1.

Nr. 1 a)

Case 1 g strictly monoton increasing

$$F_Y(y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

deravativ gives PDF [Change rule]

$$\Rightarrow \quad f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) \overset{h=g^{-1}}{=} f_X(\overset{h}{\cancel{g^{-1}}}(y)) \cdot h'(y) = f_X(h(y)) \cdot |h'(y)|$$

g strictly monotones $\Rightarrow g' \geq 0 \Rightarrow h'(y) \geq 0$

case 2 g strictly ~~streng~~ monoton decreasing

$$F_Y(y) = P(g(X) \le y) \overset{h \text{ decresing}}{=} P(X \ge h(y)) = 1 - F_X(h(y))$$

deravativ

$$\Rightarrow \quad f_Y(x) = - f_X(h(y)) \cdot h'(y) \overset{h' \le 0}{=} f_X(h(y)) \cdot |h'(y)|$$

$\square$

b) Optional

2.

a) Given: X is a continuous random variable with CDF F_X and PDF f_X. We consider the linear transformation $Y := rX + s$ for some $r, s \in \mathbb{R}$.

CDF of Y: Starting with the CDF method:

$$F_Y(y) = P(Y \leq y) = P(rX + s \leq y)$$

Case 1: $r > 0$

$$rX + s \leq y \implies X \leq \frac{y-s}{r}$$

$$F_Y(y) = F_X\left(\frac{y-s}{r}\right)$$

Case 2: $r < 0$

$$rX + s \leq y \implies X \geq \frac{y-s}{r}$$

$$F_Y(y) = 1 - F_X\left(\frac{y-s}{r}\right)$$

Case 3: $r = 0$

$$Y = s \text{ (constant)}, \quad F_Y(y) = P(Y \leq y) = P(s \leq y) = \begin{cases} 0 & \text{if } y < s \\ 1 & \text{if } y \geq s \end{cases}$$

PDF of Y: For $r \neq 0$, we obtain the PDF by differentiating the CDF:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Using the chain rule:

- For $r > 0$: $f_Y(y) = f_X\left(\frac{y-s}{r}\right) \cdot \frac{1}{r} = \frac{1}{|r|} f_X\left(\frac{y-s}{r}\right)$
- For $r < 0$: $f_Y(y) = -f_X\left(\frac{y-s}{r}\right) \cdot \frac{1}{r} = \frac{1}{|r|} f_X\left(\frac{y-s}{r}\right)$

Answer:

$$\boxed{F_Y(y) = \begin{cases} F_X\left(\frac{y-s}{r}\right) & \text{if } r > 0 \\ 1 - F_X\left(\frac{y-s}{r}\right) & \text{if } r < 0 \\ \mathbb{1}_{y \geq s} & \text{if } r = 0 \end{cases}}$$

$$\boxed{f_Y(y) = \frac{1}{|r|} f_X\left(\frac{y-s}{r}\right) \quad \text{for } r \neq 0}$$

b) Given:

- Random variable X with PDF $f_X(x) = e^{-x}$ for $x > 0$
- Transformation $Y := g(X) = \log X$ (natural logarithm)

To verify: $f_Y(y) = e^y e^{-e^y}$ for $y \in \mathbb{R}$

Since $X > 0$:

- As $X \to 0^+$: $Y = \log X \to -\infty$
- As $X \to \infty$: $Y = \log X \to +\infty$

Therefore, the support of Y is $\mathbb{R}$ (all real numbers).

Given $y = \log x$, we solve for x:

$$x = g^{-1}(y) = e^y$$

We can verify: $\log(e^y) = y$

The function $g(x) = \log x$ is strictly increasing for $x > 0$, so:

$$g'(x) = \frac{1}{x} > 0 \quad \text{for all } x > 0$$

This confirms that g is monotone increasing, which allows us to use the transformation formula.

For a monotone transformation, the PDF transformation formula is:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy} e^y = e^y$$

Since $e^y > 0$ for all $y \in \mathbb{R}$:

$$\left| \frac{d}{dy} e^y \right| = e^y$$

$$f_Y(y) = f_X(e^y) \cdot e^y$$

We have $f_X(x) = e^{-x}$, so substituting $x = e^y$:

$$f_X(e^y) = e^{-e^y}$$

Therefore:

$$f_Y(y) = e^{-e^y} \cdot e^y = e^y e^{-e^y}$$

c) Given:

- $X \sim \text{Unif}(0,1)$, so $f_X(x) = 1$ for $x \in (0,1)$
- Transformation $Y := \sqrt{X}$

Task:

1. Find the distribution of Y
2. Verify using simulation in R
3. Compute $\mathbb{E}[Y]$ using both LOTUS and the derived density

Since $X \in (0,1)$:

- As $X \to 0^+$: $Y = \sqrt{X} \to 0$
- As $X \to 1$: $Y = \sqrt{X} \to 1$

Therefore, the support of Y is $(0,1)$.

Given $y = \sqrt{x}$, we solve for x:

$$y = \sqrt{x} \implies y^2 = x$$

So the inverse function is:

$$x = g^{-1}(y) = y^2$$

The function $g(x) = \sqrt{x}$ is strictly increasing for $x > 0$:

$$g'(x) = \frac{1}{2\sqrt{x}} > 0 \quad \text{for all } x > 0$$

This confirms monotonicity, allowing to use the transformation formula.

