ECH 267 Nonlinear Control Theory
Final Project Report DRAFT

Jonathan Dorsey: Department of Mechanical & Aerospace Engineering

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Github Repo Hosted at:

https://github.com/JonnyD1117/ECH-267-Adv.-Proc.-Control

Denavit-Hartenberg Parameters

The Denavit-Hartenberg (DH) parameters are an important tool in analyzing the geometry of any given robot as well as in the formulation of joint transformations which enable a concise and universal means of deriving important quanties for kinematic and dynamic analysis of the robot.

Purpose and Appliction of DH Parameters

The purpose of DH parameters is to standardize the description of geometry of robots into a universal parameterization such that arbitary construction of different robots can all be described concisely in a single and intuitive notation.

The DH Parameters are defined to be...

$$
\begin{aligned}
a_i &= \text{ the distance from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured along } \hat{X}_i \\
\alpha_i &= \text{ the angle from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured about } \hat{X}_i \\
d_i &= \text{ the distance from } \hat{X}_{i-1} \text{ to } \hat{X}_i \text{ measured along } \hat{Z}_i; \text{ and} \\
\theta_i &= \text{ the angle from } \hat{X}_{i-1} \text{ to } \hat{X}_i \text{ measured about } \hat{Z}_i
\end{aligned}
$$

Where α_i is the angle of twist between axis of actuation from one link to the next, a_i is the offset distance measured along the x axis required to locate the frame of the next link, d_i is the distance along the axis of actuation from which is required to locate the frame of the next linbk, and finally, θ_i is the angle about the axis of actuation required to locate the frame of the next link.

It is important to note that these parameters need not be constant and are allowed to be degrees of freedom of the system such that the robot can change its geometry. Often only one of these parameters will be variable while the other three remain constants. This occurs due to the constraints acting on each joint frame, of the robot.

Rules for Obtaining DH Parameterization

While DH Parameterization is by no means the only parameterization possible, its universal use and acceptance in the robotics community means that one is more likely to see it in the literature. However there are rules which this parameterization requires in order to be conistent and valid. These are...

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$)

2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the i th axis, assign the link-frame origin.

3. Assign the $\hat{Z}_i$ axis pointing along the i th joint axis.

4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.

5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.

6. Assign $\{0\}$ to match $\{1\}$ when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

PUMA 560 Robot DH Parameter

For the PUMA 560 robot being used in this project, we can show that the DH parameters are shown as follows. Note that the angles θ_i are not included as they are degree of freedom that we can control arbitarily. This is because the system contains only 3 revolute joints of actuation and not prismatic members.

α_{i-1}	A_{i-1}	D_i
0	0	0
-90	0	243.5
0	431.8	-93.4
90	-20.3	433.1
-90	0	0
90	0	0

These parameter values have previously been determined by B. Armstrong, by using the prescribed rules and the geometry of the PUMA 560 robot.

Homogeneous Transform

The homogeneous transformation is a construct in robotics which enables both the rototation and position of any vector within a given coordinate frame to be viewed with respect to another frame. This construction provides a compact matrix notation for changing coordinate frames of reference during computation.

Rotation Matrices

Before discussing more general transformation, it is important to understand how one can translate between coordinate frames. To accomplish this task we use rotation matrices.

As mentioned previously, a rotation matrix is a transformation which rotates a vector described in one coordinate sytem into components of another coordinate system. These are extremely useful for rigid body dynamics in general but they are required for the study of robotics as the number and orientation of joint reference frames becomes too cumbersum to derive geometrically for each frame of the system.

From the following definitions of rotations about the x, y, and z axis of 3D coordinate system, can chain together the rotations necessary for a system of coordinate frames such that we can describe any single frame from any other frame mrely by multiplying the seperate rotations together, as desired.

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General Homogeneous Transformation Definition

The general homogeneous transformation is equivalent to performing the following operation.

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

However, in order to create a general matrix transformation from this vector expression we can form the following matrix by the addition of the second equation such that $1 = 1$. While this is a trivial operation, this enables us to use the matrix as a general transform on position vector, given only the rotation matrix from one frame to another and the position vector from the origin of the given frame as viewed from the other.

