FoS_HW_04_Group_II

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1.

Nr.1

a) $Z \overset{g(x)}{=} \frac{X-\mu}{\sigma} \Rightarrow g^{-1}(Z) = z\sigma + \mu$

$$\frac{dx}{dz} = \frac{d}{dz}(\sigma z + \mu) = \sigma$$

Transformation

$$f_Z(z) = f_X(g^{-1}(z)) \cdot \left|\frac{dx}{dz}\right| = f_X(\sigma z + \mu) \cdot |\sigma|$$

$$= \left[\frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left(-\frac{1}{2}\left(\frac{\sigma z + \mu - \mu}{\sigma}\right)^2\right)\right] |\sigma|$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\cancel{\sigma}}{\cancel{\sigma}} \cdot \exp\left(-\tfrac{1}{2} z^2\right)$$

$$\overset{1^2=1}{\underset{z=z-0}{=}} \frac{1}{\sqrt{2\pi}\cdot 1} \cdot \exp\left(-\tfrac{1}{2}\left(\frac{z-0}{1}\right)^2\right. \sim N(0, 1^2)$$

$\Rightarrow$ Standard Normal Distribution. $\square$

b) $P(X \le b) = P\left(\frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) = P\left(Z \le \frac{b-\mu}{\sigma}\right)$

$$\overset{1a) + \text{Def } \Phi_{0,1}(x)}{=} \Phi_{0,1}\left(\frac{b-\mu}{\sigma}\right)$$

$P(X<a) = P(X \le a) \Leftarrow P(X=a) = 0$

$$P(a \le X \le b) = P(X \le b) - P(X < a) = \underline{\underline{\Phi_{0,1}\left(\frac{b-\mu}{\sigma}\right) - \Phi_{0,1}\left(\frac{a-\mu}{\sigma}\right)}}$$ $\square$

a) & b)

c-e)

c) ① $\phi_{0,1}(x) = P(Z \le x)$ Normalverteilung is symetrical

② $\phi_{0,1}(-x) = P(Z \le -x) = P(Z \ge x) = 1 - P(Z < x) = 1 - P(Z \le x)$

$\phi_{0,1}(x) + \phi_{0,1}(-x) \overset{①}{\underset{②}{=}} P(Z \le x) + 1 - P(Z \le x) = 1$ $\square$

d) b <- qnorm(0,9) $\Longrightarrow$ $b \approx 1{,}282$

e) $P(\mu - \sigma \le X \le \mu + \sigma) \overset{b}{=} \phi_{01}\left(\frac{\mu + \sigma - \mu}{\sigma}\right) - \phi_{01}\left(\frac{\mu - \sigma - \mu}{\sigma}\right)$

$= \phi_{01}(1) - \phi_{01}(-1)$ $\leftarrow$ ~~intepend~~ does not use μ, σ $\square$

R: pnorm(1) - pnorm(-1) $\approx 0{,}683$

$\leftarrow P(X=a)=0$

```r
#f
m <- sample(-100:100, 1)
s <- sample(1:20, 1)
n <- 10000
X_samples <- rnorm(n, mean = m, sd = s)
inBoundery <- (X_samples >= m - s) & (X_samples <= m + s)
simulated_prob <- sum(inBoundery) / n
print(simulated_prob)
```

f)

```
## [1] 0.6828
```

2.

$\boxed{3}$

a) $f_Z(z) = f_{X+Y}(z) \overset{\text{Definition PMF}}{=} \begin{cases} \mathbb{P}((X+Y)=z_i) & \text{if } z=z_i \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} \mathbb{P}(Y=u, X=(z_i-u)) \ \forall u\in\mathbb{Z} & \text{if } z=z_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \sum_{u\in\mathbb{Z}} \mathbb{P}(Y=u, X=(z_i-u)) & \text{if } z=z_i \\ 0 & \text{otherwise} \end{cases}$

