Tensor Calculus Equations and Syntax

Coordinate Transformations (6)

$Z^i = Z^i(Z')$ — i is a free or live index.

$Z^{i'} = Z^{i'}(Z)$ — i' is a free or live index.

Summation Convention (9)

$\sum_{i=1}^{n} \frac{dZ^i}{ds}\frac{\partial \vec{R}}{\partial Z^i} = \frac{dZ^i}{ds}\frac{\partial \vec{R}}{\partial Z^i}$ — i is a dummy index.

Summation is implied by the syntax.

Covariant Basis Vectors (9)

$$\vec{Z}_i = \frac{\partial \vec{R}}{\partial Z^i}$$

$$\vec{Z}_{i'} = \frac{\partial \vec{R}}{\partial Z^{i'}}$$

Linear Combinations (9)

$$\vec{V} = V^i \vec{Z}_i$$

$$\vec{V} = V^{i'} \vec{Z}_{i'}$$

Jacobian and Inverse (11)

$$J_i^{i'} = \frac{\partial Z^{i'}}{\partial Z^i}$$

$$J_{i'}^{i} = \frac{\partial Z^{i}}{\partial Z^{i'}}$$

Covariant Transformation (11)

$$A_i = J_i^{i'} A_{i'}$$

$$A_{i'} = J_{i'}^{i} A_i$$

Contravariant Transformation (11)

$$A^i = J_{i'}^{i} A^{i'}$$

$$A^{i'} = J_i^{i'} A^i$$

Kronecker Delta Definition (13)

$$\delta_j^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Jacobian Inversion (13)

$$\delta_j^i = J_{i'}^i J_j^{i'} = J_j^{i'} J_{i'}^i$$

$$\delta_{j'}^{i'} = J_i^{i'} J_{j'}^i = J_{j'}^i J_i^{i'}$$

Kronecker Delta Index Absorption (13)

$$\delta_j^i A_i = A_j$$

$$\delta_j^i A^j = A^i$$

Covariant Metric Tensor (15)

$$Z_{ij} = Z_{ji} = \vec{Z}_i \cdot \vec{Z}_j$$

Metric Equation (15)

$$ds^2 = Z_{ij}\, dZ^i dZ^j$$

Arc Length (15)

$$L = \int_a^b \sqrt{Z_{ij} \frac{dZ^i}{dt} \frac{dZ^j}{dt}}\, dt$$

Contravariant Metric Tensor Definition (17)

$$Z^{ij} Z_{jk} = \delta_k^i$$

Contravariant Basis Vector Definition (17)

$$\vec{Z}^i = Z^{ij} \vec{Z}_j$$

Contravariant Basis Properties (17)

$$\vec{Z}^i \cdot \vec{Z}_k = \delta_k^i$$

$$\vec{Z}^i \cdot \vec{Z}^j = Z^{ij} = Z^{ji}$$

Linear Combinations (17)

$$\vec{V} = V_i \vec{Z}^i = V^i \vec{Z}_i$$

$$\vec{V} = V_{i'} \vec{Z}^{i'} = V^{i'} \vec{Z}_{i'}$$

Tensor Form (19)

$$T_{j_1 j_2 \dots j_b}^{i_1 i_2 \dots i_a}$$

Tensor Definition (19)

$$P(Z) = \left(T_{j_1 j_2 \dots j_b}^{i_1 i_2 \dots i_a}\right) \left(A_{i_1}^{(1)} A_{i_2}^{(2)} \cdots A_{i_a}^{(a)}\right) \left(B_{(1)}^{j_1} B_{(2)}^{j_2} \cdots B_{(b)}^{j_b}\right)$$

Tensor Transformation Rule (19)

$$\left(T^{i_1 i_2 \ldots i_a}_{j_1 j_2 \ldots j_b}\right) = \left(T^{i'_1 i'_2 \ldots i'_a}_{j'_1 j'_2 \ldots j'_b}\right)\left(J^{i_1}_{i'_1} J^{i_2}_{i'_2} \cdots J^{i_a}_{i'_a}\right)\left(J^{j'_1}_{j_1} J^{j'_2}_{j_2} \cdots J^{j'_b}_{j_b}\right)$$

Tensor Equality (20)

$A^{ijk}_{rst} = B^{ijk}_{rst} = C^{ikj}_{tsr}$ Free index structure must be the same.