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

This expression can be written as shown below, in a very abreviated and compact form, while clearly conveying the operation being done by this generation transformation.

$$P = {}^{A}_{B}T^{B}P$$

This form is known as the homogeneous transform.

DH Parameter Based Transformations

Since the DH Parameters provide a universal notation for describing the the position and orientation of a robot, it is natural to want to express the homogeneous transformation of each coordinate frame using this parameterization. In fact, we can chain a series of these transformation together, based on the DH table previously shown will will allow us to utilize the general nature of the homogeneous transformation and account for all of the twist, offsets, and linear and angular discplacements required to view a joint frame from another (usually the reference frame).

For a single transformation from a frame $\{i\}$ to frame $\{i-1\}$, we can use the homogeneous transformation for explicit rotations about a joint axis or specific translations along a joint axis as dictated by the DH parameters.The matrices R and D represent the homogeneous transformation for rotation and translations repsectively, with the subscript of each providing the axis upon which the operation should be performed.

$${}^{i-1}_{i}T = R_X\left(\alpha_{i-1}\right) D_X\left(a_{i-1}\right) R_Z\left(\theta_i\right) D_Z\left(d_i\right)$$

By using the universal, DH paramterization with homogeneous transformation we can now describe the relationships to or from any joint of the robot to any other joint of the robot. This is a powerful concept which facilitates the analysis of any robot whose joints are based on revolute or prismatic members.

DH Transforms for PUMA 560

By using DH transformations, we can extract the rotation matrices for each frame of the link with respect to $\{0\}$. We can use these rotation matrics later when deriving the kinematics of the robot. After computing these rotation matrices by applying the DH transformations, we get...

Frame 0-1 Transformation

This transform expresses vectors from the frame $\{1\}$ as components of the reference frame $\{0\}$.

$$
{}^{0}_{1}R = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Frame 0-2 Transformation

This transform expresses vectors from the frame $\{2\}$ as components of the reference frame $\{0\}$.

$$
{}^{0}_{2}R = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 \\ s_1c_2 & -s_1s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix}
$$

Frame 0-3 Transformation

This transform expresses vectors from the frame $\{3\}$ as components of the reference frame $\{0\}$.

$$
{}^{0}_{3}R = \begin{bmatrix} c_1c_{2+3} & -c_1s_{2+3} & -s_1 \\ s_1c_{2+3} & -s_1s_{2+3} & c_1 \\ -s_{2+3} & -c_{2+3} & 0 \end{bmatrix}
$$

These equations will be handy later when change of coordinates is required to compute velocities of each link.

PUMA 560 Dynamics

In order to derive the dynamics of the PUMA 560, this project will use the Euler-Lagrange Equation. This method is more compatible with the use of symbolic math engines like SymPy or Matlab's Symbolic toolbox. By using the Lagrange equations, we can compute the forward dynamics of the system by computing the kinetic and potential energy of the system, computing the Lagrangian, and performing a series of symbolic differentiations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Where the Lagrangian is defined as the difference between the kinetic energy T and the potential energy V of the system, and where Q_i is the generalized forces acting on the system.

$$L = T - V$$

By repeating this computation for each generalize coordinate q_i, we can derive the governing rigid body dynamics of the system.

Required Transformations and Parameters

Before diving into the analysis, it is important to collect all of the parameters and transformations required to construct the expressions which will be used to derive the dynamics.

DH Parameters

α_{i-1}	A_{i-1}	D_i
0	0	0
-90	0	243.5
0	431.8	-93.4
90	-20.3	433.1
-90	0	0
90	0	0

