Selected solutions to exercises from Pavel Grinfeld's *Introduction to Tensor Analysis and the Calculus of Moving Surfaces*

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Contents

Introduction

Included in this text are solutions to various exercises from *Introduction to Tensor Analysis and the Calculus of Moving Surfaces,* by Dr. Pavel Grinfeld.

Part I

Part I

Chapter 1

Chapter 1

Ex. 1: We have $x = 2x'$, $y = 2y'$. Thus

$$\begin{aligned} F'(x', y') &= F(2x', 2y') \\ &= (2x')^2 e^{2y'} \\ &= 4(x')^2 e^{2y'}. \end{aligned}$$

Ex. 2: Note that the above implies $x' = \frac{1}{2}x$, $y' = \frac{1}{2}y$. We check

$$\begin{aligned} \frac{\partial F'}{\partial x'}(x', y') &= 8(x') e^{2y'} \\ &= 8\left(\frac{1}{2}x\right) e^{2\left(\frac{1}{2}\right)y} \\ &= 4xe^{y} \\ \frac{\partial F}{\partial x}(x, y) &= 2xe^{y}. \end{aligned}$$

Thus, $\frac{\partial F'}{\partial x'}(x', y') = 2\frac{\partial F}{\partial x}(x, y)$ as desired.

Ex. 3: Let $a, b \in \mathbb{R}$, $a, b \neq 0$, and consider the "re-scaled" coordinate basis

$$\begin{aligned} \mathbf{i}' &= \begin{pmatrix} a \\ 0 \end{pmatrix} \\ \mathbf{j}' &= \begin{pmatrix} 0 \\ b \end{pmatrix}, \end{aligned}$$

where each of the above vectors is taken to be with respect to the standard basis for $\mathbb{R}^2$. Thus, given point (x, y) in standard coordinates, we have $x = ax'$, $y = by'$, where (x', y') is the same point in our new coordinate system. Now, let $T(x, y)$ be a differentiable function. Then,

$$\nabla T = \left(\frac{\partial F}{\partial x}(x, y), \frac{\partial F}{\partial y}(x, y)\right)$$

in standard coordinates

$$= \left(\frac{\partial F}{\partial x'}(x', y') \frac{\partial x'}{\partial x}(x, y), \frac{\partial F}{\partial y'}(x', y') \frac{\partial y'}{\partial y}(x, y) \right)$$

(think of x' as a function of x).

$$\begin{aligned}
&= \left(\frac{\partial F}{\partial x'}(x', y') \frac{1}{a}, \frac{\partial F}{\partial y'}(x', y') \frac{1}{b} \right) \\
&= \left(\frac{1}{\sqrt{a^2 + 0}} \frac{\partial F}{\partial x'}, \frac{1}{b^2 + 0} \frac{\partial F}{\partial y'} \right) \\
&= \left(\frac{1}{\sqrt{\mathbf{i}' \cdot \mathbf{i}'}} \frac{\partial F}{\partial x'}, \frac{1}{\sqrt{\mathbf{j}' \cdot \mathbf{j}'}} \frac{\partial F}{\partial y'}, \right)
\end{aligned}$$

as desired.

Ex. 4: Assume

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then,

$$\begin{aligned}
x' &= a + (\cos\alpha)\, x - (\sin\alpha)\, y \\
y' &= b + (\sin\alpha)\, x + (\cos\alpha)\, y
\end{aligned}$$

Also,

$$\begin{aligned}
\begin{pmatrix} x' - a \\ y' - b \end{pmatrix} &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}^{-1} \begin{pmatrix} x' - a \\ y' - b \end{pmatrix} \\
&= \begin{bmatrix} \frac{\cos\alpha}{\cos^2\alpha + \sin^2\alpha} & \frac{\sin\alpha}{\cos^2\alpha + \sin^2\alpha} \\ -\frac{\sin\alpha}{\cos^2\alpha + \sin^2\alpha} & \frac{\cos\alpha}{\cos^2\alpha + \sin^2\alpha} \end{bmatrix} \begin{pmatrix} x' - a \\ y' - b \end{pmatrix} \\
&= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} x' - a \\ y' - b \end{pmatrix}
\end{aligned}$$

Thus,

$$\begin{aligned}
x &= (\cos\alpha)\,(x' - a) + (\sin\alpha)\,(y' - b) \\
y &= -(\sin\alpha)\,(x' - a) + (\cos\alpha)\,(y' - b).
\end{aligned}$$

Further notice that we obtain $\mathbf{i}', \mathbf{j}'$ from the standard basis [Note: this "basis" would describe points be with respect to this point (a, b)]

$$\begin{aligned}
\mathbf{i}' &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j} \\
\mathbf{j}' &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}
\end{aligned}$$

Now, we have, given a function F, we compute

$$\begin{aligned}
\frac{\partial F}{\partial x}(x,y)\,\mathbf{i} + \frac{\partial F}{\partial y}(x,y)\,\mathbf{j} &= \left(\frac{\partial F}{\partial x'}(x',y')\,\frac{\partial x'}{\partial x}(x,y) + \frac{\partial F}{\partial y'}(x',y')\,\frac{\partial y'}{\partial x}(x,y) \right) \mathbf{i} + \left(\frac{\partial F}{\partial x'}(x',y')\,\frac{\partial x'}{\partial y}(x,y) + \frac{\partial F}{\partial y'}(x',y \right. \\
&= \left(\frac{\partial F}{\partial x'}(x',y')\cos\alpha + \frac{\partial F}{\partial y'}(x',y')\sin\alpha \right) \mathbf{i} + \left(-\frac{\partial F}{\partial x'}(x',y')\sin\alpha + \frac{\partial F}{\partial y'}(x',y')\cos\alpha \right) \mathbf{j} \\
&= \frac{\partial F}{\partial x'}(x',y')\cos\alpha\mathbf{i} - \frac{\partial F}{\partial x'}(x',y')\sin\alpha\mathbf{j} + \frac{\partial F}{\partial y'}(x',y')\sin\alpha\mathbf{i} + \frac{\partial F}{\partial y'}(x',y')\cos\alpha\mathbf{j} \\
&= \frac{\partial F}{\partial x'}(x',y')\,(\cos\alpha\mathbf{i} - \sin\alpha\mathbf{j}) + \frac{\partial F}{\partial y'}(x',y')\,(\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}) \\
&= \frac{\partial F}{\partial x'}(x',y')\begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix} + \frac{\partial F}{\partial y'}(x',y')\begin{bmatrix} \mathbf{i} & \mathbf{j} \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix} \\
&= \frac{\partial F}{\partial x'}(x',y')\,\mathbf{i}' + \frac{\partial F}{\partial y'}(x',y')\,\mathbf{j}'
\end{aligned}$$

[NOT SURE - I will ask about this one tomorrow]

(a, b) cooresponds to a shifted "origin," α corresponds to angle for which the whole coordinate system is rotated.

Ex. 5: We may obtain any affine orthogonal coordinate system by rotating the "standard" Cartesian coordinates via (1.7) and then applying a rescaling.

Chapter 2

Chapter 2

Ex. 6: See diagram.

Note: Diagrams will be added later for Ex. 7-12

Note: For Ex 7-12, let h denote the distance from P^* to P, where P^* is a point arbitrarily close to P along the appropriate direction for which we are taking each directional derivative. Define $f(h) := F(P^*)$, i.e. parametrize along the unit vector emanating from P in the direction of l (note $f(0) = F(P)$). Also, for points A,B, AB indicates the (unsigned) length of the vector from A to B.

Ex. 7:

$$\begin{aligned}
f(h) &= \sqrt{F(P)^2 + h^2} \\
f'(h) &= \frac{h}{\sqrt{F(P)^2 + h^2}} \\
\frac{dF(p)}{dl} &= f'(0)
\end{aligned}$$

Ex. 8: We have

$$\begin{aligned}
f(h) &= \frac{1}{AP - h} \\
f'(h) &= -\frac{-1}{(AP-h)^2} \\
&= \frac{1}{(AP-h)^2}
\end{aligned}$$

so

$$\begin{aligned}
\frac{dF(p)}{dl} &= f'(0) \\
&= \frac{1}{(AP)^2}
\end{aligned}$$

Ex. 9: Let ϕ denote the measure of angle OP^*P. By the Law of Sines, we have

$$\begin{aligned}
\frac{\sin(F(P^*))}{AP-h} &= \frac{\sin(\pi-\phi)}{OA} \\
&= \frac{\sin(\phi)}{OA} \\
\frac{\sin\phi}{OP} &= \frac{\sin(F(P)-F(P^*))}{h}
\end{aligned}$$

From the second equation, we obtain

$$\begin{aligned}
\sin\phi &= \frac{OP\sin(F(P)-F(P^*))}{h} \\
&= \frac{OP\left[\sin(F(P))\cos(F(P^*))-\cos(F(P))\sin(F(P^*))\right]}{h}
\end{aligned}$$

The, from the first equation, we have

$$\begin{aligned}
\frac{\sin(F(P^*))}{AP-h} &= \frac{OP\,[\sin(F(P))\cos(F(P^*))-\cos(F(P))\sin(}{(OA)\,h} \\
(OA)\,h\sin(F(P^*)) &= (OP)\,(AP-h)\,[\sin(F(P))\cos(F(P^*))-\cos(\\
(OA)\,h\tan(F(P^*)) &= (OP)\,(AP-h)\sin(F(P))-(OP)\,(AP-h)\,\mathrm{co} \\
((OA)\,h+(OP)\,(AP-h)\cos(F(P)))\tan(F(P^*)) &= (OP)\,(AP-h)\sin(F(P)) \\
\tan(F(P^*)) &= \frac{(OP)\,(AP-h)\sin(F(P))}{(OA)\,h+(OP)\,(AP-h)\cos(F(P))} \\
F(P^*) &= \arctan\left[\frac{(OP)\,(AP-h)\sin(F(P))}{(OA)\,h+(OP)\,(AP-h)\cos(F(P))}\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
f(h) &= \arctan\left[\frac{(OP)\,(AP-h)\sin(F(P))}{(OA)\,h+(OP)\,(AP-h)\cos(F(P))}\right] \\
f'(h) &= \frac{-(OP)\sin(F(P))\,[(OA)\,h+(OP)\,(AP-h)\cos(F(P))]-(OP)\sin(F(P))\,(AP-h)\,[(OA}{[(OA)\,h+(OP)\,(AP-h)\cos(F(P))]^2} \\
&= \frac{-(OP)\sin(F(P))\,[(OA)\,h+(OP)\,(AP-h)\cos(F(P))]-(OP)\sin(F(P))\,(AP-h)\,[(OA}{[(OA)\,h+(OP)\,(AP-h)\cos(F(P))]^2+[(OP)\,(AP-h)\sin(F(}
\end{aligned}$$

$$\begin{aligned}
\frac{dF(p)}{dl} &= f'(0) \\
&= \frac{-(OP)\sin(F(P))\left[(OP)(AP)\cos(F(P))\right]-(OP)\sin(F(P))(AP)\left[(OA)-(OP)(AP)\cos(F(P))\right]}{\left[(OP)(AP)\cos(F(P))\right]^2+\left[(OP)(AP)\sin(F(P))\right]^2} \\
&= \frac{-(OP)\sin(F(P))(OP)(AP)\cos(F(P))-(OP)\sin(F(P))(AP)(OA)+(OP)\sin(F(P))(AP)(OP)}{(OP)^2(AP)^2\left[\cos^2(F(P))+\sin^2(F(P))\right]} \\
&= \frac{-(OP)\sin(F(P))(AP)(OA)}{(OP)^2(AP)^2} \\
&= \frac{-(OA)}{(OP)(AP)}\sin(F(P))
\end{aligned}$$

Ex. 10: Clearly $F(P^*) = F(P)$ for any choice P^* in such a direction. Thus, f is constant, and we have

$$\frac{dF(p)}{dl} = 0$$

Ex. 11: Put d as the distance between P and the line from A to B. As with the previous problem, the distance from P^* to line $\overleftrightarrow{AB}$ is also d. Thus,

$$\begin{aligned}
F(P) &= \frac{1}{2}(AB)\,d \\
F(P^*) &= \frac{1}{2}(AB)\,d,
\end{aligned}$$

and we have $F(P^*) = F(P)$, so $\frac{dF(p)}{dl} = 0$ as before.

Ex. 12: Drop a perpendicular from P to $\overleftrightarrow{AB}$. Let K be this point of intersection. Note that the length $AK = F(P) + h$. Then,

$$\begin{aligned}
f(h) &= \frac{1}{2}(AB)(F(P)+h) \\
f'(h) &= \frac{1}{2}AB
\end{aligned}$$

$$\begin{aligned}
\frac{dF(p)}{dl} &= f'(0) \\
&= \frac{1}{2}AB
\end{aligned}$$

Ex. 13: (7) The gradient will point in direction $\overrightarrow{AP}$, and will have magnitude 1.

(8) [Not sure]

(9) The gradient will point in direction $\overrightarrow{AP}$ (in the same direction as was asked for the directional derivative), and thus will have magnitude

$$\frac{(OA)}{(OP)\,(AP)}\sin\left(F\left(P\right)\right)$$

(note $F\left(P\right)$ is assumed to satisfy $F\left(P\right)\leq\pi$)
(10) The gradient will point in direction perpendicular to $\overleftrightarrow{AB}$, and will have magnitude 1.

(11),(12) The gradient will point in the direction perpendicular to $\overleftrightarrow{AB}$ (in the same direction as was asked for the directional derivative in Ex.12), and thus will have magnitude

$$\frac{1}{2}AB.$$

Ex. 14: The directional derivative in direction L would then correspond to the projection of $\triangledown f$ onto L.

Ex. 15: [See diagram]

$$\left\|R\left(\alpha+h\right)-R\left(\alpha\right)\right\|^2=1+1-2\cos\left(h\right)$$

by the Law of Cosines. So,

$$\begin{aligned}
\left\|R\left(\alpha+h\right)-R\left(\alpha\right)\right\|^2 &= 2-2\cos\left(h\right)\\
\left\|R\left(\alpha+h\right)-R\left(\alpha\right)\right\| &= \sqrt{2-2\cos\left(h\right)}\\
&= \sqrt{2-2\cos\left(2\frac{h}{2}\right)}\\
&= \sqrt{2-2\left(\sin^2\frac{h}{2}\right)}\\
&= \sqrt{4\sin^2\frac{h}{2}}\\
&= 2\sin\frac{h}{2}.
\end{aligned}$$

Ex. 16:

$$\lim_{h\to0}\frac{2\sin\frac{h}{2}}{h}=\lim_{h\to0}\frac{2\left(\frac{1}{2}\right)\cos\frac{h}{2}}{1},$$

by L'Hospital's rule,

$$\begin{aligned} &= \lim_{h\to 0} \cos \frac{h}{2} \\ &= 1. \end{aligned}$$

Ex. 17:

$$\begin{aligned} \lim_{h\to 0} \frac{2\sin\frac{h}{2}}{h} &= \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \\ &= \lim_{h\to 0} \frac{\sin\left(0+\frac{h}{2}\right) - \sin(0)}{\frac{h}{2}} \\ &= \sin'(0) \\ &= \cos(0) \\ &= 1. \end{aligned}$$

Ex. 18: We have

$$R(\alpha) R'(\alpha) = 0.$$

Differentiating both sides, we obtain

$$R'(\alpha) \cdot R'(\alpha) + R(\alpha) \cdot R''(\alpha) = 0.$$

But, $R'(\alpha)$ is of unit length, so we have

$$\begin{aligned} 1 + R(\alpha) \cdot R''(\alpha) &= 0 \\ R(\alpha) \cdot R''(\alpha) &= -1. \end{aligned}$$

Now, let θ be the angle between $R(\alpha)$, $R''(\alpha)$. We thus have

$$\begin{aligned} \|R(\alpha)\| \|R''(\alpha)\| \cos\theta &= -1 \\ \|R''(\alpha)\| \cos\theta &= -1, \end{aligned}$$

since $R(\alpha)$ is of unit length. Now, let h be arbitrarily small. Since $\|R(\alpha)\| = \|R(\alpha+h)\| = \|R'(\alpha+h)\| = \|R'(\alpha)\| = 1$, we have by congruent triangles that $\|R'(\alpha+h) - R'(\alpha)\| = \|R(\alpha+h) - R(\alpha)\|$. Thus,

$$\begin{aligned} \|R''(\alpha)\| &= \lim_{h\to 0} \frac{\|R'(\alpha+h) - R'(\alpha)\|}{h} \\ &= \lim_{h\to 0} \frac{\|R(\alpha+h) - R(\alpha)\|}{h} \\ &= \|R'(\alpha)\| \\ &= 1. \end{aligned}$$

Thus, $R''(\alpha)$ is of unit length. We then have

$$\cos\theta = -1,$$

which implies that $\theta = \pi$, or that $R''(\alpha)$ points in the opposite direction as $R(\alpha)$.

Chapter 3

Chapter 3

Ex. 19: We may construct a three-dimensional Cartesian coordinate system as follows: Fix an origin O, then pick three points A, B, C such that the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ form an orthonormal system. Define $\mathbf{i} = \overrightarrow{OA}$, $\mathbf{j} = \overrightarrow{OB}$, $\mathbf{k} = \overrightarrow{OC}$. Note that in this coordinate system, A, B, C have coordinates

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

respectively, and a vector $\mathbf{V}$ connecting the origin to a point with coordinates

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

can be expreessed by the linear combination

$$\mathbf{V} = x_0 \mathbf{i} + y_0 \mathbf{j} + z_0 \mathbf{k}.$$

Ex. 20: Since our space is three-dimensional, there are three continuous degrees of freedom associated with our choice of origin O. The choice of the direction of the "x"-axis yield another two continuous degrees of freedom (note the bijection between the direction of the x-axis and a point on the unit sphere centered at O). Finally, the "y"-axis may be chosen to lie along any line orthogonal to the x-axis; the set of all such lines lie in a plane, hence our choice of direction for the y-axis yields the sixth continuous degree of freedom (there is a bijection between the set of all such directions and points on the unit circle which lies in this plane orthogonal to the x-axis.

Ex. 21: Let P be an arbitrary point in a two-dimensional Euclidean space with polar coordinates r, θ. Assume P has cartesian coordinates (x, y). Define P' to be the point along the pole that is distance x from the origin. Note that by the orthogonality of the x, y axes, we may form a right triangle with P, the origin O, and P'. Note that $OP' = x$ and $PP' = y$; hence by the properties of right triangles, we have

$$\begin{aligned} \frac{x}{r} &= \cos\theta \\ \frac{y}{r} &= \sin\theta, \end{aligned}$$

or

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

Ex. 22: We see from the above that

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2, \end{aligned}$$

hence we may solve for r (taken to be non-negative):

$$r = \sqrt{x^2 + y^2}.$$

Also,

$$\begin{aligned} \frac{y}{x} &= \frac{r\sin\theta}{r\cos\theta} \\ &= \tan\theta, \end{aligned}$$

so

$$\theta = \arctan\frac{y}{x}.$$

Ex. 23: Define the x and y coordinate of some arbitrary point P to be the Cartesian system of coordinates defined by applying Ex. 21 to the coordinate plane fixed in the definition of our cylindrical coordinates. Simply define the z (Cartesian) coordinate to be the signed distance from P to the coordinate plane (note the orthogonality of of x,y,z by the definition of distance to a plane - and also that x, y do not depend on z). The equations for x, y then follow from Ex. 21, and the z (Cartesian) coordinate is equal to the z (cylindrical) by definition.

Ex. 24: The inverse relationships for r,θ follow from Ex. 22, and the identity $z(x, y, z) = z$ follows trivially from 23.

Ex. 25: Let P be a point with spherical coordinates r, θ, ϕ. Let the x-axis be the polar axis, and the y-axis lie in the coordinate plane and point in the direction orthogonal to the polar axis (chosen in accordance to the right-hand rule). Finally, let hte z-axis be the longitudinal axis. Since the z-coordinate length OP', where P' is the orthogonal projection of P onto the longitudinal axis, we have by the properties of right triangles,

$$z = r \cos \theta.$$

Now, project P onto the coordinate plane, and denote this point P''. We clearly have the length $OP'' = r \sin \theta$. Thus, by considering the right triangle determined by the points O, P'', and the polar axis, we have

$$\begin{aligned} x &= (OP'') \cos \phi \\ &= r \sin \theta \cos \phi \\ y &= (OP'') \sin \phi \\ &= r \sin \theta \sin \phi. \end{aligned}$$

Ex. 26: From Ex. 25, we have

$$\begin{aligned} \sqrt{x^2 + y^2 + z^2} &= \sqrt{r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi + r^2 \cos^2 \theta} \\ &= r\sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta} \\ &= r\sqrt{\sin^2 \theta \left(\cos^2 \phi + \sin^2 \phi\right) + \cos^2 \theta} \\ &= r\sqrt{\sin^2 \theta + \cos^2 \theta} \\ &= r, \end{aligned}$$

so

$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Also, $z/r = \cos \theta$, so

$$\begin{aligned} \theta(x, y, z) &= \arccos \frac{z}{r} \\ &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}. \end{aligned}$$

Finally,

$$\begin{aligned}\frac{y}{x} &= \frac{r\sin\theta\sin\phi}{r\sin\theta\cos\phi} \\ &= \tan\phi,\end{aligned}$$

so

$$\phi\left(x,y,z\right)=\arctan\frac{y}{x}.$$

Chapter 4

Chapter 4

Ex. 27:

$$\begin{aligned}
\det J &= \det \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \\
&= \frac{x^2+y^2}{\left(x^2+y^2\right)\sqrt{x^2+y^2}} \\
&= \frac{1}{\sqrt{x^2+y^2}}.
\end{aligned}$$

Ex. 28:

$$\begin{aligned}
J(1,1) &= \begin{bmatrix} \frac{1}{\sqrt{1^2+1^2}} & \frac{1}{\sqrt{1^2+1^2}} \\ \frac{-1}{1^2+1^2} & \frac{1}{1^2+1^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.
\end{aligned}$$

Ex. 29:

$$\begin{aligned}
\det J' &= \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\
&= r\left(\cos^2\theta + \sin^2\theta\right) \\
&= r,
\end{aligned}$$

Using the relationship $r = \sqrt{x^2+y^2}$, we have $\det J \det J' = 1$.

Ex. 30:

$$\begin{aligned} J'\left(\sqrt{2}, \frac{\pi}{4}\right) &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sqrt{2}\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \sqrt{2}\cos\frac{\pi}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & 1 \end{bmatrix}. \end{aligned}$$

Ex. 31: We evaluate the product

$$\begin{aligned} JJ' &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

as desired.

Ex. 32:

$$\begin{aligned} J'(x,y) &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \frac{x}{r} & -y \\ \frac{y}{r} & x \end{bmatrix} \\ &= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{bmatrix}, \end{aligned}$$

so

$$\begin{aligned} JJ' &= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{bmatrix} \\ &= \begin{bmatrix} \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} & 0 \\ 0 & \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} \end{bmatrix} \\ &= I, \end{aligned}$$

similarly, $J'J = I$. Thus, J, J' are inverses of each other.

Ex. 33: We use

$$\begin{aligned} r(x,y,z) &= \sqrt{x^2+y^2+z^2} \\ \theta(x,y,z) &= \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \phi(x,y,z) &= \arctan\frac{y}{x} \end{aligned}$$

Note from our computation of the Laplacian in spherical coordinates, we have (after substituing expressions for x, y, z to obtain these results in terms of r, θ, ϕ):

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \sin\theta\cos\phi \\
\frac{\partial r}{\partial y} &= \sin\theta\sin\phi \\
\frac{\partial r}{\partial z} &= \cos\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \theta}{\partial x} &= \frac{\cos\theta\cos\phi}{r} \\
\frac{\partial \theta}{\partial y} &= \frac{\cos\phi}{r\sin\theta} \\
\frac{\partial \theta}{\partial z} &= -\frac{\sin\theta}{r}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= -\frac{\sin\phi}{r\sin\theta} \\
\frac{\partial \phi}{\partial y} &= \cos\phi\sin\phi \\
\frac{\partial \phi}{\partial z} &= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
J(r,\theta,\phi) &= \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{bmatrix} \\
&= \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\phi}{r\sin\theta} & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \cos\phi\sin\phi & 0 \end{bmatrix}.
\end{aligned}$$

We then compute

$$\begin{aligned}
J'(r,\theta,\phi) &= \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} \\
&= \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix},
\end{aligned}$$

So

$$
\begin{aligned}
JJ' &= \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\phi}{r\sin\theta} & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \cos\phi\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta + \cos^2\phi\sin^2\theta + \sin^2\theta\sin^2\phi & -r\cos\theta\sin\theta + r\cos\theta\cos^2\phi\sin\theta + r\cos\theta\,\mathrm{s} \\ -\frac{1}{r}\cos\theta\sin\theta + \frac{1}{r}\cos\phi\sin\phi + \frac{1}{r}\cos\theta\cos^2\phi\sin\theta & \sin^2\theta + (\cos\theta)\frac{\cos\phi}{\sin\theta}\sin\phi + \cos^2\theta\,\mathrm{cos} \\ -\frac{1}{r}\cos\phi\sin\phi + \cos\phi\sin\theta\sin^2\phi & r\cos\theta\cos\phi\sin^2\phi - (\cos\theta)\frac{\cos\phi}{\sin\theta}\,\mathrm{sin} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

[Note: Something may be off with the computation of J]

Ex. 34: We compute

$$
\begin{aligned}
\frac{\partial^2 f(\mu,\nu)}{\partial\mu^2} &= \frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial a}\frac{\partial A}{\partial\mu} + \frac{\partial F}{\partial b}\frac{\partial B}{\partial\mu} + \frac{\partial F}{\partial c}\frac{\partial C}{\partial\mu}\right] \\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial a}\right]\frac{\partial A}{\partial\mu} + \frac{\partial F}{\partial a}\frac{\partial}{\partial\mu}\left[\frac{\partial A}{\partial\mu}\right] \\
&\quad + \frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial b}\right]\frac{\partial B}{\partial\mu} + \frac{\partial F}{\partial b}\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\mu}\right] \\
&\quad + \frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial c}\right]\frac{\partial C}{\partial\mu} + \frac{\partial F}{\partial c}\frac{\partial}{\partial\mu}\left[\frac{\partial C}{\partial\mu}\right]
\end{aligned}
$$

by the product rule. We continue:

$$
\begin{aligned}
\frac{\partial f(\mu,\nu)}{\partial\mu^2} &= \left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu} + \frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu} + \frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\mu} + \frac{\partial F}{\partial a}\frac{\partial^2 A}{\partial\mu^2} \\
&\quad + \left[\frac{\partial^2 F}{\partial a\partial b}\frac{\partial A}{\partial\mu} + \frac{\partial^2 F}{\partial b^2}\frac{\partial B}{\partial\mu} + \frac{\partial^2 F}{\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial B}{\partial\mu} + \frac{\partial F}{\partial b}\frac{\partial^2 B}{\partial\mu^2} \\
&\quad + \left[\frac{\partial^2 F}{\partial a\partial c}\frac{\partial A}{\partial\mu} + \frac{\partial^2 F}{\partial b\partial c}\frac{\partial B}{\partial\mu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial\mu}\right]\frac{\partial C}{\partial\mu} + \frac{\partial F}{\partial c}\frac{\partial^2 C}{\partial\mu^2}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 f(\mu,\nu)}{\partial\mu\partial\nu} &= \frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial a}\frac{\partial A}{\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial B}{\partial\nu}+\frac{\partial F}{\partial c}\frac{\partial C}{\partial\nu}\right] \\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial a}\right]\frac{\partial A}{\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial}{\partial\mu}\left[\frac{\partial A}{\partial\nu}\right] \\
&\quad +\frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial b}\right]\frac{\partial B}{\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\nu}\right] \\
&\quad +\frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial c}\right]\frac{\partial C}{\partial\nu}+\frac{\partial F}{\partial c}\frac{\partial}{\partial\mu}\left[\frac{\partial C}{\partial\nu}\right] \\
&= \left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^2 A}{\partial\mu\partial\nu} \\
&\quad +\left[\frac{\partial^2 F}{\partial a\partial b}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial b^2}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial B}{\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^2 B}{\partial\mu\partial\nu} \\
&\quad +\left[\frac{\partial^2 F}{\partial a\partial c}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial b\partial c}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial\mu}\right]\frac{\partial C}{\partial\nu}+\frac{\partial F}{\partial c}\frac{\partial^2 C}{\partial\mu\partial\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f(\mu,\nu)}{\partial\nu^2} &= \left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\nu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial A}{\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^2 A}{\partial\nu^2} \\
&\quad +\left[\frac{\partial^2 F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2 F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial B}{\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^2 B}{\partial\nu^2} \\
&\quad +\left[\frac{\partial^2 F}{\partial a\partial c}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial b\partial c}\frac{\partial B}{\partial\nu}+\frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial\nu}\right]\frac{\partial C}{\partial\nu}+\frac{\partial F}{\partial c}\frac{\partial^2 C}{\partial\nu^2}.
\end{aligned}$$