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy} y^2 = 2y$$

Since $y > 0$ on the support:

$$\left| \frac{d}{dy} y^2 \right| = 2y$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

Substituting:

- $f_X(y^2) = 1$ (since $y^2 \in (0,1)$ when $y \in (0,1)$)
- $\left| \frac{d}{dy} y^2 \right| = 2y$

Therefore:

$$f_Y(y) = 1 \cdot 2y = 2y \quad \text{for } y \in (0,1)$$

$$\int_0^1 2y \, dy = \left[y^2 \right]_0^1 = 1 - 0 = 1$$

Computing $\mathbb{E}[Y]$

Method 1: Using LOTUS (Law of the Unconscious Statistician)

$$\mathbb{E}[Y] = \mathbb{E}[\sqrt{X}] = \int_0^1 \sqrt{x} \cdot f_X(x)\,dx = \int_0^1 \sqrt{x} \cdot 1\,dx$$

$$= \int_0^1 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_0^1 = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3}$$

Method 2: Using the derived density $f_Y(y) = 2y$

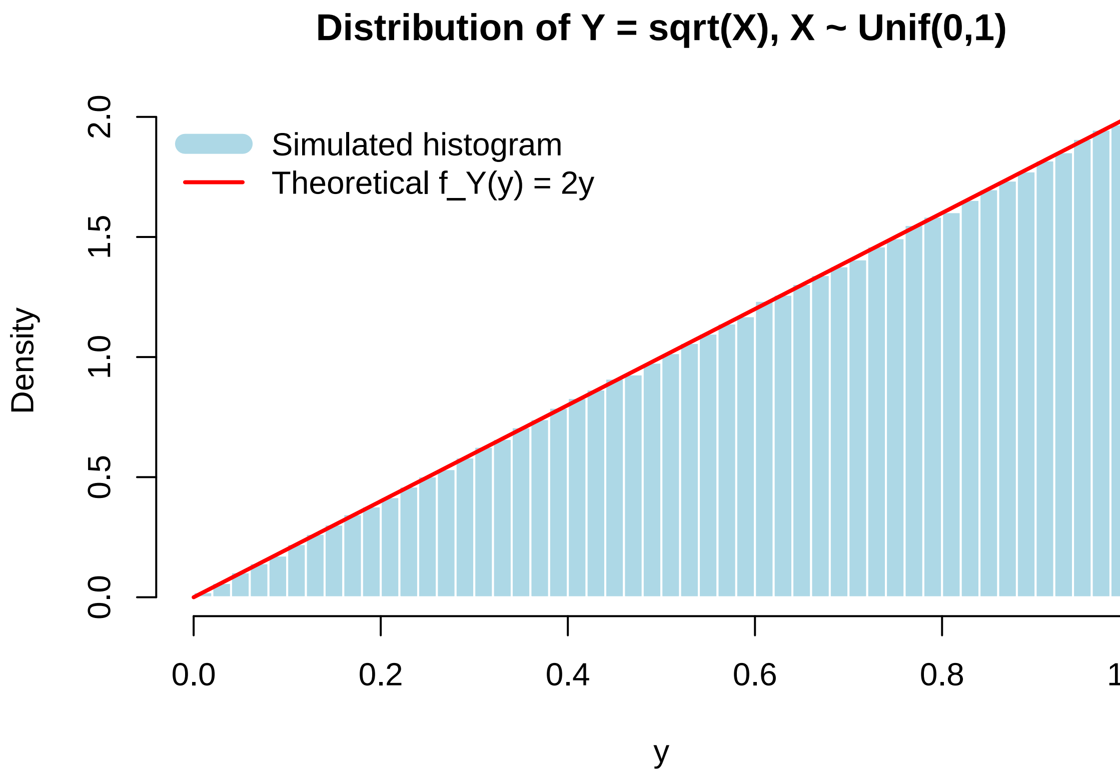
$$\mathbb{E}[Y] = \int_0^1 y \cdot f_Y(y)\,dy = \int_0^1 y \cdot 2y\,dy = \int_0^1 2y^2\,dy$$

$$= \left[\frac{2y^3}{3}\right]_0^1 = \frac{2}{3}$$

Both methods give:

$$\boxed{\mathbb{E}[Y] = \frac{2}{3}}$$

```r
set.seed(123)

# samples
n <- 10^6
X <- runif(n, 0, 1)
Y <- sqrt(X)

# histogram with theoretical density overlay
hist(Y, freq = FALSE, breaks = 50,
     main = "Distribution of Y = sqrt(X), X ~ Unif(0,1)",
     xlab = "y", ylab = "Density",
     col = "lightblue", border = "white")

# theoretical density f_Y(y) = 2y
y_vals <- seq(0, 1, length.out = 1000)
f_Y <- 2 * y_vals
lines(y_vals, f_Y, col = "red", lwd = 2)
legend("topleft",
       legend = c("Simulated histogram", "Theoretical f_Y(y) = 2y"),
       col = c("lightblue", "red"),
       lwd = c(10, 2),
       bty = "n")
```



R Code for Simulation

```
E_Y_lotus <- mean(Y)
cat("E[Y] using simulation (LOTUS):", E_Y_lotus, "\n")
```

```
## E[Y] using simulation (LOTUS): 0.6662478
```

The simulation closely matches the theoretical value of $\frac{2}{3} \approx 0.6667$, confirming the derivation.

3.

