ECH 267 Nonlinear Control Theory
Midterm Exam

Jonathan Dorsey: Department of Mechanical & Aerospace Engineering

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Problem #1

(Required for the exam to be graded.) As a student at UC Davis, I hold myself to a high standard of integrity, and by signing the statement below I reaffirm my pledge to act ethically by honoring the UC Davis Code of Academic Conduct. I will also encourage other students to avoid academic misconduct.

I acknowledge that the work I submit is my individual effort. I did not consult or receive any help from any person or other source (i.e., beyond the book and your notes). I also did not provide help to others.

I understand that suspected misconduct on this exam will be reported to the Office of Student Support and Judicial Affairs and, if established, will result in disciplinary sanctions up through Dismissal from the University and a grade penalty up to a grade of " F " for the course.

I understand that if I fail to acknowledge or sign this statement, the course instructors may not grade this work and may assign a grade of " 0 " for the exam. Please sign the top of the first page of your exam solutions to certify that you acknowledge and agree with the terms of this statement.

Signature: ______________________________

Jonathan Thomas Dorsey

Problem #2

Consider the system

$$\ddot{z} + \dot{z} + z - \frac{1}{3}z^3 = 0$$

- (a) Choose state variables and write the system in state space.
- (b) Considering the domain $z \in [-2, 2]$ and $\dot{z} \in [-1, 1]$, does a solution to the ordinary differential equation exist? If so, is it unique? Justify your answer.
- (c) Find all equilibrium points and determine the type (e.g., stable/unstable node or focus) of each one.

Problem 2 Solution

Part A.

By assigning the states $q_1 = z$, and $q_2 = \dot{z}$, we can rewrite the governing differential equation as a system of single order ODEs as follows.

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= -q_2 - q_1 + \tfrac{1}{3}q_1^3 \end{aligned}$$

Part B.

According to **Theorem 3.1** we know that if a system satisfies the Lipschitz condition for domain $D \subset \mathbb{R}^n$ that a solution for the system exists, but further more that this solution is unique. In order to show this property it must be shown that ...

$$\|f(x) - f(y)\| \leq L\,\|x - y\|$$

On the domain where $z \in [-2, 2]$ and $\dot{z} \in [-1, 1]$, we can see that domain is symetrically bounded above and below for both z and $\dot{z}$. Since the each function of the state space are continuously differentiable (e.g. polynomials) then it must follow that the system is **locally Lipschitz Continuous**. This implies not only that there exists a solution but also that there is a unique solution.

Part C.

In order to detemine all of the equilibrium points, we must first set the state rates equal to zero and then solve for the states that satisfy this condition.

$$\begin{aligned} 0 &= q_2 \\ 0 &= -q_2 - q_1 + \tfrac{1}{3}q_1^3 \end{aligned}$$

We can immediately tell by observation that the system must have an equilibrium point such that $q_2 \equiv 0$. By carrying the algebra through, we can conclude that the system has equilibrium points at ...

$$q_2 = 0$$
$$q_1 = 0, \pm\sqrt{3}$$

We can check this by plugging in each of these values into the state space equation and showing that $\dot{q}_1$ and $\dot{q}_2$ equal zero for each combination of q_1 and q_2 above.

Further more we can determine the <u>type</u> of the equilibrium by **linearizing** the state-space. This is accomplished by computing the Jacobian matrix of the state equations about the equilibrium points.

$$A = \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right]\Bigg|_{x=x_{eq}} = \left[\begin{array}{cc} 0 & 1 \\ -1+q_1^2 & -1 \end{array} \right]$$

By plugging in each of the equilibrium points above, we can compute the Jacobian of each individual point and after computing the eigenvalues of each point, we can infer the local type of the equilibrium point.

$$A_1 = \left[\begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array} \right]\Bigg|_{q=(0,0)}$$

$$A_2 = A_3 = \left[\begin{array}{cc} 0 & 1 \\ 2 & -1 \end{array} \right]\Bigg|_{q=(\pm\sqrt{3},0)}$$

By computing the eigenvalues for each matrix we find that $eig(A_1) = -0.5 \pm 0.8660j$, and that $eig(A_2) = eig(A_3) = [1 - 2]^T$.