Ex. 35: We compute

$$\begin{aligned}
\frac{\partial^3 f(\mu,\nu)}{\partial^2\mu\partial\nu} &= \frac{\partial}{\partial\mu}\left(\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^2 A}{\partial\mu\partial\nu}\right) \\
&\quad +\frac{\partial}{\partial\mu}\left(\left[\frac{\partial^2 F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2 F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial B}{\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^2 B}{\partial\mu\partial\nu}\right) \\
&\quad +\frac{\partial}{\partial\mu}\left(\left[\frac{\partial^2 F}{\partial a\partial c}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial b\partial c}\frac{\partial B}{\partial\nu}+\frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial\nu}\right]\frac{\partial C}{\partial\nu}+\frac{\partial F}{\partial c}\frac{\partial^2 C}{\partial\mu\partial\nu}\right) \\
&= I+J+K,
\end{aligned}$$

for sake of clarity,

$$
\begin{aligned}
I &= \frac{\partial}{\partial\mu}\left(\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^2 A}{\partial\mu\partial\nu}\right)\\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}\\
&\quad +\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial}{\partial\mu}\left[\frac{\partial A}{\partial\nu}\right]+\frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial a}\right]\frac{\partial^2 A}{\partial\mu\partial\nu}+\frac{\partial F}{\partial a}\left[\frac{\partial^3 A}{\partial^2\mu\partial\nu}\right]\\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}\\
&\quad +\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2 A}{\partial\mu\partial\nu}\\
&\quad +\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2 A}{\partial\mu\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^3 A}{\partial^2\mu\partial\nu}\\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}\\
&\quad +2\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2 A}{\partial\mu\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^3 A}{\partial^2\mu\partial\nu}\\
&= \left(\frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a^2}\right]\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a^2}\frac{\partial}{\partial\mu}\left[\frac{\partial A}{\partial\mu}\right]+\frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a\partial b}\right]\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\mu}\right]+\frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a\partial c}\right]\frac{\partial C}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\right.\\
&\quad +2\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2 A}{\partial\mu\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^3 A}{\partial^2\mu\partial\nu}\\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a^2}\right]\frac{\partial A}{\partial\mu}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial a^2}\frac{\partial}{\partial\mu}\left[\frac{\partial A}{\partial\mu}\right]\frac{\partial A}{\partial\nu}\\
&\quad +\frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a\partial b}\right]\frac{\partial B}{\partial\mu}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\mu}\right]\frac{\partial A}{\partial\nu}\\
&\quad +\frac{\partial}{\partial\mu}\left[\frac{\partial^2 F}{\partial a\partial c}\right]\frac{\partial C}{\partial\mu}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial}{\partial\mu}\left[\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}\\
&\quad +2\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2 A}{\partial\mu\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^3 A}{\partial^2\mu\partial\nu}\\
&= \left[\frac{\partial^3 F}{\partial a^3}\frac{\partial A}{\partial\mu}+\frac{\partial^3 F}{\partial a^2\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^3 F}{\partial a^2\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\mu}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial a^2}\frac{\partial^2 A}{\partial\mu^2}\frac{\partial A}{\partial\nu}\\
&\quad +\left[\frac{\partial^3 F}{\partial a^2\partial b}\frac{\partial A}{\partial\mu}+\frac{\partial^3 F}{\partial a\partial b^2}\frac{\partial B}{\partial\mu}+\frac{\partial^3 F}{\partial a\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial B}{\partial\mu}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial^2 B}{\partial\mu^2}\frac{\partial A}{\partial\nu}\\
&\quad +\left[\frac{\partial^3 F}{\partial a^2\partial c}\frac{\partial A}{\partial\mu}+\frac{\partial^3 F}{\partial a\partial b\partial c}\frac{\partial B}{\partial\mu}+\frac{\partial^3 F}{\partial a\partial c^2}\frac{\partial C}{\partial\mu}\right]\frac{\partial C}{\partial\mu}\frac{\partial A}{\partial\nu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial^2 C}{\partial\mu^2}\frac{\partial A}{\partial\nu}\\
&\quad +2\left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial b}\frac{\partial B}{\partial\mu}+\frac{\partial^2 F}{\partial a\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2 A}{\partial\mu\partial\nu}+\frac{\partial F}{\partial a}\frac{\partial^3 A}{\partial^2\mu\partial\nu}
\end{aligned}
$$

$$\begin{aligned}
J &= \frac{\partial}{\partial\mu}\left(\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial B}{\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^2B}{\partial\mu\partial\nu}\right)\\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial B}{\partial\nu}\\
&\quad+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\nu}\right]+\frac{\partial}{\partial\mu}\left[\frac{\partial F}{\partial b}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial}{\partial\mu}\left[\frac{\partial^2B}{\partial\mu\partial\nu}\right]\\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial B}{\partial\nu}\\
&\quad+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\mu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\mu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^3B}{\partial\mu^2\partial\nu}\\
&= \left(\frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial a\partial b}\right]\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial a\partial b}\frac{\partial}{\partial\mu}\left[\frac{\partial A}{\partial\nu}\right]+\frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial b^2}\right]\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\nu}\right]+\frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial b\partial c}\right]\frac{\partial C}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial}{\partial\mu}\left[\frac{\partial C}{\partial\nu}\right]\right)\\
&\quad+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\mu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\mu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^3B}{\partial\mu^2\partial\nu}\\
&= \frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial a\partial b}\right]\frac{\partial A}{\partial\nu}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial a\partial b}\frac{\partial}{\partial\mu}\left[\frac{\partial A}{\partial\nu}\right]\frac{\partial B}{\partial\nu}\\
&\quad+\frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial b^2}\right]\left(\frac{\partial B}{\partial\nu}\right)^2+\frac{\partial^2F}{\partial b^2}\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\nu}\right]\frac{\partial B}{\partial\nu}\\
&\quad+\frac{\partial}{\partial\mu}\left[\frac{\partial^2F}{\partial b\partial c}\right]\frac{\partial C}{\partial\nu}\frac{\partial B}{\partial\nu}+\frac{\partial}{\partial\mu}\left[\frac{\partial C}{\partial\nu}\right]\frac{\partial B}{\partial\nu}\\
&\quad+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\mu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\mu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^3B}{\partial\mu^2\partial\nu}\\
&= \left[\frac{\partial^3F}{\partial a^2\partial b}\frac{\partial A}{\partial\mu}+\frac{\partial^3F}{\partial a\partial b^2}\frac{\partial B}{\partial\mu}+\frac{\partial^3F}{\partial a\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial A}{\partial\nu}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial a\partial b}\frac{\partial^2A}{\partial\mu\partial\nu}\frac{\partial B}{\partial\nu}\\
&\quad+\left[\frac{\partial^3F}{\partial a\partial b^2}\frac{\partial A}{\partial\mu}+\frac{\partial^3F}{\partial b^3}\frac{\partial B}{\partial\mu}+\frac{\partial^3F}{\partial b^2\partial c}\frac{\partial C}{\partial\mu}\right]\left(\frac{\partial B}{\partial\nu}\right)^2+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\mu\partial\nu}\frac{\partial B}{\partial\nu}\\
&\quad+\left[\frac{\partial^3F}{\partial a\partial b\partial c}\frac{\partial A}{\partial\mu}+\frac{\partial^3F}{\partial b^2\partial c}\frac{\partial B}{\partial\mu}+\frac{\partial^3F}{\partial b\partial c^2}\frac{\partial C}{\partial\mu}\right]\frac{\partial C}{\partial\nu}\frac{\partial B}{\partial\nu}+\frac{\partial^2C}{\partial\mu\partial\nu}\frac{\partial B}{\partial\nu}\\
&\quad+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\nu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\nu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\nu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\left[\frac{\partial^2F}{\partial a\partial b}\frac{\partial A}{\partial\mu}+\frac{\partial^2F}{\partial b^2}\frac{\partial B}{\partial\mu}+\frac{\partial^2F}{\partial b\partial c}\frac{\partial C}{\partial\mu}\right]\frac{\partial^2B}{\partial\mu\partial\nu}+\frac{\partial F}{\partial b}\frac{\partial^3B}{\partial\mu^2\partial\nu}
\end{aligned}$$

$$
\begin{aligned}
K &= \frac{\partial}{\partial \mu}\left(\left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial C}{\partial \nu} + \frac{\partial F}{\partial c}\frac{\partial^2 C}{\partial \mu \partial \nu}\right) \\
&= \frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial C}{\partial \nu} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial}{\partial \mu}\left[\frac{\partial C}{\partial \nu}\right] + \frac{\partial}{\partial \mu}\left[\frac{\partial F}{\partial c}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \frac{\partial F}{\partial c}\frac{\partial}{\partial \mu}\left[\frac{\partial^2 C}{\partial \mu \partial \nu}\right] \\
&= \frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial C}{\partial \nu} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \frac{\partial F}{\partial c}\frac{\partial}{\partial \mu} \\
&= \left(\frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial a \partial c}\right]\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial a \partial c}\frac{\partial}{\partial \mu}\left[\frac{\partial A}{\partial \nu}\right] + \frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial b \partial c}\right]\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial}{\partial \mu}\left[\frac{\partial B}{\partial \nu}\right] + \frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial c^2}\right]\frac{\partial C}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\right. \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \frac{\partial F}{\partial c}\frac{\partial}{\partial \mu} \\
&= \frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial a \partial c}\right]\frac{\partial A}{\partial \nu}\frac{\partial C}{\partial \nu} + \frac{\partial^2 F}{\partial a \partial c}\frac{\partial}{\partial \mu}\left[\frac{\partial A}{\partial \nu}\right]\frac{\partial C}{\partial \nu} \\
&\quad + \frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial b \partial c}\right]\frac{\partial B}{\partial \nu}\frac{\partial C}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial}{\partial \mu}\left[\frac{\partial B}{\partial \nu}\right]\frac{\partial C}{\partial \nu} \\
&\quad + \frac{\partial}{\partial \mu}\left[\frac{\partial^2 F}{\partial c^2}\right]\left(\frac{\partial C}{\partial \nu}\right)^2 + \frac{\partial^2 F}{\partial c^2}\frac{\partial}{\partial \mu}\left[\frac{\partial C}{\partial \nu}\right]\frac{\partial C}{\partial \nu} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \frac{\partial F}{\partial c}\frac{\partial}{\partial \mu} \\
&= \left[\frac{\partial^3 F}{\partial a^2 \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^3 F}{\partial a \partial b \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^3 F}{\partial a \partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial A}{\partial \nu}\frac{\partial C}{\partial \nu} + \frac{\partial^2 F}{\partial a \partial c}\frac{\partial^2 A}{\partial \mu \partial \nu}\frac{\partial C}{\partial \nu} \\
&\quad + \left[\frac{\partial^3 F}{\partial a \partial b \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^3 F}{\partial b^2 \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^3 F}{\partial b \partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial B}{\partial \nu}\frac{\partial C}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial^2 B}{\partial \mu \partial \nu}\frac{\partial C}{\partial \nu} \\
&\quad + \left[\frac{\partial^3 F}{\partial a \partial c^2}\frac{\partial A}{\partial \mu} + \frac{\partial^3 F}{\partial b \partial c^2}\frac{\partial B}{\partial \mu} + \frac{\partial^3 F}{\partial c^3}\frac{\partial C}{\partial \mu}\right]\left(\frac{\partial C}{\partial \nu}\right)^2 + \frac{\partial^2 F}{\partial c^2}\frac{\partial^2 C}{\partial \mu \partial \nu}\frac{\partial C}{\partial \nu} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \nu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \nu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \nu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial^2 C}{\partial \mu \partial \nu} + \frac{\partial F}{\partial c}\frac{\partial}{\partial \mu}
\end{aligned}
$$

Ex. 36

$$\begin{aligned}
\frac{\partial f(\mu,\nu)}{\partial \mu^2} &= \left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial a \partial b}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial a \partial c}\frac{\partial C}{\partial \mu}\right]\frac{\partial A}{\partial \mu} + \frac{\partial F}{\partial a}\frac{\partial^2 A}{\partial \mu^2} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial b}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b^2}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial C}{\partial \mu}\right]\frac{\partial B}{\partial \mu} + \frac{\partial F}{\partial b}\frac{\partial^2 B}{\partial \mu^2} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial C}{\partial \mu} + \frac{\partial F}{\partial c}\frac{\partial^2 C}{\partial \mu^2} \\
&= \frac{\partial^2 F}{\partial a^i \partial a^j}\frac{\partial A^j}{\partial \mu}\frac{\partial A^i}{\partial \mu} + \frac{\partial F}{\partial a^i}\frac{\partial^2 A^i}{\partial \mu^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f(\mu,\nu)}{\partial \mu \partial \nu} &= \left[\frac{\partial^2 F}{\partial a^2}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial a \partial b}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial a \partial c}\frac{\partial C}{\partial \mu}\right]\frac{\partial A}{\partial \nu} + \frac{\partial F}{\partial a}\frac{\partial^2 A}{\partial \mu \partial \nu} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial b}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b^2}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial C}{\partial \mu}\right]\frac{\partial B}{\partial \nu} + \frac{\partial F}{\partial b}\frac{\partial^2 B}{\partial \mu \partial \nu} \\
&\quad + \left[\frac{\partial^2 F}{\partial a \partial c}\frac{\partial A}{\partial \mu} + \frac{\partial^2 F}{\partial b \partial c}\frac{\partial B}{\partial \mu} + \frac{\partial^2 F}{\partial c^2}\frac{\partial C}{\partial \mu}\right]\frac{\partial C}{\partial \nu} + \frac{\partial F}{\partial c}\frac{\partial^2 C}{\partial \mu \partial \nu} \\
&= \frac{\partial^2 F}{\partial a^i \partial a^j}\frac{\partial A^j}{\partial \mu}\frac{\partial A^i}{\partial \nu} + \frac{\partial F}{\partial a^i}\frac{\partial^2 A^i}{\partial \mu \partial \nu}
\end{aligned}$$

$$\frac{\partial f(\mu,\nu)}{\partial \nu^2} = \frac{\partial^2 F}{\partial a^i \partial a^j}\frac{\partial A^j}{\partial \nu}\frac{\partial A^i}{\partial \nu} + \frac{\partial F}{\partial a^i}\frac{\partial^2 A^i}{\partial \nu^2}$$

Ex. 37 We may generalize the above three equations, setting $\mu^1 = \mu$, $\mu^2 = \nu$, to yield

$$\frac{\partial^2 f(\mu,\nu)}{\partial \mu^\alpha \partial \mu^\beta} = \frac{\partial^2 F}{\partial a^i \partial a^j}\frac{\partial A^j}{\partial \mu^\alpha}\frac{\partial A^i}{\partial \mu^\beta} + \frac{\partial F}{\partial a^i}\frac{\partial^2 A^i}{\partial \mu^\alpha \partial \mu^\beta}.$$

This encompasses three separate identities, since we have been assuming that we may switch the order of partial differentiation throughout.

Ex. 38 [Not finished]

Ex. 39 Begin with

$$\cos \arccos x = x$$

and differentiate both sides:

$$\begin{aligned}
\frac{d}{dx}[\cos \arccos x] &= 1 \\
-(\sin \arccos x)\frac{d}{dx}[\arccos x] &= 1 \\
\frac{d}{dx}[\arccos x] &= \frac{-1}{(\sin \arccos x)}
\end{aligned}$$

By examining triangles with unit hypotenuse, we obtain

$$\sin \arccos x = \pm\sqrt{1-x^2},$$

so

$$\frac{d}{dx}\left[\arccos x\right] = \pm\frac{1}{\sqrt{1-x^2}}$$

Ex. 40: We know that f, g satisfy

$$g'\left(f\left(x\right)\right) f'\left(x\right) = 1.$$

Differentiating both sides, we obtain

$$\begin{aligned}
\frac{d}{dx}\left[g'\left(f\left(x\right)\right)\right] f'\left(x\right) + g'\left(f\left(x\right)\right)\frac{d}{dx}\left[f'\left(x\right)\right] &= 0 \\
g''\left(f\left(x\right)\right) f'\left(x\right) f'\left(x\right) + g'\left(f\left(x\right)\right) f''\left(x\right) &= 0
\end{aligned}$$

$$g''\left(f\left(x\right)\right)\left[f'\left(x\right)\right]^2 + g'\left(f\left(x\right)\right) f''\left(x\right) = 0 \tag{4.1}$$

as desired.

Ex. 41: We compute

$$\begin{aligned}
f'\left(x\right) &= e^x \\
f''\left(x\right) &= e^x \\
g'\left(x\right) &= \frac{1}{x} \\
g''\left(x\right) &= -\frac{1}{x^2},
\end{aligned}$$

So

$$\begin{aligned}
g''\left(f\left(x\right)\right)\left[f'\left(x\right)\right]^2 + g'\left(f\left(x\right)\right) f''\left(x\right) &= -\frac{1}{e^{2x}}e^{2x} + \frac{1}{e^x}e^x \\
&= -1+1 \\
&= 0,
\end{aligned}$$

as desired.

Ex. 42: We compute

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-x^2}} \\ f''(x) &= -2x\frac{1}{2}\left(1-x^2\right)^{-3/2} \\ &= -x\left(1-x^2\right)^{-3/2} \\ g'(x) &= -\sin(x) \\ g''(x) &= -\cos(x), \end{aligned}$$

So

$$\begin{aligned} g''(f(x))\left[f'(x)\right]^2 + g'(f(x))\,f''(x) &= -\cos(\arccos x)\frac{1}{1-x^2} - \sin(\arccos(x))\left(-x\left(1-x^2\right)^{-3/2}\right) \\ &= -\frac{x}{1-x^2} + \frac{x\sqrt{1-x^2}}{\sqrt{1-x^2}^3} \\ &= -\frac{x}{1-x^2} + \frac{x}{1-x^2} \\ &= 0, \end{aligned}$$

as desired.

Ex. 43: We differentiate both sides of the second-order relationship to obtain

$$\begin{aligned} \frac{d}{dx}\left(g''(f(x))\left[f'(x)\right]^2 + g'(f(x))\,f''(x)\right) &= 0 \\ \frac{d}{dx}\left[g''(f(x))\right]\left[f'(x)\right]^2 + g''(f(x))\frac{d}{dx}\left[\left[f'(x)\right]^2\right] + \frac{d}{dx}\left[g'(f(x))\right]f''(x) + g'(f(x))\frac{d}{dx}f''(x) &= 0 \\ g^{(3)}(f(x))\left[f'(x)\right]^3 + g''(f(x))\cdot 2f'(x)\,f''(x) + g''(f(x))\,f'(x)\,f''(x) + g'(f(x))\,f^{(3)}(x) &= 0 \\ g^{(3)}(f(x))\left[f'(x)\right]^3 + 3g''(f(x))\,f'(x)\,f''(x) + g'(f(x))\,f^{(3)}(x) &= 0 \end{aligned}$$

Ex. 44:
Ex. 45:
Ex. 46:
Ex. 47:

Ex. 48: We begin with the identity (note the top indices should be considered "first")

$$J^i_{i'}J^{i'}_j = \delta^i_j,$$

and write out the dependences on unprimed coordinates:

$$J^i_{i'}\left(Z'(Z)\right)J^{i'}_j(Z) = \delta^i_j(Z) \tag{4.2}$$

(note, however, that the Krönicker delta is constant with respect to the unprimed coordinates Z). We differentiate both sides of (4.2) with respect to Z^k:

$$\begin{aligned}
\frac{\partial}{\partial Z^k}\left[J^i_{i'}\left(Z'\left(Z\right)\right)J^{i'}_j\left(Z\right)\right] &= \frac{\partial}{\partial Z^k}\left[\delta^i_j\left(Z\right)\right] \\
\frac{\partial}{\partial Z^k}\left[J^i_{i'}\left(Z'\left(Z\right)\right)J^{i'}_j\left(Z\right)\right] &= 0 \\
\frac{\partial}{\partial Z^k}\left[J^i_{i'}\left(Z'\left(Z\right)\right)\right]J^{i'}_j\left(Z\right) + J^i_{i'}\left(Z'\left(Z\right)\right)\frac{\partial}{\partial Z^k}\left[J^{i'}_j\left(Z\right)\right] &= 0,
\end{aligned}$$

since differentiation passes through the implied summation over i'. Then, using the definition of the Jacobian,

$$\frac{\partial}{\partial Z^k}\left[\frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\right]\frac{\partial Z^{i\prime}}{\partial Z^j}\left(Z\right) + \frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\frac{\partial}{\partial Z^k}\left[\frac{\partial Z^{i\prime}}{\partial Z^j}\left(Z\right)\right] = 0 \tag{4.3}$$

$$\frac{\partial^2 Z^i}{\partial Z^{k'}\partial Z^{i'}}\left(Z'\left(Z\right)\right)\frac{\partial Z^{k'}}{\partial Z^k}\left(Z\right)\frac{\partial Z^{i\prime}}{\partial Z^j}\left(Z\right) + \frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\frac{\partial^2 Z^{i'}}{\partial Z^k \partial Z^j}\left(Z\right) = 0, \tag{4.4}$$

applying the chain rule to the first term, and implying summation over new index k'. Then, if we define the "Hessian" object

$$J^i_{k',i'} := \frac{\partial^2 Z^i}{\partial Z^{k'}\partial Z^{i'}}\left(Z'\right)$$

with an analogous definition for $J^{i'}_{k,i}$, we write (4.3) concisely:

$$J^i_{k',i'}J^{k'}_k J^{i'}_j + J^i_{i'}J^{i'}_{k,j} = 0,$$

or, using a renaming of dummy indicex k' to j' and a reversing of the order of partial derivatives,

$$J^i_{i'j'}J^{i'}_j J^{j'}_k + J^i_{i'}J^{i'}_{jk} = 0.$$

This above tensor relationship represents n^3 identities.

Ex. 49: Since each $J^i_{i'}$ is constant for a transformation from one affine coordinate system to another, each second derivative vanishes, and hence each $J^i_{i'j'} = 0$.

Ex. 50: Begin with the identity derived in Ex. 48:

$$J^i_{i'j'}J^{i'}_j J^{j'}_k + J^i_{i'}J^{i'}_{jk} = 0,$$

Then, letting k' be arbitrary, we multiply both sides by $J^k_{k'}$, implying summation over k:

$$\begin{aligned}\left[J^i_{i'j'}J^{i'}_jJ^{j'}_k + J^i_{i'}J^{i'}_{jk}\right]J^k_{k'} &= 0\\ J^i_{i'j'}J^{i'}_jJ^{j'}_kJ^k_{k'} + J^i_{i'}J^{i'}_{jk}J^k_{k'} &= 0\end{aligned}$$

but, $J^{j'}_kJ^k_{k'} = \delta^{j'}_{k'}$, so

$$J^i_{i'j'}J^{i'}_j\delta^{j'}_{k'} + J^i_{i'}J^{i'}_{jk}J^k_{k'} = 0.$$

Note that we have $\delta^{j'}_{k'} = 1$ if and only if $j' = k'$, so the first term is equal to $J^i_{i'k'}J^{i'}_j$. After re-naming $k' = j'$, we obtain

$$J^i_{i'j'}J^{i'}_j + J^i_{i'}J^{i'}_{jk}J^k_{j'} = 0. \tag{4.5}$$

Ex. 51: Let k' be arbitrary, and multiply both sides of (4.5) by $J^j_{k'}$, implying summation over j:

$$\begin{aligned}\left[J^i_{i'j'}J^{i'}_j + J^i_{i'}J^{i'}_{jk}J^k_{j'}\right]J^j_{k'} &= 0\\ J^i_{i'j'}J^{i'}_jJ^j_{k'} + J^i_{i'}J^{i'}_{jk}J^k_{j'}J^j_{k'} &= 0\\ J^i_{i'j'}\delta^{i'}_{k'} + J^i_{i'}J^{i'}_{jk}J^k_{j'}J^j_{k'} &= 0\\ J^i_{i'j'}\delta^{i'}_{k'} + J^{i'}_{jk}J^i_{i'}J^k_{j'}J^j_{k'} &= 0.\end{aligned}$$

Rename the dummy index in the second term $i' = h'$. Then,

$$J^i_{i'j'}\delta^{i'}_{k'} + J^{h'}_{jk}J^i_{h'}J^k_{j'}J^j_{k'} = 0.$$

Noting that the first term is zero for all $i' \neq k'$, and setting $k' = i'$:

$$J^i_{i'j'} + J^{h'}_{jk}J^i_{h'}J^k_{j'}J^j_{i'} = 0.$$

We then may re-introduce k' as a dummy index:

$$J^i_{i'j'} + J^{k'}_{jk}J^i_{k'}J^k_{j'}J^j_{i'} = 0.$$

Then, switch the roles of j, k as dummy indices:

$$J^i_{i'j'} + J^{k'}_{kj}J^i_{k'}J^j_{j'}J^k_{i'} = 0$$

Ex. 52: Return to

$$J^i_{i'j'}J^{i'}_jJ^{j'}_k + J^i_{i'}J^{i'}_{jk} = 0,$$

and write out the dependences

$$\begin{aligned}
J^i_{i'j'}\left(Z'\left(Z\right)\right)J^{i'}_j\left(Z\right)J^{j'}_k\left(Z\right) + J^i_{i'}\left(Z'\left(Z\right)\right)J^{i'}_{jk}\left(Z\right) &= 0\\
\frac{\partial^2 Z^i}{\partial Z^{i'}\partial Z^{j'}}\left(Z'\left(Z\right)\right)\frac{\partial Z^{i'}}{\partial Z^j}\left(Z\right)\frac{\partial Z^{j'}}{\partial Z^k}\left(Z\right) + \frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\frac{\partial^2 Z^{i'}}{\partial Z^j\partial Z^k}\left(Z\right) &= 0
\end{aligned}$$

Then, differentiate both sides with respect to Z^m:

$$\begin{aligned}
0 &= \frac{\partial}{\partial Z^m}\left[\frac{\partial^2 Z^i}{\partial Z^{i'}\partial Z^{j'}}\left(Z'\left(Z\right)\right)\frac{\partial Z^{i'}}{\partial Z^j}\left(Z\right)\frac{\partial Z^{j'}}{\partial Z^k}\left(Z\right) + \frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\frac{\partial^2 Z^{i'}}{\partial Z^j\partial Z^k}\left(Z\right)\right]\\
&= \frac{\partial}{\partial Z^m}\left[\frac{\partial^2 Z^i}{\partial Z^{i'}\partial Z^{j'}}\left(Z'\left(Z\right)\right)\right]\frac{\partial Z^{i'}}{\partial Z^j}\left(Z\right)\frac{\partial Z^{j'}}{\partial Z^k}\left(Z\right) + \frac{\partial^2 Z^i}{\partial Z^{i'}\partial Z^{j'}}\left(Z'\left(Z\right)\right)\frac{\partial}{\partial Z^m}\left[\frac{\partial Z^{i'}}{\partial Z^j}\left(Z\right)\right]\frac{\partial Z^{j'}}{\partial Z^k}\left(Z\right) + \\
&\quad + \frac{\partial}{\partial Z^m}\left[\frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\right]\frac{\partial^2 Z^{i'}}{\partial Z^j\partial Z^k}\left(Z\right) + \frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\frac{\partial}{\partial Z^m}\left[\frac{\partial^2 Z^{i'}}{\partial Z^j\partial Z^k}\left(Z\right)\right]\\
&= \left[\frac{\partial^3 Z^i}{\partial Z^{i'}\partial Z^{j'}\partial Z^{m'}}\left(Z'\left(Z\right)\right)\frac{\partial Z^{m'}}{\partial Z^m}\left(Z\right)\right]\frac{\partial Z^{i'}}{\partial Z^j}\left(Z\right)\frac{\partial Z^{j'}}{\partial Z^k}\left(Z\right) + \frac{\partial^2 Z^i}{\partial Z^{i'}\partial Z^{j'}}\left(Z'\left(Z\right)\right)\frac{\partial^2 Z^{i'}}{\partial Z^m\partial Z^j}\left(Z\right)\frac{\partial Z^{j'}}{\partial Z^k}\\
&\quad + \left[\frac{\partial^2 Z^i}{\partial Z^{i'}\partial Z^{m'}}\left(Z'\left(Z\right)\right)\frac{\partial Z^{m'}}{\partial Z^m}\left(Z\right)\right]\frac{\partial^2 Z^{i'}}{\partial Z^j\partial Z^k}\left(Z\right) + \frac{\partial Z^i}{\partial Z^{i'}}\left(Z'\left(Z\right)\right)\frac{\partial^3 Z^{i'}}{\partial Z^j\partial Z^k\partial Z^m}\left(Z\right)\\
&= J^i_{i'j'm'}J^{m'}_mJ^{i'}_jJ^{j'}_k + J^i_{i'j'}J^{i'}_{jm}J^{j'}_k + J^i_{i'j'}J^{i'}_jJ^{j'}_{km} + J^i_{i'm'}J^{m'}_mJ^{i'}_{jk} + J^i_{i'}J^{i'}_{jkm},
\end{aligned}$$

so, setting $k' = m'$ as a dummy index:

$$J^i_{i'j'k'}J^{i'}_jJ^{j'}_kJ^{k'}_m + J^i_{i'j'}J^{j'}_kJ^{i'}_{jm} + J^i_{i'j'}J^{i'}_jJ^{j'}_{km} + J^i_{i'k'}J^{i'}_{jk}J^{k'}_m + J^i_{i'}J^{i'}_{jkm} = 0. \quad (4.6)$$

