1 Perturbation Functions

We consider the following two perturbation functions, normalised to a maximum of 1. They both originate from the function:

$$f(t) = \frac{e^{-a/t}}{e^{-a/t} + e^{-a/(1-t)}}.$$

The first perturbation is in time only and is defined as:

$$\begin{aligned} g(t) &= \frac{1}{2} f(t - t_0, a) \times f(t - t_0, -a) \\ &= \frac{1}{2} \frac{e^{-a/(t-t_0)}}{e^{-a/(t-t_0)} + e^{-a/(1-t-t_0)}} \times \frac{e^{a/(t-t_0)}}{e^{a/(t-t_0)} + e^{a/(1-t-t_0)}}. \end{aligned}$$

The normalised version is then

$$\tilde{g}(t) = \frac{g(t)}{\max|g(t)|},$$

so that $\max \tilde{g}(t) = 1$. With $a = 0.7$ and $t_0 = -0.01$, this looks like Figure 1 (left).

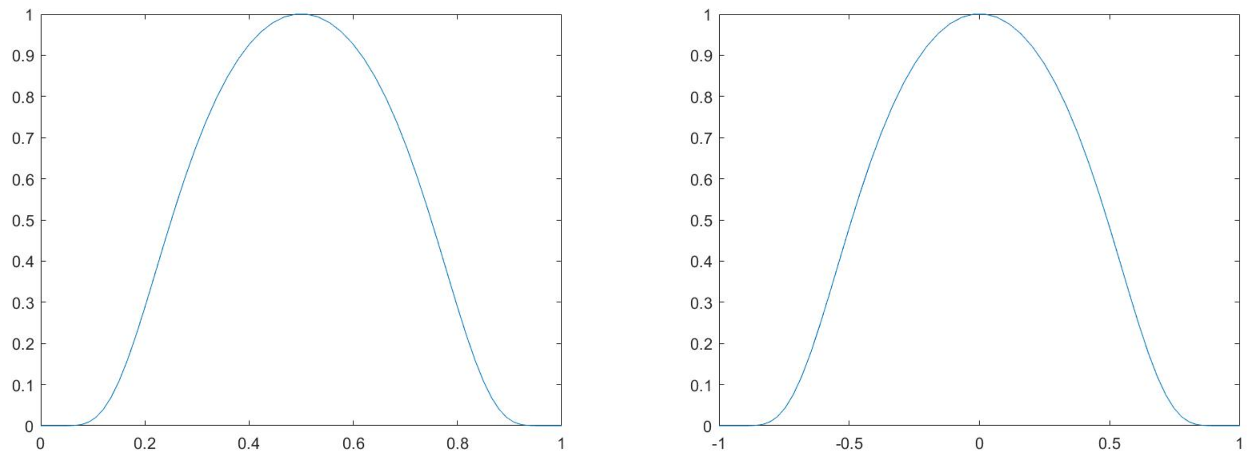

Figure 1: Perturbation $\tilde{g}(t)$ (left) and $\tilde{h}(x)$ (right) with $a = 0.7$ and $t_0 = -0.01$.

A similar perturbation can be done in space:

$$\begin{aligned} h(x) &= \frac{1}{2} f(x - x_0, 2a) \times f(x - x_0, -2a) \\ &= \frac{1}{2} \frac{e^{-2a/(x-x_0)}}{e^{-2a/(x-x_0)} + e^{-2a/(1-x-x_0)}} \times \frac{e^{2a/(x-x_0)}}{e^{2a/(x-x_0)} + e^{2a/(1-x-x_0)}}. \end{aligned}$$

Again, the normalised version is then

$$\tilde{h}(x) = \frac{h(x)}{\max|h(x)|}.$$

With $a = 0.7$ and $t_0 = -0.01$, this looks like Figure 1 (right).
The considered perturbations are applied as follows:

$$w_{pert} = w_{ex}(1 + \epsilon\tilde{g}(t))$$
$$w_{pert} = w_{ex}(1 + \epsilon\tilde{g}(t)\tilde{h}(x)).$$

The perturbations considered below are all with $\epsilon = 1$, however, when testing convergence, ϵ will be varied.
Applying the time perturbation to Neumann (plus2, e^t) Flow Control, the error in w is up to 25, see Figure 2. For the Dirichlet Case (e^t), the maximum error in w with the same perturbation is 3. When choosing the linear in time Neumann (plus 2) problem, the maximum error of

