AST 2000 - Part 5
Satellite Launch

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In this study we have studied the Hohmann transfer and seen how this method helps our shuttle on its interplanetary voyage from planet 0 to planet 6. We found that the best time to launch for this transfer orbit to be most effective was after $t = 2.059$, and we managed to perform the maneuver, but we were unsuccessful in reaching orbit due to unforeseen circumstances.

I. INTRODUCTION

In this study we will make a simulation of our interplanetary trajectory in order for us to better plan the actual launch and voyage. We will be looking at ways to efficiently launch, and make way, towards our destination planet. To do the actual trajectory we are going to use a Hohmann transfer orbit. We choose this transfer method due to it being very fuel efficient compared to other methods.

With the Hohmann transfer there are some requirements we need to fulfill, i.e finding the point in time where the two planets in the transfer orbit are perfectly aligned, as well as knowing how much of a boost is needed to put us on the correct trajectory. All of this will be explained further in the study.

II. THEORY

See Appendix and references for more detailed mathematics behind the Hohmann transfer orbit, and behind linear interpolation.

III. METHOD

The program we are creating will be generalized, in order to give us the option for trial and error. This is because we've made alot of previous assumptions and simplifications, and thus we are not going to be able to make stable orbit on our first attempt!

First thing first we need to revisit our planetary simulations for part 2 of our project. In this part we simulated the planets orbital trajectories in discrete timesteps over a time period of 20 years. This is not accurate enough for us to use when we are now looking to use this so simulate the interplanetary travel that will take at most a few months. To solve this issue we are going to interpolate the planetary positions given to us by our friends at the ast2000 research center to be able to access the positions at any arbitrary point in time within the interval 0 to 20 years.

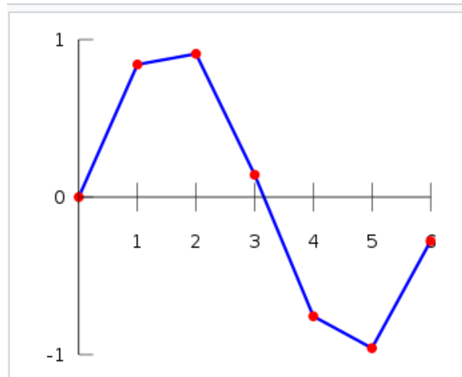

Figure 1. Linear interpolation

To do this interpolation we will use Scipy modules interpolation method interp1d. This is by default a linear interpolation method, and the method behind this is illustrated in the figure (1). The program will "draw" a straight curve through every point in the list of points that are being interpolated See Scipy documentation in references for more detailed version.

Furthermore the program will also allow us to give boosts to our shuttle midflight, in order to compensate for miscalculations in our planned trajectory.

To simplify things a bit we will make some assumptions about our shuttle, namely

1. We are free to adjust our orbital orientation without the use of fuel.
2. The time used boosting is minuscule compared to the time used for our voyage, and so we are considering the boost to be *instantaneous*.
3. From previous study we invoke the same assumptions, most importantly that the entirety of the voyage and planetary orbits exist purely in the xy-plane.

To create the program, and for it to be trial and error viable we need to be able to adjust the initial parameteres as we please. Thus the structure for our inputs will be

- Initial time (yr)
- Initial position (AU)

- Initial velocity (AU/yr)
- simulation time
- Time step length

With these inputs, our program will provide us with final time (yr), final position (AU) and final velocity (AU/yr) for the shuttles voyage.

To actually simulate the spaceflight we are going to use the differential equations for motion using Newtons 2nd law

$$m\frac{d^2\mathbf{x}}{dt^2} = \sum \mathbf{F}$$

to solve for the position x. This means we need to find all external forces acting on the shuttle.

