Exact Solutions for the Full Problem
with Force Control and with Flow Control

The Equations: Force Control

Forward Problem

$$\partial_t \rho = \nabla^2 \rho - \nabla \cdot (\rho \mathbf{w}_{Flow}) + \nabla \cdot (\rho \nabla V_{ext}) + w_{Force} + f$$
$$\rho = \rho_0 \quad \text{at} \quad t = 0.$$

No-Flux
$$\frac{\partial \rho}{\partial n} - \rho \mathbf{w}_{Flow} \cdot \mathbf{n} + \rho \frac{\partial V_{ext}}{\partial n} = 0$$
Dirichlet
$$\rho = 0$$

Adjoint Equation

$$\partial_t p = -\rho + \hat{\rho} - \nabla p \cdot \mathbf{w}_{Flow} - \nabla^2 p + \nabla p \cdot \nabla V_{ext}$$
$$p(r, T) = 0$$

No-Flux
$$\frac{\partial p}{\partial n} = 0,$$
Dirichlet
$$p = 0.$$

Gradient Equation

$$w_{Force} = -\frac{1}{\beta} p$$

Exact Solutions: Force Control

(+++ Just a very small adaptation from John's notes!+++)

Dirichlet BCs

We choose:

$$\begin{aligned}\rho &= 2e^t\cos(\frac{\pi x}{2})\\ p &= (e^T - e^t)\cos(\frac{\pi x}{2})\\ V_{ext} &= \frac{1}{2}\cos(\frac{\pi x}{2})\end{aligned}$$

The expression for V_{ext} is chosen such that $V_{ext} = \frac{1}{2}V_{extOld}$, where $V_{extOld} = \cos(\frac{\pi x}{2})$ is taken from John's notes. Then a valid choice of $\mathbf{w}_{Flow}$ is $\frac{1}{2}\nabla V_{extOld}$:

$$\mathbf{w}_{Flow} = -\frac{1}{2}\nabla\cos(\frac{\pi x}{2}) = \frac{\pi}{4}\sin(\frac{\pi x}{2}).$$

Note that it can be verified that $\mathbf{w}_{Flow} - \nabla V_{ext} = -\nabla V_{extOld}$. Therefore, the expressions for w_{Force}, f and $\hat{\rho}$ are the same as in John's notes:

$$w_{Force} = -\frac{1}{\beta}(e^T - e^t)\cos(\frac{\pi x}{2}),$$

which follows from the gradient equation and the choice of p. Also, from computations and choice of ρ,p and V_{ext}, we get:

$$\begin{aligned}f &= \left((2+\frac{\pi^2}{2}-\frac{1}{\beta})e^t + \frac{1}{\beta}e^T\right)\cos(\frac{\pi x}{2}) - \frac{\pi^2}{2}e^t\left(\sin^2(\frac{\pi x}{2}) - \cos^2(\frac{\pi x}{2})\right)\\ \hat{\rho} &= e^t\cos(\frac{\pi x}{2}) - \frac{\pi^2}{4}(e^T - e^t)\cos(\frac{\pi x}{2}) - \frac{\pi^2}{4}(e^T - e^t)\sin^2(\frac{\pi x}{2})\end{aligned}$$

Neumann BCs

We choose:

$$\begin{aligned}\rho &= 2e^t\cos(\pi x)\\ p &= (e^T - e^t)\cos(\pi x)\\ V_{ext} &= \frac{1}{2}\cos(\pi x)\end{aligned}$$

The expression for V_{ext} is again chosen such that $V_{ext} = \frac{1}{2}V_{extOld}$, where $V_{extOld} = \cos(\pi x)$ is taken from John's notes. Then a valid choice of $\mathbf{w}_{Flow}$ is $\frac{1}{2}\nabla V_{extOld}$:

$$\mathbf{w}_{Flow} = -\frac{1}{2}\nabla\cos(\pi x) = \frac{\pi}{2}\sin(\pi x).$$

Note that it can be verified that $\mathbf{w}_{Flow} - \nabla V_{ext} = -\nabla V_{extOld}$. Therefore, the expressions for w_{Force}, f and $\hat{\rho}$ are again the same as in John's notes:

$$w_{Force} = -\frac{1}{\beta}(e^T - e^t)\cos(\pi x),$$

which follows from the gradient equation and the choice of p. Also, from computations and choice of ρ,p and V_{ext}, we get:

$$f = \left((2 + 2\pi^2 - \frac{1}{\beta})e^t + \frac{1}{\beta}e^T\right)\cos(\pi x) - 2\pi^2 e^t\left(\sin^2(\pi x) - \cos^2(\pi x)\right)$$
$$\hat{\rho} = e^t\cos(\pi x) - \pi^2(e^T - e^t)\cos(\pi x) - \pi^2(e^T - e^t)\sin^2(\pi x)$$

The Equations: Flow Control

Forward Problem

$$\partial_t \rho = \nabla^2\rho - \nabla\cdot(\rho \mathbf{w}_{Flow}) + \nabla\cdot(\rho\nabla V_{ext}) + w_{Force} + f$$
$$\rho = \rho_0 \quad \text{at} \quad t = 0.$$

No-Flux

$$\frac{\partial\rho}{\partial n} - \rho\mathbf{w}_{Flow}\cdot\mathbf{n} + \rho\frac{\partial V_{ext}}{\partial n} = 0$$