Nr.3 a) $g(x,y) = \begin{pmatrix} x+y \\ x \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$g^{-1}\begin{pmatrix} u \\ v \end{pmatrix} \overset{!}{=} \begin{pmatrix} x \\ y \end{pmatrix}$ $\cancel{x=v}$ $\Rightarrow \begin{pmatrix} v \\ u-v \end{pmatrix} = g^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$

$J_{g^{-1}}(u,v) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad \det(J_{g^{-1}}) = 0\cdot(-1) - 1^2 = -1$

$g(x,y) = \begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow f_{u,v}(u,v) = 1 \cdot f_{X,Y}\begin{pmatrix} v \\ u-v \end{pmatrix} \overset{\text{indipendent}}{=} f_X(v)\cdot f_Y(u-v)$

$\Rightarrow f_U(u) = \int_{-\infty}^{\infty} f_X(v) f_Y(u-v)\, dv$ $\quad$ same es in Exercice 4 th. $\square$

b) $U = X+Y \quad V = X-Y \quad g^{-1}(u,v) = \begin{pmatrix} (u+v)\cdot\frac{1}{2} \\ (u-v)\cdot\frac{1}{2} \end{pmatrix}$

$J = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \Rightarrow \det J = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

$f_{u,v}(u,v) = \frac{1}{2} f_{X,Y}\left(\frac{(u+v)}{2}, \left(\frac{u-v}{2}\right)\right) = \cancel{\frac{1}{2} \exp\left(-\frac{1}{2}\left[\left(\frac{u+v}{2}\right), \left(\frac{u-v}{2}\right)\right]\right)}$

$= \frac{1}{2} \exp\left(-\frac{1}{2} \left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot I \cdot \begin{pmatrix} \frac{u+v}{2} \\ \frac{u-v}{2} \end{pmatrix}\right) \cdot \frac{1}{\sqrt{(2\pi)^2}}$

$= \frac{1}{4\pi} \exp\left(-\frac{1}{2} \left(\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right)\right) = \frac{1}{4\pi} \exp\left(-\frac{1}{2}\left[\frac{1}{4}(u^2 + 2uv + v^2 + u^2 - 2uv + v^2)\right]\right)$

$= \frac{1}{4\pi} \exp\left(-\frac{1}{2}\left(\frac{1}{2}(u^2+v^2)\right)\right) = \frac{1}{4\pi} \exp\left(-\frac{1}{4}u^2\right) \cdot \exp\left(\frac{1}{4}v^2\right)$

$= \underbrace{\frac{1}{\sqrt{4\pi}} \exp\left(-\frac{1}{4}u^2\right)}_{f_u(u)} \cdot \underbrace{\frac{1}{\sqrt{4\pi}} \exp\left(-\frac{1}{4}v^2\right)}_{f_v(v)} \quad \square$

4.

Probability Integral Transform **Goal:** Show that $Y = F_X(X) \sim \text{Unif}(0,1)$

Proof: For any $y \in [0,1]$, we compute the CDF of Y:

$$P(Y \leq y) = P(F_X(X) \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

Since $P(Y \leq y) = y$ for all $y \in [0,1]$, we have $Y \sim \text{Unif}(0,1) \quad \square$.

Intuition: Applying a CDF to its own random variable "uniformizes" it.

Inverse Transform **Goal:** Show that if $U \sim \text{Unif}(0,1)$ and $Z = F_X^{-1}(U)$, then $F_Z = F_X$

Proof: For any $z \in I$:

$$P(Z \leq z) = P(F_X^{-1}(U) \leq z) = P(U \leq F_X(z)) = F_X(z)$$

The last equality holds because $U \sim \text{Unif}(0,1)$. Thus $F_Z = F_X \quad \square$.

Conclusion: We can generate any continuous distribution by applying F_X^{-1} to uniform random numbers!

Simulation in R Given density: $f(x) = \frac{2}{(x+1)^3}$ for $x > 0$

the CDF by integration:

$$F(x) = \int_0^x \frac{2}{(t+1)^3} dt = 1 - \frac{1}{(x+1)^2}$$

inverse CDF by solving $u = F(x)$ for x:

