FoS_HW_03_Group_II

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Part I

1.

a) $S = \{$two friends choose the same number$\}$ $\quad S^c = \{$every friend chooses a different number$\}$

$\mathbb{P}(S) = 1 - \mathbb{P}(S^c)$

There are $5^4 = 640$ possibilities on how the numbers are choosen.

There are $5 \cdot 4 \cdot 3 \cdot 2 = 120$ possible distribution that every friend has a different number

The first person is not restricted, the second has only 4 options to still have distinct numbers and so on.

Therefore $\mathbb{P}(S^c) = \frac{5!}{5^4} = \frac{120}{625} = \frac{24}{125} = \underline{\underline{0,192}}$ and $\mathbb{P}(S) = 1 - 0,192 = 0,808$

a)

```r
set.seed(420)
n <- 1000
count <- 0

for (i in 1:n) {
  picks <- sample(1:5, 4, replace = TRUE)
  if (length(unique(picks)) < 4) count <- count + 1
}

estimate <- count / n
print(estimate)
```

b)

```
## [1] 0.824
```

2.

Possible family compositions:

- $P(first \quad boy) = 0.5$
- $P(first \quad girl) = 0.5$
- $P(girl \cap boy) = 0.5 \cdot 0.5 = 0.25$
- $P(girl \cap girl) = 0.25$

Having;

- $P(boy) = P(first \quad boy \cup girl \cap boy) = 0.5 + 0.25 = 0.75$
- $P(girl) = P(first \quad girl \cup girl \cap girl) = 0.5 + 0.25 = 0.75$

Ratio

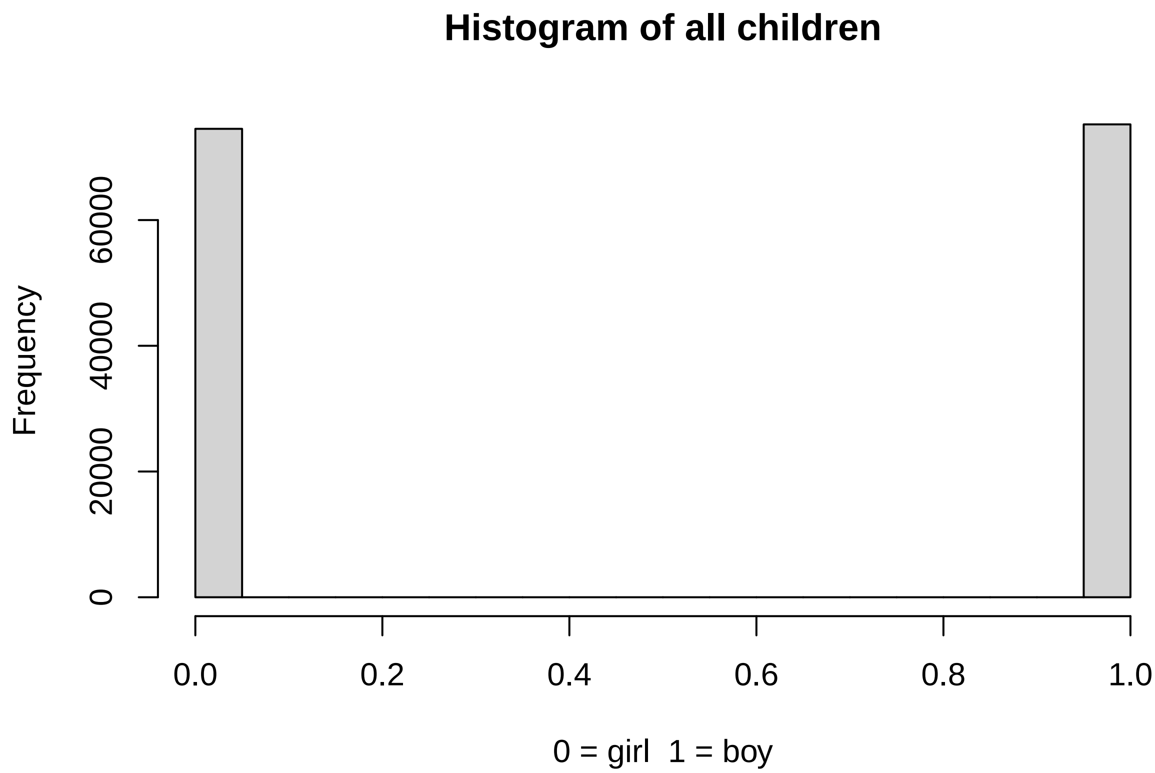
is 1:1

```r
sim <- function() {
  first_child <- sample(c(1, 0), 1, prob = c(0.5,0.5))  # 1=boy

  if (first_child == 1) {
    return(1)
  } else {
    second_child <- sample(c(1, 0), 1, prob = c(0.5,0.5))
    return(c(0, second_child))
  }
}

allc <- c()
for (i in 1:100000) {
  allc <- c(allc, sim())
}

hist(allc, main = "Histogram of all children", xlab = "0 = girl  1 = boy")
```




```r
sum(allc)/length(allc)
```

