ECH 267 Nonlinear Control Theory
Homework #2

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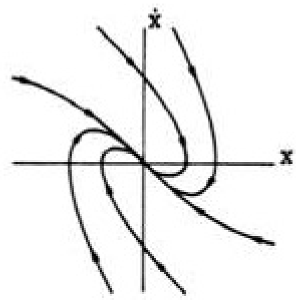
Problem 1

1. Stable Node: **Asymptotically Stable**
2. Unstable node: **Unstable**
3. Stable Focus: **Asymptotically Stable**
4. Unstable Focus: **Unstable**
5. Center: **Stable**

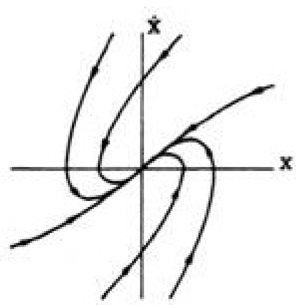
Part 1

A stable node is asymptotically stable since it all trajectories will trend to the origin (aka node) as **t** goes to ∞. While stability states that given an initial condition the trajectory of the solution will be bounded, asymptotic stability states that the trjectory will trend towards zero as time goes to ∞



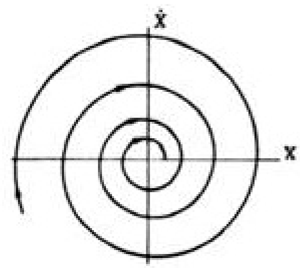
Part 2

By definition, an unstable node, is unstable. However this can be seen via it's phase portrait where any trajectory starting from the initial condition deviates away from the origin (not asymptotically stable) and further more does not stay in a bounded neighborhood (no stable). Since neither of these traits are present, the system is **unstable**.



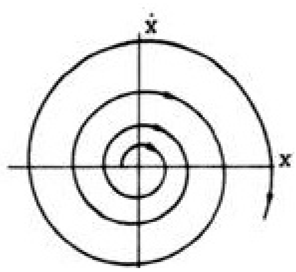
Part 3

A stable focus is asymptotically stable since it all trajectories will trend to the origin. This assertion is clear from it Phase Portrait, that any trajectory started from any initial condition will spiral into the origin. This agrees with the definition of **asymptotic stability** all trjectories will trend towards zero (the origin) as time goes to ∞



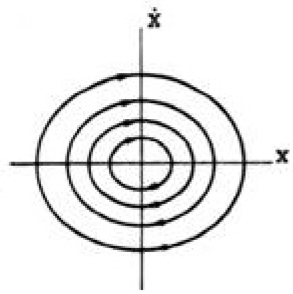
Part 4

By definition, an unstable focus is unstable. However, it is clear from the phase portrait that all trajecories from an initial condition will invariably trend away from the origin. Since the trajectories do not converge to the origin, and since the tracjectories do not stay bounded in some neighborhood they are unstable



Part 5

A center is stable since it stays we can find some neighborhood about the origin in which the trajectory of the center is orbiting. However, as is clear from the phase portrait, since the center does not converge to the origin but oscillates around it, it cannot be asymptotically stable since the trajectory of the center does not converge to the origin, but merely orbits it.



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Problem 2

Given the scalar system...

$$\dot{x} = ax^p + g(x)$$

Where p is a positive integer such that...

$$|g(x)| \leq k|x|^{p+1}$$

Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$

Solution:

Inorder to determine stability, we should begin by establishing a Lyapunov candidate function. Since the function is a scalar equation, we can use...

$$V(x) = \frac{1}{2} \cdot x^2$$

In order to establish stability using this Lyapunov function, we must differentiate the function with respect to x.

$$\dot{V}(x) = \frac{dV}{dx} \cdot \dot{x}$$

Using this definition $\dot{V}$ can be found by differeniating the function, with respect to x, and then substituting in the given system.