During the spaceflight the shuttle will be affected by the gravitational force from other astronomical bodies in the solarsystem. To simplify the calculations, we will assume that the gravitational pull from lesser astronomical bodies (i.e asteroids, moons, etc.) are so small compared to the larger bodies (star and planets) that we can ignore them in our simulated trajectory. Therefore we can apply the superposition principle to find the net sum of external forces acting on our shuttle

$$\mathbf{F} = \mathbf{F}_s + \sum_i \mathbf{F}_{p,i}$$

Where $\mathbf{F_s}$ is the gravitational force on the spacecraft from the sun, and $\mathbf{F}_{p,i}$ is the gravitational force from planet i. We know from Newtons Gravitational law that the gravitational force from an object with mass M onto a smaller object with mass m is given by the expression

$$\mathbf{F}_G = G\frac{mM}{r^3}\mathbf{r}$$

meaning we can generalize our expression to

$$\vec{F} = -G\frac{mM_s}{|\vec{r}|^3}\vec{r} - \sum_{i=1}^{N}\frac{GmM_i}{|\vec{r}-\vec{r_i}|^3}(\vec{r}-\vec{r_i})$$

Where G is the gravitational constant, m is the mass of the rocket, M_s is the mass of the sun, $\vec{r}$ is the position of the rocket, M_i is the mass of the planet with index i, and $\vec{r_i}$ is the position of the planet with index i.

We know that the gravitational force is an *attractive* force, and that is the reason why each of the forces are negative since each of the vectors $\vec{r}$ and $(\vec{r}-\vec{r_i})$ points outwards from the mass M to the shuttle with mass m.

This can be calculated anywhere, and for anytime, using the interpolated positions of the planets. This means we can use Newtons 2nd law

$$\sum \mathbf{F} = m\mathbf{a}$$

$$a = -G\frac{M_s}{|\mathbf{r}|^3}\mathbf{r} - \sum_i^N G\frac{M_i}{|\mathbf{r}-\mathbf{r}_i|}(\mathbf{r}-\mathbf{r}_i)$$

to find an expression for acceleration.

We have what we need to setup the differential equation for motion as we mentioned previously, and we will solve this numerically using leapfrog integration method as we've previously explained in part 2 of the project, since we are still moving in a conservative, gravitational field and leapfrog (see reference for more detailed explanation) will allow us to conserve mechanical energy as we would expect from motion in a conservative vectorfield.
This is an initial value problem, so all we need in order to initialize the numerical solver are initial values for our trajectory.

We now have a way to simulate an arbitrary trajectory given proper initial conditions, and our next step is to find the trajectory we want for our shuttle, i.e the initial values for our numerical solution.

As mentioned in the introduction section, we will be using Hohmann's transfer orbit for our trajectory. The benefits of this is that we are utilising the gravitational pull of our star to travel through the solarsystem. This is also a rather fuel efficient way of interplanetary travel. To perform this Hohmann maneuver we will again assume circular orbits for the planets in our solarsystem. See figure (2)

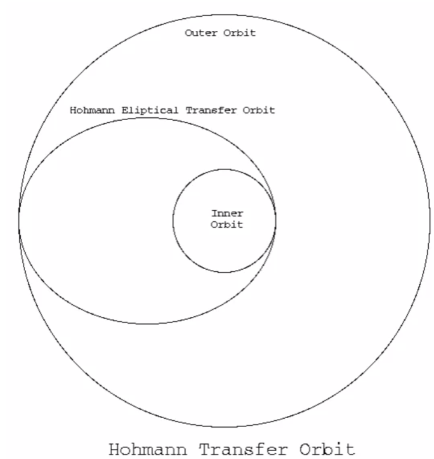

Figure 2. Hohmann transfer idea

The idea here is that, if we give our shuttle enough boost, we can increase our elliptical orbit enough so that we are in a sense being transferred from the inner orbit (homeplanet) to the outer orbit (our destination). See figure (2).

There are three main aspects we will be focusing from Hohmann's method.