% of boys is;

```
## [1] 0.502364
```

Part II

3.

3)
$$\sum_{n\geq 1}^{\infty} \mathbb{P}(X \geq n) = \sum_{n\geq 1}^{\infty} \mathbb{P}(X=n \vee X=n+1 \vee \dots)$$

$$= \sum_{n\geq 1}^{\infty} \mathbb{P}(X=n) \vee \mathbb{P}(X=n+1) \vee \dots)$$

$$= \sum_{n\geq 1}^{\infty} \sum_{i=n}^{\infty} \mathbb{P}(X=i)$$

$$= \sum_{n=1}^{\infty} n \cdot \mathbb{P}(X=i) = \mathbb{E}[X] \quad \blacksquare$$

4.

a) i) There are two cases that can occur:

Case 1: A occurs $\Rightarrow \mathbb{1}_A$ evaluates to 1

$\mathbb{1}_A^2 = \mathbb{1}_A \cdot \mathbb{1}_A = 1 \cdot 1 = 1 = \mathbb{1}_A$

Case 2: A does not occur $\Rightarrow \mathbb{1}_A$ evaluates to 0

$\mathbb{1}_A^2 = \mathbb{1}_A \cdot \mathbb{1}_A = 0 \cdot 0 = \mathbb{1}_A$

$\Rightarrow \mathbb{1}_A^2 = \mathbb{1}_A$

ii) We again make a case distinction

Case 1: A occurs, B not

$\mathbb{1}_{A \cap B} = 0 = 1 \cdot 0 = \mathbb{1}_A \cdot \mathbb{1}_B$

since B doesn't occur

Case 2: A occurs, B occurs

$\mathbb{1}_{A \cap B} = 1 = 1 \cdot 1 = \mathbb{1}_A \cdot \mathbb{1}_B$

Case 3: A not, B occurs

$\mathbb{1}_{A \cap B} = 0 = 1 \cdot 0 = \mathbb{1}_A \cdot \mathbb{1}_B$

since A doesn't occur

Case 4: A not, B not

$\mathbb{1}_{A \cap B} = 0 = 0 \cdot 0 = \mathbb{1}_A \cdot \mathbb{1}_B$

$\Rightarrow \mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$

iii) We again make a case distinction

Case 1: A occurs, B not

$\mathbb{1}_{A \cup B} = 1 = 1 + 0 - 1 \cdot 0 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$

Case 2: A occurs, B occurs

$\mathbb{1}_{A \cup B} = 1 = 1 + 1 - 1 \cdot 1 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$

Case 3: A not, B occurs

$\mathbb{1}_{A \cup B} = 1 = 0 + 1 - 0 \cdot 1 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$

Case 4: A not, B not

$\mathbb{1}_{A \cup B} = 0 = 0 + 0 - 0 \cdot 0 = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$