Tensor Addition (20)

$A^{ij}_{rs} + B^{ij}_{rs} = C^{ij}_{rs}$ If A and B are tensors, then C is a tensor.

$C^{ij}_{rs} = A^{ij}_{rs} + B^{ij}_{rs}$ Converse is not necessarily true.

Scalar Multiplication (20)

$5P^{i}_{jk} = Q^{i}_{jk}$ If P is a tensor, then Q is a tensor.

Linear Property (20)

$3A^{ij}_{rs} + 5B^{ij}_{rs} - 4C^{ij}_{rs} = D^{ij}_{rs}$ If A, B and C are tensors, then D is a tensor.

Outer or Tensor Product (20)

$P^{i}_{j} Q^{rs}_{k} = R^{irs}_{jk}$ If P and Q are tensors, then R is a tensor.

Decomposition of Tensors into Vector Products (20)

$W^{i}_{j} = A^{i}X_j + B^{i}Y_j + C^{i}Z_j$ n^{r-1} terms of r vectors each.

$$W^{ij}_{k} = D^iG^jX_k + E^iH^jX_k + F^iI^jX_k + J^iM^jY_k + K^iN^jY_k + L^iP^jY_k + Q^iT^jZ_k + R^iU^jZ_k + S^iV^jZ_k$$

Inner Product or Contraction (21)

$P^{i}_{j} Q^{js}_{k} = R^{is}_{k}$ If P and Q are tensors, then R is a tensor.

Tensor Contraction (21)

$Q^{ijs}_{jk} = R^{is}_{k}$ If Q is a tensor, then R is a tensor.

Quotient Theorem (21)

$R^{is}_{k} = P^{i}_{j} Q^{js}_{k}$ If P and R are tensors, then Q is a tensor, provided $P \neq 0$.

Cancellation of Common Factors (21)

$S_j^{ir} Q_s^k = T_j^{ir} Q_s^k$ Where $Q_s^k \neq 0$ (outer product).

$S_j^{ir} = T_j^{ir}$

$S_j^{ir} Q_s^j = T_j^{ir} Q_s^j$ Where Q_s^j is arbitrary (inner product).

$S_j^{ir} = T_j^{ir}$

Raising the Index (22)

$$\vec{Z}^i = Z^{ij} \vec{Z}_j$$

$$V^i = Z^{ij} V_j$$

Lowering the Index (22)

$$\vec{Z}_i = Z_{ij} \vec{Z}^j$$

$$V_i = Z_{ij} V^j$$

Raising and Lowering Indexes for Higher Rank Tensors (22)

$$Z^{ri} T_{ij} = T_{\cdot j}^{r} = T_j^r$$

$$Z^{sj} T_j^r = T^{rs}$$

$$Z^{sj} T_{ij} = T_i^{\cdot s}$$

$$Z^{ri} T_i^{\cdot s} = T^{rs}$$

$$Z^{ri} Z^{sj} T_{ij} = T^{rs}$$

$$Z_{mj} T_{rst}^{ijk} = Z_{mj} T_{\cdots rst}^{ijk\cdots} = T_{\cdot m \cdot rst}^{i \cdot k \cdots} = T_{m \cdot rst}^{i \cdot k}$$

$$Z^{jm} T_{m \cdot rst}^{i \cdot k} = Z^{jm} T_{\cdot m \cdot rst}^{i \cdot k \cdots} = T_{\cdots rst}^{ijk\cdots} = T_{rst}^{ijk}$$