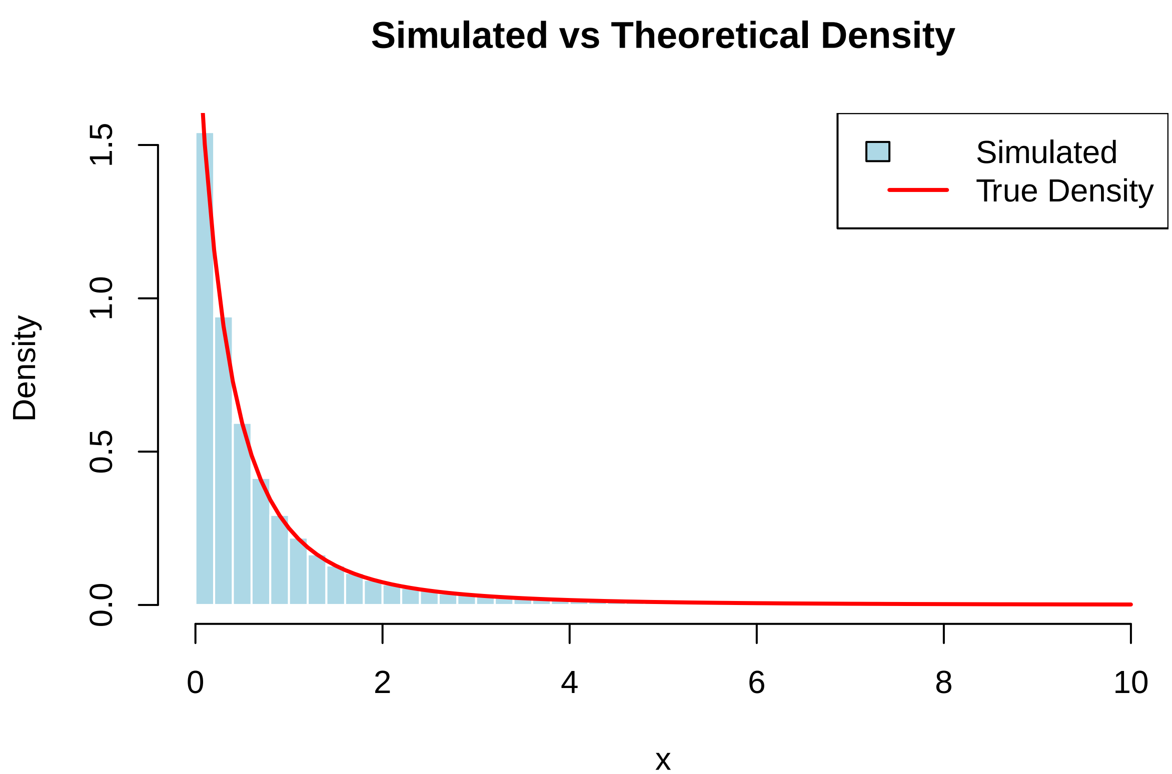
$$F^{-1}(u) = \frac{1}{\sqrt{1-u}} - 1$$

```r
set.seed(123)

# inverse CDF
F_inv <- function(u) {
  1 / sqrt(1 - u) - 1
}

# true density
f <- function(x) {
  2 / (x + 1)^3
}

# n = 10^5 samples using inverse transform
n <- 1e5
U <- runif(n)
X <- F_inv(U)
```

```r
# Filter values < 10 for better visualization (as suggested)
X_filtered <- X[X < 10]
cat("Percentage of values < 10:",
    round(100 * length(X_filtered) / n, 2), "%\n")
```

```
## Percentage of values < 10: 99.18 %
```

```r
# histogram with theoretical density overlay
hist(X_filtered, breaks = 50, probability = TRUE,
     main = "Simulated vs Theoretical Density",
     xlab = "x", col = "lightblue", border = "white")

# the true density
curve(f(x), from = 0, to = 10, add = TRUE,
      col = "red", lwd = 2)
legend("topright", legend = c("Simulated", "True Density"),
       fill = c("lightblue", NA), border = c("black", NA),
       lty = c(NA, 1), col = c(NA, "red"), lwd = c(NA, 2))
```



Verification: The histogram closely matches the theoretical density, confirming the implementation! The heavy tail explains why most values are concentrated near 0.

5 a) Show: $\mathbb{P}\left(\left|\bar{X}_n - \frac{1}{n}\sum_{i=1}^{n}\mu_i\right| > a\right) \leq \frac{\sum_{i=1}^{n}\sigma_i^2}{n^2 a^2}$

$$\mathbb{P}\left(\left|\bar{X}_n - \frac{1}{n}\sum_{i=1}^{n}\mu_i\right| > a\right) = \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i - \frac{1}{n}\sum_{i=1}^{n}\mu_i\right| > a\right)$$

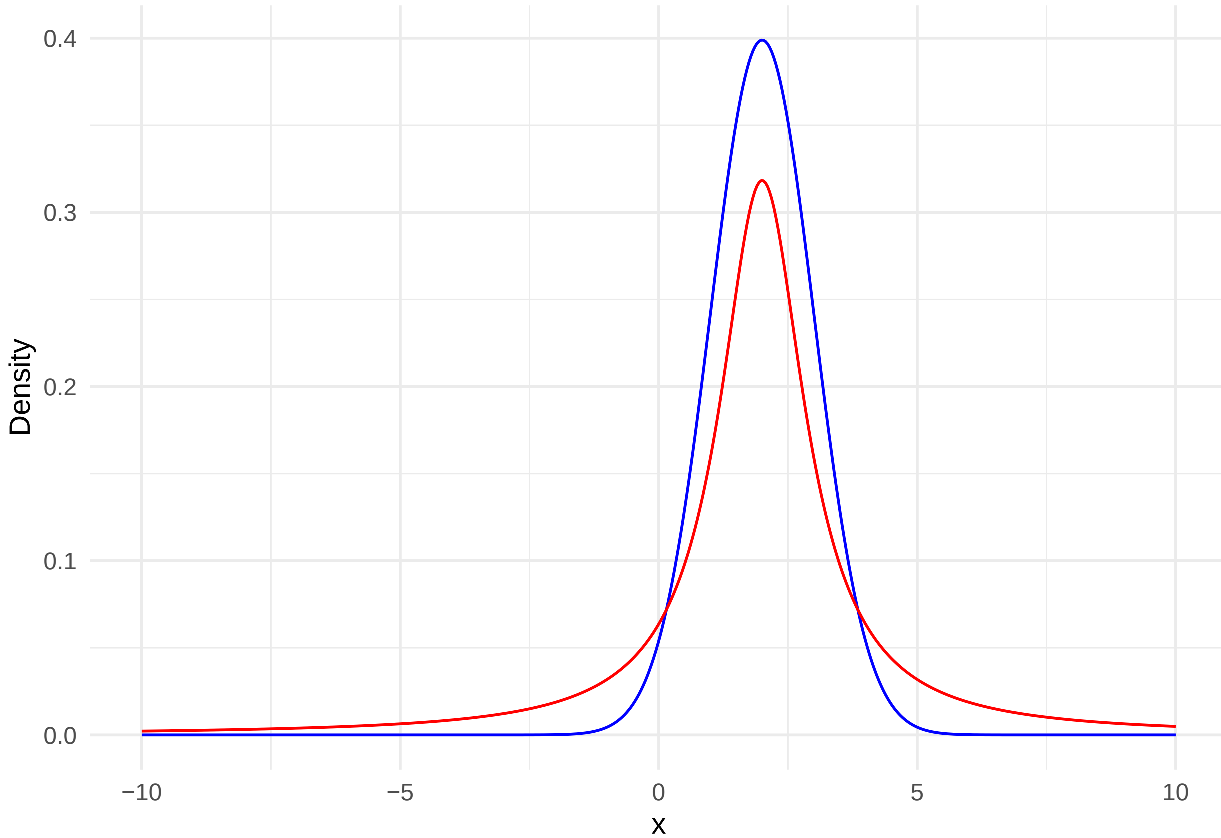
$$= \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_i)\right| > a\right) = \mathbb{P}\left(\left|\sum_{i=1}^{n}(X_i-\mu_i)\right| > an\right) \overset{*}{\leq} \frac{\sum_{i=1}^{n}\sigma_i^2}{n^2 a^2} \quad \blacksquare$$

