FoS_HW_03_Group_II

anon

2025-11-05

Part I

1.

a) $S = \{$two friends choose the same number$\}$ $S^c = \{$every friend chooses a different number$\}$

$\mathbb{P}(S) = 1 - \mathbb{P}(S^c)$

There are $5^4 = 640$ possibilities on how the numbers are choosen.

There are $5 \cdot 4 \cdot 3 \cdot 2 = 120$ possible distribution that every friend has a different number

The first person is not restricted, the second has only 4 options to still have distinct numbers and so on.

Therefore $\mathbb{P}(S^c) = \frac{5!}{5^4} = \frac{120}{625} = \frac{24}{125} = \underline{\underline{0{,}192}}$ and $\mathbb{P}(S) = 1 - 0{,}192 = 0{,}808$

a)

```r
set.seed(420)
n <- 1000
count <- 0

for (i in 1:n) {
  picks <- sample(1:5, 4, replace = TRUE)
  if (length(unique(picks)) < 4) count <- count + 1
}

estimate <- count / n
print(estimate)
```

b)

```
## [1] 0.824
```

2.

Possible family compositions:

- $P(first \quad boy) = 0.5$
- $P(first \quad girl) = 0.5$
- $P(girl \cap boy) = 0.5 \cdot 0.5 = 0.25$
- $P(girl \cap girl) = 0.25$

Having;

- $P(boy) = P(first \quad boy \cup girl \cap boy) = 0.5 + 0.25 = 0.75$
- $P(girl) = P(first \quad girl \cup girl \cap girl) = 0.5 + 0.25 = 0.75$

Ratio

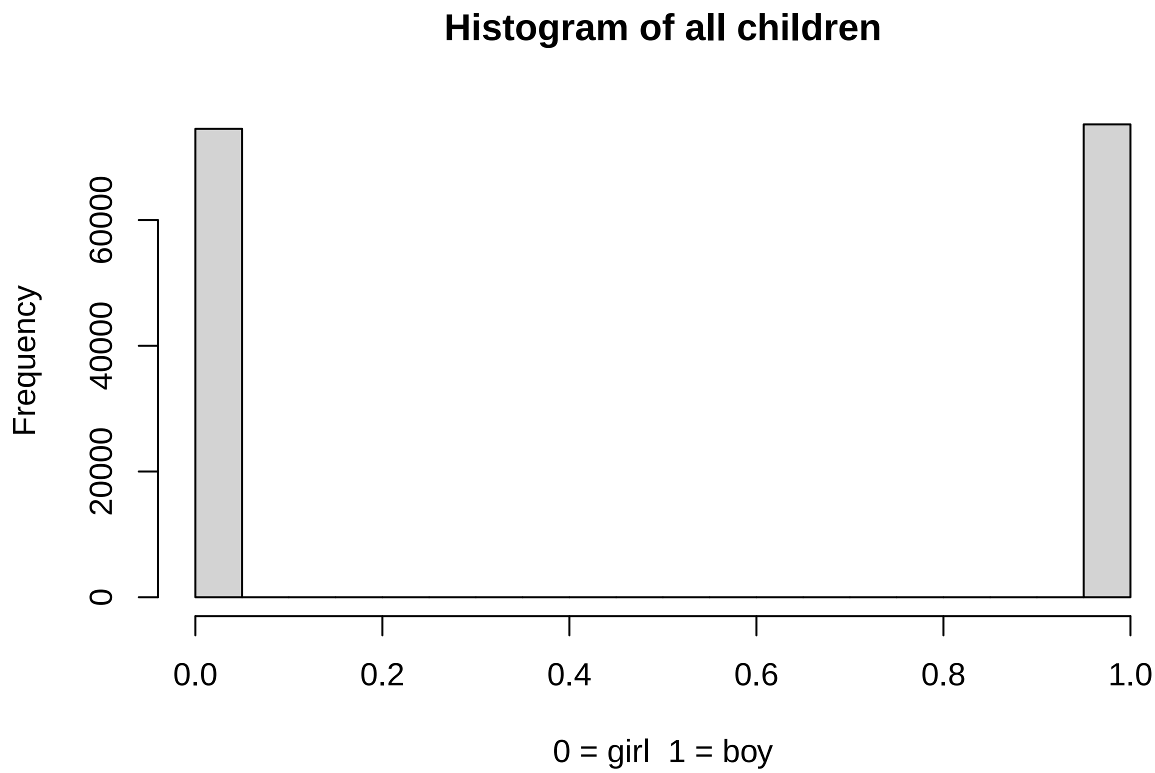
is 1:1

```r
sim <- function() {
  first_child <- sample(c(1, 0), 1, prob = c(0.5,0.5))  # 1=boy

  if (first_child == 1) {
    return(1)
  } else {
    second_child <- sample(c(1, 0), 1, prob = c(0.5,0.5))
    return(c(0, second_child))
  }
}

allc <- c()
for (i in 1:100000) {
  allc <- c(allc, sim())
}

hist(allc, main = "Histogram of all children", xlab = "0 = girl  1 = boy")
```




```r
sum(allc)/length(allc)
```