$$\frac{dV}{dx} = x$$

$$\dot{V} = x[ax^p + g(x)] \leq ax^{p+1} + k|x|^{p+2}$$

Case 1: "P" is odd & "a" less than 0

Given the inequality provided in the problem statement, we can see that for ax^p dominates the function around the origin. Due to the absolute value in the inequality for $g(x)$, any value of x put into $g(x)$, will return a positive number. When p is odd, we can see that the exponent of x is even. This means that even for negative values of x the output will be positive. Therefore since both terms of the Lyapunov function derivative yeild positive numbers, $\dot{V}(x)$ can only be **negative semi definite** or better when the constant $a < 0$.

Case 2: "P" is odd & "a" greater than 0

As with Case #1, when the p is odd. We can show that the sign of the Lyapunov derivative is negative **only** when the constant $a < 0$. In this situation, where $a > 0$, that means that $\dot{V}(x)$ is not negative semi definite or better and therefore must be unstable.

Case 3: “P” is even & “a” is not equal to 0

In the even that p is an even number, due to the formulation of $\dot{V}(x)$ we can see that the dominate part of the function ax^{p+1} will be an exponent to an odd power because of the plus one. Since odd functions retain the sign of their argument, there will always be some region of this function (under these conditions) where the function is not negative semi definite or better therefore must be unstable. This result holds since even though the constant a may change the sign of the function, the resultant output will carry the sign of its argument. Since the condition for stability does not hold all inputs in the statespace, it must be unstable. Naturally the case when $a = 0$ is trivial since the resulting expression is valued at zero.

Problem 3

Given the following systems, use a quadratic Lyapunov function candidate to show that the orgin isymptotically stable. Then investigate whether the origin is globally asymptotically stable

Part 1

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2$$

Part 1 - Asymptotic Stability

Since the first section of this question is asking to verify that the equilibrium point is asymptotically stable, it will suffice to find the local domain D where the conditions $V(0) = 0$ and $\dot{V}(x) < 0 \ \forall x \in D - \{0\}$.

Assuming a quadratic function, we can compute the Lyapunov Function as...

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$
$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2$$

such that...

$$\dot{V}(x) = -x_1^2 + x_1^2 x_2 - x_2^2$$

To show that function, $V(x)$ is asymptotically stable, we can show that (1) $V(0) = 0$ and that (2) $\dot{V}(x) < 0, \ \forall x \in D - \{0\}$, according to Lyapunov Theorem. It is trivial to show that the function $V(0) = 0$, and that $V(x) > 0$, since the function candidate is given as quadratic. Therefore the problem is to demonstrate that the time derivative of V is **negative definite**.

In order to establish asymptotical stability, we only need to demonstrate that the stability exists for some domain $D \subset \mathbb{R}^2$. To test the function in domain D we can assume that there exists a ball such that...

$$B_r = \{x \in \mathbb{R}^2 | \, \|x\| < r\}$$

If we can show that if there exists a radius of r for which $\dot{V}(x)$ is negative definite, then have have effectively shown that the domain D exists and proven asymptotic stability via Lyapunov Theorem.

Given the condition...

$$\dot{V}(x) = -x_1^2 + x_1^2 x_2 - x_2^2 < 0$$

For the inequality to hold...

$$x_1^2 x_2 < x_1^2 + x_2^2$$

By using the definition of the ball B_r, we can show that for a given radius r, $\|x_1\| < r$ & $\|x_2\| < r$, for which we can rewrite the inequality as ...

$$r^3 < r^2 + r^2$$

After simplification we find that...

$$r < 2$$

Therefore for a given radius of $r < 2$ the system is **asymptotically stable**.