Dirichlet

$$\rho = 0$$

Adjoint Equation

$$\partial_t p = -\rho + \hat{\rho} - \nabla p\cdot\mathbf{w}_{Flow} - \nabla^2 p + \nabla p\cdot\nabla V_{ext}$$
$$p(r, T) = 0$$

No-Flux

$$\frac{\partial p}{\partial n} = 0,$$

Dirichlet

$$p = 0.$$

Gradient Equation

$$\mathbf{w}_{Flow} = -\frac{1}{\beta}\rho\nabla p$$

Exact Solutions: Flow Control

(+++ Just a very small adaptation from John's notes!+++)

Dirichlet BCs

We choose:

$$\rho = 2e^t \cos(\frac{\pi x}{2})$$
$$p = (e^T - e^t)\cos(\frac{\pi x}{2})$$

Then, from the gradient equation we find that:

$$\mathbf{w}_{Flow} = \frac{\pi}{\beta} e^t (e^T - e^t) \cos(\frac{\pi x}{2}) \sin(\frac{\pi x}{2}).$$

Then we need to solve the forward equation using these quantities. That is:

$$\partial_t \rho - \nabla^2 \rho + \nabla \cdot (\rho \mathbf{w}_{Flow}) = \nabla \cdot (\rho \nabla V_{ext}) + w_{Force} + f.$$

One choice is to set:

$$\nabla \cdot (\rho \mathbf{w}_{Flow}) = \nabla \cdot (\rho \nabla V_{ext}),$$

which implies that

$$\begin{aligned} \mathbf{w}_{Flow} &= \nabla V_{ext} \\ &= \frac{\pi}{\beta} e^t (e^T - e^t) \cos(\frac{\pi x}{2}) \sin(\frac{\pi x}{2}). \end{aligned}$$

Then, integrating this, we find V_{ext}:

$$\begin{aligned} V_{ext} &= \int_\Omega \frac{\pi}{\beta} e^t (e^T - e^t) \cos(\frac{\pi x}{2}) \sin(\frac{\pi x}{2}) dx \\ &= \frac{1}{\beta} e^t (e^T - e^t) \sin^2(\frac{\pi x}{2}). \end{aligned}$$

Since we know that $\nabla \cdot (\rho \mathbf{w}_{Flow}) = \nabla \cdot (\rho \nabla V_{ext})$, the remaining terms to balance are:

$$\partial_t \rho - \nabla^2 \rho = w_{Force} + f.$$

Then

$$w_{Force} + f = (2 + \frac{\pi^2}{2}) e^t \cos(\frac{\pi x}{2}),$$

where $w_{Force} + f$ is one force term. Now $\hat{\rho}$ can be found using the adjoint equation:

$$\partial_t p = -\rho + \hat{\rho} - \nabla p \cdot \mathbf{w}_{Flow} - \nabla^2 p + \nabla p \cdot \nabla V_{ext},$$

which reduces to:

$$\partial_t p = \rho - \hat{\rho} - \nabla^2 p,$$

due to the choices of $\mathbf{w}_{Flow}$ and V_{ext}, which gives:

$$\begin{aligned} \hat{\rho} &= \partial_t p + \rho + \nabla^2 p \\ &= (1 - \frac{\pi^2}{4}) e^t \cos(\frac{\pi x}{2}) + \frac{\pi^2}{4} e^t \cos(\frac{\pi x}{2}). \end{aligned}$$

Neumann BCs

Choose scaling factors $c\beta$ in $\mathbf{w}_{Flow}$ to damp the advection dominance. We choose:

$$\begin{aligned}\rho &= c^{1/2}\beta^{1/2}2e^t\cos(\pi x)\\ p &= c^{1/2}\beta^{1/2}(e^T - e^t)\cos(\pi x).\end{aligned}$$

Then, from the gradient equation, we get:

$$\begin{aligned}\mathbf{w}_{Flow} &= c\beta\frac{2\pi}{\beta}e^t(e^T - e^t)\cos(\pi x)\sin(\pi x)\\ &= c2\pi e^t(e^T - e^t)\cos(\pi x)\sin(\pi x),\end{aligned}$$

where it can be noted that β is cancelled out. Again the choice is to set:

$$\nabla\cdot(\rho\mathbf{w}_{Flow}) = \nabla\cdot(\rho\nabla V_{ext}),$$

which implies that

$$\begin{aligned}\mathbf{w}_{Flow} &= \nabla V_{ext}\\ &= c2\pi e^t(e^T - e^t)\cos(\pi x)\sin(\pi x).\end{aligned}$$

Then, integrating this, we find V_{ext}:

$$\begin{aligned}V_{ext} &= \int_\Omega c2\pi e^t(e^T - e^t)\cos(\pi x)\sin(\pi x)dx\\ &= ce^t(e^T - e^t)\sin^2(\pi x).\end{aligned}$$

The remaining terms to balance in the forward equation are again:

$$\partial_t\rho - \nabla^2\rho = w_{Force} + f.$$

This is:

$$w_{Force} + f = c^{1/2}\beta^{1/2}2e^t\cos(\pi x) + c^{1/2}\beta^{1/2}2\pi^2\cos(\pi x),$$

where $w_{Force} + f$ is again one force term. Then, from the adjoint equation, as before, we can derive $\hat{\rho}$:

$$\begin{aligned}\hat{\rho} &= \partial_t p + \rho + \nabla^2 p\\ &= c^{1/2}\beta^{1/2}(1-\pi^2)e^t\cos(\pi x) + c^{1/2}\beta^{1/2}\pi^2 e^T\cos(\pi x).\end{aligned}$$