Curve Equations and Syntax

Parametric Description of Curves (88) - (ref 59)

$$Z^i = Z^i(U)$$

Covariant Basis Vectors (88) - (ref 59)

$$\vec{U}_\Phi = \frac{\partial \vec{R}}{\partial U^\Phi} \qquad \vec{U}_1 = \frac{d\vec{R}}{dU^1}$$

$$\vec{U}_{\Phi'} = \frac{\partial \vec{R}}{\partial U^{\Phi'}} \qquad \vec{U}_{1'} = \frac{d\vec{R}}{dU^{1'}}$$

Jacobian and Inverse (88) - (ref 59)

$$J_\Phi^{\Phi'} = \frac{\partial U^{\Phi'}}{\partial U^\Phi} \qquad J_1^{1'} = \frac{dU^{1'}}{dU^1}$$

$$J_{\Phi'}^{\Phi} = \frac{\partial U^{\Phi}}{\partial U^{\Phi'}} \qquad J_{1'}^{1} = \frac{dU^{1}}{dU^{1'}}$$

Covariant Transformation (88) - (ref 59)

$$T_\Phi = J_\Phi^{\Phi'} T_{\Phi'} \qquad T_1 = \frac{dU^{1'}}{dU^1} T_{1'}$$

Contravariant Transformation (88) - (ref 59)

$$T^\Phi = J_{\Phi'}^{\Phi} T^{\Phi'} \qquad T^1 = \frac{dU^1}{dU^{1'}} T^{1'}$$

Jacobian Inversion (88) - (ref 59)

$$\delta_\Psi^\Phi = J_{\Phi'}^{\Phi} J_\Psi^{\Phi'} = J_\Psi^{\Phi'} J_{\Phi'}^{\Phi} \qquad 1 = \frac{dU^1}{dU^{1'}} \frac{dU^{1'}}{dU^1}$$

Covariant Metric Tensor (88) - (ref 59)

$$U_{\Phi\Psi} = U_{\Psi\Phi} = \vec{U}_\Phi \cdot \vec{U}_\Psi U_{11} = \vec{U}_1 \cdot \vec{U}_1$$

Contravariant Metric Tensor Definition (88) - (ref 59)

$$U^{\Phi\Psi} U_{\Psi\Theta} = \delta_\Theta^\Phi \qquad U^{11} U_{11} = 1$$

Contravariant Basis Vector Definition (88) - (ref 59)

$$\vec{U}^\Phi = U^{\Phi\Psi} \vec{U}_\Psi \qquad \vec{U}^1 = U^{11} \vec{U}_1$$

Contravariant Basis Properties (88) - (ref 59)

$$\vec{U}^\Phi \cdot \vec{U}_\Psi = \delta_\Psi^\Phi \qquad \vec{U}^1 \cdot \vec{U}_1 = 1$$

$$\vec{U}^\Phi \cdot \vec{U}^\Psi = U^{\Phi\Psi} = U^{\Psi\Phi} \qquad \vec{U}^1 \cdot \vec{U}^1 = U^{11}$$

Linear Combinations for Vectors Tangent to Curve (88) - (ref 59)

$$\vec{T} = T_\Phi \vec{U}^\Phi = T^\Phi \vec{U}_\Phi \qquad \vec{T} = T_1 \vec{U}^1 = T^1 \vec{U}_1$$

$$\vec{T} = T_{\Phi'} \vec{U}^{\Phi'} = T^{\Phi'} \vec{U}_{\Phi'} \qquad \vec{T} = T_{1'} \vec{U}^{1'} = T^{1'} \vec{U}_{1'}$$