$\Rightarrow \mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$

a)

b) $\mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(\text{"A occurs"}) = p$

$\mathbb{P}(\mathbb{1}_A = 0) = \mathbb{P}(\text{"A does not occur"}) = 1 - \mathbb{P}(\text{"A occurs"}) = 1 - p =: q$

$\Big\}\ \mathbb{1}_A \sim Ber(p)$

$$F_{\mathbb{1}_A}(x) = \mathbb{P}(\mathbb{1}_A \leq x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

c) $\mathbb{E}[\mathbb{1}_A] = \sum_{i \geq 1} x_i \, \mathbb{P}(X = x_i) = 1 \cdot p + 0 \, (1-p) = p = \mathbb{P}(A)$

d) $$\text{CDF for } \mathcal{U}(0,1) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$\mathbb{P}(X = 1) = \mathbb{P}(\mathcal{U}(\omega) < p) = F_{\mathcal{U}}(p) = p$

$\mathbb{P}(X = 0) = \mathbb{P}(\mathcal{U}(\omega) \geq p) = 1 - \mathbb{P}(\mathcal{U}(\omega) < p) = 1 - F_{\mathcal{U}}(p) = 1 - p = q$

$\Big\}$ $X \sim Ber(p)$ and it allows the representation $X = \mathbb{1}_A$

b), c), d)

5.

a) Find: The CDF of $Y = X^+ = max(0, X)$

Case 1: $y < 0$

Since $Y = max(0, X) \geq 0 \ \forall x$, we have $P(Y \leq y) = 0$ for $y < 0$.

Case 2: $y \geq 0$

For $y \geq 0$:

- $Y \leq y \iff max(0, X) \leq y$
- This is equivalent to $X \leq y$ (since both $X \leq 0$ and $0 < X \leq y$ satisfy the condition)
- Therefore: $P(Y \leq y) = P(X \leq y) = F_X(y)$

Result:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ F_X(y) & \text{if } y \geq 0 \end{cases}$$

---

b) Find: The CDF and PDF of Y = -X

CDF of Y:

$$F_Y(y) = P(Y \leq y) = P(-X \leq y)| \cdot -1$$

$$P(-X \leq y) = P(X \geq -y)$$

$$P(X \geq -y) = 1 - P(X < -y)$$

Since X is continuous,

$$P(X < -y) = P(X \leq -y) = F_X(-y)$$

$$F_Y(y) = 1 - F_X(-y)$$

PDF of Y: We obtain the PDF by differentiating the CDF with respect to y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}[1 - F_X(-y)]$$

Using the chain rule:

$$f_Y(y) = -F'_X(-y) \cdot (-1) = F'_X(-y) = f_X(-y)$$

6.

~~Exakt zu einer Gruppe~~

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = 0 \int_{-\infty}^{a} x\, dx + \int_{a}^{b} x \cdot \frac{1}{b-a}\, dx + \int_{b}^{\infty} x \cdot 0\, dx$$

$$\overset{*}{=} \int_{a}^{b} x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \cdot \left[\tfrac{1}{2} x^2 \right]_a^b = \left[\tfrac{1}{2} b^2 - \tfrac{1}{2} a^2 \right] \cdot \frac{1}{b-a}$$

$$= \tfrac{1}{2} \frac{b^2 - a^2}{b-a} = \tfrac{1}{2} \frac{(b+a)(b-a)}{(b-a)} = \tfrac{1}{2} (b+a) \quad \square$$

$$Var(X) = E[X^2] - E[X]^2 \overset{*}{=} \int_{a}^{b} x^2 \frac{1}{b-a}\, dx - \left(\frac{b+a}{2} \right)^2$$

$$= \tfrac{1}{3} (b^3 - a^3) \frac{1}{b-a} - \frac{b^2 + 2ab + a^2}{4} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} - \frac{b^2 + 2ab + a^2}{4}$$