* here we used chebyshevs inequality and the fact that the variance of the sum of independent RV's is the sum of variances.

b) $\lim_{n\to\infty} \mathbb{P}\left(\left|\bar{X}_n - \frac{1}{n}\sum_{i=1}^{n}\mu_i\right| > a\right) \leq \lim_{n\to\infty} \frac{\sum_{i=1}^{n}\sigma_i^2}{n^2 a^2} \leq \lim_{n\to\infty} \frac{n \cdot M}{n^2 a^2} \xrightarrow{n\to\infty} 0 \quad \blacksquare$

6.

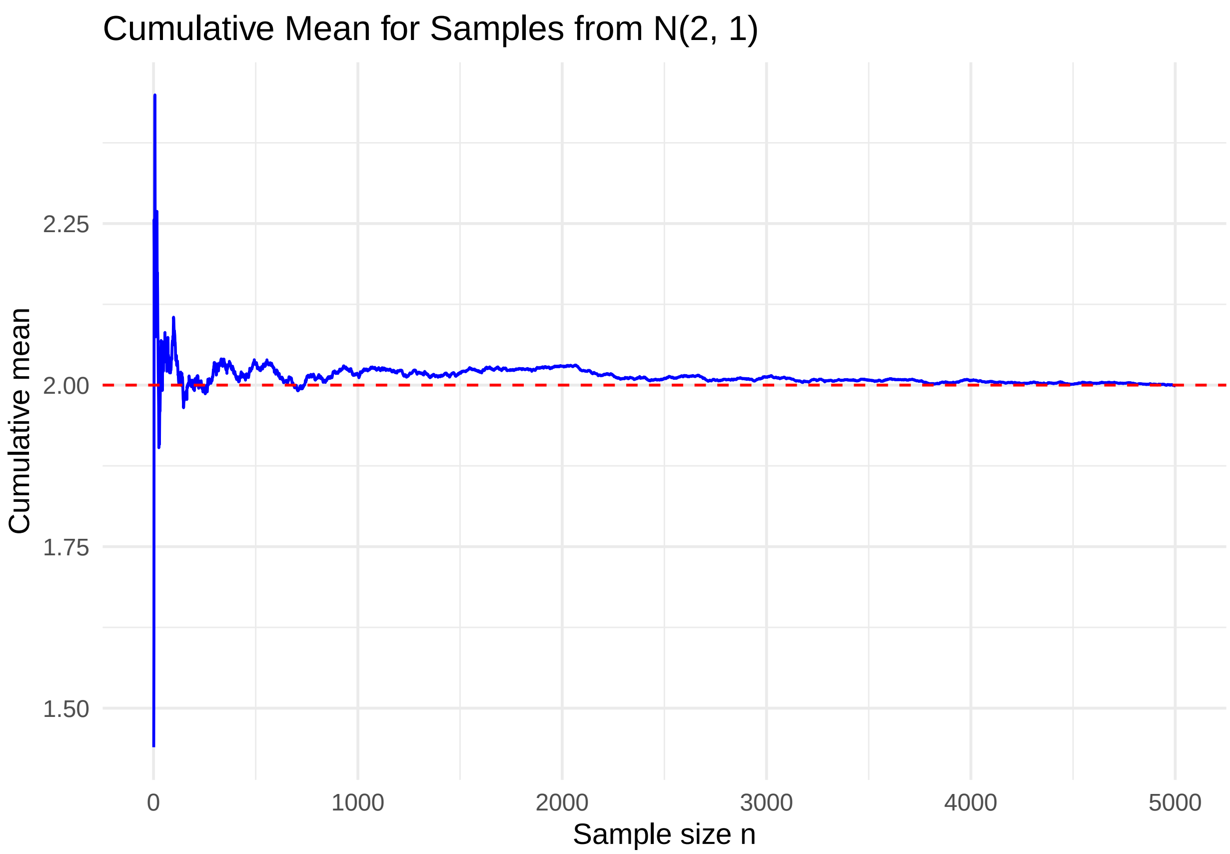
```r
library(ggplot2)

x_a <- seq(-10, 10, length.out=500)

df_a <- data.frame(
  x = x_a,
  normal = dnorm(x_a, mean = 2, sd = 1),
  cauchy = dcauchy(x_a, location = 2, scale = 1)
)

ggplot() +
  geom_line(data=df_a, aes(x=x, y=normal), color='blue') +
  geom_line(data=df_a, aes(x=x, y=cauchy), color='red') +
  labs(
    y = "Density"
  ) +
  theme_minimal()
```



a)