Raising the Index (88) - (ref 59)

$$\vec{U}^\Phi = U^{\Phi\Psi} \vec{U}_\Psi \qquad \vec{U}^1 = U^{11} \vec{U}_1$$

$$T^\Phi = U^{\Phi\Psi} T_\Psi \qquad T^1 = U^{11} T_1$$

Lowering the Index (88) - (ref 59)

$$\vec{U}_\Phi = U_{\Phi\Psi} \vec{U}^\Psi \qquad \vec{U}_1 = U_{11} \vec{U}^1$$

$$T_\Phi = U_{\Phi\Psi} T^\Psi \qquad T_1 = U_{11} T^1$$

Raising and Lowering Indexes for Higher Rank Tensors (88) - (ref 59)

$$U^{\Sigma\Phi} T_{\Phi\Psi} = T^{\Sigma}_{\cdot\Psi} = T^{\Sigma}_{\Psi}$$

$$U^{\Theta\Psi} T^{\Sigma}_{\Psi} = T^{\Sigma\Theta}$$

$$U^{\Theta\Psi} T_{\Phi\Psi} = T_{\Phi}^{\cdot\Theta}$$

$$U^{\Lambda\Phi} T_{\Phi}^{\cdot\Theta} = T^{\Lambda\Theta}$$

$$U^{\Lambda\Phi} U^{\Theta\Psi} T_{\Phi\Psi} = T^{\Lambda\Theta}$$

$$U_{\Theta\Psi} T^{\Phi\Psi\Sigma}_{\Lambda\Omega\Delta} = U_{\Theta\Psi} T^{\Phi\Psi\Sigma\cdots}_{\cdots\Lambda\Omega\Delta} = T^{\Phi\cdot\Sigma\cdots}_{\cdot\Theta\cdot\Lambda\Omega\Delta} = T^{\Phi\cdot\Sigma}_{\Theta\cdot\Lambda\Omega\Delta}$$

$$U^{\Psi\Theta} T^{\Phi\cdot\Sigma}_{\Theta\cdot\Lambda\Omega\Delta} = U^{\Psi\Theta} T^{\Phi\cdot\Sigma\cdots}_{\cdot\Theta\cdot\Lambda\Omega\Delta} = T^{\Phi\Psi\Sigma\cdots}_{\cdots\Lambda\Omega\Delta} = T^{\Phi\Psi\Sigma}_{\Lambda\Omega\Delta}$$

Flipping Dummy Indexes (88) - (ref 59)

$$P^\Phi T_\Phi = P_\Phi T^\Phi \qquad P^1 T_1 = P_1 T^1$$

$$T_{\Phi\Psi} P^\Phi R^\Psi = T^{\Phi\Psi} P_\Phi R_\Psi \qquad T_{11} P^1 R^1 = T^{11} P_1 R_1$$

Dot Product for Vectors Tangent to Curve (88) - (ref 59)

$$\vec{P} \cdot \vec{Q} = U_{\Phi\Psi} P^\Phi Q^\Psi = P^\Phi Q_\Phi = P_\Phi Q^\Phi = U^{\Phi\beta} P_\Phi Q_\Psi$$

$$\vec{P} \cdot \vec{Q} = U_{11} P^1 Q^1 = P^1 Q_1 = P_1 Q^1 = U^{11} P_1 Q_1$$

Permutation Symbols (88) - (ref 59)

$$e^\Phi \qquad e^\Phi = 1$$

$$e_\Phi \qquad e_\Phi = 1$$

Determinant Relationships (88) - (ref 59)

$$Ae^{\Phi} = e^{\Psi} a_{\Psi}^{\Phi} \qquad A = a_1^1$$

$$Ae_{\Phi} = e^{\Psi} a_{\Phi\Psi} \qquad A = a_{11}$$

$$Ae^{\Phi} = e_{\Psi} a^{\Phi\Psi} \qquad A = a^{11}$$

Explicit Determinant Forms (88) - (ref 59)

$$A = \frac{1}{1!} \delta_{\Psi}^{\Phi} a_{\Phi}^{\Psi} \qquad A = a_1^1$$

$$A = \frac{1}{1!} e^{\Phi} e^{\Psi} a_{\Phi\Psi} \qquad A = a_{11}$$

$$A = \frac{1}{1!} e_{\Phi} e_{\Psi} a^{\Phi\Psi} \qquad A = a^{11}$$

Relative Tensors (88) - (ref 59)

$$\left| a_{\cdot'}^{\cdot'} \right| = |a_{\cdot}^{\cdot}|$$

$$|a_{\cdot'\cdot'}| = J^2 |a_{\cdot\cdot}|$$

$$\left| a^{\cdot'\cdot'} \right| = J^{-2} |a^{\cdot\cdot}|$$

Volume Element Transformation (88) - (ref 59)

