ECH 267 Nonlinear Control Theory
Homework #2

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Github Repo Hosted at:

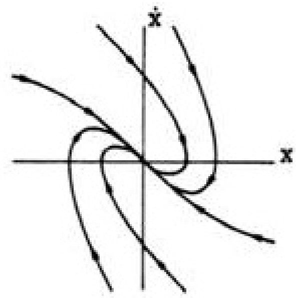
https://github.com/JonnyD1117/ECH-267-Adv.-Proc.-Control

Problem 1

1. Stable Node: **Asymptotically Stable**
2. Unstable node: **Unstable**
3. Stable Focus: **Asymptotically Stable**
4. Unstable Focus: **Unstable**
5. Center: **Stable**

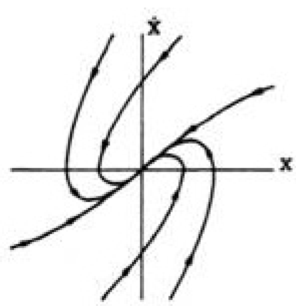
Part 1

A stable node is asymptotically stable since it all trajectories will trend to the origin (aka node) as **t** goes to ∞. While stability states that given an initial condition the trajectory of the solution will be bounded, asymptotic stability states that the trjectory will trend towards zero as time goes to ∞



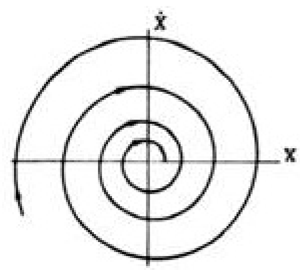
Part 2

By definition, an unstable node, is unstable. However this can be seen via it's phase portrait where any trajectory starting from the initial condition deviates away from the origin (not asymptotically stable) and further more does not stay in a bounded neighborhood (no stable). Since neither of these traits are present, the system is **unstable**.



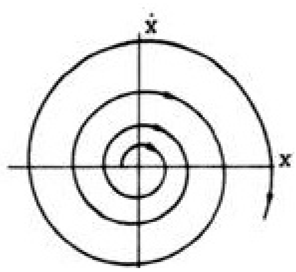
Part 3

A stable focus is asymptotically stable since it all trajectories will trend to the origin. This assertion is clear from it Phase Portrait, that any trajectory started from any initial condition will spiral into the origin. This agrees with the definition of **asymptotic stability** all trjectories will trend towards zero (the origin) as time goes to ∞



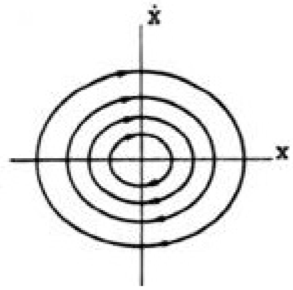
Part 4

By definition, an unstable focus is unstable. However, it is clear from the phase portrait that all trajecories from an initial condition will invariably trend away from the origin. Since the trajectories do not converge to the origin, and since the tracjectories do not stay bounded in some neighborhood they are unstable



Part 5

A center is stable since it stays we can find some neighborhood about the origin in which the trajectory of the center is orbiting. However, as is clear from the phase portrait, since the center does not converge to the origin but oscillates around it, it cannot be asymptotically stable since the trajectory of the center does not converge to the origin, but merely orbits it.


$\dot{x}$
x

Problem 2

Given the scalar system...

$$\dot{x} = ax^p + g(x)$$

Where p is a positive integer such that...

$$|g(x)| \leq k|x|^{p+1}$$

Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$

Solution:

Inorder to determine stability, we should begin by establishing a Lyapunov candidate function. Since the function is a scalar equation, we can use...

$$V(x) = \frac{1}{2} \cdot x^2$$

In order to establish stability using this Lyapunov function, we must differentiate the function with respect to x.

$$\dot{V}(x) = \frac{dV}{dx} \cdot \dot{x}$$

Using this definition $\dot{V}$ can be found by differeniating the function, with respect to x, and then substituting in the given system.

$$\frac{dV}{dx} = x$$

$$\dot{V} = x[ax^p + g(x)] \leq ax^{p+1} + k|x|^{p+2}$$

Case 1: "P" is odd & "a" less than 0

Given the inequality provided in the problem statement, we can see that for ax^p dominates the function around the origin. Due to the absolute value in the inequality for $g(x)$, any value of x put into $g(x)$, will return a positive number. When p is odd, we can see that the exponent of x is even. This means that even for negative values of x the output will be positive. Therefore since both terms of the Lyapunov function derivative yeild positive numbers, $\dot{V}(x)$ can only be **negative semi definite** or better when the constant $a < 0$.

Case 2: "P" is odd & "a" greater than 0

As with Case #1, when the p is odd. We can show that the sign of the Lyapunov derivative is negative **only** when the constant $a < 0$. In this situation, where $a > 0$, that means that $\dot{V}(x)$ is not negative semi definite or better and therefore must be unstable.

Case 3: "P" is even & "a" is not equal to 0

In the even that p is an even number, due to the formulation of $\dot{V}(x)$ we can see that the dominate part of the function ax^{p+1} will be an exponent to an odd power because of the plus one. Since odd functions retain the sign of their argument, there will always be some region of this function (under these conditions) where the function is not negative semi definite or better therefore must be unstable. This result holds since even though the constant a may change the sign of the function, the resultant output will carry the sign of its argument. Since the condition for stability does not hold all inputs in the statespace, it must be unstable. Naturally the case when $a = 0$ is trivial since the resulting expression is valued at zero.

Problem 3

Given the following systems, use a quadratic Lyapunov function candidate to show that the orgin isymptotically stable. Then investigate whether the origin is globally asymptotically stable

Part 1

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = -x_2$$

Part 1 - Asymptotic Stability

Since the first section of this question is asking to verify that the equilibrium point is asymptotically stable, it will suffice to find the local domain D where the conditions $V(0) = 0$ and $\dot{V}(x) < 0 \ \forall x \in D - \{0\}$.

Assuming a quadratic function, we can compute the Lyapunov Function as...

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$
$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2$$

such that...

$$\dot{V}(x) = -x_1^2 + x_1^2 x_2 - x_2^2$$

To show that function, $V(x)$ is asymptotically stable, we can show that (1) $V(0) = 0$ and that (2) $\dot{V}(x) < 0, \ \forall x \in D - \{0\}$, according to Lyapunov Theorem. It is trivial to show that the function $V(0) = 0$, and that $V(x) > 0$, since the function candidate is given as quadratic. Therefore the problem is to demonstrate that the time derivative of V is **negative definite**.

In order to establish asymptotical stability, we only need to demonstrate that the stability exists for some domain $D \subset \mathbb{R}^2$. To test the function in domain D we can assume that there exists a ball such that...

$$B_r = \{x \in \mathbb{R}^2 | \, \|x\| < r\}$$

If we can show that if there exists a radius of r for which $\dot{V}(x)$ is negative definite, then have have effectively shown that the domain D exists and proven asymptotic stability via Lyapunov Theorem.

Given the condition...

$$\dot{V}(x) = -x_1^2 + x_1^2 x_2 - x_2^2 < 0$$

For the inequality to hold...

$$x_1^2 x_2 < x_1^2 + x_2^2$$

By using the definition of the ball B_r, we can show that for a given radius r, $\|x_1\| < r$ & $\|x_2\| < r$, for which we can rewrite the inequality as ...

$$r^3 < r^2 + r^2$$

After simplification we find that...

$$r < 2$$

Therefore for a given radius of $r < 2$ the system is **asymptotically stable**.

Part 1 - Global Stability

In order to demonstrate Globally Asymptotic Stability, we can use the Lyapunov Theorem for Global Stability which can be stated as follows.

$$\begin{aligned} V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0 \\ \|x\| \to \infty \Rightarrow V(x) \to \infty \\ \dot{V}(x) < 0, \quad \forall x \neq 0 \end{aligned} \tag{1}$$

We have shown in the previous problem that there exist conditions where $\dot{V}$ is negative definite $\dot{V}(x) < -x_1^2 - x_2^2 + r|x_1||x_2| < 0$, when $r < 2$ and by inspection we can see that the Lyapunov candiate function is 'radially unbounded'. By demonstrating these condition, we can use the theorem and say that the function must be globally asymptotically stable.

