AST2000 - Part 2
Planetary Orbits

(Dated: December 17, 2020)

In this study we have managed to calculate the orbits of the planets in our solar system. Both by simulating numerically, and plotting analytically. In Addition, we analysed how planet 6 with mass of 0.000062 solar mass affects the sun's motion from a distance of 0.5 AU, and how this gives the sun a radial velocity of 0.0015 AU per year. In addition to that, we have shown how an alien life could use least squared method to analyse the radial velocity curve of the star to deduce the mass of our planet.We also managed to capture a video of an event where 3 planets orbit one large planet.

I. INTRODUCTION

In our solar system, the planets are locket in their orbital trajectories around the star at the center of the system. This is the result of the gravitational pull towards the mass center. In most solar system, the bulk the total mass in the system is located in the star. We want to simulate and plot these orbital trajectories. In order to do this, we will be modeling and simulating the orbits of the planets in our solar system by using Newton gravitational law and Newtons second law to find an expression for acceleration that we then integrate using leapfrog method. We choose this method over any other methods such as Euler's integration because with leapfrog, we know that energy is conserved.

We will also plot a sketch of these orbital trajectories using analytical expressions in order to have a way to verify our simulated results. These analytical expression are found using the properties of ellipses together with known values of the orbital trajectories.

By knowing these orbital trajectories, we open up the possibilities for easier planning of interplanetary travels since we will know where the different planets are at given points in time.

We are also interested of how the planets around our sun affects the sun itself. In addition, we want to know if those effects are noticeable from other distant solar-systems. The reason why we are interested in this, is because we can then know if the alien civilisation can see our planet, and might come visit!

II. METHOD

Doing these simulations will result in heavy computational load if we were to do them perfectly, and we need to make the following assumptions to simplify our model:

i) Ignore planet-planet interactions

ii) Ignore relativistic effects

iii) Ignore gravitational pull *on* the star *by* the plants

iv) All planets, and the star lie in the xy-plane

v) Planets both orbit the star, and rotate about the z-axis counterclockwise

vi) There are no other astronomical bodies in the solar system

In addition we will be using AU unit to prevent calculating with large numbers that could cause potential computational errors.

1. *Analytical plot of the Orbits*

We know that the planets follow elliptical trajectory around the sun (even circular obits, circles is just a special case of an ellipse.) In order to plot our planets motion, we will therefore be using the definition of the ellipse by inputting semi-major axis and eccentricity

$$x(f) = a\cos(f)$$

$$y(f) = ab\sin(f)$$

where a is the semimajor axis, and b is the semiminor axis which also can be expressed with e (eccentricity) as

$$\sqrt{1-e^2}$$

see references for more details on the properties of the ellipse.

We also assume that the object m_1 is located in principle focus. And therefore the angle between the position vector r and x-axis is f.(see figure (1))

This expression assumes that the focus point m_1 is the sun, and we are attempting to find the position r of the a given object with the right parameters ref I of the planets around the sun. Figure (1) illustrates the idea of this expression visually.

Our great scientist have managed to observe the values of semi-major axis and eccentricity of every planet in our solar system, which is represented in the following table

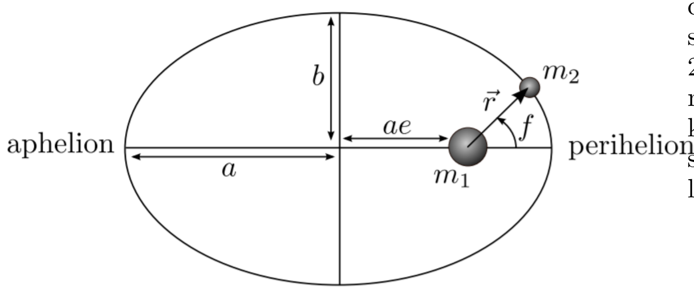

Figure 1: The illustrated ellipse of the expression

Table I: Collected data of the planets

Planet Index (i)	**Semi-major axis** (a in AU)	**Eccentricity** (e in AU)
0	0.124267	0.03077
1	0.16706	0.0031755
2	0.92620	0.042781
3	0.31065	0.022062
4	1.19198	0.03332
5	0.555823	0.00139016
6	0.234912	0.064521

We then have the possibility to use the expressions of $x(f)$ and $y(f)$ to plot the orbit of the planets with a give value of a and e from table above for f in the range from 0 to 2π.