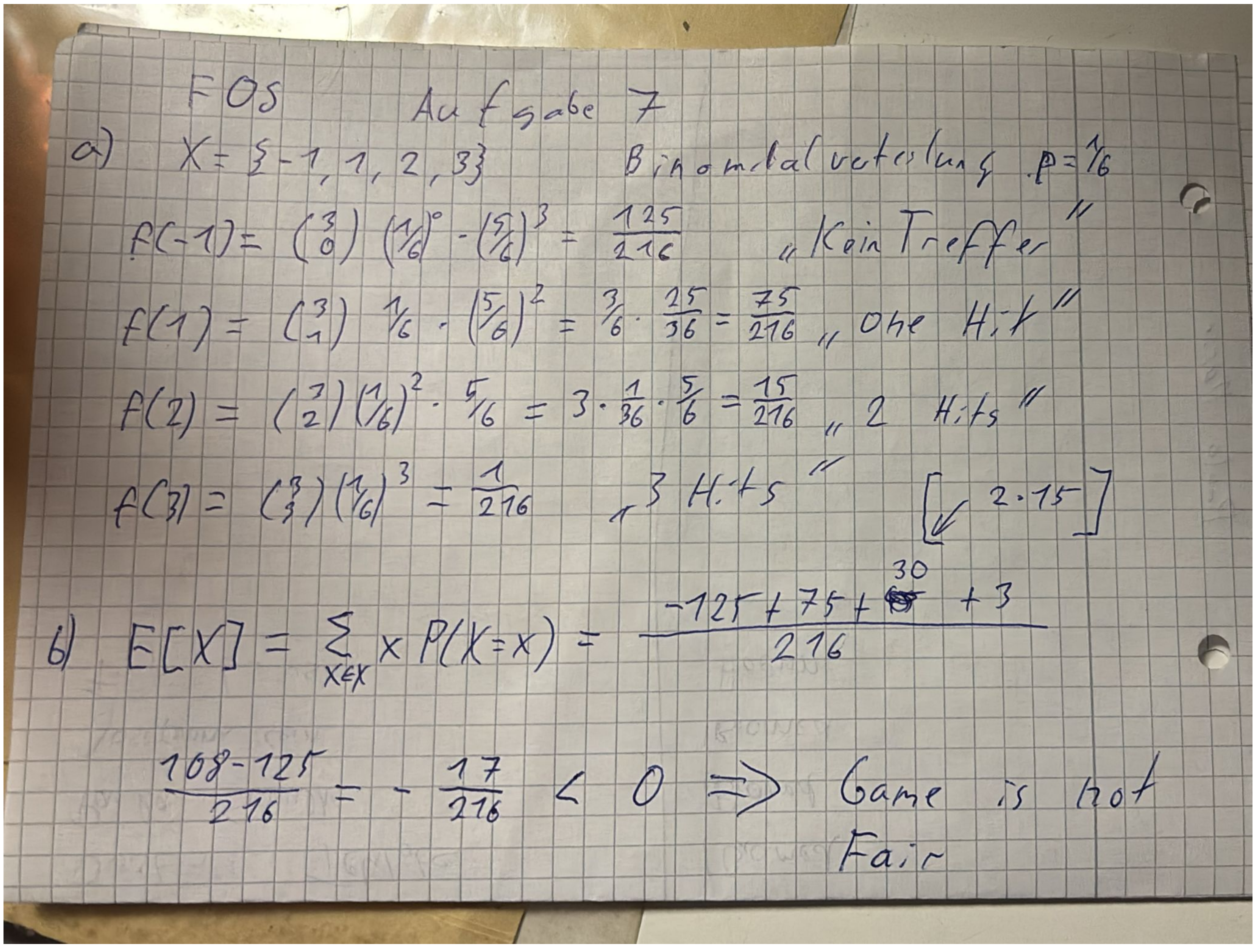
$$= \frac{4b^2 - 3b^2 + 4ab - 6ab + 4a^2 - 3a^2}{12} = \frac{b^2 - 2ab\ a^2}{12} = \frac{(b-a)^2}{12}$$

$$\square$$

Part III

7.

