ECH 267 Nonlinear Control Theory
Lecture Notes

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Lecture # 6: Lyapunov Stability of Equilbium

Given the system...

$$\dot{x} = f(x) \qquad \text{and input } u = k \cdot x$$

Such that...

$$\dot{x} = f(x,u) = f(x,k(x)) := f(x)$$

Where the function is defined from

$$f : D \to \mathbb{R}^n$$

Where $D \subset \mathbb{R}^n$. Addition, the system is understood to be **locally Lipschitz Continuous**, such that.

$$\|f(x) - f(y)\| \leq L \|x - y\|$$

Without loss of generality, we can write that $\bar{x} = 0$, is an equilibrium point of the system $\bar{f}(x)$ such that when evaluated at the point $\bar{x} = 0$, then $\bar{f}(\bar{x}) = f(0) = 0$

Conditions for Stability

We can call the given system stable if..

$$\forall \epsilon > 0 \text{ , } \exists \delta = \delta(\epsilon) > 0$$

Such that...

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon \text{ , } \forall t \geq 0$$

Stability Explaination

In order for the system to be stable we must be able to demonstrate in a rigorous $\epsilon - \delta$ fashion that, there exists some domain in which, if a trajectory is begun inside of ball δ, the trajectory will be maintained inside of ball ϵ, for all positive time greater than the initial condition.

Conditions for Instability

Conversely, the system is unstable if...

$$\forall \epsilon > 0 \text{ , } \exists \delta = \delta(\epsilon) > 0$$

Such that...

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| \geq \epsilon \text{ , } \forall t \geq 0$$

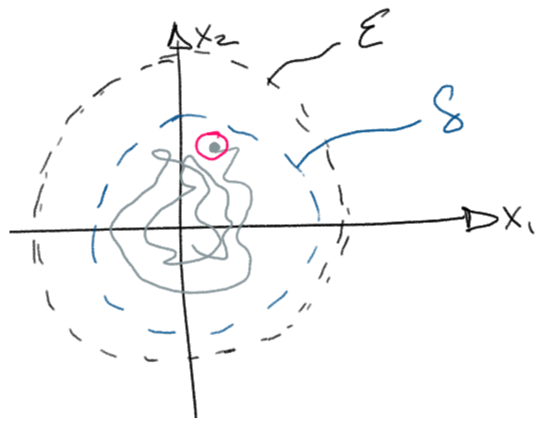
Instability Explaination

Unlike the case for stability, we can show using the same mathematical formulation that a system is unstable if inside a ball of radius δ, the trajectory cannot be maintained inside of a finite ball of radius ϵ, for all time.

Examples of Stability vs Instability

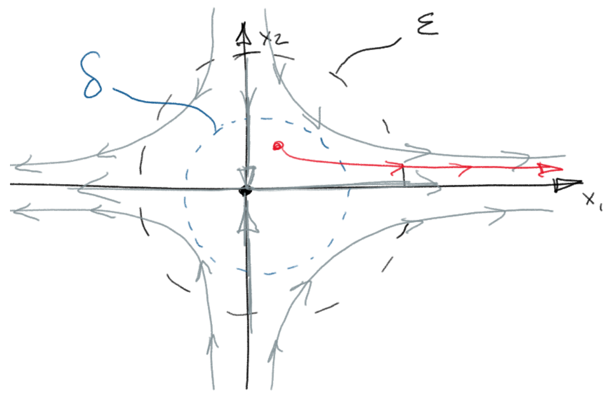
Stable Trajectory

As shown using the $\epsilon - \delta$ balls, any trajectory which starts with initial conditions inside the ball δ, there exists an ϵ such that the solution is bounded by ϵ. This is an example of stable behavior.



Saddle Point

A saddle point is clearly unstable visually, when inspecting the phase portrait of the system, since there exist trajectories which diverge asymptotically to $\pm\infty$. Even though the origin, demonstrates stable behavior there does not exist a $\epsilon - \delta$ neighborhood, such that the trajectories in that neighborhood converge to the origin, or stay inside of the defined bounds.