1. How much time does our rocket use in order to get from the homeplanet to the destination planet.
2. How much do we need to boost in order to get that new elliptical orbit that would align with trajectory of our destination planet.
3. the angle θ between r_1 (from figure 3) and the vector r_i of the destination planet at time i.

The first aspect finds the time that our rocket uses in order to reach our destination planet. This is important because we can then trace the position of our destination planet. By knowing that position, we can then find when we should launch the rocket from our home planet.

By using the Kepler's third law, we can derive the expression for the time that our rocket would use in order to meet our destination planet.

$$T = \frac{\left[\frac{r_2+r_1}{2}\right]^{3/2}}{2}$$

Where r_1 and r_2 are represented in illustration in figure 3. See appendix A for full derivation. As for the timing of the launch, we mentioned that we need to have the two planets aligned perfectly before we launch. This alignment is defined by an angle Θ illustrated in figure (3).

In this figure we illustrate what the angle θ represents,

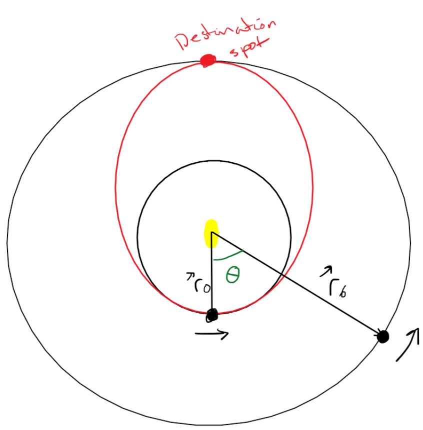

Figure 3. Sketch of the angle θ

it is the angle between the position vector for our home planet, $\mathbf{r}_0$, and our destination planet, $\mathbf{r}_6$, so that when we launch our hohmann transfer, the destination planet will be at the "red" point at the same time as our shuttle.

We have derived in appendix C an analytical expression for this angle

$$\Theta = \left[1 - \frac{\sqrt{\left[\frac{r_1+r_2}{2}\right]^3}}{P_6}\right]$$

Where P_6 is the full period of the orbit for our destination planet. We are going to find the point in time where the angle between the two position vectors illustrated in figure (3) fulfills the equation over. If we perform the Hohman maneuver at this point in time, we will (in theory) be successful in reaching our destination.

The next step is to find at which point in time the two planets are aligned this way, and to do so we will use our interpolated planetary positions, and use basic trigonometry to find the angle between two vectors

$$\theta = \cos^{-1}\left(\frac{\mathbf{r}_0 \cdot \mathbf{r}_6}{r_0 r_6}\right)$$

and we want to find at which timestep i this is in fact, equal to θ in the expression we derived in appendix C.

Lastly, and perhaps most importantly, we have to determine the size of our boost in order to change into the right trajectory.

To find this boost we will begin with the velocity of our home planet, which we call v_1 as the initial velocity for our planet. In order to change the rockets orbit, we need to boost at the perihelion position of the planets orbit. The boost has to be orthogonal to the position vector of our planet. This way we use least amount of fuel, we also do not need to reoriented our rocket. The desired velocity, v_p, will be of a magnitude such the elliptical orbit resulting from having this velocity in the perihelion, will make the aphelion align with the elliptical orbit of our destionation planet (as illustrated in figure (2)). Meaning that

$$v_1 + \Delta v = v_p$$

where the index p stands for perihelion. We find the expression for v_p with the use of conservation laws in appendix B. We then are left with full expression for how much we have to boost our rocket.

$$\Delta v = \sqrt{2GM_o(r_2/(r_1(r_1+r_2)))} - v_1$$

Equivalently we can find the boost, or retardation, we need to perform when we are at the aphelion to align our shuttles velocity with the velocity of the destinations planet.