Part 2

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= x_1 - x_2(1 - x_1^2 - x_2^2) \end{aligned}$$

Part 2 - Asymptotic Stability

The Lyapunov function candidate and its time derivate can be given by...

$$\begin{aligned} &V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 \\ &\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\left(-x_2 - x_1\left[1 - x_1^2 - x_2^2\right]\right) + \ldots \\ &\ldots + x_2\left(x_1 - x_2\left[1 - x_1^2 - x_2^2\right]\right) \end{aligned} \tag{2}$$

When simplifed this functions yield the expression....

$$\dot{V}(x) = \left(-x_1^2 - x_2^2\right)\left[1 - x_1^2 - x_2^2\right]$$

The first expression in this term is always negative definite due to the coefficients in front of x_1 and x_2. This implies that in order for the function to be negative definite the term $1 - x_1^2 - x_2^2 > 0$ since the multiplicitive negative would negate the previous term. This means that we need to show the second term is positive definite, for the complete inequality to hold.
By rearrangement...

$$x_1^2 + x_2^2 < 1 \;\; \forall x \in \mathbb{R}^+$$

The only way for this to be true is if the values of x_1 and x_2 are bounded from above by $\sqrt{.5}$. This can also be shown by using the definition of the function V and substituting it into the expression for $\dot{V}(x)$. Which yields.

$$\dot{V}(x) = -2 \cdot V\left[1 - 2 \cdot V\right] < 0$$

By simplification the result must be that the value of $V < \frac{1}{2}$. This is equivalent to the bound placed on x_1 and x_2 above.

Part 2 - Global Stability

By the very definition of the previous stability result, we showed that $V \ngeq \frac{1}{2}$. Since this is the requirement for obtaining the negative definite $\dot{V}$. Therefore the solution is **not** globally asymptotically stable, since it is not radially unbounded on $\mathbb{R}^2$.

Part 3

$$\begin{aligned} \dot{x}_1 &= x_2(1 - x_2^2) \\ \dot{x}_2 &= -(x_1 + x_2)(1 - x_2^2) \end{aligned}$$

Part 3 - Asymptotic Stability

Given the quadratic Lyapunov function, and its corresponding time derivative, we obtain the following expression for the given system.

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1\left(x_2\left[1 - x_1^2\right]\right) + x_2\left(-\left(x_1 + x_2\right)\left[1 - x_1^2\right]\right) \end{aligned} \tag{3}$$

This expression can be reduced to..

$$V\dot{(x)} = -x_2^2\left[1 - x_1^2\right]$$

Since this first part of this expression is always negative definite $x \in \mathbb{R}^2$, we know that the second term must therefore be positive definite for the function to remain negative due to the multiplicitive negative. This requires...

$$1 - x_1^2 > 0$$

Which futher simplies to the fact that $x_1 < 1$ for the function to remain negative definite. Under this condition, the Lyapunov function candiate follows all of the conditions for given in Lyapunov's Stability Theorem and therefore is **asymptotically stable**.

Part 3 - Global Stability

Since the time derivate of V is not negative definite for $x \in \mathbb{R}^2$, due to the restriction that $x_1 < 1$. The system is **not globally asymptotically stable**.

Part 4

$$\begin{aligned}
\dot{x}_1 &= -x_1 - x_2 \\
\dot{x}_2 &= 2x_1 - x_2^3
\end{aligned}$$

By using the following Lyapunov Candidate function...

$$V(x) = x_1^2 + \frac{1}{2}x_2^2$$

We can show that its time derivative will be ...

$$\begin{aligned}
\dot{V} &= 2x_1\dot{x}_1 + x_2\dot{x}_2 \\
\dot{V} &= 2x_1\left(-x_1 - x_2\right) + x_2\left(2x_1 - x_2^3\right)
\end{aligned}$$

After simplication the Lyapunov Function is shown to be...

$$\dot{V} = -2x_1^2 - x_1^4$$

Since this function is negative definite such that $\forall x \in D - \{0\} | \dot{V} < 0$ we can conclude (via Lyapunov Stability Theorem) that the origin is **asymptotically stable** However, we can go beyond this conclusion, since this previous statement is valid for when $\forall x \in \mathbb{R}^2$, and since the quadratic candidate function is radially unbounded with respect to x, we can conlude (via Lyapunov Global Stability Theorem) that the system is **Globally Asymptotically Stable** aout the origin.

Problem 4

Use $V(x) = x_1^2 + x_2^2$ to study the stability of the origin of the system.

$$\begin{aligned}
\dot{x}_1 &= x_1 \left(k^2 - x_1^2 - x_2^2\right) + x_2 \left(x_1^2 + x_2^2 + k^2\right) \\
\dot{x}_2 &= -x_1 \left(k^2 + x_1^2 + x_2^2\right) + x_2 \left(k^2 - x_1^2 - x_2^2\right)
\end{aligned}$$

We can compute $\dot{V}(s)$ from the given Lyapunov candidate to be ...

$$\begin{aligned}
V(x) &= x_1^2 + x_2^2 \\
\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\
&= 2x_1 \left(x_1 \left[k^2 - x_1^2 - x_2^2\right] + x_2 \left[x_1^2 + x_2^2 + k^2\right]\right) + \dots \\
\dots &+ 2x_2 \left(-x_1 \left[k^2 + x_1^2 + x_2^2\right] + x_2 \left[k - x_1^2 - x_2\right]\right)
\end{aligned} \tag{4}$$

We can reduce that expression as follows...

$$\dot{V}(x) = \left(2x_1^2 + 2x_2^2\right) \left[k^2 - x_1^2 - x_2^2\right]$$

The first term in this expression is clearly positive definite $\forall x \in \mathbb{R}^2$. Therefore in order to determine the stability of the system, we must look at the second term and determine its behavior for the arbitrary constant k. The two cases for k are written below.

Case: $K = 0$

In order to garuntee that $\dot{V}(x) < 0$, we must show that the term $k^2 - x_1^2 - x_2^2 \leq 0$. For the cases when $k = 0$. It is clear that the term $-x_1^2 - x_2^2$ is **must** be negative definite $\forall x \in \mathbb{R}^2$. Therefore the system is **globally asymptotically stable** when $k = 0$, due to the fact that system time derivative of the Lyapunov function is negative definite $\forall x \in \mathbb{R}^2$, according to Lyapunov Global Stability Theorem.

Case: $K \neq 0$

For the case when $x \neq 0$, we must garuntee that the term $k^2 - x_1^2 - x_2^2 \leq 0$. By substituting the expression for the candidate function into this derivate form ...

$$\begin{aligned}
&k^2 - x_1^2 - x_2^2 \leq 0 \\
&k^2 - (x_1^2 + x_2^2) \leq 0 \\
&k^2 - V(x) < 0
\end{aligned} \tag{5}$$

Such that...

$$V < k^2$$

This restrictions shows that there is some restriction on the state space in order to maintain the inequality. This infer that the system is **asymptotically**

stable, but that the system cannot be globally asymptotically stable since by this very condition there exist limits on the function where its derivative are no longer negative definite.

Problem #5

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - \text{sat}\,(2x_1 + x_2)$$

Assuming a quadratic Lyapunov candidate function, we can show that...

$$\dot{V}(x) = 2x_1x_2 - x_2\,\text{sat}\,(2x_1 + x_2)$$

Part A

In order to show that origin of the system is asymptotically stable, we must show that it complies with Lyapunov Stability Theorem. For this to be valid, the function $\dot{V}(x) < 0$. This is slightly more difficult to determine in this case due to the presence of the saturation function. Which is defined as ...

$$\text{sat}(y) = \begin{cases} -ka & \text{if } y < a \\ ky & \text{if } -a \leq y \leq a \\ ka & \text{if } y > a \end{cases}$$

Where k is the value of the slope of the linear region when the input $y \in [-a, a]$. Using this definition, we can breakdown the analysis of the inequality into each segment of the piecewise saturation function and determine whether the system is stable given each segment.