$$\sqrt{U'} = |J| \sqrt{U}$$

Levi-Civita Symbols (88) - (ref 59)

$$\varepsilon^{\Phi} = \frac{1}{\sqrt{U}} e^{\Phi} \qquad \varepsilon^{\Phi} = \frac{1}{\sqrt{U}}$$

$$\varepsilon_{\Phi} = \sqrt{U} e_{\Phi} \qquad \varepsilon_{\Phi} = \sqrt{U}$$

Shift Tensor Definition (88) - (ref 60)

$$Z_{\Phi}^{i} = \frac{\partial Z^i}{\partial U^{\Phi}}$$

Shift Tensor Transformation (88) - (ref 60)

$$Z_{\Phi}^{i} = Z_{\Phi'}^{i'} J_{i'}^{i} J_{\Phi}^{\Phi'}$$

Shift Tensor Inversion (88) - (ref 60)

$$Z_i^{\Phi} = Z_{ij} U^{\Phi\Psi} Z_{\Psi}^{j}$$

Shift Tensor Contraction (88) - (ref 60)

$$\delta_{\Psi}^{\Phi} = Z_i^{\Phi} Z_{\Psi}^{i} = Z_{\Psi}^{i} Z_i^{\Phi}$$

Curve Vector Relationships (88) - (ref 60)

$$\vec{U}_\Phi = Z^i_\Phi \vec{Z}_i$$

$$\vec{U}^\Phi = Z^\Phi_i \vec{Z}^i$$

$$T^i = Z^i_\Phi T^\Phi$$

$$T_i = Z^\Phi_i T_\Phi$$

$$T_\Phi = Z^i_\Phi T_i$$

$$T^\Phi = Z^\Phi_i T^i$$

Curve Metric Tensor Relationships (88) - (ref 62)

$$U_{\Phi\Psi} = Z^i_\Phi Z^j_\Psi Z_{ij}$$

$$U^{\Phi\Psi} = Z^\Phi_i Z^\Psi_j Z^{ij}$$

Curve Metric Equation (88) - (ref 62)

$$dS^2 = U_{\Phi\Psi} dU^\Phi dU^\Psi \qquad dS = \sqrt{U} dU^1$$

Curve Arc Length (88) - (ref 62)

$$L = \int_a^b \sqrt{U_{\Phi\Psi} \frac{dU^\Phi}{dt} \frac{dU^\Psi}{dt}} dt \qquad L = \int_a^b \sqrt{U} dU^1$$

Vector Curvature Normal (89)

$$\vec{B}^\Phi_\Phi = \nabla^\Phi \vec{U}_\Phi = \nabla^\Phi \nabla_\Phi \vec{R} = \nabla^2 \vec{R}$$

$$\vec{B}^\Phi_\Phi = \frac{1}{\sqrt{U}} \frac{d}{dU^1} \left(\frac{1}{\sqrt{U}} \vec{U}_1 \right) = \frac{d^2 \vec{R}}{ds^2} = \mathrm{K}\hat{P} = B^\Phi_\Phi \hat{P}$$

Curve Christoffel Symbol Properties (90) - (ref 68)

$$\Gamma^\Phi_{\Psi\Theta} = \vec{U}^\Phi \cdot \frac{\partial \vec{U}_\Psi}{\partial U^\Theta} \qquad \text{explicit definition}$$

$$\Gamma^\Phi_{\Psi\Theta} = J^\Phi_{\Phi'} J^{\Psi'}_\Psi J^{\Theta'}_\Theta \Gamma^{\Phi'}_{\Psi'\Theta'} + J^\Phi_{\Phi'} J^{\Phi'}_{\Psi\Theta} \text{transformation}$$

$$\Gamma_{\Phi\Psi\Theta} = U_{\Phi\Sigma} \Gamma^\Sigma_{\Psi\Theta} = \vec{U}_\Phi \cdot \frac{\partial \vec{U}_\Psi}{\partial U^\Theta} \qquad \text{first kind}$$

$$\Gamma^\Phi_{\Psi\Theta} = U^{\Phi\Sigma} \Gamma_{\Sigma\Psi\Theta} = \vec{U}^\Phi \cdot \frac{\partial \vec{U}_\Psi}{\partial U^\Theta} \qquad \text{second kind}$$