Flipping Dummy Indexes (22)

$$S^i T_i = S_i T^i$$

$$T_{ij} S^i R^j = T^{ij} S_i R_j$$

Tensor Equations (23)

$$A_{rs}^{ij} = 2 B_s^{kj} C_{rk}^i + D_r^t E_{tm}^j F_s^{im} - 3 G_{mskr}^{jkmi}$$

Each free index must appear exactly once in the same position (upper or lower) in each term.
Dummy indexes must be uniquely paired (upper and lower) within each term.
Each side of the equation must be a known tensor.

Invariant Relationships (23)

$$P = 2B_i^{kj} C_{jk}^i + D_j^t E_{tm}^j F_i^{im} - 3G_{mikj}^{jkmi}$$

Dot Product (24)

$$\vec{U} \cdot \vec{V} = Z_{ij} U^i V^j = U^i V_i = U_i V^i = Z^{ij} U_i V_j$$

Transformation for Partial Derivative of Scalar Function (27)

$$\frac{\partial T}{\partial Z^i} = J_i^{i'} \frac{\partial T}{\partial Z^{i'}} \qquad \text{tensor}$$

Transformations for Partial Derivative of Vector Component (27)

$$\frac{\partial T^i}{\partial Z^j} = J_{i'}^i J_j^{j'} \frac{\partial T^{i'}}{\partial Z^{j'}} + J_j^{j'} J_{i'j'}^i T^{i'} \qquad \text{non-tensor}$$

$$\frac{\partial T_i}{\partial Z^j} = J_i^{i'} J_j^{j'} \frac{\partial T_{i'}}{\partial Z^{j'}} + J_{ij}^{i'} T_{i'} \qquad \text{non-tensor}$$

Christoffel Symbol (28)

$$\frac{\partial \vec{Z}_i}{\partial Z^j} = \Gamma_{ij}^k \vec{Z}_k$$

Christoffel Symbol Properties (28)

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

$$\Gamma_{ij}^k = 0 \qquad \text{for affine coordinates}$$

Covariant Derivative for Contravariant Vector (28)

$$\nabla_j T^i = \frac{\partial T^i}{\partial Z^j} + \Gamma_{jk}^i T^k$$

Divergence (28)

$$\operatorname{div} \vec{T} = \nabla \cdot \vec{T} = \nabla_i T^i$$

Christoffel Symbol by Kind (29)

$$\Gamma_{jk}^i = \vec{Z}^i \cdot \frac{\partial \vec{Z}_j}{\partial Z^k} = Z^{im} \Gamma_{mjk} \qquad \text{Second kind}$$

$$\Gamma_{ijk} = \vec{Z}_i \cdot \frac{\partial \vec{Z}_j}{\partial Z^k} = Z_{im} \Gamma_{jk}^m \qquad \text{First kind}$$

Transformation for Christoffel Symbol (29)

$$\Gamma_{jk}^i = J_{i'}^i J_j^{j'} J_k^{k'} \Gamma_{j'k'}^{i'} + J_{i'}^i J_{jk}^{i'} \qquad \text{non-tensor}$$

Partial Derivative of Covariant Metric Tensor (30)

$$\frac{\partial Z_{ij}}{\partial Z^k} = \Gamma_{jik} + \Gamma_{ijk}$$

Evaluation of Christoffel Symbol of the First Kind (30)

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial Z_{ij}}{\partial Z^k} + \frac{\partial Z_{ki}}{\partial Z^j} - \frac{\partial Z_{jk}}{\partial Z^i}\right)$$

Evaluation of Christoffel Symbol of the Second Kind (30)

$$\Gamma_{jk}^i = \frac{1}{2} Z^{im}\left(\frac{\partial Z_{mj}}{\partial Z^k} + \frac{\partial Z_{km}}{\partial Z^j} - \frac{\partial Z_{jk}}{\partial Z^m}\right)$$

