ECH 267 Nonlinear Control Theory
Homework #2

Jonathan Dorsey: Department of Mechanical & Aerospace Engineering

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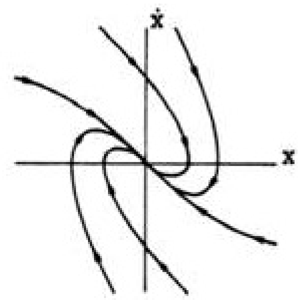
Problem 1

1. Stable Node: **Asymptotically Stable**
2. Unstable node: **Unstable**
3. Stable Focus: **Asymptotically Stable**
4. Unstable Focus: **Unstable**
5. Center: **Stable**

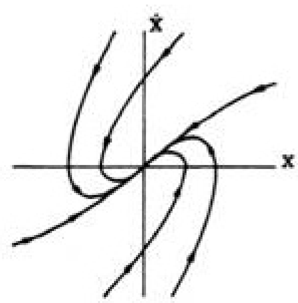
Part 1

A stable node is asymptotically stable since it all trajectories will trend to the origin (aka node) as **t** goes to ∞. While stability states that given an initial condition the trajectory of the solution will be bounded, asymptotic stability states that the trjectory will trend towards zero as time goes to ∞



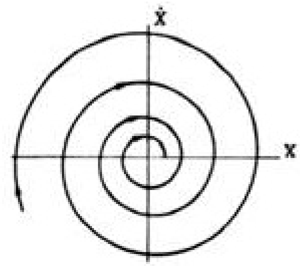
Part 2

By definition, an unstable node, is unstable. However this can be seen via it's phase portrait where any trajectory starting from the initial condition deviates away from the origin (not asymptotically stable) and further more does not stay in a bounded neighborhood (no stable). Since neither of these traits are present, the system is **unstable**.



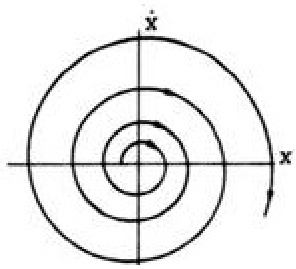
Part 3

A stable focus is asymptotically stable since it all trajectories will trend to the origin. This assertion is clear from it Phase Portrait, that any trajectory started from any initial condition will spiral into the origin. This agrees with the definition of **asymptotic stability** all trjectories will trend towards zero (the origin) as time goes to ∞



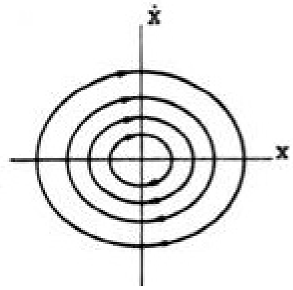
Part 4

By definition, an unstable focus is unstable. However, it is clear from the phase portrait that all trajecories from an initial condition will invariably trend away from the origin. Since the trajectories do not converge to the origin, and since the tracjectories do not stay bounded in some neighborhood they are unstable



Part 5

A center is stable since it stays we can find some neighborhood about the origin in which the trajectory of the center is orbiting. However, as is clear from the phase portrait, since the center does not converge to the origin but oscillates around it, it cannot be asymptotically stable since the trajectory of the center does not converge to the origin, but merely orbits it.


$\dot{x}$
x

Problem 2

Given the scalar system...

$$\dot{x} = ax^p + g(x)$$

Where p is a positive integer such that...

$$|g(x)| \leq k|x|^{p+1}$$

Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$

Solution:

Inorder to determine stability, we should begin by establishing a Lyapunov candidate function. Since the function is a scalar equation, we can use...

$$V(x) = \frac{1}{2} \cdot x^2$$

In order to establish stability using this Lyapunov function, we must differentiate the function with respect to x.

$$\dot{V}(x) = \frac{dV}{dx} \cdot \dot{x}$$

Using this definition $\dot{V}$ can be found by differeniating the function, with respect to x, and then substituting in the given system.

$$\frac{dV}{dx} = x$$

$$\dot{V} = x[ax^p + g(x)] \leq ax^{p+1} + k|x|^{p+2}$$

Case 1: "P" is odd & "a" less than 0

Given the inequality provided in the problem statement, we can see that for ax^p dominates the function around the origin. Due to the absolute value in the inequality for $g(x)$, any value of x put into $g(x)$, will return a positive number. When p is odd, we can see that the exponent of x is even. This means that even for negative values of x the output will be positive. Therefore since both terms of the Lyapunov function derivative yeild positive numbers, $\dot{V}(x)$ can only be **negative semi definite** or better when the constant $a < 0$.

Case 2: "P" is odd & "a" greater than 0

As with Case #1, when the p is odd. We can show that the sign of the Lyapunov derivative is negative **only** when the constant $a < 0$. In this situation, where $a > 0$, that means that $\dot{V}(x)$ is not negative semi definite or better and therefore must be unstable.

Case 3: "P" is even & "a" is not equal to 0

In the even that p is an even number, due to the formulation of $\dot{V}(x)$ we can see that the dominate part of the function ax^{p+1} will be an exponent to an odd power because of the plus one. Since odd functions retain the sign of their argument, there will always be some region of this function (under these conditions) where the function is not negative semi definite or better therefore must be unstable. This result holds since even though the constant a may change the sign of the function, the resultant output will carry the sign of its argument. Since the condition for stability does not hold all inputs in the statespace, it must be unstable. Naturally the case when $a = 0$ is trivial since the resulting expression is valued at zero.

Problem 3

Given the following systems, use a quadratic Lyapunov function candidate to show that the orgin isymptotically stable. Then investigate whether the origin is globally asymptotically stable

Part 1

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2$$

Part 2

$$\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

Part 3

$$\dot{x}_1 = x_2(1 - x_2^2)$$
$$\dot{x}_2 = -(x_1 + x_2)(1 - x_2^2)$$

Part 4

$$\dot{x}_1 = -x_1 - x_2$$
$$\dot{x}_2 = 2x_1 - x_2^3$$

1 Problem 4

Use $V(x) = x_1^2 + x_2^2$ to study the stability of the origin of the system.

$$\begin{aligned}
\dot{x}_1 &= x_1 \left(k^2 - x_1^2 - x_2^2\right) + x_2 \left(x_1^2 + x_2^2 + k^2\right) \\
\dot{x}_2 &= -x_1 \left(k^2 + x_1^2 + x_2^2\right) + x_2 \left(k^2 - x_1^2 - x_2^2\right)
\end{aligned}$$

2 Problem #5

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 - \operatorname{sat}(2x_1 + x_2)
\end{aligned}$$

3 Problem 6

Part 1

$$\dot{x}_1 = x_1^3 + x_1^2 x_2$$
$$\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

Part 2

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1^6 - x_2^3$$

4 Problem 7

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= (x_1 + x_2)\sin x_1 - 3x_2
\end{aligned}$$

5 Problem 8

6 Problem 9

$$\dot{z} = \hat{f}(z), \qquad \text{where } \hat{f}(z) = \left.\frac{\partial T}{\partial x} f(x)\right|_{x=T^{-1}(z)}$$

7 Problem 10

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \tfrac{1}{3}x_1^3 - x_2
\end{aligned}
$$