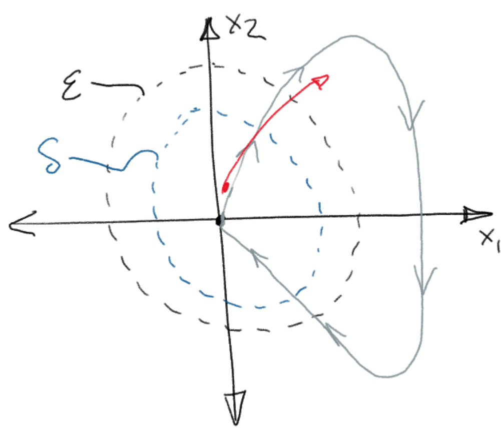
Homoclinic Orbit

While the previous examples are fairly obvious, there are a great many number of nonlinear system which exhibit strange behavior, and require further description.

For example, the concepts of domain of attraction and asymptotic stability, do not always go hand in hand.

- **Attractive:**
  A system is attractive if...
  $\exists\ \bar{\delta} > 0$ s.t. $\|x(0)\| < \bar{\delta} \Rightarrow x(t) \to 0$ as $t \to 0$

- **Asymptotically Stable:** Stable & Attractive



The image above, is the phase portrait of a homoclinic orbit that is attractive but **NOT** stable. For this system, even though the system is attractive, it is impossible to define a $\epsilon - \delta$ neighborhood which is stable since the trajectories shoot out before coming back to the origin. Therefore, even though the system is not stable it is simultaneously not attractive.

Global Stability

There also exist systems which are globally stable. This property is defined as follows.
A system is globally stable if, in addition to being stable over the domain $D \subset \mathbb{R}^n$ it follows that ...

$$\text{Given } \delta > 0,\ \exists\, \epsilon = \epsilon(\delta) > 0$$

$$\text{s.t. } \|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \quad \forall t \geq 0$$

$$\text{Such that } B_r = \{x \in \mathbb{R}^n : \|x\| \leq \delta\}$$

Global Stability Explaination

While local stability required the $\epsilon - \delta$ neighborhood to exist for some domain D which is a subset of the real-numbers, global stability, dictates that there must exist an $\epsilon - \delta$ neighborhood for the entire domain of the state space.

Types of Global Stability

- **Globally Stable**

  A system is globally stable if it satisfies the definition given above, such that any for any initial state in the domain $\mathbb{R}^n$ the trajectories of the system are bounded.

- **Globally Attractive**

  A system is globally attractive if $x(t) \to 0$ as $t \to \infty$ for all $x(0) \in \mathbb{R}^n$. (If the system is attractive throughout the entire state space)

- **Globally Asymptotically Stable**

  A system is globally asymptotically stable if it is ...

  1. Globally Attractive
  2. Globally Stable

- **Exponentially Stable**

  A system, is exponentially stable if ...

$$\exists\, C,\ K,\ \lambda > 0 \text{ s.t.}$$

$$\|x(t)\| \leq K \, \|x_0\| \, e^{(-\lambda \cdot t)}$$

Assessing Stability

Given the system...

$$\dot{x} = f(x)$$

About the equilibrium point $x_0 = 0$ such that..

$$\dot{x} = f(x_0) = 0$$

We can establish local stability on the basis of linearization of the system, about the equilibrium point such that...

$$\dot{y} = Ay \qquad \text{where } y := x - x_0$$

Where the matrix A, is the jacobian of the system and is defined as ...

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \cdots \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Lyapunov's Indirect Method

This linear approach to finding the stability of a nonlinear system is known as Lyapunov's Indirect Method. In effect, this method is making local generalization about the stability of the nonlinear system due to its linear characteristics.

Since linear systems can be characterized via eigenvalues, we can use standard definitions of stability using classical control theory as follows...