Case: Linear Opertion $\quad -a \leq y \leq a$

By simplifying the expression for $\dot{V} < 0$, and plugging in the expression for the saturation function in the linear region we obtain...

$$\dot{V}(x) = 2x_1^2x_2^2 - x^2\,\text{sat}(2x_1 + x_2) < 0$$
$$2x_1 < \text{sat}(2x_1 + x_2)$$

Such that...

$$2x_1 < k\,(2x_1 + x_2)$$

The restriction imposed by this shows that so long as the following expression is valid.

$$\frac{2x_1}{2x_1 + x_2} < k$$

This means that as long as the stability of the origin is restricted by the value of the slope of the saturation function. In this case, this means that the system is **asymptotically stable**, while the condition is satisfied since this condition keeps $\dot{V}$ as a negative definite function, as required my Lyapunov Stability Theorem.

Case: Negative Saturation $y < -a$

By applying the same logic as the previous case, with the appropriate substitution for the region of operation of the saturation function, we obtain the condition to keep $\dot{V}(x)$ negative definite is...

$$2x_1 < -ka$$

This function fundamentally restricts the value of the state space due to the fact that saturation function platues at $sat(y) = -ka$, $\forall y < -a$.

Case: Positive Saturation $y > a$

By applying the same logic as the previous case, with the appropriate substitution for the region of operation of the saturation function, we obtain the condition to keep $\dot{V}(x)$ negative definite is...

$$2x_1 < ka$$

This function fundamentally restricts the value of the state space due to the fact that saturation function platues at $sat(y) = ka$, $\forall y < a$.

Part B

As can be seen in the positive and negative operating regions of the saturation function above, the values of the state space are fundamentally limited by the constant $|x| < .5ka$. Since this condition is required for the function to be stable/asymptotically stable, we must conclude that there exists regions of the state space where the time derivative of the candidate function is not negative semi definite and thereby is not stable. This infers that trajectories starting from this region of the state space are not stable and do not converge towards the origin.

Part C

Due to the conclusions drawn above, it is plain that while there exists a set $D \subset \mathbb{R}^2$, for which the origin is stable, via the appropriate conditions of the Lyapunov candidate function, according to the Lyapunov Stability Theorem, the restriction on the stable domain of the state space means that V cannot map $\mathbb{R}^2 \to \mathbb{R}$, such that the state space is radially unbounded, for all time. Since this restriction exists the system cannot be **globally asymptotically stable**.

Problem 6

Part 1

$$\dot{x}_1 = x_1^3 + x_1^2 x_2$$
$$\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

In order to show that the given system is unstable, according the Lyapunov Stability Theorem, it is sufficient to show that conditions for stability do not hold in the domain about the origin. This can be seen computing the time derivate of the Lyapunov function and substituting in the state equations.

$$V = x_1 \left(x_1^3 + x_1^2 x_2 \right) + x_2 \left(-x_2 + x_2^2 + x_1 x_2 - x_1^3 \right)$$

Given this formulation, we know that according to Lyapunov's stability theorem that the function must be negative semi definite for the system to be stable. Under this condition, the previous equations simplifies to the following expression.

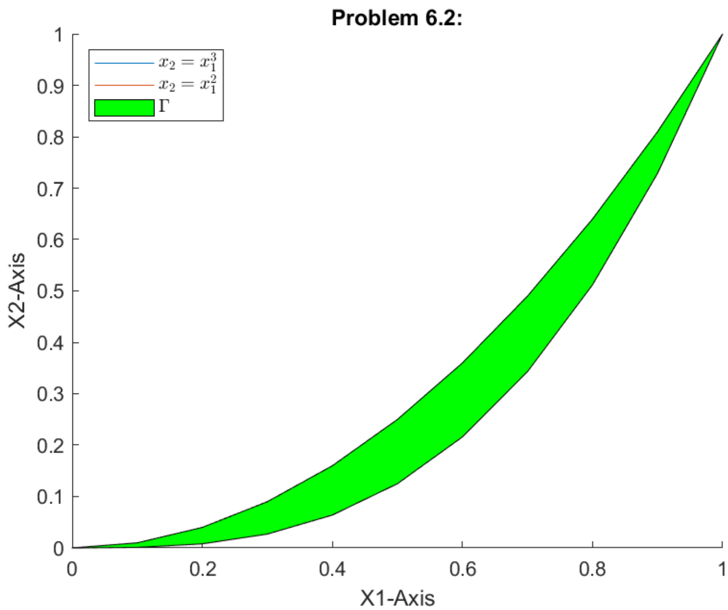
$$x_1^4 - x_2^2 + x_2^3 + x_1 x_2^2 \leq 0$$

Since it is rather difficult to analytically show the regions where this inequality holds, we can show numerically that there does not since exist a neighborhood around the origin, such that the ball $B_r = \{x \in D \subset \mathbb{R}^2 | \, \|x\| < r\}$ maintains the previous inequality. This is the case for point $x = (.5, .5)$ for $r < 1$. When evaluated at this point the inequality does not hold. Since we cannot define a ball around the origin such that the equality holds for **all** points contained inside of that ball, we must conclude that the system is unstable. In particular, I believe that the points in the 1st quadrant of the of statespace do not hold for the inequality.

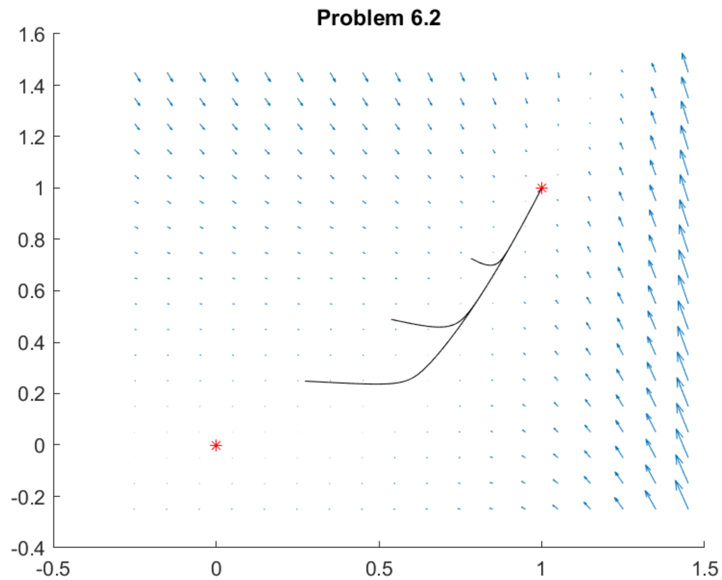
Part 2

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1^6 - x_2^3$$

In the problem statement, we are told to investigate the set defined by $\Gamma = \{0 \leq x_1 \leq 1\} \cap \left\{x_2 \geq x_1^3\right\} \cap \left\{x_2 \leq x_1^2\right\}$, which would give us insight into the stability of the system. We can graphically show this set by plotting the two curves over the valid domain shown. The area between these curves represented the graphical description of the set Γ.



By investigating the behavior of the functions at both of its borders, we can develope an intuitiion about the behavior of the system and whether the origin is stable or not. We can accomplish this by setting each state equation independently to 0 and determining the behavior of the remaining component of the trajectory; however, since we know that the set Γ is an invariant set, we know that any trajectory started inside of this set will remain in this set, can investigate the stability of the origin, but observing how trajectories behave. We can visualize this behavior numerically by looking at the vector field of the problem and noticing that even initial conditions very close to the origin, diverge away from it and converge to the point (1,1), as shown in the figure below.



However, we can show this same behavior analytically such that ...

$$\dot{x}_2 = 0 \rightarrow \dot{x}_1 > 0$$

Similarly...

$$\dot{x}_1 = 0 \rightarrow \dot{x}_2 > 0$$

This means any tracjectory starting inside of Γ, the values of x_1 and x_2 will continue to increase, unless the system is already at an equilibirum point. However, if the trajectory is not begun at an equilibrium point, then the value of the trajectory will continue to grow, until it reaches $x = 1$. This means that the origin of the system is unstable, since the only valid values of the system are greater than $x = 0$, and since the trajectory will only increase in value over time, there does not exist any point (other than $x = 0$), which will converge to the origin.