2. *The simulation of the Orbits*

We will be using Leap Frog integration method (sometimes called varlet integration method as well) to solve the differential equations for each planets motion. This method will conserve energy throughout the motion, as expected when our planets are moving purely through a gravitational field where the force is conservative. Any other method such as Euler's method, or Euler-Cromer's method could distort the trajectories of the planets, and we would end up with an inaccurate simulation model for our solar system. We refer the reader to read more in depth explanation of leapfrog method in the references

The integration will be set up as follows

$$a = F_g/m$$

$$v_{i+\frac{1}{2}} = v_{i-\frac{1}{2}} + a\frac{dt}{2}$$

$$x_{i+1} = x_i + v_{i+\frac{1}{2}}dt$$

Firstly, we find the acceleration a of the planet by calculating the gravitational force F_g acting towards the sun, and dividing it by the planets mass using Newtons 2nd law $\sum F = ma$. We can then use the leapfrog method and find half next step of the velocity v. By knowing that velocity step, we can then find the next step in the position x. Figure (2) illustrates the idea of leapfrog method.

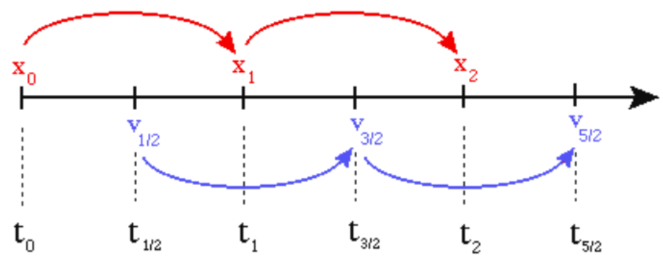

Figure 2: We can see that the velocity is updated by adding half time leap to $v_{1/2}$, and the updating the position by adding that halv velocity leap to the previous position

However, in order to "kickstart" this method procces, we will be needing some initial values. We will be needing the mass of the sun and the initial velocities, initial positions and mass of the planets. Once again, our great scientist were able to observe these values. Table II provides us with these initial values and needed information.

Table II: Collected data of the planets

Body	**Mass** (M_{sun})	**Initial velocity** ((x,y)AU/yr)	**Initial position** ((x,y)AU)
sun	0.410444	(0,0)	(0,0)
planet 0	5.1836e-06	(0,11.072)	(0.128,0)
planet 1	5.8222e-07	(9.373,-2.982)	(-0.0502,-0.1595)
planet 2	1.3250e-08	(1.422,-3.838)	(-0.8744,-0.3612)
planet 3	1.0383e-06	(6.187,3.607)	(0.163,-0.267)
planet 4	0.00027879	(3.218,-1.750)	(-0.607,-1.034)
planet 5	6.2184e-05	(-4.077,3.537)	(0.364,0.419)
planet 6	7.7384e-07	(7.157,-4.269)	(-0.132,-0.192)

As for the timestep, we will make sure that we use atleast 10 000 time steps per year. Meaning that $dt = 0.00001$ years. We need to have enough of timesteps in order for the simulation to be precise enough. The more timesteps, the better it is. Figure fig:timesteps illustrates the reason why we need many timesteps.

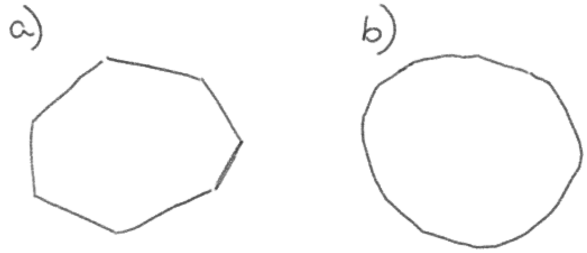

Figure 3: a) shows a simulated cirlce with 6 timesteps. While b) shows circle with 70 timesteps. We can therefore see that more timesteps gives more precise results

However, we cannot allow us to have to many timesteps, because then our computations would take too long time to compute all these timesteps. We therefore choose 10 000 timesteps, this way we can receive precise enough results, without overcomputing.

Now that we have all the information we need. We just let the leapfrog integration method run for all the planets.