1. $q = (0, 0)$ is a <u>Stable Focus</u>, since its real component is in the left-half-plane.
2. $q = (0, \sqrt{3}$ is a <u>Saddle Point</u>, since one of its eigenvalues is negative while the other is positive.
3. $q = (0, -\sqrt{3}$ is a <u>Saddle Point</u>, since one of its eigenvalues is negative while the other is positive.

Problem #3

Consider the nonlinear system

$$\dot{x}_1 = h(t)x_2 - g(t)x_1^3$$
$$\dot{x}_2 = -h(t)x_1 - g(t)x_2^3$$

where $h(t)$ and $g(t)$ are bounded continuously differentiable functions, with $0 < g_0 \leq g(t)$.

- (a) Is the equilibrium $x = 0$ uniformly asymptotically stable? Is it globally uniformly asymptotically stable?
- (b) Is it (globally/locally) exponentially stable?

Solution Problem 3

To investigate the stability properties of this system, we can use Lyapunov Stability Theory. We can construct a Lyapunov Candidate function $V(x)$ such that.

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

By taking the time derivate of V...

$$\dot{V}(x) = x_1\dot{x_1} + x_2\dot{x_2}$$

We can further expand this expression by substituting the state equations.

$$\begin{aligned}\dot{V}(x) &= x_1\left(h(t)x_2 - g(t)x_1^3\right) + x_2\left(-h(t)x_1 - g(t)x_2^3\right)\\ &= x_1x_2h(t) - x_1x_2h(t) - x_1^4g(t) - x_2^4g(t)\end{aligned}$$

After canceling terms, we get that the Lyanpunov function for this problem is...

$$\dot{V}(x) = -x_1^4g(t) - x_2^4g(t)$$

According to Theorem 4.8 (Khalil), we state that a equilibrium point is uniformly stable, if we can show that

$$W_1(x) \leq V(t,x) \leq W_2(x)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) \leq 0$$

$\forall t \geq t_0$ and $\forall x \in D$

Where the function W_1 and W_2 are positive definite functions. The reasoning is that for V to obey this property V must itself be positive definite, while for time derivative of V must be at negative semi-definite. Additionally, inorder for this theorem to hold, it must be shown that these properties are independent of time.

However, in order to show that the system is Uniformly Asymptotically Stable, we need a stronger condition. This condition is...

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x)$$

Where W_3 is also a positive definite function. This condition implies that there must exist a bound on the values of $\dot{V}(t,x)$ such that $\dot{V}$ is **always** be strictly less than zero. Using this information we can show whether or not the system is uniformly asymptotically stable.

Part A

It is trivial to show that the candidate function $V(x)$ is positive definite $\forall x \in \mathbb{R}^2$, since by its constructed of even powers and since each term is negated the function must be $V(x) > 0 \quad \forall x \in \mathbb{R}^2 - \{0\}$ and $V(0) = 0$. However, inorder to demonstrate that the theorem above holds, we mut show that the function $\dot{V}(x)$ is strictly negative definite.

Given the Lyapunov function...

$$\dot{V}(x) = -x_1^4 g(t) - x_2^4 g(t)$$

In order to ensure that this function stays negative definite over the entire domain $\forall x \in \mathbb{R}^2$, it must be shown that $g(t)$ term does not impact the negative definite-ness of $\dot{V}$. Since we are given that...

$$0 < g_0 \leq g(t)$$

We know that the value of $g(t)$ will always remain greater than zero, by definition. This tells us that $\dot{V}(x) > 0$, $\forall x \in \mathbb{R}^2$ and that $\dot{V}(0) = 0$ at the equilibrium point of the system. Since this function is negative definite, it is possible to find a positive definite function that satisfies the relationship for W_3 presented above.

Therefore according to **Theorem 4.8** and **Theorem 4.9** (Khalil), we have shown that there exists a positive definite function $V(x)$which can be upper and lower bounded by other PD functions, and that there exists a function $\dot{V}$ that is strictly negative definite, BOTH of which are independent of the initial time of the system t_0. Therefore we can conclude that the system is **Uniformly Asymptotically Stable**!