Then, multiply both sides by $J^m_{m'}$, implying summation over m:

$$\begin{aligned}
J^i_{i'j'k'}J^{i'}_jJ^{j'}_kJ^{k'}_mJ^m_{m'} + J^i_{i'j'}J^{j'}_kJ^{i'}_{jm}J^m_{m'} + J^i_{i'j'}J^{i'}_jJ^{j'}_{km}J^m_{m'} + J^i_{i'k'}J^{i'}_{jk}J^{k'}_mJ^m_{m'} + J^i_{i'}J^{i'}_{jkm}J^m_{m'} &= 0\\
J^i_{i'j'k'}J^{i'}_jJ^{j'}_k\delta^{k'}_{m'} + J^i_{i'j'}J^{j'}_kJ^{i'}_{jm}J^m_{m'} + J^i_{i'j'}J^{i'}_jJ^{j'}_{km}J^m_{m'} + J^i_{i'k'}J^{i'}_{jk}\delta^{k'}_{m'} + J^i_{i'}J^{i'}_{jkm}J^m_{m'} &= 0,
\end{aligned}$$

This holds for all m', so specifically for $m' = k'$, the above identity reads

$$J^i_{i'j'k'}J^{i'}_jJ^{j'}_k + J^i_{i'j'}J^{j'}_kJ^{i'}_{jm}J^m_{k'} + J^i_{i'j'}J^{i'}_jJ^{j'}_{km}J^m_{k'} + J^i_{i'k'}J^{i'}_{jk} + J^i_{i'}J^{i'}_{jkm}J^m_{k'} = 0 \quad (4.7)$$

Next, in an analogous manner, multiply both sides by $J^k_{m'}$ for arbitrary m':

$$\begin{aligned}
J^i_{i'j'k'}J^{i'}_jJ^{j'}_kJ^k_{m'}+J^i_{i'j'}J^{j'}_kJ^{i'}_{jm}J^m_{k'}J^k_{m'}+J^i_{i'j'}J^{i'}_jJ^{j'}_{km}J^m_{k'}J^k_{m'}+J^i_{i'k'}J^{i'}_{jk}J^k_{m'}+J^i_{i'}J^{i'}_{jkm}J^m_{k'}J^k_{m'} &= 0\\
J^i_{i'j'k'}J^{i'}_j\delta^{j'}_{m'}+J^i_{i'j'}J^{j'}_kJ^{i'}_{jm}J^m_{k'}J^k_{m'}+J^i_{i'j'}J^{i'}_jJ^{j'}_{km}J^m_{k'}J^k_{m'}+J^i_{i'k'}J^{i'}_{jk}J^k_{m'}+J^i_{i'}J^{i'}_{jkm}J^m_{k'}J^k_{m'} &= 0,
\end{aligned}$$

rename the dummy index $h'=j'$ in all but the first term:

$$J^i_{i'j'k'}J^{i'}_j\delta^{j'}_{m'}+J^i_{i'h'}J^{h'}_kJ^{i'}_{jm}J^m_{k'}J^k_{m'}+J^i_{i'h'}J^{i'}_jJ^{h'}_{km}J^m_{k'}J^k_{m'}+J^i_{i'k'}J^{i'}_{jk}J^k_{m'}+J^i_{i'}J^{i'}_{jkm}J^m_{k'}J^k_{m'}=0,$$

then, as in the previous exercises, set $m'=j'$:

$$J^i_{i'j'k'}J^{i'}_j+J^i_{i'h'}J^{h'}_kJ^{i'}_{jm}J^m_{k'}J^k_{j'}+J^i_{i'h'}J^{i'}_jJ^{h'}_{km}J^m_{k'}J^k_{j'}+J^i_{i'k'}J^{i'}_{jk}J^k_{j'}+J^i_{i'}J^{i'}_{jkm}J^m_{k'}J^k_{j'}=0, \tag{4.8}$$

Finally, multiply both sides by $J^j_{m'}$:

$$\begin{aligned}
J^i_{i'j'k'}J^{i'}_jJ^j_{m'}+J^i_{i'h'}J^{h'}_kJ^{i'}_{jm}J^m_{k'}J^k_{j'}J^j_{m'}+J^i_{i'h'}J^{i'}_jJ^{h'}_{km}J^m_{k'}J^k_{j'}J^j_{m'}+J^i_{i'k'}J^{i'}_{jk}J^k_{j'}J^j_{m'}+J^i_{i'}J^{i'}_{jkm}J^m_{k'}J^k_{j'}J^j_{m'} &= 0\\
J^i_{i'j'k'}\delta^{i'}_{m'}+J^i_{i'h'}J^{h'}_kJ^{i'}_{jm}J^m_{k'}J^k_{j'}J^j_{m'}+J^i_{i'h'}J^{i'}_jJ^{h'}_{km}J^m_{k'}J^k_{j'}J^j_{m'}+J^i_{i'k'}J^{i'}_{jk}J^k_{j'}J^j_{m'}+J^i_{i'}J^{i'}_{jkm}J^m_{k'}J^k_{j'}J^j_{m'} &= 0,
\end{aligned}$$

rename the dummy index i' to g' in all but the first term:

$$J^i_{i'j'k'}\delta^{i'}_{m'}+J^i_{g'h'}J^{h'}_kJ^{g'}_{jm}J^m_{k'}J^k_{j'}J^j_{m'}+J^i_{g'h'}J^{g'}_jJ^{h'}_{km}J^m_{k'}J^k_{j'}J^j_{m'}+J^i_{g'k'}J^{g'}_{jk}J^k_{j'}J^j_{m'}+J^i_{g'}J^{g'}_{jkm}J^m_{k'}J^k_{j'}J^j_{m'}=0.$$

Then, set $m'=i'$:

$$J^i_{i'j'k'}+J^i_{g'h'}J^{h'}_kJ^{g'}_{jm}J^m_{k'}J^k_{j'}J^j_{i'}+J^i_{g'h'}J^{g'}_jJ^{h'}_{km}J^m_{k'}J^k_{j'}J^j_{i'}+J^i_{g'k'}J^{g'}_{jk}J^k_{j'}J^j_{i'}+J^i_{g'}J^{g'}_{jkm}J^m_{k'}J^k_{j'}J^j_{i'}=0. \tag{4.9}$$

Ex. 53: We have the following relationship

$$Z\left(Z'\left(Z''\left(Z\right)\right)\right)=Z,$$

or for each i,

$$Z^i\left(Z'\left(Z''\left(Z\right)\right)\right)=Z^i.$$

Differentiating both sides with respect to Z^j, we obtain

$$\frac{\partial}{\partial Z^j}\left[Z^i\left(Z'\left(Z''\left(Z\right)\right)\right)\right]=\delta^i_j.$$

Then, we apply the chain rule twice:

$$\begin{aligned}\frac{\partial Z^i}{\partial Z^{i'}}\frac{\partial}{\partial Z^j}\left[Z'\left(Z''\left(Z\right)\right)\right] &= \delta^i_j \\ \frac{\partial Z^i}{\partial Z^{i'}}\frac{\partial Z^{i'}}{\partial Z^{i''}}\frac{\partial}{\partial Z^j}\left[Z^{i''}\left(Z\right)\right] &= \delta^i_j \\ \frac{\partial Z^i}{\partial Z^{i'}}\frac{\partial Z^{i'}}{\partial Z^{i''}}\frac{\partial Z^{i''}}{\partial Z^j} &= \delta^i_j,\end{aligned}$$

or

$$J^i_{i'}J^{i'}_{i''}J^{i''}_j = \delta^i_j.$$

Chapter 5

Chapter 5

Ex. 61: δ^i_j.

Ex. 62: Assume U is an arbitrary nontrivial linear combination $U^i\mathbf{Z}_i$ of coordinate bases $\mathbf{Z}_i$. Since

$$U \cdot U > 0$$

and

$$U \cdot U = Z_{ij}U^iU^j,$$

or in matrix notation

$$U \cdot U = U^T Z U,$$

this condition implies $Z = Z_{ij}$ is positive definite.

Ex. 63:

$$\begin{aligned} \|V\| &= \sqrt{V \cdot V} \\ &= \sqrt{Z_{ij}V^iV^j} \end{aligned}$$

Ex. 64: Put $Z = Z_{ij}$. Thus, $Z^{-1} = Z^{ij}$ by definition. Let x be an arbitrary nontrivial vector. Then, define $y = Z^{-1}x$. We have

$$\begin{aligned} y^T &= \left(Z^{-1}x\right)^T \\ &= x^T\left(Z^{-1}\right)^T \\ &= x^T Z^{-1}, \end{aligned}$$

since Z^{-1} is symmetric. Then, since Z is positive definite, note

$$\begin{aligned} 0 &< y^T Z y \\ &= x^T Z^{-1} Z Z^{-1} x \\ &= x^T Z^{-1} x. \end{aligned}$$

Since x was arbitrary, this implies that $Z = Z^{ij} > 0$.

Ex. 65:

$$\begin{aligned} \mathbf{Z}^i \cdot \mathbf{Z}_j &= Z^{ik} \mathbf{Z}_k \cdot \mathbf{Z}_j \\ &= Z^{ik} Z_{kj} \end{aligned}$$

by definition. But,

$$Z^{ik} Z_{kj} = \delta^i_j,$$

so we have

$$\mathbf{Z}^i \cdot \mathbf{Z}_j = \delta^i_j$$

Ex. 66, 67: [Not sure - which coordinate system are we in (if any?)]

Ex. 68: Use the definition

$$\begin{aligned} \mathbf{Z}^i &= Z^{ij} \mathbf{Z}_j \\ Z_{ik} \mathbf{Z}^i &= Z_{ik} Z^{ij} \mathbf{Z}_j \\ &= \delta^j_k \mathbf{Z}_j \\ &= \mathbf{Z}_k. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{Z}_k &= Z_{ik} \mathbf{Z}^i \\ &= Z_{ki} \mathbf{Z}^i, \end{aligned}$$

since Z_{ik} is symmetric.

Ex. 69: Examine

$$\begin{aligned} Z_{ki} \mathbf{Z}^i \cdot \mathbf{Z}^j &= Z_{ki} Z^{in} \mathbf{Z}_n \cdot \mathbf{Z}^j \\ &= \delta^n_k \delta^j_n \\ &= \delta^j_k, \end{aligned}$$

since $\delta_k^n \delta_n^j = 1$ iff $k = n$ and $n = j$, or by transitivity, iff $k = j$. Thus, $\mathbf{Z}^i \cdot \mathbf{Z}^j$ determines the matrix inverse of Z_{ki}, which must be Z^{ij} by uniqueness of matrix inverse.

Ex. 70: Because the inverse of a matrix is uniquely determined, we have that $\mathbf{Z}^i$ are uniquely determined.

Ex. 71:

$$\begin{aligned} Z^{ij} Z_{jk} &= \mathbf{Z}^i \cdot \mathbf{Z}^j Z_{jk} \\ &= \mathbf{Z}^i \cdot \mathbf{Z}_k, \end{aligned}$$

from 5.17

$$= \delta_k^i$$

from 5.16.

Ex. 72: We compute

$$\begin{aligned} \mathbf{Z}^1 \cdot \mathbf{Z}_2 &= \left(\frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j}\right) \cdot \mathbf{j} \\ &= \frac{1}{3}\mathbf{i} \cdot \mathbf{j} - \frac{1}{3}\mathbf{j} \cdot \mathbf{j} \\ &= \frac{1}{3} \|\mathbf{i}\| \|\mathbf{j}\| \cos \frac{\pi}{3} - \frac{1}{3} \|\mathbf{j}\|^2 \\ &= \frac{1}{3} (2)(1) \left(\frac{1}{2}\right) - \frac{1}{3} 1^2 \\ &= 0; \end{aligned}$$

thus, $\mathbf{Z}^1, \mathbf{Z}_2$ are orthogonal. We further compute

$$\begin{aligned} \mathbf{Z}^2 \cdot \mathbf{Z}_1 &= \left(-\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j}\right) \cdot \mathbf{i} \\ &= -\frac{1}{3} \|\mathbf{i}\|^2 + \frac{4}{3} \|\mathbf{i}\| \|\mathbf{j}\| \cos \frac{\pi}{3} \\ &= -\frac{1}{3} (4) + \frac{4}{3} (2)(1) \left(\frac{1}{2}\right) \\ &= 0, \end{aligned}$$

so $\mathbf{Z}^2, \mathbf{Z}_1$ are orthogonal.

Ex. 73:

$$\begin{aligned}
\mathbf{Z}^1 \cdot \mathbf{Z}_1 &= \left(\frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j}\right) \cdot \mathbf{i} \\
&= \frac{1}{3}\|\mathbf{i}\|^2 - \frac{1}{3}\|\mathbf{i}\|\,\|\mathbf{j}\|\cos\frac{\pi}{3} \\
&= \frac{4}{3} - \frac{2}{3}\left(\frac{1}{2}\right) \\
&= 1. \\
\mathbf{Z}^2 \cdot \mathbf{Z}_2 &= \left(-\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j}\right) \cdot \mathbf{j} \\
&= -\frac{1}{3}\|\mathbf{i}\|\,\|\mathbf{j}\|\cos\frac{\pi}{3} + \frac{4}{3}\|\mathbf{j}\|^2 \\
&= -\frac{2}{3}\left(\frac{1}{2}\right) + \frac{4}{3} \\
&= 1.
\end{aligned}$$

Ex. 74:

$$\begin{aligned}
\mathbf{Z}^1 \cdot \mathbf{Z}^2 &= \left(\frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j}\right) \cdot \left(-\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j}\right) \\
&= -\frac{1}{9}\|\mathbf{i}\|^2 + \frac{4}{9}\|\mathbf{i}\|\,\|\mathbf{j}\|\cos\frac{\pi}{3} + \frac{1}{9}\|\mathbf{i}\|\,\|\mathbf{j}\|\cos\frac{\pi}{3} - \frac{4}{9}\|\mathbf{j}\|^2 \\
&= -\frac{4}{9} + \frac{4}{9} + \frac{1}{9} - \frac{4}{9} \\
&= -\frac{3}{9} \\
&= -\frac{1}{3}.
\end{aligned}$$

Ex. 75: Let $\mathbf{R}$ denote the position vector. We compute

$$\begin{aligned}
\mathbf{Z}_3 &= \frac{\partial \mathbf{R}(\mathbf{Z})}{\mathbf{Z}_3} \\
&= \frac{\partial \mathbf{R}(r, \theta, z)}{\partial z}
\end{aligned}$$

for cylindrical coordinates

$$= \lim_{h \to 0} \frac{\mathbf{R}(r, \theta, z+h) - \mathbf{R}(r, \theta, z)}{h}.$$

But, $\mathbf{R}(r,\theta,z+h)-\mathbf{R}(r,\theta,z)$ is clearly a vector of length h pointing in the z direction; thus, for any h,

$$\frac{\mathbf{R}(r,\theta,z+h)-\mathbf{R}(r,\theta,z)}{h}$$

is the unit vector pointing in the z direction. This implies $\mathbf{Z}_3$ is the unit vector pointing in the z direction.

Ex. 76: The computations of the diagonal elements Z_{11} and Z_{22} are the same as for polar coordinates; moreover the zero off-diagonal entries Z_{12}, Z_{21} follow from the orthogonality of $\mathbf{Z}_1$, $\mathbf{Z}_2$. By definition of cylindrical coordinates, the z axis is perpendicular to the coordinate plane (upon which $\mathbf{Z}_1$, $\mathbf{Z}_2$ lie); thus, since $\mathbf{Z}_3$ points in the z direction, we have that $\mathbf{Z}_3$ is perpendicular to both $\mathbf{Z}_1$, $\mathbf{Z}_2$. This implies that the off-diagonal entries in row 3 and column 3 of Z_{ij} are zero. Morover, since $\mathbf{Z}_3$ is of unit length; we have $Z_{33}=\mathbf{Z}_3\cdot\mathbf{Z}_3=1$. Thus, we have

$$Z_{ij}=\begin{bmatrix}1&0&0\\0&r^2&0\\0&0&1\end{bmatrix}.$$

Now, since Z^{ij} is defined to be the inverse of Z_{ij}, we may easily compute

$$Z^{ij}=\begin{bmatrix}1&0&0\\0&r^{-2}&0\\0&0&1\end{bmatrix},$$

since the inverse of a diagonal matrix (with non-zero diagonal entries, of course) is the diagonal matrix with corresponding reciprocal diagonal entries.

Ex. 77: We have

$$\begin{aligned}\mathbf{Z}^3 &= Z^{3j}\mathbf{Z}_j\\ &= 0\mathbf{Z}_1+0\mathbf{Z}_2+1\mathbf{Z}_3\\ &= \mathbf{Z}_3\end{aligned}$$

Chapter 6

Chapter 6

Ex. 87: Look at

$$Z^{ij}J_i^{i'}J_j^{j'}Z_{j'k'} = Z^{ij}J_i^{i'}J_j^{j'}Z_{jk}J_{j'}^{j}J_{k'}^{k}$$

by the tensor property of Z_{jk}

$$\begin{aligned}
&= Z^{ij}Z_{jk}J_i^{i'}J_{k'}^{k} \\
&= \delta_k^i J_i^{i'}J_{k'}^{k} \\
&= J_k^{i'}J_{k'}^{k} \\
&= \delta_{k'}^{i'},
\end{aligned}$$

so, in linear algebra terms, we have that $Z^{ij}J_i^{i'}J_j^{j'}$ is the matrix inverse of $Z_{j'k'}$. By uniqueness of matrix inverses, this forces $Z^{ij}J_i^{i'}J_j^{j'} = Z^{i'j'}$, as desired.

Ex. 88: Let Z,Z' be two coordinate systems. Write the unprimed coordinates in terms of the primed coordinates

$$Z = Z(Z').$$

Then,

$$\begin{aligned}
\frac{\partial F(Z)}{\partial Z^{i'}} &= \frac{\partial F(Z(Z'))}{\partial Z^{i'}} \\
&= \frac{\partial F}{\partial Z^{i}}\frac{\partial Z^{i}}{\partial Z^{i'}} \\
&= \frac{\partial F}{\partial Z^{i}}J_{i'}^{i},
\end{aligned}$$

so $\frac{\partial F}{\partial Z^i}$ is a covariant tensor.

Ex. 89: We show the general case (since by the previous exercise, we know that the collection of first partial derivatives is a covariant tensor). Define, given a covariant tensor field T_i

$$\begin{aligned} S_{ij} &= \frac{\partial T_i}{Z^j} \\ S_{i'j'} &= \frac{\partial T_{i'}}{Z^{j'}} \end{aligned}$$

so

$$\begin{aligned} S_{i'j'} &= \frac{\partial T_{i'}}{\partial Z^{j'}} \\ &= \frac{\partial}{\partial Z^{j'}} \left[T_i J^i_{i'} \right], \end{aligned}$$

since T is a covariant tensor,

$$\begin{aligned} &= \frac{\partial}{Z^{j'}} \left[T_i \left(Z' \left(Z \right) \right) J^i_{i'} \left(Z' \right) \right] \\ &= \frac{\partial T^i}{\partial Z^j} \frac{\partial Z^j}{\partial Z^{j'}} J^i_{i'} + T_i J^i_{i'j'} \\ &= S_{ij} J^j_{j'} J^i_{i'} + T_i J^i_{i'j'} \\ &\neq S_{ij} J^j_{j'} J^i_{i'} \end{aligned}$$

(except in the trivial case where $T_i = 0$). Thus, in general, the collection

$$\frac{\partial T_i}{Z^j}$$

is not a covariant tensor.

Ex. 90: Compute

$$\begin{aligned} S_{i'j'} &= \frac{\partial T_{i'}}{\partial Z^{j'}} - \frac{\partial T_{j'}}{\partial Z^{i'}} \\ &= \frac{\partial T_i}{\partial Z^j} J^j_{j'} J^i_{i'} + T_i J^i_{i'j'} - \left[\frac{\partial T_j}{\partial Z^i} J^i_{i'} J^j_{j'} + T_i J^i_{j'i'} \right] \end{aligned}$$

by the above, interchanging the rolls of i', j' for the second term:

$$= \frac{\partial T_i}{\partial Z^j} J^j_{j'} J^i_{i'} + T_i J^i_{i'j'} - \left[\frac{\partial T_j}{\partial Z^i} J^j_{j'} J^i_{i'} + T_i J^i_{i'j'} \right],$$

since $J^i_{j'i'} = J^i_{i'j'}$,

$$\begin{aligned}
&= \frac{\partial T_i}{\partial Z^j} J^j_{j'} J^i_{i'} + T_i J^i_{i'j'} - \frac{\partial T_j}{\partial Z^i} J^j_{j'} J^i_{i'} - T_i J^i_{i'j'} \\
&= \left(\frac{\partial T_i}{\partial Z^j} - \frac{\partial T_j}{\partial Z^i} \right) J^j_{j'} J^i_{i'} \\
&= S_{ij} J^i_{i'} J^j_{j'},
\end{aligned}$$

so this skew-symmetric part S_{ij} is indeed a covariant tensor.

Ex. 91: Put

$$S^{ij} = \frac{\partial T^i}{\partial Z^j},$$

so

$$\begin{aligned}
S^{i'j'} &= \frac{\partial T^{i'}}{\partial Z^{j'}} \\
&= \frac{\partial}{\partial Z^{j'}} \left[T^i J^{i'}_i \right],
\end{aligned}$$

since T is a contravariant tensor,

$$\begin{aligned}
&= \frac{\partial}{\partial Z^{j'}} \left[T^i \left(Z' \left(Z \right) \right) \right] J^{i'}_i + T^i \frac{\partial}{\partial Z^{j'}} \left[J^{i'}_i \left(Z \left(Z' \right) \right) \right] \\
&= \frac{\partial T^i}{\partial Z^j} J^{j'}_j J^{i'}_i + T^i \frac{\partial J^{i'}_i}{\partial Z^j} \frac{\partial Z^j}{\partial Z^{j'}} \\
&= S^{ij} J^{i'}_i J^{j'}_j + T^i J^{i'}_{ij} J^j_{j'} \\
&\neq S^{ij} J^{i'}_i J^{j'}_j
\end{aligned}$$

except in the trivial case.

Ex. 92: We have

$$\Gamma^k_{ij} = \mathbf{Z}^k \cdot \frac{\partial \mathbf{Z}_i}{\partial Z^j},$$

so in primed coordinates,

$$\begin{aligned}
\Gamma^{k'}_{i'j'} &= \mathbf{Z}^{k'} \cdot \frac{\partial \mathbf{Z}_{i'}}{\partial Z^{j'}} \\
&= \mathbf{Z}^{k'} \cdot \left(\frac{\partial \mathbf{Z}_i}{\partial Z^j} J^i_{i'} J^j_{j'} + \mathbf{Z}_i J^i_{i'j'} \right)
\end{aligned}$$

by our work done earlier (note that $\mathbf{Z}_i$ is a covariant tensor)

$$\left(\mathbf{Z}^k J_k^{k'}\right) \cdot \left(\frac{\partial \mathbf{Z}_i}{\partial Z^j} J_{i'}^i J_{j'}^j + \mathbf{Z}_i J_{i'j'}^i\right)$$

since Z^k is a contravariant tensor,

$$\begin{aligned}
&= \left(\mathbf{Z}^k \cdot \frac{\partial \mathbf{Z}_i}{\partial Z^j}\right) J_k^{k'} J_{i'}^i J_{j'}^j + \left(\mathbf{Z}^k \cdot \mathbf{Z}_i\right) J_k^{k'} J_{i'j'}^i \\
&= \left(\mathbf{Z}^k \cdot \frac{\partial \mathbf{Z}_i}{\partial Z^j}\right) J_k^{k'} J_{i'}^i J_{j'}^j + \delta_i^k J_k^{k'} J_{i'j'}^i \\
&= \left(\mathbf{Z}^k \cdot \frac{\partial \mathbf{Z}_i}{\partial Z^j}\right) J_k^{k'} J_{i'}^i J_{j'}^j + J_i^{k'} J_{i'j'}^i \\
&\neq \left(\mathbf{Z}^k \cdot \frac{\partial \mathbf{Z}_i}{\partial Z^j}\right) J_k^{k'} J_{i'}^i J_{j'}^j = \Gamma_{ij}^k J_k^{k'} J_{i'}^i J_{j'}^j
\end{aligned}$$

except in the trivial case.

Ex. 93: Compute

$$\begin{aligned}
\frac{\partial T_{i'j'}}{\partial Z^{k'}} &= \frac{\partial}{\partial Z^{k'}} \left[T_{ij} J_{i'}^i J_{j'}^j\right] \\
&= \frac{\partial}{\partial Z^{k'}} \left[T_{ij}\right] J_{i'}^i J_{j'}^j + T_{ij} \frac{\partial}{\partial Z^{k'}} \left[J_{i'}^i\right] J_{j'}^j + T_{ij} J_{i'}^i \frac{\partial}{\partial Z^{k'}} \left[J_{j'}^j\right] \\
&= \frac{\partial}{\partial Z^{k'}} \left[T_{ij}\right] J_{i'}^i J_{j'}^j + T_{ij} J_{i'k'}^i J_{j'}^j + T_{ij} J_{i'}^i J_{j'k'}^j \\
&= \frac{\partial}{\partial Z^{k'}} \left[T_{ij}\left(Z\left(Z'\right)\right)\right] J_{i'}^i J_{j'}^j + T_{ij} J_{i'k'}^i J_{j'}^j + T_{ij} J_{i'}^i J_{j'k'}^j \\
&= \frac{\partial T_{ij}}{\partial Z^k} \frac{\partial Z^k}{\partial Z^{k'}} J_{i'}^i J_{j'}^j + T_{ij} J_{i'k'}^i J_{j'}^j + T_{ij} J_{i'}^i J_{j'k'}^j \\
&= \frac{\partial T_{ij}}{\partial Z^k} J_{k'}^k J_{i'}^i J_{j'}^j + T_{ij} J_{i'k'}^i J_{j'}^j + T_{ij} J_{i'}^i J_{j'k'}^j .
\end{aligned}$$

Thus, from 5.66,

$$
\begin{aligned}
\Gamma^{k'}_{i'j'} &= \frac{1}{2}Z^{k'm'}\left(\frac{\partial Z_{m'i'}}{\partial Z^{j'}}+\frac{\partial Z_{m'j'}}{\partial Z^{i'}}-\frac{\partial Z_{i'j'}}{\partial Z^{m'}}\right)\\
&= \frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(\frac{\partial Z_{m'i'}}{\partial Z^{j'}}+\frac{\partial Z_{m'j'}}{\partial Z^{i'}}-\frac{\partial Z_{i'j'}}{\partial Z^{m'}}\right)\\
&= \frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}(\frac{\partial Z_{mi}}{\partial Z^{j}}J^j_{j'}J^m_{m'}J^i_{i'}+Z_{mi}J^m_{m'j'}J^i_{i'}+Z_{mi}J^m_{m'}J^i_{i'j'}\\
&\quad +\frac{\partial Z_{mj}}{\partial Z^{i}}J^i_{i'}J^m_{m'}J^j_{j'}+Z_{mj}J^m_{m'i'}J^j_{j'}+Z_{mj}J^m_{m'}J^j_{i'j'}\\
&\quad -\frac{\partial Z_{ij}}{\partial Z^{m}}J^m_{m'}J^i_{i'}J^j_{j'}-Z_{ij}J^i_{i'm'}J^j_{j'}-Z_{ij}J^i_{i'}J^j_{j'm'})\\
&= \frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(\frac{\partial Z_{mi}}{\partial Z^{j}}J^j_{j'}J^m_{m'}J^i_{i'}+\frac{\partial Z_{mj}}{\partial Z^{i}}J^i_{i'}J^m_{m'}J^j_{j'}-\frac{\partial Z_{ij}}{\partial Z^{m}}J^m_{m'}J^i_{i'}J^j_{j'}\right)\\
&\quad +\frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(Z_{mi}J^m_{m'j'}J^i_{i'}+Z_{mi}J^m_{m'}J^i_{i'j'}\right)\\
&\quad +\frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(Z_{mj}J^m_{m'i'}J^j_{j'}+Z_{mj}J^m_{m'}J^j_{i'j'}\right)\\
&\quad -\frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(Z_{ij}J^i_{i'm'}J^j_{j'}+Z_{ij}J^i_{i'}J^j_{j'm'}\right)\\
&= \frac{1}{2}Z^{km}J_k^{k'}J^j_{j'}J^i_{i'}\left(\frac{\partial Z_{mi}}{\partial Z^{j}}+\frac{\partial Z_{mj}}{\partial Z^{i}}-\frac{\partial Z_{ij}}{\partial Z^{m}}\right)\\
&\quad +\frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(Z_{mi}J^m_{m'j'}J^i_{i'}+Z_{mi}J^m_{m'}J^i_{i'j'}\right)\\
&\quad +\frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(Z_{mj}J^m_{m'i'}J^j_{j'}+Z_{mj}J^m_{m'}J^j_{i'j'}\right)\\
&\quad -\frac{1}{2}Z^{km}J_k^{k'}J_m^{m'}\left(Z_{ij}J^i_{i'm'}J^j_{j'}+Z_{ij}J^i_{i'}J^j_{j'm'}\right)\\
&= \Gamma^k_{ij}J_k^{k'}J^j_{j'}J^i_{i'}\\
&\quad +\frac{1}{2}Z^{km}Z_{mi}J_k^{k'}J_m^{m'}\left(J^m_{m'j'}J^i_{i'}+J^m_{m'}J^i_{i'j'}\right)\\
&\quad +\frac{1}{2}Z^{km}Z_{mj}J_k^{k'}J_m^{m'}\left(J^m_{m'i'}J^j_{j'}+J^m_{m'}J^j_{i'j'}\right)\\
&\quad -\frac{1}{2}Z^{km}Z_{ij}J_k^{k'}J_m^{m'}\left(J^i_{i'm'}J^j_{j'}+J^i_{i'}J^j_{j'm'}\right)\\
&= \Gamma^k_{ij}J_k^{k'}J^j_{j'}J^i_{i'}\\
&\quad +\frac{1}{2}\delta^k_iJ_k^{k'}J_m^{m'}\left(J^m_{m'j'}J^i_{i'}+J^m_{m'}J^i_{i'j'}\right)\\
&\quad +\frac{1}{2}\delta^k_jJ_k^{k'}J_m^{m'}\left(J^m_{m'i'}J^j_{j'}+J^m_{m'}J^j_{i'j'}\right)\\
&\quad -\frac{1}{2}Z^{km}Z_{ij}J_k^{k'}J_m^{m'}\left(J^i_{i'm'}J^j_{j'}+J^i_{i'}J^j_{j'm'}\right)\\
&= \Gamma^k_{ij}J_k^{k'}J^j_{j'}J^i_{i'}\\
&\quad +\frac{1}{2}J_i^{k'}J_m^{m'}\left(J^m_{m'j'}J^i_{i'}+J^m_{m'}J^i_{i'j'}\right)\\
&\quad +\frac{1}{2}J_j^{k'}J_m^{m'}\left(J^m_{m'i'}J^j_{j'}+J^m_{m'}J^j_{i'j'}\right)\\
&\quad -\frac{1}{2}Z^{km}Z_{ij}J_k^{k'}J_m^{m'}\left(J^i_{i'm'}J^j_{j'}+J^i_{i'}J^j_{j'm'}\right)\\
&= \Gamma^k_{ij}J_k^{k'}J^j_{j'}J^i_{i'}\\
&\quad +\frac{1}{2}J_i^{k'}J_m^{m'}J^m_{m'j'}J^i_{i'}+\frac{1}{2}J_i^{k'}J_m^{m'}J^m_{m'}J^i_{i'j'}\\
&\quad +\frac{1}{2}J_j^{k'}J_m^{m'}J^m_{m'i'}J^j_{j'}+\frac{1}{2}J_j^{k'}J_m^{m'}J^m_{m'}J^j_{i'j'}
\end{aligned}
$$

$$\begin{aligned}
&= \Gamma^k_{ij} J^{k'}_k J^j_{j'} J^i_{i'} \\
&\quad + \frac{1}{2} J^{k'}_i J^i_{i'} J^{m'}_m J^m_{m'j'} + \frac{1}{2} J^{k'}_i J^i_{i'j'} \\
&\quad + \frac{1}{2} J^{k'}_j J^j_{j'} J^{m'}_m J^m_{m'i'} + \frac{1}{2} J^{k'}_j J^j_{i'j'} \\
&\quad - \frac{1}{2} Z^{km} Z_{ij} J^{k'}_k J^{m'}_m J^i_{i'm'} J^j_{j'} - \frac{1}{2} Z^{km} Z_{ij} J^{k'}_k J^{m'}_m J^i_{i'} J^j_{j'm'} \\
&= \Gamma^k_{ij} J^{k'}_k J^j_{j'} J^i_{i'} + J^{k'}_i J^i_{i'j'} + \frac{1}{2} \delta^{k'}_{i'} J^{m'}_m J^m_{m'j'} + \frac{1}{2} \delta^{k'}_{j'} J^{m'}_m J^m_{m'i'} - \frac{1}{2} Z^{km} Z_{ij} J^j_{j'} J^{k'}_k J^{m'}_m J^i_{i'm'} - \frac{1}{2} Z^{km} Z_{ij} J^i_{i'} J \\
&= \Gamma^k_{ij} J^{k'}_k J^j_{j'} J^i_{i'} + J^{k'}_i J^i_{i'j'} + 0.
\end{aligned}$$

Thus, we have

$$\Gamma^{k'}_{i'j'} = \Gamma^k_{ij} J^{k'}_k J^j_{j'} J^i_{i'} + J^{k'}_i J^i_{i'j'}.$$

Ex. 94: We show the result for degree-one covariant tensors. The generalization to other tensors is then evident.