3. *Testing the simulation with Kepler's laws*

We can implement Keplers laws into our program to verify if numerical solution holds water. These three laws must be fulfilled for the orbit to be realistic:

i) The orbits must be elliptic, with the solar mass in one of the focal points

ii) The area of the triangle spun by the planets change in position for a given timestep is the same throughout the motion.

$$\frac{dA}{dt} = constant$$

iii) The square of a planets orbital period is proportional to the cube of the semi-major of its orbit

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3$$

These points will form the base of our test to see if the planetary orbits we simulate are consistent with real orbital trajectories.
The first point we can easily check by plotting our planets motion, and the star, in an xy plot and see if our stars mass is located in the focal point of the planets elliptic orbit.
The two remaining points will be checked numerically by:

1. Checking the area spun in each timestep is easily implemented into your numerical integration loop, where we can implement so that the program crashes should the area change by too much (can't expect this to truly be constant when we solve it numerically).

2. We can also measure our planets orbital period and check that our equation (Keplers third law) holds true within a given tolerance (again, we cant expect exact numbers when using numerical methods). We know that we can rewrite keplers third law as

$$P^2 - \frac{4\pi^2}{G(m_1 + m_2)} a^3 = 0$$

As for the rotation period time, we will make a simulation that detects when the planet is back on its starting position. We will be starting to take time when the simulation starts, til the planet is back on the same position.

4. *Simulation of what extraterrestrials would observe of our solar system*

When we are checking the possibility for extraterrestrial life to detect our planets, we will need to look at the elliptic orbit the star follows around the center of mass. We do this to see if the "tangential wobbling" of the star can be observed with present day technology.
We assume that extraterrestrial life also have access to a Hubble-like spacetelescope with an accuracy of 0.1 arcseconds and that our solarsystem consist of equalsized planets.

To do this we need to construct a radial velocity curve for our star, from a given set of parameters

i) i = inclination of the plane relative to the line of sight

ii) v_p = peculiar velocity of mass center (the velocity that the mass center has parallell to the observer)

However, we firstly have to start to simulate a simple model simulation of our own system. For simplicity sake, we will choose the planet with the most mass (from table II we can see that planet 4 has the largest mass), and let it orbit around the sun. We will now let the sun move freely, without fixating it in the origin. This way we can observe the "wobbling" of the sun around the mass center.

In order to simulate this, we will again use leapfrog integration method, in order to make sure that the energy is conserved.

When we have the simulation in our hand, we will then test its accuracy using the analytical approach. This analytical approach is discussed more in detail in appendix:C . In addition, we are ready to create the radial velocity curve. For simplicty sake, we choose our peculiar velocity to be 0. We also choose the inclination angle to be 0. We then measure the velocity towards positive direction of the x-axis. The situation is illustrated in figure fig:curve

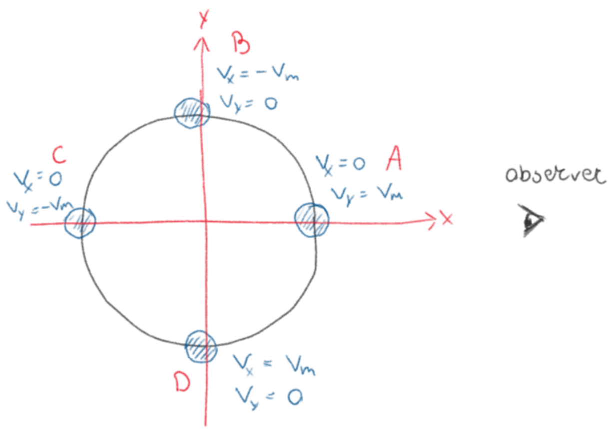

Figure 4: Sun (blue circle) orbiting mass center (origin). v_m is the maximum velocity

By plotting the velocity only in x-direction, we should should receive graph as illustrated in figure fig:graph

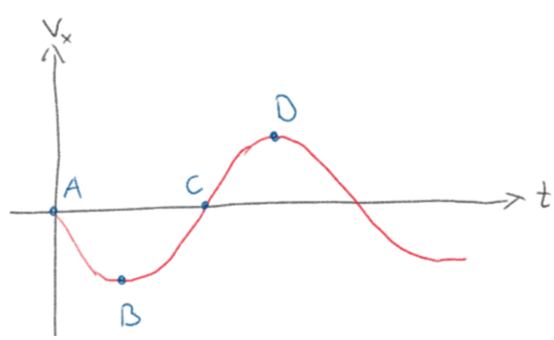

Figure 5: The graphical representation of the velocity of the sun towards the positive direction of x-axis