X,Y independent

$= \sum_{u\in\mathbb{Z}} \begin{cases} \mathbb{P}(Y=u)\,\mathbb{P}(X=(z_i-u)) & \text{if } z=z_i \\ 0 & \text{otherwise} \end{cases} = \sum_{u\in\mathbb{Z}} f_X(z-u)\cdot f_Y(u)$ ■

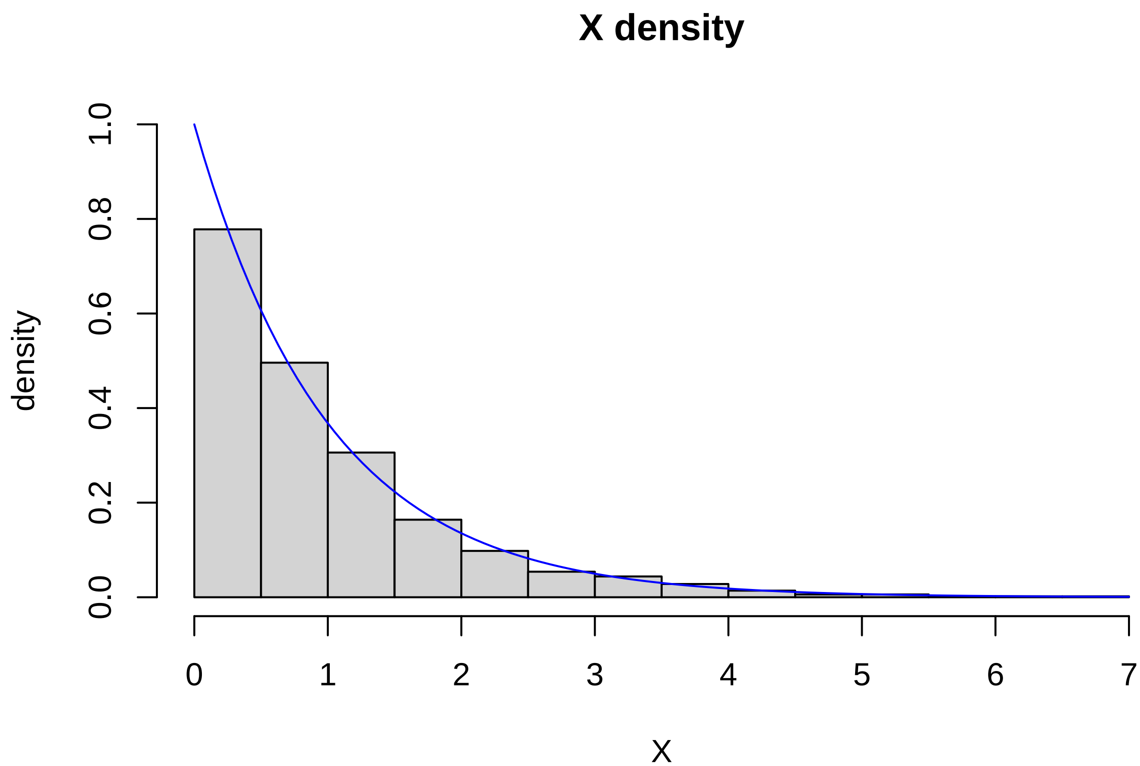
b) $\mathbb{P}(Z\le z) = \mathbb{P}((X+Y)\le z) = \int_{-\infty}^{\infty} \mathbb{P}((X+Y)\le z \mid X=u)\cdot \mathbb{P}(X=u)\,du = \int_{-\infty}^{\infty} \mathbb{P}(Y\le z-u \mid X=u)\cdot f_X(u)\,du$

Independence

$= \int_{-\infty}^{\infty} \mathbb{P}(Y\le z-u)\cdot f_X(u)\,du = \int_{-\infty}^{\infty}\int_{-\infty}^{z} f_Y(z-u)\,dz\cdot f_X(u)\,du = \int_{-\infty}^{z}\underbrace{\int_{-\infty}^{\infty} f_Y(z-u)\cdot f_X(u)\,du}_{f_Z(z)}\,dz$ ■