Assume $T_i = 0$. Then, since $T_{i'} = T_i J^i_{i'}$, we have $T_{i'} = 0$.

Ex. 95: Note that in such a coordinate change,

$$J^{i'}_i = A^{i'}_i,$$

so the "Hessian" object

$$J^{i'}_{ij} = 0,$$

since $A^{i'}_i$ is assumed to be constant with respect to Z. This means that all but the first terms in the above computations of $\frac{\partial T_{i'}}{Z^{j'}}$, $\frac{\partial T^{i'}}{\partial Z^{j'}}$, $\frac{\partial T_{i'j'}}{\partial Z^{k'}}$, and even $\Gamma^{k'}_{i'j'}$ are zero, and hence we have that each of these objects have this "tensor property" with respect to coordinate changes that are linear transformations.

Ex. 96: Since the sum of two tensors is a tensor, we may inductively show that the sum of finitely many tensors is a tensor. We must show that for any constant c,

$$cA^i_{jk}$$

is a tensor. Compute
$$cA^{i'}_{j'k'} = cA^{i}_{jk}J^{i'}_{i}J^{j}_{j'}J^{k}_{k'},$$
since A^i_{jk} is a tensor. Thus, by the above, we have if each $A(n)^i_{jk}$ is a tensor, then the sum
$$\sum_{n=1}^{N} c_n A(n)^i_{jk}$$
is a tensor. Thus, linear combinations of tensors are tensors.

Ex. 97: We have that
$$\begin{aligned} S_i T^{ij} &= S_i \delta^i_k T^{kj} \\ &= \delta^i_k S_i T^{kj}, \end{aligned}$$
which is a tensor, since both δ^i_k and $S_i T^{kj}$ are tensors by the previous section and by the fact that the product of two tensors is a tensor.

Ex. 98: $\delta^i_i = n$ by the summation convention. Thus, δ^i_i returns the dimension of the ambient space.

Ex. 99: We have
$$\mathbf{V}_{ij} = V^k_{ij}\mathbf{Z}_k,$$
so
$$\begin{aligned} \mathbf{V}_{ij} \cdot \mathbf{Z}^m &= V^k_{ij}\mathbf{Z}_k \cdot \mathbf{Z}^m \\ &= V^k_{ij}\delta^m_k \\ &= V^m_{ij} \end{aligned}$$
so substituting $m = k$, we have an expression for the components
$$V^k_{ij} = \mathbf{V}_{ij} \cdot \mathbf{Z}^k$$
So,
$$\begin{aligned} V^{k'}_{i'j'} &= \mathbf{V}_{i'j'} \cdot \mathbf{Z}^{k'} \\ &= \mathbf{V}_{ij}J^i_{i'}J^j_{j'} \cdot \mathbf{Z}^k J^{k'}_k, \end{aligned}$$
since both $\mathbf{V}_{ij}, \mathbf{Z}^k$ are tensors
$$= \mathbf{V}_{ij} \cdot \mathbf{Z}^k J^i_{i'}J^j_{j'}J^{k'}_k$$

by linearity
$$= V^k_{ij} J^i_{i'} J^j_{j'} J^{k'}_k$$

as desired.

Ex. 100: Fix a coordinate system $Z^{\bar{i}}$
$$T^{ij}_k = T^{\bar{i}\bar{j}}_{\bar{k}} J^i_{\bar{i}} J^j_{\bar{j}} J^{\bar{k}}_k,$$

so
$$\begin{aligned}
T^{ij}_k J^{i'}_i J^{j'}_j J^k_{k'} &= T^{\bar{i}\bar{j}}_{\bar{k}} J^i_{\bar{i}} J^j_{\bar{j}} J^{\bar{k}}_k J^{i'}_i J^{j'}_j J^k_{k'} \\
&= T^{\bar{i}\bar{j}}_{\bar{k}} \delta^{i'}_{\bar{i}} \delta^{j'}_{\bar{j}} \delta^{\bar{k}}_{k'} \\
&= T^{i'j'}_{k'},
\end{aligned}$$

as desired.

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Ex. 183: Assume $n = 3$. Given a_{ij}, put A as the determinant. We define

$$A = e^{ijk} a_{i1} a_{j2} a_{k3}.$$

Note that switching the roles of $1, 2$ in the above equation yields

$$\begin{aligned} e^{ijk} a_{i2} a_{j1} a_{k3} &= e^{ijk} a_{j1} a_{i2} a_{k3} \\ &= e^{jik} a_{i1} a_{j2} a_{k3} \\ &= -e^{ijk} a_{i1} a_{j2} a_{k3} \\ &= -A \end{aligned}$$

Generalizing, we let (r, s, t) be a permutation of $(1, 2, 3)$. We may then see that

$$A = e^{ijk} e^{rst} a_{ir} a_{js} a_{kt}$$

(note that the summation convention is not implied in the above line). Then, since there are $3!$ permutations of $(1, 2, 3)$, we may write

$$3! A = \sum_{\substack{\text{permutations} \\ (r,s,t)}} e^{ijk} e_{rst} a_{ir} a_{js} a_{kt}$$

But, $e^{rst} = 0$ for (r, s, t) that is not a permutation; hence we may sum over all $0 \leq r, s, t \leq 3$, and apply the Einstein summation convention:

$$3! A = e^{ijk} e_{rst} a_{ir} a_{js} a_{kt},$$

or

$$A = \frac{1}{3!} e^{ijk} e_{rst} a_{ir} a_{js} a_{kt}$$

We may similarly show that for a^{ij}, we have

$$A = \frac{1}{3!}e_{ijk}a^{i1}a^{j2}a^{k3}.$$

Ex. 184: We have

$$\delta^{123}_{rst}a^r_2a^s_2a^t_3 = \delta^{123}_{srt}a^s_2a^r_2a^t_3$$

after index renaming

$$\begin{aligned} &= \delta^{123}_{srt}a^r_2a^s_2a^t_3 \\ &= -\delta^{123}_{rst}a^r_2a^s_2a^t_3. \end{aligned}$$

Since

$$\delta^{123}_{rst}a^r_2a^s_2a^t_3 = -\delta^{123}_{rst}a^r_2a^s_2a^t_3,$$

we need

$$\delta^{123}_{rst}a^r_2a^s_2a^t_3 = 0.$$

The result for $\delta^{132}_{srt}a^s_2a^r_3a^t_2$ follows similarly.

Ex. 185: Note

$$\delta^{123}_{rst} = e^{123}e_{rst} = 1 \cdot e_{rst} = e_{rst},$$

so

$$\delta^{123}_{rst}a^r_1a^s_2a^t_3 = e_{rst}a^r_1a^s_2a^t_3 = A.$$

Also,

$$\begin{aligned} \delta^{132}_{rst}a^r_2a^s_3a^t_1 &= \delta^{132}_{rst}a^t_1a^r_2a^s_3 \\ &= \delta^{132}_{str}a^r_1a^s_2a^t_3 \\ &= -\delta^{123}_{str}a^r_1a^s_2a^t_3 \\ &= -e_{str}a^r_1a^s_2a^t_3 \\ &= A. \end{aligned}$$

[Note: Is there an error somewhere - should this be $-A$?]

Ex. 186: Define
$$A^{ir} = \frac{1}{2!} e^{ijk} e^{rst} a_{js} a_{tk}.$$

We check that
$$\frac{\partial A}{\partial a_{ir}} = A^{ir}.$$

Check
$$\begin{aligned}
\frac{\partial A}{\partial a_{lu}} &= \frac{1}{3!} e^{ijk} e^{rst} \frac{\partial \left(a_{ir} a_{js} a_{kt}\right)}{\partial a_{lu}} \\
&= \frac{1}{3!} e^{ijk} e^{rst} \left[\frac{\partial a_{ir}}{\partial a_{lu}} a_{js} a_{kt} + a_{ir} \frac{\partial a_{js}}{\partial a_{lu}} a_{kt} + a_{ir} a_{js} \frac{\partial a_{kt}}{\partial a_{lu}}\right] \\
&= \frac{1}{3!} e^{ijk} e^{rst} \left[\delta_i^l \delta_r^u a_{js} a_{kt} + a_{ir} \delta_j^l \delta_s^u a_{kt} + a_{ir} a_{js} \delta_k^l \delta_t^u\right] \\
&= \frac{1}{3!} \left[e^{ijk} \delta_i^l e^{rst} \delta_r^u a_{js} a_{kt} + a_{ir} e^{ijk} \delta_j^l e^{rst} \delta_s^u a_{kt} + a_{ir} a_{js} e^{ijk} \delta_k^l e^{rst} \delta_t^u\right] \\
&= \frac{1}{3!} \left[e^{ljk} e^{ust} a_{js} a_{kt} + a_{ir} e^{ilk} e^{rut} a_{kt} + a_{ir} a_{js} e^{ijl} e^{rsu}\right] \\
&= \frac{1}{3!} \left[e^{ljk} e^{ust} a_{js} a_{kt} + e^{ilk} e^{rut} a_{ir} a_{kt} + e^{ijl} e^{rsu} a_{ir} a_{js}\right] \\
&= \frac{1}{3!} \left[3 e^{ljk} e^{ust} a_{js} a_{kt}\right],
\end{aligned}$$

after an index renaming,
$$\begin{aligned}
&= \frac{1}{2!} e^{ljk} e^{ust} a_{js} a_{kt} \\
&= A^{lu}
\end{aligned}$$

as desired. Similarly, if we define
$$A_{ir} = \frac{1}{2!} e_{ijk} e_{rst} a^{js} a^{tk},$$

we have
$$\frac{\partial A}{\partial a^{ir}} = A_{ir}$$

by a similar argument.

Ex. 187: In cartesian coordinates,
$$Z_{ij} = \delta^i_{j,}$$

so

$$\begin{aligned} Z &= |Z_{\cdot\cdot}| \\ &= |I| \\ &= 1. \end{aligned}$$

Thus,

$$\sqrt{Z} = 1.$$

In polar coordinates,

$$[Z_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix},$$

so

$$\begin{aligned} Z &= \begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix} \\ &= r^2; \end{aligned}$$

hence

$$\sqrt{Z} = r.$$

In spherical coordinates,

$$[Z_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix},$$

so

$$\begin{aligned} Z &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix} \\ &= r^4 \sin^2\theta; \end{aligned}$$

thus,

$$\sqrt{Z} = r^2 \sin\theta.$$

Ex. 188: We compute, using the Voss-Weyl formula,

$$
\begin{aligned}
\nabla_i \nabla^i F &= \frac{1}{\sqrt{Z}} \frac{\partial}{\partial Z^i} \left(\sqrt{Z} Z^{ij} \frac{\partial F}{\partial Z^j} \right) \\
&= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin\theta \,(1)\, \frac{\partial F}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(r^2 \sin\theta r^2 \frac{\partial F}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(r^2 \sin\theta r^2 \sin^2\theta \frac{\partial F}{\partial \phi} \right) \right] \\
&= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin\theta \frac{\partial F}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(r^4 \sin\theta \frac{\partial F}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(r^4 \sin^3\theta \frac{\partial F}{\partial \phi} \right) \right] \\
&= \frac{1}{r^2 \sin\theta} \left[\sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial F}{\partial r} \right) + r^4 \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial F}{\partial \theta} \right) + r^4 \sin^3\theta \frac{\partial}{\partial \phi} \left(\frac{\partial F}{\partial \phi} \right) \right] \\
&= \frac{1}{r^2 \sin\theta} \left[\sin\theta \left(2r \frac{\partial F}{\partial r} + r^2 \frac{\partial^2 F}{\partial r^2} \right) + r^4 \left(\cos\theta \frac{\partial F}{\partial \theta} + \sin\theta \frac{\partial^2 F}{\partial \theta^2} \right) + r^4 \sin^3\theta \frac{\partial^2 F}{\partial \phi^2} \right] \\
&= \frac{1}{r^2} \left(2r \frac{\partial F}{\partial r} + r^2 \frac{\partial^2 F}{\partial r^2} \right) + \frac{r^2}{\sin\theta} \left(\cos\theta \frac{\partial F}{\partial \theta} + \sin\theta \frac{\partial^2 F}{\partial \theta^2} \right) + r^2 \sin^2\theta \frac{\partial^2 F}{\partial \phi^2} \\
&= \frac{2}{r} \frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial r^2} + r^2 \cot\theta \frac{\partial F}{\partial \theta} + r^2 \frac{\partial^2 F}{\partial \theta^2} + r^2 \sin^2\theta \frac{\partial^2 F}{\partial \phi^2} \\
&= \frac{\partial^2 F}{\partial r^2} + \frac{2}{r} \frac{\partial F}{\partial r} + r^2 \frac{\partial^2 F}{\partial \theta^2} + r^2 \cot\theta \frac{\partial F}{\partial \theta} + r^2 \sin^2\theta \frac{\partial^2 F}{\partial \phi^2}
\end{aligned}
$$

Ex. 189: We compute, for cylindrical coordinates

$$
\begin{aligned}
\nabla_i \nabla^i F &= \frac{1}{\sqrt{Z}} \frac{\partial}{\partial Z^i} \left(\sqrt{Z} Z^{ij} \frac{\partial F}{\partial Z^j} \right) \\
&= \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \,(1)\, \frac{\partial F}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(r \cdot r^2 \frac{\partial F}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(r \,(1)\, \frac{\partial F}{\partial z} \right) \right] \\
&= \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial F}{\partial r} \right) + r^3 \frac{\partial^2 F}{\partial \theta^2} + r \frac{\partial^2 F}{\partial z^2} \right] \\
&= \frac{1}{r} \left[\frac{\partial F}{\partial r} + r \frac{\partial^2 F}{\partial r^2} + r^3 \frac{\partial^2 F}{\partial \theta^2} + r \frac{\partial^2 F}{\partial z^2} \right] \\
&= \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + r^2 \frac{\partial^2 F}{\partial \theta^2} + \frac{\partial^2 F}{\partial z^2}.
\end{aligned}
$$

Part II

Part II

Chapter 10

Chapter 10

Ex. 213: Note that

$$\begin{aligned} Z^i_\alpha Z^\alpha_j &= \left(\mathbf{S}_\alpha \cdot \mathbf{Z}^i\right)\left(\mathbf{S}^\alpha \cdot \mathbf{Z}_j\right) \\ &= \left(\mathbf{S}_\alpha \left(\mathbf{S}^\alpha \cdot \mathbf{Z}_j\right) \cdot \mathbf{Z}^i\right) \\ &= \left(\mathbf{S}^\alpha \cdot \mathbf{Z}_j\right) \mathbf{S}_\alpha \cdot \mathbf{Z}^i \\ &\neq \delta^i_j, \end{aligned}$$

since

$$\left(\mathbf{S}^\alpha \cdot \mathbf{Z}_j\right) \mathbf{S}_\alpha$$

is merely the projection of $\mathbf{Z}_j$ onto the tangent space. [ASK]

EARLIER ATTEMPT:

$$\begin{aligned} Z^i_\alpha Z^\alpha_j &= \delta^i_j \\ \left(\mathbf{S}_\alpha \cdot \mathbf{Z}^i\right) Z_{j\beta} S^{\alpha\beta} &= \delta^i_j \\ \mathbf{S}_\alpha S^{\alpha\beta} \cdot \mathbf{Z}^i Z_{j\beta} &= \delta^i_j \\ \mathbf{S}^\beta Z_{j\beta} \cdot \mathbf{Z}^i &= \delta^i_j \\ \mathbf{S}^\beta Z_{j\beta} \cdot \mathbf{Z}^i &= \mathbf{Z}_j \cdot \mathbf{Z}^i, \end{aligned}$$

which forces

$$\mathbf{S}^\beta Z_{j\beta} = \mathbf{Z}_j$$

??? [Not sure - maybe a dimensional argument?]

Ex. 214: We have

$$T^i = T^\alpha Z_\alpha^i.$$

Now,

$$\begin{aligned} T^i Z_i^\alpha &= T^\beta Z_\beta^i Z_i^\alpha \\ &= T^\beta \delta_\beta^\alpha \\ &= T^\alpha, \end{aligned}$$

as desired.

Ex. 215: We show that $(\mathbf{V}-\mathbf{P})\cdot\mathbf{N}=0$. Compute

$$\begin{aligned} (\mathbf{V}-\mathbf{P})\cdot\mathbf{N} &= (\mathbf{V}-(\mathbf{V}\cdot\mathbf{N})\,\mathbf{N})\cdot\mathbf{N} \\ &= \mathbf{V}\cdot\mathbf{N}-(\mathbf{V}\cdot\mathbf{N})\,(\mathbf{N}\cdot\mathbf{N}) \\ &= \mathbf{V}\cdot\mathbf{N}-\mathbf{V}\cdot\mathbf{N} \\ &= 0, \end{aligned}$$

since

$$\mathbf{N}\cdot\mathbf{N}=1.$$

Ex. 216: We compute

$$\begin{aligned} P_j^i P_k^j &= N^i N_j N^j N_k \\ &= N^i\,(1)\,N_k \\ &= P_k^i, \end{aligned}$$

as desired.

Ex. 217: We show that $\mathbf{V}-\mathbf{T}$ is orthogonal to the tangent plane. We compute

$$\begin{aligned} (\mathbf{V}-\mathbf{T})\cdot\mathbf{S}^\beta &= (\mathbf{V}-(\mathbf{V}\cdot\mathbf{S}^\alpha)\,\mathbf{S}_\alpha)\cdot\mathbf{S}^\beta \\ &= \mathbf{V}\cdot\mathbf{S}^\beta-(\mathbf{V}\cdot\mathbf{S}^\alpha)\,\mathbf{S}_\alpha\cdot\mathbf{S}^\beta \\ &= \mathbf{V}\cdot\mathbf{S}^\beta-(\mathbf{V}\cdot\mathbf{S}^\alpha)\,\delta_\alpha^\beta \\ &= \mathbf{V}\cdot\mathbf{S}^\beta-\mathbf{V}\cdot\mathbf{S}^\beta \\ &= 0. \end{aligned}$$

Ex. 218: Similarly to 216, we have, given definition $T^i_j = N^i N_j$

$$\begin{aligned} T^i_j T^j_k &= N^i N_j N^j N_k \\ &= N^i N_k \\ &= T^i_k. \end{aligned}$$

[Note: This seems like the exact same problem - do we mean to define $T^i_j = T^i T_j$?]

Ex. 219: We have [Note that this implies 213 additionally]

$$N^i N_j + Z^i_\alpha Z^\alpha_j = \delta^i_j.$$

Contract both sides with N_i :

$$\begin{aligned} N^i N_j N_i + Z^i_\alpha Z^\alpha_j N_i &= \delta^i_j N_i \\ N_i N^i N_j + N_i Z^i_\alpha Z^\alpha_j &= N_j \\ N_i N^i N_j + 0 &= N_j \\ N_i N^i N_j &= N_j, \end{aligned}$$

where the third line follows from $N_i Z^i_\alpha = 0$. Now, this holds for all N_j, for which at least one is nonzero (we cannot have the normal vector be zero). Hence, we have

$$N_i N^i = 1,$$

as desired.

Ex. 220: Using similar manipulations of indices to the earlier discussion of the Levy-Civita symbols, we derive

$$\begin{aligned} -\frac{1}{4}\delta^{ijk}_{rst} T^t_j T^s_k &= -\frac{1}{4}\delta^{ijk}_{rts} T^s_j T^t_k \\ &= \frac{1}{4}\delta^{ijk}_{rst} T^s_j T^t_k, \end{aligned}$$

so

$$\begin{aligned} N^i N_r &= \frac{1}{4}\delta^{ijk}_{rst} T^s_j T^t_k - \frac{1}{4}\delta^{ijk}_{rst} T^t_j T^s_k \\ &= 2\left(\frac{1}{4}\delta^{ijk}_{rst} T^s_j T^t_k\right) \\ &= \frac{1}{2}\delta^{ijk}_{rst} T^s_j T^t_k. \end{aligned}$$

Ex. 221: This result follows exactly as was done earlier, except we use the new definition of the Jacobian for surface coordinates

$$J_{\alpha}^{\alpha'} = \frac{\partial S^{\alpha'}}{\partial S^{\alpha}}.$$

Ex. 222: From before, we have

$$\frac{\partial Z_{ij}}{\partial Z^k} = Z_{li}\Gamma_{jk}^l + Z_{lj}\Gamma_{ik}^l.$$

From the analogous definitions of $S_{\alpha\beta}$, we have

$$\frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} = S_{\delta\alpha}\Gamma_{\beta\gamma}^{\delta} + S_{\delta\beta}\Gamma_{\alpha\gamma}^{\delta}$$

compute

$$\begin{aligned}
& \frac{1}{2}S^{\alpha\omega}\left(\frac{\partial S_{\omega\beta}}{\partial S^{\gamma}} + \frac{\partial S_{\omega\gamma}}{\partial S^{\beta}} - \frac{\partial S_{\beta\gamma}}{\partial S^{\omega}}\right) \\
= & \frac{1}{2}S^{\alpha\omega}\left(S_{\delta\omega}\Gamma_{\beta\gamma}^{\delta} + S_{\delta\beta}\Gamma_{\omega\gamma}^{\delta} + S_{\delta\omega}\Gamma_{\beta\gamma}^{\delta} + S_{\delta\gamma}\Gamma_{\omega\beta}^{\delta} - \left(S_{\delta\beta}\Gamma_{\gamma\omega}^{\delta} + S_{\delta\gamma}\Gamma_{\beta\omega}^{\delta}\right)\right) \\
= & \frac{1}{2}S^{\alpha\omega}\left(S_{\delta\omega}\Gamma_{\beta\gamma}^{\delta} + S_{\delta\beta}\Gamma_{\omega\gamma}^{\delta} + S_{\delta\omega}\Gamma_{\beta\gamma}^{\delta} + S_{\delta\gamma}\Gamma_{\omega\beta}^{\delta} - S_{\delta\beta}\Gamma_{\gamma\omega}^{\delta} - S_{\delta\gamma}\Gamma_{\beta\omega}^{\delta}\right) \\
= & \frac{1}{2}\left(\delta_{\delta}^{\alpha}\Gamma_{\beta\gamma}^{\delta} + S^{\alpha\omega}S_{\delta\beta}\Gamma_{\alpha\gamma}^{\delta} + \delta_{\delta}^{\alpha}\Gamma_{\beta\gamma}^{\delta} + S^{\alpha\omega}S_{\delta\gamma}\Gamma_{\omega\beta}^{\delta} - S^{\alpha\omega}S_{\delta\beta}\Gamma_{\gamma\omega}^{\delta} - S^{\alpha\omega}S_{\delta\gamma}\Gamma_{\beta\omega}^{\delta}\right) \\
= & \frac{1}{2}\left(2\delta_{\delta}^{\alpha}\Gamma_{\beta\gamma}^{\delta}\right) \\
= & \Gamma_{\beta\gamma}^{\alpha},
\end{aligned}$$

as desired.

Ex. 223: Assume the ambient space is reffered to affine coordiantes. We have

$$\begin{aligned}
\Gamma_{\beta\gamma}^{\alpha} &= Z_i^{\alpha}\frac{\partial Z_{\beta}^i}{\partial S^{\gamma}} + \Gamma_{jk}^i Z_{\iota}^{\alpha} Z_{\beta}^j Z_{\beta}^k \\
&= Z_i^{\alpha}\frac{\partial Z_{\beta}^i}{\partial S^{\gamma}} + 0,
\end{aligned}$$

since $\Gamma_{jk}^i = 0$ in affine coordinates.