Additionally, since we can find a function W_1 that is radially unbounded (**Theorem 4.9**), and since all of the former properties hold for the entire domain $\forall x \in R^2$ and not just a subset $D \subset \mathbb{R}^2$, we can conclude that the system is also **Uniformly Globally Asymptotically Stable**.

Part B

In order to determine whether the system is (globally/locally) exponentially stable, we must establish by **Theorem 4.10** (Khalil), that there exists constants k_1, k_2, k_3, and a such that ...

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a$$

In order for to evaluate this, we only need to determine whether such constants exist. We can start by rewriting our candidate function in terms of its norm.

$$V(x) = \frac{1}{2}\|x\|_2^2$$

It is relatively straightforward to demonstrate that we can satisfy the first condition, with $k_1 = .25$, $k_2 = 1$, $a = 2$.

$$\frac{1}{2}\|x\|_2^2 \leq V(x) \leq \|x\|_2^2$$

The final task to determine whether there is a constant k_3 such that ...

$$-x_1^4 g(t) - x_2^4 g(t) \leq -k_3\|x\|_2^2$$

While it is possible that I am not seeing the pattern or simplifying the form the correct way, I cannot find a constant k_3 for this specific Lyapunov function that demonstrates that the system is exponentially stable, therefore I think that the system is **NOT <u>Exponentially Stable</u>**.

My reasoning for this decision that for the Lyapunov function that I constructed to solve this problem, it is very hard to relate the 2-norm of the x to $-x_1^4 g(t) - x_2^4 g(t)$. This makes it hard to reduce the problem down to its core. Since as far as I can tell, this expression does not simplify (such defining a ball of a certain radius and using that to get the form of eqach side of the inequality to be more or less comparable), then it appears to me that the there are obvious values of $\forall x \in \mathbb{R}^2$ that would require a different k_3 to satisfy the inequality. In particular, the when the value of $x_1 < 1, x_2 < 1$, would require a significantly different k_3 than when $x_1 > 1, x_2 > 1$.

Problem #4

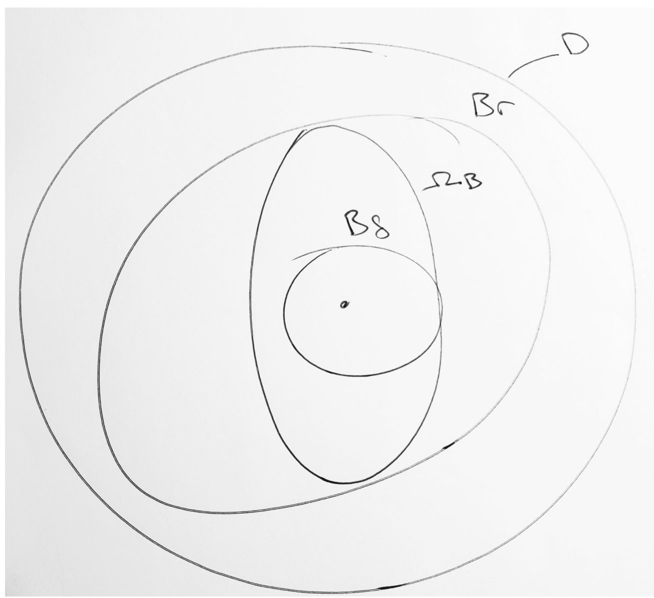
Consider the nonlinear system:

$$\dot{x} = f(x), \quad x(0) = x_0$$

where $f(0) = 0$. Assuming that $f : D \to \mathbb{R}^n$ is Lipschitz on its domain and let solutions of the system be bounded by:

$$\|x(t)\| \le \beta(\|x(0)\|, t)$$

for $\|x(0)\| < c$ for some $c > 0$ where $\beta \in \mathcal{KL}$. Show that the origin is asymptotically stable in the sense of the classical asymptotic stability ($\epsilon - \delta$) definition.