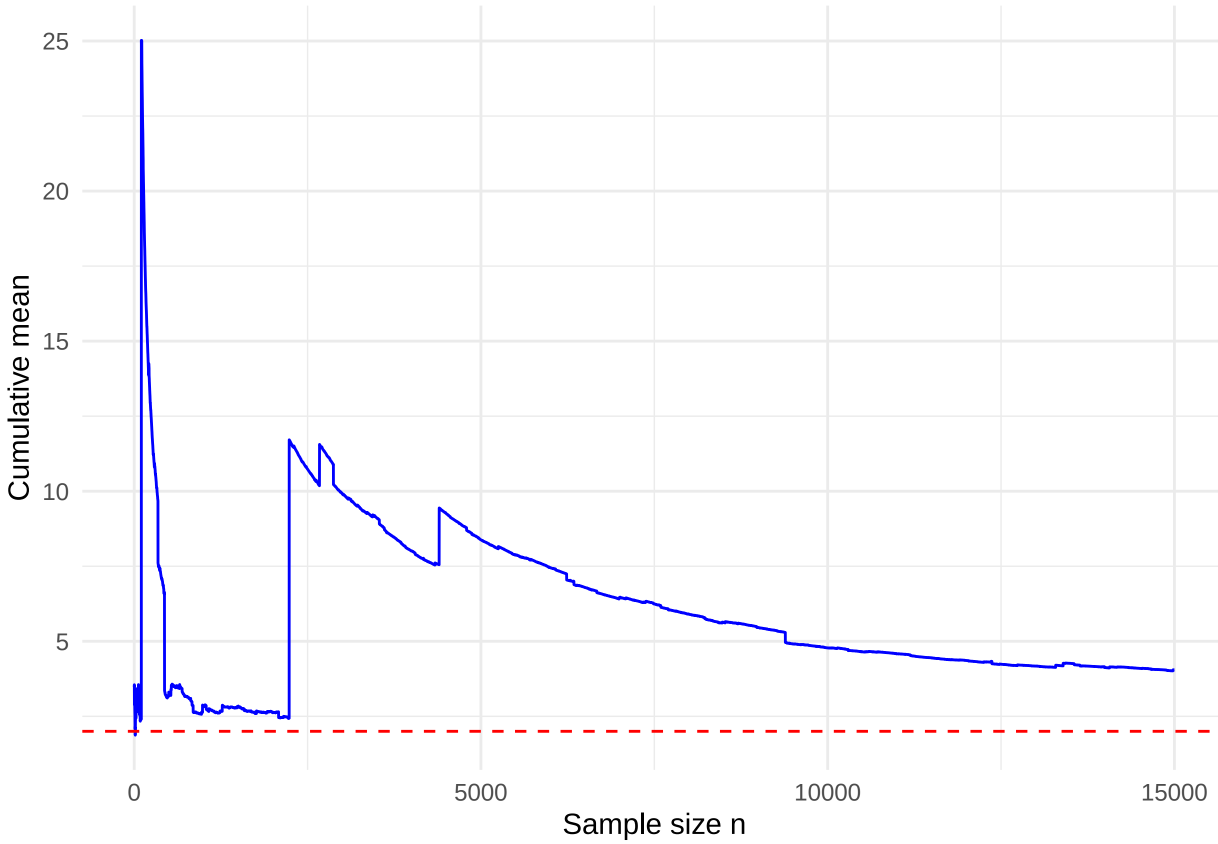
The chauchy distribution has heavier tails then the normal distribution

```
set.seed(123)

x_b <- rnorm(5000, mean = 2, sd = 1)

cum_mean <- cumsum(x_b) / seq_along(x_b)

df_b <- data.frame(
  n = seq_along(x_b),
  cum_mean = cum_mean
)

ggplot(df_b, aes(x = n, y = cum_mean)) +
  geom_line(color = "blue") +
  geom_hline(yintercept = 2, color = "red", linetype = "dashed") +
  labs(
    x = "Sample size n",
    y = "Cumulative mean",
    title = "Cumulative Mean for Samples from N(2, 1)"
  ) +
  theme_minimal()
```



b)

```r
x_c <- rcauchy(15000, location=2, scale=1)

cum_mean <- cumsum(x_c) / seq_along(x_c)

df_c <- data.frame(
  n = seq_along(x_c),
  cum_mean = cum_mean
)

ggplot(df_c, aes(x = n, y = cum_mean)) +
  geom_line(color = "blue") +
  geom_hline(yintercept = 2, color = "red", linetype = "dashed") +
  labs(
    x = "Sample size n",
    y = "Cumulative mean",
  ) +
  theme_minimal()
```



c)

We cannot observe a similar convergence. The chauchy distribution has no mean which is neccessary for R.V.'s to follow the LLN.

7.

Theory: The Central Limit Theorem Let X_i, $i \geq 1$ be i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 > 0$.

