Homework 1

ECE 345 Algorithms and Data Structures
Fall Semester, 2017

Due: Friday,September 29 2017, 12 noon

This homework is designed so you can practice your background in introductory discrete mathematics and combinatorics. You need to have this background material practiced well because future homeworks will use it extensively!

- All page numbers are from 2009 edition of Cormen, Leiserson, Rivest and Stein.
- For each algorithm you asked to design you should give a detailed *description* of the idea, proof of algorithm correctness, termination, analysis of time and space complexity. If not, your answer will be incomplete and you will miss credit. You are allowed to refer to pages in the textbook.
- Do not write C code! When asked to describe an algorithm give analytical pseudocode.
- Staple your homework properly. Use a stapler; do not use glue or other weird material to put it together. If you are missing pages, we are not responsible for it but you are!
- Write *clearly*, if we cannot understand what you write you may not get credit for the question. Be as formal as possible in your answers. Don't forget to include your name(s) and student number(s) on the front page!

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1. **[Permutations and Combinations, 10+10 points]**

   (a) A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

   (b) Prove that
   $$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$$
   using a *combinatorial argument*. That is, pose a combinatorial question and argue that the answer to your question can be either side of the equation. Non-combinatorial proofs will receive no credit.

2. **[Recurrences, 15 points]**

   Solve the following recurrences by giving tight Θ-notation bounds. (*Hint*: apply the Master Theorem on the first four.)

   (a) $T(n) = 2T(n/2) + n^3$

   (b) $T(n) = 4T(n/4) + \sqrt{n}$

(c) $T(n) = 7T(n/2) + n^2$

(d) $T(n) = 2T(n/4) + \sqrt{n}$

(e) $T(n) = T(n/5) + T(4n/5) + \Theta(n)$

3. **[Asymptotics, 25 points]**

Sort the following 24 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. Do as much as you can, you will receive partial credit if you do not sort all of them. You will also need to turn in proofs that *justify your answers* to receive full credit! All logarithms are base 2 unless otherwise stated.

$n^{4.5} - (n-2)^{4.5}$	n	$n^{1+(1/\log n)}$	$\log^*(n/2)$	$\sum_{i=1}^n \log i$
$\sum_{i=2}^n \left(\frac{1}{i-1} - \frac{1}{i+1}\right) + 2$	$\log^*(\log^* n)$	2^n	$(\log n)^{\log^* n}$	n^5
$\log^* 2^n$	$2^{\log^* n}$	e^n	$\lfloor \log\log(n!) \rfloor$	
$\left(1 - \log \frac{1}{1-1/n}\right)^n$	$n^{(\log\log n)/(\log n)}$	$(\log n)^{(n/2)}$	$(\log n)^{\log n}$	$\left(1 + \frac{1}{200n}\right)^{200n}$
$n^{1/\log\log n}$	$n^{\log\log n}$	$\log^{(200)} n$	$\log^2 n$	$n(\log n)^2$

To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = O(g(n))$, $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$, and $f(n) \gg g(n)$ to mean $f(n) = \Omega(g(n))$. For example, the functions n^2, n, $\binom{n}{2}$, n^3 could be sorted either as $n^3 \gg \binom{n}{2} \equiv n^2 \gg n$ or as $n^3 \gg n^2 \equiv \binom{n}{2} \gg n$.

4. **[Induction, 10 points]**

Use mathematical induction to prove the following statement. Make sure to show clearly all three steps of induction (inductive basis, inductive hypothesis and inductive step) or you will miss credit.

Every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

5. **[Probability, 2+4+4 points]**

A deck of playing cards has 52 unique cards, each with one of 13 ranks $(A, 2, 3, ..., 10, J, Q, K)$ and one of four suits $(\heartsuit, \diamondsuit, \clubsuit, \spadesuit)$.

(a) How many different hands of 5 cards are there?

(b) A four-of-a-kind refers to a hand of 5 cards where 4 are of the same rank (*e.g.,* $2\heartsuit, 2\diamondsuit, 2\clubsuit, 2\spadesuit, 3\diamondsuit$). If you draw 5 cards at random, what is the probability of getting four-of-a-kind?

(c) A full house refers to a hand of five 5 cards where 3 belong to one rank and 2 belong to another rank (*e.g.,* $2\heartsuit, 2\diamondsuit, 2\clubsuit, 3\heartsuit, 3\diamondsuit$). If you draw 5 cards at random, what is the probability of getting a full house?

6. **[Graphs, Proof by contradiction, 10 points]**

There are 21 people in a party. Suppose that each of them shakes hands with at least one person. Prove that there must be someone who shakes hands with at least two persons. (*Hint*: model the people as vertices and handshakes as edges.)

7. **[Trees, Proof by induction, 10 points]**

   A perfect binary tree is a binary tree where all nodes have either 0 or 2 children and all leaves are at the same depth. Prove by induction that a perfect tree of height h has $2^{h+1} - 1$ nodes.