

ECH 267 Nonlinear Control Theory

Homework #2

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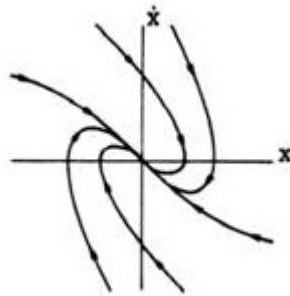
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Problem 1

1. Stable Node: **Asymptotically Stable**
2. Unstable node: **Unstable**
3. Stable Focus: **Asymptotically Stable**
4. Unstable Focus: **Unstable**
5. Center: **Stable**

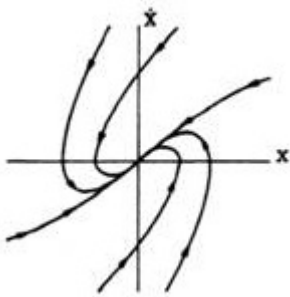
Part 1

A stable node is asymptotically stable since all trajectories will trend to the origin (aka node) as t goes to ∞ . While stability states that given an initial condition the trajectory of the solution will be bounded, asymptotic stability states that the trajectory will trend towards zero as time goes to ∞ .



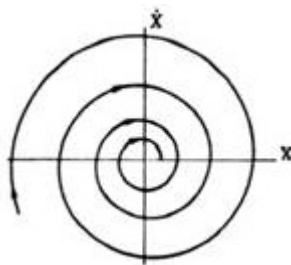
Part 2

By definition, an unstable node, is unstable. However this can be seen via its phase portrait where any trajectory starting from the initial condition deviates away from the origin (not asymptotically stable) and furthermore does not stay in a bounded neighborhood (no stable). Since neither of these traits are present, the system is **unstable**.



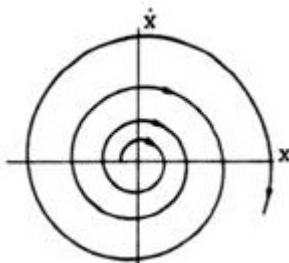
Part 3

A stable focus is asymptotically stable since all trajectories will trend to the origin. This assertion is clear from its Phase Portrait, that any trajectory started from any initial condition will spiral into the origin. This agrees with the definition of **asymptotic stability** all trajectories will trend towards zero (the origin) as time goes to ∞



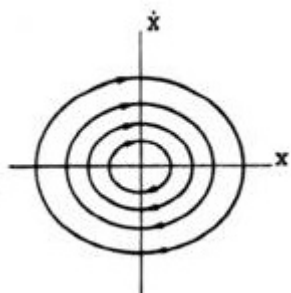
Part 4

By definition, an unstable focus is unstable. However, it is clear from the phase portrait that all trajectories from an initial condition will invariably trend away from the origin. Since the trajectories do not converge to the origin, and since the trajectories do not stay bounded in some neighborhood they are unstable



Part 5

A center is stable since it stays we can find some neighborhood about the origin in which the trajectory of the center is orbiting. However, as is clear from the phase portrait, since the center does not converge to the origin but oscillates around it, it cannot be asymptotically stable since the trajectory of the center does not converge to the origin, but merely orbits it.



Problem 2

Given the scalar system...

$$\dot{x} = ax^p + g(x)$$

Where p is a positive integer such that...

$$|g(x)| \leq k|x|^{p+1}$$

Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$

Solution:

In order to determine stability, we should begin by establishing a Lyapunov candidate function. Since the function is a scalar equation, we can use...

$$V(x) = \frac{1}{2} \cdot x^2$$

In order to establish stability using this Lyapunov function, we must differentiate the function with respect to x .

$$\dot{V}(x) = \frac{dV}{dx} \cdot \dot{x}$$

Using this definition \dot{V} can be found by differentiating the function, with respect to x , and then substituting in the given system.

$$\frac{dV}{dx} = x$$

$$\dot{V} = x[ax^p + g(x)] \leq ax^{p+1} + k|x|^{p+2}$$

Case 1: “P” is odd & “a” less than 0

Given the inequality provided in the problem statement, we can see that for ax^p dominates the function around the origin. Due to the absolute value in the inequality for $g(x)$, any value of x put into $g(x)$, will return a positive number. When p is odd, we can see that the exponent of x is even. This means that even for negative values of x the output will be positive. Therefore since both terms of the Lyapunov function derivative yield positive numbers, $\dot{V}(x)$ can only be **negative semi definite** or better when the constant $a < 0$.