Link Masses

m_1	$-$
m_2	17.40
m_3	4.80
m_4	0.82
m_5	0.35
m_6	0.09

Link Moments of Inertia

I_{xx_1}	$-$
I_{yy_1}	$-$
I_{zz_1}	0.350
I_{xx_2}	0.130
I_{yy_2}	0.524
I_{zz_2}	0.539
I_{xx_3}	$66.0\mathrm{e}-3$
I_{yy_3}	$12.5\mathrm{e}-3$
I_{zz_3}	$86.0\mathrm{e}-3$
I_{xx_4}	$1.80\mathrm{e}-3$
I_{yy_4}	$1.80\mathrm{e}-3$
I_{zz_1}	$1.30\mathrm{e}-3$
I_{xx_8}	$300\mathrm{e}-6$
I_{yy_5}	$300\mathrm{e}-6$
I_{zz_5}	$400\mathrm{e}-6$
I_{xx_6}	$150\mathrm{e}-6$
I_{yy_6}	$150\mathrm{e}-6$
I_{zz_6}	$40\mathrm{e}-6$

Frame i to Frame i-CG Vectors

s_{x1}	$-$
s_{y1}	$-$
s_{z1}	$-$
s_{x2}	68
s_{y2}	6
s_{z2}	-16
s_{x3}	0
s_{y3}	-70
s_{z3}	14
s_{x4}	0
s_{y4}	0
s_{z4}	-19
s_{x5}	0
s_{y5}	0
s_{z5}	0
s_{x6}	0
s_{y6}	0
s_{z6}	32

Computing Kinetic and Potential Energies

For the case of a chain of rigid bodies, we can compute the individual kinetic energy of each link to be ...

$$k_i = \frac{1}{2} m_i v_{C_i}^T v_{C_i} + \frac{1}{2}{}^i\omega_i^T C_i I_i^i \omega_i$$

From this expression we can see that the kinetic energy of each rigid body is a function of its linear and angular velocities. Furthermore, we can show that the total kinetic energy is merely the sum of the kinetic energy of each body as shown below.

$$k = \sum_{i=1}^{n} k_i$$

This provides a scalar expression for the total kinetic energy associated with the motion of the system of rigid bodies. Furthermore we can show that the potential energy of a rigid body is the work done by gravity on the body itself. In a similar fashion to the kinetic energy, we can show that the total potential energy is the sum of the individual potential energies of each rigid body in the system. This can be shown as ...

$$u_i = -m_i^0 g^{T0} P_{C_i} + u_{ref_i}$$

$$u = \sum_{i=1}^{n} u_i$$

It should be noted that the only nonconstant terms that we need to define are the height vectors P_i for the potential energy, the linear velocity of the center of mass V_{C_i}, and the angular velocities ω_i. All of these quantities need to be computed with respect to the reference frame $\{0\}$, before we can continue to deriving the equations of motion for the PUMA 560.

Defining Velocities in Couple Frames of Motion

The task of determining the linear and angular velocities of each link of the robot is a non-trival task. Since our robot is constructed as chain of connected rigid bodies, each capable of independent motion and actuation, we can see that positions and velocities (aka kinematics) of the robot are coupled. This means that kinematic analysis must be very cautiuous to define and reference dependent motion of the robot.

In order to rigorously define the motion of the robot, we can make use of DH parameters, homogeneous transforms to aid in garunteeing that the each component of the system is computed correct with respect to the reference frame. The reference frame of the system is $\{0\}$. The next step is compute the non-zero velocities the primary 3-links in the PUMA 560. The following general equations are useful for converting between dependent motion of couple coordinate frames.

$$^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R\dot{\theta}_{i+1}{}^{i+1}\hat{Z}_{i+1}$$

$$^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}$$

Both of these equations express how we can translate the velocity as seen in one frame into the another frame which is presummed to also be in motion. By applying these general expressions, with respect to the reference frame $\{0\}$, we can compute the individual velocities of each link in the system.

Coupled Linear Velocities

The linear velocities of the the system are arguably the most difficult terms in the kinetic energy to obtain since, in order to compute the velocity at the center of mass of each link, we must first compute the velocity of the coordinate system for each link. This effectively means that two seperate transformations must be computed for the velocity of the CG is determined.

Inorder to compute the velocity to the origin of the joint frame (w.r.t. $\{0\}$) we can use the following expression.

$$^{0}V_{i+1} = {}^{0}V_i + {}^{0}\omega_i \times {}^{0}P_{i+1}$$

Where ${}^{0}P_{i+1}$ is the position vector from the origin of the reference frame to the current of interest. This vector can itself be broken down further using frame transformations, such that...

$$^{0}P_{i+1} = \left({}^{0}_{i}R\right)\left({}^{i}P_{i+1}\right)$$

This transformation enables us to leverage the geometry of the robot to determine the vector ${}^{i}P_{i+1}$, which can then be transformed into the correct frame of reference.