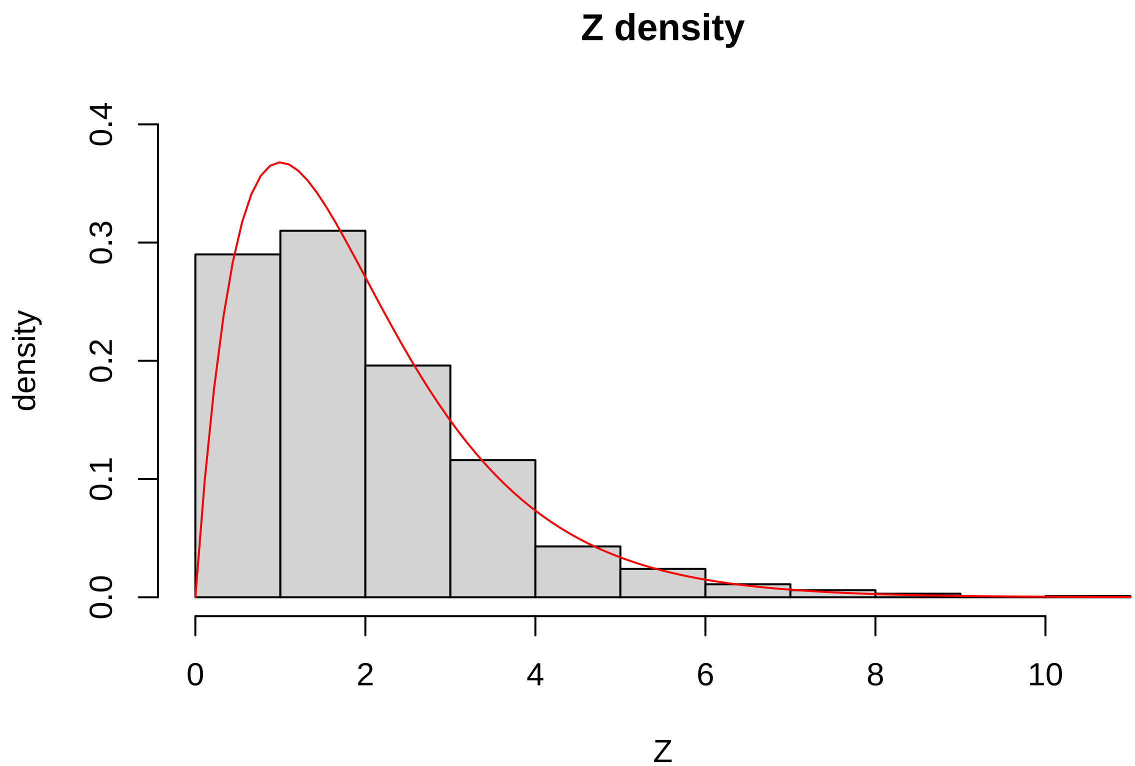
a), b)

```r
# set a seed for reproducibility
set.seed(420)

# ?rexp()
lambda <- 1
values_X <- rexp(n=1000, rate=lambda)
values_Y <- rexp(n=1000, rate=lambda)

# ?hist()
# ?curve()
hist(values_X, probability=TRUE, main="X density", xlab="X", ylab="density", ylim=c(0,1))
curve(dexp(x, rate=1), col="blue", add=TRUE)
```



c)

```
# The resulting RV follows a gamma distribution with shape 1 and rate 1.
values_Z <- values_X + values_Y

hist(values_Z, probability=TRUE, main="Z density", xlab="Z", ylab="density", ylim=c(0,0.4))
curve(dgamma(x, shape=2, rate=1), col="red", add=TRUE)
```


Z density
density
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
10
Z

3.