In order to demonstrate that the system (as defined using Comparison functions) is asymptotically stable, we need to demonstrate the $\epsilon - \delta$ proof of asymptotic stability and relate that to the usage of comparison functions to show the equivalence between the two.

For $\epsilon > 0$ such that $r \in (0, \epsilon]$

$$B_r = \{x \in \mathbb{R}^n | \, \|x\| < r\} \subset D$$

Then there is an $\alpha = min_{\|x\|=r} V(x)$, non-negative $\alpha > 0$, such that $\beta \in (0, \alpha)$ we can define a positive invariant set Ω_B such that

$$\Omega_B = \{x \in B_r | V(x) \le \beta\}$$

Note that since, Ω_B is an invariant set, it has the property that any trajectory starting in it $x(t_0)$ remains in it such that $\|x(t)\| \in \Omega_B$. Due to this property we know that there must exist a function which is strictly decreasing such that ...

$$V(x(t)) \leq V(x(t_0)) \leq \beta \qquad \forall t \leq t_0$$

In order for this to be true

$$\dot{V}(x) \leq 0$$

It should be noted that Ω_B is a compact set by construction since it is a subset of B_r. From this fact, and under the assumptions listed in the problem statement, we can say that $\dot{x} = f(x)$ has a unique solution $\forall t \geq t_0$ for $X(t_0) \in \Omega_B$.

If $V(x)$ is continuous and $V(0) = 0$, we can define a δ such that..

$$\|x\| \leq \delta \Rightarrow V(x) < \beta$$

Since we know that ball of radius δ infers a value of $V(x)$ inside of ball of radius β, we can show that...

$$B_\delta \subset \Omega_B \subset B_r$$

If we start a trajectory inside of the ball of radius δ we can show that

$$x(t_0) \in B_\delta \Rightarrow x(t_0) \in \Omega_B \Rightarrow x(t) \in \Omega_B \Rightarrow x(t) \in B_r$$

This means that given this setup, from a given δ there must exist a ϵ (recall: B_r defines the ϵ) such that we can state.

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < r \leq \epsilon, \qquad \forall t \geq t_0$$

If this is true such this means that the system is stable according to the $\epsilon - \delta$ definition. However, this does not show that system is attractive. In order to show that we need to prove that...

$$x(t) \to 0 \quad \text{as } t \to \infty$$

Since the function $V(x(t))$ is decreasing we can show that if ...

$$V(x(t)) \to c \geq 0 \text{ as } t \to \infty$$

It is sufficient to show that the function $V(x) \to 0$ as $t \to \infty$.

Solution Problem 4

With the structure of the main proof out of the way, we need to show how the comparison functions can be linked to this definition for asymptotic stabiilty. We can show that we can define the functions $V(t,x)$ and $\dot{V}(t,x)$ in terms of the comparison function $\alpha_i \in K$ and $\beta_i \in KL$. From the definitions of the classes (K, KL) we can show that in order for the function V to be positive definite it must be able to be upper and lower bounded by functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$.

In addition to this, we must show that the function $\dot{V}(x)$ is negative definite, which can be stated as $\dot{V} \leq -\alpha_3(\cdot)$. We can observe that via Lemma 4.2, that by using properties of these functions we can rewrite

$$V(t,x) \leq \sigma(V(t_0, x(t_0)), t - t_0), \quad \forall V(t_0, x(t_0)) \in [0, c]$$

which can be written

$$\|x\| \leq \beta(\|x(0)\|, t)$$

Since it is possible to rewrite the original statement of asymptotic stability in terms of comparison functions, we can show the direct connection between the form of the problem given and the direct $\epsilon - \delta$ proof of asymptotic stability.

Problem #5

Show that the origin of the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable, using a suitable Lyapunov function.

Solution Problem 5

In order to determine the stability of the origin of this system, we must first develope a Lyapunov Candidate function $V(x)$ that is postive definite.