Case 2: “P” is odd & “a” greater than 0

As with Case #1, when the p is odd. We can show that the sign of the Lyapunov derivative is negative **only** when the constant $a < 0$. In this situation, where $a > 0$, that means that $\dot{V}(x)$ is not negative semi definite or better and therefore must be unstable.

Case 3: “P” is even & “a” is not equal to 0

In the even that p is an even number, due to the formulation of $\dot{V}(x)$ we can see that the dominate part of the function ax^{p+1} will be an exponent to an odd power because of the plus one. Since odd functions retain the sign of their argument, there will always be some region of this function (under these conditions) where the function is not negative semi definite or better therefore must be unstable. This result holds since even though the constant a may change the sign of the function, the resultant output will carry the sign of its argument. Since the condition for stability does not hold all inputs in the statespace, it must be unstable. Naturally the case when $a = 0$ is trivial since the resulting expression is valued at zero.

Problem 3

Given the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then investigate whether the origin is globally asymptotically stable

Part 1

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1x_2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

Part 1 - Asymptotic Stability

Since the first section of this question is asking to verify that the equilibrium point is asymptotically stable, it will suffice to find the local domain D where the conditions $V(0) = 0$ and $\dot{V}(x) < 0 \forall x \in D - \{0\}$.

Assuming a quadratic function, we can compute the Lyapunov Function as...

$$\begin{aligned}V(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ \text{such that...}\end{aligned}$$

$$\dot{V}(x) = -x_1^2 + x_1^2x_2 - x_2^2$$

To show that function, $V(x)$ is asymptotically stable, we can show that (1) $V(0) = 0$ and that (2) $\dot{V}(x) < 0, \forall x \in D - \{0\}$, according to Lyapunov Theorem. It is trivial to show that the function $V(0) = 0$, and that $V(x) > 0$, since the function candidate is given as quadratic. Therefore the problem is to demonstrate that the time derivative of V is **negative definite**.

In order to establish asymptotical stability, we only need to demonstrate that the stability exists for some domain $D \subset \mathbb{R}^2$. To test the function in domain D we can assume that there exists a ball such that...

$$B_r = \{x \in \mathbb{R}^2 \mid \|x\| < r\}$$

If we can show that if there exists a radius of r for which $\dot{V}(x)$ is negative definite, then we have effectively shown that the domain D exists and proven asymptotic stability via Lyapunov Theorem.

Given the condition...

$$\dot{V}(x) = -x_1^2 + x_1^2 x_2 - x_2^2 < 0$$

For the inequality to hold...

$$x_1^2 x_2 < x_1^2 + x_2^2$$

By using the definition of the ball B_r , we can show that for a given radius r , $\|x_1\| < r$ & $\|x_2\| < r$, for which we can rewrite the inequality as ...

$$r^3 < r^2 + r^2$$

After simplification we find that...

$$r < 2$$

Therefore for a given radius of $r < 2$ the system is **asymptotically stable**.

Part 1 - Global Stability

In order to demonstrate Globally Asymptotic Stability, we can use the Lyapunov Theorem for Global Stability which can be stated as follows.

$$\begin{aligned} V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0 \\ \|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty \\ \dot{V}(x) < 0, \quad \forall x \neq 0 \end{aligned} \tag{1}$$

We have shown in the previous problem that there exist conditions where \dot{V} is negative definite $\dot{V}(x) < -x_1^2 - x_2^2 + r|x_1||x_2| < 0$, when $r < 2$ and by inspection we can see that the Lyapunov candidate function is ‘radially unbounded’. By demonstrating these condition, we can use the theorem and say that the function must be globally asymptotically stable.

Part 2

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= x_1 - x_2(1 - x_1^2 - x_2^2) \end{aligned}$$

Part 2 - Asymptotic Stability

The Lyapunov function candidate and its time derivate can be given by...

$$\begin{aligned} V(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_2 - x_1[1 - x_1^2 - x_2^2]) + \dots \\ &\dots + x_2(x_1 - x_2[1 - x_1^2 - x_2^2]) \end{aligned} \tag{2}$$

When simplified this functions yield the expression....

$$\dot{V}(x) = (-x_1^2 - x_2^2) [1 - x_1^2 - x_2^2]$$

The first expression in this term is always negative definite due to the coefficients in front of x_1 and x_2 . This implies that in order for the function to be negative definite the term $1 - x_1^2 - x_2^2 > 0$ since the multiplicative negative would negate the previous term. This means that we need to show the second term is positive definite, for the complete inequality to hold.