Problem 7

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = (x_1 + x_2)\sin x_1 - 3x_2$$

Part A

In order to determine whether the origin is a unique equilibrium point, we must determine all possible equilibrium points of the system by setting the state equations equal to zero, such that...

$$0 = -x_1 + x_2$$
$$0 = (x_1 + x_2)\sin x_1 - 3x_2$$

This expression simplifies to...

$$x_1 = x_2$$
$$0 = [2x_2 \sin(x_1) - 3x_2]\left(\frac{1}{x_2}\right)$$

After substituting the first state state equation in the second we get...

$$0 = 2\sin(x_1) - 3$$

Which reduces to ...

$$\frac{3}{2} = sin(x_1)$$

Solution

: In order to solve for equilibrium points (other than the origin) there must exist solutions to the previous equations. However the *Sine* function is bouded by $[-1, 1]]$. This means that it is impossible for there to exist a solution which satisfies this expression. Therefore the origin is a **unique equilibrium point**.

Part B

To linearize the system, we must determine the Jacobian of the given system about the equilibrium point...

$$A_j = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ sin(x_1) + x_1 cos(x_1 + x_2 cos(x_2)) & \sin(x) - 3 \end{bmatrix}\Bigg|_{x=0}$$

After evaluating the system at its origin, we can show that...

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$$

By taking the eigenvalues of this matrix, we can see that the matrix is Hurwitz, since it yields the following eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$ in which both exist in the left half side of the complex plane.

Part C

Problem 8

Suppose the set M in LaSalle's theorem consists of a finite number of isolated points. Show that $lim_{t\to\infty}x(t)$ exists and equals one of these points.

Solution:

In LaSalle's Theorem, the set M is the largest invariant set in the set E. Where E is the set of all points in the domain σ where $\dot{V}(x) = 0$.

We know from Lyapunov's Theorem, that $\dot{V}(x(t))$ is negative semi definite function, which means that the Lyapunov candidate function V decreases with time. Due to these properties, and since the solution $x(t)$ starts in the **invariant set** σ, the solution $x(t)$ must exist and be bounded inside this set, due to the property that all any trajectory starting at $x(0)$ in the set σ must say in σ.

According to, Lemma 4.1, the solution $x(t)$ will approach its positive limit set as $t \to \infty$. Since the limit set L^+ must comply with the Lyapunov's Theorem such that $\dot{V}(x) \leq 0$, we know can show that the largest limit set will occur at equality when when the function is not decreasing, $v(x) = 0$. Since we know from Lemma 4.1 that the solution $x(t)$ must approach its limit set L^+ as $t \to \infty$, since it is bounded, and since we defined M to be the largest invariant set of E where $v(x) = 0$ holds, then we can show that the solution $x(t)$ must converge to M as $t \to \infty$.

Problem 9

$$\dot{z} = \hat{f}(z), \quad \text{where } \hat{f}(z) = \frac{\partial T}{\partial x} f(x)\bigg|_{x=T^{-1}(z)}$$

Part A

In order to prove that $x = 0$ is an isolated equilibrium point if and only if $z = 0$, it is necessary to show that if we assume multiple equilibrium we will encounter a contradiction.

Given that $x = 0$ is an <u>isolated</u> equilibrium point, we will assume that it is not. This means that we can find some value of z such that $f(z) = 0$. According to the transformation provided, we can state that for an function of z there must exist a seperate function of x that are equivalent.

Therefore given.. $f(z) = \frac{\partial T}{\partial x} f(x)$, we can show via the assumed properties that the inverse of this function is $f(x) = [\frac{\partial T}{\partial x}]^{-1} \hat{f}(z)$. If $z \neq 0$ is an equilibrium point of the system then it follows that...

$$f(x) = [\frac{\partial T}{\partial x}]^{-1} \hat{f}(z) = 0$$

This implies that there is an equivalent x that is also an equilibrium point of the system. By using the continuity of $T(\cdot)^{-1}$, we can show that we can arbitrarily find x values close to the origin that satisfy the problem, since this behavior contradicts the fact that the system is actually an isolated equilibrium, it must **not** be true that arbitrary values of z can yield arbitrary values of x that are equilibrium points. This implies that $x = 0$ must only be an isolated equilibrium point when $z = 0$ is an isolated equilibrium point.

Part B

We can show that $x = 0$ is asymptotically stable, stable, or unstable using the same proof. For the case that the equilibrium point is stable, we know that inorder for that statement to hold, there must exist an $\epsilon > 0$ and a δ such that the following is true...

$$\|x(0)\| < \delta \rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

Due to the continuity of the transformation $T(\cdot)$, it holds that there exists..

$$\|x\| < r \Rightarrow \|z\| < \gamma$$

Likewise, we can show that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \Rightarrow \|z(t)\| < \gamma, \quad \forall t \geq 0$$

This demonstrates that there exists parameters for the original and transformed system such that their norms are bounded as expected. This allows us to translate from the norm of one coordinate system to the norm of the other coordinate system by using this bounds.

In the similar manner, by the continuity of the inverse transformation T^{-1} we can show that there exists an $\eta > 0$ such that

$$\|z\| < \eta \Rightarrow \|x\| < \delta$$

This statement states that there is an equivalent inverse property such that the bound of the transformed system implies the original bound on the original system. By compiling all of these into a single equivalent statement we can write that

$$\|z(0)\| < \eta \Rightarrow \|x(0)\| < \delta \Rightarrow \|z(t)\| < \gamma, \quad \forall t \geq 0$$

In order to account for the case where the system is asymptotically stable, we apply the same chain of logic used above, to demonstrate that the $x(t) \rightarrow$ as $t \rightarrow \infty$ is equivalent to the transformed system such that $z(t) \rightarrow$ as $t \rightarrow \infty$

Problem 10

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$

In order to determine the equilibrium points of the system we must set the state equations equal and solve.

$$0 = x_2$$
$$0 = -x_1 + \frac{1}{3}x_1^3 - x_2$$

Therefore we can show that since $x_2 = 0$, we can solve for the remaining values of x_1, such that..

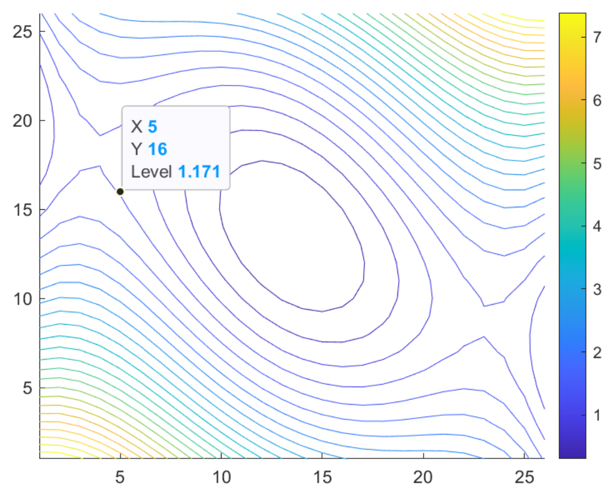
$$x_1^2 = 3$$

This yields the following solutions.

1. **Eq. Pt. #1:** $x_1 = x_2 = 0$
2. **Eq. Pt. #2:** $x_2 = 0$ and $x_1 = \sqrt{3}$
3. **Eq. Pt. #3:** $x_2 = 0$ and $x_1 = -\sqrt{3}$

Contour Plots

Given the Lyapunov Candidate function $V(x) = \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2$, we want to show graphically the region of attraction and that this point is only stable when $V(x) < \frac{9}{8} < 1.125$.



We can see in the contour plot that the trajectory of the system greater than 1.125 will not converge to the origin , but will be attracted to other points in the system.