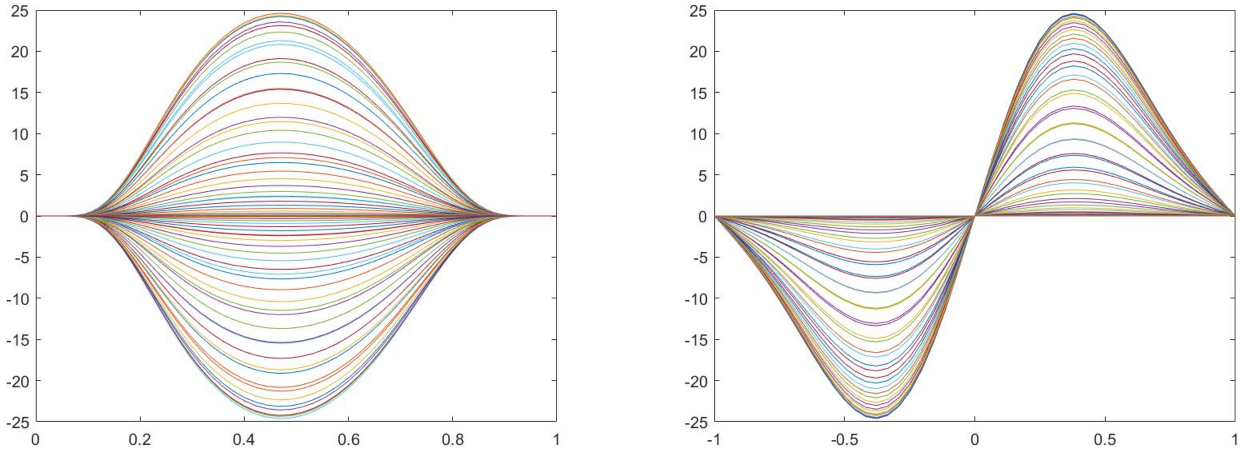

Figure 2: Error in w (Neumann plus2 e^t) due to $\tilde{g}(t)$, with $a = 0.7$ and $t_0 = -0.01$.

the same perturbation, $\tilde{g}(t)$, is 1.8, see Figure 4. For the linear in time Dirichlet problem, the maximum error is only 0.4 for the same perturbation, see Figure 5.
When perturbing in time and space with $\tilde{g}(t)\tilde{h}(x)$, Figure 6 shows that with Neumann (plus2) and e^t the maximum magnitude of the error is lowered from 25 to 20, and for linear t from 1.8 to 1.5. In Figure 7, similar observations can be made for the Dirichlet case. In the e^t case, the maximum error decreases from 3 to 2 and in the linear t case from 0.4 to 0.3.

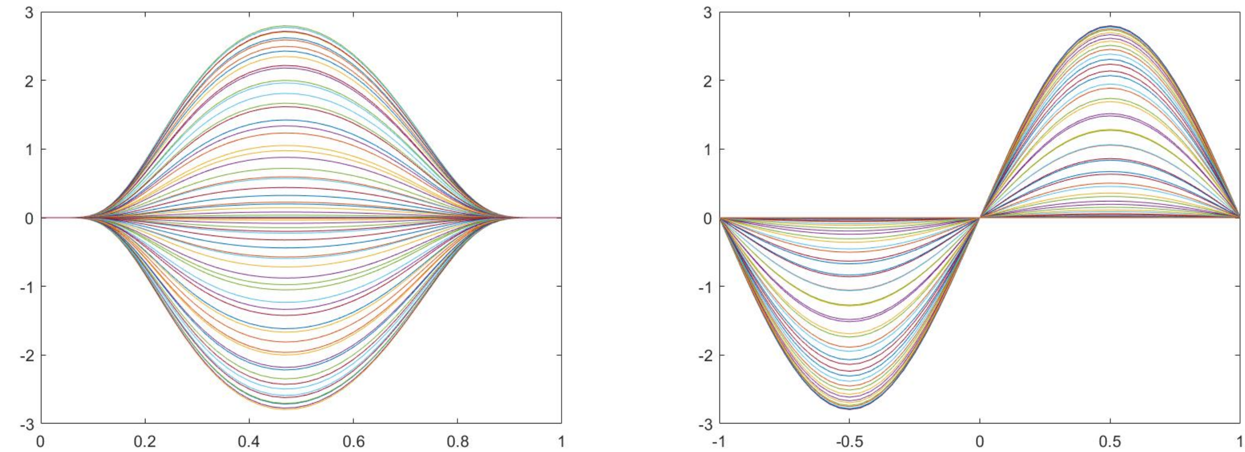

Figure 3: Error in w (Dirichlet e^t) due to $\tilde{g}(t)$, with $a = 0.7$ and $t_0 = -0.01$.