The **standardized sum** is defined as:

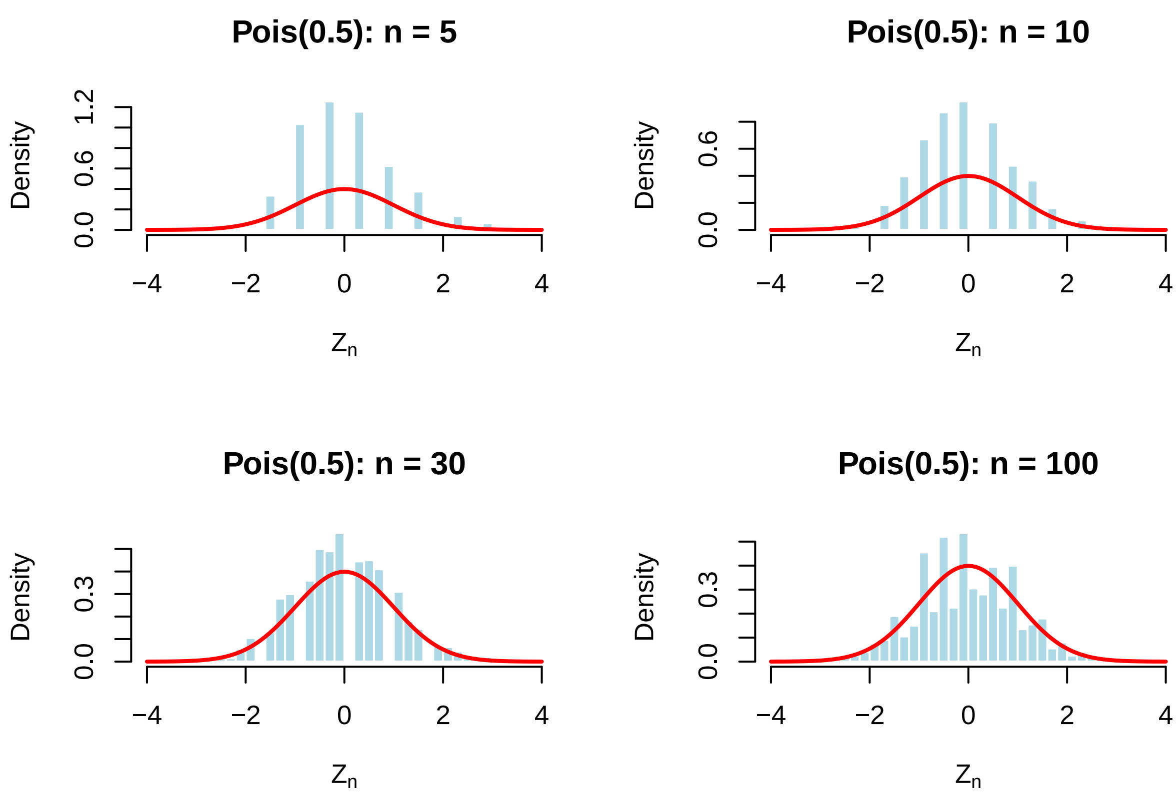
$$Z_n := \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$$

CLT Statement: As $n \to \infty$, we have $Z_n \xrightarrow{d} \mathcal{N}(0, 1)$.

Poisson(0.5) Distribution For $X_i \sim \mathrm{Pois}(0.5)$: $\mu = 0.5$, $\sigma^2 = 0.5$

```
# parameters
set.seed(42)
k <- 1000  # simulations
mu_pois <- 0.5
sigma_pois <- sqrt(0.5)

# to generate k samples of Z_n
simulate_CLT <- function(n, k, mu, sigma, dist_func) {
```

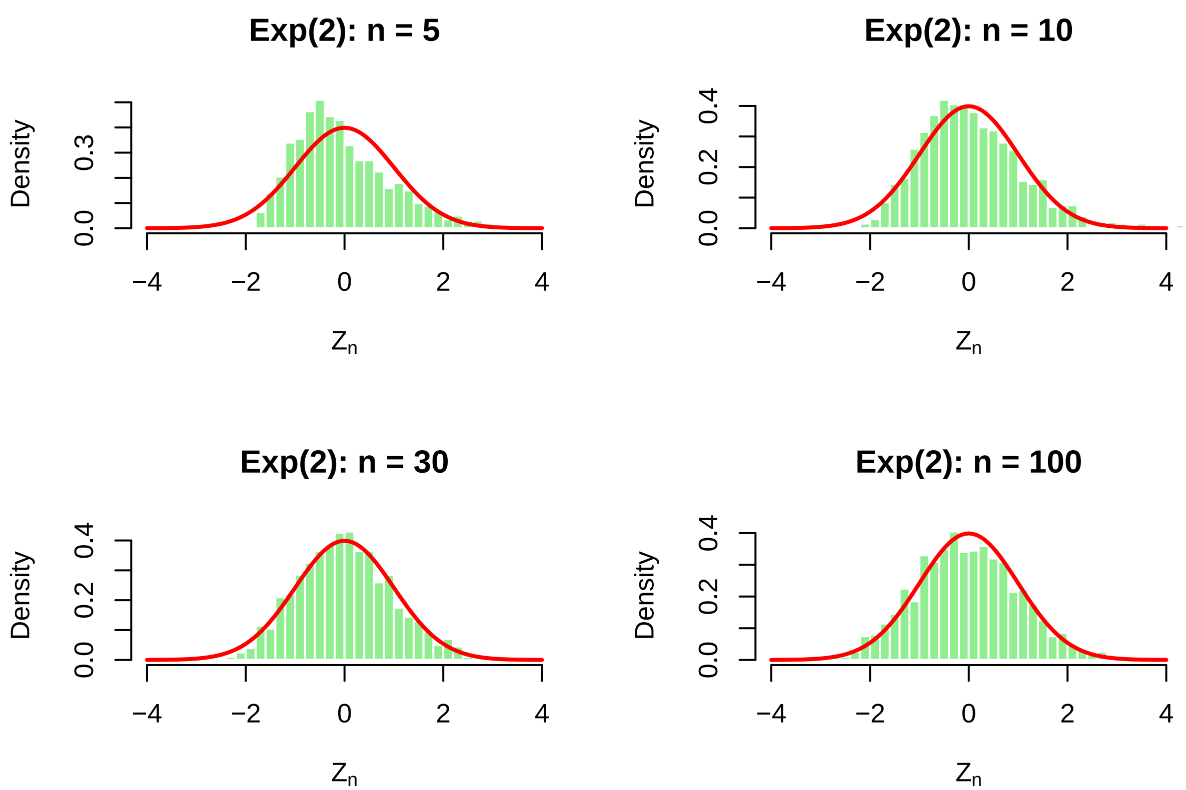
```r
  Z_samples <- numeric(k)
  for (i in 1:k) {
    X <- dist_func(n)  # Generate n i.i.d. samples
    X_bar <- mean(X)
    Z_samples[i] <- (X_bar - mu) / (sigma / sqrt(n))
  }
  return(Z_samples)
}

n_values <- c(5, 10, 30, 100)

par(mfrow = c(2, 2))
for (n in n_values) {
  Z_n <- simulate_CLT(n, k, mu_pois, sigma_pois,
                      function(n) rpois(n, lambda = 0.5))

  hist(Z_n, breaks = 30, probability = TRUE,
       main = paste("Pois(0.5): n =", n),
       xlab = expression(Z[n]), col = "lightblue",
       border = "white", xlim = c(-4, 4))

  # standard normal density
  curve(dnorm(x, 0, 1), from = -4, to = 4,
        add = TRUE, col = "red", lwd = 2)
}
```