Problem 11

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x} &= -x_1 - x_2 \sec\left(x_2^2 - x_3^2\right) \\
\dot{x}_3 &= x_3 \,\text{sat}\left(x_2^2 - x_3^2\right)
\end{aligned}$$

Part 1: Unique Eq. Pt.

Given the previous system, we want to detemine want to determine whether the origin is an unique equilibrium point. To solve this, we need to set the state equations equal to zero. After a bit of simplication we arrive at the following expressions.

$$\begin{aligned}
0 &= x_2 \\
0 &= x_1 \\
0 &= x_3 sat(x_2^2 - x_3^2)
\end{aligned}$$

The saturation function can only scale values between $[-1, +1]$. Sincee this scaling constant could be divided against zero, we can show that the presence of the saturation function does not change the result of the last state equation since we could cancel the scaling of the saturation by dividing it out.

This means that $x_1 = x_2 = x_3 = 0$ is a unique equilibrium point of the system, since there are no other values possible which satisfy the expression.

Part 2: Asymptotic Stability

In the problem statement we are told to solve the problem given the Lyapunov Candidate function $V(x) = x^T X$. This expands to...

$$V(x) = x_1^2 + x_1^2 + x_1^2$$

The time derivative of this function is...

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 + 2x_3\dot{x}_3$$

By substituting the state equations into this expression we get...

$$\dot{V}(x) = -2x_2^2 \,\text{sat}\left(x_2^2 - x_3^2\right) + 2x_3^2 \,\text{sat}\left(x_2^2 - x_3^2\right)$$

However, solving this equation and showing that $\dot{V} < 0$, for all $x \in \mathbb{R}^3$ is complicated by the presence of the Saturation function which is define as shown below.

$$Sat(x) = \begin{cases} 1, & x \geq a \\ x, & -1 \geq x \geq x \\ -1, & x \leq a \end{cases}$$

Note that with this definition we can show that the saturation function is always positive for positive inputs and is always negative for negative inputs. This means that we only need to reason about the sign of the input and then we can conclude the sign of the saturation which is the only thing that the presence of the saturation function would change given the previous definition.

For the expression for $\dot{V}(x)$ given above, we can see that it is **not** obvious whether the function is negative definite or positive definite due to the alternating of signs as well as the presence of the saturation functions. However, as mentioned above, we know that the saturation function is merely a scaling function between the range of $[-1, 1]$. This means that we can if we can reason about the inputs to the saturation function we can reason about the sign of the output of the saturation function. This will allow use to investigate whether the system is ND or PD.

Since the inputs for both saturation functions is $x_2^2 - x_3^2$ we only need to investigate the two cases where one variable is larger than the other.

Case: $x_2 > x_3$

If we assume that $x_2 > x_3$ we can see from the function $sat(x_2^2 - x_3^2)$ that no matter the sign of x_2 or x_3 the system will only be negative of $x_3 > x_2$ and that the system will only be positive of $x_2 > x_3$. Using this reasoning, we can state that ...

If $x_2 > x_3$:

Then ...

$$Sat(x_2^2 - x_3^2) > 0$$

By applying this fact to our Lyapunov function, we can show that Saturation must result in a multiplicative constant that is greater than zero such that. $0 \leq C_i \leq 1$

$$\dot{V}(x) = -2x_2^2 \operatorname{sat}\left(x_2^2 - x_3^2\right) + 2x_3^2 \operatorname{sat}\left(x_2^2 - x_3^2\right)$$

$$\dot{V}(x) = -2C_1x_2^2 + 2C_2x_3^2$$

Since we are assumming that $x_2 > x_3$ we can state that the function $\dot{V} < 0$ since any squared value of x_2 will be larger than any squared value of x_3. Additionally since the constants C_1 and C_2 are identical, they scale each term identically and do not alter the conclusion that $\dot{V}(x)$ is Negative Definite.

Case: $x_3 > x_2$

Using the same logic as before. We can show that ...
If $x_3 > x_2$:

Then ...

$$Sat(x_2^2 - x_3^2) < 0$$

Likewise

$$\dot{V}(x) = 2C_2x_2^2 - 2C_3x_3^2$$

Since we are assumming that $x_3 > x_2$ we can state that the function $\dot{V} < 0$ since any squared value of x_3 will be larger than any squared value of x_2. Additionally since the constants C_3 and C_4 are identical, they scale each term identically and do not alter the conclusion that $\underline{\dot{V}(x) \text{ is Negative Definite}}$.

Conclusion

Since the function $\dot{V}(x) < 0$ for all $\forall X \in \mathbb{R}^3$, and since $V(x)$ is Positive definite, and radially unbounded, we can conclude that the system is $\underline{\text{Globally Asympototically Stable}}$.

Problem 12

Given that the origin $x = 0$ is an equilibrium point of the system

$$\dot{x}_1 = -kh(x)x_1 + x_2$$
$$\dot{x}_2 = -h(x)x_2 - x_1^3$$

let $D = \{x \in \mathbb{R}^2 | \, \|x\|_2 < 1\}$. Using $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, investigate stability of the origin of the following cases.
Note the derivative of the function $V(x)$ can be written as...

$$\dot{V}(x) = x_1^3 \dot{(x)}_1 + x_2 \dot{x}_2$$

(1) $k > 0, h(x) > 0, \forall x \in D$

$$\dot{V}(x) = x_1^3[-kh(x)x_1 + x_2] + x_2[-h(x)x_2 - x_1^3]$$

After simplification...

$$\dot{V}(x) = -kh(x)x_1^4 - h(x)x_2^2$$

Solution

Given that $\forall x \in D$, the function $\dot{V}(x)$ is Negative Semi-Definite (due to $\dot{V}(0) = 0$), this system is **Stable** according to the Lyapunov Stability Theorem.

(2) $k > 0, h(x) > 0, \forall x \in \mathbb{R}^2$

$$\dot{V}(x) = -kh(x)x_1^4 - h(x)x_2^2$$

Solution

Given that $\forall x \in \mathbb{R}^2$, the function $\dot{V}(x)$ is Negative Semi-Definite (due to $\dot{V}(0) = 0$), this system is **Globally Stable** according to the Lyapunov Global Stability Theorem, since in addition to the Lyapunov function being NSD, the function $V(x)$ is radially unbounded with respect to $\|x\|$. However since Lyapunov function is not negative definite, we cannot conclude asymptotic stability.

(3) $k > 0, h(x) < 0, \forall x \in D$

$$\dot{V}(x) = -kh(x)x_1^4 - h(x)x_2^2$$

Solution

Given that $\forall x \in D$, the system is **unstable**. This is because both terms in the function would be negative semi-definite only as long as both k and $h(x)$ are greater than zero. Since $h(x) < 0$ for this case, this has the effect of negating a NSD function, which in turn produces a Positive Semi-Definite function. This violates the Lyapunov Stability Theorem and we can conclude that under this condition the system must be unstable.

(4) $k > 0, h(x) = 0, \forall x \in D$

$$\dot{V}(x) = -kh(x)x_1^4 - h(x)x_2^2$$

Solution

Given that $\forall x \in D$, the function $\dot{V}(x)$ is <u>Negative Semi-Definite</u> (due to $\dot{V}(0) = 0$), this system is **Stable**. However, while technically this Lyapunov function is NSD, since $h(x) = 0$ it is clear that this Lyapunov function is zero $\forall x \in D$. While this doesn't violate the stability theorem, to me this would appear to be a <u>marginally stable</u> system and even though it technically might be classified as stable, is so close to instability that it could practically be considered unstable.

(5) $k = 0, h(x) > 0, \forall x \in D$

$$\dot{V}(x) = -h(x)x_2^2$$

Solution

Given that $\forall x \in D$, the function $\dot{V}(x)$ is <u>Negative Semi-Definite</u> (due to $\dot{V}(0) = 0$), this system is **Stable**. This is due to the fact that when $k = 0$, the only term in the remaining expression is NSD. Since we are given that the term $h(x) > 0$ this means that the function $\dot{V}(x)$ will never flip signs over the defined domain.