Whenever we observe this kind of motion from other star, the data we collect is full of noises. We therefore have to use methods such as least square in order to clean the data from those noises in order to have a readable data. This method This method consist of changing the parameters in our analytical model given as

$$v_r(t) = v_* \cos\frac{2\pi}{P}(t - t_0)$$

Where v_* is the maximum valelocity amplitude of the curve (v_m from figure above), and P is the full period of the curve,

until the net sum of all the variances from the analytical model to the observed curve has its minimum i.e

$$S(v_*, P, t_0) = \sum(v_r^{obs} - v_r^{model})^2$$

when the function S has its minimum. We can now use this radial velocity to try compute the mass of our planet, and number of planets in our system, with the expression

$$m_p = (\frac{P}{2\pi G})^{\frac{1}{3}} m_* v_{*,r}$$

Where m_* is the mass of the sun.
These three expression above are intorduced and derived in lecture notes Part 1C.

Now knowing the method to find the mass of the planet and how we find clean curve out of noisy curve, we can go backwards knowin all the information about our sun and make such noicy curve. The results will give use a observation curve that the alien life would probably observe by looking at our solarsystem.

However, we will generate a random noise curve with expected values, and try to end up with a clean curve that we could gain information about the motion of the sun. We will be using then the least square method that we have introduced above.

5. *Upgrading the velocity curve of the star*

We have created a method of making a radial velocity curve of the sun by making one plane influence sun's motion. However, we will upgrade this by adding 2 more planets.

We will again use leapfrog integration method and let the sun be influenced by all 3 planets. However, we will choose not to make other planets influence each other. Every planet affects sun with a force F. We will now sum all these forces

$$F_{total} = \Sigma_{i=1}^{3} F_i = \Sigma_{i=1}^{3} G\frac{m_{sun} m_{planet(i)}}{r_i^3}\vec{r_i}$$

Where m is the mass of the bodie, r is the position of the planet, and i is the index of the planet.

III. RESULTS

By using the analytical approach, we plot the orbit around the sun for our home planet 0 as shown in figure 6.

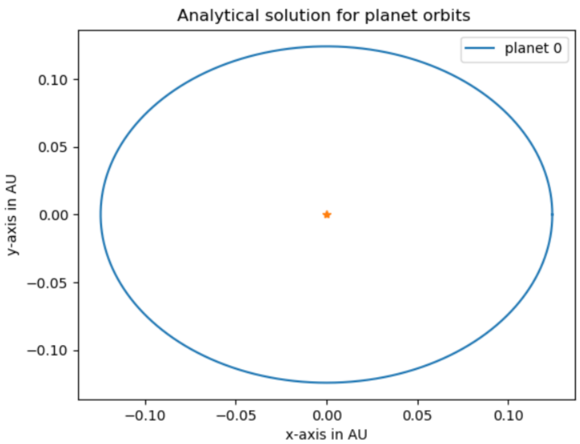

Figure 6: Visualisation of our homeplanets orbit around the sun plotted by analytical solution

By plotting all of the planets in one plot, we would receive figure 7 as the result.

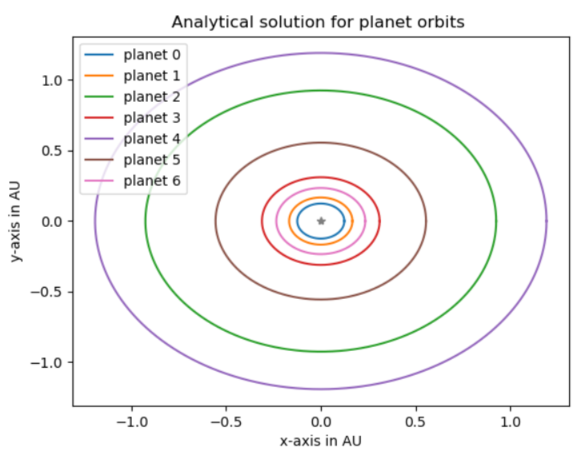

Figure 7: visual representation of analytically plotted orbits of all 6 planets

From the numerical simulation, figure 8 shows a plot of the orbit of our home planet. Here we have used 10 000 timesteps for 1 year.