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

Which results in the following Lyapunov function...

$$\dot{V}(x) = x_1^3\dot{x}_1 + x_2\dot{x}_2$$

By substituing the state equations back into $\dot{V}$, we obtain...

$$\begin{aligned}\dot{V}(x) &= x_1^3(x_2) + x_2(-x_1^3 - x_2^3) \\ &= -x_2^4\end{aligned}$$

Unfortunately, we can only state that this function is negative definite since, given $x_2 = 0$ we can show that $\forall x_1 \in \mathbb{R}$, the Lyapunov function is equal to zero. Since this function doesn't have the property to that $\dot{V} < 0$ and $V(0) = 0$ at the origin $x = (0,0)$, we must conclude that the function is only negative semi-definite.

This is unfortunate since we cannot use Lyapunov's Global Stability Theorem inorder to demonstrate that the system is globally asymptotically stable. In stead we must use **4.4 La Salle's Theorem** and its following **Corolarry 4.2** (Khalil), to demonstrate any further properties about the system.

From the previous analysis we have shown the following properties

- $V : \mathbb{R}^2 \to \mathbb{R}$ (Defined on a global domain)
- $V \in C_1$ (Continuously differentiable)
- $V(x) \to \infty$ as $\|x\| \to \infty$ (Radially unbounded)
- $V(x) > 0$ and $V(0) = 0$ (Positive Definite)

The only limitation we have is that $\dot{V}(x)$ is only negative semi-definite and NOT negative definite. By applying La Salle's Theorem (and its Corolarry) we can show that...

$$S = \{x \in \mathbb{R}^2 | \dot{V}(x) = 0\}$$

By solving the equation $\dot{V}(x) = -x_2^4 = 0$, we can show that $x_2 \equiv 0$ and therefore...

$$S = \{x_2 = 0\}$$

However, we can show that by plugging this result into the state equations that

$$x_2 \equiv 0 \Rightarrow x_1 = 0$$

Since there is no solution that can stay identically in S other than the origin $x(t) = (0, 0)$, then by **Corolarry 4.2** we can conclude that the system is **Globally Asymptotically Stable**.

Problem #6

Consider the following nonlinear system:

$$\begin{aligned}\dot{x}_1 &= f_1\left(x_1, x_2, x_3\right) + g_1\left(x_1\right) u \\ \dot{x}_2 &= f_2\left(x_1, x_2, x_3\right) \\ \dot{x}_3 &= f_3\left(x_1, x_2, x_3\right) \\ y &= h\left(x_2\right)\end{aligned}$$

Where $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, $x_3 \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $f_1(0,0,0) = f_2(0,0,0) = f_3(0,0,0) = 0$ and $g_1(0) \neq 0$.

- (a) Determine the relative degree of the controlled output y with respect to the manipulated input u.
- (b) Design an input/output feedback linearizing controller to stabilize the input/output dynamics.
- (c) State the conditions that ensure this controller enforces local asymptotically stability of the origin.

Solution Problem 6

Part A

To determine the relative degree of the system we need to determine how many derivatives of the output y must be taken inorder to see the effect of the input on the system.

This can be show by taking the generalized system form $\dot{x} = f(x) + g(x)u$ and computing the Lie Derivaties.

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0$$

This relation determines whether the derives of the output are independent of the input to the system. To begin this process for the given system, we need to take the Lie Derivative L_g for $i = 1$ which gives ...

$$L_g h(x) = \underbrace{\frac{\partial h}{\partial x_1} g_1(\cdot)}_{0} + \underbrace{\frac{\partial h}{\partial x_2} g_2(\cdot)}_{0} + \underbrace{\frac{\partial h}{\partial x_3} g_3(\cdot)}_{0}$$

Therefore ...

$$L_g h(x) = 0$$

This indicates that the first derivative is independent of the input. This means that we need to take the second derivate of the output and evaluate the system again, this time for $i = 2$.

$$L_g L_f h(x) = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} \cdot f\right) \cdot g$$

In order to evluate this expression we need to compute $L_f h(x)$ and then compute the $L_g(L_f h(x))$.