Part 1 - Global Stability

In order to demonstrate Globally Asymptotic Stability, we can use the Lyapunov Theorem for Global Stability which can be stated as follows.

$$\begin{aligned} V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0 \\ \|x\| \to \infty \Rightarrow V(x) \to \infty \\ \dot{V}(x) < 0, \quad \forall x \neq 0 \end{aligned} \tag{1}$$

We have shown in the previous problem that there exist conditions where $\dot{V}$ is negative definite $\dot{V}(x) < -x_1^2 - x_2^2 + r|x_1||x_2| < 0$, when $r < 2$ and by inspection we can see that the Lyapunov candiate function is 'radially unbounded'. By demonstrating these condition, we can use the theorem and say that the function must be globally asymptotically stable.

Part 2

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= x_1 - x_2(1 - x_1^2 - x_2^2) \end{aligned}$$

Part 2 - Asymptotic Stability

The Lyapunov function candidate and its time derivate can be given by...

$$\begin{aligned} &V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 \\ &\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\left(-x_2 - x_1\left[1 - x_1^2 - x_2^2\right]\right) + \dots \\ &\dots + x_2\left(x_1 - x_2\left[1 - x_1^2 - x_2^2\right]\right) \end{aligned} \tag{2}$$

When simplifed this functions yield the expression....

$$\dot{V}(x) = \left(-x_1^2 - x_2^2\right)\left[1 - x_1^2 - x_2^2\right]$$

The first expression in this term is always negative definite due to the coefficients in front of x_1 and x_2. This implies that in order for the function to be negative definite the term $1 - x_1^2 - x_2^2 > 0$ since the multiplicitive negative would negate the previous term. This means that we need to show the second term is positive definite, for the complete inequality to hold.
By rearrangement...

$$x_1^2 + x_2^2 < 1 \ \forall x \in \mathbb{R}^+$$

The only way for this to be true is if the values of x_1 and x_2 are bounded from above by $\sqrt{.5}$. This can also be shown by using the definition of the function V and substituting it into the expression for $\dot{V}(x)$. Which yields.

$$\dot{V}(x) = -2 \cdot V\left[1 - 2 \cdot V\right] < 0$$

By simplification the result must be that the value of $V < \frac{1}{2}$. This is equivalent to the bound placed on x_1 and x_2 above.

Part 2 - Global Stability

By the very definition of the previous stability result, we showed that $V \ngeq \frac{1}{2}$. Since this is the requirement for obtaining the negative definite $\dot{V}$. Therefore the solution is **not** globally asymptotically stable, since it is not radially unbounded on $\mathbb{R}^2$.

Part 3

$$\begin{aligned}\dot{x}_1 &= x_2(1 - x_2^2)\\ \dot{x}_2 &= -(x_1 + x_2)(1 - x_2^2)\end{aligned}$$

Part 3 - Asymptotic Stability

Given the quadratic Lyapunov function, and its corresponding time derivative, we obtain the following expression for the given system.

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2\\ &= x_1\left(x_2\left[1 - x_1^2\right]\right) + x_2\left(-\left(x_1 + x_2\right)\left[1 - x_1^2\right]\right)\end{aligned} \tag{3}$$

This expression can be reduced to..

$$V\dot{(}x) = -x_2^2\left[1 - x_1^2\right]$$

Since this first part of this expression is always negative definite $x \in \mathbb{R}^2$, we know that the second term must therefore be positive definite for the function to remain negative due to the multiplicitive negative. This requires...

$$1 - x_1^2 > 0$$

Which futher simplies to the fact that $x_1 < 1$ for the function to remain negative definite. Under this condition, the Lyapunov function candiate follows all of the conditions for given in Lyapunov's Stability Theorem and therefore is **asymptotically stable**.

Part 3 - Global Stability

Since the time derivate of V is not negative definite for $x \in \mathbb{R}^2$, due to the restriction that $x_1 < 1$. The system is **not globally asymptotically stable**.