$\boxed{3}$

a) Poisson distribution: $\mathbb{P}(X=k) = \frac{\lambda^k}{k!} \cdot e^{-\lambda}$ Binomial distribution: $\mathbb{P}(X=k) = \binom{n}{k} p^k \cdot (1-p)^{n-k}$

$$f_{X,Y}(x,y) = \mathbb{P}(X=x \text{ and } Y=y) = \mathbb{P}(Y=y \mid X=x) \cdot \mathbb{P}(X=x) = \binom{x}{y} p^y \cdot (1-p)^{x-y} \cdot \frac{\lambda^x}{x!} \cdot e^{-\lambda} \quad \blacksquare$$

b)

$$f_Y(y) = \mathbb{P}(Y=y) = \sum_{x=0}^{\infty} \mathbb{P}(X=x \text{ and } Y=y) = \sum_{x=0}^{\infty} \frac{x!}{(x-y)!\,y!} p^y (1-p)^{x-y} \frac{\lambda^x}{x!} e^{-\lambda}$$

$\lambda^x = \lambda^y \cdot \lambda^{x-y}$

$$= \sum_{x=0}^{\infty} \frac{(\lambda - p\lambda)^{x-y}}{x-y!} \cdot \frac{(p\lambda)^y}{y!} \cdot e^{-\lambda} = e^{\lambda - p\lambda} \cdot \frac{(p\lambda)^y}{y!} e^{-\lambda}$$

$$= \frac{(p\lambda)^y}{y!} \cdot e^{-p\lambda} \sim \mathrm{Pois}(p\lambda) \quad \blacksquare$$

c)

$$f_{X+X'}(z) = \sum_{u=0}^{z} f_X(z-u) \cdot f_{X'}(u) = \sum_{u}^{z} \frac{\lambda^{z-u}}{(z-u)!} e^{-\lambda} \cdot \frac{\mu^u}{u!} \cdot e^{-\mu}$$

$$= e^{-(\lambda+\mu)} \cdot \sum_{u}^{z} \frac{\lambda^{z-u} \mu^u}{(z-u)!\,u!} = \frac{e^{-(\lambda+\mu)}}{z!} \cdot \sum_{u=0}^{z} \binom{z}{u} \cdot \lambda^{(z-u)} \cdot \mu^u$$

binomial Theorem

$$= \frac{-e^{(\lambda+\mu)}}{z!} \cdot (\lambda+\mu)^z \quad \blacksquare$$

a), b), c)

4.

$$P(X > t+s | X > t) = \frac{P(X > t+s \cap X > t)}{X > t} \quad (1)$$

CDF of the exponential distribution is $1 - e^{-\lambda x}$:

$$P(X > x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x} \quad (2)$$

The denominater of (1) has to be;

$$P(X > t+s \cap X > t) = P(X > t+s)$$

since we have $>$ and $\cap$

Putting (2) in (1) gives;

$$P(X > t+s | X > t) = \frac{P(X > t+s)}{e^{-\lambda t}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$

5.

a) We use the **Poisson distribution** because:

- Events occur randomly and independently
- Constant average rate ($\lambda = 3$ per month)
- We count discrete events in a fixed time interval

Model: $X \sim \text{Poisson}(\lambda = 3)$

For a Poisson distribution with parameter λ:

$$\mathbb{E}[X] = \lambda = 3$$
$$\text{Var}(X) = \lambda = 3$$

```r
# P(X >= 6) = 1 - P(X <= 5)
p_geq_6 <- 1 - ppois(5, lambda = 3)
p_geq_6
```

b)

```
## [1] 0.08391794
```

Result: $P(X \geq 6) \approx 0.0839$

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c)

Chebyshev's Inequality For any random variable with mean μ and variance σ^2:

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

We have:

- $\mu = 3$
- $\sigma^2 = 3$

$$P(X \geq 6) \leq P(|X - \mu| \geq 3) \leq \frac{\sigma^2}{k^2} = \frac{3}{3^2} = \frac{3}{9} = \frac{1}{3}$$

Comparison The Chebyshev bound gives us $P(X \geq 6) \leq \frac{1}{3} \approx 0.333$

From part b, we calculated the exact probability: $P(X \geq 6) \approx 0.0839$

Conclusion: The Chebyshev bound is much looser than the exact probability, which is expected since Chebyshev's inequality is a general bound that works(both the left and right sides considered) for any distribution and only uses mean and variance information.