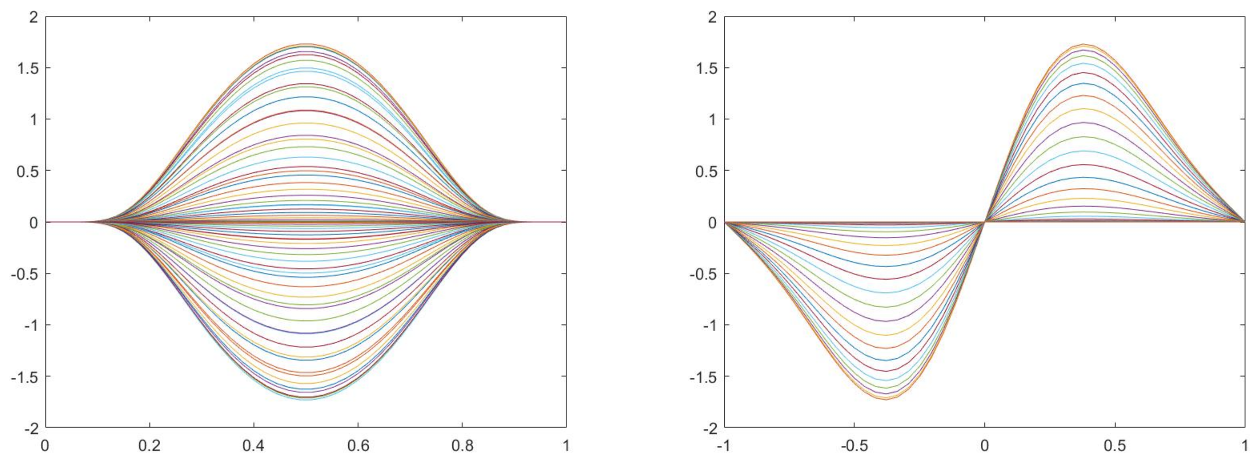

Figure 4: Error in w (Neumann plus2, linear t) due to $\tilde{g}(t)$, with $a = 0.7$ and $t_0 = -0.01$.

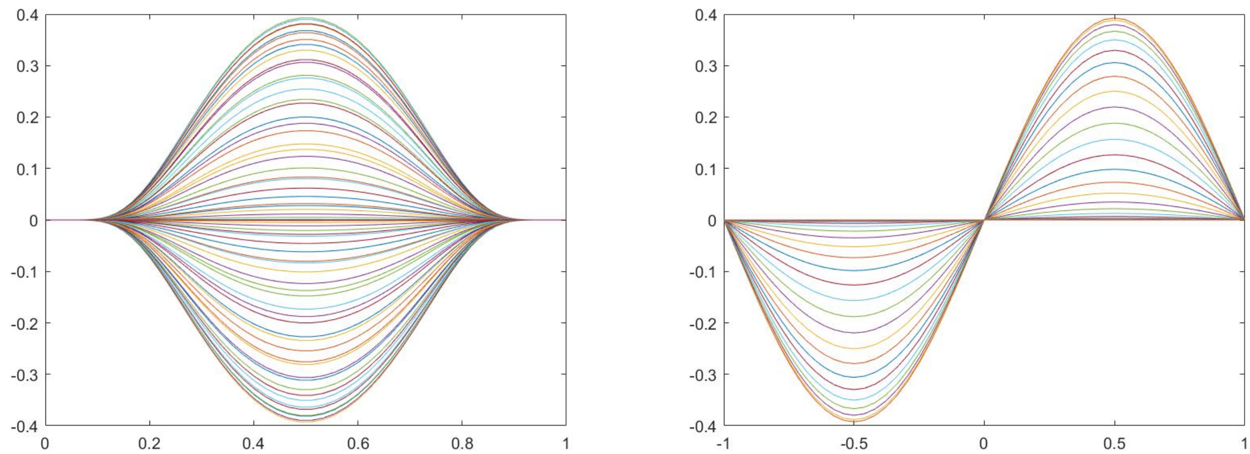

Figure 5: Error in w (Dirichlet, linear t) due to $\tilde{g}(t)$, with $a = 0.7$ and $t_0 = -0.01$.