Part 4

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= 2x_1 - x_2^3 \end{aligned}$$

Problem 4

Use $V(x) = x_1^2 + x_2^2$ to study the stability of the origin of the system.

$$\begin{aligned}\dot{x}_1 &= x_1\left(k^2 - x_1^2 - x_2^2\right) + x_2\left(x_1^2 + x_2^2 + k^2\right)\\ \dot{x}_2 &= -x_1\left(k^2 + x_1^2 + x_2^2\right) + x_2\left(k^2 - x_1^2 - x_2^2\right)\end{aligned}$$

We can compute $\dot{V}(s)$ from the given Lyapunov candidate to be ...

$$\begin{aligned}V(x) &= x_1^2 + x_2^2\\ \dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2\\ &= 2x_1\left(x_1\left[k^2 - x_1^2 - x_2^2\right] + x_2\left[x_1^2 + x_2^2 + k^2\right]\right) + \ldots\\ \ldots &+ 2x_2\left(-x_1\left[k^2 + x_1^2 + x_2^2\right] + x_2\left[k - x_1^2 - x_2\right]\right)\end{aligned} \tag{4}$$

We can reduce that expression as follows...

$$\dot{V}(x) = \left(2x_1^2 + 2x_2^2\right)\left[k^2 - x_1^2 - x_2^2\right]$$

The first term in this expression is clearly positive definite $\forall x \in \mathbb{R}^2$. Therefore in order to determine the stability of the system, we must look at the second term and determine its behavior for the arbitrary constant k. The two cases for k are written below.

Case: $K = 0$

In order to garuntee that $\dot{V}(x) < 0$, we must show that the term $k^2 - x_1^2 - x_2^2 \leq 0$. For the cases when $k = 0$. It is clear that the term $-x_1^2 - x_2^2$ is **must** be negative definite $\forall x \in \mathbb{R}^2$. Therefore the system is **globally asymptotically stable** when $k = 0$, due to the fact that system time derivative of the Lyapunov function is negative definite $\forall x \in \mathbb{R}^2$, according to Lyapunov Global Stability Theorem.

Case: $K \neq 0$

For the case when $x \neq 0$, we must garuntee that the term $k^2 - x_1^2 - x_2^2 \leq 0$. By substituting the expression for the candidate function into this derivate form ...

$$\begin{aligned}&k^2 - x_1^2 - x_2^2 \leq 0\\ &k^2 - (x_1^2 + x_2^2) \leq 0\\ &k^2 - V(x) < 0\end{aligned} \tag{5}$$

Such that...

$$V < k^2$$

This restrictions shows that there is some restriction on the state space in order to maintain the inequality. This infer that the system is **asymptotically stable**, but that the system cannot be globally asymptotically stable since by this very condition there exist limits on the function where its derivative are no longer negative definite.

Problem #5

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \text{sat}\,(2x_1 + x_2)\end{aligned}$$

Assuming a quadratic Lyapunov candidate function, we can show that...

$$\dot{V}(x) = 2x_1x_2 - x_2\,\text{sat}\,(2x_1 + x_2)$$

Part A

In order to show that origin of the system is asymptotically stable, we must show that it complies with Lyapunov Stability Theorem. For this to be valid, the function $\dot{V}(x) < 0$. This is slightly more difficult to determine in this case due to the presence of the saturation function. Which is defined as ...

$$\text{sat}(y) = \begin{cases} -ka & \text{if } \; y < a \\ ky & \text{if } \; -a \le y \le a \\ ka & \text{if } \; y > a \end{cases}$$

Where k is the value of the slope of the linear region when the input $y \in [-a, a]$. Using this definition, we can breakdown the analysis of the inequality into each segment of the piecewise saturation function and determine whether the system is stable given each segment.