6.

Part I: $-1 \leq \rho(X, Y) \leq 1$ By definition:

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] = \mathbb{E}(\tilde{X} \cdot \tilde{Y})$$

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}(\tilde{X}^2)$$

$$\mathrm{Var}(Y) = \mathbb{E}[(Y - \mathbb{E}(Y))^2] = \mathbb{E}(\tilde{Y}^2)$$

centered variables

define the centered random variables:

$$\tilde{X} = X - \mathbb{E}(X), \quad \tilde{Y} = Y - \mathbb{E}(Y)$$

These have mean zero: $\mathbb{E}(\tilde{X}) = 0$ and $\mathbb{E}(\tilde{Y}) = 0$.

Applying the Cauchy-Schwarz inequality to $\tilde{X}$ and $\tilde{Y}$:

$$|\mathbb{E}(\tilde{X} \cdot \tilde{Y})| \leq \sqrt{\mathbb{E}(\tilde{X}^2)} \cdot \sqrt{\mathbb{E}(\tilde{Y}^2)}$$

which is;

$$|\mathrm{Cov}(X, Y)| \leq \sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}$$

the bounds

Dividing both sides by $\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}$ (assuming both variances are positive):

$$\frac{|\mathrm{Cov}(X, Y)|}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}} \leq \frac{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}} = 1$$

This gives us:

$$|\rho(X, Y)| \leq 1$$

Which is equivalent to:

$$-1 \leq \rho(X, Y) \leq 1$$

Part II: Equality condition $|\rho(X,Y)| = 1 \iff Y = cX + d$ **Direction 1:** $|\rho(X,Y)| = 1 \Rightarrow Y = cX + d$

From the Cauchy-Schwarz inequality, equality holds if and only if there exists a constant $a \in \mathbb{R}$ such that:

$$\tilde{X} = a\tilde{Y}$$

This means:

$$X - \mathbb{E}(X) = a(Y - \mathbb{E}(Y))$$

Rearranging:

$$X = aY + [\mathbb{E}(X) - a\mathbb{E}(Y)]$$

Or equivalently (solving for Y):

$$Y = \frac{1}{a}X + \left[\mathbb{E}(Y) - \frac{1}{a}\mathbb{E}(X)\right]$$

Setting $c = \frac{1}{a}$ and $d = \mathbb{E}(Y) - c\mathbb{E}(X)$, we get:

$$Y = cX + d$$

Determining the sign of c:

From $\tilde{X} = a\tilde{Y}$, taking variances:

$$\text{Var}(X) = a^2\text{Var}(Y)$$

Therefore:

$$a = \pm\sqrt{\frac{\text{Var}(X)}{\text{Var}(Y)}}$$

And:

$$c = \frac{1}{a} = \pm\sqrt{\frac{\text{Var}(Y)}{\text{Var}(X)}}$$

Now, the covariance is:

$$\text{Cov}(X,Y) = \mathbb{E}(\tilde{X} \cdot \tilde{Y}) = \mathbb{E}(a\tilde{Y} \cdot \tilde{Y}) = a\mathbb{E}(\tilde{Y}^2) = a \cdot \text{Var}(Y)$$

The correlation coefficient becomes:

$$\rho(X,Y) = \frac{a \cdot \text{Var}(Y)}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}} = \frac{a \cdot \sqrt{\text{Var}(Y)}}{\sqrt{\text{Var}(X)}}$$

- If $a > 0$: $\rho(X,Y) = +1$ and $c = +\sqrt{\frac{\text{Var}(Y)}{\text{Var}(X)}}$
- If $a < 0$: $\rho(X,Y) = -1$ and $c = -\sqrt{\frac{\text{Var}(Y)}{\text{Var}(X)}}$

Direction 2: $Y = cX + d \Rightarrow |\rho(X,Y)| = 1$

Assume $Y = cX + d$ for constants $c \neq 0$ and d.