(6) $k = 0, h(x) > 0, \forall x \in \mathbb{R}^2$

$$\dot{V}(x) = -h(x)x_2^2$$

Solution

Given that $\forall x \in \mathbb{R}^2$, the function $\dot{V}(x)$ is <u>Negative Semi-Definite</u> (due to $\dot{V}(0) = 0$), this system is **Globally Stable**. We can conclude that this function is globally stable since, its NSD as well as being <u>radially unbounded</u>. This means that over the entire domain of $\forall x \in \mathbb{R}^2$, the sign of the Lyapunov function will never flip and that the candidate function is well-conditioned as $\|x\| \to \infty$ such that $V(x) \to \infty$.

Problem 13

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - kx_1 - dx_2 - cx_3 \\ \dot{x}_3 &= -x_3 + x_2\end{aligned}$$

where all coefficients are positive and $k > a$. Using

$$V(x) = 2a\int_0^{x_1} \sin y dy + kx_1^2 + x_2^2 + px_3^2$$

Solution

$$V(x) = 2a[-cos(x)]_0^{x_1} + kx_1^2 + x_2^2 + px_3^2$$

$$\dot{V}(x) = 2aSin(x_1)\dot{x}_1 + 2kx_1\dot{x}_1 + 2x_2\dot{x}_2 + 2Px_3\dot{x}_3$$

After substitution and simplications using the state equations...

$$\dot{V}(x) = -2dx_2^2 - 2cx_2x_3 - 2Px_3^2 + 2Px_2x_3$$

If we select P such that $P = c$ (which is valid since $p > 0$) we can reduce this function to...

$$\dot{V}(x) = -2dx_2^2 - 2Px_3^2$$

Under this restriction on P we can see that $\dot{V}(x) < 0$ over the complete domain $\forall x \in \mathbb{R}^3$. In addition to this, since the function $V(x)$ is radially unbounded we can conclude via Lyapunov Global Stability Theorem that the system is **Globally Asymptotically Stable**.

Problem 14

Given the governing equation...

$$M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$$

We can create the following state space system, given the following state definitions, $x_1 = y - \frac{Mg}{k}$, and $x_2 = \dot{y}$.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= g - \tfrac{ky}{m} - \tfrac{c_1 x_2}{m} - c_2 x_2 |x_2|\end{aligned}$$

Such that...

$$g = (y - x_1)\frac{k}{m}$$

Which simplifies to...

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{kx_1}{m} - \tfrac{c_1 x_2}{m} - c_2 x_2 |x_2|\end{aligned}$$

Using candidate function, $V(x) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2$, we can check the stability of the system by evaluating the time derivative of the candidate function, and drawing conclusions via Lyapunov Stability Theory. By taking the time derivative we obtain...

$$\begin{aligned}\dot{V}(x) &= ax_1\dot{x}_1 + bx_2\dot{x}_2 \\ \dot{V}(x) &= ax_1x_2 + bx_2\left(-\tfrac{k}{m}x_1 - \tfrac{c_1}{m}x_2 - \tfrac{c_2}{m}x_2|x_2|\right) \\ \dot{V}(x) &= \left(a - \tfrac{bk}{m}\right)x_2 - \tfrac{c_1}{m}x_2^2 - \tfrac{c_2}{m}x_2^2|x_2|\end{aligned}$$

This function will be negative semidefinite only when the x_2 term is cancelled. In order to achieve this, we can define the arbitrary constants such that $a = \frac{k}{m}$, and $b = 1$. Under these conditions the Lyapunov function simplifies to the following NSD expression.

$$\dot{V}(x) = 2 - \frac{c_1}{m}x_2^2 - \frac{c_2}{m}x_2^2|x_2|$$

By using La Salle's Theorem (Corollary 4.2)...

$$\dot{V} \equiv 0$$

For this to be valid, we must infer that $x_2 = 0$. Further, we can show that under this condition, $\dot{x_1} = 0$, which means that $x_1 = 0$. Since the only place where the function can be stationary is at the equilibrium $x = 0$, we can show that since the Lyapunov function is NSD that with the exception of the equilibrium point the system must be decaying, the system must at least be asymptotically stable. However, since this is true for all $\mathbb{R}^2$ and since the function $V(x)$ is radially unbounded with respect to x, we can conclude that the system is Globally Asymptotically Stable.

Problem 15

Given the system in the problem statement we wish to show that it is globally asymptotically stable for the following cases.

Case: $Q > 0$

Given the Lyapunov Candidate $V(X) = x^t P x$ we can show that $\dot{V}(x)$ can be dervied by differeniating each term according to the product rule of calculus.

$$\dot{V}(x) = \dot{x^T} P x + x^T \dot{P} x + x^T P \dot{x}$$

By substituting the state equation and its transpose into this expression we obtain the equation..

$$x^\top \left(A - BR^{-1}B^\top P\right)^\top Px + x^\top P \left(A - BR^{-1}B^T P\right) x$$

By substituting the Riccati Equation into this expression we can simplify the equation

$$\dot{V} = x^\top \left[PA - PBR^{-1}B^T P + A^T P - P\left(BR^{-1}B^T P\right)^\top\right] X$$

Which simplies to ...

$$\dot{V} = -x^\top \left[Q + PBR^{-1}B^\top P\right] x$$

Since the problem statement told us that P, and Q, and R are positive definite and since (A, B) is controllable, we can conclude that the expression inside of the square brackets are Positive Definite. Since negating a PD function yields a ND function, we can show that according to Lyapunov Global Stability Theorem, that the system is **Globally Asymptotically Stable** since $\dot{V} < 0$ and the function V is PD, defined for all $\mathbb{R}^n$, and radially unbounded.

Case: $Q = C^T C$

In the case where $Q = C^t C$, we obtain the following Lyapunov function...

$$\dot{V} = -x^\top \left[C^T C + PBR^{-1}B^\top P\right] x$$

We know from problem statement that $Ce^{At} \equiv 0$ only when $x = 0$. This means that if we look at at the case when $\dot{V}(x) = 0$...

$$\dot{V} = -x^\top \left[Q + PBR^{-1}B^\top P\right] x \equiv 0$$

Since we know that $Cx \equiv 0$ we know that $R^{-1}B^T P x(t) \equiv 0$ for the system to be valid. This means our system symplifies to...

$$\dot{x} = Ax - BR^{-1}B^T P x = Ax$$

Since this is a linear equation we know we can compute the solution by integrating both sides (since it is observable), such that ...

$$x(t) = Ce^{At}x_0$$

According the Corollary 4.2 from La Salle's Theorem, when $V(x)$ is positive definite and radially unbounded, and $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$ and the only solution which can stay identically 0 is the trivial solution, then the origin is globally asymptotically stable. We can draw this conclusion since we know that (for the given Lyapunov function) at $V(x) = 0$ the only point which at which $Ce^{At}x_0 = 0$ is at the origin. Therefore this system is **globally asymptotically stable**.

Problem 16

Consider the system...

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= (x_1 x_2 - 1)\, x_2^3 + \left(x_1 x_2 - 1 + x_1^2\right) x_2\end{aligned}$$

Part A

By setting the state equations to zero the equilibrium points can be computed as shown below...

$$\begin{aligned}0 &= -x_1 \\ 0 &= (x_1 x_2 - 1)\, x_2^3 + \left(x_1 x_2 - 1 + x_1^2\right) x_2\end{aligned}$$

It is clear that $x_1 = 0$ and that by substituting this result into the second state equation, that $x_2 = 0$. Since the system is not valid for any other points which keep the system at equilibrium $x_1 = 0 \& x_2 = 0$ are a unique equilibrium.

Part B

By linearizing the state equations we obtain the following Jacobian Matrix.

$$A_j = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ (x_2^4 + x_2^2 + 2x_1 x_2) & (4x_1 x_2^3 - 3x_2^2 + 2x_1 x_2 - 1x_1^2) \end{bmatrix}\Bigg|_{x_{eq}=0}$$

When the Jacobian is evaluated at the equilibrium point $x = 0$ it results in the following matrix.

$$A_j = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since $Re(\lambda_i) < 0$, via local analysis, we can show that the sytem is asympotically stable.