a and b)



FOS Aufgabe 7

a) $X = \{-1, 1, 2, 3\}$ Binomialverteilung $p = 1/6$

$f(-1) = \binom{3}{0}(1/6)^0 \cdot (5/6)^3 = \frac{125}{216}$ „Kein Treffer"

$f(1) = \binom{3}{1} 1/6 \cdot (5/6)^2 = 3/6 \cdot \frac{25}{36} = \frac{75}{216}$ „One Hit"

$f(2) = \binom{3}{2}(1/6)^2 \cdot 5/6 = 3 \cdot \frac{1}{36} \cdot \frac{5}{6} = \frac{15}{216}$ „2 Hits"

$f(3) = \binom{3}{3}(1/6)^3 = \frac{1}{216}$ „3 Hits"

$[2 \cdot 15]$

b) $E[X] = \sum_{x \in X} x P(X=x) = \frac{-125+75+30+3}{216}$

$\frac{108-125}{216} = -\frac{17}{216} < 0 \Rightarrow$ Game is not Fair

```r
nloop<-10000
a<-5
Win<-rep(NA,nloop)
for (k in 1:nloop){
Dice<-sample(1:6,size=3,replace=TRUE)
Count_a<-sum(Dice==a)
Win[k]<-ifelse(Count_a==0,-1,Count_a)
}
sum(Win) ## overall
```

c)

```
## [1] -636
```

```r
sum(Win)/nloop ## on average per game
```

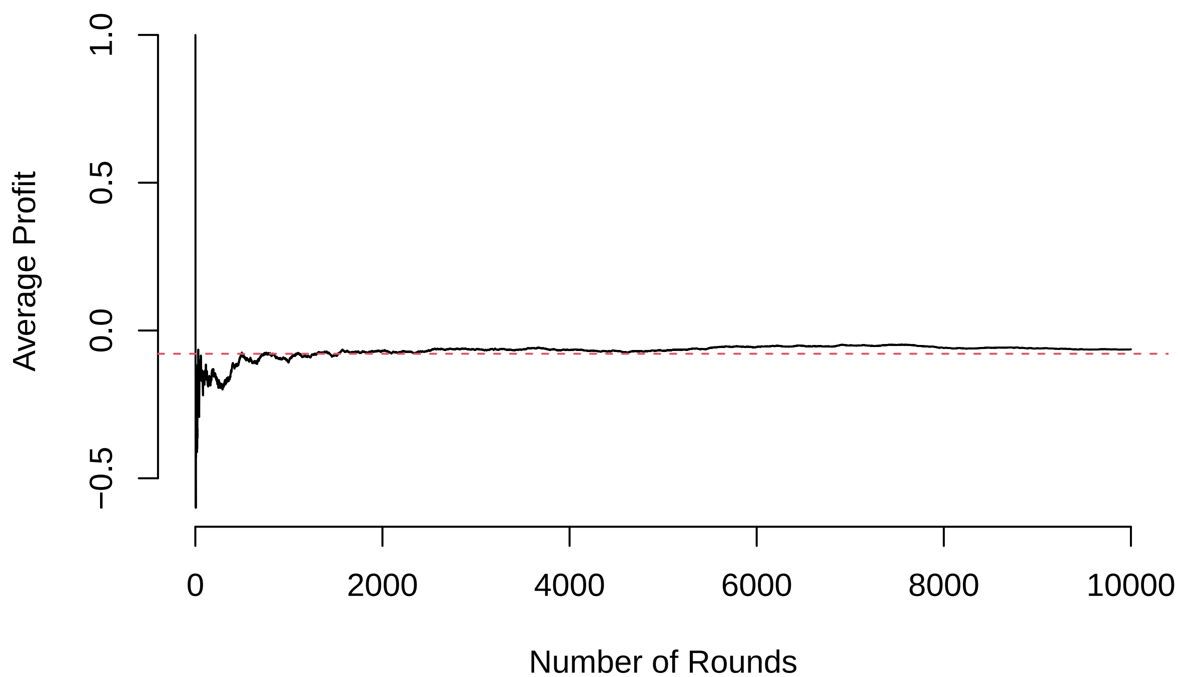
```
## [1] -0.0636
```

```
# outcome for nloop <- 10000:
#sum(Win) = -759
#sum(Win)/nloop = -0.0759  (on average per game)

# outcome for nloop <- 100000:
#sum(Win) = -8007
#sum(Win)/nloop = -0.08007  (on average per game)
```