By rearrangement...

$$x_1^2 + x_2^2 < 1 \quad \forall x \in \mathbb{R}^+$$

The only way for this to be true is if the values of x_1 and x_2 are bounded from above by $\sqrt{5}$. This can also be shown by using the definition of the function V and substituting it into the expression for $\dot{V}(x)$. Which yields.

$$\dot{V}(x) = -2 \cdot V [1 - 2 \cdot V] < 0$$

By simplification the result must be that the value of $V < \frac{1}{2}$. This is equivalent to the bound placed on x_1 and x_2 above.

Part 2 - Global Stability

By the very definition of the previous stability result, we showed that $V \not\geq \frac{1}{2}$. Since this is the requirement for obtaining the negative definite \dot{V} . Therefore the solution is **not** globally asymptotically stable, since it is not radially unbounded on \mathbb{R}^2 .

Part 3

$$\begin{aligned}\dot{x}_1 &= x_2(1 - x_2^2) \\ \dot{x}_2 &= -(x_1 + x_2)(1 - x_2^2)\end{aligned}$$

Part 3 - Asymptotic Stability

Given the quadratic Lyapunov function, and its corresponding time derivative, we obtain the following expression for the given system.

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1(x_2[1 - x_1^2]) + x_2(-(x_1 + x_2)[1 - x_1^2])\end{aligned}\tag{3}$$

This expression can be reduced to..

$$\dot{V}(x) = -x_2^2 [1 - x_1^2]$$

Since this first part of this expression is always negative definite $x \in \mathbb{R}^2$, we know that the second term must therefore be positive definite for the function to remain negative due to the multiplicative negative. This requires...

$$1 - x_1^2 > 0$$

Which further simplifies to the fact that $x_1 < 1$ for the function to remain negative definite. Under this condition, the Lyapunov function candidate follows all of the conditions for given in Lyapunov's Stability Theorem and therefore is **asymptotically stable**.

Part 3 - Global Stability

Since the time derivative of V is not negative definite for $x \in \mathbb{R}^2$, due to the restriction that $x_1 < 1$. The system is **not globally asymptotically stable**.

Part 4

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= 2x_1 - x_2^3\end{aligned}$$

Problem 4

Use $V(x) = x_1^2 + x_2^2$ to study the stability of the origin of the system.

$$\begin{aligned}\dot{x}_1 &= x_1 (k^2 - x_1^2 - x_2^2) + x_2 (x_1^2 + x_2^2 + k^2) \\ \dot{x}_2 &= -x_1 (k^2 + x_1^2 + x_2^2) + x_2 (k^2 - x_1^2 - x_2^2)\end{aligned}$$

We can compute $\dot{V}(s)$ from the given Lyapunov candidate to be ...

$$\begin{aligned}V(x) &= x_1^2 + x_2^2 \\ \dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1 (x_1 [k^2 - x_1^2 - x_2^2] + x_2 [x_1^2 + x_2^2 + k^2]) + \dots \\ &\dots + 2x_2 (-x_1 [k^2 + x_1^2 + x_2^2] + x_2 [k^2 - x_1^2 - x_2^2])\end{aligned} \tag{4}$$

We can reduce that expression as follows...

$$\dot{V}(x) = (2x_1^2 + 2x_2^2) [k^2 - x_1^2 - x_2^2]$$

The first term in this expression is clearly positive definite $\forall x \in \mathbb{R}^2$. Therefore in order to determine the stability of the system, we must look at the second term and determine its behavior for the arbitrary constant k . The two cases for k are written below.

Case: $K = 0$

In order to guarantee that $\dot{V}(x) < 0$, we must show that the term $k^2 - x_1^2 - x_2^2 \leq 0$. For the cases when $k = 0$. It is clear that the term $-x_1^2 - x_2^2$ is **must** be negative definite $\forall x \in \mathbb{R}^2$. Therefore the system is **globally asymptotically stable** when $k = 0$, due to the fact that system time derivative of the Lyapunov function is negative definite $\forall x \in \mathbb{R}^2$, according to Lyapunov Global Stability Theorem.