% of boys is;

```
## [1] 0.502364
```

Part II

3.

3)

$$\sum_{n\geq 1}^{\infty} \mathbb{P}(X \geq n) = \sum_{n\geq 1}^{\infty} \mathbb{P}(X=n \vee X=n+1 \vee \dots)$$

$$= \sum_{n\geq 1}^{\infty} \mathbb{P}(X=n) \vee \mathbb{P}(X=n+1) \vee \dots)$$

$$= \sum_{n\geq 1}^{\infty} \sum_{i=n}^{\infty} \mathbb{P}(X=i)$$

$$= \sum_{n=1}^{\infty} n \cdot \mathbb{P}(X=i) = \mathbb{E}[X] \quad \blacksquare$$

4.

a) i) There are two cases that can occur:

Case 1: A occurs $\Rightarrow \mathbb{1}_A$ evaluates to 1

$$\mathbb{1}_A^2 = \mathbb{1}_A \cdot \mathbb{1}_A = 1 \cdot 1 = 1 = \mathbb{1}_A$$

Case 2: A does not occur $\Rightarrow \mathbb{1}_A$ evaluates to 0

$$\mathbb{1}_A^2 = \mathbb{1}_A \cdot \mathbb{1}_A = 0 \cdot 0 = \mathbb{1}_A$$

$$\Rightarrow \mathbb{1}_A^2 = \mathbb{1}_A$$

ii) We again make a case distinction

Case 1: A occurs, B not

$$\mathbb{1}_{A\cap B} = 0 = 1 \cdot 0 = \mathbb{1}_A \cdot \mathbb{1}_B$$

since B doesn't occurs

Case 2: A occurs, B occurs

$$\mathbb{1}_{A\cap B} = 1 = 1 \cdot 1 = \mathbb{1}_A \cdot \mathbb{1}_B$$

Case 3: A not, B occurs

$$\mathbb{1}_{A\cap B} = 0 = 1 \cdot 0 = \mathbb{1}_A \cdot \mathbb{1}_B$$

since A doesn't occurs

Case 4: A not, B not

$$\mathbb{1}_{A\cap B} = 0 = 0 \cdot 0 = \mathbb{1}_A \cdot \mathbb{1}_B$$

$$\Rightarrow \mathbb{1}_{A\cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$$

iii) We again make a case distinction

Case 1: A occurs, B not

$$\mathbb{1}_{A\cup B} = 1 = 1 + 0 - 1 \cdot 0 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

Case 2: A occurs, B occurs

$$\mathbb{1}_{A\cup B} = 1 = 1 + 1 - 1 \cdot 1 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

Case 3: A not, B occurs

$$\mathbb{1}_{A\cup B} = 1 = 0 + 1 - 0 \cdot 1 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