Part C

Show that the following set is positively invariant.

$$\Gamma = \left\{ x \in \mathbb{R}^2 \mid x_1 x_2 \geq 2 \right\}$$

Solution

This next problem requires rethinking what it would require for the given set to be positively invariant. Unlike previous problems that typically provide an upper bound on the elements of the set, this set places a lower bound on the elements of the set. Since we know that inorder for the system to actually be positively invariant, any trajectory starting in Γ must stay in Γ for all $t \geq 0$.

Since the set is not defined at the equilibrium point which is the origin, we can reason that we do not want the system to converge towards the origin as that would violate the definition of the set which is assumed to be invariant. For this specific case though, we can see that if the trajectory goes to infinity $\forall x \in \mathbb{R}^2$, the value is still maintained in the definition of the invariant set.

In this case then, we wish to drive the system away from the origin, anywhere between $[2, \infty)$. Since this is the goal, we are interested in making sure that the system is unstable with respect to the origin of the system. This means that we want to ensure that $\dot{V}(x) > 0$ (for all valid point of the set) since this condition is what allows the system to diverge away from any the origin.

To test this mathematically we can define...

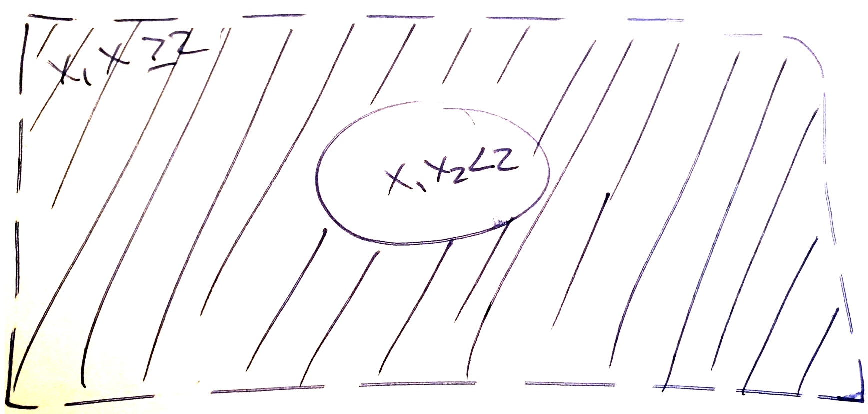
$$V(x) = x_1 x_2$$

$$\begin{aligned}
&\dot{V}(x) = \dot{x}_1 x_2 + \dot{x}_2 x_1 \\
&\dot{V}(x) = -x_1 x_2 + x_1 \big((x_1 x_2 - 1)x_2^3 + (x_1 x_2 - 1 + x_1^2)x_2\big) \\
&\dot{V}(x) = -x_1 x_2 + x_1^2 x_2^4 - x_1 x_2^3 + x_1^2 x_2^2 - x_1 x_2 + x_1^3 x_2
\end{aligned}$$

After simplification we can show that...

$$\dot{V}(x) \mid_{x_1, x_2 = 2} > 0$$

Since this shows that the system is unstable for $x = 2$, we can conclude that the set will not converge towards the origin, and hence will be maintained inside of the set, which is unbounded.



In this picture, we can see that that positively invariant set Γ includes everything except set where $x_1 x_2 < 2 \quad \forall x \in \mathbb{R}^2$. And that the set Γ is an open set since there is no upper limit on it.

Part D

Using the conclusion from the previous problem, since we have just shown that the system does not converge towards the origin if initialized in set Γ, the system is **not** globally asymptotically stable at $x = 0$.

Problem 17

For each of the following systems, use lineraization to show that the origin is aymptotically stable. Then sxhow that the origin is globally asymptoticially stable.

Part 1

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = (x_1 + x_2)\sin x_1 - 3x_2$$

To linearize the system we need to take its Jacobian.

$$A_j = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ sin(x_1) + x_1 cos(x_1 + x_2 cos(x_2)) & \sin(x) - 3 \end{bmatrix}\Bigg|_{x=0}$$

After evaluating the system at its origin, we can show that...

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$$

Since the eigenvalues of this system are $\lambda_1 = -1$ and $\lambda_2 = -3$ we can conclude that locally this system is <u>stable</u>.

Part 2

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = -ax_1 - bx_2, a, b > 0$$

$$A_j = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -3x_1 & 1 \\ -a & -b \end{bmatrix}\Bigg|_{x=0}$$

By taking the eigenvalues of this Jacobian we can determine the local stability condition of the system. These are given by $\lambda_{1,2} = .5 \pm .886j$, when the constant a and b equal 1. Therefore the system is a <u>stable focus</u>.

We can show that this is globally asymptoticially stable, using a standard quadratic Lyapunov candidate function.

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

Whose time derivative is...

$$\dot{V}(x) = x_1\dot{x_1} + x_2\dot{x_2}$$

After substituting in the state equations...

$$\dot{V}(x) = -x_1^4 + x_1x_2 - ax_1x_2 - bx_2^2$$

Note that if the constants a and b both equal 1, then the Lyapunov function reduces to..

$$\dot{V}(x) = -x_1^4 - x_2^2$$

Since this function is negative definite, and since the candidate function is positive definite and radially unbounded we can say that according to the Global Lyapunov stability theorem that the system must be <u>globally asymptotically stable</u> about the origin.

Problem 18

$$\dot{X}(t) = \frac{-x}{t+1} \quad \forall t \geq 0$$

We can solve this equation directly using integration. This leads to...

$$x(t) = x(t_0)exp\left(\int_{t_0}^{t} \frac{-1}{1+\tau} d\tau\right)$$

$$\frac{dx}{dt} = \frac{-x}{1+t}$$

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} \left(\frac{-1}{1+t}\right) dt$$

$$\ln(x)|_{x_0}^{x} = \left(\int_{t_0}^{t} \frac{-1}{1+\tau} d\tau\right)$$

$$X(t) = x\,(t_0) \exp\left\{\left(\int_{-\infty}^{t} \frac{-1}{1+\tau} d\tau\right)\right\}$$

Using this solution, we can say that the any trajectory in the system will ...

$$x(t) = x(t_0)e^{-\alpha}$$

We know that due to the sign of the exponent in the solution of the problem that the system must be decaying over time... such that

$$|x(t)| \leq |x(t_0)|$$

This means that the solution is stable for all $t \geq 0$; however, if we want to determine global stability we must demonstrate that the system $X(t) \to 0$ as $t \to \infty$. In order to show these, we can perform the integral in the exponent of the previous solution and then analyze the system at $t = \infty$.

$$x(t) = x(t_0)\left[\frac{1+t_0}{1+t}\right] = 0 \qquad t \to \infty$$

Using this solution, we can clearly see that the value of x trends towards zero as time trends towards infinity. Because of this, we can conclude that the system is globally asymptoticially stable.

Problem 19

Part 1

$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ \alpha(t) & -2 \end{bmatrix} x, |\alpha(t)| \leq 1$$

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$
$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2$$

$$\dot{V}(x) = -x_1^2 - 2x_2^2 + 2x_1x_2\alpha(t)$$

We can check whether the system is exponentially stable if it satisfies **Theorem 4.10**

1. $V is C_1$
2. $V(0) = 0$ and $V(x) > 0, \quad \forall x \in D - \{0\}$
3. $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^2$

Theorem 4.10

$$K_1 \left\|^2 x\right\| \leq V(x) \leq K_2 \|x\|^2$$

$$\dot{V} \leq -K_3 \|x\|^2$$

This is satisfied given $K_1 = .25$, $K_2 = 1$, $K_3 = 1$, and $a = 2$.

Part 2

$$\dot{x} = \begin{bmatrix} -1 & \alpha(t) \\ -\alpha(t) & -2 \end{bmatrix} x$$

$$\dot{V}(x) = -x_1^2 - 2x_2^2$$

1. $V is C_1$
2. $V(0) = 0$ and $V(x) > 0, \quad \forall x \in D - \{0\}$
3. $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^2$

Theorem 4.10

$$K_1 \left\|^2 x\right\| \leq V(x) \leq K_2 \|x\|^2$$

$$\dot{V} \leq -K_3 \|x\|^2$$

This is satisfied given $K_1 = .25$, $K_2 = 1$, $K_3 = 1$, and $a = 2$.