$$\frac{\partial U_{\Phi\Psi}}{\partial U^\Theta} = \Gamma_{\Psi\Phi\Theta} + \Gamma_{\Phi\Psi\Theta} \qquad \frac{dU_{11}}{dU^1} = 2\Gamma_{111}$$

$$\Gamma_{\Phi\Psi\Theta} = \frac{1}{2} \left(\frac{\partial U_{\Phi\Psi}}{\partial U^\Theta} + \frac{\partial U_{\Theta\Phi}}{\partial U^\Psi} - \frac{\partial U_{\Psi\Theta}}{\partial U^\Phi} \right) \qquad \Gamma_{111} = \frac{1}{2} \frac{dU_{11}}{dU^1}$$

$$\Gamma^\Phi_{\Psi\Theta} = \frac{1}{2} \mathrm{U}^{\Phi\Sigma} \left(\frac{\partial \mathrm{U}_{\Sigma\Psi}}{\partial \mathrm{U}^\Theta} + \frac{\partial \mathrm{U}_{\Theta\Sigma}}{\partial \mathrm{U}^\Psi} - \frac{\partial \mathrm{U}_{\Psi\Theta}}{\partial \mathrm{U}^\Sigma} \right) \qquad \Gamma^1_{11} = \frac{1}{2} \mathrm{U}^{11} \frac{\mathrm{dU}_{11}}{\mathrm{dU}^1}$$

Curve Christoffel Symbol Conversion (90) - (ref 69)

$$\Gamma^{\Phi}_{\Psi\Theta} = Z^{\Phi}_{i} \frac{\partial Z^{i}_{\Psi}}{\partial U^{\Theta}} + \Gamma^{i}_{jk} Z^{\Phi}_{i} Z^{j}_{\Psi} Z^{k}_{\Theta}$$

$$\Gamma^{\Phi}_{\Psi\Theta} = Z^{\Phi}_{i} \frac{\partial Z^{i}_{\Psi}}{\partial U^{\Theta}}$$ where Z^{i} is affine

Covariant Derivative Definition (90) - (ref 73)

$$\nabla_{\Psi} T^{\Phi} = \frac{\partial T^{\Phi}}{\partial U^{\Psi}} + \Gamma^{\Phi}_{\Psi\Theta} T^{\Theta}$$ only when $\vec{T}$ is tangent to curve

$$\nabla_{\Psi} T_{\Phi} = \frac{\partial T_{\Phi}}{\partial U^{\Psi}} - \Gamma^{\Theta}_{\Psi\Phi} T_{\Theta}$$ only when $\vec{T}$ is tangent to curve

$$\nabla_{\Psi} T^{i} = \frac{\partial T^{i}}{\partial U^{\Psi}} + Z^{k}_{\Psi} \Gamma^{i}_{km} T^{m}$$

$$\nabla_{\Psi} T_{i} = \frac{\partial T_{i}}{\partial U^{\Psi}} - Z^{k}_{\Psi} \Gamma^{m}_{ki} T_{m}$$

Covariant Dervative for Higher Rank Tensors (90) - (ref 73)

$$\nabla_{\Sigma} T^{i\Phi}_{j\Psi} = \frac{\partial T^{i\Phi}_{j\Psi}}{\partial U^{\Sigma}} + Z^{k}_{\Sigma} \Gamma^{i}_{km} T^{m\Phi}_{j\Psi} - Z^{k}_{\Sigma} \Gamma^{m}_{kj} T^{i\Phi}_{m\Psi} + \Gamma^{\Phi}_{\Sigma\Theta} T^{i\Theta}_{j\Psi} - \Gamma^{\Theta}_{\Sigma\Psi} T^{i\Phi}_{j\Theta}$$

