Exact Solution to the Flow Control Problem with Mixed Boundary Conditions - 1D

Note: 2D should just be $(x^2-1)^2(y^2-1)^2$.
PDECO Problem:

$$J = \frac{1}{2}||\rho - \hat{\rho}||_{L_2}^2 + \frac{\beta}{2}||w||_{L_2}^2$$

$$\begin{aligned}
&\partial_t \rho = \Delta \rho - \nabla \cdot (w\rho) + f \\
&\alpha(\frac{\partial \rho}{\partial n} - \rho w \cdot n) + \gamma \rho = 0,
\end{aligned}$$

where α, γ are constant. Note that the below holds for any choice of α and γ.

Optimality System (by linearity and previous results):

$$\begin{aligned}
&\partial_t \rho = \Delta \rho - \nabla \cdot (w\rho) + f \\
&\partial_t p = -\Delta p - \rho + \hat{\rho} - w \cdot \nabla p \\
&w = -\frac{1}{\beta}\rho \nabla p \\
&\text{BCs:} \\
&\alpha(\frac{\partial \rho}{\partial n} - \rho w \cdot n) + \gamma \rho = 0 \\
&\alpha \frac{\partial p}{\partial n} + \gamma p = 0
\end{aligned}$$

An exact solution

$$\begin{aligned}
&p = \beta(e^T - e^t)(x^2-1)^2 \\
&\rho = e^t(x^2-1)^2 \\
&w = -4e^t(e^T - e^t)x(x^2-1)^3
\end{aligned}$$

These are chosen such that p, ρ, $\frac{\partial p}{\partial x}$ and $\frac{\partial \rho}{\partial x}$ are zero at $x = -1$ and $x = 1$. Furthermore, $p(T) = 0$ is satisfied.
Then f and $\hat{\rho}$ are:

$$\begin{aligned}
&f = e^t\Big((x^2-1)^2 - 12x^2 + 4\Big) - e^{2t}(e^T - e^t)\Big(4(x^2-1)^4(11x^2-1)\Big) \\
&\hat{\rho} = (1-\beta)e^t(x^2-1)^2 + \beta(e^T - e^t)(12x^2-4) - 16\beta e^t(e^T - e^t)^2 x^2 (x^2-1)^4
\end{aligned}$$

A different exact solution

Here, β is split between ρ and p because it seemed above for the other mixed solution that the Kalise algorithm changed the error in exact solution with β, but this may not matter so much. The term $(x^2 - 1)$ is replaced by $(1 - x^2)$ only to create a positive exact solution for w but I don't think it really matters.

$$\begin{aligned} p &= \beta^{1/2}(T-t)(1-x^2)^2 \\ \rho &= \beta^{1/2}e^t(1-x^2)^2 \\ w &= 4t(T-t)x(1-x^2)^3 \end{aligned}$$

These are chosen such that p, ρ, $\frac{\partial p}{\partial x}$ and $\frac{\partial \rho}{\partial x}$ are zero at $x = -1$ and $x = 1$. Furthermore, $p(T) = 0$ is satisfied.

Then f and $\hat{\rho}$ are:

$$\begin{aligned} f &= \beta^{1/2}(1-x^2)^2 + \beta^{1/2}t(4-12x^2) + 4\beta^{1/2}t^2(T-t)\Big((1-x^2)^4(1-11x^2)\Big) \\ \hat{\rho} &= -\beta^{1/2}(1-x^2)^2 + \beta^{1/2}t(1-x^2)^2 - \beta^{1/2}(T-t)(4-12x^2) - 16\beta^{1/2}t(T-t)^2x^2(1-x^2)^4 \end{aligned}$$