$$v_a = v_2 - \Delta v$$

where from appendix B we had the relation

$$v_a = \frac{r_1}{r_2} v_p = \frac{r_1}{r_2}\sqrt{2GM_o(r_2/(r_1(r_1+r_2)))}$$

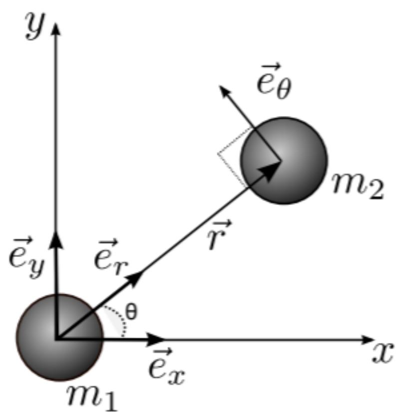

Figure 4. Unit vector illustration

Both of these boost needs to be made perpendicular to the positionvector from planet to shuttle. Using vector calculus, and lecture notes 1B (see references) we can easily find this unit vector $\hat{e}_\theta$ (keep in mind, this is a different θ than the one mentioned earlier) as

$$e_\theta = -\sin\theta\hat{e}_x + \cos\theta\hat{e}_y$$

where the angle θ is found using basic trigonometry from the two-body problem in lecture notes 1B (see references)

$$\theta = \cos^{-1}\left(\frac{\mathbf{r}\cdot\hat{e}_x}{r}\right)$$

as seen in the illustration in figure (4)

As we can see, with the Hohmann transfer we are only required to boost twice, once when leaving the home plants orbit, and one last time when we are entering the destination planets orbit to slow, or accelerate, our shuttle to match the velocity of the destination planet. However, this is only valid when the planetary orbits are perfectly circular and also the gravitionall pull from other astronomical bodies are negligible.

IV. RESULTS

Doing the calculations shown in the method section gives us the following values:

- The time it takes to complete the transfer

$$T = \frac{\left(\frac{r_1+r_2}{2}\right)^{\frac{3}{2}}}{2} = 13.9 \text{ days}$$

- The velocity boost needed to initiate the transfer

$$\Delta v = \sqrt{2GM\frac{r_2}{r_1(r_1+r_2)}} - v_1 = [5.0465, 8.0135]\ [AU/yr]$$

where we've used in our calculations, v_1 as the velocity of the shuttle after launch.

- The point in time where the two planets align with the angle θ

$$t = 2.049 \text{ [yr]}$$

after checking through every timestep in the interval 0 to 20 years

Using these values for T and Δv along the unit vector $\hat{e}_\theta$ gives us the simulated trajectory shown i figure (5) From this we can clearly see that we've done something

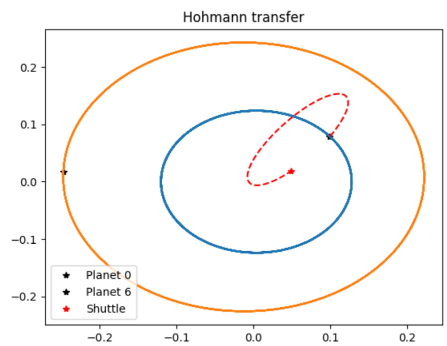

Figure 5. Poor simulation

wrong in our implementation. But since our genius programmers work fast, we've used a different approach to finding these values, from the bright minds over at Instructables seen in the references.

Using their methods we find the following

- Semi-major axis in transfer orbit

$$a = \frac{r_1+r_2}{2} = 0.17959\,AU$$

- Orbital period of transfer orbit

$$P = \sqrt{\frac{4\pi^2a^3}{GM_s}} = 0.11880\,yr$$

- Velocity required in perihelion

$$v_p = \frac{2\pi a}{P}\sqrt{\left(\frac{2a}{r1} - 1\right)}$$
$$\Delta v = v_p - v_1 = [0.85730238, -0.09056247]\ AU/yr$$

Using these new values for the boost, again along the unit vector $\hat{e}_\theta$ that we derived earlier, we get the simulated trajectory as seen in figure (6)