Case 4: A not, B not

$$\mathbb{1}_{A\cup B} = 0 = 0 + 0 - 0 \cdot 0 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

$$\Rightarrow \mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

a)

$$\left.\begin{aligned} &b)\quad \mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(\text{" A occurs"}) = p \\ &\quad\ \ \mathbb{P}(\mathbb{1}_A = 0) = \mathbb{P}(\text{" A does not occur"}) = 1 - \mathbb{P}(\text{" A occurs"}) = 1 - p =: q \end{aligned}\right\} \mathbb{1}_A \sim Ber(p)$$

$$F_{\mathbb{1}_A}(x) = \mathbb{P}(\mathbb{1}_A \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

c) $\mathbb{E}[\mathbb{1}_A] = \sum_{i\geq 1} x_i \, \mathbb{P}(X = x_i) = 1 \cdot p + 0\,(1-p) = p = \mathbb{P}(A)$

d) CDF for $\mathcal{U}(0,1) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$

$$\left.\begin{aligned} &\mathbb{P}(X = 1) = \mathbb{P}(\mathcal{U}(\omega) < p) = F_{\mathcal{U}}(p) = p \\ &\mathbb{P}(X = 0) = \mathbb{P}(\mathcal{U}(\omega) \geq p) = 1 - \mathbb{P}(\mathcal{U}(\omega) < p) = 1 - F_{\mathcal{U}}(p) = 1 - p = q \end{aligned}\right\} \begin{aligned} &X \sim Ber(p) \text{ and it allows} \\ &\text{the representation } X = \mathbb{1}_A \end{aligned}$$

b), c), d)

5.

a) Find: The CDF of $Y = X^+ = max(0, X)$

Case 1: $y < 0$

Since $Y = max(0, X) \geq 0 \ \forall x$, we have $P(Y \leq y) = 0$ for $y < 0$.

Case 2: $y \geq 0$

For $y \geq 0$:

- $Y \leq y \iff max(0, X) \leq y$
- This is equivalent to $X \leq y$ (since both $X \leq 0$ and $0 < X \leq y$ satisfy the condition)
- Therefore: $P(Y \leq y) = P(X \leq y) = F_X(y)$

Result:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ F_X(y) & \text{if } y \geq 0 \end{cases}$$

---

b) Find: The CDF and PDF of Y = -X

CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(-X \leq y)| \cdot -1$$

$$P(-X \leq y) = P(X \geq -y)$$

$$P(X \geq -y) = 1 - P(X < -y)$$

Since X is continuous,

$$P(X < -y) = P(X \leq -y) = F_X(-y)$$

$$F_Y(y) = 1 - F_X(-y)$$

PDF of Y: We obtain the PDF by differentiating the CDF with respect to y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}[1 - F_X(-y)]$$

Using the chain rule:

$$f_Y(y) = -F_X'(-y) \cdot (-1) = F_X'(-y) = f_X(-y)$$

6.

~~Exakt zu einer Gruppe~~

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = 0 \int_{-\infty}^{a} x\, dx + \int_a^b x \cdot \frac{1}{b-a}\, dx + \int_b^{\infty} x \cdot 0\, dx$$

$$\overset{*}{=} \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \cdot \left[\frac{1}{2} x^2 \right]_a^b = \left[\frac{1}{2} b^2 - \frac{1}{2} a^2\right] \cdot \frac{1}{b-a}$$

$$= \frac{1}{2} \frac{b^2-a^2}{b-a} = \frac{1}{2} \frac{(b+a)(b-a)}{(b-a)} = \frac{1}{2} (b+a) \quad \square$$

$$Var(X) = E[X^2] - E[X]^2 \overset{*}{=} \int_a^b x^2 \frac{1}{b-a}\, dx - \left(\frac{b+a}{2}\right)^2$$