Part 3

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha(t) \end{bmatrix} x, \alpha(t) \geq 2$$

$$\dot{V}(x) = -x_2\alpha(t)$$

1. $V is C_1$
2. $V(0) = 0$ and $V(x) > 0, \quad \forall x \in D - \{0\}$
3. $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^2$

Theorem 4.10

$$K_1 \left\|^2 x\right\| \leq V(x) \leq K_2 \left\|x\right\|^2$$

$$\dot{V} \leq -K_3 \left\|x\right\|^2$$

This is satisfied given $K_1 = .25$, $K_2 = 1$, $K_3 \leq \frac{x_2\alpha(t)}{\|x\|^2}$, and $a = 2$.

Part 4

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ \alpha(t) & -2 \end{bmatrix} x$$

$$\dot{V} = -x_1^2 + x_1x_2\alpha(t) - 2x_2^2$$

1. $V is C_1$
2. $V(0) = 0$ and $V(x) > 0, \quad \forall x \in D - \{0\}$
3. $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^2$

Theorem 4.10

$$K_1 \left\|^2 x\right\| \leq V(x) \leq K_2 \left\|x\right\|^2$$

$$\dot{V} \leq -K_3 \left\|x\right\|^2$$

This is satisfied given $K_1 = .25$, $K_2 = 1$, $K_3 \leq \frac{\dot{V}(x)}{\|x\|^2}$, and $a = 2$.

Problem 20

Given the system...

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2\end{aligned}$$

We can simplify that this expression such that...

(a) Verify that $x_1(t) = t$, $x_2 = 1$ is a solution.

(b) Show that if $x(0)$ is sufficiently close to $[01]^T$, then $x(t)$ approaches $[t1]^T$ as $t \to \infty$

Part A We can show that $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$ is a solution by differentiating these values with respect to time and then substituting into the state equations.

$$\begin{aligned}\tfrac{d}{dt}\left(x_1(t)\right) &= \tfrac{d}{dt}\left(t\right) \qquad \rightarrow \quad \dot{x_1} = 1 \\ \tfrac{d}{dt}\left(x_2(t)\right) &= \tfrac{d}{dt}\left(1\right) \qquad \rightarrow \quad \dot{x_2} = 0\end{aligned}$$

After subbing these into the state equations we obtain...

$$\begin{aligned}\dot{x}_1 &= 1 \\ \dot{x}_2 &= 0\end{aligned}$$

Since this answer is consistent with the state equations we know that it is a valid solution of the system.

Part B The main task of this problem is take a the system whose equilibrium is not at the origin and to transform the system such that we have an equivalent system about the origin. After performing this transformation, we can then use the rules and techniques that we have learned to prove whether or not the system is asymptoticially stable and approaches the origin as $t \to \infty$. However, even though we redefine the problem to be about the origin, the result will hold the original system we are interested in investigating.

In order to transform this system we must define new state variables z_i such that the system appears to be centered on the origin $z = (0, 0)$. Since we are told in the problem statement that we want to know whether the system will converge to $[t, 1]^T$, we know that we want to shift the x_i variables such that the system in z will converge to $[0, 0]^T$.

$$\begin{aligned}z_1 &= x_1 - t \\ z_2 &= x_2 - 1\end{aligned}$$

By using this change of variables, we can see that at $x = (t, 1)$ that $z = (0, 0)$. We can now carry this substitution all the way through and rewrite the state equations in terms of z_i.

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = 2z_1z_2 - z_1 - 2z_2$$

Note that these state equations are Autonomous, unlike the original set of state equations that we first were given. This means that we can now use our standard toolkit to solve for the stability of the system, since this system is now defined to be independent of time. We can now determine the stability by linearization and evaluating about the equilibrium point of the system.

$$A_j = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_1} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2z_2 & -2 + 2z_1 \end{bmatrix}\Bigg|_{z=0} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Since the linearized equation has negative eigenvalues, we know that the system must be Uniformly asymptoticially Stable. We know that this result is **uniform** since it is independent of time.

Problem 21

$$\dot{x}_1 = -x_1 + x_2 + \left(x_1^2 + x_2^2\right)\sin t$$
$$\dot{x}_2 = -x_1 - x_2 + \left(x_1^2 + x_2^2\right)\cos t$$

Given the system we want to show that it is exponentially stable and to provide an estimate for the region of attraction. In order to do this we begin with a simple quadratic Lyapunov candidate function, as shown below, take its time derivative and then substitute the state equations back into the Lyapunov function.

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

This results in the system shown below.

$$\begin{aligned}\dot{V} &= -x_1^2 + x_1x_2 + x_1\left(x_1^2 + x_2^2\right)\sin t - x_1x_2 - x_2^2 + x_2\left(x_1^2 + x_2^2\right)\cos t \\ &= -\left(x_1^2 + x_2^2\right) + \left(x_1^2 + x_2^2\right)\left(x_1 \sin t + x_2 \cos t\right)\end{aligned}$$

In order to evaluate the bound of the system, we can rewrite the system in terms of its norms. This form allows us to determine a ball with a specific radius such that the Lyapunov function is valid within. This is important in the estimation of the region of attraction. For the given Lyapunov system we can write the equation as an inequality using 2-norms.

$$\dot{V} \le -\|x\|_2^2 + \|x\|_2^3 \sqrt{(\sin t)^2 + (\cos t)^2}$$

We can reduce this system using the identity $1^2 = Cos^2(t) + Sin^2(t)$, which simplifies this

$$\dot{V} \le -\|x\|_2^2 + \|x\|_2^3$$

We can further simplify this statement by creating an equivalent expression described by the same norm.

$$\dot{V}(x) \le -(1-r)\|x\|_2^2, \quad \forall \|x\|_2 \le r$$

With this expression we can show show that the Lyapunov function is negative definite for any $r < 1$. This means that our system is <u>stable</u> under this condition. However, in order to show that the system is exponentially stable, we must apply Theorem 4.10, and show that it satisfies its requirements.

1. $V is C_1$
2. $V(0) = 0$ and $V(x) > 0, \quad \forall x \in D - \{0\}$
3. $\dot{V}(x) < 0 \quad \forall x \in D$

Theorem 4.10

$$K_1 \|x\|^2 \leq V(x) \leq K_2 \|x\|^2$$

$$\dot{V} \leq -K_3 \|x\|^2$$

This is satisfied given $K_1 = .25$, $K_2 = 1$, $K_3 = r$, and $a = 2$.
Under these conditions, we can show that the system is exponentially stable and that the region of attraction can be estimated with $\|x\| \leq r$, for $r < 1$.

Problem 22

$$\dot{x} = f(x) \quad , \dot{x} = \underbrace{h(x)f(x)}_{g(x)}$$

Given these two related systems and the fact that $h(x)$ is positive definite, we can evaluate the stability of each system by linearization. In order to linearize the function $g(x)$ we must apply the chain-rule, as shown below.

$$\left.\frac{\partial g}{\partial x}\right|_{x=0} = h(0)\frac{\partial f(0)}{\partial x} + \frac{\partial h(0)}{\partial x} f(0)$$

By evaluating the system at its equlibrium point, we see that the function $f(0) = 0$ leaving us with the expression

$$\left.\frac{\partial g(0)}{\partial x}\right| = h(0)\frac{\partial f(g)}{\partial x}$$

Since this is the linearized system, we can rewrite this system in terms of the linear matrices which drop out from the equation above.

$$A_2 = h(0)A_1$$

Since we know that $h(x)$ is positive definite and that $h(0) > 0$, we can see that second system A_2 will be stable if and only if the first system A_1 is stable since $h(0)$ merely scales the values and cannot flip the signs. This means that if the eigenvalues of the system are in the left half plane, and then eigenvalues of the second must also be in the left half plane. Since this is the case, we can show using similar logic that if the first system is <u>exponentially stable</u> then the second system **must** also be exponentially stable.