Case: Linear Opertion $\quad -a \le y \le a$

By simplifying the expression for $\dot{V} < 0$, and plugging in the expression for the saturation function in the linear region we obtain...

$$\begin{aligned}\dot{V}(x) = 2x_1^2x_2^2 - x^2\,\text{sat}(2x_1 + x_2) < 0 \\ 2x_1 < \text{sat}(2x_1 + x_2)\end{aligned}$$

Such that...

$$2x_1 < k\,(2x_1 + x_2)$$

The restriction imposed by this shows that so long as the following expression is valid.

$$\frac{2x_1}{2x_1 + x_2} < k$$

This means that as long as the stability of the origin is restricted by the value of the slope of the saturation function. In this case, this means that the system is **asymptotically stable**, while the condition is satisfied since this condition keeps $\dot{V}$ as a negative definite function, as required my Lyapunov Stability Theorem.

Case: Negative Saturation $\quad y < -a$

By applying the same logic as the previous case, with the appropriate substitution for the region of operation of the saturation function, we obtain the condition to keep $\dot{V}(x)$ negative definite is...

$$2x_1 < -ka$$

This function fundamentally restricts the value of the state space due to the fact that saturation function platues at $sat(y) = -ka$, $\forall y < -a$.

Case: Positive Saturation $\quad y > a$

By applying the same logic as the previous case, with the appropriate substitution for the region of operation of the saturation function, we obtain the condition to keep $\dot{V}(x)$ negative definite is...

$$2x_1 < ka$$

This function fundamentally restricts the value of the state space due to the fact that saturation function platues at $sat(y) = ka$, $\forall y < a$.

Part B

As can be seen in the positive and negative operating regions of the saturation function above, the values of the state space are fundamentally limited by the constant $|x| < .5ka$. Since this condition is required for the function to be stable/asymptotically stable, we must conclude that there exists regions of the state space where the time derivative of the candidate function is not negative semi definite and thereby is not stable. This infers that trajectories starting from this region of the state space are not stable and do not converge towards the origin.

Part C

Due to the conclusions drawn above, it is plain that while there exists a set $D \subset \mathbb{R}^2$, for which the origin is stable, via the appropriate conditions of the Lyapunov candidate function, according to the Lyapunov Stability Theorem, the restriction on the stable domain of the state space means that V cannot map $\mathbb{R}^2 \to \mathbb{R}$, such that the state space is radially unbounded, for all time. Since this restriction exists the system cannot be **globally asymptotically stable**.

1 Problem 6

Part 1

$$\dot{x}_1 = x_1^3 + x_1^2 x_2$$
$$\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

In order to show that the given system is unstable, according the Lyapunov Stability Theorem, it is sufficient to show that conditions for stability do not hold in the domain about the origin. This can be seen computing the time derivate of the Lyapunov function and substituting in the state equations.

$$V = x_1 \left(x_1^3 + x_1^2 x_2\right) + x_2 \left(-x_2 + x_2^2 + x_1 x_2 - x_1^3\right)$$

Given this formulation, we know that according to Lyapunov's stability theorem that the function must be negative semi definite for the system to be stable. Under this condition, the previous equations simplifies to the following expression.

$$x_1^4 - x_2^2 + x_2^3 + x_1 x_2^2 \leq 0$$

Since it is rather difficult to analytically show the regions where this inequality holds, we can show numerically that there does not since exist a neighborhood around the origin, such that the ball $B_r = \{x \in D \subset \mathbb{R}^2 | \, \|x\| < r\}$ maintains the previous inequality. This is the case for point $x = (.5, .5)$ for $r < 1$. When evaluated at this point the inequality does not hold. Since we cannot define a ball around the origin such that the equality holds for **all** points contained inside of that ball, we must conclude that the system is <u>unstable</u>.

Part 2

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1^6 - x_2^3$$

Problem 8

Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $lim_{t\to\infty}x(t)$ exists and equals one of these points.

Solution:

In LaSalle's Theorem, the set M is the largest invariant set in the set E. Where E is the set of all points in the domain σ where $\dot{V}(x) = 0$.