Once the velocity of the joint frame is established, we can then compute the velocity of the center of mass of the link using

$$^{0}V_{CG_i} = {}^{0}V_i + {}^{0}\omega_i \times r_i$$

Where ${}^{0}V_i$ is the velocity (as seen from $\{0\}$) of the i-th link, and where ${}^{0}\omega_i$ is the angular velocity (as seen from $\{0\}$), and where the position vector r_i, is the vector from the origin of the i-th link frame to the center of gravity of the same link.

Coupled Angular Velocities

Unlike linear velocities, angular velocities can be simply described from different coordinate frames by using the following expression.

$$ {}^{0}\omega_{i} = {}^{0}\omega_{i-1} + \dot{q}_{i} \left({}^{0}_{i}R\right) \left({}^{i}\hat{Z}_{i}\right) $$

Where $\dot{q}_i$ is the magnitude of the angular velocity of the i-th joint, and where ${}^{i}\hat{Z}_{i}$ is the unit vector in the z direction of the i-th joint, since by DH convention, this is the only axis allowed to rotate. By using this expression, we can describe the angular velocity of the i-th joint, in such a way that we can correctly relate the angular velocities of i-th and (i-1) frames, since the motion between these two frames is coupled, and must be accounted for under transformations.

Computing Linear and Angular Velocities

Frame {0} Velocities

Since frame {0} is the reference frame of the robot, we know that both the linear and angular velocities must be zero, such that...

$$ {}^{0}V_{0} = \langle 0, 0, 0 \rangle $$

$$ {}^{0}\omega_{0} = \langle 0, 0, 0 \rangle $$

Frame {1} Velocities

We know from DH parameters and PUMA 560 diagram that the origin frame {1} is coincident to the origin of frame {0}; however, while the reference frame is stationary, frame {1} can rotate about the $\hat{Z}_1$ axis. By using the velocity transforms show above, we can show that...

$$ {}^{0}V_{1} = {}^{0}V_{0} + {}^{0}\omega_{0} \times {}^{0}P_{1} $$

Since both the ${}^{0}\omega_{0} = {}^{0}P_{1} = 0$, we know that ...

$$ {}^{0}V_{1} = 0 $$

$$ {}^{0}\omega_{1} = {}^{0}\omega_{0} + \dot{q}_{1} \left({}^{0}_{1}R\right) \left({}^{1}\hat{Z}_{1}\right) $$

By plugging in the appropriate rotation matrix we get...

$$ {}^{0}\omega_{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \dot{q}_{1} \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \left({}^{1}\hat{Z}_{1}\right) $$

Since $\hat{Z}_1$ is a unit vector in the Z_1 direction, we can reduce this to...

$$ {}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \dot{q}_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

Therefore ...

$$ {}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} $$

$$ {}^0V_1 = 0 $$

Frame $\{2\}$ Velocities

From the previous section, we now know that, ${}^0V_1 = 0$ and that ${}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$, by applying the same transforms for the $i = 2$ frame, we can show that...

$$ {}^0V_2 = {}^0V_1 + {}^0\omega_1 \times {}^0P_2 $$

$$ {}^0\omega_2 = {}^0\omega_1 + \dot{q}_2 \left({}^0_2R\right) \left({}^2\hat{Z}_2\right) $$

Where ...

$$ {}^0P_2 = {}^0_1R \left({}^1P_2\right) = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ .2435 \\ 0 \end{bmatrix} $$

Therefore ...

$$ {}^0P_2 = .2435 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} $$

Where ${}^1P_2 = \begin{bmatrix} 0 \\ .2435 \\ 0 \end{bmatrix}$ is the position vector from frame $\{1\}$ to frame $\{2\}$. The numerical value of this vector is given by the DH parameters.

$$ {}^0V_2 = {}^0V_1 + {}^0\omega_1 \times .2435 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} $$