1. **Local Asymptotic Stability:**

   We can conclude a linearized system is L.A.S. if...

$$\text{Re}(\lambda_i) < 0 \qquad \forall i \in [i, \cdots n]$$

2. **Unstable:**

   We can condlude a linearize system is unstable if at least one $\text{Re}(\lambda_i) > 0$, at the equilibrium point $x = 0$.

3. **No Determination:**

We cannot determine the stability of the system if lat least one $\text{Re}(\lambda_i) = 0$

Alternative Approach for Assessing Stability

The motivating example for this alternative approach is to understand the physical implications of the system. To understand this, we will look at a Pendulum with Friction, where the states of the system are $x_1 = \theta$ and $x_2 = \dot{\theta}$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\left(\frac{g}{l}\right) \cdot Sin(x_1) - \frac{k}{m}x_2$$

We know that energy of physical systems must be conserved. Therefore, we know that give that our system has friction, a form of energy dissipation, that the system must inherently trend towards 0. We can use this intuition from the physical system and compute the energy of the system which takes the form of a scalar equation from which will act as the stand-in for a larger more well defined class of functions that will facilitate more rigorous analysis.

$$\text{Energy} = \text{PE} + \text{KE}$$
$$= \int_0^{x_1} \left(\frac{g}{l}\right) \cdot Sin(x_1)ds + \frac{1}{2}x_2^2$$
$$= \frac{g}{l}\left[1 - Cos(x_1)\right] + \frac{1}{2}x_2^2$$

This type of analysis will provide a valuable perspective later when analyzing the stability of more complicated and non-trivial systems.

Lecture #7: Lyapunov Direct Method Stability

Definition of Stability($\epsilon - \delta$)

Recall that we can a stable system to be one in where

$$\forall\epsilon > 0, \exists\, \delta = \delta(\epsilon) > 0$$

Such that for a given

$$\|x(0)\| < \delta$$

There exists a ϵ such that

$$\|x(t)\| < \epsilon$$

Definition of Asymptoticlly Stable

Further, recall that we can classify a system as asymptotically stable when ...

$$x(t) \to 0,\, t \to \infty$$

This is the condition that the system is attractive.

Example: Pendulum with Friction

$$E(x) = \frac{f}{l}[a - Cos(x_1)] + \frac{1}{2}x_2^2$$

Properties

1. $E(0,0) = 0$ Function is lower bounded by zero
2. $E(x_1, x_2) \geq 0$

$$\frac{dE}{dt} < 0 \quad \forall t \to \infty \quad x = 0 \text{ (Asymptoticlly Stable)}$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial E}{\partial x_2}\frac{dx_2}{dt}$$

Where $\dot{x_1}$ and $\dot{x_2}$ are the state equations of the system.

$$\therefore \frac{dE}{dt} = -\frac{k}{m}d_2^2 \leq 0\ \forall x \in \mathbb{R}^2$$

Since $\frac{dE}{dt} \leq 0$, show that $x = 0$ is stable...

Remarks on Pendulum:

1. This energy function is nonincreasing but not necessarily decreasing over time.
2. From physics, we know that the origin is **asymptotically stable**.

Lyapunov Direction Method for Assessing Stability

Given the system...

$$\dot{x} = f(x) \qquad f : D \to \mathbb{R}^n \text{ at, } f(0) = 0$$

is locally Lipschitz (solution exists & is unique), let's define

$$V : D \to \mathbb{R}^+ \quad \text{where } D \subset \mathbb{R}^n$$

The time derivative of the V is given by

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} \cdot f(x)$$

<u>NOTE</u>: $\frac{dV}{dx} = \left[\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n} \right]$

Lie Derivative Notation

The Lie derivative notation for the Lyapunov Function is given by

$$\dot{V} = \frac{\partial V}{\partial x} f(x) = L_f V$$

Lyapunov Stability Theorem:

1 *Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$ and let V be a function $V : D \to \mathbb{R}$, $V \in C_1$ [continuously differentiable] on D (a neighborhood of $X = 0$), $V(0) = 0$, $V(x) > 0$, $\forall x \in D - \{0\}$*

1. If $\dot{V} = \frac{\partial V}{\partial x} f(x) \leq 0$, $\forall x \in D$, then $x = 0$ is <u>stable</u>.
2. If $\dot{V} \leq 0$, $\forall x \in D$, then $x = 0$ is <u>asymptotically stable</u>.