Covariant Derivative for Covariant Vector (32)

$$\nabla_j T_i = \frac{\partial T_i}{\partial Z^j} - \Gamma_{ji}^k T_k$$

Covariant Derivative Product Rule (34)

$$\nabla_k\left(S_i T^j\right) = (\nabla_k S_i) T^j + S_i\left(\nabla_k T^j\right)$$

$$\nabla_k\left(S_i T^j\right) = \frac{\partial\left(S_i T^j\right)}{\partial Z^k} - \Gamma_{ki}^m S_m T^j + \Gamma_{km}^j S_i T^m$$

Covariant Derivative of a Scalar Function (34)

$$\nabla_j\left(S_i T^i\right) = \frac{\partial\left(S_i T^i\right)}{\partial Z^j}$$

$$\nabla_i f = \frac{\partial f}{\partial Z^i}$$

Covariant Derivative with Multiple Factors (34)

$$\nabla_r\left(A^i B^j C_k D_i E_m\right) = \frac{\partial\left(A^i B^j C_k D_i E_m\right)}{\partial Z^r} + \Gamma_{rs}^j A^i B^s C_k D_i E_m - \Gamma_{rk}^s A^i B^j C_s D_i E_m - \Gamma_{rm}^s A^i B^j C_k D_i E_s$$

Covariant Derivative for Second Rank Mixed Tensor (35)

$$\nabla_k Q_j^i = \frac{\partial Q_j^i}{\partial Z^k} + \Gamma_{km}^i Q_j^m - \Gamma_{kj}^m Q_m^i$$

Covariant Derivative for Higher Rank Tensor (35)

$$\nabla_k T_{rsi}^{ij} = \frac{\partial T_{rsi}^{ij}}{\partial Z^k} + \Gamma_{km}^j T_{rsi}^{im} - \Gamma_{kr}^m T_{msi}^{ij} - \Gamma_{ks}^m T_{rmi}^{ij}$$

Sum Rule (Linearity) (36)

$$\nabla_k\left(2P_j^i + 5Q_j^i\right) = 2\nabla_k P_j^i + 5\nabla_k Q_j^i$$

Product Rule (36)

$$\nabla_k\left(S_j^i T_{si}^r\right) = \left(\nabla_k S_j^i\right) T_{si}^r + S_j^i\left(\nabla_k T_{si}^r\right)$$

Commutativity (36)

$$\left(\nabla_i \nabla_j - \nabla_j \nabla_i\right) T^k = 0$$

Only true for flat manifolds (e.g. Euclidean)

Tensors with Vector Components (37)

$$\vec{T}^i = T^{ij}\vec{Z}_j = T^{i}_{\cdot j}\vec{Z}^j$$

$$\vec{T}_i = T_{ij}\vec{Z}^j = T_i^{\cdot j}\vec{Z}_j$$

$$\nabla_j\vec{T}^i = \frac{\partial\vec{T}^i}{\partial Z^j} + \Gamma^i_{jk}\vec{T}^k$$

$$\nabla_j\vec{T}_i = \frac{\partial\vec{T}_i}{\partial Z^j} - \Gamma^k_{ji}\vec{T}_k$$

Metrinilic Property with Basis Vectors (37)

$$\nabla_j\vec{Z}_i = 0$$

$$\nabla_j\vec{Z}^i = 0$$

$$\nabla_j\left(T^i\vec{Z}_i\right) = \left(\nabla_j T^i\right)\vec{Z}_i$$

$$\nabla_j\left(T_i\vec{Z}^i\right) = \left(\nabla_j T_i\right)\vec{Z}^i$$

Metrinilic Property with Kronecker Delta (38)

$$\nabla_k\delta^i_j = 0$$

$$\delta^i_j\nabla_k T^j = \nabla_k T^i$$

$$S^j\nabla_k\left(\delta^i_j T^m\right) = S^i\nabla_k T^m$$

Metrinilic Property with Metric Tensors (38)

