ECH 267 Nonlinear Control Theory
Homework #3

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Github Repo Hosted at:

https://github.com/JonnyD1117/ECH-267-Adv.-Proc.-Control

Problem 1.1

The first task is to check whether the **unforced** system is stable.

$$\dot{x} = -x^3$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = -x^4$$

Obviously, the candidate function is positive definite and radially unbounded while the Lyapunov function is negative definite. Therefore the unforced system is **Globally Asymptotically Stable**.
Now we can look at the forced behavior of the system given by

$$\dot{x} = -(1+u)x^3$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = -(1+u)x^4$$

It is clear from observation that, the system is stable **only** so long as the inputs to the system are bounded above $u \geq -1$. When this bounded is obvserved the system is **locally ISS**.

Problem 1.2

The first task is to check whether the **unforced** system is stable.

$$\dot{x} = -(1)x^3 - x^5$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = -x^4 - x^6$$

Obviously, the candidate function is positive definite and radially unbounded while the Lyapunov function is negative definite. Therefore the unforced system is **Globally Asymptotically Stable**.
Now we can look at the forced behavior of the system given by

$$\dot{x} = -(1+u)x^3 - x^5$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = x(-(1+u)x^3 - x^5)$$

$$\dot{V}(x) = -(1+u)x^4 - x^6$$

Which expands to...

$$\dot{V}(x) = -x^4 - ux^4 - x^6$$

Using this expression we want to show that the conditions under which we can find the bounds of the inputs such that the system is asymptotically stable. To show this we want to show that $\dot{V}(x)$ will remain negative definite. We can determine the conditions this will occur if we upper bound the Lyapunov function by$-x^4$ and determine the bounds on the input such that the inequality holds.

$$\dot{V}(x) = -(1+u)x^4 - x^6 \leq 0$$

It is clear from this expression that the system will only be negative definite (hence G.A.S.) if the term $-(1+u)x^4$ never becomes zero or positive. This is achieved when...

$$u > -1$$

As long as this bound on the input is obeyed, the forced system is stable we can show from this result by Theorem 4.19 that the system is actually **ISS**.

Problem 1.3

The first task is to check whether the **unforced** system is stable.

$$\dot{x} = -x$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = -x^2$$

Obviously, the candidate function is positive definite and radially unbounded while the Lyapunov function is negative definite. Therefore the unforced system is **Globally Asymptotically Stable**.
Now we can look at the forced behavior of the system given by

$$\dot{x} = -x + x^2 u$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = x(-x + x^2 u)$$

This expands to ..

$$\dot{V}(x) = -x^2 + x^3 u$$

To determine the bound such that the input to the system never causes the system to become unstable we use $\theta \in (0, 1)$ to parameterize the system as follows.

$$\dot{V}(x) = -(1 - \theta)x^2 - \theta x^2 + x^3 u$$

Since we want this function to be negative definite, we want to ensure that $\dot{V}(x) \leq -(1 - \theta)x^2$. Which enables us to write the expression as follows...

$$\dot{V}(x) = -(1 - \theta)x^2 - \theta x^2 + x^3 u \leq -(1 - \theta)x^2$$

This expression simplifies to...

$$\dot{V}(x) = -\theta x^2 + x^3 u \leq 0$$

By using this expression we can detemine the bounds for which the inputs of the system will keep the system asymptotically stable.

$$-\theta + ux \leq 0$$

$$ux \leq 0$$

$$|x| \leq \frac{\theta}{|u|}$$

Under this condition the system is globally asymptotically stable, and therefore we can show via Theorem 4.19 that the system is **<u>ISS</u>**.

Problem 1.4

The first task is to check whether the **<u>unforced</u>** system is stable.

$$\dot{x} = x - x^3$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = x^2 - x^4$$

Clearly, the candidate function the system is **<u>not globally stable</u>**, about the origin, since we can find a value for x^2 and $-x^4$, such that there sum is zero for non-zero values of x. This means that the Lyapunov function is only Negative Semi-definite, at best. Since ISS implies global asymptotically stability, we obviously can conclude, that since the unforced system is not G.A.S that the system is not ISS.

Problem #2.1

The first task is to check whether the **unforced** system is stable.

$$\dot{x}_1 = -x_1 + x_1^2 x_2$$
$$\dot{x}_2 = -x_1^3 - x_2 + u$$

Where $V(x) = .5\,\|x\|_2^2$, such that $\dot{V} = x_1\dot{x_1} + x_2\dot{x_2}$. By plugging in the state equation we get...

$$\dot{V}(x) = -x_1(-x_1 + x_1^2 x_2) + x_2(-x_1^3 - x_2 + u)$$

$$\dot{V}(x) = -x_1^2 - x_2^2 + x_2 u$$

For the unforced system wher $u \equiv 0$, it is obvious that the candidate function is positive definite and radially unbounded while the Lyapunov function is negative definite. Therefore the origin of the unforced system is **Globally Asymptotically Stable**.