Remarks of Lyapunov's Theorem:

- This theorem does not prove "necessary condition", its only provides sufficient condition for proving $x = 0$ is stable or A.S.
- No connection between the domain D and the $\epsilon - \delta$ definition of stability.

Pendulum with Friction

Goal: Show that $x = 0$ is asymptotically stable.

$$V(x) = \frac{1}{2}x^T Px + \frac{g}{x}(1 - Cos(x_1))$$

Where P is a positive definite matrix

$$\therefore \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = P^T \succ 0$$

Such that $P_{22} = 1$, $P_{11} = \frac{k}{m} \cdot P_{12}$, $P_{12} = P_{21} = .5\frac{k}{m}$, which define the Lyapunov Candidate function. After taking the time derivative of this function, we obtain that

$$\dot{V}(x) = -.5\frac{g}{l}\frac{k}{m}x_1 Sin(x_1) - .5\frac{k}{m}x_2^2$$

Which is the function that we want to verify is negative semidefinite(stable) or negative definite (asymptotically stable).

$$x_1 sin(x_1) > 0 \quad \text{ for } 0 < |x_1| < \pi$$

Such that

$$D = \{x \in \mathbb{R}^2 | \, \|x_1\| < \pi\}$$

1. V is a positive definite function
2. $\dot{V}$ is a negative definite function over the specified domain. Since $x = 0$ the system is asymptotically stable, via the Lyapunov Theorem.

Important Definitions

1. Lyapunov function candidate: $V : D \to \mathbb{R}$, $V \in C_1$ on D, $V(0) = 0$, $V(x) > 0$, $\forall x \in D - \{0\}$
2. Lyapunov Function: $\dot{V} \leq 0$, $\forall x \in D$
3. Level Set of V: $\Omega_c := \{x \in D | V(x) \leq C\}$

Proof of Lyapunov Stability:

1. **Stability:**
   Given $\epsilon > 0$, choose $r \in (0, \epsilon)$ s.t. $B_r := \{x \in \mathbb{R}^n | \, \|x\| < r\}$, where $B_r \subset D$.
2. Let $\alpha = min_{\|x\|=r} V(x) \Rightarrow \alpha > 0$
3. Choos $\beta \in (0, \alpha)$ such that we have level set Ω_β such that $\Omega \subset B_r$

4. Any trajectory starting in Ω_β at $t = 0$, staus in Ω_β for $t \geq 0$.

$$\dot{V}(x) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta \qquad \forall \geq 0$$

Ω_β is a compact set (1) Closed & (2) Bounded. This implies $\dot{x} = f(x)$ has a unique solution if $x(0) \in \Omega_\beta$.

$$V iscontinuous \& V(0) = 0 \quad \exists \delta > 0$$

$$\|x(0)\| \leq \delta \Rightarrow V(x) < \beta$$

$$B_\delta \subset \Omega_\beta \subset B_r$$

$x(0) \in B_8 \rightarrow x(0) \in \Omega_\delta \quad \therefore x(t) \in \Omega_B$
$x(t) \in B_r, \forall t \geq 0$
$\|x(0)\ | < \delta \rightarrow \|x(t)\| < r \leqslant \varepsilon \quad \forall t \geq 0$
$x = 0$ is stable

Lecture 8

Proof of Lyapunov Stability Theorem

In order to begin prove Lyapunov Stability, it helps to first review a few salient definitions.

1. **Stability**

$$\|\|, \delta \Rightarrow x(t) \in \Omega_C \forall t \geq 0$$

   Such that there exists...

$$\|x(t)\| < \epsilon$$

2. **Asymptotic Stability**

$$lim_{t\to\infty}\, x(t) \to 0 \Rightarrow \forall a > 0$$

$$\exists T > 0 \text{ s.t } \|x(t)\| < a \forall f \geq T$$