ECH 267 Nonlinear Control Theory
Final Project Report DRAFT

Jonathan Dorsey: Department of Mechanical & Aerospace Engineering

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Github Repo Hosted at:

https://github.com/JonnyD1117/ECH-267-Adv.-Proc.-Control

Denavit-Hartenberg Parameters

The Denavit-Hartenberg (DH) parameters are an important tool in analyzing the geometry of any given robot as well as in the formulation of joint transformations which enable a concise and universal means of deriving important quanties for kinematic and dynamic analysis of the robot.

Purpose and Appliction of DH Parameters

The purpose of DH parameters is to standardize the description of geometry of robots into a universal parameterization such that arbitary construction of different robots can all be described concisely in a single and intuitive notation.

The DH Parameters are defined to be...

$$\begin{aligned}
a_i &= \text{ the distance from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured along } \hat{X}_i \\
\alpha_i &= \text{ the angle from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured about } \hat{X}_i \\
d_i &= \text{ the distance from } \hat{X}_{i-1} \text{ to } \hat{X}_i \text{ measured along } \hat{Z}_i; \text{ and} \\
\theta_i &= \text{ the angle from } \hat{X}_{i-1} \text{ to } \hat{X}_i \text{ measured about } \hat{Z}_i
\end{aligned}$$

Where α_i is the angle of twist between axis of actuation from one link to the next, a_i is the offset distance measured along the x axis required to locate the frame of the next link, d_i is the distance along the axis of actuation from which is required to locate the frame of the next linbk, and finally, θ_i is the angle about the axis of actuation required to locate the frame of the next link.

It is important to note that these parameters need not be constant and are allowed to be degrees of freedom of the system such that the robot can change its geometry. Often only one of these parameters will be variable while the other three remain constants. This occurs due to the constraints acting on each joint frame, of the robot.

Rules for Obtaining DH Parameterization

While DH Parameterization is by no means the only parameterization possible, its universal use and acceptance in the robotics community means that one is more likely to see it in the literature. However there are rules which this parameterization requires in order to be conistent and valid. These are...

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i + 1$)

2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the i th axis, assign the link-frame origin.

3. Assign the $\hat{Z}_i$ axis pointing along the i th joint axis.

4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.

5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.

6. Assign $\{0\}$ to match $\{1\}$ when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

PUMA 560 Robot DH Parameter

For the PUMA 560 robot being used in this project, we can show that the DH parameters are shown as follows. Note that the angles θ_i are not included as they are degree of freedom that we can control arbitarily. This is because the system contains only 3 revolute joints of actuation and not prismatic members.

α_{i-1}	A_{i-1}	D_i
0	0	0
-90	0	243.5
0	431.8	-93.4
90	-20.3	433.1
-90	0	0
90	0	0

These parameter values have previously been determined by B. Armstrong, by using the prescribed rules and the geometry of the PUMA 560 robot.

Homogeneous Transform

The homogeneous transformation is a construct in robotics which enables both the rototation and position of any vector within a given coordinate frame to be viewed with respect to another frame. This construction provides a compact matrix notation for changing coordinate frames of reference during computation.

Rotation Matrices

Before discussing more general transformation, it is important to understand how one can translate between coordinate frames. To accomplish this task we use rotation matrices.

As mentioned previously, a rotation matrix is a transformation which rotates a vector described in one coordinate sytem into components of another coordinate system. These are extremely useful for rigid body dynamics in general but they are required for the study of robotics as the number and orientation of joint reference frames becomes too cumbersum to derive geometrically for each frame of the system.

From the following definitions of rotations about the x, y, and z axis of 3D coordinate system, can chain together the rotations necessary for a system of coordinate frames such that we can describe any single frame from any other frame mrely by multiplying the seperate rotations together, as desired.

$$\mathbf{R}_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General Homogeneous Transformation Definition

The general homogeneous transformation is equivalent to performing the following operation.

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

However, in order to create a general matrix transformation from this vector expression we can form the following matrix by the addition of the second equation such that $1 = 1$. While this is a trivial operation, this enables us to use the matrix as a general transform on position vector, given only the rotation matrix from one frame to another and the position vector from the origin of the given frame as viewed from the other.

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{Borg} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

This expression can be written as shown below, in a very abreviated and compact form, while clearly conveying the operation being done by this generation transformation.

$$P = {}^{A}_{B}T^{B}P$$

This form is known as the homogeneous transform.

DH Parameter Based Transformations

Since the DH Parameters provide a universal notation for describing the the position and orientation of a robot, it is natural to want to express the homogeneous transformation of each coordinate frame using this parameterization. In fact, we can chain a series of these transformation together, based on the DH table previously shown will will allow us to utilize the general nature of the homogeneous transformation and account for all of the twist, offsets, and linear and angular discplacements required to view a joint frame from another (usually the reference frame).