Covariant Derivative Chain Rule (90) - (ref 72)

$$\nabla_{\Psi} T^{i} = Z^{k}_{\Psi} \nabla_{k} T^{i}$$

$$\nabla_{\Psi} T_{i} = Z^{k}_{\Psi} \nabla_{k} T_{i}$$

$$\nabla_{\Psi} T^{i}_{j} = Z^{k}_{\Psi} \nabla_{k} T^{i}_{j}$$

Differentiation for Vectors Tangent to Curve (90) - (ref 71)

$$\frac{\partial \vec{T}}{\partial U^{\Psi}} = \nabla_{\Psi} \vec{U}_{\Phi} + T^{\Phi} B_{\Phi\Psi} \hat{P}$$

$$\frac{\partial \vec{T}}{\partial U^{\Psi}} = \nabla_{\Psi} \vec{U}^{\Phi} + \mathrm{T}_{\Phi} B^{\Phi}_{\Psi} \hat{P}$$

Covariant Derivative Metrinilic Property (one exception) (90) - (ref 73)

$$\nabla_{\Psi} \vec{U}_{\Phi} = B_{\Phi\Psi} \hat{P} \; \nabla_{\Psi} \vec{U}^{\Phi} = B^{\Phi}_{\Psi} \hat{P}$$

$\nabla_{\Theta} U_{\Phi\Psi} = 0$ $\quad$ $\nabla_{\Theta} U^{\Phi\Psi} = 0$

$\nabla_{\Theta} \delta^{\Phi}_{\Psi} = 0$

$\nabla_{\Theta} \varepsilon_{\Phi} = 0$ $\quad$ $\nabla_{\Theta} \varepsilon^{\Phi} = 0$

$\nabla_{\Theta} \vec{Z}_{i} = 0$ $\quad$ $\nabla_{\Theta} \vec{Z}^{i} = 0$

$\nabla_{\Theta} Z_{ij} = 0$ $\quad$ $\nabla_{\Theta} Z^{ij} = 0$

$\nabla_{\Theta} \delta^{i}_{j} = 0$ $\quad$ $\nabla_{\Theta} \delta^{ijk}_{rst} = 0$

$\nabla_{\Theta} \varepsilon_{ijk} = 0$ $\quad$ $\nabla_{\Theta} \varepsilon^{ijk} = 0$

Contravariant Derivative (90) - (ref 73)

$$\nabla^{\Phi} = U^{\Phi\Psi}\nabla_{\Psi}$$

Laplacian (90) - (ref 73)

$$\nabla^2 f = \nabla_{\Phi}\nabla^{\Phi} f = \nabla^{\Phi}\nabla_{\Phi} f \qquad \nabla^2 f = \frac{d^2 f}{ds^2}$$

Riemann-Christoffel Tensor (90) - (ref 82)

$$R^{\Theta}_{\Sigma\Phi\Psi} = \frac{\partial\Gamma^{\Theta}_{\Psi\Sigma}}{\partial U^{\Phi}} - \frac{\partial\Gamma^{\Theta}_{\Phi\Sigma}}{\partial U^{\Psi}} + \Gamma^{\Theta}_{\Phi\Delta}\Gamma^{\Delta}_{\Psi\Sigma} - \Gamma^{\Theta}_{\Psi\Delta}\Gamma^{\Delta}_{\Phi\Sigma} \qquad R^{1}_{111} = 0$$

$$R_{\Theta\Sigma\Phi\Psi} = \frac{\partial\Gamma_{\Theta\Psi\Sigma}}{\partial U^{\Phi}} - \frac{\partial\Gamma_{\Theta\Phi\Sigma}}{\partial U^{\Psi}} + \Gamma_{\Delta\Theta\Psi}\Gamma^{\Delta}_{\Phi\Sigma} - \Gamma_{\Delta\Theta\Phi}\Gamma^{\Delta}_{\Psi\Sigma} \qquad R_{1111} = 0$$

Frenet Formulas (91)

$$\frac{d\hat{s}}{ds} = \kappa\hat{p}$$

$$\frac{d\hat{p}}{ds} = -\kappa\hat{s} + \tau\hat{q}$$

$$\frac{d\hat{q}}{ds} = -\tau\hat{p}$$

Frenet Formulas for Higher Dimensions (91)

$$\frac{d\hat{T}_0}{ds} = \kappa_1\hat{T}_1$$

$$\frac{d\hat{T}_m}{ds} = \kappa_{m+1}\hat{T}_{m+1} - \kappa_m\hat{T}_{m-1} \qquad \text{where} \quad 0 < m < n-1$$

$$\frac{d\hat{T}_{n-1}}{ds} = -\kappa_{n-1}\hat{T}_{n-2}$$

Shift Tensor Chain Relationships (92)