Observation: As n increases, the histogram converges to $\mathcal{N}(0,1)$ (red curve).

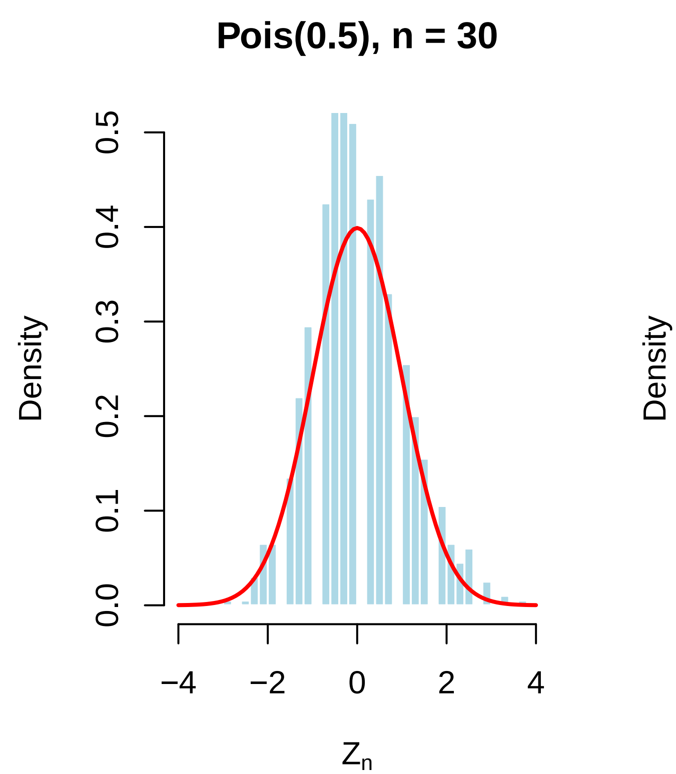
Exponential(2) Distribution For $X_i \sim \text{Exp}(2)$: $\mu = 1/2 = 0.5$, $\sigma^2 = 1/4 = 0.25$

```
mu_exp <- 1/2
sigma_exp <- 1/2

par(mfrow = c(2, 2))
for (n in n_values) {
  Z_n <- simulate_CLT(n, k, mu_exp, sigma_exp,
                      function(n) rexp(n, rate = 2))

  hist(Z_n, breaks = 30, probability = TRUE,
       main = paste("Exp(2): n =", n),
       xlab = expression(Z[n]), col = "lightgreen",
       border = "white", xlim = c(-4, 4))

  # standard normal density
  curve(dnorm(x, 0, 1), from = -4, to = 4,
        add = TRUE, col = "red", lwd = 2)
}
```



Observation: Despite the exponential being highly skewed (unlike Poisson), Z_n still converges to $\mathcal{N}(0,1)$ as n increases!

```r
layout(matrix(1:2, nrow = 1))

# Poisson
Z_pois <- simulate_CLT(30, k, mu_pois, sigma_pois,
                       function(n) rpois(n, lambda = 0.5))
hist(Z_pois, breaks = 30, probability = TRUE,
     main = "Pois(0.5), n = 30",
     xlab = expression(Z[n]), col = "lightblue",
     border = "white", xlim = c(-4, 4), ylim = c(0, 0.5))
curve(dnorm(x, 0, 1), add = TRUE, col = "red", lwd = 2)

# Exponential
Z_exp <- simulate_CLT(30, k, mu_exp, sigma_exp,
                      function(n) rexp(n, rate = 2))
hist(Z_exp, breaks = 30, probability = TRUE,
     main = "Exp(2), n = 30",
     xlab = expression(Z[n]), col = "lightgreen",
     border = "white", xlim = c(-4, 4), ylim = c(0, 0.5))
curve(dnorm(x, 0, 1), add = TRUE, col = "red", lwd = 2)
```



Comparison: Does the Result Depend on Distribution?

```r
# Reset
layout(1)
```

Answer: **No** The CLT is universal:

- The **limiting distribution** $\mathcal{N}(0,1)$ is always the same, regardless of the original distribution
- Only the **speed of convergence** depends on the distribution:
  - Symmetric distributions (like Poisson) converge faster
  - Skewed distributions (like Exponential) need larger n for good approximation
- As long as $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$ exist, CLT holds!