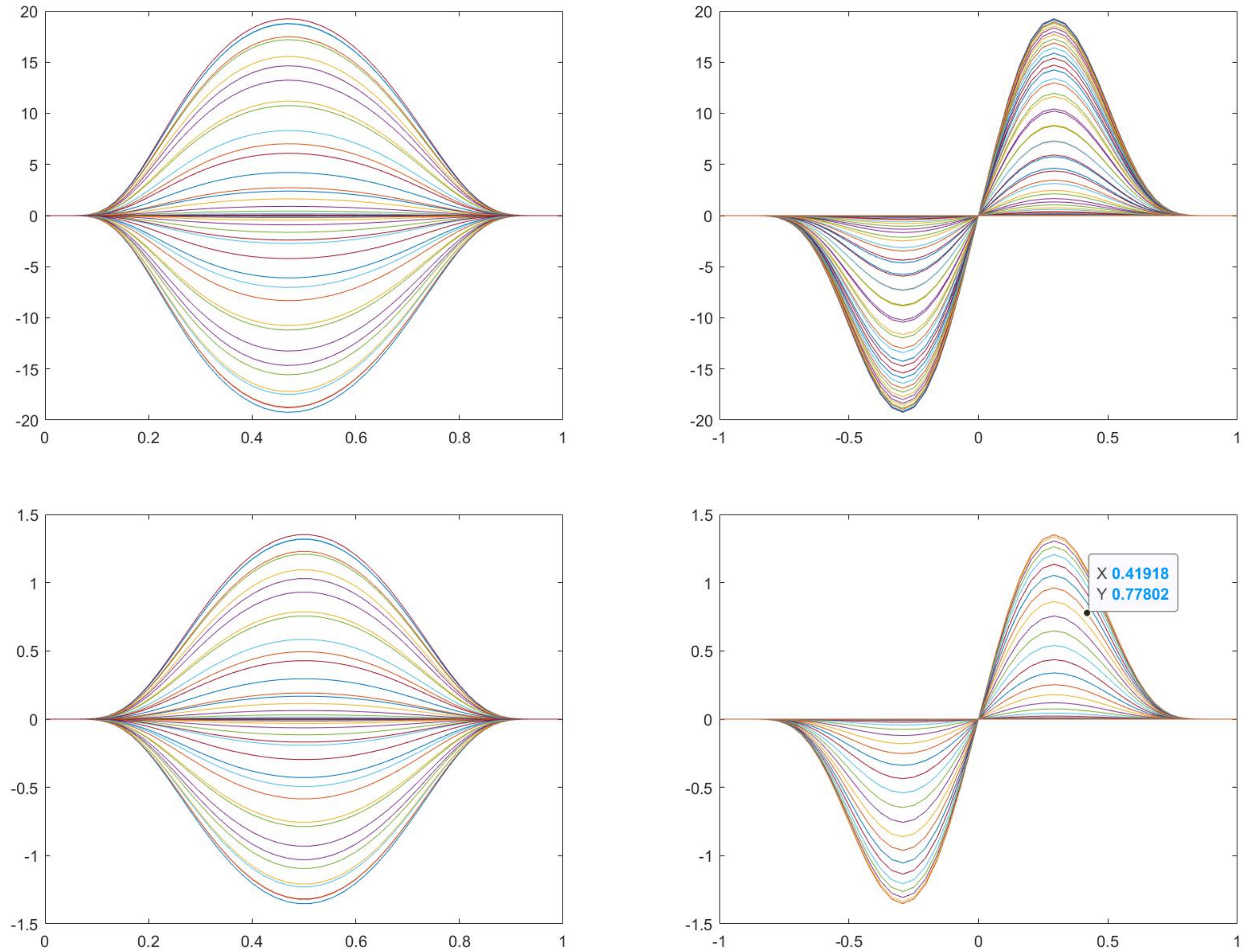

Figure 6: Error in w (Neumann plus2) due to $\tilde{g}(t)\tilde{h}(x)$, top e^t, bottom linear t, with $a = 0.7$ and $t_0 = -0.01$.