Ex. 224 [Still Working]

Ex. 225 We compute, given

$$\begin{aligned} Z^1(\theta,\phi) &= R \\ Z^2(\theta,\phi) &= \theta \\ Z^3(\theta,\phi) &= \phi \end{aligned}$$

$$\begin{aligned} Z^i_\alpha &= \frac{\partial Z^i}{\partial S^\alpha} \\ &= \begin{bmatrix} \frac{\partial Z^1}{\partial S^1} & \frac{\partial Z^1}{\partial S^2} \\ \frac{\partial Z^2}{\partial S^1} & \frac{\partial Z^2}{\partial S^2} \\ \frac{\partial Z^3}{\partial S^1} & \frac{\partial Z^3}{\partial S^2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

then, note that since

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have

$$Z^\alpha_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} N^i &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \mathbf{i} \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{S}_1 &= Z^i_1 \mathbf{Z}_i \\ &= Z^2_1 \mathbf{Z}_2 \\ &= R\cos\theta\cos\phi\mathbf{i} + R\cos\theta\sin\phi\mathbf{j} - R\sin\theta\mathbf{k} \\ \mathbf{S}_2 &= Z^i_2 \mathbf{Z}_i \\ &= Z^3_2 \mathbf{Z}_3 \\ &= -R\sin\theta\sin\phi\mathbf{i} + R\sin\theta\cos\phi\mathbf{j} \end{aligned}$$

$$
\begin{aligned}
S_{\alpha\beta} &= \begin{bmatrix} R^2\cos^2\theta\cos^2\phi + R^2\cos^2\theta\sin^2\phi + R^2\sin^2\theta & -R^2\cos\theta\cos\phi\sin\theta\sin\phi + R^2\cos\theta\sin\phi \\ -R^2\cos\theta\cos\phi\sin\theta\sin\phi + R^2\cos\theta\sin\phi\sin\theta\cos\phi & R^2\sin^2\theta\sin^2\phi + R^2\sin^2\theta\cos^2\phi \end{bmatrix} \\
&= \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix} \quad \text{[ASK - should this be the same as when the ambient coordinates are Cart.?} \\
S^{\alpha\beta} &= \begin{bmatrix} R^{-2} & 0 \\ 0 & R^{-2}\sin^{-2}\theta \end{bmatrix} \\
\sqrt{S} &= \sqrt{\begin{vmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{vmatrix}} \\
&= R^2\sin\theta.
\end{aligned}
$$

Now, recall the Christoffel symbols for the ambient space (in spherical coords):

$$
\begin{aligned}
\Gamma^1_{22} &= -r \\
\Gamma^1_{33} &= -r\sin^2\theta \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^2_{23} &= \Gamma^2_{32} = \cot\theta.
\end{aligned}
$$

Now, setting θ as coord. 1 and ϕ as coord. 2, and using

$$
\Gamma^\alpha_{\beta\gamma} = Z^\alpha_i \frac{\partial Z^i_\beta}{\partial S^\gamma} + \Gamma^i_{jk} Z^\alpha_\iota Z^j_\beta Z^k_\beta,
$$

we compute

$$\begin{aligned}
\Gamma_{11}^{1} &= Z_{i}^{1}\frac{\partial Z_{1}^{i}}{\partial S^{1}} + \Gamma_{jk}^{i} Z_{\iota}^{1} Z_{1}^{j} Z_{1}^{k} \\
&= Z_{2}^{1}\frac{\partial Z_{1}^{2}}{\partial S^{1}} + \Gamma_{jk}^{2} Z_{2}^{1} Z_{1}^{j} Z_{1}^{k} \\
&= Z_{2}^{1}\frac{\partial Z_{1}^{2}}{\partial S^{1}} + \Gamma_{22}^{2} Z_{2}^{1} Z_{1}^{2} Z_{1}^{2} \\
&= \frac{\partial Z_{1}^{2}}{\partial S^{1}} \\
&= 0 \\
\Gamma_{21}^{1} &= \Gamma_{12}^{1} = 0 + \Gamma_{jk}^{i} Z_{\iota}^{1} Z_{2}^{j} Z_{1}^{k} \\
&= \Gamma_{jk}^{2} Z_{2}^{1} Z_{2}^{j} Z_{1}^{k} \\
&= \Gamma_{32}^{2} Z_{2}^{1} Z_{2}^{3} Z_{1}^{2} \\
&= \cot\theta\,(1)\,(1)\,(1) \\
&= \cot\theta \\
\Gamma_{22}^{1} &= \Gamma_{jk}^{i} Z_{\iota}^{1} Z_{2}^{j} Z_{2}^{k} \\
&= \Gamma_{jk}^{2} Z_{2}^{1} Z_{2}^{j} Z_{2}^{k} \\
&= \Gamma_{33}^{2} Z_{2}^{1} Z_{2}^{3} Z_{2}^{3} \\
&= -\sin\theta\cos\theta
\end{aligned}$$

$$\begin{aligned}
\Gamma_{11}^{2} &= \Gamma_{jk}^{i} Z_{\iota}^{2} Z_{1}^{j} Z_{1}^{k} \\
&= \Gamma_{jk}^{3} Z_{3}^{2} Z_{1}^{j} Z_{1}^{k} \\
&= \Gamma_{22}^{3} Z_{3}^{2} Z_{1}^{2} Z_{1}^{2} \\
&= 0 \\
\Gamma_{21}^{2} &= \Gamma_{12}^{2} = \Gamma_{jk}^{i} Z_{\iota}^{2} Z_{2}^{j} Z_{1}^{k} \\
&= \Gamma_{jk}^{3} Z_{3}^{2} Z_{2}^{j} Z_{1}^{k} \\
&= \Gamma_{32}^{3} Z_{3}^{2} Z_{2}^{3} Z_{1}^{2} \\
&= 0 \\
\Gamma_{22}^{2} &= \Gamma_{jk}^{i} Z_{\iota}^{2} Z_{2}^{j} Z_{2}^{k} \\
&= \Gamma_{33}^{3} Z_{3}^{2} Z_{2}^{3} Z_{2}^{3} \\
&= 0,
\end{aligned}$$

(note $\frac{\partial Z_{\beta}^{i}}{\partial S^{\gamma}}$ vanishes in each computation).

Ex. 226: We have

$$\sqrt{(x'(s))^2 + (y'(s))^2} = 1,$$

since this is an arc-length parametrization. Thus,

$$\begin{aligned} N^i &= \begin{bmatrix} y'(s) \\ -x'(s) \end{bmatrix} \\ S_{\alpha\beta} &= (x'(s))^2 + (y'(s))^2 \\ &= 1 \\ S^{\alpha\beta} &= 1 \\ \sqrt{S} &= 1 \end{aligned}$$

[ASK why $x'x'' + y'y'' = 0$].

Ex. 227: Simply denote $t = x$, and then we have the parametrization

$$\begin{aligned} x(t) &= t \\ y(t) &= y(t), \end{aligned}$$

and ccompute these objects in the preceding section, noting that $x'(t) = 1$. The, re-substitute $x = t$.

Ex. 228: Again, we have (in polar coordinates)

$$\sqrt{r'(s)^2 + r(s)^2 \theta'(s)^2} = 1,$$

so the results follow similarly to the above cases.

Chapter 11

Chapter 11

Ex. 229: This follows similarly as with the ambient covariant derviative, using the tensor properties of $T_{\alpha}^{\beta}(S)$ given in surface coordinates, and using the analogous Jacobians $J_{\alpha}^{\alpha'}$.

Ex. 230: The sum rule is clear from the sum rule of the partial derivative, and the properties of contraction. Also, the product rule follows as with the ambient case.

Ex. 231: We compute, using

$$\Gamma_{\beta\gamma}^{\alpha} = \frac{1}{2} S^{\alpha\omega} \left(\frac{\partial S_{\omega\beta}}{\partial S^{\gamma}} + \frac{\partial S_{\omega\gamma}}{\partial S^{\beta}} - \frac{\partial S_{\beta\gamma}}{\partial S^{\omega}} \right)$$

$$\begin{aligned}
\nabla_{\gamma} S_{\alpha\beta} &= \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} - \Gamma_{\gamma\alpha}^{\delta} S_{\delta\beta} - \Gamma_{\gamma\beta}^{\delta} S_{\alpha\delta} \\
&= \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} - \Gamma_{\alpha\gamma}^{\delta} S_{\delta\beta} - \Gamma_{\beta\gamma}^{\delta} S_{\alpha\delta} \\
&= \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} - \frac{1}{2} S^{\delta\omega} \left(\frac{\partial S_{\omega\alpha}}{\partial S^{\gamma}} + \frac{\partial S_{\omega\gamma}}{\partial S^{\alpha}} - \frac{\partial S_{\alpha\gamma}}{\partial S^{\omega}} \right) S_{\delta\beta} - \frac{1}{2} S^{\delta\omega} \left(\frac{\partial S_{\omega\beta}}{\partial S^{\gamma}} + \frac{\partial S_{\omega\gamma}}{\partial S^{\beta}} - \frac{\partial S_{\beta\gamma}}{\partial S^{\omega}} \right) S_{\alpha\delta} \\
&= \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} - \frac{1}{2} \delta_{\beta}^{\omega} \left(\frac{\partial S_{\omega\alpha}}{\partial S^{\gamma}} + \frac{\partial S_{\omega\gamma}}{\partial S^{\alpha}} - \frac{\partial S_{\alpha\gamma}}{\partial S^{\omega}} \right) - \frac{1}{2} \delta_{\alpha}^{\omega} \left(\frac{\partial S_{\omega\beta}}{\partial S^{\gamma}} + \frac{\partial S_{\omega\gamma}}{\partial S^{\beta}} - \frac{\partial S_{\beta\gamma}}{\partial S^{\omega}} \right) \\
&= \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} - \frac{1}{2} \left(\frac{\partial S_{\beta\alpha}}{\partial S^{\gamma}} + \frac{\partial S_{\beta\gamma}}{\partial S^{\alpha}} - \frac{\partial S_{\alpha\gamma}}{\partial S^{\beta}} \right) - \frac{1}{2} \left(\frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} + \frac{\partial S_{\alpha\gamma}}{\partial S^{\beta}} - \frac{\partial S_{\beta\gamma}}{\partial S^{\alpha}} \right) \\
&= \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} - \frac{1}{2} \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} - \frac{1}{2} \frac{\partial S_{\alpha\beta}}{\partial S^{\gamma}} \\
&= 0.
\end{aligned}$$

Similarly, we may show that in the contravariant case,

$$\nabla_{\gamma} S^{\alpha\beta} = 0.$$

For the Levy-Civita symbols, note

$$\begin{aligned}\varepsilon_{\alpha\beta} &= \sqrt{S}e_{\alpha\beta}\\ \varepsilon^{\alpha\beta} &= \frac{1}{\sqrt{S}}e^{\alpha\beta}\end{aligned}$$

The result follows similarly to the ambient case, carefully noting that

$$\Gamma^{\alpha}_{\beta\gamma} = \mathbf{S}^{\alpha}\cdot\frac{\partial \mathbf{S}_{\beta}}{\partial S^{\gamma}}.$$

The delta systems follow from the product rule and the fact that $\nabla_\gamma S_{\alpha\beta} = \nabla_\gamma \varepsilon_{\alpha\beta} = \nabla_\gamma \varepsilon^{\alpha\beta} = 0$.

Ex. 232: Commutativity with contraction follows exactly as in the ambient case.

Ex. 233: We compute, using

$$S^{\alpha\beta} = \begin{bmatrix} R^{-2} & 0\\ 0 & R^{-2}\sin^{-2}\theta\end{bmatrix},$$

the following:

$$\begin{aligned}\nabla_\alpha\nabla^\alpha F &= \frac{1}{\sqrt{S}}\frac{\partial}{\partial S^\alpha}\left(\sqrt{S}S^{\alpha\beta}\frac{\partial F}{\partial S^\beta}\right)\\
&= \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(R^2\sin\theta S^{1\beta}\frac{\partial F}{\partial S^\beta}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\phi}\left(R^2\sin\theta S^{2\beta}\frac{\partial F}{\partial S^\beta}\right)\\
&= \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(R^2\sin\theta S^{11}\frac{\partial F}{\partial\theta}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\phi}\left(R^2\sin\theta S^{22}\frac{\partial F}{\partial\phi}\right)\\
&= \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(R^2\sin\theta R^{-2}\frac{\partial F}{\partial\theta}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\phi}\left(R^2\sin\theta R^{-2}\sin^{-2}\theta\frac{\partial F}{\partial\phi}\right)\\
&= \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial F}{\partial\phi}\right)\\
&= \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}.\end{aligned}$$

Ex. 234: For the surface of a cylinder, note

$$\begin{aligned}S^{\alpha\beta} &= \begin{bmatrix} R^{-2} & 0\\ 0 & 1\end{bmatrix}\\ \sqrt{S} &= R,\end{aligned}$$

so

$$\begin{aligned}
\nabla_\alpha \nabla^\alpha F &= \frac{1}{\sqrt{S}} \frac{\partial}{\partial S^\alpha} \left(\sqrt{S} S^{\alpha\beta} \frac{\partial F}{\partial S^\beta} \right) \\
&= \frac{1}{R} \frac{\partial}{\partial \theta} \left(R S^{11} \frac{\partial F}{\partial \theta} \right) + \frac{1}{R} \frac{\partial}{\partial z} \left(R S^{22} \frac{\partial F}{\partial z} \right) \\
&= \frac{1}{R^2} \frac{\partial^2 F}{\partial \theta^2} + \frac{\partial^2 F}{\partial z^2}.
\end{aligned}$$

Ex. 235: Note

$$\begin{aligned}
S^{\alpha\beta} &= \begin{bmatrix} (R + r\cos\phi)^{-2} & 0 \\ 0 & r^{-2} \end{bmatrix} \\
\sqrt{S} &= r\,(R + r\cos\phi)\,,
\end{aligned}$$

and compute

$$\begin{aligned}
\nabla_\alpha \nabla^\alpha F &= \frac{1}{\sqrt{S}} \frac{\partial}{\partial S^\alpha} \left(\sqrt{S} S^{\alpha\beta} \frac{\partial F}{\partial S^\beta} \right) \\
&= \frac{1}{r\,(R + r\cos\phi)} \frac{\partial}{\partial \theta} \left(r\,(R + r\cos\phi)\, S^{1\beta} \frac{\partial F}{\partial S^\beta} \right) \\
&\quad + \frac{1}{r\,(R + r\cos\phi)} \frac{\partial}{\partial \phi} \left(r\,(R + r\cos\phi)\, S^{2\beta} \frac{\partial F}{\partial S^\beta} \right) \\
&= \frac{1}{r\,(R + r\cos\phi)} \frac{\partial}{\partial \theta} \left(r\,(R + r\cos\phi)\,(R + r\cos\phi)^{-2} \frac{\partial F}{\partial \theta} \right) \\
&\quad + \frac{1}{r\,(R + r\cos\phi)} \frac{\partial}{\partial \phi} \left(r\,(R + r\cos\phi)\, r^{-2} \frac{\partial F}{\partial \phi} \right) \\
&= \frac{1}{(R + r\cos\phi)^2} \frac{\partial^2 F}{\partial \theta^2} + \frac{1}{r^2\,(R + r\cos\phi)} \frac{\partial}{\partial \phi} \left((R + r\cos\phi) \frac{\partial F}{\partial \phi} \right).
\end{aligned}$$

Ex. 236: We have

$$\begin{aligned}
S^{\alpha\beta} &= \begin{bmatrix} r\,(z)^{-2} & 0 \\ 0 & \frac{1}{1 + r'(z)^2} \end{bmatrix} \\
\sqrt{S} &= r\,(z) \sqrt{1 + r'\,(z)^2};
\end{aligned}$$

Thus,

$$
\begin{aligned}
\nabla_\alpha \nabla^\alpha F &= \frac{1}{\sqrt{S}} \frac{\partial}{\partial S^\alpha} \left(\sqrt{S} S^{\alpha\beta} \frac{\partial F}{\partial S^\beta} \right) \\
&= \frac{1}{r(z)\sqrt{1+r'(z)^2}} \frac{\partial}{\partial \theta} \left(\sqrt{S} S^{11} \frac{\partial F}{\partial \theta} \right) + \frac{1}{r(z)\sqrt{1+r'(z)^2}} \frac{\partial}{\partial z} \left(\sqrt{S} S^{22} \frac{\partial F}{\partial z} \right) \\
&= \frac{1}{r(z)\sqrt{1+r'(z)^2}} \frac{\partial}{\partial \theta} \left(r(z) \sqrt{1+r'(z)^2} r(z)^{-2} \frac{\partial F}{\partial \theta} \right) \\
&\quad + \frac{1}{r(z)\sqrt{1+r'(z)^2}} \frac{\partial}{\partial z} \left(r(z) \sqrt{1+r'(z)^2} \frac{1}{1+r'(z)^2} \frac{\partial F}{\partial z} \right) \\
&= \frac{1}{r(z)\sqrt{1+r'(z)^2}} \frac{\partial}{\partial z} \left(\frac{r(z)}{\sqrt{1+r'(z)^2}} \frac{\partial F}{\partial z} \right) + \frac{1}{r(z)^2} \frac{\partial^2 F}{\partial \theta^2}.
\end{aligned}
$$

Ex. 237: These were computed earlier.

Ex. 238: We compute:

$$
\begin{aligned}
\nabla_\gamma \mathbf{Z}_i &= \frac{\partial \mathbf{Z}_i}{\partial S^\gamma} - Z^j_\gamma \Gamma^k_{ij} \mathbf{Z}_k \\
&= \frac{\partial \mathbf{Z}_i}{\partial Z^m} \frac{\partial Z^m}{\partial S^\gamma} - Z^j_\gamma \Gamma^k_{ij} \mathbf{Z}_k \\
&= \frac{\partial \mathbf{Z}_i}{\partial Z^m} Z^m_\gamma - Z^j_\gamma \Gamma^k_{ij} \mathbf{Z}_k \\
&= \Gamma^k_{im} \mathbf{Z}_k Z^m_\gamma - Z^j_\gamma \Gamma^k_{ij} \mathbf{Z}_k \\
&= \left(Z^m_\gamma \Gamma^k_{im} - Z^j_\gamma \Gamma^k_{ij} \right) \mathbf{Z}_k \\
&= 0,
\end{aligned}
$$

after index renaming. The contravariant case follows similarly. Also, since $Z_{ij} = \mathbf{Z}_i \cdot \mathbf{Z}_j$, we have

$$\nabla_\gamma Z_{ij} = 0$$

by the product rule. Similarly,

$$\nabla_\gamma Z^{ij} = 0.$$

[Levy-Civita Symbols to come]

Ex. 239: Begin with equation 10.41:

$$N_i N^i = 1,$$

and compute take the surface covariant derivative of both sides:

$$\begin{aligned} 0 &= \nabla_\alpha \left(N_i N^i\right) \\ &= \nabla_\alpha N_i N^i + N_i \nabla_\alpha N^i \\ &= \nabla_\alpha \left(N^j Z_{ij}\right) N_k Z^{ik} + N_i \nabla_\alpha N^i \\ &= \left(\nabla_\alpha N^j Z_{ij}\right) N_k Z^{ik} + N_i \nabla_\alpha N^i, \end{aligned}$$

by the metrinilic property,

$$\begin{aligned} &= \nabla_\alpha N^j N_k \delta_j^k + N_i \nabla_\alpha N^i \\ &= \nabla_\alpha N^k N_k + N_i \nabla_\alpha N^i \\ &= N_k \nabla_\alpha N^k + N_i \nabla_\alpha N^i \\ &= 2 N_i \nabla_\alpha N^i, \end{aligned}$$

after index renaming. Thus,

$$N_i \nabla_\alpha N^i = 0.$$

Ex. 240: We compute

$$\begin{aligned}
\varepsilon^{ijk}\varepsilon_{\alpha\beta}Z_j^{\beta}N_k &= \varepsilon^{ijk}\varepsilon_{\alpha\beta}Z_j^{\beta}\left(\frac{1}{2}\varepsilon_{kmn}\varepsilon^{\gamma\delta}Z_\gamma^m Z_\delta^n\right) \\
&= \frac{1}{2}\varepsilon^{ijk}\varepsilon_{kmn}\varepsilon^{\gamma\delta}\varepsilon_{\alpha\beta}Z_j^{\beta}Z_\gamma^m Z_\delta^n \\
&= \frac{1}{2}\delta^{ijk}_{kmn}\delta^{\gamma\delta}_{\alpha\beta}Z_j^{\beta}Z_\gamma^m Z_\delta^n \\
&= -\frac{1}{2}\delta^{ijk}_{mkn}\delta^{\gamma\delta}_{\alpha\beta}Z_j^{\beta}Z_\gamma^m Z_\delta^n \\
&= \frac{1}{2}\delta^{ijk}_{mnk}\delta^{\gamma\delta}_{\alpha\beta}Z_j^{\beta}Z_\gamma^m Z_\delta^n \\
&= \frac{1}{2}2\delta^{ij}_{mn}\delta^{\gamma\delta}_{\alpha\beta}Z_j^{\beta}Z_\gamma^m Z_\delta^n \\
&= \delta^{ij}_{mn}\delta^{\gamma\delta}_{\alpha\beta}Z_j^{\beta}Z_\gamma^m Z_\delta^n \\
&= \left(\delta^i_m\delta^j_n - \delta^j_m\delta^i_n\right)\left(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta\right)Z_j^{\beta}Z_\gamma^m Z_\delta^n \\
&= \delta^i_m\delta^j_n\delta^\gamma_\alpha\delta^\delta_\beta Z_j^{\beta}Z_\gamma^m Z_\delta^n - \delta^j_m\delta^i_n\delta^\gamma_\alpha\delta^\delta_\beta Z_j^{\beta}Z_\gamma^m Z_\delta^n - \delta^i_m\delta^j_n\delta^\delta_\alpha\delta^\gamma_\beta Z_j^{\beta}Z_\gamma^m Z_\delta^n + \delta^j_m\delta^i_n\delta^\delta_\alpha\delta^\gamma_\beta Z_j^{\beta}Z_\gamma^m Z \\
&= \delta^\gamma_\alpha\delta^\delta_\beta Z_j^{\beta}Z_\gamma^i Z_\delta^j - \delta^\gamma_\alpha\delta^\delta_\beta Z_j^{\beta}Z_\gamma^j Z_\delta^i - \delta^\delta_\alpha\delta^\gamma_\beta Z_j^{\beta}Z_\gamma^i Z_\delta^j + \delta^\delta_\alpha\delta^\gamma_\beta Z_j^{\beta}Z_\gamma^j Z_\delta^i \\
&= \delta^\gamma_\alpha\delta^\delta_\beta\delta^\beta_\delta Z_\gamma^i - \delta^\gamma_\alpha\delta^\delta_\beta\delta^\beta_\gamma Z_\delta^i - \delta^\delta_\alpha\delta^\gamma_\beta\delta^\beta_\delta Z_\gamma^i + \delta^\delta_\alpha\delta^\gamma_\beta\delta^\beta_\gamma Z_\delta^i \\
&= Z_\alpha^i - Z_\alpha^i - Z_\alpha^i + Z_\alpha^i
\end{aligned}$$

[Some factors of 2 needed?]

Ex. 241: Note that for a general covariant-contravariant tensor, we have

$$\nabla_\alpha T^i_j = Z^k_\alpha \nabla_k T^i_j.$$

Thus,

$$\nabla_\alpha u = Z^k_\alpha \nabla_k u$$

and

$$\nabla^\alpha u = Z^{\alpha k}\nabla_k u.$$

Thus,

$$\begin{aligned}
\nabla_\gamma\nabla^\alpha u &= \nabla_\gamma\left(Z^{\alpha k}\nabla_k u\right) \\
&= \nabla_\gamma Z^{\alpha k}\nabla_k u + Z^{\alpha k}\nabla_\gamma\nabla_k u \\
&= B^\alpha_\gamma N^k\nabla_k u + Z^{\alpha k}Z^m_\gamma\nabla_m\nabla_k u. \\
&= B^\alpha_\gamma N^k\nabla_k u + Z^{\alpha k}Z_{n\gamma}Z^{mn}\nabla_m\nabla_k u \\
&= B^\alpha_\gamma N^k\nabla_k u + Z^{\alpha k}Z^\delta_n S_{\delta\gamma}Z^{mn}\nabla_m\nabla_k u
\end{aligned}$$

Now, set $\gamma = \alpha$ and contract:

$$\begin{aligned}\nabla_\alpha \nabla^\alpha u &= B_\alpha^\alpha N^k \nabla_k u + Z^{\alpha k} Z_n^\delta S_{\delta\alpha} Z^{mn} \nabla_m \nabla_k u \\ &= B_\alpha^\alpha N^k \nabla_k u + \delta_\delta^\beta Z_\beta^k Z_n^\delta Z^{mn} \nabla_m \nabla_k u \\ &= B_\alpha^\alpha N^k \nabla_k u + Z_\beta^k Z_n^\beta Z^{mn} \nabla_m \nabla_k u.\end{aligned}$$

Now,

$$N^k N_n + Z_\beta^k Z_n^\beta = \delta_n^k,$$

so

$$Z_\beta^k Z_n^\beta = \delta_n^k - N^k N_n.$$

We substitute in the above:

$$\begin{aligned}\nabla_\alpha \nabla^\alpha u &= B_\alpha^\alpha N^k \nabla_k u + \left(\delta_n^k - N^k N_n\right) Z^{mn} \nabla_m \nabla_k u \\ &= B_\alpha^\alpha N^k \nabla_k u + Z^{km} \nabla_m \nabla_k u - N^k N_n Z^{mn} \nabla_m \nabla_k u \\ &= B_\alpha^\alpha N^k \nabla_k u + \nabla_m \nabla^m u - N^m N^k \nabla_m \nabla_k u,\end{aligned}$$

or after renaming dummy indices,

$$\nabla_\alpha \nabla^\alpha u = B_\alpha^\alpha N^i \nabla_i u + \nabla_i \nabla^i u - N^i N^j \nabla_i \nabla_j u,$$

or

$$N^i N^j \nabla_i \nabla_j u = \nabla_i \nabla^i u - \nabla_\alpha \nabla^\alpha u + B_\alpha^\alpha N^i \nabla_i u$$

Ex. 242: Let

$$Z^i(s)$$

be the parametrization of the line normal to the surface, emanating from point Z_0^i. Note that we have

$$\frac{dZ^i}{ds}(0) = \lim_{h\to 0} \frac{Z^i(h) - Z_0^i}{h} = N^i.$$

also, compute

$$\begin{aligned}\frac{d}{ds}\left[\frac{dZ^i}{ds}\mathbf{Z}_i\right] &= \frac{d^2 Z^i}{ds^2}\mathbf{Z}_i + \frac{dZ^i}{ds}\frac{d\mathbf{Z}_i}{ds}(Z(s)) \\ &= \frac{d^2 Z^i}{ds^2}\mathbf{Z}_i + \frac{dZ^i}{ds}\frac{\partial \mathbf{Z}_i}{\partial Z^k}\frac{dZ^k}{ds} \\ &= \frac{d^2 Z^i}{ds^2}\mathbf{Z}_i + \frac{dZ^i}{ds}\Gamma_{ik}^n \mathbf{Z}_n \frac{dZ^k}{ds},\end{aligned}$$

so at $s = 0$,

$$\begin{aligned}
\frac{d}{ds}\left[\frac{dZ^i}{ds}\mathbf{Z}_i\right]|_{s=0} &= \frac{d^2Z^i}{ds^2}\mathbf{Z}_i + N^i\Gamma^n_{ik}\mathbf{Z}_n N^k \\
&= \frac{d^2Z^i}{ds^2}\mathbf{Z}_i + N^i N^k\Gamma^n_{ik}\mathbf{Z}_n \\
&= \frac{d^2Z^i}{ds^2}\mathbf{Z}_i + N^j N^k\Gamma^n_{jk}\mathbf{Z}_i \\
&= \left(\frac{d^2Z^i}{ds^2} + N^j N^k\Gamma^i_{jk}\right)\mathbf{Z}_i
\end{aligned}$$

Now, examine the LHS of the above. Since

$$\frac{d}{ds}\left[\frac{dZ^i}{ds}\mathbf{Z}_i\right]$$

represents the second derivative of a line, the LHS vanishes. Thus,

$$\frac{d^2Z^i}{ds^2} = -N^j N^k\Gamma^i_{jk},$$

Then, define

$$F(s) = u(Z(s)),$$

so

$$\begin{aligned}
F'(s) &= \frac{\partial u}{\partial Z^i}(Z(s))\frac{dZ^i}{ds}(s) \\
F''(s) &= \frac{d}{ds}\left[\frac{\partial u}{\partial Z^i}(Z(s))\right]\frac{dZ^i}{ds}(s) + \frac{\partial u}{\partial Z^i}(Z(s))\frac{d}{ds}\left[\frac{dZ^i}{ds}(s)\right] \\
&= \frac{\partial^2 u}{\partial Z^i\partial Z^j}(Z(s))\frac{dZ^i}{ds}(s)\frac{dZ^j}{ds}(s) + \frac{\partial u}{\partial Z^i}(Z(s))\frac{d^2Z^i}{ds^2}(s).
\end{aligned}$$

at $s = 0$:

$$\begin{aligned}
F''(0) &= \frac{\partial^2 u}{\partial Z^i\partial Z^j}(Z(0))N^iN^j + \frac{\partial u}{\partial Z^i}(Z(0))\frac{d^2Z^i}{ds^2}(0) \\
&= \frac{\partial}{\partial Z^j}[\nabla_i u]N^iN^j + \nabla_i u\frac{d^2Z^i}{ds^2}(0)
\end{aligned}$$

now, note

$$\begin{aligned}
\nabla_j \nabla_i u &= \frac{\partial \nabla_i u}{\partial Z^j} - \Gamma_{ij}^k \nabla_k u \\
\frac{\partial \nabla_i u}{\partial Z^j} &= \nabla_j \nabla_i u + \Gamma_{ij}^k \nabla_k u,
\end{aligned}$$

so

$$\begin{aligned}
F''(0) &= \left(\nabla_j \nabla_i u + \Gamma_{ij}^k \nabla_k u\right) N^i N^j + \nabla_i u \frac{d^2 Z^i}{ds^2}(0) \\
&= \nabla_j \nabla_i u N^i N^j + \Gamma_{ij}^k \nabla_k u N^i N^j + \nabla_i u \frac{d^2 Z^i}{ds^2}(0) \\
&= N^i N^j \nabla_j \nabla_i u + N^i N^j \Gamma_{ij}^k \nabla_k u - N^j N^k \Gamma_{jk}^i \nabla_i u \\
&= N^i N^j \nabla_j \nabla_i u;
\end{aligned}$$

thus

$$\frac{\partial^2 u}{\partial n^2} = F''(0) = N^i N^j \nabla_i \nabla_j u$$

after renaming indices.

Chapter 12

Chapter 12

Ex. 243: This follows from the definition and from lowering the index γ.

Ex. 244: We have

$$R^{\gamma}_{\cdot\delta\alpha\beta} = \frac{\partial\Gamma^{\gamma}_{\beta\delta}}{\partial S^{\alpha}} - \frac{\partial\Gamma^{\gamma}_{\alpha\delta}}{\partial S^{\beta}} + \Gamma^{\gamma}_{\alpha\omega}\Gamma^{\omega}_{\beta\delta} - \Gamma^{\gamma}_{\beta\omega}\Gamma^{\omega}_{\alpha\delta},$$

so

$$\begin{aligned}
R^{\delta}_{\cdot\delta\alpha\beta} &= \frac{\partial\Gamma^{\delta}_{\beta\delta}}{\partial S^{\alpha}} - \frac{\partial\Gamma^{\delta}_{\alpha\delta}}{\partial S^{\beta}} + \Gamma^{\delta}_{\alpha\omega}\Gamma^{\omega}_{\beta\delta} - \Gamma^{\delta}_{\beta\omega}\Gamma^{\omega}_{\alpha\delta} \\
&= \frac{\partial\Gamma^{\delta}_{\beta\delta}}{\partial S^{\alpha}} - \frac{\partial\Gamma^{\delta}_{\alpha\delta}}{\partial S^{\beta}} + \Gamma^{\omega}_{\beta\delta}\Gamma^{\delta}_{\alpha\omega} - \Gamma^{\delta}_{\beta\omega}\Gamma^{\omega}_{\alpha\delta} \\
&= \frac{\partial\Gamma^{\delta}_{\beta\delta}}{\partial S^{\alpha}} - \frac{\partial\Gamma^{\delta}_{\alpha\delta}}{\partial S^{\beta}} \\
&= 0.
\end{aligned}$$

Ex. 245: This was done for the final exam.

