if the given solution is exact. Check that no-flux interacting is correct.

We have the following problems, differences measured with the L_1/L_2 norm, with $N = 16$, and with optimality tolerance 10^{-4} in the old code:

Dirichlet exact, Error = 2.1631×10^{-16} , time = 120 s,

Dirichlet with $\kappa = 1$, Error = 4.8164×10^{-6} , time = 120 s va 480 s,

Dirichlet exact with V_{ext} , Error = 0.0027, time = 130 s.

Then for the Neumann problems we have, with $N = 20$:

Neumann exact, Error = 2.3286×10^{-15} , time = 1.0824×10^3 s,

I ran the original code for this, which gives 228 s. I think I messed with the timing of the algorithm somehow...

(++ note: we fix the slowness by choosing to eliminate the flags. We get 4.1841×10^{-9} for the interacting Neumann case)

2 Hodge Helmholtz Decomposition

We choose a basis of Lagrange polynomials (dimension $2(p-1)p$) and choose an element v from this basis. We then note that if v satisfies $\nabla \times v = 0$, we can write $Dv = 0$, where D is the discretized curl. Then we can split this into two parts, such that $Dv = D_1v_1 + D_2v_2 = 0$. We determine that $(p-1)^2$ is the degree of interpolation polynomial and furthermore, this implies that $(p-1)^2$ rows of D are linearly independent. Therefore, we split up D such that D_2 has $(p-1)^2$ rows and is invertible. We can then see that

$$v_2 = -D_2^{-1}D_1u_1,$$

which provides a formula for creating a vector $\mathbf{v} = (v_1, v_2)$ satisfying that $\nabla \times v = 0$. We can then choose $p^2 - 1$ of such vectors as a basis of the curl free vector space. Any curl free vector field \mathbf{w} can then be described as a linear combination of these, i.e. $\mathbf{w} = \sum_i \alpha_i v_i$.

A similar argument can be used to construct a divergence free basis. The harmonic part of the vector field is found by subtracting the curl free and divergence free parts from the original field.

3 Curl free control

We consider

$$\min_{\rho, \mathbf{w}} \frac{1}{2} \int_0^T \int_{\Omega} (\rho - \widehat{\rho})^2 d\mathbf{x}dt + \frac{\beta}{2} \int_0^T \int_{\Omega} \mathbf{w}^2 d\mathbf{x}dt + \frac{\eta}{2} \int_0^T \int_{\Omega} (\nabla \times \mathbf{w})^2 d\mathbf{x}dt$$

subject to:

$$\frac{\partial \rho}{\partial t} = \nabla^2 \rho - \nabla \cdot (\rho \mathbf{w})$$

$$\frac{\partial \rho}{\partial n} - \rho \mathbf{w} \cdot \mathbf{n} = 0$$

We know that in two dimensions

$$\nabla \times \mathbf{w} = \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2},$$

and so the Lagrangian is

$$\begin{aligned} \mathcal{L}(\rho, \mathbf{w}, q_1, q_2) = & \frac{1}{2} \int_0^T \int_{\Omega} (\rho - \widehat{\rho})^2 d\mathbf{x}dt + \frac{\beta}{2} \int_0^T \int_{\Omega} \mathbf{w}^2 d\mathbf{x}dt + \frac{\eta}{2} \int_0^T \int_{\Omega} \left(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right)^2 d\mathbf{x}dt \\ & - \int_0^T \int_{\Omega} q_1 \left(\frac{\partial \rho}{\partial t} - \nabla^2 \rho + \nabla \cdot (\rho \mathbf{w}) \right) d\mathbf{x}dt \\ & - \int_0^T \int_{\partial \Omega} q_2 \left(\frac{\partial \rho}{\partial n} - \rho \mathbf{w} \cdot \mathbf{n} \right) d\mathbf{x}dt. \end{aligned}$$

Then, since we know that $q_1 = q_2$, we get

$$\begin{aligned} \mathcal{L}(\rho, \mathbf{w}, q) = & \frac{1}{2} \int_0^T \int_{\Omega} (\rho - \widehat{\rho})^2 d\mathbf{x}dt + \frac{\beta}{2} \int_0^T \int_{\Omega} \mathbf{w}^2 d\mathbf{x}dt + \frac{\eta}{2} \int_0^T \int_{\Omega} \left(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right)^2 d\mathbf{x}dt \\ & - \int_0^T \int_{\Omega} -\rho \frac{\partial q}{\partial t} - \rho \nabla^2 q - \nabla q \cdot (\rho \mathbf{w}) d\mathbf{x}dt - \int_{\Omega} q(T) \rho(T) - q(0) \rho(0) d\mathbf{x} \\ & - \int_0^T \int_{\partial \Omega} -\rho \nabla q \cdot \mathbf{n} d\mathbf{x}dt. \end{aligned}$$