Given that ${}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$

$$ {}^0V_2 = 0 + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \times .2435 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} $$

$$ {}^0V_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \times .2435 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} $$

By performing the Cross Product we will get the velocity.

$$ {}^0V_2 = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix} $$

We can now turn our attention to the angular velocity of the second frame.

$$ {}^0\omega_2 = {}^0\omega_1 + \dot{q}_2 \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 \\ s_1c_2 & -s_1s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix} \left({}^2\hat{Z}_2 \right) $$

Since ${}^2\hat{Z}_2$ is a unit vector we can simplify the expression to....

$$ {}^0\omega_2 = {}^0\omega_1 + \dot{q}_2 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} $$

We know from the previous section that ${}^0\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$

Therefore we can show that...

$$ {}^0\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} + \begin{bmatrix} -\dot{q}_2s_1 \\ \dot{q}_2c_1 \\ 0 \end{bmatrix} $$

$$ {}^0\omega_2 = \begin{bmatrix} -\dot{q}_2s_1 \\ \dot{q}_2c_1 \\ \dot{q}_1 \end{bmatrix} $$

$$ {}^0V_2 = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix} $$

Frame $\{3\}$ Velocities

$$ {}^0V_3 = {}^0V_2 + {}^0\omega_2 \times {}^0P_3 $$

$$ {}^0\omega_3 = {}^0\omega_2 + \dot{q}_3 \left({}^0_3R \right) \left({}^3\hat{Z}_3 \right) $$

Since we know from the last section that ${}^0V_2 = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix}$,

We know that

$$^0P_3 = {}^0_2R\left({}^2P_3\right)$$

$$^0P_3 = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 \\ s_1c_2 & -s_1s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix} {}^2P_3$$

Where ${}^2P_3 = \begin{bmatrix} .4318 \\ 0 \\ -.093 \end{bmatrix}$. The the values for this vector have been obtained by the DH parameters.

$$^0P_3 = \begin{bmatrix} c_1c_{2+3} & -c_1s_{2+3} & -s_1 \\ s_1c_{2+3} & -s_1s_{2+3} & c_1 \\ -s_{2+3} & -c_{2+3} & 0 \end{bmatrix} \begin{bmatrix} .4318 \\ 0 \\ -.093 \end{bmatrix}$$

Since we know that ${}^0\omega_2 = \begin{bmatrix} -\dot{q}_2s_1 \\ \dot{q}_2c_1 \\ \dot{q}_1 \end{bmatrix}$, and that ${}^0V_2 = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix}$

$$^0V_3 = {}^0V_2 + {}^0\omega_2 \times {}^0P_3$$

$$^0V_3 = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{q}_2s_1 \\ \dot{q}_2c_1 \\ \dot{q}_1 \end{bmatrix} \times {}^0P_3$$

$$^0V_3 = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{q}_2s_1 \\ \dot{q}_2c_1 \\ \dot{q}_1 \end{bmatrix} \times \begin{bmatrix} c_1c_{2+3} & -c_1s_{2+3} & -s_1 \\ s_1c_{2+3} & -s_1s_{2+3} & c_1 \\ -s_{2+3} & -c_{2+3} & 0 \end{bmatrix} \begin{bmatrix} .4318 \\ 0 \\ -.093 \end{bmatrix}$$

We know that ${}^0\omega_2 = \begin{bmatrix} -\dot{q}_2s_1 \\ \dot{q}_2c_1 \\ \dot{q}_1 \end{bmatrix}$.

$$^0\omega_3 = {}^0\omega_2 + \dot{q}_3\left({}^0_3R\right)\left({}^3\hat{Z}_3\right)$$

$$^0\omega_3 = {}^0\omega_2 + \dot{q}_3 \begin{bmatrix} c_1c_{2+3} & -c_1s_{2+3} & -s_1 \\ s_1c_{2+3} & -s_1s_{2+3} & c_1 \\ -s_{2+3} & -c_{2+3} & 0 \end{bmatrix} \left({}^3\hat{Z}_3\right)$$

Where ${}^3\hat{Z}_3$ is a unit vector in the Z direction for the third frame.