Case: $K \neq 0$

For the case when $x \neq 0$, we must guarantee that the term $k^2 - x_1^2 - x_2^2 \leq 0$. By substituting the expression for the candidate function into this derivative form ...

$$\begin{aligned}k^2 - x_1^2 - x_2^2 &\leq 0 \\ k^2 - (x_1^2 + x_2^2) &\leq 0 \\ k^2 - V(x) &< 0\end{aligned} \tag{5}$$

Such that...

$$V < k^2$$

This restriction shows that there is some restriction on the state space in order to maintain the inequality. This infer that the system is **asymptotically stable**, but that the system cannot be globally asymptotically stable since by this very condition there exist limits on the function where its derivative are no longer negative definite.

Problem #5

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - \text{sat}(2x_1 + x_2)\end{aligned}$$

Assuming a quadratic Lyapunov candidate function, we can show that...

$$\dot{V}(x) = 2x_1x_2 - x_2 \text{sat}(2x_1 + x_2)$$

Part A

In order to show that origin of the system is asymptotically stable, we must show that it complies with Lyapunov Stability Theorem. For this to be valid, the function $\dot{V}(x) < 0$. This is slightly more difficult to determine in this case due to the presence of the saturation function. Which is defined as ...

$$\text{sat}(y) = \begin{cases} -ka & \text{if } y < -a \\ ky & \text{if } -a \leq y \leq a \\ ka & \text{if } y > a \end{cases}$$

Where k is the value of the slope of the linear region when the input $y \in [-a, a]$. Using this definition, we can breakdown the analysis of the inequality into each segment of the piecewise saturation function and determine whether the system is stable given each segment.

Case: Linear Operation $-a \leq y \leq a$

By simplifying the expression for $\dot{V} < 0$, and plugging in the expression for the saturation function in the linear region we obtain...

$$\begin{aligned}\dot{V}(x) &= 2x_1^2x_2^2 - x_2^2 \text{sat}(2x_1 + x_2) < 0 \\ 2x_1 &< \text{sat}(2x_1 + x_2)\end{aligned}$$

Such that...

$$2x_1 < k(2x_1 + x_2)$$

The restriction imposed by this shows that so long as the following expression is valid.

$$\frac{2x_1}{2x_1 + x_2} < k$$

This means that as long as the stability of the origin is restricted by the value of the slope of the saturation function. In this case, this means that the system is **asymptotically stable**, while the condition is satisfied since this condition keeps \dot{V} as a negative definite function, as required by Lyapunov Stability Theorem.

Case: Negative Saturation $y < -a$

By applying the same logic as the previous case, with the appropriate substitution for the region of operation of the saturation function, we obtain the condition to keep $\dot{V}(x)$ negative definite is...

$$2x_1 < -ka$$

This function fundamentally restricts the value of the state space due to the fact that saturation function platues at $sat(y) = -ka, \forall y < -a$.

Case: Positive Saturation $y > a$

By applying the same logic as the previous case, with the appropriate substitution for the region of operation of the saturation function, we obtain the condition to keep $\dot{V}(x)$ negative definite is...

$$2x_1 < ka$$

This function fundamentally restricts the value of the state space due to the fact that saturation function platues at $sat(y) = ka, \forall y > a$.

Part B

As can be seen in the positive and negative operating regions of the saturation function above, the values of the state space are fundamentally limited by the constant $|x| < .5ka$. Since this condition is required for the function to be stable/asymptotically stable, we must conclude that there exists regions of the state space where the time derivative of the candidate function is not negative semi definite and thereby is not stable. This infers that trajectories starting from this region of the state space are not stable and do not converge towards the origin.

Part C

Due to the conclusions drawn above, it is plain that while there exists a set $D \subset \mathbb{R}^2$, for which the origin is stable, via the appropriate conditions of the Lyapunov candidate function, according to the Lyapunov Stability Theorem, the restriction on the stable domain of the state space means that V cannot map $\mathbb{R}^2 \rightarrow \mathbb{R}$, such that the state space is radially unbounded, for all time. Since this restriction exists the system cannot be **globally asymptotically stable**.