We know from Lyapunov's Theorem, that $\dot{V}(x(t))$ is negative semi definite function, which means that the Lyapunov candidate function V decreases with time. Due to these properties, and since the solution $x(t)$ starts in the **invariant set** σ, the solution $x(t)$ must exist and be bounded inside this set, due to the property that all any trajectory starting at $x(0)$ in the set σ must say in σ.

According to, Lemma 4.1, the solution $x(t)$ will approach its positive limit set as $t \to \infty$. Since the limit set L^+ must comply with the Lyapunov's Theorem such that $\dot{V}(x) \leq 0$, we know can show that the largest limit set will occur at equality when when the function is not decreasing, $v(x) = 0$. Since we know from Lemma 4.1 that the solution $x(t)$ must approach its limit set L^+ as $t \to \infty$, since it is bounded, and since we defined M to be the largest invariant set of E where $v(x) = 0$ holds, then we can show that the solution $x(t)$ must converge to M as $t \to \infty$.

Problem 9

$$\dot{z} = \hat{f}(z), \qquad \text{where } \hat{f}(z) = \frac{\partial T}{\partial x} f(x)\bigg|_{x=T^{-1}(z)}$$

Part A

In order to prove that $x = 0$ is an isolated equilibrium point if and only if $z = 0$, it is necessary to show that if we assume multiple equilibrium we will encounter a contradiction.

Given that $x = 0$ is an isolated equilibrium point, we will assume that it is not. This means that we can find some value of z such that $f(z) = 0$. According to the transformation provided, we can state that for an function of z there must exist a seperate function of x that are equivalent.

Therefore given.. $f(z) = \frac{\partial T}{\partial x} f(x)$, we can show via the assumed properties that the inverse of this function is $f(x) = [\frac{\partial T}{\partial x}]^{-1} \hat{f}(z)$. If $z \neq 0$ is an equilibrium point of the system then it follows that...

$$f(x) = [\frac{\partial T}{\partial x}]^{-1} \hat{f}(z) = 0$$

This implies that there is an equivalent x that is also an equilibrium point of the system. By using the continuity of $T(\cdot)^{-1}$, we can show that we can arbitrarily find x values close to the origin that satisfy the problem, since this behavior contradicts the fact that the system is actually an isolated equilibrium, it must **not** be true that arbitrary values of z can yield arbitrary values of x that are equilibrium points. This implies that $x = 0$ must only be an isolated equilibrium point when $z = 0$ is an isolated equilibrium point.

Part B

We can show that $x = 0$ is asymptotically stable, stable, or unstable using the same proof. For the case that the equilibrium point is stable, we know that inorder for that statement to hold, there must exist an $\epsilon > 0$ and a δ such that the following is true...

$$\|x(0)\| < \delta \rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

Due to the continuity of the transformation $T(\cdot)$, it holds that there exists..

$$\|x\| < r \Rightarrow \|z\| < \gamma$$

Likewise, we can show that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \Rightarrow \|z(t)\| < \gamma, \quad \forall t \geq 0$$

This demonstrates that there exists parameters for the original and transformed system such that their norms are bounded as expected. This allows us to translate from the norm of one coordinate system to the norm of the other coordinate system by using this bounds.

In the similar manner, by the continuity of the inverse transformation T^{-1} we can show that there exists an $\eta > 0$ such that

$$\|z\| < \eta \Rightarrow \|x\| < \delta$$

This statement states that there is an equivalent inverse property such that the bound of the transformed system implies the original bound on the original system. By compiling all of these into a single equivalent statement we can write that

$$\|z(0)\| < \eta \Rightarrow \|x(0)\| < \delta \Rightarrow \|z(t)\| < \gamma, \quad \forall t \geq 0$$

In order to account for the case where the system is asymptotically stable, we apply the same chain of logic used above, to demonstrate that the $x(t) \rightarrow$ as $t \rightarrow \infty$ is equivalent to the transformed system such that $z(t) \rightarrow$ as $t \rightarrow \infty$