Then:

$$\mathbb{E}(Y) = c\mathbb{E}(X) + d$$
$$\tilde{Y} = Y - \mathbb{E}(Y)$$

$$= (cX + d) - (c\mathbb{E}(X) + d)$$

$$= cX + d - c\mathbb{E}(X) - d$$

$$= cX - c\mathbb{E}(X)$$

$$= c(X - \mathbb{E}(X))$$

$$= c\tilde{X}$$

Therefore:

$$\text{Cov}(X, Y) = \mathbb{E}(\tilde{X} \cdot \tilde{Y}) = \mathbb{E}(\tilde{X} \cdot c\tilde{X}) = c\mathbb{E}(\tilde{X}^2) = c \cdot \text{Var}(X)$$

$$\text{Var}(Y) = \mathbb{E}(\tilde{Y}^2) = c^2\mathbb{E}(\tilde{X}^2) = c^2 \cdot \text{Var}(X)$$

The correlation coefficient is:

$$\rho(X, Y) = \frac{c \cdot \text{Var}(X)}{\sqrt{\text{Var}(X) \cdot c^2\text{Var}(X)}} = \frac{c \cdot \text{Var}(X)}{|c| \cdot \text{Var}(X)} = \frac{c}{|c|} = \text{sign}(c)$$

Therefore:

- If $c > 0$: $\rho(X, Y) = +1$
- If $c < 0$: $\rho(X, Y) = -1$

Thus, $|\rho(X, Y)| = 1$.

7.

Nr. 7 $\quad Y = X - \mu \quad E(Y) = 0 \quad \sigma^2 = Var(Y) = E[Y^2] - E[0 \cdot Y] \Rightarrow E[Y^2] = \sigma^2$

a) $P((Y+c)^2 > (a+c)^2) = P(Y > a) + P(Y < a)$

$$\Rightarrow P(X-\mu > a) \le P((Y+c)^2 > (a+c)^2) \overset{\text{Marc Ineq}}{\le} \frac{E(Y^2 + 2cY + c^2)}{(a+c)^2}$$

$$= \frac{\sigma^2 + 2c \cdot \overset{E[Y]=0}{0} + c^2}{a^2 + 2ac + c^2} = \frac{\sigma^2}{2ac + a^2}$$

$c = \frac{\sigma^2}{2a} > 0$

$$\Rightarrow P(X-\mu > a) \le \frac{\sigma^2}{\sigma^2 + a^2} \quad \square$$

b) $Z = -Y \quad E[Z] = 0 \quad Var(Z) = \sigma^2$

$$\Rightarrow P(X-\mu < -a) = P(-Y > a) = P(Z > a) \overset{a)}{\le} \frac{\sigma^2}{\sigma^2 + a^2} \quad \square$$

a and b)

below is a alternativ solution but i am not sure if it is ok to argument like this could i get a feedback on the idee

R: pnorm(1) - pnorm(-1) $\approx 0{,}683$

Aufgabe 7 $\quad <$ same as $\leq$ $\leftarrow P(X=a)=0$

$Y = X - \mu \implies E[Y] = 0 \; , \; Var(Y) = \sigma^2 \overset{*}{\geq} 0$

$Var(Y) = E(Y^2) - E(Y)^2 \iff \sigma^2 = E(Y^2) \qquad a^2 \overset{*}{\geq} 0$

i) $P(X-\mu \geq a) = P(Y \geq a) \overset{\text{Mar. inequ.}}{\leq} \frac{E(Y)}{a} = 0 \overset{*}{\leq} \frac{\sigma^2}{\sigma^2+a^2}$ ✓

ii) $P(X-\mu < -a) = P(Y < -a) = P(-Y \geq a)$ $\quad \underline{|\, E(-Y) = -1 \cdot E(Y) = 0}$

$\overset{\text{Mar. inequ.}}{\leq} \frac{E(-Y)}{a} = 0 \leq \frac{\sigma^2}{\sigma^2+a^2}$ ✓

$\square$