$$Z^{i}_{\Phi} = Z^{i}_{\alpha}S^{\alpha}_{\Phi}$$

$$Z^{\Phi}_{i} = Z^{\alpha}_{i}S^{\Phi}_{\alpha}$$

Curves Embedded in Surface Relationships (92)

$$U_{\Phi\Psi} = S_{\alpha\beta}S^{\alpha}_{\Phi}S^{\beta}_{\Psi}$$

$$\Gamma^{\Phi}_{\Psi\Theta} = S^{\Phi}_{\alpha}\frac{\partial S^{\alpha}_{\Psi}}{\partial U^{\Theta}} + \Gamma^{\alpha}_{\beta\omega}S^{\Phi}_{\alpha}S^{\beta}_{\Psi}S^{\omega}_{\Theta}$$

$$\Gamma^{\Phi}_{\Psi\Theta} = \frac{1}{2}\mathrm{U}^{\Phi\Sigma}\left(\frac{\partial \mathrm{U}_{\Sigma\Psi}}{\partial \mathrm{U}^{\Theta}} + \frac{\partial \mathrm{U}_{\Theta\Sigma}}{\partial \mathrm{U}^{\Psi}} - \frac{\partial \mathrm{U}_{\Psi\Theta}}{\partial \mathrm{U}^{\Sigma}}\right)$$

Curve Normal (93)

$$\hat{n} = n^{\alpha}\vec{S}_{\alpha} = n_{\alpha}\vec{S}^{\alpha}$$

$$n^{\alpha} = \varepsilon^{\alpha\beta}\varepsilon_{\Phi}S_{\beta}^{\Phi}$$

$$n_{\alpha} = \varepsilon_{\alpha\beta}\varepsilon^{\Phi}S_{\Phi}^{\beta}$$

Geodesic Curvature Tensor (93)

$$\nabla_{\Psi}S_{\Phi}^{\alpha} = b_{\Phi\Psi}n^{\alpha}$$

$$b_{\Phi\Psi} = n_{\alpha}\nabla_{\Psi}S_{\Phi}^{\alpha}$$

Vector Curvature Normal Projections (93)

$$\vec{B}_{\Phi}^{\Phi} = \hat{N}\left(B_{\alpha}^{\alpha} - B_{\alpha\beta}n^{\alpha}n^{\beta}\right) + \hat{n}b_{\Phi}^{\Phi}$$

Geodesic Curves (95)

$$b_{\Phi}^{\Phi} = 0$$

$$\frac{\partial^2 S^{\alpha}}{\partial U^{\Psi}\partial U^{\Phi}} + \Gamma_{\beta\omega}^{\alpha}\frac{\partial S^{\beta}}{\partial U^{\Psi}}\frac{\partial S^{\omega}}{\partial U^{\Phi}} - \Gamma_{\Psi\Phi}^{\Sigma}\frac{\partial S^{\alpha}}{\partial U^{\Sigma}} = 0$$

$$\frac{d^2 S^{\alpha}}{d\lambda^2} + \Gamma_{\beta\omega}^{\alpha}\frac{dS^{\beta}}{d\lambda}\frac{dS^{\omega}}{d\lambda} = 0$$ where λ is an affine parameter

Parallel Transport (96)

$$\frac{DT^{\alpha}}{D\lambda} = 0$$

$$\frac{dS^{\beta}}{d\lambda}\nabla_{\beta}T^{\alpha} = 0$$

$$\frac{dT^{\alpha}}{d\lambda} + \frac{dS^{\beta}}{d\lambda}\Gamma_{\beta\omega}^{\alpha}T^{\omega} = 0$$

$$dT^{\alpha} = -\Gamma_{\beta\omega}^{\alpha}T^{\omega}du^{\beta}$$

Gauss-Bonnet Theorem (98)

$$\Phi = \int_{\Omega}\mathrm{K}dA$$

$$\int_{\mathrm{S}}\mathrm{K}dA = 4\pi(1-g)$$

Holonomy Differential (99)

$$dT^{\lambda} = \mathrm{R}_{\omega\alpha\beta}^{\lambda}T^{\omega}du^{\alpha}du^{\beta}$$