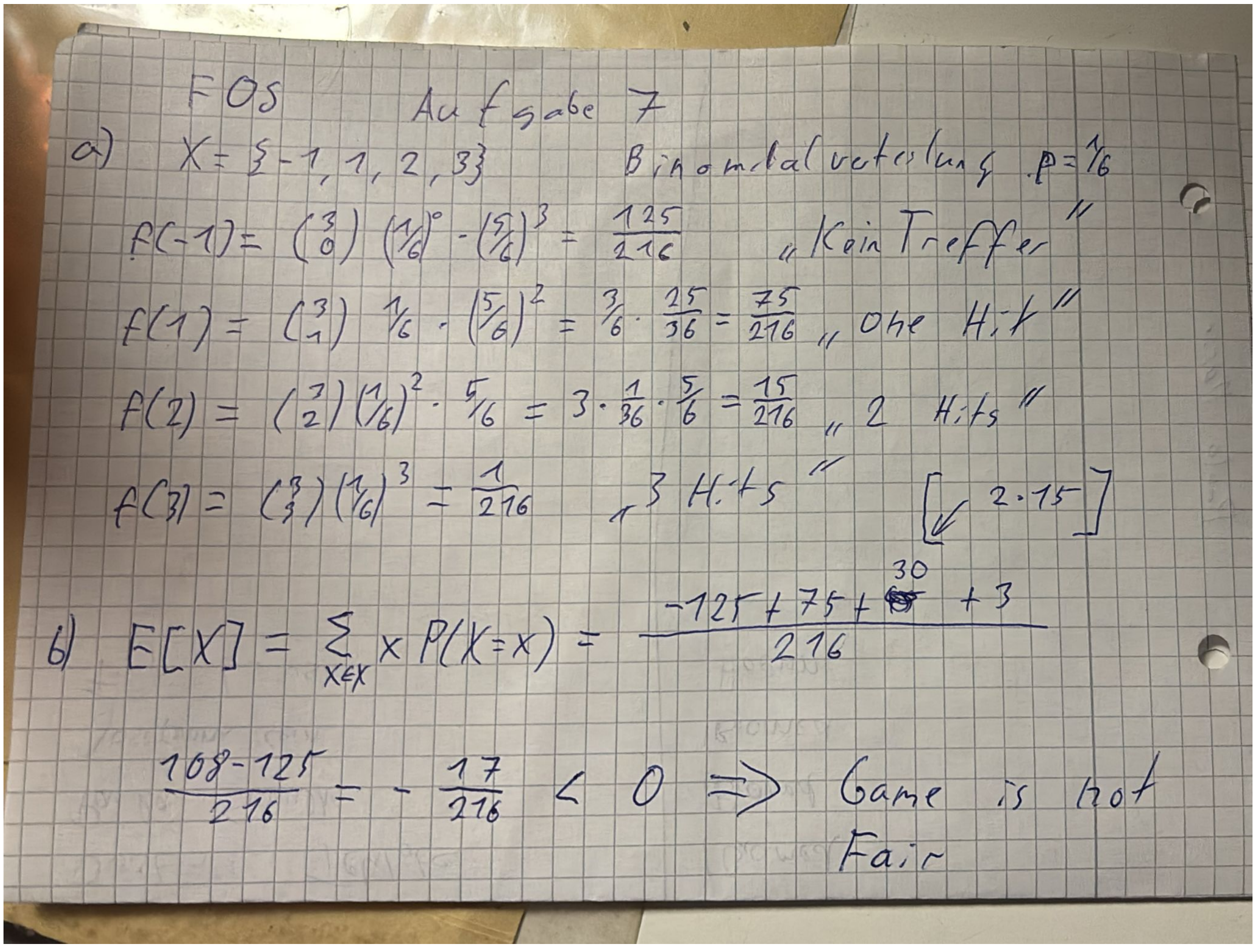
$$= \frac{1}{3}(b^3 - a^3) \frac{1}{b-a} - \frac{b^2+2ab+a^2}{4} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4b^2-3b^2+4ab-6ab+4a^2-3a^2}{12} = \frac{b^2-2ab\ a^2}{12} = \frac{(b-a)^2}{12}$$

$\square$

Part III

7.