$$\nabla_k Z_{ij} = 0$$

$$\nabla_k Z^{ij} = 0$$

$$Z_{im}\nabla_k T^m = \nabla_k T_i$$

$$S_m\nabla_k\left(Z^{im}T_j\right) = S^i\nabla_k T_j$$

$$S^i\nabla_k T_i = S_i\nabla_k T^i$$

Contravariant Derivative (38)

$$\nabla^i = Z^{ij}\nabla_j$$

$$\operatorname{div}\vec{T} = \nabla\cdot\vec{T} = \nabla_i T^i = \nabla^i T_i$$

Mixed Metric Tensor (38)

$$Z^i_j = Z^{im}Z_{mj} = \delta^i_j$$

Gradient (39)

$$\nabla f = \nabla_i f \vec{Z}^i = \nabla^i f \vec{Z}_i$$

Directional Derivative (39)

$$\frac{df}{ds} = \nabla f \cdot \hat{s}$$

Laplacian (39)

$$\nabla^2 f = \nabla_i \nabla^i f = \nabla^i \nabla_i f$$

Intrinsic, Absolute or Total Derivative (41)

$$\frac{DT^{ij\cdot\cdot}_{rs\cdot\cdot}}{D\lambda} = \frac{dZ^m}{d\lambda} \nabla_m T^{ij\cdot\cdot}_{rs\cdot\cdot}$$

$$\frac{DT}{D\lambda} = \frac{dT}{d\lambda}$$

$$\frac{DT^i}{D\lambda} = \frac{dT^i}{d\lambda} + \frac{dZ^m}{d\lambda} \Gamma^i_{mk} T^k$$

$$\frac{DT^i_r}{D\lambda} = \frac{dT^i_r}{d\lambda} + \frac{dZ^m}{d\lambda} \Gamma^i_{mk} T^k_r - \frac{dZ^m}{d\lambda} \Gamma^k_{mr} T^i_k$$

Trajectory Objects (41)

$\vec{R}(t)$		Position
$\vec{V}(t) = \frac{d\vec{R}}{dt} = V^i \vec{Z}_i$	$V^i = \frac{dZ^i}{dt}$	Velocity
$\vec{A}(t) = \frac{d\vec{V}}{dt} = A^i \vec{Z}_i$	$A^i = \frac{DV^i}{Dt} = \frac{dV^i}{dt} + \Gamma^i_{jk} V^j V^k$	Acceleration
$\vec{J}(t) = \frac{d\vec{A}}{dt} = J^i \vec{Z}_i$	$J^i = \frac{DA^i}{Dt} = \frac{D^2 V^i}{D^2 t}$	Jolt or Surge

Total Symmetry (43)

$$T_{ijk} = T_{jik} = T_{ikj} = T_{kji}$$

Total Anti-symmetry or Skew Symmetry (43)

$$S_{ijk} = -S_{jik} = -S_{ikj} = -S_{kji}$$

Permutation Symbols (43)

e^{ijk}	Skew symmetric and $e^{123} = 1$
e_{ijk}	Skew symmetric and $e_{123} = 1$

Determinant Relationships (44)

$$Ae^{rst} = e^{ijk} a_i^r a_j^s a_k^t$$

$$Ae_{rst} = e^{ijk} a_{ir} a_{js} a_{kt}$$

$$Ae^{rst} = e_{ijk} a^{ir} a^{js} a^{kt}$$

Explicit Determinant Forms (44)

$$A = \frac{1}{3!} e^{ijk} e_{rst} a_i^r a_j^s a_k^t$$

$$A = \frac{1}{3!} e^{ijk} e^{rst} a_{ir} a_{js} a_{kt}$$

$$A = \frac{1}{3!} e_{ijk} e_{rst} a^{ir} a^{js} a^{kt}$$

Delta System Definition (45)

$$\delta^{ijk}_{rst} = \begin{vmatrix} \delta^i_r & \delta^i_s & \delta^i_t \\ \delta^j_r & \delta^j_s & \delta^j_t \\ \delta^k_r & \delta^k_s & \delta^k_t \end{vmatrix}$$

$$\delta^{ij}_{rs} = \begin{vmatrix} \delta^i_r & \delta^i_s \\ \delta^j_r & \delta^j_s \end{vmatrix}$$