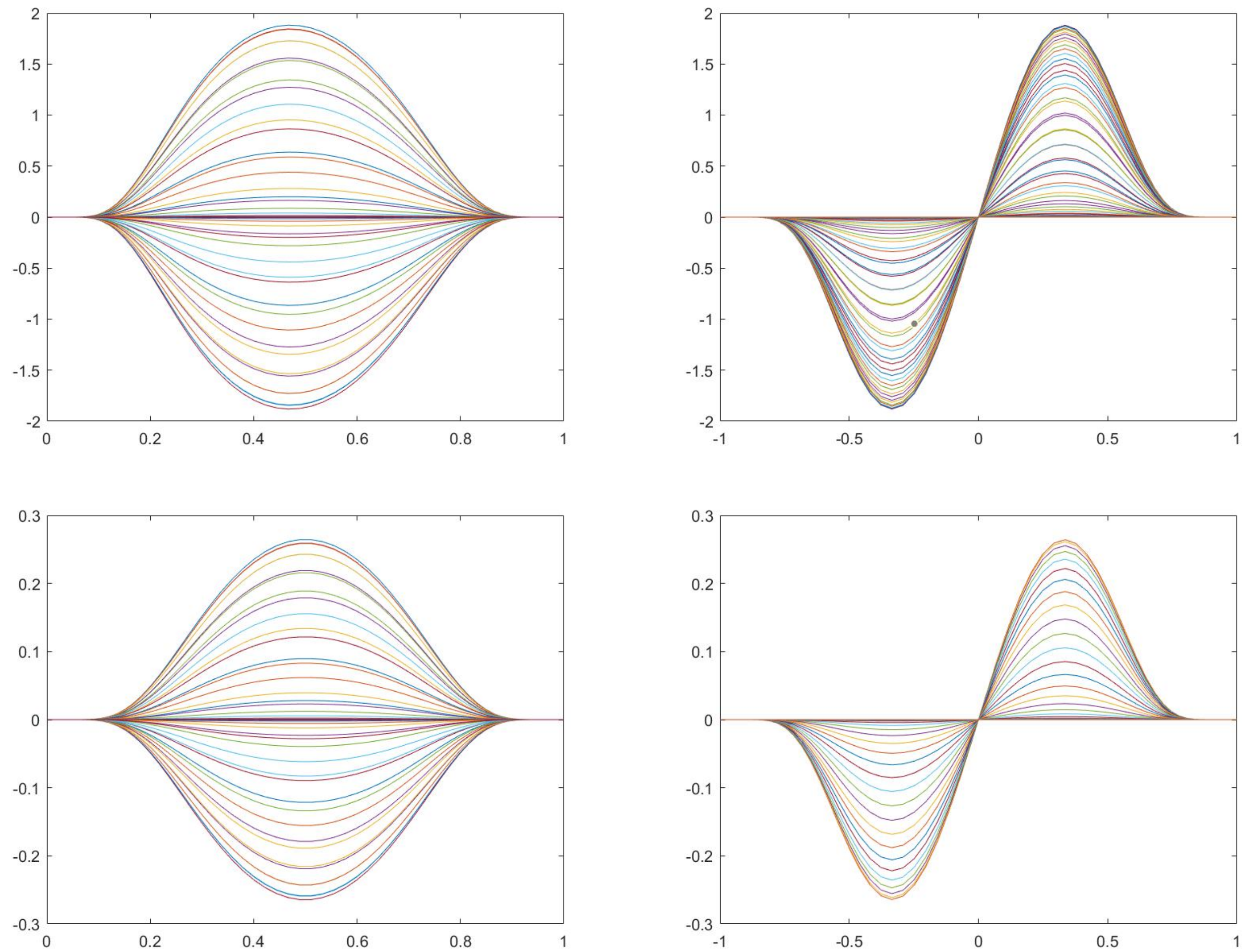

Figure 7: Error in w (Dirichlet) due to $\tilde{g}(t)\tilde{h}(x)$, top e^t, bottom linear t, with $a = 0.7$ and $t_0 = -0.01$.