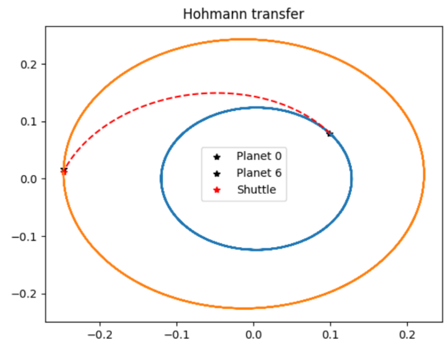

Figure 6. Better simulation

We can clearly see that this trajectory gives us a better shot at reaching orbit around our destination planet. And so, these are the values we are going to use when we are performing the actual launch using the launch facility over at the ast2000 center. The beauty of this facility is that they can also help us track our rocket our its voyage.

After launch we have about 1000kg of remaining fuel and when we perform the first part of the Hohmann transfer we only use about half our remaining fuel for the initial boost to get our shuttle onto our transfer orbit.
However, after travelling for a time $T = P/2 = 0.0594$yr and getting ready to slow our shuttle down to align with our destination we are unable to find our planet. Taking a picture of our surroundings we see absolutely nothing but our star, as seen in the picture

Figure 7. Our beautiful star

V. DISCUSSION

As we see from the plot of the Hohman transfer in figure (6), we are not perfectly aligned on top of our destination, but we can deduce that this is sufficient to use for our actual launch since we retain a rather big amount of fuel even after starting the Hohman trajectory. During the actual voyage we will need to perform correctional boost due to unforeseen, or rather, unaccounted for factors anyways. Furthermore, even if the simulation landed us perfectly in orbit, the actual voyage would most likely not, since the simulation used simulated orbits that are not exact as well as the fact that we assumed that other astronomical bodies had a negligible gravitational effect on our shuttle.
This is not the case, as we cannot be certain that we aren't travelling through a large asteroid belt, or passing multiple large moons on our interplanetary voyage that would change our trajectory slightly, but enough for us to "miss" our target destination.

This was obvious when we made our voyage without making adjustments other than the two planned boost from our hohman transfer, we ended up in deep space devoid of any company! This is likely caused by unaccounted for astronomical bodies exerting gravtational pull on our shuttle, as well as the fact that our Hohmann transfer is based on planets following perfectly circular orbits. As seen in the picture our poor astronauts took in figure (7) they are all alone, and our destination is nowhere in sight!

VI. CONCLUSION

We managed to find a very efficient way to execute the orbital transfer from home planet to destination, but we weren't successful in actually reaching orbit around our destination planet. This is likely caused by poor implementation of the ideas we've explained in the method section of this study, and also from a lack of time. However, given more time for trial and error we believe that our methods would prove quite successful seeing as how they are in principle very simple and robust.

ACKNOWLEDGMENTS

I would like to thank the brave astronauts that made the voyage, and our sincerest apoligies to their family and friends.

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Appendix A: Derivation of time taken to reach aphelion

We will be using Kepler's third law in order to derive the expression.

We know that the Kepler's third law is expressed as

$$P^2 = a^3$$

Where P is the period of full resolution in elliptical orbit, and a is the semi-major axis for the elliptical orbit.

Figure **??** shows the relationship between the length r_1 and r_2 and semi-major axis a.
FIGURE Detta veid eg ikej kå ska vær dom
We can therefore express a as

$$a = \frac{r_1 + r_2}{2}$$

$$P^2 = (\frac{r_1 + r_2}{2})^3$$

The period for the ellipse must therefore be equal to

$$P = \sqrt{\left[\frac{r_1+r_2}{2}\right]^3}$$

The P is the time taken to travel a full resolution around its orbit. However, we want to know how long our rocket exactly travels to the aphelion position from perihelion position. We can see that this must be only half orbit. Meaning that we can divide P by two.