When the input is not identically zero, we can show that ...

$$\dot{V}(x) = -\|x\|_2^2 + \|x_2\|\,|u|$$

We can show with the following parameterization that that this system is stable under the condition the following condition

$$-(1-\theta)\,\|x\|_2^2 - \theta\,\|x\|_2^2 + \|x_2\|\,|u|$$

We can show that this function can be upper bounded by ...

$$-(1-\theta)\,\|x\|_2^2 - \theta\,\|x\|_2^2 + \|x\|_2\,|u|$$

We can now detemine the condition such that the positive term in the previous function will always remain less than or equal to zero.

$$-(1-\theta)\,\|x\|_2^2 - \theta\,\|x\|_2^2 + \|x\|_2\,|u| \leq -(1-\theta)\,\|x\|_2^2$$

After simplication we can conclude that...

$$\forall\,\|x\| \geq \frac{|u|}{\theta}$$

From which we can infer that the system is **locally ISS**; however in order to determine the nature of globally ISS, we need to show that, $f \in C_1$, globally Lipschitz in (x, u), uniformly in $t, x = 0$ is G.E.S. for $u \equiv 0 \rightarrow \dot{x} = f(t, x, u)$

Given this definition we can show using Theorem 4.10 that the system is actually G.E.S. when the input $u \equiv 0$, since we can show that we can bound $V(x)$ using K_1 and K_2, and since we can further show that we can bound $\dot{V}(x)$ with a constant K_3 with a given constant $a = 2$. Therefore we can conlcude that the system is **globally ISS**.

Problem 2.2

The first task is to check whether the **unforced** system is stable.

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_1^3 - x_2 + u$$

Where $V(x) = .5x^2$, such that $\dot{V} = x\dot{x}$. By plugging in the state equation we get...

$$\dot{V}(x) = -x_1^2 + x_1 x_2 - x_1^3 x_2 - x_2^2$$

Clearly, the candidate function the system is **not globally stable**, about the origin, since we can find a value x_1 and x_2 such that previous equations results in zero for non-zero values of x. This means that the Lyapunov function is only Negative Semi-definite, at best. Since ISS implies global asymptotically stability, we obviously can conclude, that since the unforced system is not G.A.S that the system is not ISS.

Problem 2.3

The first task is to check whether the **unforced** system is stable.

$$\dot{x}_1 = (x_1 - x_2 + u)\left(x_1^2 - 1\right)$$
$$\dot{x}_2 = (x_1 + x_2 + u)\left(x_1^2 - 1\right)$$

Assuming...

$$V = \frac{1}{2}\|x\|_2^2$$

We can show that ...

$$\dot{V} = x_1\left[(x_1 - x_2)\left(x_1^2 - 1\right)\right] + x_2\left[(x_1 + x_2)\left(x_1^2 - 1\right)\right] \leqslant 0$$

Which reduces to..

$$\left(x_1^2 + x_2^2\right)\left(x_1^2 - 1\right) \leq 0$$

$$x_1^4 - x_1^2 + x^2 x_2^2 - x_2^2 \leq 0$$

It is clear that this Lyapunov function is not stable at the origin; however, before we can state state this system is no globally stable, we need to show a stonger condition of instability since, lack of existence of stable Lyapunov function does not infer instability. To address this we can use La Salle's Theorem, and evaluate $\dot{V}(x) = 0$ and see if the only solution that can stay identically at the origin IS the origin.

$$x_1^4 - x_1^2 + x^2 x_2^2 - x_2^2 = 0$$

$$x_1^4 + x^2 x_2^2 = x_1^2 + x_2^2$$

By using corollary 4.2, we can show that the previous equation is valid for the origin; however, it is also valid for the case where $x_1 = x_2 = 1$. According to this corollary, we can state that since there exists a non-trivial solution to $\dot{V}(x) = 0$, the system is **NOT** locally or globally asymptotically stable. Under this conclusion, we can immediate state that the system is **not ISS** (locally or globally), since ISS infers at least uniform asymptotic stability about the origin.

Problem 3

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A\xi + Bu$$

Where $\eta \in \mathbb{R}^{n-\tau}, \xi \in \mathbb{R}^r$ for some $1 \leq r < n, (A, B)$ is controllable, and the system $\eta = f_0(\eta, \xi)$ with ξ viewed as the input, is (globally) input-to-state stable. Find a state feedback control $u = \gamma(\eta, \xi)$ that stabilizes the origin of the full system.

Solution:

In order to solve this problem, we need to analyze this casaded system and each of its component systems such that under control input $u = \gamma(\eta, \xi)$, the origin of the combined system is stablilized.

Since we are given that the first state equation is Globally IIS, we can use Lemma 4.7 to investigate the requirements for the other state equation such that the combined system is garunteed to be stable. Under this lemma if we can show that, $\dot{\xi}$ is globally uniformly asymptotically stable, then we can show this property.

However, since $\dot{\xi} = A\xi + Bu$ is (assumed) to be a linear system and is given as controllable, we can show that it must globally exponentially stable if and only if the closed loop system has negative eigenvalues. Since G.U.E.S. is a stronger condition than G.U.A.S., we can state that so long as the control input to the second state is garunteed to produce a <u>stable</u> closed-loop system, that the second state equation must be G.U.E.S. and therefore satisfies the condition for Lemma 4.7, for the casaded system.

This means that if we choose a control input $u = -K\xi$..

$$\dot{\xi} = (A - BK)\xi$$

So long as the clossed-loop system $(A - BK)$ is <u>Hurwitz</u>, we can show that the origin of the system is stabilized.