Curvature Equations and Syntax

Curvature Normal (79)

$$\nabla^2 \vec{R} = \nabla_\alpha \nabla^\alpha \vec{R} = \nabla^\alpha \nabla_\alpha \vec{R} = B_\alpha^\alpha \hat{N}$$

$$B_\beta^\alpha = \nabla_\beta \nabla^\alpha \vec{R} \cdot \hat{N}$$

Mean Curvature (79)

$$B_\alpha^\alpha = \mathrm{K}_1 + \mathrm{K}_2$$

Gaussian Curvature (79)

$$|B_\cdot^\cdot| = \mathrm{K} = \mathrm{K}_1 \mathrm{K}_2$$

Principal Curvatures (80)

$$\mathrm{K}_1 = \frac{1}{2}(B_1^1 + B_2^2) + \frac{1}{2}\sqrt{(B_1^1 - B_2^2)^2 + 4B_2^1 B_1^2}$$

$$\mathrm{K}_2 = \frac{1}{2}(B_1^1 + B_2^2) - \frac{1}{2}\sqrt{(B_1^1 - B_2^2)^2 + 4B_2^1 B_1^2}$$

Commutative Operator (82)

$$[\nabla_\alpha, \nabla_\beta] T^\lambda = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) T^\lambda$$

Riemann-Christoffel Tensor (82)

$$R_{\omega\alpha\beta}^\lambda = \frac{\partial \Gamma_{\beta\omega}^\lambda}{\partial S^\alpha} - \frac{\partial \Gamma_{\alpha\omega}^\lambda}{\partial S^\beta} + \Gamma_{\alpha\sigma}^\lambda \Gamma_{\beta\omega}^\sigma - \Gamma_{\beta\sigma}^\lambda \Gamma_{\alpha\omega}^\sigma$$

$$R_{\lambda\omega\alpha\beta} = \frac{\partial \Gamma_{\lambda\beta\omega}}{\partial S^\alpha} - \frac{\partial \Gamma_{\lambda\alpha\omega}}{\partial S^\beta} + \Gamma_{\sigma\lambda\beta} \Gamma_{\alpha\omega}^\sigma - \Gamma_{\sigma\lambda\alpha} \Gamma_{\beta\omega}^\sigma$$

Commutation for Contravariant Vector (82)

$$[\nabla_\alpha, \nabla_\beta] T^\lambda = R_{\omega\alpha\beta}^\lambda T^\omega$$

Riemann-Christoffel Tensor in Riemann Normal Coordinates (83)

$$R_{\lambda\omega\alpha\beta} = \frac{\partial \Gamma_{\lambda\beta\omega}}{\partial S^\alpha} - \frac{\partial \Gamma_{\lambda\alpha\omega}}{\partial S^\beta}$$

$$R_{\lambda\omega\alpha\beta} = \frac{1}{2}\left(\frac{\partial^2 S_{\lambda\beta}}{\partial S^\alpha \partial S^\omega} - \frac{\partial^2 S_{\beta\omega}}{\partial S^\alpha \partial S^\lambda} - \frac{\partial^2 S_{\lambda\alpha}}{\partial S^\beta \partial S^\omega} + \frac{\partial^2 S_{\alpha\omega}}{\partial S^\beta \partial S^\lambda}\right)$$

Riemann-Christoffel Tensor Symmetries (84)

$$R_{\lambda\omega\alpha\beta} = -R_{\omega\lambda\alpha\beta}$$

$$R_{\lambda\omega\alpha\beta} = -R_{\lambda\omega\beta\alpha}$$

$$R_{\lambda\omega\alpha\beta} = R_{\alpha\beta\omega\lambda}$$

Bianchi Identities (84)

$$R_{\lambda\omega\alpha\beta} + R_{\lambda\alpha\beta\omega} + R_{\lambda\beta\omega\alpha} = 0$$

$$\nabla_\sigma R_{\lambda\omega\alpha\beta} + \nabla_\lambda R_{\omega\sigma\alpha\beta} + \nabla_\omega R_{\sigma\lambda\alpha\beta} = 0$$

Commutation for Vectors and Scalar Functions (85)

$$[\nabla_\alpha, \nabla_\beta] T^\lambda = R^\lambda_{\omega\alpha\beta} T^\omega$$

$$[\nabla_\alpha, \nabla_\beta] T_\lambda = -R^\omega_{\lambda\alpha\beta} T_\omega$$

$$[\nabla_\alpha, \nabla_\beta] f = 0$$

$$[\nabla_\alpha, \nabla_\beta] T^i = 0$$

$$[\nabla_\alpha, \nabla_\beta] T_i = 0$$

Commutation Sum and Product Rules (85)

$$[\nabla_\alpha, \nabla_\beta]\left(3S^{\omega i}_{\lambda j} + 5T^{\omega i}_{\lambda j}\right) = 3[\nabla_\alpha, \nabla_\beta] S^{\omega i}_{\lambda j} + 5[\nabla_\alpha, \nabla_\beta] T^{\omega i}_{\lambda j}$$

$$[\nabla_\alpha, \nabla_\beta]\left(S^\omega_j T^i_\lambda\right) = \left([\nabla_\alpha, \nabla_\beta] S^\omega_j\right) T^i_\lambda + S^\omega_j \left([\nabla_\alpha, \nabla_\beta] T^i_\lambda\right)$$

Commutation for Tensors of Higher Rank (85)

$$[\nabla_\alpha, \nabla_\beta]\left(S^\omega T_\lambda U^i V_j\right) = R^\omega_{\sigma\alpha\beta} S^\sigma T_\lambda U^i V_j - R^\sigma_{\lambda\alpha\beta} S^\omega T_\sigma U^i V_j$$

$$[\nabla_\alpha, \nabla_\beta] T^{\omega i}_{\lambda j} = R^\omega_{\sigma\alpha\beta} T^{\sigma i}_{\lambda j} - R^\sigma_{\lambda\alpha\beta} T^{\omega i}_{\sigma j}$$

Codazzi Equation (86)

$$\nabla_\alpha B_{\beta\omega} = \nabla_\beta B_{\alpha\omega}$$

Theorema Egregium (86)

$$R_{\lambda\omega\alpha\beta} = B_{\alpha\lambda} B_{\beta\omega} - B_{\beta\lambda} B_{\alpha\omega}$$

Gaussian Curvature – Riemann Tensor Relationships (86)

$$\frac{R_{1212}}{S} = |B^{\cdot}_{\cdot}| = \mathrm{K}$$

$$\mathrm{K}\, \varepsilon_{\lambda\omega} \varepsilon_{\alpha\beta} = R_{\lambda\omega\alpha\beta}$$

$$\mathrm{K} = \frac{1}{4} \varepsilon^{\lambda\omega} \varepsilon^{\alpha\beta} R_{\lambda\omega\alpha\beta}$$

Ricci Tensor (87)

$$R_{\alpha\beta} = R_{\beta\alpha} = R^\omega_{\alpha\omega\beta}$$

Ricci Scalar (87)

$$R = R_{\alpha}^{\alpha} = R_{\alpha\beta}^{\alpha\beta}$$

$$\mathrm{K} = \frac{1}{2} R$$

Contracted Bianchi Identity (87)

$$\nabla_{\alpha}\left(R^{\alpha\beta} - \frac{1}{2} R S^{\alpha\beta}\right) = 0$$

Einstein Tensor (87)

$$G_{\alpha\beta} = G_{\beta\alpha} = R_{\alpha\beta} - \frac{1}{2} R S_{\alpha\beta}$$

$$\nabla_{\alpha} G^{\alpha\beta} = 0$$