Ex. 246: Compute

$$\begin{aligned}
R_{\delta\gamma\alpha\beta} &= R_{\alpha\beta\delta\gamma} \quad (12.5) \\
&= -R_{\alpha\beta\gamma\delta} \quad (12.3) \\
&= -R_{\gamma\delta\alpha\beta} \quad (12.5).
\end{aligned}$$

Ex. 247: Examine

$$\begin{aligned}
R_{\alpha\beta} &= R^{\gamma}_{\cdot\alpha\gamma\beta} \\
&= S^{\delta\gamma} R_{\delta\alpha\gamma\beta} \\
&= S^{\delta\gamma} R_{\gamma\beta\delta\alpha} \\
&= S^{\gamma\delta} R_{\gamma\beta\delta\alpha} \\
&= R^{\delta}_{\cdot\beta\delta\alpha} \\
&= R^{\gamma}_{\cdot\beta\gamma\alpha} \\
&= R_{\beta\alpha}.
\end{aligned}$$

Ex. 248: We may easily see the symmetry of the Einstein tensor from the fact that both $R_{\alpha\beta}$ and $S_{\alpha\beta}$ are symmetric.

Ex. 249: Note

$$\begin{aligned}
G^{\beta}_{\alpha} &= R_{\alpha\gamma} S^{\gamma\beta} - \frac{1}{2} R S_{\alpha\gamma} S^{\gamma\beta} \\
&= R_{\alpha\gamma} S^{\gamma\beta} - \frac{1}{2} R \delta^{\beta}_{\alpha},
\end{aligned}$$

so

$$\begin{aligned}
G^{\alpha}_{\alpha} &= R_{\alpha\gamma} S^{\gamma\alpha} - R \\
&= R^{\alpha}_{\alpha} - R \\
&= R - R \\
&= 0,
\end{aligned}$$

since $R^{\alpha}_{\alpha} = R$ by definition.

Ex. 250: We compute

$$\begin{aligned}
(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) T_\gamma &= (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) T^{\delta} S_{\delta\gamma} \\
&= R^{\delta}_{\cdot\varepsilon\alpha\beta} T^{\varepsilon} S_{\delta\gamma} \\
&= R^{\delta}_{\cdot\varepsilon\alpha\beta} S_{\delta\gamma} T^{\varepsilon} \\
&= R_{\gamma\varepsilon\alpha\beta} T^{\varepsilon} \\
&= R_{\gamma\varepsilon\alpha\beta} T_{\omega} S^{\varepsilon\omega} \\
&= -R_{\varepsilon\gamma\alpha\beta} T_{\omega} S^{\varepsilon\omega} \\
&= -R_{\varepsilon\gamma\alpha\beta} S^{\varepsilon\omega} T_{\omega} \\
&= -R^{\omega}_{\cdot\gamma\alpha\beta} T_{\omega} \\
&= -R^{\delta}_{\cdot\gamma\alpha\beta} T_{\delta},
\end{aligned}$$

with index renaming at the last step.

Ex. 251: The invariant case follows from the commutativity of partial derivatives. Now, we consider the covariant case:

$$
\begin{aligned}
\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) T^i &= \nabla_\alpha \nabla_\beta T^i - \nabla_\beta \nabla_\alpha T^i \\
&= \frac{\partial\left(\nabla_\beta T^i\right)}{\partial S^\alpha} - \Gamma^\gamma_{\alpha\beta} \nabla_\gamma T^i - \left(\frac{\partial\left(\nabla_\alpha T^i\right)}{\partial S^\beta} - \Gamma^\gamma_{\beta\alpha} \nabla_\gamma T^i\right) \\
&= \frac{\partial\left(\nabla_\beta T^i\right)}{\partial S^\alpha} - \frac{\partial\left(\nabla_\alpha T^i\right)}{\partial S^\beta} \\
&= \frac{\partial}{\partial S^\alpha}\left(\frac{\partial T^i}{\partial S^\beta} + Z^k_\beta \Gamma^i_{km} T^m\right) - \frac{\partial}{\partial S^\beta}\left(\frac{\partial T^i}{\partial S^\alpha} + Z^k_\alpha \Gamma^i_{km} T^m\right) \\
&= \frac{\partial^2 T^i}{\partial S^\alpha \partial S^\beta} + \frac{\partial}{\partial S^\alpha}\left(Z^k_\beta \Gamma^i_{km} T^m\right) - \frac{\partial^2 T^i}{\partial S^\alpha \partial S^\beta} - \frac{\partial}{\partial S^\beta}\left(Z^k_\alpha \Gamma^i_{km} T^m\right) \\
&= \frac{\partial}{\partial S^\alpha}\left(Z^k_\beta \Gamma^i_{km} T^m\right) - \frac{\partial}{\partial S^\beta}\left(Z^k_\alpha \Gamma^i_{km} T^m\right) \\
&= \frac{\partial Z^k_\beta}{\partial S^\alpha} \Gamma^i_{km} T^m + Z^k_\beta \frac{\partial \Gamma^i_{km}}{\partial S^\alpha} + Z^k_\beta \Gamma^i_{km} \frac{\partial T^m}{\partial S^\alpha} \\
&\quad - \frac{\partial Z^k_\alpha}{\partial S^\beta} \Gamma^i_{km} T^m - Z^k_\alpha \frac{\partial \Gamma^i_{km}}{\partial S^\beta} T^m - Z^k_\alpha \Gamma^i_{km} \frac{\partial T^m}{\partial S^\beta} \\
&= 0 \ \ \text{[not sure yet]}
\end{aligned}
$$

Ex. 252: Look at

$$\begin{aligned}
R_{\alpha\beta\gamma\delta}+R_{\alpha\gamma\delta\beta}+R_{\alpha\delta\beta\gamma} &= \frac{\partial\Gamma_{\alpha,\delta\beta}}{\partial S^\gamma}-\frac{\partial\Gamma_{\alpha,\gamma\beta}}{\partial S^\delta}+\Gamma_{\omega,\alpha\delta}\Gamma^\omega_{\gamma\beta}-\Gamma_{\omega,\beta\delta}\Gamma^\omega_{\gamma\alpha}\\
&\quad+\frac{\partial\Gamma_{\alpha,\beta\gamma}}{\partial S^\delta}-\frac{\partial\Gamma_{\alpha,\delta\gamma}}{\partial S^\beta}+\Gamma_{\omega,\alpha\beta}\Gamma^\omega_{\delta\gamma}-\Gamma_{\omega,\gamma\beta}\Gamma^\omega_{\delta\alpha}\\
&\quad+\frac{\partial\Gamma_{\alpha,\gamma\delta}}{\partial S^\beta}-\frac{\partial\Gamma_{\alpha,\beta\delta}}{\partial S^\gamma}+\Gamma_{\omega,\alpha\gamma}\Gamma^\omega_{\beta\delta}-\Gamma_{\omega,\delta\gamma}\Gamma^\omega_{\beta\alpha}\\
&= \frac{\partial\Gamma_{\alpha,\beta\delta}}{\partial S^\gamma}-\frac{\partial\Gamma_{\alpha,\beta\gamma}}{\partial S^\delta}+\Gamma_{\omega,\alpha\delta}\Gamma^\omega_{\beta\gamma}-\Gamma_{\omega,\beta\delta}\Gamma^\omega_{\alpha\gamma}\\
&\quad+\frac{\partial\Gamma_{\alpha,\beta\gamma}}{\partial S^\delta}-\frac{\partial\Gamma_{\alpha,\gamma\delta}}{\partial S^\beta}+\Gamma_{\omega,\alpha\beta}\Gamma^\omega_{\gamma\delta}-\Gamma_{\omega,\beta\gamma}\Gamma^\omega_{\alpha\delta}\\
&\quad+\frac{\partial\Gamma_{\alpha,\gamma\delta}}{\partial S^\beta}-\frac{\partial\Gamma_{\alpha,\beta\delta}}{\partial S^\gamma}+\Gamma_{\omega,\alpha\gamma}\Gamma^\omega_{\beta\delta}-\Gamma_{\omega,\gamma\delta}\Gamma^\omega_{\alpha\beta}\\
&= \Gamma_{\omega,\alpha\delta}\Gamma^\omega_{\beta\gamma}-\Gamma_{\omega,\beta\delta}\Gamma^\omega_{\alpha\gamma}\\
&\quad+\Gamma_{\omega,\alpha\beta}\Gamma^\omega_{\gamma\delta}-\Gamma_{\omega,\beta\gamma}\Gamma^\omega_{\alpha\delta}\\
&\quad+\Gamma_{\omega,\alpha\gamma}\Gamma^\omega_{\beta\delta}-\Gamma_{\omega,\gamma\delta}\Gamma^\omega_{\alpha\beta}\\
&= S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\delta}\Gamma^\omega_{\beta\gamma}-S_{\omega\varepsilon}\Gamma^\varepsilon_{\beta\delta}\Gamma^\omega_{\alpha\gamma}\\
&\quad+S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\beta}\Gamma^\omega_{\gamma\delta}-S_{\omega\varepsilon}\Gamma^\varepsilon_{\beta\gamma}\Gamma^\omega_{\alpha\delta}\\
&\quad+S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\gamma}\Gamma^\omega_{\beta\delta}-S_{\omega\varepsilon}\Gamma^\varepsilon_{\gamma\delta}\Gamma^\omega_{\alpha\beta}\\
&= S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\delta}\Gamma^\omega_{\beta\gamma}-S_{\varepsilon\omega}\Gamma^\omega_{\beta\delta}\Gamma^\varepsilon_{\alpha\gamma}\\
&\quad+S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\beta}\Gamma^\omega_{\gamma\delta}-S_{\varepsilon\omega}\Gamma^\omega_{\beta\gamma}\Gamma^\varepsilon_{\alpha\delta}\\
&\quad+S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\gamma}\Gamma^\omega_{\beta\delta}-S_{\varepsilon\omega}\Gamma^\omega_{\gamma\delta}\Gamma^\varepsilon_{\alpha\beta}\\
&= S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\delta}\Gamma^\omega_{\beta\gamma}-S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\gamma}\Gamma^\omega_{\beta\delta}\\
&\quad+S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\beta}\Gamma^\omega_{\gamma\delta}-S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\delta}\Gamma^\omega_{\beta\gamma}\\
&\quad+S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\gamma}\Gamma^\omega_{\beta\delta}-S_{\omega\varepsilon}\Gamma^\varepsilon_{\alpha\beta}\Gamma^\omega_{\gamma\delta}\\
&= 0,
\end{aligned}$$

as desired.

Ex. 253: Compute

$$\begin{aligned}
\nabla_\varepsilon R_{\alpha\beta\gamma\delta}+\nabla_\gamma R_{\alpha\beta\delta\varepsilon}+\nabla_\delta R_{\alpha\beta\varepsilon\gamma} &= \frac{\partial R_{\alpha\beta\gamma\delta}}{\partial S^\varepsilon}-\Gamma^\omega_{\alpha\varepsilon}R_{\omega\beta\gamma\delta}-\Gamma^\omega_{\beta\varepsilon}R_{\alpha\omega\gamma\delta}-\Gamma^\omega_{\gamma\varepsilon}R_{\alpha\beta\omega\delta}-\Gamma^\omega_{\delta\varepsilon}R_{\alpha\beta\gamma\omega}\\
&\quad+\frac{\partial R_{\alpha\beta\delta\varepsilon}}{\partial S^\gamma}-\Gamma^\omega_{\alpha\gamma}R_{\omega\beta\delta\varepsilon}-\Gamma^\omega_{\beta\gamma}R_{\alpha\omega\delta\varepsilon}-\Gamma^\omega_{\delta\gamma}R_{\alpha\beta\omega\varepsilon}-\Gamma^\omega_{\varepsilon\gamma}R_{\alpha\beta\delta\omega}\\
&\quad+\frac{\partial R_{\alpha\beta\varepsilon\gamma}}{\partial S^\delta}-\Gamma^\omega_{\alpha\delta}R_{\omega\beta\varepsilon\gamma}-\Gamma^\omega_{\beta\delta}R_{\alpha\omega\varepsilon\gamma}-\Gamma^\omega_{\varepsilon\delta}R_{\alpha\beta\omega\gamma}-\Gamma^\omega_{\gamma\delta}R_{\alpha\beta\varepsilon\omega}\\
&= \frac{\partial R_{\alpha\beta\gamma\delta}}{\partial S^\varepsilon}-\Gamma^\omega_{\alpha\varepsilon}R_{\omega\beta\gamma\delta}-\Gamma^\omega_{\beta\varepsilon}R_{\alpha\omega\gamma\delta}-\Gamma^\omega_{\gamma\varepsilon}R_{\alpha\beta\omega\delta}-\Gamma^\omega_{\delta\varepsilon}R_{\alpha\beta\gamma\omega}\\
&\quad+\frac{\partial R_{\alpha\beta\delta\varepsilon}}{\partial S^\gamma}-\Gamma^\omega_{\alpha\gamma}R_{\omega\beta\delta\varepsilon}-\Gamma^\omega_{\beta\gamma}R_{\alpha\omega\delta\varepsilon}-\Gamma^\omega_{\delta\gamma}R_{\alpha\beta\omega\varepsilon}+\Gamma^\omega_{\varepsilon\gamma}R_{\alpha\beta\omega\delta}\\
&\quad+\frac{\partial R_{\alpha\beta\varepsilon\gamma}}{\partial S^\delta}-\Gamma^\omega_{\alpha\delta}R_{\omega\beta\varepsilon\gamma}-\Gamma^\omega_{\beta\delta}R_{\alpha\omega\varepsilon\gamma}+\Gamma^\omega_{\varepsilon\delta}R_{\alpha\beta\gamma\omega}+\Gamma^\omega_{\gamma\delta}R_{\alpha\beta\omega\varepsilon}\\
&= \frac{\partial R_{\alpha\beta\gamma\delta}}{\partial S^\varepsilon}-\Gamma^\omega_{\alpha\varepsilon}R_{\omega\beta\gamma\delta}-\Gamma^\omega_{\beta\varepsilon}R_{\alpha\omega\gamma\delta}\\
&\quad+\frac{\partial R_{\alpha\beta\delta\varepsilon}}{\partial S^\gamma}-\Gamma^\omega_{\alpha\gamma}R_{\omega\beta\delta\varepsilon}-\Gamma^\omega_{\beta\gamma}R_{\alpha\omega\delta\varepsilon}\\
&\quad+\frac{\partial R_{\alpha\beta\varepsilon\gamma}}{\partial S^\delta}-\Gamma^\omega_{\alpha\delta}R_{\omega\beta\varepsilon\gamma}-\Gamma^\omega_{\beta\delta}R_{\alpha\omega\varepsilon\gamma}\\
&= \frac{\partial R_{\alpha\beta\gamma\delta}}{\partial S^\varepsilon}-\Gamma^\omega_{\alpha\varepsilon}\left(\frac{\partial\Gamma_{\omega,\delta\beta}}{\partial S^\gamma}-\frac{\partial\Gamma_{\omega,\gamma\beta}}{\partial S^\delta}+\Gamma_{\phi,\omega\delta}\Gamma^\phi_{\gamma\beta}-\Gamma_{\phi,\beta\delta}\Gamma^\phi_{\gamma\omega}\right)-\Gamma^\omega_{\beta\varepsilon}\left(\frac{\partial\Gamma_{\alpha,\delta\omega}}{\partial S^\gamma}-\right.\\
&\quad+\ldots
\end{aligned}$$

Ex. 254: Compute

$$\begin{aligned}
\frac{1}{4}\varepsilon^{\gamma\delta}\varepsilon^{\alpha\beta}R_{\gamma\delta\alpha\beta} &= \frac{1}{4}\varepsilon^{\gamma\delta}\varepsilon^{\alpha\beta}\frac{R_{1212}}{S}\varepsilon_{\gamma\delta}\varepsilon_{\alpha\beta}\\
&= \frac{1}{4}(2)(2)\frac{R_{1212}}{S}\\
&= \frac{R_{1212}}{S}\\
&= K,
\end{aligned}$$

as desired.

Ex. 255: Compute

$$
\begin{aligned}
K\left(S_{\alpha\gamma}S_{\beta\delta}-S_{\alpha\delta}S_{\beta\gamma}\right) &= \frac{1}{4}\varepsilon^{\gamma\delta}\varepsilon^{\alpha\beta}R_{\gamma\delta\alpha\beta}\left(S_{\alpha\gamma}S_{\beta\delta}-S_{\alpha\delta}S_{\beta\gamma}\right)\\
&= \frac{1}{4}\varepsilon^{\gamma\delta}\varepsilon^{\alpha\beta}S_{\alpha\gamma}S_{\beta\delta}R_{\gamma\delta\alpha\beta}-\frac{1}{4}\varepsilon^{\gamma\delta}\varepsilon^{\alpha\beta}S_{\alpha\delta}S_{\beta\gamma}\\
&= \frac{1}{4}\varepsilon^{\gamma\delta}\varepsilon^{\alpha\beta}S_{\alpha\gamma}S_{\beta\delta}R_{\gamma\delta\alpha\beta}+\frac{1}{4}\varepsilon^{\gamma\delta}\varepsilon^{\beta\alpha}S_{\alpha\delta}S_{\beta\gamma}\\
&= \frac{1}{4}\delta^{\delta}_{\alpha}\delta^{\alpha}_{\delta}R_{\gamma\delta\alpha\beta}+\frac{1}{4}\delta^{\gamma}_{\alpha}\delta^{\alpha}_{\gamma}R_{\gamma\delta\alpha\beta}\\
&= \frac{1}{4}\left(2\right)R_{\gamma\delta\alpha\beta}+\frac{1}{4}\left(2\right)R_{\gamma\delta\alpha\beta}\\
&= R_{\gamma\delta\alpha\beta}.
\end{aligned}
$$

Ex. 256: Note

$$\frac{1}{2}R^{\alpha\beta}_{\cdot\gamma\delta}=\frac{1}{2}R_{\omega\xi\gamma\delta}S^{\omega\alpha}S^{\xi\beta},$$

so

$$
\begin{aligned}
\frac{1}{2}R^{\alpha\beta}_{\cdot\alpha\beta} &= \frac{1}{2}R_{\omega\xi\alpha\beta}S^{\omega\alpha}S^{\xi\beta}\\
&= \frac{1}{2}K\varepsilon_{\omega\xi}\varepsilon_{\alpha\beta}S^{\omega\alpha}S^{\xi\beta}\\
&= \frac{1}{2}K\delta^{\alpha}_{\xi}\delta^{\xi}_{\alpha}\\
&= K
\end{aligned}
$$

Ex. 257: This follows since

$$\nabla_{\alpha}\mathbf{S}_{\beta}=\nabla_{\beta}\mathbf{S}_{\alpha},$$

hence

$$\mathbf{N}B_{\alpha\beta}=\mathbf{N}B_{\beta\alpha},$$

or

$$B_{\alpha\beta}=B_{\beta\alpha}.$$

Ex. 258: Note that since $B^{\alpha}_{\alpha} = 0$, we have that both eigenvalues of $B^{\cdot}_{\cdot}$ are equal in absolute value and are negatives of each other; denote them $\lambda, -\lambda$. Thus,

$$|B^{\cdot}_{\cdot}| = -\lambda^2.$$

Now,

$$B^{\alpha}_{\beta} B^{\beta}_{\gamma} := C^{\alpha}_{\gamma}.$$

In linear algebra terms, we have

$$C^{\cdot}_{\cdot} = B^{\cdot 2}_{\cdot}$$

then,

$$\begin{aligned} B^{\alpha}_{\beta} B^{\beta}_{\alpha} &= \text{tr } B^{\cdot 2}_{\cdot} \\ &= \mu_1 + \mu_2, \end{aligned}$$

where μ_1, μ_2 are the eigenvalues of $B^{\cdot 2}_{\cdot}$. But, since $\mu_1 = \lambda^2$ and $\mu_2 = (-\lambda)^2 = \lambda^2$ by the properties of eigenvalues, we have

$$\begin{aligned} B^{\alpha}_{\beta} B^{\beta}_{\alpha} &= 2\lambda^2 \\ &= -2\,|B^{\cdot}_{\cdot}| \end{aligned}$$

by the above.

Ex. 259: We compute, given

$$\begin{aligned} r(z) &= a\cosh\left(\frac{z-b}{a}\right) \\ &= a\left(\frac{e^{(z-b)/a} + e^{(b-z)/a}}{2}\right) \\ &= \frac{a}{2}e^{(z-b)/a} + \frac{a}{2}e^{(b-z)/a} \end{aligned}$$

$$r'(z) = \frac{1}{2}e^{(z-b)/a} - \frac{1}{2}e^{(b-z)/a}$$

$$r''(z) = \frac{1}{2a}e^{(z-b)/a} + \frac{1}{2a}e^{(b-z)/a}.$$

Compute

$$\begin{aligned}
r''(z)\,r(z) - r'(z)^2 &= \left(\frac{1}{2a}e^{(z-b)/a} + \frac{1}{2a}e^{(b-z)/a}\right)\left(\frac{a}{2}e^{(z-b)/a} + \frac{a}{2}e^{(b-z)/a}\right) - \left(\frac{1}{2}e^{(z-b)/a} - \frac{1}{2}e^{(b-z)/a}\right. \\
&= \frac{1}{4}e^{2(z-b)/a} - \frac{1}{4}e^{2(b-z)/a} - \left[\frac{1}{4}e^{2(z-b)/a} - 1 + \frac{1}{4}e^{2(b-z)/a}\right] \\
&= 1
\end{aligned}$$

Thus,

$$r''(z)\,r'(z) - r'(z)^2 - 1 = 0$$

$$\begin{aligned}
B^{\alpha}_{\alpha} &= \frac{r''(z)\,r'(z) - r'(z)^2 - 1}{r(z)\sqrt{1 + r'(z)^2}} \\
&= 0,
\end{aligned}$$

as desired.

Ex. 260: We have

$$\begin{aligned}
\mathbf{V} &= \frac{d\mathbf{R}}{dt}(S(t)) \\
&= \frac{\partial \mathbf{R}}{\partial S^{\alpha}} \frac{dS^{\alpha}}{dt} \\
&= \mathbf{S}_{\alpha} V^{\alpha} \\
&= V^{\alpha} \mathbf{S}_{\alpha}
\end{aligned}$$

as desired.

Ex. 261: Compute

$$\begin{aligned}
\mathbf{A} &= \frac{d\mathbf{V}}{dt} \\
&= \frac{d}{dt}\left[V^\alpha \mathbf{S}_\alpha\right] \\
&= \frac{dV^\alpha}{dt}\mathbf{S}_\alpha + V^\alpha \frac{d\mathbf{S}_\alpha}{dt}\left(S\left(t\right)\right) \\
&= \frac{dV^\alpha}{dt}\mathbf{S}_\alpha + V^\alpha \frac{\partial \mathbf{S}_\alpha}{\partial S^\beta}\frac{dS^\beta}{dt} \\
&= \frac{dV^\alpha}{dt}\mathbf{S}_\alpha + V^\alpha V^\beta \frac{\partial \mathbf{S}_\alpha}{\partial S^\beta} \\
&= \frac{dV^\alpha}{dt}\mathbf{S}_\alpha + V^\alpha V^\beta \left(\nabla_\beta \mathbf{S}_\alpha + \Gamma^\gamma_{\alpha\beta}\mathbf{S}_\gamma\right) \\
&= \frac{dV^\alpha}{dt}\mathbf{S}_\alpha + V^\alpha V^\beta \Gamma^\gamma_{\alpha\beta}\mathbf{S}_\gamma + V^\alpha V^\beta \nabla_\beta \mathbf{S}_\alpha \\
&= \frac{dV^\alpha}{dt}\mathbf{S}_\alpha + V^\beta V^\gamma \Gamma^\alpha_{\beta\gamma}\mathbf{S}_\alpha + V^\alpha V^\beta \nabla_\beta \mathbf{S}_\alpha \\
&= \frac{\delta V^\alpha}{\delta t}\mathbf{S}_\alpha + V^\alpha V^\beta \nabla_\beta \mathbf{S}_\alpha \\
&= \frac{\delta V^\alpha}{\delta t}\mathbf{S}_\alpha + \mathbf{N} V^\alpha V^\beta B_{\alpha\beta} \\
&= \frac{\delta V^\alpha}{\delta t}\mathbf{S}_\alpha + \mathbf{N} B_{\alpha\beta} V^\alpha V^\beta,
\end{aligned}$$

as desired.

Ex. 262: We define

$$\frac{\delta T^\alpha_\beta}{\delta t} = \frac{dT^\alpha_\beta}{dt} + V^\gamma \Gamma^\alpha_{\gamma\omega} T^\omega_\beta - V^\gamma \Gamma^\omega_{\gamma\beta} T^\alpha_\omega.$$

Ex. 263: Look at

$$\begin{aligned}
\frac{\delta T^{\alpha'}_{\beta'}}{\delta t} &= \frac{dT^{\alpha'}_{\beta'}}{dt} + V^\gamma \Gamma^{\alpha'}_{\gamma\omega} T^\omega_{\beta'} - V^\gamma \Gamma^\omega_{\gamma\beta'} T^{\alpha'}_\omega \\
&= \frac{d}{dt}\left(T^\alpha_\beta J^{\alpha'}_\alpha \left(S\left(t\right)\right) J^\beta_{\beta'}\left(S'\left(t\right)\right)\right) + V^\gamma \Gamma^{\alpha'}_{\gamma\omega} T^\omega_\beta J^\beta_{\beta'} - V^\gamma \Gamma^\omega_{\gamma\beta'} T^\alpha_\omega J^{\alpha'}_\alpha \\
&= \frac{dT^\alpha_\beta}{dt} J^{\alpha'}_\alpha J^\beta_{\beta'} + T^\alpha_\beta \frac{\partial J^{\alpha'}_\alpha}{\partial S^\gamma}\frac{dS^\gamma}{dt} J^\beta_{\beta'} + T^\alpha_\beta \frac{\partial J^\beta_{\beta'}}{\partial S^{\gamma'}} J^{\alpha'}_\alpha \frac{dS^{\gamma'}}{dt} + V^\gamma \Gamma^{\alpha'}_{\gamma\omega} T^\omega_\beta J^\beta_{\beta'} - V^\gamma \Gamma^\omega_{\gamma\beta'} T^\alpha_\omega J^{\alpha'}_\alpha \\
&= \frac{dT^\alpha_\beta}{dt} J^{\alpha'}_\alpha J^\beta_{\beta'} + T^\alpha_\beta J^{\alpha'}_{\gamma\alpha} V^\gamma J^\beta_{\beta'} + T^\alpha_\beta J^\beta_{\gamma'\beta'} V^{\gamma'} J^{\alpha'}_\alpha + V^\gamma \Gamma^{\alpha'}_{\gamma\omega} T^\omega_\beta J^\beta_{\beta'} - V^\gamma \Gamma^\omega_{\gamma\beta'} T^\alpha_\omega J^{\alpha'}_\alpha \\
&= \ldots \quad ??? \\
&= \frac{\delta T^\alpha_\beta}{\delta t} J^{\alpha'}_\alpha J^\beta_{\beta'}
\end{aligned}$$

Ex. 264: These follow from the properties of the standard derivative.

Ex. 265: This also follows from the properties of the standard derivative.

Ex. 266: Not sure - are we considering the surface metrics as functions of time? In that case, would this be a moving surface, and then we would require the derivative in Part III?

Ex. 267: Compute

$$\begin{aligned}
\frac{\delta \mathbf{S}_\alpha}{\delta t} &= \frac{d\mathbf{S}_\alpha\left(S\left(t\right)\right)}{dt} - V^\gamma \Gamma^\omega_{\gamma\beta}\mathbf{S}_\omega \\
&= \frac{\partial \mathbf{S}_\alpha}{\partial S^\beta}\frac{dS^\beta}{dt} - V^\gamma \Gamma^\omega_{\gamma\beta}\mathbf{S}_\omega \\
&= \left(\nabla_\beta \mathbf{S}_\alpha + \Gamma^\omega_{\alpha\beta}\mathbf{S}_\omega\right)V^\beta - V^\gamma \Gamma^\omega_{\gamma\beta}\mathbf{S}_\omega \\
&= \nabla_\beta \mathbf{S}_\alpha V^\beta + V^\beta \Gamma^\omega_{\alpha\beta}\mathbf{S}_\omega - V^\gamma \Gamma^\omega_{\gamma\beta}\mathbf{S}_\omega \\
&= \nabla_\beta \mathbf{S}_\alpha V^\beta \\
&= \mathbf{N} B_{\alpha\beta} V^\beta \\
&= \mathbf{N} V^\beta B_{\alpha\beta},
\end{aligned}$$

as desired.

Ex. 268: This follows from the sum and product rules.

