1 Averaging the advection-diffusion equation

We consider the standard advection-diffusion optimality system, with flow control and Neumann boundary conditions:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \nabla^2 \rho - \nabla \cdot (\rho \mathbf{w}) + f \\
0 &= (-\nabla \rho + \rho \mathbf{w}) \cdot \mathbf{n}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial q}{\partial t} &= -\nabla^2 q - \mathbf{w} \cdot \nabla q - \rho + \widehat{\rho} \\
0 &= \nabla q \cdot \mathbf{n}
\end{aligned}$$

$$\mathbf{w} = -\frac{1}{\beta} \rho \nabla q$$

We now have the following operators in polar coordinates acting on a function f and a vector field $\mathbf{w}$:

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial f}{\partial z^2} \\
\nabla f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r} \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right) \\
\nabla \cdot \mathbf{w} &= \frac{1}{r} \frac{\partial}{\partial r} (r \mathbf{w}_r) - \frac{1}{r} \frac{\partial \mathbf{w}_\theta}{\partial \theta} + \frac{\partial \mathbf{w}_z}{\partial z}
\end{aligned}$$

Implementing these, we get:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \rho}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial \rho}{\partial z^2} - \frac{1}{r} \frac{\partial}{\partial r} (r \rho \mathbf{w}_r) + \frac{1}{r} \frac{\partial \rho \mathbf{w}_\theta}{\partial \theta} - \frac{\partial \rho \mathbf{w}_z}{\partial z} + f \\
0 &= \left(- \left(\frac{\partial \rho}{\partial r}, \frac{1}{r} \frac{\partial \rho}{\partial \theta}, \frac{\partial \rho}{\partial z} \right) + \rho \mathbf{w} \right) \cdot \mathbf{n}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial q}{\partial t} &= -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial q}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 f q}{\partial \theta^2} - \frac{\partial q}{\partial z^2} - \mathbf{w} \cdot \left(\frac{\partial q}{\partial r}, \frac{1}{r} \frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial z} \right) - \rho + \widehat{\rho} \\
0 &= \left(\frac{\partial q}{\partial r}, \frac{1}{r} \frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial z} \right) \cdot \mathbf{n}
\end{aligned}$$

$$\mathbf{w} = -\frac{1}{\beta} \rho \left(\frac{\partial q}{\partial r}, \frac{1}{r} \frac{\partial q}{\partial \theta}, \frac{\partial q}{\partial z} \right)$$

Then we set θ to be constant, so that all derivatives in θ are zero and we multiply out some derivatives, to get:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{1}{r}\frac{\partial \rho}{\partial r} + \frac{\partial^2 \rho}{\partial r^2} + \frac{\partial \rho}{\partial z^2} - \frac{\rho \mathbf{w}_r}{r} - \rho\frac{\partial \mathbf{w}_r}{\partial r} - \mathbf{w}_r\frac{\partial \rho}{\partial r} - \rho\frac{\partial \mathbf{w}_z}{\partial z} - \mathbf{w}_z\frac{\partial \rho}{\partial z} + f \\
0 &= \left(-\left(\frac{\partial \rho}{\partial r}, \frac{\partial \rho}{\partial z}\right) + \rho\mathbf{w}\right)\cdot \mathbf{n}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= -\frac{1}{r}\frac{\partial q}{\partial r} - \frac{\partial^2 q}{\partial r^2} - \frac{\partial q}{\partial z^2} - \mathbf{w}\cdot\left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z}\right) - \rho + \widehat{\rho} \\
0 &= \left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z}\right)\cdot \mathbf{n}
\end{aligned}
$$

$$
\mathbf{w} = -\frac{1}{\beta}\rho\left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z}\right)
$$

We can rewrite some of the terms in the forward equation:

$$
\begin{aligned}
-\mathbf{w}_r\frac{\partial \rho}{\partial r} - \mathbf{w}_z\frac{\partial \rho}{\partial z} &= -\left(\mathbf{w}\cdot\nabla\right)\rho \\
-\rho\frac{\partial \mathbf{w}_r}{\partial r} - \rho\frac{\partial \mathbf{w}_z}{\partial z} &= -\left(\nabla\cdot\mathbf{w}\right)\rho
\end{aligned}
$$

Using the vector identity $\mathbf{a}(\nabla\cdot\mathbf{b}) + (\mathbf{b}\cdot\nabla)\mathbf{a} = \nabla\cdot(\mathbf{b}\mathbf{a}^{\mathbf{T}})$, we get that the above two terms become:

$$
-\left(\mathbf{w}\cdot\nabla\right)\rho - \left(\nabla\cdot\mathbf{w}\right)\rho = -\nabla\cdot(\rho\mathbf{w}).
$$

Then the optimality system is:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{1}{r}\frac{\partial \rho}{\partial r} + \frac{\partial^2 \rho}{\partial r^2} + \frac{\partial \rho}{\partial z^2} - \frac{\rho \mathbf{w}_r}{r} - \nabla\cdot(\rho\mathbf{w}) + f \\
0 &= \left(-\left(\frac{\partial \rho}{\partial r}, \frac{\partial \rho}{\partial z}\right) + \rho\mathbf{w}\right)\cdot \mathbf{n}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial q}{\partial t} &= -\frac{1}{r}\frac{\partial q}{\partial r} - \frac{\partial^2 q}{\partial r^2} - \frac{\partial q}{\partial z^2} - \mathbf{w}\cdot\left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z}\right) - \rho + \widehat{\rho} \\
0 &= \left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z}\right)\cdot \mathbf{n}
\end{aligned}
$$

$$
\mathbf{w} = -\frac{1}{\beta}\rho\left(\frac{\partial q}{\partial r}, \frac{\partial q}{\partial z}\right)
$$

1.1 Exact Solution

We are choosing an exact solution which satisfies the boundary conditions, matches the final time condition for q and is invariant in θ. We choose:

$$\begin{aligned} \rho &= \beta^{1/2} e^{t} \cos(\pi r) \cos(\pi z) \\ q &= \beta^{1/2} (e^{T} - e^{t}) \cos(\pi r) \cos(\pi z), \end{aligned}$$

and use these to determine the values of $\mathbf{w}$, f and $\widehat{\rho}$.

There must be a mistake somewhere because the exact solution is still not exact. I am not sure what this mistake is yet.