For a single transformation from a frame $\{i\}$ to frame $\{i-1\}$, we can use the homogeneous transformation for explicit rotations about a joint axis or specific translations along a joint axis as dictated by the DH parameters.The matrices R and D represent the homogeneous transformation for rotation and translations repsectively, with the subscript of each providing the axis upon which the operation should be performed.

$${}^{i-1}_{i}T = R_X\left(\alpha_{i-1}\right) D_X\left(a_{i-1}\right) R_Z\left(\theta_i\right) D_Z\left(d_i\right)$$

By using the universal, DH paramterization with homogeneous transformation we can now describe the relationships to or from any joint of the robot to any other joint of the robot. This is a powerful concept which facilitates the analysis of any robot whose joints are based on revolute or prismatic members.

Dynamics Model of RRR-Robot Manipulator

Lagrange-Euler Equations of Motion

SymPy-Symbolic Math Library

EOM Derivation

Equation 1

$$
\begin{aligned}
&1.0 I_1 \frac{d^2}{dt^2} \mathrm{q}_1(t) + 0.25 L_2^2 m_2 \frac{d^2}{dt^2} \mathrm{q}_1(t) + 1.0 L_2^2 m_3 \frac{d^2}{dt^2} \mathrm{q}_1(t) \\
&\quad + 0.5 L_2 L_3 m_3 \sin(\mathrm{q}_1(t) - \mathrm{q}_2(t)) \left(\frac{d}{dt} \mathrm{q}_2(t)\right)^2 \\
&\quad + 0.5 L_2 L_3 m_3 \cos(\mathrm{q}_1(t) - \mathrm{q}_2(t)) \frac{d^2}{dt^2} \mathrm{q}_2(t) \\
&\quad + 0.5 L_2 g m_2 \cos(\mathrm{q}_1(t)) + 1.0 L_2 g m_3 \cos(\mathrm{q}_1(t))
\end{aligned} \tag{1}
$$

Equation 2

$$
\begin{aligned}
&1.0 I_2 \frac{d^2}{dt^2} \mathrm{q}_2(t) - 0.5 L_2 L_3 m_3 \sin(\mathrm{q}_1(t) - \mathrm{q}_2(t)) \left(\frac{d}{dt} \mathrm{q}_1(t)\right)^2 \\
&\quad + 0.5 L_2 L_3 m_3 \cos(\mathrm{q}_1(t) - \mathrm{q}_2(t)) \frac{d^2}{dt^2} \mathrm{q}_1(t) \\
&\quad + 0.25 L_3^2 m_3 \frac{d^2}{dt^2} \mathrm{q}_2(t) + 0.5 L_3 g m_3 \cos(\mathrm{q}_2(t))
\end{aligned} \tag{2}
$$

Equation 3

$$
T_3 = 1.0 I_3 \frac{d^2}{dt^2} \mathrm{q}_3(t) \tag{3}
$$

RRR Forward Dynamics Matrices

$$
\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + K(\theta)
$$

$$
M(\theta) = \begin{bmatrix} I_1 + 0.25 L_2^2 m_2 + L_2^2 m_3 & 0.5 L_2 L_3 m_3 \cos(\mathrm{q}_1(t) - \mathrm{q}_2(t)) & 0 \\ 0.5 L_2 L_3 m_3 \cos(\mathrm{q}_1(t) - \mathrm{q}_2(t)) & I_2 + 0.25 L_3^2 m_3 & 0 \\ 0 & 0 & I_3 \end{bmatrix}
$$

$$
V(\theta, \dot{\theta}) = \begin{bmatrix} 0.5 L_2 L_3 m_3 \sin(\mathrm{q}_1(t) - \mathrm{q}_2(t)) \left(\frac{d}{dt} \mathrm{q}_2(t)\right)^2 \\ -0.5 L_2 L_3 m_3 \sin(\mathrm{q}_1(t) - \mathrm{q}_2(t)) \left(\frac{d}{dt} \mathrm{q}_1(t)\right)^2 \\ 0 \end{bmatrix}
$$

$$K(\theta) = \begin{bmatrix} 0.5L_2 g m_2 \cos\left(\mathrm{q}_1(t)\right) + L_2 g m_3 \cos\left(\mathrm{q}_1(t)\right) \\ 0.5L_3 m_3 \cos\left(\mathrm{q}_2(t)\right) \\ 0 \end{bmatrix}$$

Inverse Dynamics

$$\ddot{\theta} = M^{-1}(\theta)[\tau - V(\theta, \dot{\theta}) - K(\theta)]$$