Ex. 269: [Not finished]

Ex. 270: For a cylinder, we have

$$B^\alpha_\beta = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{bmatrix},$$

so

$$\begin{aligned}
K &= \left|B^\cdot_\cdot\right| \\
&= 0\left(-\frac{1}{R}\right) \\
&= 0.
\end{aligned}$$

Ex. 271: For a cone, we have

$$B_{\beta}^{\alpha} = \begin{bmatrix} -\frac{\cot\Theta}{r} & 0 \\ 0 & 0 \end{bmatrix},$$

which has determinant zero. Thus,

$$K = 0.$$

Ex. 272: For a sphere, we have

$$B_{\beta}^{\alpha} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix},$$

so

$$\begin{aligned} K &= |B_{\cdot}^{\cdot}| = \left(-\frac{1}{R}\right)\left(-\frac{1}{R}\right) \\ &= \frac{1}{R^2} \end{aligned}$$

Ex. 273: For a torus, we have

$$B_{\beta}^{\alpha} = \begin{bmatrix} -\frac{\cos\phi}{R+r\cos\phi} & 0 \\ 0 & -\frac{1}{r} \end{bmatrix},$$

so

$$K = \frac{\cos\phi}{r\left(R + r\cos\phi\right)}$$

Ex. 274: For a surface of revolution, we have

$$B_{\beta}^{\alpha} = \begin{bmatrix} -\frac{1}{r(z)\sqrt{1+r'(z)^2}} & 0 \\ 0 & \frac{r''(z)}{\left(1+r'(z)^2\right)^{3/2}} \end{bmatrix},$$

so

$$\begin{aligned} K &= -\frac{1}{r\left(z\right)\sqrt{1+r'\left(z\right)^2}}\frac{r''\left(z\right)}{\left(1+r'\left(z\right)^2\right)^{3/2}} \\ &= -\frac{r''\left(z\right)}{r\left(z\right)\left(1+r'\left(z\right)^2\right)^2} \end{aligned}$$

Ex. 275: We integrate

$$\begin{aligned}\int_S K dS &= \int_S \frac{1}{R^2} dS \\ &= \frac{1}{R^2}\int_S dS \\ &= \frac{1}{R^2}4\pi R^2 \\ &= 4\pi\end{aligned}$$

Ex. 276: We integrate

$$\begin{aligned}\int_S \frac{\cos\phi}{r\left(R + r\cos\phi\right)} dS &= \int_0^{2\pi}\int_0^{2\pi} \frac{\cos\phi}{r\left(R + r\cos\phi\right)}\sqrt{S} d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi} \frac{\cos\phi}{r\left(R + r\cos\phi\right)} r\left(R + r\cos\phi\right) d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{2\pi} \cos\phi d\phi d\theta \\ &= 2\pi\left(\sin\left(2\pi\right) - \sin\left(0\right)\right) \\ &= 0.\end{aligned}$$

Part III

Part III

Ex. 291: We have

$$\theta(\alpha) = \operatorname{arccot}(At\cot\alpha),$$

so

$$\begin{aligned} \theta &= \operatorname{arccot}(At\cot\alpha) \\ \cot\theta &= At\cot\alpha \\ \frac{1}{At}\cot\theta &= \cot\alpha \\ \alpha(\theta) &= \operatorname{arccot}\left(\frac{1}{At}\cot\theta\right) \end{aligned}$$

$$\begin{aligned} J_t^{\alpha} &= \frac{\partial S^{\alpha}(t, S')}{\partial t} \\ &= \frac{\partial}{\partial t}\operatorname{arccot}\left(\frac{1}{At}\cot\theta\right) \\ &= -\frac{1}{1+\frac{1}{A^2t^2}\cot^2\theta}\cdot -\frac{\cot\theta}{At^2} \\ &= \frac{\cot\theta}{At^2+\frac{1}{A}\cot^2\theta} \\ &= \frac{A\cot\theta}{A^2t^2+\cot^2\theta} \end{aligned}$$

$$\begin{aligned} J_t^{\alpha'} &= \frac{\partial}{\partial t}\operatorname{arccot}(At\cot\alpha) \\ &= -\frac{A\cot\alpha}{1+A^2t^2\cot^2\alpha} \end{aligned}$$

Ex. 292:

$$V^{\iota} = \frac{\partial Z^i}{\partial t} = \begin{bmatrix} A\cos\alpha \\ 0 \end{bmatrix}$$

Ex. 293:

$$
\begin{aligned}
V^i &= \frac{\partial Z^i}{\partial t} = \begin{bmatrix} \frac{\partial}{\partial t}\left(\frac{A\cos\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}}\right) \\ \frac{\partial}{\partial t}\left(\frac{A\sin\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}}\right) \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2A^2t\sin^2\theta\cdot A\cos\theta}{2\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}^3} \\ -\frac{2A^2t\sin^2\theta\cdot A\sin\theta}{2\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}^3} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{A^3t\sin^2\theta\cos\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}^3} \\ -\frac{A^3t\sin^3\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}^3} \end{bmatrix}.
\end{aligned}
$$

Ex. 294: Clearly, the above expressions do not show the tensor property with respect to changes in surface coordinates.

Ex. 295: First, note our parametrization:

$$
Z^i(\alpha) = \begin{bmatrix} At\cos\alpha \\ \sin\alpha \end{bmatrix}
$$

Then, compute the shift tensors:

$$
\begin{aligned}
Z^i_\alpha &= \frac{\partial Z^i}{\partial S^\alpha} \\
&= \begin{bmatrix} -At\sin\alpha \\ \cos\alpha \end{bmatrix},
\end{aligned}
$$

so

$$
Z^\alpha_i = \begin{bmatrix} -\frac{1}{At}\sin\alpha & \cos\alpha \end{bmatrix},
$$

since we need

$$
Z^\alpha_i Z^i_\beta = \delta^\alpha_\beta.
$$

Thus,

$$
\begin{aligned}
V^\alpha &= V^i Z^\alpha_i \\
&= \begin{bmatrix} -\frac{1}{At}\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} A\cos\alpha \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{t}\sin\alpha \end{bmatrix}.
\end{aligned}
$$

Ex. 296: Recall

$$Z^i(\theta) = \begin{bmatrix} \frac{A\cos\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}} \\ \frac{A\sin\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}} \end{bmatrix},$$

and note that

$$\begin{aligned} \frac{\partial}{\partial\theta}\sqrt{\cos^2\theta + A^2t^2\sin^2\theta} &= \frac{-2\cos\theta\sin\theta + 2A^2t^2\sin\theta\cos\theta}{2\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}} \\ &= \frac{A^2t^2\sin\theta\cos\theta - \cos\theta\sin\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}} \end{aligned}$$

so

$$\begin{aligned} Z^i_\alpha &= \frac{\partial Z^i}{\partial S^\alpha} \\ &= \frac{1}{\cos^2\theta + A^2t^2\sin^2\theta}\begin{bmatrix} \sqrt{\cos^2\theta + A^2t^2\sin^2\theta}\,(-A\sin\theta) - A\cos\theta\frac{A^2t^2\sin\theta\cos\theta - \cos\theta\sin\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}} \\ \sqrt{\cos^2\theta + A^2t^2\sin^2\theta}\,(A\cos\theta) - A\sin\theta\frac{A^2t^2\sin\theta\cos\theta - \cos\theta\sin\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}} \end{bmatrix}, \end{aligned}$$

The result is $V^\alpha = 0$, since the motion of a particle on the surface with constant θ is normal to tangent space; hence the projection $V^\alpha = V^i Z^\alpha_i = 0$.

Ex. 297: We see that V^α is not a tensor with respect to changes in ambient coordinates.

Ex. 298: We have

$$\mathbf{V} = V^i \mathbf{Z}_i,$$

so

$$\mathbf{V}\cdot\mathbf{Z}^j = V^j,$$

confirming that V^i is a tensor ($\mathbf{V}$ is an invariant and $\mathbf{Z}^j$ is a tensor).

Ex. 299: Both V^i and Z^α_i are tensors with regard to ambient coordinate changes; hence the contraction $V^\alpha = V^i Z^\alpha_i$ is a tensor with regard to ambient coordinate changes.

Ex. 300: This was done before.

Ex. 301: Both parametrizations use Cartesian ambient coordinates. Thus, the Jacobians $J^{i'}_i$ are the identity. We compute

$$\begin{aligned} V^i J^{i'}_i + Z^i_\alpha J^{i'}_i J^\alpha_t &= V^i + Z^i_\alpha J^\alpha_t \\ &= \begin{bmatrix} A\cos\alpha \\ 0 \end{bmatrix} + \frac{A\cot\theta}{A^2t^2 + \cot^2\theta}\begin{bmatrix} -\frac{1}{At}\sin\alpha \\ \cos\alpha \end{bmatrix}. \end{aligned}$$

[To be continued]

Ex. 302: Write

$$\begin{aligned}
V^{\alpha'} &= V^{i'} Z_{i'}^{\alpha'} \\
&= \left(V^i J_i^{i'} + Z_\alpha^i J_i^{i'} J_t^\alpha\right) Z_j^\beta J_{i'}^j J_\beta^{\alpha'} \\
&= V^i J_i^{i'} Z_j^\beta J_{i'}^j J_\beta^{\alpha'} + Z_\alpha^i J_i^{i'} J_t^\alpha Z_j^\beta J_{i'}^j J_\beta^{\alpha'} \\
&= V^i Z_j^\beta J_\beta^{\alpha'} \delta_i^j + Z_\alpha^i J_t^\alpha Z_j^\beta J_\beta^{\alpha'} \delta_i^j \\
&= V^j Z_j^\beta J_\beta^{\alpha'} + Z_\alpha^j J_t^\alpha Z_j^\beta J_\beta^{\alpha'} \\
&= V^\beta J_\beta^{\alpha'} + \delta_\beta^\alpha J_t^\alpha J_\beta^{\alpha'} \\
&= V^\beta J_\beta^{\alpha'} + J_\beta^{\alpha'} J_t^\beta,
\end{aligned}$$

as desired.

Ex. 303: Let the unprimed coordinates denote the first parametrization. Then, note

$$\begin{aligned}
J_\beta^{\alpha'} &= \frac{d}{d\alpha} \left(\operatorname{arccot}\left(At \cot \alpha\right)\right) \\
&= -\frac{1}{1 + A^2 t^2 \cot^2 \alpha} \left(-At \csc^2 \alpha\right) \\
&= \frac{At \csc^2 \alpha}{1 + A^2 t^2 \cot^2 \alpha}
\end{aligned}$$

$$\begin{aligned}
V^\beta J_\beta^{\alpha'} + J_\beta^{\alpha'} J_t^\beta &= -\frac{1}{t} \sin \alpha \frac{At \csc^2 \alpha}{1 + A^2 t^2 \cot^2 \alpha} + \frac{A \cot \theta}{A^2 t^2 + \cot^2 \theta} \frac{At \csc^2 \alpha}{1 + A^2 t^2 \cot^2 \alpha} \\
&=
\end{aligned}$$

[perhaps I am using the wrong Jacobians]

Ex. 304: We have

$$Z^i = \begin{bmatrix} t \cos \alpha \\ \sin \alpha \end{bmatrix},$$

so

$$\mathbf{S}_\alpha = \frac{d}{d\alpha} \left(t \cos \alpha\right) \mathbf{i} + \frac{d}{d\alpha} \left(\sin \alpha\right) \mathbf{j},$$

since our ambient space is in Cartesian coordinates. So,

$$\mathbf{S}_\alpha = -t \sin \alpha \mathbf{i} + \cos \alpha \mathbf{j},$$

and choose the outward normal

$$\mathbf{N} = \cos\alpha\mathbf{i} + t\sin\alpha\mathbf{j},$$

and thus

$$N^i = \begin{bmatrix} \cos\alpha \\ t\sin\alpha \end{bmatrix},$$

and

$$N_i = \begin{bmatrix} \cos\alpha & t\sin\alpha \end{bmatrix},$$

since our ambient space is in Cartesian coordinates. Thus,

$$\begin{aligned} C &= V^i N_i \\ &= \begin{bmatrix} \cos\alpha & t\sin\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha \\ 0 \end{bmatrix} \\ &= \cos^2\alpha. \end{aligned}$$

Ex. 305: As before, compute

$$\begin{aligned} \mathbf{S}_\alpha &= \frac{d}{d\theta}\left(\frac{\cos\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}}\right)\mathbf{i} + \frac{d}{d\theta}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}}\right)\mathbf{j} \\ &= \frac{1}{\cos^2\theta + t^2\sin^2\theta}\left(\left(\sqrt{\cos^2\theta + t^2\sin^2\theta}\,(-\sin\theta) - \cos\theta\frac{t^2\sin\theta\cos\theta - \cos\theta\sin\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}}\right)\mathbf{i} + \sqrt{\cos^2\theta + t^2\sin^2\theta}\,(\cos \right. \\ &= \left(-\frac{\sin\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}} - \frac{t^2\sin\theta\cos^2\theta - \cos^2\theta\sin\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}^3}\right)\mathbf{i} + \left(\frac{\cos\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}} - \frac{t^2\sin^2\theta\cos\theta - \cos\theta\sin^2\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}^3}\right. \end{aligned}$$

so

$$N_i = \left[\frac{\cos\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}} - \frac{-\cos\theta\sin^2\theta + t^2\cos\theta\sin^2\theta}{(\cos^2\theta + t^2\sin^2\theta)^{\frac{3}{2}}} \quad \frac{\sin\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}} + \frac{-\cos^2\theta\sin\theta + t^2\cos^2\theta\sin\theta}{(\cos^2\theta + t^2\sin^2\theta)^{\frac{3}{2}}}\right],$$

and

$$\begin{aligned} C &= V^i N_i \\ &= \left[\frac{\cos\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}} - \frac{-\cos\theta\sin^2\theta + t^2\cos\theta\sin^2\theta}{(\cos^2\theta + t^2\sin^2\theta)^{\frac{3}{2}}} \quad \frac{\sin\theta}{\sqrt{\cos^2\theta + t^2\sin^2\theta}} + \frac{-\cos^2\theta\sin\theta + t^2\cos^2\theta\sin\theta}{(\cos^2\theta + t^2\sin^2\theta)^{\frac{3}{2}}}\right] \begin{bmatrix} -\frac{t\sin^2\theta\cos\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}^3} \\ -\frac{t\sin^3\theta}{\sqrt{\cos^2\theta + A^2t^2\sin^2\theta}^3} \end{bmatrix} \\ &= \end{aligned}$$

[Note: Shouldn't both be 1 by geometric considerations?]

Ex. 306: We have

$$\begin{aligned} V^{\alpha'} &= V^{\alpha} J^{\alpha'}_{\alpha} + J^{\alpha'}_{\alpha} J^{\alpha}_{t} \\ \frac{\partial T(t, S')}{\partial t} &= \frac{\partial T(t, S)}{\partial t} + J^{\alpha}_{t} \nabla_{\alpha} t, \end{aligned}$$

so compute

$$\begin{aligned} \dot{\nabla} T(t, S') &= \frac{\partial T(t, S')}{\partial t} - V^{\alpha'}(t, S') \nabla_{\alpha'} T \\ &= \frac{\partial T(t, S)}{\partial t} + J^{\alpha}_{t} \nabla_{\alpha} t - V^{\alpha'} \nabla_{\beta} T J^{\alpha'}_{\beta}, \end{aligned}$$

since $\nabla_{\alpha'} T$ has the tensor property. But,

$$V^{\alpha'} = V^{\gamma} J^{\alpha'}_{\gamma} + J^{\alpha'}_{\gamma} J^{\gamma}_{t},$$

so

$$\begin{aligned} \dot{\nabla} T(t, S') &= \frac{\partial T(t, S)}{\partial t} + J^{\alpha}_{t} \nabla_{\alpha} t - \left(V^{\gamma} J^{\alpha'}_{\gamma} + J^{\alpha'}_{\gamma} J^{\gamma}_{t} \right) \nabla_{\beta} T J^{\beta}_{\alpha'} \\ &= \frac{\partial T(t, S)}{\partial t} + J^{\alpha}_{t} \nabla_{\alpha} t - \left(V^{\gamma} \delta^{\beta}_{\gamma} + J^{\gamma}_{t} \delta^{\beta}_{\gamma} \right) \nabla_{\beta} T \\ &= \frac{\partial T(t, S)}{\partial t} + J^{\alpha}_{t} \nabla_{\alpha} t - V^{\beta} \nabla_{\beta} T - J^{\beta}_{t} \nabla_{\beta} T \\ &= \frac{\partial T(t, S)}{\partial t} - V^{\beta} \nabla_{\beta} T \\ &= \frac{\partial T(t, S)}{\partial t} - V^{\alpha} \nabla_{\alpha} T, \end{aligned}$$

so $\dot{\nabla} T$ does not depend on changes in suface coordinates.

Ex. 307: This follows from the sum rule for partial derivatives and the covariant derivative

Ex. 308: This follows from the product rule for partial derivatives and the covariant derivative.

Ex. 309: Same as above

Ex. 310: This follows because the numerator of 15.33 would be zero.

Ex. 311: This follows from the definition of C, since we take our $\mathbf{R}(S_t + h)$ in the normal direction.

Ex. 312: We have

$$\mathbf{S}_\alpha = Z^i_\alpha \mathbf{Z}_i.$$

Ex. 313: Begin with

$$\dot{\nabla}\mathbf{R} = \left(V^i - V^\alpha Z^i_\alpha\right)\mathbf{Z}_i$$

and

$$V^\alpha = V^j Z^\alpha_j,$$

so

$$\begin{aligned}\dot{\nabla}\mathbf{R} &= \left(V^i - V^j Z^\alpha_j Z^i_\alpha\right)\mathbf{Z}_i\\ &= \left(V^i - V^j\left(\delta^i_j - N^i N_j\right)\right)\mathbf{Z}_i\\ &= \left(V^i - V^i + V^j N^i N_j\right)\mathbf{Z}_i\\ &= V^j N^i N_j \mathbf{Z}_i,\end{aligned}$$

as desired.

Ex. 314: Compute

$$\begin{aligned}\frac{d}{dt}\int_S \mathbf{N} B^\alpha_\alpha dS &= \frac{d}{dt}\int_S \nabla^\alpha \mathbf{S}_\alpha dS\\ &= \int_S \dot{\nabla}\left(\nabla^\alpha \mathbf{S}_\alpha\right) dS - \int_S C B^\alpha_\alpha \nabla^\beta \mathbf{S}_\beta dS\\ &= \int_S \frac{\partial\left(\nabla^\alpha \mathbf{S}_\alpha\right)}{\partial t} - V^\beta \nabla_\beta \nabla^\alpha \mathbf{S}_\alpha dS - \int_S C B^\alpha_\alpha \nabla^\beta \mathbf{S}_\beta dS\\ &= \int_S \frac{\partial\left(\nabla^\alpha \mathbf{S}_\alpha\right)}{\partial t} - V^i Z^\beta_i \nabla_\beta \nabla^\alpha \mathbf{S}_\alpha dS - \int_S C B^\alpha_\alpha \nabla^\beta \mathbf{S}_\beta dS\end{aligned}$$

[ask about integral problems]

Ex. 315: By the above,

$$\int_S \mathbf{N} B^\alpha_\alpha dS$$

is constant. Since our surface is of genus zero, we may smoothly append our surface evolution so that for all $t \geq T$ for some T, S is a sphere of constant radius 1. Since the above quantity is constant for all t, then it must be equal to

$$-2\int_S \mathbf{N}dS$$

for all t, since for a sphere,

$$B_\alpha^\alpha = \frac{-2}{R}.$$

But, $\int_S \mathbf{N}dS = 0$ (our surface is closed), so we have that $\int_S \mathbf{N}B_\alpha^\alpha dS$ vanishes.

Ex. 316: Need to show:

$$\frac{d}{dt}\int_\Omega Fd\Omega = \int_\Omega \frac{\partial F}{\partial t}d\Omega + \int_S CFdS,$$

i.e.

$$\frac{d}{dt}\int_{A_1}^{A_2}\int_0^{b(t,x)} F(x,y)\,dydx = \int_S CFdS,$$

since $\frac{\partial F}{\partial t} = 0$.
Clearly, C is zero on all of S except for the portion given by the graph of b. Let B denote the surface given by this graph. Then,:

$$\int_S CFdS = \int_B CFdB$$

Clearly, $\mathbf{S}_\alpha$ is the vector (given relative to the ambient Cartesian basis)

$$\mathbf{S}_\alpha = \mathbf{i} + b_x\mathbf{j},$$

so

$$\mathbf{N} = \frac{1}{\sqrt{1+b_x^2}}\left(-b_x\mathbf{i} + \mathbf{j}\right),$$

and

$$V^i = \begin{bmatrix} 0 \\ b_t \end{bmatrix},$$

so

$$\begin{aligned} C &= V^iN_i \\ &= \frac{b_t}{\sqrt{1+b_x^2}}. \end{aligned}$$

Now, at each t, our surface has line element $\sqrt{1+b_x^2}$, so

$$\begin{aligned}\int_B CFdB &= \int_{A_1}^{A_2} b_t F\left(x, b\left(t,x\right)\right) dx \\ &= \int_{A_1}^{A_2} \frac{d}{dt}\int_0^{b(t,x)} F\left(x,y\right) dy dx,\end{aligned}$$

by FTC,

$$= \frac{d}{dt}\int_{A_1}^{A_2}\int_0^{b(t,x)} F\left(x,y\right) dy dx$$

Ex. 317: We have

$$\begin{aligned}U^{i'} &= \frac{\partial T^{i'}\left(t,S'\right)}{\partial t} \\ &= \frac{\partial}{\partial t}\left(T^i\left(t,S\right) J_i^{i'}\left(Z\left(t,S\right)\right)\right) \\ &= \frac{\partial}{\partial t} T^i\left(t,S\left(t,S'\right)\right) J_i^{i'} + T^i \frac{\partial}{\partial t}\left(J_i^{i'}\left(Z\left(t,S\right)\right)\right) \\ &= \left(\frac{\partial T^i}{\partial t} + \frac{\partial T^i}{\partial S^\alpha}\frac{\partial}{\partial t}S^\alpha\right) J_i^{i'} + T^i \frac{\partial J_i^{i'}}{\partial Z^j}\frac{\partial}{\partial t} Z\left(t,S\right) \\ &= \left(\frac{\partial T^i}{\partial t} + \frac{\partial T^i}{\partial S^\alpha}\frac{\partial S^\alpha}{\partial t}\right) J_i^{i'} + T^i J_{ji}^{i'}\left(\frac{\partial Z^j}{\partial t} + \frac{\partial Z^j}{\partial S^\alpha}\frac{\partial S^\alpha}{\partial t}\right) \\ &= \left(U^i + \frac{\partial T^i}{\partial S^\alpha} J_t^\alpha\right) J_i^{i'} + T^i J_{ji}^{i'}\left(V^j + Z_\alpha^j J_t^\alpha\right) \\ &= U^i J_i^{i'} + \frac{\partial T^i}{\partial S^\alpha} J_t^\alpha J_i^{i'} + T^i J_{ji}^{i'} V^j + T^i J_{ji}^{i'} Z_\alpha^j J_t^\alpha,\end{aligned}$$

which is the desired result.

Ex. 318: This follows similarly to the above.

Ex. 319: Write

$$\begin{aligned}U^{\alpha'} &= \frac{\partial T^{\alpha'}\left(t,S'\right)}{\partial t} \\ &= \frac{\partial}{\partial t}\left(T^\alpha\left(t,S\right) J_\alpha^{\alpha'}\left(t,S\right)\right) \\ &= \frac{\partial}{\partial t} T^\alpha\left(t,S\right) J_\alpha^{\alpha'} + T^\alpha \frac{\partial}{\partial t} J_\alpha^{\alpha'}\left(t,S\right) \\ &= \left(\frac{\partial T^\alpha}{\partial t} + \frac{\partial T^\alpha}{\partial S^\beta}\frac{\partial S^\beta}{\partial t}\right) J_\alpha^{\alpha'} + T^\alpha\left(\frac{\partial J_\alpha^{\alpha'}}{\partial t} + \frac{\partial J_\alpha^{\alpha'}}{\partial S^\beta}\frac{\partial S^\beta}{\partial t}\right) \\ &= U^\alpha J_\alpha^{\alpha'} + \frac{\partial T^\alpha}{\partial S^\beta} J_t^\beta J_\alpha^{\alpha'} + T^\alpha J_{\alpha t}^{\alpha'} + T^\alpha J_{\beta\alpha}^{\alpha'} J_t^\beta.\end{aligned}$$

Ex. 320: The covariant case is analogous to the above.

Ex. 321: Note

$$
\begin{aligned}
V^\alpha \nabla_\alpha T^{i'} &= V^\alpha \nabla_\alpha T^i J^{i'}_i \\
&= V^\alpha \left(\frac{\partial T^i}{\partial S^\alpha}(t,S) + \Gamma^i_{jk} T^j Z^j_\alpha \right) J^{i'}_i \\
&= V^{j'} Z^\alpha_{j'} \left(\frac{\partial T^i}{\partial S^\alpha}(t,S) + \Gamma^i_{jk} T^k Z^j_\alpha \right) J^{i'}_i \\
&= \left(V^j J^{j'}_j + Z^j_\beta J^{j'}_j J^\beta_t \right) \left(\frac{\partial T^i}{\partial S^\alpha}(t,S) + \Gamma^i_{jk} T^k Z^j_\alpha \right) J^{i'}_i \\
&= \left(V^j J^{j'}_j \frac{\partial T^i}{\partial S^\alpha} + V^j J^{j'}_j \Gamma^i_{jk} T^k Z^j_\alpha + Z^j_\beta J^{j'}_j J^\beta_t \frac{\partial T^i}{\partial S^\alpha} + Z^j_\beta J^{j'}_j J^\beta_t \Gamma^i_{jk} T^k Z^j_\alpha \right) J^{i'}_i \\
&= V^j J^{j'}_j \frac{\partial T^i}{\partial S^\alpha} J^{i'}_i + V^j J^{j'}_j \Gamma^i_{jk} T^k Z^j_\alpha J^{i'}_i + Z^j_\beta J^{j'}_j J^\beta_t \frac{\partial T^i}{\partial S^\alpha} J^{i'}_i + Z^j_\beta J^{j'}_j J^\beta_t \Gamma^i_{jk} T^k Z^j_\alpha J^{i'}_i,
\end{aligned}
$$

so given our work above,

$$
\begin{aligned}
\frac{\partial T^{i'}}{\partial t} - V^\alpha \nabla_\alpha T^{i'} &= \frac{\partial T^i}{\partial t} J^{i'}_i + \frac{\partial T^i}{\partial S^\alpha} J^\alpha_t J^{i'}_i + T^i J^{i'}_{ji} V^j + T^i J^{i'}_{ji} Z^j_\alpha J^\alpha_t \\
&\quad - \left(V^j J^{j'}_j \frac{\partial T^i}{\partial S^\alpha} J^{i'}_i + V^j J^{j'}_j \Gamma^i_{jk} T^k Z^j_\alpha J^{i'}_i + Z^j_\beta J^{j'}_j J^\beta_t \frac{\partial T^i}{\partial S^\alpha} J^{i'}_i + Z^j_\beta J^{j'}_j J^\beta_t \Gamma^i_{jk} T^k Z^j_\alpha J^{i'}_i \right.
\end{aligned}
$$

[not finished]

Chapter 13

Chapter 16

Ex. 325: Assume the sum, product rules hold, in addition to commutativity with contraction and the metrinilic property with respect to the ambient basis. Then, compute

$$\begin{aligned} \dot{\nabla}\mathbf{T} &= \dot{\nabla}\left(T^i\mathbf{Z}_i\right) \\ &= \dot{\nabla}T^i\mathbf{Z}_i + T^i\dot{\nabla}\mathbf{Z}_i, \end{aligned}$$

by commutativity with contraction and the product rule,

$$= T^i\mathbf{Z}_i,$$

since the second term would be zero by the metrinilic property.

Ex. 326: Compute

$$\begin{aligned} \frac{\partial \mathbf{Z}_i}{\partial t} &= \frac{\partial \mathbf{Z}_i\left(Z\left(t\right)\right)}{\partial t} \\ &= \frac{\partial \mathbf{Z}_i}{\partial Z^j}\frac{\partial Z^j}{\partial t} \\ &= \Gamma^k_{ij}\mathbf{Z}_k V^j \\ &= V^j\Gamma^k_{ij}\mathbf{Z}_k, \end{aligned}$$

as desired.

Ex. 327: Write

$$\mathbf{T} = T_i\mathbf{Z}^i,$$

so

$$\dot{\nabla}T_i\mathbf{Z}^i = \dot{\nabla}\mathbf{T} = \frac{\partial T_i}{\partial t}\mathbf{Z}^i + T_i\frac{\partial \mathbf{Z}^i}{\partial t} - V^\gamma\nabla_\gamma T_i\mathbf{Z}^i,$$

but

$$\begin{aligned}
\frac{\partial \mathbf{Z}^i}{\partial t} &= \frac{\partial \mathbf{Z}^i}{\partial Z^j}\frac{\partial Z^j}{\partial t} \\
&= -\Gamma^i_{jk}\mathbf{Z}^k V^j \\
&= -\Gamma^i_{jk} V^j \mathbf{Z}^k,
\end{aligned}$$

so we have

$$\dot{\nabla} T_i = \frac{\partial T_i}{\partial t} - V^\gamma \nabla_\gamma T_i - V^j \Gamma^k_{ij} T_k,$$

after index renaming in the second term of the above expression.

Ex. 328: We know $\dot{\nabla}\mathbf{T}$ is invariant, and since $\dot{\nabla}\mathbf{T} = \dot{\nabla} T^i \mathbf{Z}_i$ and $\mathbf{Z}_i$ is a tensor, by the quotient law, $\dot{\nabla} T^i$ must be a tensor. An argument using changes in coordinates would follow similarly to the covariant derivative computations.