$$L_f h(x) = \frac{\partial h}{\partial x} f(x) = \frac{\partial h}{\partial x_1} f_1(\cdot) + \frac{\partial h}{\partial x_2} f_2(\cdot) + \frac{\partial h}{\partial x_3} f_3(\cdot)$$

Since the function $h(x_2)$ is only dependent on x_2 derivatives of $h(x_2)$ with respect to x_1 and x_2 are zero.

$$L_f h(x) = \underbrace{\frac{\partial h}{\partial x_1} f_1(\cdot)}_{0} + \frac{\partial h}{\partial x_2} f_2(\cdot) + \underbrace{\frac{\partial h}{\partial x_3} f_3(\cdot)}_{0}$$

Therefore

$$L_f h(x) = \frac{\partial h}{\partial x_2} f_2(\cdot)$$

To complete the process we need to take the following derivative

$$L_g L_f h(x) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x_2} f_2(\cdot)\right] g(\cdot)$$

We can expand this to...

$$L_g L_f h(x) = \frac{\partial}{\partial x_1}(\frac{\partial}{\partial x_2} f_2(\cdot)) \cdot g_1() + \underbrace{\frac{\partial}{\partial x_2}(\frac{\partial}{\partial x_2} f_2(\cdot)) \cdot g_2()}_{0} + \underbrace{\frac{\partial}{\partial x_3}(\frac{\partial}{\partial x_2} f_2(\cdot)) \cdot g_3()}_{0}$$

Since g_1 and g_2 are zero, the last two terms of the expression are cancelled out. This means that we can write.

$$L_g L_f h(x) = \frac{\partial}{\partial x_1}(\frac{\partial}{\partial x_2} f_2(\cdot)) \cdot g_1()$$

Since this term does not reduce to zero we have determined that the system has a **Relative Degee** $= 2$, since it took two time derivatives of the output to obtain an input/output relationship.

Part B

We can directly find the input/output feedback linearizing controller, from the general equation, since our system is expressed in the general form $\dot{x} = f(x) + g(x)u$. The generalize linearizing input/output controller can be shown to be...

$$u = \frac{1}{\mathrm{L_g\,L_f}\,h(x)}\left[-L_f^2 h(x) + v\right]$$

Since we already have the expression for $L_g L_f h(x)$ from the previous problem, we only need to determine $L_f^2 h(x)$ and the controller will be solved for.

$$L_f\left(L_f h\right) = \frac{\partial}{\partial x}\left(L_f h\right) = \frac{\partial}{\partial x}\left(\left[\frac{\partial h(x)}{\partial x}\right] \cdot f(x)\right) \cdot f(x)$$
$$= \frac{\partial}{\partial x}\left(\left[\frac{\partial h(x_2)}{\partial x_2}\right] \cdot f_2(x)\right) \cdot f(x)$$

By using prduct rules of calculus, this term expands to...

$$L_f\left(L_f h\right) = \frac{\partial}{\partial x_1}\left[f_2()\right] \cdot f_1() + \left[\frac{\partial^2}{\partial x_2^2}(h(x_2)) \cdot f_2() + \frac{\partial}{\partial x_2}(h(x_2)) \cdot \frac{\partial}{\partial x_2}(f_2())\right] + \frac{\partial}{\partial x_3}\left[f_2()\right] \cdot f_3()$$

By substituting the derived expressions for $L_g L_f h(x)$ and $L_f^2 h(x)$ into the feedback linearizing controller...

$$u = \frac{1}{\mathrm{L_g}\,\mathrm{L_f}\,h(x)}\left[-L_f^2 h(x) + v\right]$$

We can design a controller that linearizes the nonlinear terms in the state equations and allows us to manipulate the system as if it was linear, and enabling the use of linear control techniques such as pole placement to control the system and keep it stable.

Part C

Just linearizing the control input does not inherently make the system stable so in order to control the system and keep it stable we need to ensure the following ...

- The system is in a form conducive to linerization $\dot{x} = f(x) + g(x)u$
- The virtual input v needs to enforce stable dynamics such that the closed loop system $(A - BK)$ has stable poles in the left-half-plane (aka Hurwitz)
- The state space is controllable.