If upper/lower indexes are identical sets of distinct values,

+1 if they relate as even permutations

-1 if they relate as odd permutations

0 for everything else

Delta System Identities (45)

$$\delta^{ijk}_{rsk} = \delta^{ij}_{rs}$$

$$\delta^{ij}_{rj} = 2\delta^i_r$$

$$\delta^i_i = 3$$

$$\delta^{ijk}_{rsk} = e^{ijk}e_{rst}$$

Alternate Determinant Form (46)

$$A = \frac{1}{3!}\delta^{ijk}_{rst}a^r_i a^s_j a^t_k$$

Multiplication Property of Determinants (46)

$$a^i_j = b^i_k c^k_j$$

$$A = BC$$

Jacobian Determinant Inverse (46)

$$\left|J^{\cdot}_{\cdot'}\right|\left|J^{\cdot'}_{\cdot}\right| = 1$$

$$\left|J^{\cdot}_{\cdot'}\right| = J$$

$$\left|J^{\cdot'}_{\cdot}\right| = J^{-1}$$

Transformation of Permutation Symbols (46)

$$e^{i'j'k'} = Je^{ijk}J^{i'}_i J^{j'}_j J^{k'}_k$$

$$e_{i'j'k'} = J^{-1}e_{ijk}J^i_{i'}J^j_{j'}J^k_{k'}$$

Relative Tensors (46)

$$\left|a^{\cdot'}_{\cdot'}\right| = |a^{\cdot}_{\cdot}|$$

$$|a_{\cdot'\cdot'}| = J^2 |a_{\cdot\cdot}|$$

$$\left|a^{\cdot'\cdot'}\right| = J^{-2} |a^{\cdot\cdot}|$$

Determinant Cofactors (47)

$$A^i_r = \frac{\partial A}{\partial a^r_i} = \frac{1}{2!} \delta^{ijk}_{rst} a^s_j a^t_k$$

$$A^{ir} = \frac{\partial A}{\partial a_{ri}} = \frac{1}{2!} e^{ijk} e^{rst} a_{sj} a_{tk}$$

$$A_{ir} = \frac{\partial A}{\partial a^{ri}} = \frac{1}{2!} e_{ijk} e_{rst} a^{sj} a^{tk}$$

Cofactor Proportional to Determinant Inverse (47)

$$A^i_r a^r_m = A \delta^i_m$$

$$A^{ir} a_{rm} = A \delta^i_m$$

$$A_{ir} a^{rm} = A \delta^m_i$$

Volume Element Transformation (48)

$$\sqrt{Z'} = |J| \sqrt{Z}$$

Volume Element Derivative (48)

$$\frac{\partial \sqrt{Z}}{\partial Z^i} = \sqrt{Z} \Gamma^m_{mi}$$

Voss-Weyl Formula (49)

$$\operatorname{div} \vec{T} = \nabla_i T^i = \frac{1}{\sqrt{Z}} \frac{\partial}{\partial Z^i} \left(\sqrt{Z} T^i \right)$$

$$\nabla^2 f = \nabla_i \nabla^i f = \frac{1}{\sqrt{Z}} \frac{\partial}{\partial Z^i} \left(\sqrt{Z} Z^{ij} \frac{\partial f}{\partial Z^j} \right)$$

Levi-Civita Symbols (51)

$\varepsilon^{ijk} = \frac{1}{\sqrt{Z}} e^{ijk}$ tensor - orientation preserving transformation