FOS Aufgabe 7

a) $X = \{-1, 1, 2, 3\}$ Binomialverteilung $p = 1/6$

$f(-1) = \binom{3}{0}(1/6)^0 \cdot (5/6)^3 = \frac{125}{216}$ „Kein Treffer"

$f(1) = \binom{3}{1} 1/6 \cdot (5/6)^2 = 3/6 \cdot \frac{25}{36} = \frac{75}{216}$ „One Hit"

$f(2) = \binom{3}{2}(1/6)^2 \cdot 5/6 = 3 \cdot \frac{1}{36} \cdot \frac{5}{6} = \frac{15}{216}$ „2 Hits"

$f(3) = \binom{3}{3}(1/6)^3 = \frac{1}{216}$ „3 Hits"

[2·15]

b) $E[X] = \sum_{x \in X} x\, P(X=x) = \frac{-125 + 75 + \overset{30}{\cancel{15}} + 3}{216}$

$\frac{108-125}{216} = -\frac{17}{216} < 0 \Rightarrow$ Game is not Fair

a and b)

c) 10000 rounds

```
nloop<-10000
a<-5
Win<-rep(NA,nloop)
for (k in 1:nloop){
Dice<-sample(1:6,size=3,replace=TRUE)
Count_a<-sum(Dice==a)
Win[k]<-ifelse(Count_a==0,-1,Count_a)
}
sum(Win) ## overall
```

```
## [1] -636
```

```
avg <- sum(Win)/nloop ## on average per game
cat("For 10000 rounds we get an average profit of", avg, "compared to the theoretical value of", -17/21
```

```
## For 10000 rounds we get an average profit of -0.0636 compared to the theoretical value of -0.0787037
```

100000 rounds

```
nloop<-100000
a<-5
Win<-rep(NA,nloop)
for (k in 1:nloop){
Dice<-sample(1:6,size=3,replace=TRUE)
Count_a<-sum(Dice==a)
Win[k]<-ifelse(Count_a==0,-1,Count_a)
}
sum(Win) ## overall
```

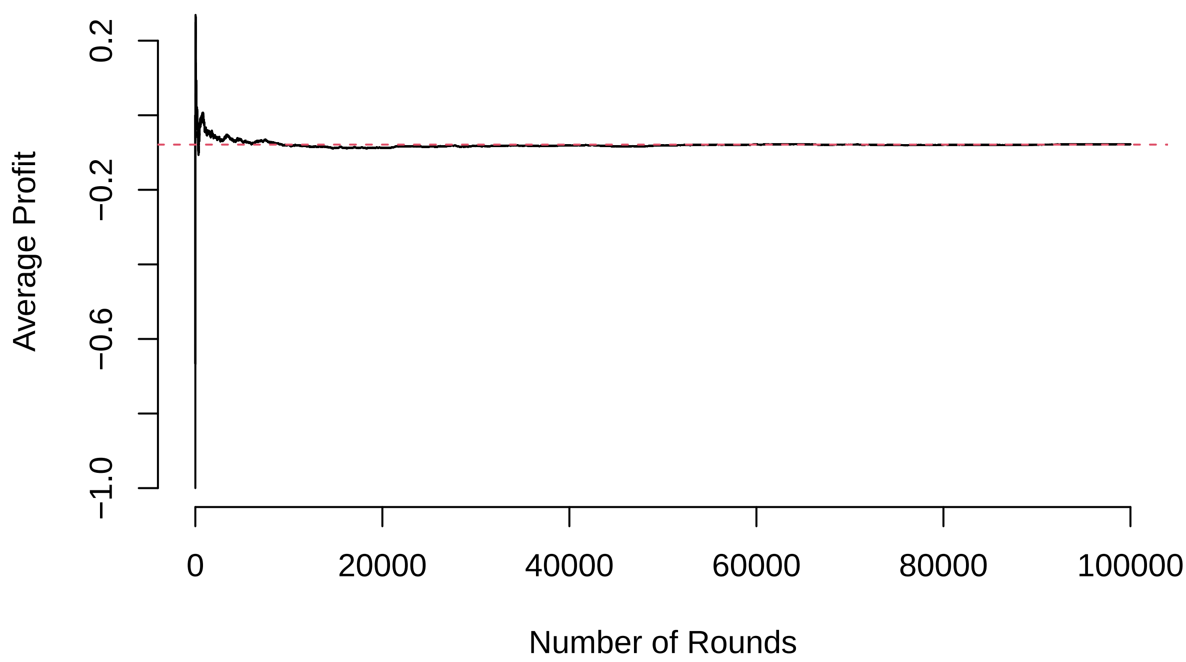
```
## [1] -7778
```

```
avg <- sum(Win)/nloop ## on average per game
cat("For 100000 rounds we get an average profit of", avg, "compared to the theoretical value of", -17/2
```

```
## For 100000 rounds we get an average profit of -0.07778 compared to the theoretical value of -0.07870
```