For the adjoint equation, we find the usual results. We take the derivative with respect to \mathbf{w}

$$\mathcal{L}_{\mathbf{w}}(\rho, \mathbf{w}, q)h = \int_0^T \int_{\Omega} \beta \mathbf{w} \cdot \mathbf{h} + \rho \nabla q \cdot \mathbf{h} + \eta \left(\frac{\partial w_2}{\partial x_1} \frac{\partial h_2}{\partial x_1} - \frac{\partial w_2}{\partial x_1} \frac{\partial h_1}{\partial x_2} - \frac{\partial h_2}{\partial x_1} \frac{\partial w_1}{\partial x_2} + \frac{\partial w_1}{\partial x_2} \frac{\partial h_1}{\partial x_2} \right) d\mathbf{x}dt.$$

Collecting terms in h_1 and h_2 we get

$$\begin{aligned}
\mathcal{L}_{\mathbf{w}}(\rho, \mathbf{w}, q)h &= \int_0^T \int_{\Omega} h_1 \left(\beta w_1 + \rho \frac{\partial q}{\partial x_1} \right) + h_2 \left(\beta w_2 + \rho \frac{\partial q}{\partial x_2} \right) \\
&\quad + \frac{\partial h_1}{\partial x_2} \eta \left(-\frac{\partial w_2}{\partial x_1} + \frac{\partial w_1}{\partial x_2} \right) + \frac{\partial h_2}{\partial x_1} \eta \left(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right) d\mathbf{x}dt \\
&= \int_0^T \int_{\Omega} h_1 \left(\beta w_1 + \rho \frac{\partial q}{\partial x_1} \right) + h_2 \left(\beta w_2 + \rho \frac{\partial q}{\partial x_2} \right) \\
&\quad - h_1 \eta \left(-\frac{\partial^2 w_2}{\partial x_1 x_2} + \frac{\partial^2 w_1}{\partial x_2^2} \right) - h_2 \eta \left(\frac{\partial^2 w_2}{\partial x_1^2} - \frac{\partial^2 w_1}{\partial x_1 x_2} \right) d\mathbf{x}dt \\
&\quad + \int_0^T \int_{\partial\Omega} h_1 \eta \left(-\frac{\partial w_2}{\partial n_1} + \frac{\partial w_1}{\partial n_2} \right) + h_2 \eta \left(\frac{\partial w_2}{\partial n_1} - \frac{\partial w_1}{\partial n_2} \right) d\mathbf{x}dt.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{L}_{\mathbf{w}}(\rho, \mathbf{w}, q)h &= \int_0^T \int_{\Omega} \mathbf{h}(\beta \mathbf{w} + \rho \nabla q) + \eta \mathbf{h} \left(\frac{\partial^2 w_2}{\partial x_1 x_2}, \frac{\partial^2 w_1}{\partial x_1 x_2} \right) - \eta \mathbf{h} \left(\frac{\partial^2 w_1}{\partial x_2^2}, \frac{\partial^2 w_2}{\partial x_1^2} \right) d\mathbf{x}dt \\
&\quad + \int_0^T \int_{\partial\Omega} \eta \mathbf{h} \left(-\frac{\partial w_2}{\partial n_1} + \frac{\partial w_1}{\partial n_2}, \frac{\partial w_2}{\partial n_1} - \frac{\partial w_1}{\partial n_2} \right) d\mathbf{x}dt
\end{aligned}$$

So we can extract

$$\begin{aligned}
\beta \mathbf{w} + \rho \nabla q + \eta \left(\frac{\partial^2 w_2}{\partial x_1 x_2} - \frac{\partial^2 w_1}{\partial x_2^2}, \frac{\partial^2 w_1}{\partial x_1 x_2} - \frac{\partial^2 w_2}{\partial x_1^2} \right) &= 0 \quad \text{in } \Omega \\
\eta \left(-\frac{\partial w_2}{\partial n_1} + \frac{\partial w_1}{\partial n_2}, \frac{\partial w_2}{\partial n_1} - \frac{\partial w_1}{\partial n_2} \right) &= 0 \quad \text{on } \partial\Omega
\end{aligned}$$

The first components give

$$\begin{aligned}
\beta w_1 - \eta \frac{\partial^2 w_1}{\partial x_2^2} &= \rho \frac{\partial q}{\partial x_1} - \eta \frac{\partial^2 w_2}{\partial x_1 x_2} \quad \text{in } \Omega \\
\eta \left(-\frac{\partial w_2}{\partial n_1} + \frac{\partial w_1}{\partial n_2} \right) &= 0 \quad \text{on } \partial\Omega
\end{aligned}$$

and the second are

$$\begin{aligned}
\beta w_2 - \eta \frac{\partial^2 w_2}{\partial x_1^2} &= \rho \frac{\partial q}{\partial x_2} - \eta \frac{\partial^2 w_1}{\partial x_1 x_2} \quad \text{in } \Omega \\
\eta \left(\frac{\partial w_2}{\partial n_1} - \frac{\partial w_1}{\partial n_2} \right) &= 0 \quad \text{on } \partial\Omega.
\end{aligned}$$