Ex. 329: This also follows from the fact that $\dot{\nabla}\mathbf{T} = \dot{\nabla} T_i \mathbf{Z}^i$ and by the quotient law. An argument using changes in coordinates would also follow similarly.

Ex. 330: Put

$$\mathbf{S}_j = T^i_j \mathbf{Z}_i.$$

then, clearly,

$$\dot{\nabla}\mathbf{S}_j = \dot{\nabla} T^i_j \mathbf{Z}_i.$$

Dot both sides with $\mathbf{Z}^k$:

$$\begin{aligned}
\dot{\nabla}\mathbf{S}_j \cdot \mathbf{Z}^k &= \dot{\nabla} T^i_j \mathbf{Z}_i \cdot \mathbf{Z}^k \\
\dot{\nabla}\mathbf{S}_j \cdot \mathbf{Z}^k &= \dot{\nabla} T^k_j.
\end{aligned}$$

Since the LHS is clearly a tensor, the RHS is as well.

Ex. 331: We may use induction with a process similar to 330 to extend this result to arbitrary indices.

Ex. 332: Assume S^i, T^j_i are arbitrary tensors. Put

$$U^j = S^i T^j_i.$$

Now, compute

$$\begin{aligned}
\dot{\nabla}\left(S^i T_i^j\right) &= \frac{\partial\left(S^i T_i^j\right)}{\partial t} - V^\gamma \nabla_\gamma \left(S^i T_i^j\right) + V^n \Gamma_{nk}^j S^i T_i^j \\
&= \frac{\partial U^j}{\partial t} - V^\gamma \nabla_\gamma U^j + V^n \Gamma_{nk}^j S^i T_i^j,
\end{aligned}$$

since both the partial derivatives and the covariant surface derivatives commute with contraction.

Ex. 333: This follows from the sum and product rules for the partial derivatives and the covariant surface derivative.

Ex. 334: Note

$$\nabla_\gamma T_j^i = Z_\gamma^k \nabla_k T_j^i,$$

and compute

$$\begin{aligned}
\dot{\nabla} T_j^i (t,S) &= \frac{\partial T_j^i (t, Z(t,S))}{\partial t} - V^\gamma \nabla_\gamma T_j^i (t, Z(t,S)) + V^k \Gamma_{km}^i T_j^m (t, Z(t,S)) - V^k \Gamma_{kj}^m T_m^i (t, Z(t,S)) \\
&= \frac{\partial T_j^i}{\partial t} + \frac{\partial T_j^i}{\partial Z^k}\frac{\partial Z^k}{\partial t} - V^\gamma Z_\gamma^k \nabla_k T_j^i + V^k \Gamma_{km}^i T_j^m - V^k \Gamma_{kj}^m T_m^i \\
&= \frac{\partial T_j^i}{\partial t} + \frac{\partial T_j^i}{\partial Z^k} V^k - V^\gamma Z_\gamma^k \nabla_k T_j^i + V^k \Gamma_{km}^i T_j^m - V^k \Gamma_{kj}^m T_m^i \\
&= \frac{\partial T_j^i}{\partial t} + \left(\frac{\partial T_j^i}{\partial Z^k} + \Gamma_{km}^i T_j^m - \Gamma_{kj}^m T_m^i\right) V^k - V^\gamma Z_\gamma^k \nabla_k T_j^i \\
&= \frac{\partial T_j^i}{\partial t} + \nabla_k T_j^i V^k - V^\gamma Z_\gamma^k \nabla_k T_j^i \\
&= \frac{\partial T_j^i}{\partial t} + \left(V^k - V^\gamma Z_\gamma^k\right) \nabla_k T_j^i \\
&= \frac{\partial T_j^i}{\partial t} + C N^k \nabla_k T_j^i,
\end{aligned}$$

where the last step follows from the observation that $V^k - V^\gamma Z_\gamma^k$ is the normal component.

Ex. 335: Note that $\frac{\partial \mathbf{Z}_i}{\partial t} = 0$, and the second term of the above is also zero by the metrinilic property for covariant derivatives. Hence, $\dot{\nabla}\mathbf{Z}_i = 0$. the other results follow similarly.

Ex. 336: Compute

$$\begin{aligned}
\frac{\partial \mathbf{S}_\beta \left(Z\left(t,S\right)\right)}{\partial t} &= \frac{\partial \mathbf{S}_\beta}{\partial Z^i}\frac{\partial Z^i}{\partial t} \\
&= \frac{\partial \mathbf{S}_\beta}{\partial Z^i} V^i \\
&= \frac{\partial \mathbf{S}_\beta}{\partial S^\alpha}\frac{\partial S^\alpha}{\partial Z^i} V^i \\
&= \frac{\partial^2 \mathbf{R}}{\partial S^\alpha \partial S^\beta} Z_i^\alpha V^i \\
&= \frac{\partial \mathbf{S}_\alpha}{\partial S^\beta} Z_i^\alpha V^i \\
&= \text{[not sure]}
\end{aligned}$$

Ex. 337: Simply decompose $\mathbf{V}$ into its tangential and normal coordinates, to obtain the substitution used for the RHS.

Ex. 338: Compute

$$\begin{aligned}
\nabla_\beta \left(V^\alpha \mathbf{S}_\alpha + C\mathbf{N}\right) &= \nabla_\beta V^\alpha \mathbf{S}_\alpha + V^\alpha \nabla_\beta \mathbf{S}_\alpha + \nabla_\beta C\mathbf{N} + C\nabla_\beta \mathbf{N} \\
&= \nabla_\beta V^\alpha \mathbf{S}_\alpha + V^\alpha \mathbf{N} B_{\beta\alpha} + \mathbf{N}\nabla_\beta C + C\left(-B_\beta^\alpha \mathbf{S}_\alpha\right) \\
&= \nabla_\beta V^\alpha \mathbf{S}_\alpha + V^\alpha \mathbf{N} B_{\beta\alpha} + \mathbf{N}\nabla_\beta C - C B_\beta^\alpha \mathbf{S}_\alpha.
\end{aligned}$$

Ex. 339: Use

$$\begin{aligned}
\dot{\nabla}\mathbf{T} &= \frac{\partial T^\alpha}{\partial t}\mathbf{S}_\alpha + T^\beta \frac{\partial \mathbf{S}_\beta}{\partial t} - V^\beta \nabla_\beta T^\alpha \mathbf{S}_\alpha - V^\beta T^\alpha \mathbf{N} B_{\beta\alpha} \\
&= \frac{\partial T^\alpha}{\partial t}\mathbf{S}_\alpha + T^\beta \left(\nabla_\beta V^\alpha \mathbf{S}_\alpha + V^\alpha \mathbf{N} B_{\beta\alpha} + \mathbf{N}\nabla_\beta C - C B_\beta^\alpha \mathbf{S}_\alpha\right) - V^\beta \nabla_\beta T^\alpha \mathbf{S}_\alpha - V^\beta T^\alpha \mathbf{N} B_{\beta\alpha} \\
&= \frac{\partial T^\alpha}{\partial t}\mathbf{S}_\alpha + \nabla_\beta V^\alpha T^\beta \mathbf{S}_\alpha + V^\alpha T^\beta \mathbf{N} B_{\beta\alpha} + T^\beta \mathbf{N}\nabla_\beta C - C B_\beta^\alpha T^\beta \mathbf{S}_\alpha - V^\beta \nabla_\beta T^\alpha \mathbf{S}_\alpha - V^\beta T^\alpha \mathbf{N} B_{\beta\alpha} \\
&= \frac{\partial T^\alpha}{\partial t}\mathbf{S}_\alpha + T^\beta \nabla_\beta V^\alpha \mathbf{S}_\alpha + T^\beta \nabla_\beta C\mathbf{N} - T^\beta C B_\beta^\alpha \mathbf{S}_\alpha - V^\beta \nabla_\beta T^\alpha \mathbf{S}_\alpha \\
&= \frac{\partial T^\alpha}{\partial t}\mathbf{S}_\alpha + T^\beta \nabla_\beta V^\alpha \mathbf{S}_\alpha + T^\alpha \nabla_\alpha C\mathbf{N} - T^\beta C B_\beta^\alpha \mathbf{S}_\alpha - V^\beta \nabla_\beta T^\alpha \mathbf{S}_\alpha.
\end{aligned}$$

Ex. 340: Simply decompose $\mathbf{T}$ with respect to the contravariant basis $\mathbf{S}^\alpha$, and use the similar decomposition

$$\mathbf{V} = V^\alpha \mathbf{S}_\alpha + C\mathbf{N}.$$

Ex. 341: Note

$$
\begin{aligned}
0 &= \dot{\nabla} S_{\alpha\beta} \\
&= \frac{\partial S_{\alpha\beta}}{\partial t} - V^{\omega}\nabla_{\omega}S_{\alpha\beta} - \left(\nabla_{\alpha}V^{\omega} - CB^{\omega}_{\alpha}\right)S_{\omega\beta} - \left(\nabla_{\beta}V^{\omega} - CB^{\omega}_{\beta}\right)S_{\alpha\omega} \\
&= \frac{\partial S_{\alpha\beta}}{\partial t} - \nabla_{\alpha}V^{\omega}S_{\omega\beta} + CB^{\omega}_{\alpha}S_{\omega\beta} - \nabla_{\beta}V^{\omega}S_{\alpha\omega} + CB^{\omega}_{\beta}S_{\alpha\omega} \\
&= \frac{\partial S_{\alpha\beta}}{\partial t} - \nabla_{\alpha}V_{\beta} - \nabla_{\beta}V_{\alpha} + CB_{\alpha\beta} + CB_{\beta\alpha} \\
&= \frac{\partial S_{\alpha\beta}}{\partial t} - \nabla_{\alpha}V_{\beta} - \nabla_{\beta}V_{\alpha} + 2CB_{\alpha\beta},
\end{aligned}
$$

so

$$\frac{\partial S_{\alpha\beta}}{\partial t} = \nabla_{\alpha}V_{\beta} + \nabla_{\beta}V_{\alpha} - 2CB_{\alpha\beta}.$$

The contravariant case follows similarly.

Ex. 342: First, examine

$$\frac{\partial S}{\partial S^{\alpha\beta}} = SS^{\alpha\beta},$$

using the properties of the cofactor matrix. Then, compute

$$
\begin{aligned}
\frac{\partial S}{\partial t} &= \frac{\partial S}{\partial S^{\alpha\beta}}\frac{\partial S^{\alpha\beta}}{\partial t} \\
&= SS^{\alpha\beta}\left(\nabla_{\alpha}V_{\beta} + \nabla_{\beta}V_{\alpha} - 2CB_{\alpha\beta}\right) \\
&= S\left(\nabla_{\alpha}V^{\alpha} + \nabla_{\beta}V^{\beta} - 2CB^{\alpha}_{\alpha}\right) \\
&= 2S\left(\nabla_{\alpha}V^{\alpha} - CB^{\alpha}_{\alpha}\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{\partial \sqrt{S}}{\partial t} &= \frac{1}{2\sqrt{S}}\frac{\partial S}{\partial t} \\
&= \sqrt{S}\left(\nabla_{\alpha}V^{\alpha} - CB^{\alpha}_{\alpha}\right),
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial\left(\sqrt{S}\right)^{-1}}{\partial t} &= -\frac{1}{\left(\sqrt{S}\right)^{2}}\frac{\partial\sqrt{S}}{\partial t} \\
&= -\frac{1}{S}\sqrt{S}\left(\nabla_\alpha V^\alpha - CB^\alpha_\alpha\right) \\
&= -\frac{1}{\sqrt{S}}\left(\nabla_\alpha V^\alpha - CB^\alpha_\alpha\right).
\end{aligned}$$

Ex. 343: Note

$$\varepsilon_{\alpha\beta} = \sqrt{S}e_{\alpha\beta},$$

so

$$\begin{aligned}
\frac{\partial\varepsilon_{\alpha\beta}}{\partial t} &= \frac{\partial\sqrt{S}}{\partial t}e_{\alpha\beta} + \sqrt{S}\frac{\partial e_{\alpha\beta}}{\partial t} \\
&= \frac{\partial\sqrt{S}}{\partial t}e_{\alpha\beta},
\end{aligned}$$

since $e_{\alpha\beta}$ does not depend on t.

$$\begin{aligned}
&= \sqrt{S}\left(\nabla_\gamma V^\gamma - CB^\gamma_\gamma\right)e_{\alpha\beta} \\
&= \varepsilon_{\alpha\beta}\left(\nabla_\gamma V^\gamma - CB^\gamma_\gamma\right).
\end{aligned}$$

Ex. 344: Note

$$\varepsilon^{\alpha\beta} = \left(\sqrt{S}\right)^{-1}e^{\alpha\beta},$$

so

$$\begin{aligned}
\frac{\partial\varepsilon^{\alpha\beta}}{\partial t} &= \frac{\partial\left(\sqrt{S}\right)^{-1}}{\partial t}e^{\alpha\beta} + \sqrt{S}\frac{\partial e^{\alpha\beta}}{\partial t} \\
&= \frac{\partial\left(\sqrt{S}\right)^{-1}}{\partial t}e^{\alpha\beta} \\
&= -\frac{1}{\sqrt{S}}\left(\nabla_\gamma V^\gamma - CB^\gamma_\gamma\right)e^{\alpha\beta} \\
&= -\varepsilon^{\alpha\beta}\left(\nabla_\gamma V^\gamma - CB^\gamma_\gamma\right).
\end{aligned}$$

Ex. 345: Simply write

$$\begin{aligned}\dot{\nabla}\varepsilon^{\alpha\beta} &= \dot{\nabla}\left(\varepsilon_{\gamma\delta}S^{\gamma\alpha}S^{\delta\beta}\right)\\ &= \dot{\nabla}\varepsilon_{\gamma\delta}S^{\gamma\alpha}S^{\delta\beta},\end{aligned}$$

by the metrinilic property,

$$= 0,$$

by the above.

Ex. 346: Use Gauss' Theorema Egregium:

$$\begin{aligned}K &= |B^{\cdot}_{\cdot}|\\ &= \frac{1}{2}\varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}B^{\beta}_{\sigma},\end{aligned}$$

so

$$\begin{aligned}2\dot{\nabla}K &= \dot{\nabla}\varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}B^{\beta}_{\sigma} + \varepsilon^{\rho\sigma}\dot{\nabla}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}B^{\beta}_{\sigma} + \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}\dot{\nabla}B^{\alpha}_{\rho}B^{\beta}_{\sigma} + \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}\dot{\nabla}B^{\beta}_{\sigma}\\ &= \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}\dot{\nabla}B^{\alpha}_{\rho}B^{\beta}_{\sigma} + \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}\dot{\nabla}B^{\beta}_{\sigma},\end{aligned}$$

since the first two terms vanish,

$$\begin{aligned}&= \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}\left(\nabla^{\alpha}\nabla_{\rho}C + CB^{\alpha}_{\gamma}B^{\gamma}_{\rho}\right)B^{\beta}_{\sigma} + \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}\left(\nabla^{\beta}\nabla_{\sigma}C + CB^{\beta}_{\gamma}B^{\gamma}_{\sigma}\right)\\ &= \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\beta}_{\sigma}\nabla^{\alpha}\nabla_{\rho}C + C\varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\beta}_{\sigma}B^{\alpha}_{\gamma}B^{\gamma}_{\rho} + \varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}\nabla^{\beta}\nabla_{\sigma}C + C\varepsilon^{\rho\sigma}\varepsilon_{\alpha\beta}B^{\alpha}_{\rho}B^{\beta}_{\gamma}B^{\gamma}_{\sigma}\\ &= \delta^{\rho\sigma}_{\alpha\beta}\left(B^{\beta}_{\sigma}\nabla^{\alpha}\nabla_{\rho}C + B^{\alpha}_{\rho}\nabla^{\beta}\nabla_{\sigma}C + B^{\beta}_{\sigma}B^{\alpha}_{\gamma}B^{\gamma}_{\rho}C + B^{\alpha}_{\rho}B^{\beta}_{\gamma}B^{\gamma}_{\sigma}C\right)\\ &= \left(\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha}\right)B^{\beta}_{\sigma}\nabla^{\alpha}\nabla_{\rho}C + B^{\alpha}_{\rho}\nabla^{\beta}\nabla_{\sigma}C + B^{\beta}_{\sigma}B^{\alpha}_{\gamma}B^{\gamma}_{\rho}C + B^{\alpha}_{\rho}B^{\beta}_{\gamma}B^{\gamma}_{\sigma}C\\ &= \delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}B^{\beta}_{\sigma}\nabla^{\alpha}\nabla_{\rho}C + \delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}B^{\alpha}_{\rho}\nabla^{\beta}\nabla_{\sigma}C + \delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}B^{\beta}_{\sigma}B^{\alpha}_{\gamma}B^{\gamma}_{\rho}C + \delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}B^{\alpha}_{\rho}B^{\beta}_{\gamma}B^{\gamma}_{\sigma}C\\ &\quad - \left(\delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha}B^{\beta}_{\sigma}\nabla^{\alpha}\nabla_{\rho}C + \delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha}B^{\alpha}_{\rho}\nabla^{\beta}\nabla_{\sigma}C + \delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha}B^{\beta}_{\sigma}B^{\alpha}_{\gamma}B^{\gamma}_{\rho}C + \delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha}B^{\alpha}_{\rho}B^{\beta}_{\gamma}B^{\gamma}_{\sigma}C\right)\\ &= B^{\sigma}_{\sigma}\nabla^{\alpha}\nabla_{\alpha}C + B^{\alpha}_{\alpha}\nabla^{\sigma}\nabla_{\sigma}C + B^{\sigma}_{\sigma}B^{\alpha}_{\gamma}B^{\gamma}_{\alpha}C + B^{\alpha}_{\alpha}B^{\sigma}_{\gamma}B^{\gamma}_{\sigma}C\\ &\quad - \left(B^{\rho}_{\sigma}\nabla^{\sigma}\nabla_{\rho}C + B^{\sigma}_{\rho}\nabla^{\rho}\nabla_{\sigma}C + B^{\rho}_{\sigma}B^{\sigma}_{\gamma}B^{\gamma}_{\rho}C + B^{\sigma}_{\rho}B^{\rho}_{\gamma}B^{\gamma}_{\sigma}C\right)\\ &= 2B^{\alpha}_{\alpha}\nabla^{\beta}\nabla_{\beta}C - 2B^{\alpha}_{\beta}\nabla^{\beta}\nabla_{\alpha}C + 2B^{\alpha}_{\alpha}B^{\beta}_{\gamma}B^{\gamma}_{\beta}C - 2B^{\alpha}_{\beta}B^{\beta}_{\gamma}B^{\gamma}_{\alpha}C\\ &= 2\left(B^{\alpha}_{\alpha}\nabla^{\beta}\nabla_{\beta}C - B^{\alpha}_{\beta}\nabla^{\beta}\nabla_{\alpha}C + \left(B^{\alpha}_{\alpha}B^{\beta}_{\gamma}B^{\gamma}_{\beta} - B^{\alpha}_{\beta}B^{\beta}_{\gamma}B^{\gamma}_{\alpha}\right)C\right)\\ &= 2\left(B^{\alpha}_{\alpha}\nabla^{\beta}\nabla_{\beta}C - B^{\alpha}_{\beta}\nabla^{\beta}\nabla_{\alpha}C + B^{\alpha}_{\alpha}KC\right),\end{aligned}$$

which gives us the desired result [Note: I believe the last equality follows from Gauss' Theorema Egregium].

Chapter 14

Chapter 17

Ex. 347: Note that u is an invariant with respect to ambient indices, and thus $\nabla_i u = \frac{\partial u}{\partial Z^i}$. Write

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_i u &= \frac{\partial}{\partial t}\frac{\partial u}{\partial Z^i} \\
&= \frac{\partial}{\partial Z^i}\frac{\partial u}{\partial t} \\
&= \nabla_i \frac{\partial}{\partial t} u,
\end{aligned}$$

under smoothness assumptions (hence the partials commute), and since $\frac{\partial u}{\partial t}$ is an invariant.

Ex. 348: First, compute, given polar coordinates

$$\begin{aligned}
\nabla_i u &= \frac{\partial u}{\partial Z^i} \\
&= \begin{bmatrix} \frac{\partial}{\partial r}\left(\frac{J_0(\rho r)}{\sqrt{\pi}J_1(\rho)}\right) \\ \frac{\partial}{\partial \theta}\left(\frac{J_0(\rho r)}{\sqrt{\pi}J_1(\rho)}\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{-\rho J_1(\rho r)}{\sqrt{\pi}J_1(\rho)} \\ 0 \end{bmatrix}
\end{aligned}$$

Now, since for polar coordinates,

$$Z^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix},$$

so

$$\begin{aligned}
\nabla^i u &= Z^{ij}\nabla_j u \\
&= \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} \begin{bmatrix} \frac{-\rho J_1(\rho r)}{\sqrt{\pi} J_1(\rho)} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-\rho J_1(\rho r)}{\sqrt{\pi} J_1(\rho)} \\ 0 \end{bmatrix}.
\end{aligned}$$

thus,

$$\nabla_i u \nabla^i u = \frac{-\rho^2 J_1\left(\rho r\right)^2}{\pi J_1\left(\rho\right)^2}.$$

Now, at $t = 0$, our surface yields $r = 1$, so

$$\nabla_i u \nabla^i u = \frac{-\rho^2}{\pi}$$

Next, compute C for our surface evolution. Consider, in Cartesian coordinates,

$$\begin{aligned}
V^i &= \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} \\
&= \begin{bmatrix} a\cos\alpha \\ b\sin\alpha \end{bmatrix}
\end{aligned}$$

With respect to the Cartesian basis,

$$\mathbf{S}_\alpha = -\left(1+at\right)\sin\alpha\mathbf{i} + \left(1+bt\right)\cos\alpha\mathbf{j}$$

$$N_i = \frac{1}{\sqrt{\left(1+at\right)^2\cos^2\alpha + \left(1+bt\right)^2\sin^2\alpha}} \begin{bmatrix} (1+bt)\cos\alpha \\ (1+at)\sin\alpha \end{bmatrix},$$

at $t = 0$,

$$N_i = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$\begin{aligned}
C &= V^i N_i \\
&= a\cos^2\alpha + b\sin^2\alpha;
\end{aligned}$$

thus, by the Hadamard formula,

$$\begin{aligned}
\lambda_1 &= \int_0^{2\pi} \left(a\cos^2\alpha + b\sin^2\alpha\right)\left(\frac{-\rho^2}{\pi}\right) d\alpha \\
&= \frac{-\rho^2}{\pi}\int_0^{2\pi} a\cos^2\alpha + b\sin^2\alpha d\alpha \\
&= \frac{-\rho^2}{\pi}\int_0^{2\pi} a\left(\cos^2\alpha\right) + b\left(1-\cos^2\alpha\right) d\alpha \\
&= \frac{-\rho^2}{\pi}\int_0^{2\pi} a\cos^2\alpha + b\cos^2\alpha d\alpha - 2b\rho^2 \\
&= \frac{-\rho^2}{\pi}\left(a+b\right)\int_0^{2\pi} \cos^2\alpha d\alpha - 2b\rho^2 \\
&= \frac{-\rho^2}{\pi}\left(a+b\right)\left[\frac{\alpha}{2} + \frac{1}{4}\sin 2\alpha\right]_0^{2\pi} - 2b\rho^2 \\
&= \frac{-\rho^2}{\pi}\left(a+b\right)\pi - 2b\rho^2 \\
&= -\rho^2\left(a+b\right) - 2b\rho^2 \\
&= -\rho^2\left(a+b\right).
\end{aligned}$$

Since $\lambda = \rho^2$, we have the desired result.

Ex. 349: [Not sure]

Ex. 350: Want to show:

$$\lambda_1 = \int_S \left(u_1 N_i \nabla^i u - N_i \nabla^i u_1\right) dS.$$

Dirichlet:

$$\lambda_1 = -\int_S C\nabla_i u \nabla^i u dS$$

Neumann:

$$\lambda_1 =$$

Ex. 354: We have

$$\begin{aligned}
\rho\left(\dot{\nabla} C + 2V^\alpha \nabla_\alpha C + B_{\alpha\beta} V^\alpha V^\beta\right) &= \sigma B^\alpha_\alpha \\
\dot{\nabla} V^\alpha + V^\beta \nabla_\beta V^\alpha - C\nabla^\alpha C - CV^\beta B^\alpha_\beta &= 0.
\end{aligned}$$

Write

$$V^{\alpha} = V^i Z_i^{\alpha}$$

and

$$C = V^i N_i,$$

Begin with the second, and contract with $\mathbf{S}_{\alpha}$:

$$\begin{aligned}
\dot{\nabla} V^{\alpha}\mathbf{S}_{\alpha} + V^{\beta}\nabla_{\beta}V^{\alpha}\mathbf{S}_{\alpha} - C\nabla^{\alpha}C\mathbf{S}_{\alpha} - CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= 0 \\
\dot{\nabla}\left(V^{\alpha}\mathbf{S}_{\alpha}\right) - V^{\alpha}\dot{\nabla}\mathbf{S}_{\alpha} + V^{\beta}\nabla_{\beta}\left(V^{\alpha}\mathbf{S}_{\alpha}\right) - V^{\beta}V^{\alpha}\nabla_{\beta}\mathbf{S}_{\alpha} - C\nabla^{\alpha}C\mathbf{S}_{\alpha} - CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= 0 \\
\dot{\nabla}\left(V^{\alpha}\mathbf{S}_{\alpha}\right) - V^{\alpha}\mathbf{N}\nabla_{\alpha}C + V^{\beta}\nabla_{\beta}\left(V^{\alpha}\mathbf{S}_{\alpha}\right) - V^{\beta}V^{\alpha}B_{\alpha\beta}\mathbf{N} - C\nabla^{\alpha}C\mathbf{S}_{\alpha} - CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= 0.
\end{aligned}$$

Then, manipulate the first:

$$\dot{\nabla}C + 2V^{\alpha}\nabla_{\alpha}C + B_{\alpha\beta}V^{\alpha}V^{\beta} = \frac{\sigma}{\rho}B_{\alpha}^{\alpha},$$

and multiply by $\mathbf{N}$:

$$\begin{aligned}
\dot{\nabla}C\mathbf{N} + 2V^{\alpha}\nabla_{\alpha}C\mathbf{N} + B_{\alpha\beta}V^{\alpha}V^{\beta}\mathbf{N} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N} \\
\dot{\nabla}\left(C\mathbf{N}\right) - C\dot{\nabla}\mathbf{N} + 2V^{\alpha}\nabla_{\alpha}C\mathbf{N} + B_{\alpha\beta}V^{\alpha}V^{\beta}\mathbf{N} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N} \\
\dot{\nabla}\left(C\mathbf{N}\right) - C\left(-\mathbf{S}_{\alpha}\nabla^{\alpha}C\right) + 2V^{\alpha}\nabla_{\alpha}C\mathbf{N} + B_{\alpha\beta}V^{\alpha}V^{\beta}\mathbf{N} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N} \\
\dot{\nabla}\left(C\mathbf{N}\right) + C\mathbf{S}_{\alpha}\nabla^{\alpha}C + 2V^{\alpha}\nabla_{\alpha}C\mathbf{N} + B_{\alpha\beta}V^{\alpha}V^{\beta}\mathbf{N} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N} \\
\dot{\nabla}\left(C\mathbf{N}\right) + 2V^{\alpha}\nabla_{\alpha}C\mathbf{N} + V^{\alpha}V^{\beta}B_{\alpha\beta}\mathbf{N} + C\nabla^{\alpha}C\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}.
\end{aligned}$$

Now, add the results of both manipulations to each other:

$$\begin{aligned}
\dot{\nabla}\left(V^{\alpha}\mathbf{S}_{\alpha}\right)+\dot{\nabla}\left(C\mathbf{N}\right)+V^{\alpha}\nabla_{\alpha}C\mathbf{N}+V^{\beta}\nabla_{\beta}\left(V^{\alpha}\mathbf{S}_{\alpha}\right)-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}C\mathbf{N}+V^{\beta}\nabla_{\beta}\left(V^{\alpha}\mathbf{S}_{\alpha}\right)-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\left(C\mathbf{N}\right)-V^{\alpha}C\nabla_{\alpha}\mathbf{N}+V^{\beta}\nabla_{\beta}\left(V^{\alpha}\mathbf{S}_{\alpha}\right)-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\mathbf{V}-V^{\alpha}C\nabla_{\alpha}\mathbf{N}-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\mathbf{V}-V^{\alpha}C\nabla_{\alpha}\left(N^{i}\mathbf{Z}_{i}\right)-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\mathbf{V}-V^{\alpha}C\nabla_{\alpha}N^{i}\mathbf{Z}_{i}-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\quad\text{[note the metrinilic property]}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\mathbf{V}+V^{\alpha}CZ_{\beta}^{i}B_{\alpha}^{\beta}\mathbf{Z}_{i}-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\mathbf{V}+V^{\alpha}CB_{\alpha}^{\beta}Z_{\beta}^{i}\mathbf{Z}_{i}-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\mathbf{V}+CV^{\alpha}B_{\alpha}^{\beta}\mathbf{S}_{\beta}-CV^{\beta}B_{\beta}^{\alpha}\mathbf{S}_{\alpha} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}\\
\dot{\nabla}\mathbf{V}+V^{\alpha}\nabla_{\alpha}\mathbf{V} &= \frac{\sigma}{\rho}B_{\alpha}^{\alpha}\mathbf{N}
\end{aligned}$$