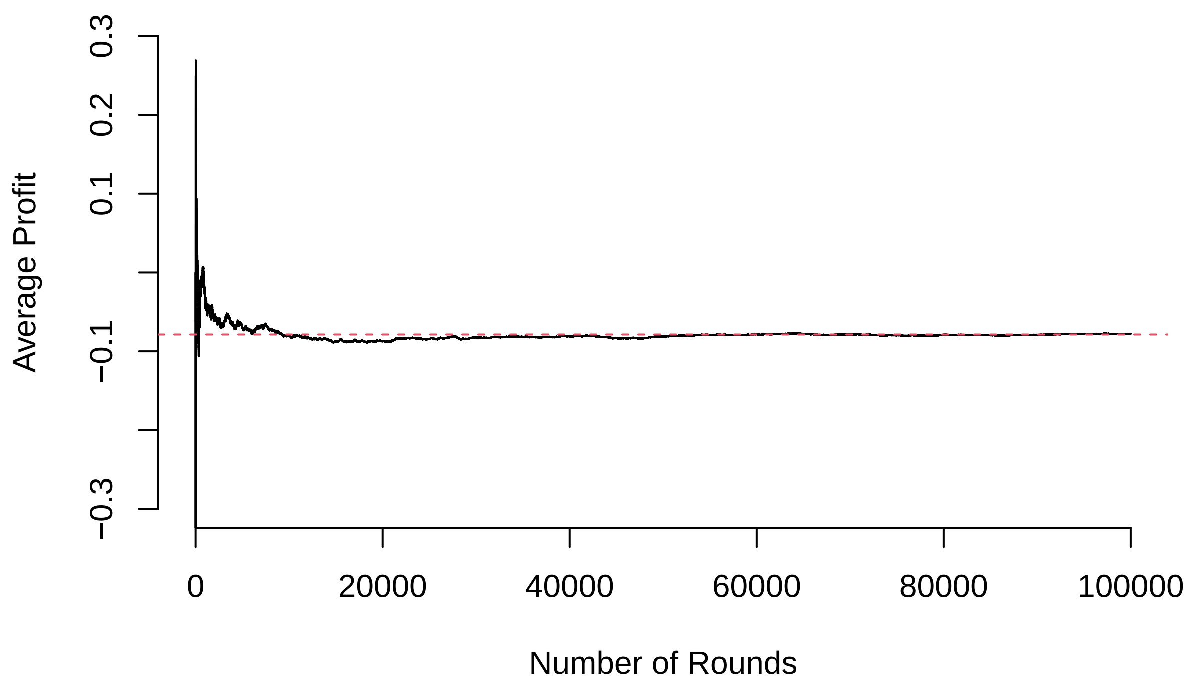
d) Code given:

```
options(scipen=999)
plot(cumsum(Win)/(1:nloop),type="l",bty="n",
ylab="Average Profit",xlab="Number of Rounds")
abline(h=-17/216,col=2,lty=2)
```



Our Code:

```
options(scipen=999)
plot(cumsum(Win)/(1:nloop),type="l",bty="n",
     ylab="Average Profit",xlab="Number of Rounds",
     ylim = c(-0.3, 0.3))
abline(h=-17/216,col=2,lty=2)
```



I changed the y axis so that it only goes from -0.3 to 0.3 because really quick it got around the expactation and this way we see the process in more detail.

From the plot, it is evident that the average profit converges to the theoretical expected value (keyword: law of large numbers).