$$^0\omega_3 = {}^0\omega_2 + \dot{q}_3 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

$$
{}^0\omega_3 = \begin{bmatrix} -\dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} + \dot{q}_3 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}
$$

$$
{}^0\omega_3 = \begin{bmatrix} -(\dot{q}_2 + \dot{q}_3) s_1 \\ (\dot{q}_2 + \dot{q}_3) c_1 \\ \dot{q}_1 \end{bmatrix}
$$

Velocities at Link CGs

As with the derivation of the linear velocities of the frame origins, we need to use the following expression to properly determine the velocity of the link at its center of gravity (CG).

$$
{}^0V_{C_i} = {}^0V_i + {}^0\omega_i \times r_i
$$

Where r_i is the vector from the i-th frames origin to its CG. By applying this to every non-static link in the robot, we can compute the velocity of the CG as seen from frame $\{0\}$.

$$
{}^0V_0 = \langle 0, 0, 0 \rangle
$$

$$
{}^0\omega_0 = \langle 0, 0, 0 \rangle
$$

Frame $\{1\}$ CG Velocity

Using the link CG vectors for the Puma 560 parameters determined by B. Armstrong, we can see that the effective CG of the first link is either difficult to measure or insignificant to the point such that no vector from the origin of frame $\{1\}$ to the CG of frame $\{1\}$ is given. Therefore we can skip this term and assume that

$$
{}^0V_{C_1} = 0
$$

Frame $\{2\}$ CG Velocity

We know from the previous derivations of velocities that ...

$$
{}^0\omega_2 = \begin{bmatrix} -\dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} \qquad {}^0V_2 = .2435 \begin{bmatrix} -c_1 \dot{q}_1 \\ -s_1 \dot{q}_1 \\ 0 \end{bmatrix}
$$

By plugging these terms into the expression below

$$^0V_{C_2} = {^0V_2} + {^0\omega_2} \times r_2$$

Where $r_2 = \begin{bmatrix} 0.068 \\ 0.006 \\ -0.016 \end{bmatrix}$, is obtained from B. Armstrong.

$$^0V_{C_2} = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} \times \begin{bmatrix} 0.068 \\ 0.006 \\ -0.016 \end{bmatrix}$$

Frame $\{3\}$ CG Velocity

We know from the previous derivations of velocities that ...

$$^0V_3 = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} \times \begin{bmatrix} c_1 c_{2+3} & -c_1 s_{2+3} & -s_1 \\ s_1 c_{2+3} & -s_1 s_{2+3} & c_1 \\ -s_{2+3} & -c_{2+3} & 0 \end{bmatrix} \begin{bmatrix} .4318 \\ 0 \\ -.093 \end{bmatrix}$$

$$^0\omega_3 = \begin{bmatrix} -(\dot{q}_2 + \dot{q}_3)s_1 \\ (\dot{q}_2 + \dot{q}_3)c_1 \\ \dot{q}_1 \end{bmatrix}$$

By plugging these terms into the expression below

$$^0V_{C_3} = {^0V_3} + {^0\omega_3} \times r_3$$

Where $r_3 = \begin{bmatrix} 0 \\ -0.07 \\ 0.014 \end{bmatrix}$, is obtained from B. Armstrong.

$$^0V_{C_2} = .2435 \begin{bmatrix} -c_1\dot{q}_1 \\ -s_1\dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\dot{q}_2 s_1 \\ \dot{q}_2 c_1 \\ \dot{q}_1 \end{bmatrix} \times \begin{bmatrix} c_1 c_{2+3} & -c_1 s_{2+3} & -s_1 \\ s_1 c_{2+3} & -s_1 s_{2+3} & c_1 \\ -s_{2+3} & -c_{2+3} & 0 \end{bmatrix} \begin{bmatrix} .4318 \\ 0 \\ -.093 \end{bmatrix} + \begin{bmatrix} -(\dot{q}_2 + \dot{q}_3)s_1 \\ (\dot{q}_2 + \dot{q}_3)c_1 \\ \dot{q}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -0.07 \\ 0.014 \end{bmatrix}$$

Derivation of Equations of Motion

Equations of Motion

Coding of Dynamics