$$T = \frac{P}{2} = \frac{\sqrt{\left[\frac{r_1+r_2}{2}\right]^3}}{2}$$

Appendix B: Derivation of the boost expression

In order to derive the expression for the boost needed to gain a new elliptical orbit for our rocket, we use the conservation of two physical entities. The conservation of energy, and the conservation of angular momentum.

$$E = \frac{1}{2}mv_p^2 - \frac{GmM_o}{r_1} = \frac{1}{2}mv_a^2 - \frac{GmM_o}{r_2}$$

$$L = mv_p r_1 = mv_a r_2$$

Where v_p is the velocity needed at perihelion, v_a the velocity need at aphelion, m as for the mass of the rocket, and M_o as the mass of the sun itself. We have discussed that the relation between the velocity of the planet, the boost and the goal velocity is

$$v_1 + \Delta v = v_p$$

The only value we need to find here, is the value of v_p. We use those two conservation in order to express v_p

We can see that we can express the v_a from the conservation of angular momentum

$$v_a = \frac{r_1}{r_2} v_p$$

We use this expression in the expression of conservation of energy.

$$\frac{1}{2} m v_p^2 - \frac{GmM_o}{r_1} = \frac{1}{2} m (\frac{r_1}{r_2} v_p)^2 - \frac{GmM_o}{r_2}$$

Now all we are left to do, is solve this expression for v_p

$$\frac{1}{2} v_p^2 - \frac{GM_o}{r_1} = \frac{1}{2} (\frac{r_1}{r_2} v_p)^2 - \frac{GM_o}{r_2}$$

$$\frac{1}{2} v_p^2 - \frac{1}{2} (\frac{r_1}{r_2})^2 v_p^2 = GM_o (\frac{1}{r_1} - \frac{1}{r_2})$$

$$v_p^2 \left[1 - (\tfrac{r_1}{r_2})^2\right] = 2GM_o (\frac{1}{r_1} - \frac{1}{r_2})$$

$$v_p^2 \left[\tfrac{r_2^2 - r_1^2}{r_2^2}\right] = 2GM_o \left[\tfrac{r_2 - r_1}{r_1 r_2}\right]$$

$$v_p^2 = 2GM_o \left[\tfrac{(r_2 - r_1) r_2}{r_1 (r_2^2 - r_1^2)}\right]$$

$$v_p = \sqrt{2GM_o \left[\tfrac{(r_2 - r_1) r_2}{r_1 (r_2^2 - r_1^2)}\right]}$$

Now that we have the expression of v_p, we can finally express the expression of the boost

$$v_1 + \Delta v = v_p$$

$$\Delta v = v_p - v_1$$

$$\Delta v = \sqrt{2GM_o \left[\tfrac{(r_2 . r_1) r_2}{r_1 (r_2^2 - r_1^2)}\right]} - v_1$$

Appendix C: Derivation of the angle θ

In order to find this angle θ, we first look at the relation between the T time it takes for the rocket to reach aphelion, and the full period of the destination planet.

$$\frac{T}{P_6}$$

as the index 6 indicates that our destination planet is planet 6. This time relation must relate to the relation between angles.

$$\frac{180 - \theta}{360}$$

. The expression of the relation would be

$$\frac{T}{P_6} = \frac{180 - \theta}{360}$$

We choose to express this in radians.

$$\frac{T}{P_6} = \frac{\pi - \theta}{2\pi}$$

$$\frac{2T\pi}{P_6} = \pi - \theta$$

$$\theta = \pi - \frac{2T\pi}{P_6} = \pi (1 - \frac{2T}{P_6})$$

We have found the expression for T in appendix A.

$$\theta = \pi - \frac{2T\pi}{P_6} = \pi (1 - \frac{2(\frac{r_1 + r_2}{2})^{3/2}}{2P_6})$$

$$\theta = \pi \left[1 - \frac{\sqrt{(\frac{r_1 + r_2}{2})^3}}{P_6}\right]$$