$\varepsilon_{ijk} = \sqrt{Z} e_{ijk}$ tensor - orientation preserving transformation

Levi-Civita Symbol Properties (51)

$$\delta^{ijk}_{rst} = \varepsilon^{ijk}\varepsilon_{rst}$$

$$\varepsilon^{ijk} Z_{ir} Z_{js} Z_{kt} = \varepsilon_{rst}$$

$$\varepsilon_{ijk} Z^{ir} Z^{js} Z^{kt} = \varepsilon^{rst}$$

$$\nabla_m \varepsilon^{ijk} = 0$$

$$\nabla_m \varepsilon_{ijk} = 0$$

Cross Product Definition (52)

$$W^i = \varepsilon^{ijk} U_j V_k$$

$$W_i = \varepsilon_{ijk} U^j V^k$$

$$\vec{W} = \varepsilon^{ijk} U_j V_k \vec{Z}_i = \varepsilon_{ijk} U^j V^k \vec{Z}^i$$

$$\vec{W} = \vec{U} \times \vec{V}$$

Cross Product Properties (52)

$$\varepsilon_{ijk} U^j V^k = -\varepsilon_{ijk} V^j U^k$$

$$\left(\vec{U} \times \vec{V}\right) = -\left(\vec{V} \times \vec{U}\right)$$

$$\left(\varepsilon_{ijk} U^j V^k\right) U^i = \left(\varepsilon_{ijk} U^j V^k\right) V^i = 0$$

$$\vec{W} \cdot \vec{U} = \vec{W} \cdot \vec{V} = 0$$

Cross Product Magnitude (52)

$$W_i W^i = \varepsilon_{ijk} U^j V^k \varepsilon^{ist} U_s V_t = U^s V^t U_s V_t - U^t V^s U_s V_t$$

$$\left|\vec{W}\right|^2 = \left|\vec{U}\right|^2 \left|\vec{V}\right|^2 - \left(\vec{U} \cdot \vec{V}\right)^2$$

$$\left|\vec{W}\right| = \left|\vec{U}\right| \left|\vec{V}\right| \sin\theta$$

Cross Product Identity (52)

$$U^i\left(\varepsilon_{ijk} V^j W^k\right) = V^i\left(\varepsilon_{ijk} W^j U^k\right) = W^i\left(\varepsilon_{ijk} U^j V^k\right)$$

$$\vec{U} \cdot \left(\vec{V} \times \vec{W}\right) = \vec{V} \cdot \left(\vec{W} \times \vec{U}\right) = \vec{W} \cdot \left(\vec{U} \times \vec{V}\right)$$

Curl Definition (53)

$$V^i = \varepsilon^{ijk} \nabla_j U_k$$

$$V_i = \varepsilon_{ijk} \nabla^j U^k$$

$$\vec{V} = \varepsilon^{ijk} \nabla_j U_k \vec{Z}_i = \varepsilon_{ijk} \nabla^j U^k \vec{Z}^i$$

$$\vec{V} = \mathrm{curl}\, \vec{U} = \nabla \times \vec{U}$$

Alternate Curl Definition (53)

$$V^i = \varepsilon^{ijk} \frac{\partial U_k}{\partial Z^j}$$

Curl Identities (53)

$$\mathrm{curl}\,(\mathrm{grad}\, f) = \nabla \times \nabla f = 0$$

$$\mathrm{div}\left(\mathrm{curl}\, \vec{U}\right) = \nabla \cdot \left(\nabla \times \vec{U}\right) = 0$$

Invariant Volume Differential (54)

$$d\Omega = \sqrt{Z}\, dZ^1 dZ^2 dZ^3$$

Physical Integral Evaluation (54)

$$\int_\Omega f \, d\Omega = \iiint_\Omega f(Z) \sqrt{Z}\, dZ^1 dZ^2 dZ^3$$