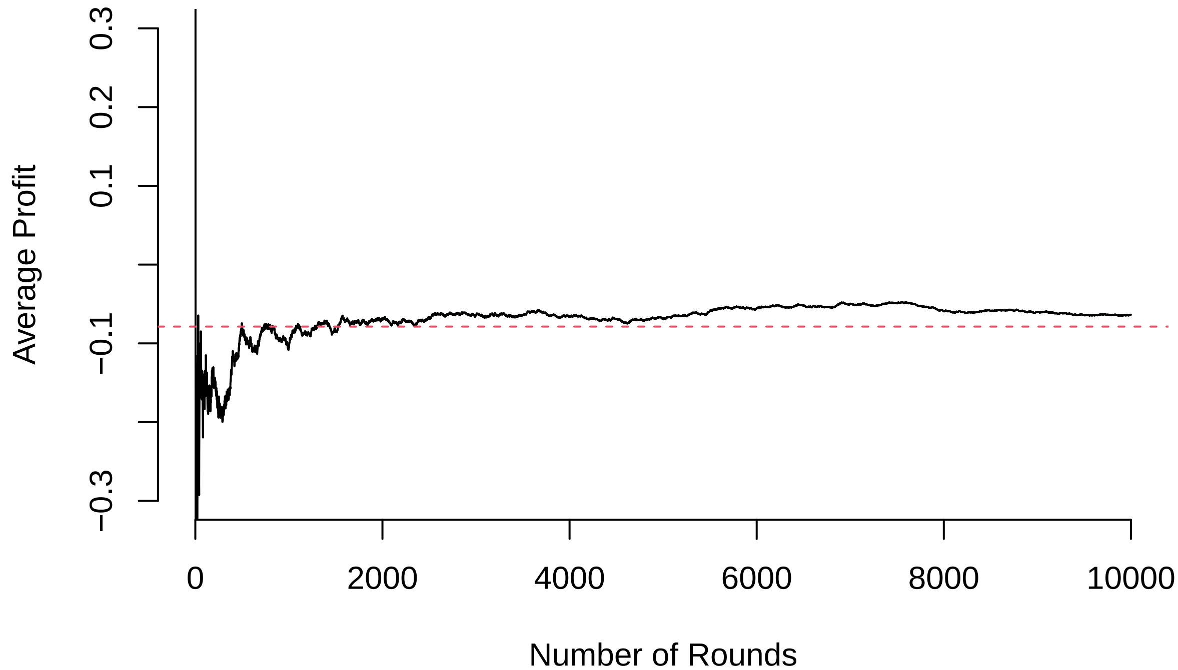
d) Code given:

```
options(scipen=999)
plot(cumsum(Win)/(1:nloop),type="l",bty="n",
ylab="Average Profit",xlab="Number of Rounds")
abline(h=-17/216,col=2,lty=2)
```



Our Code:

```
options(scipen=999)
plot(cumsum(Win)/(1:nloop),type="l",bty="n",
     ylab="Average Profit",xlab="Number of Rounds",
     ylim = c(-0.3, 0.3))
abline(h=-17/216,col=2,lty=2)
```



I changed the y axis so that it only goes from -0.3 to 0.3 because really quick it got around the expactation and this way we see the process in more detail