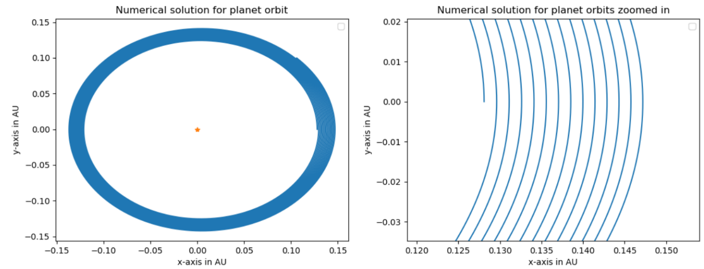

Figure 8: Numerical solution for homeplanet orbit with 10 000 timesteps.

By changing timesteps to 1 000 000, we receive figure 9 as the result.

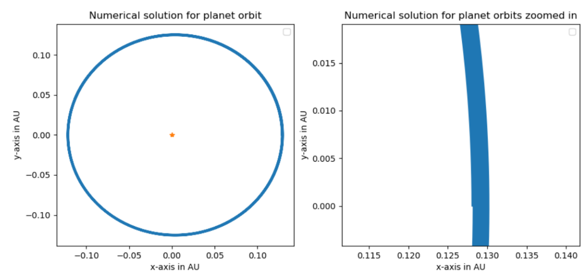

Figure 9: Same as figure 8, but with timesteps of 1 000 000

For comparing the analytical and numerical solution, we plot our homeplanet orbit on top of each other. Figure 10 shows the results.

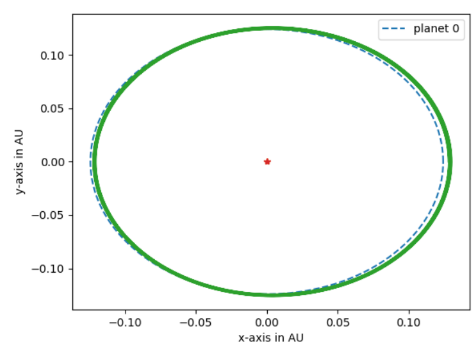

Figure 10: Comparison between numerical and analytical orbit solution. Green orbit is numerical orbit for planet 0, and blue dashed line is analytical solution for planet 0.

If we now let sun not be a fixated point, we can observe how it it affected by a planet with the most mass in our solar system. Figure 11 shows the plotted results of this.

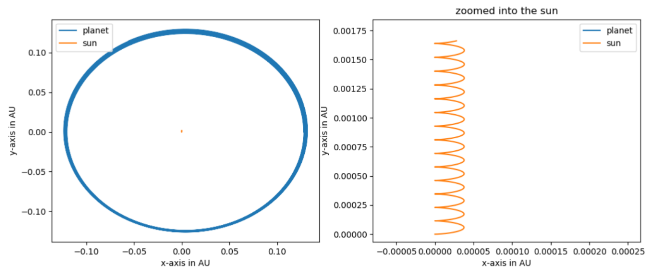

Figure 11: On the left we can see planet orbiting around the sun, on the right we have zoomed into the sun, we can see that it gains peculiar velocity into the negativ y-direction

If we plot the velocity in x-direction with respect of time, we receive figure 12 as the result.

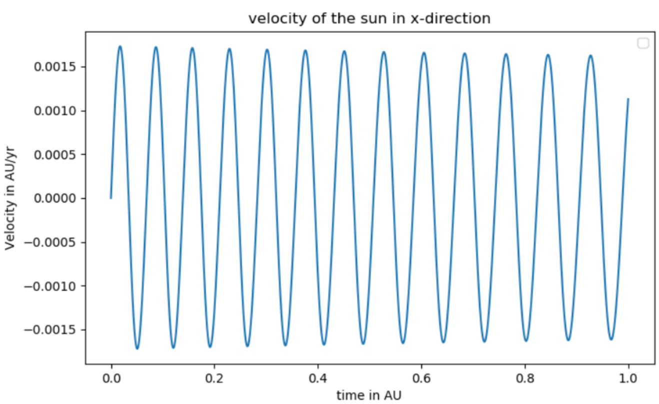

Figure 12: cosinosoidal velocity fluxation of the sun affected by gravitation of the planet

In order to get a radial velocity curve with noice, we chose to use $v_* = 26$ km/s, and timeperiod of 30000 days. These are not the values we receive from our simulation. Figure 13 shows the resulting plot.

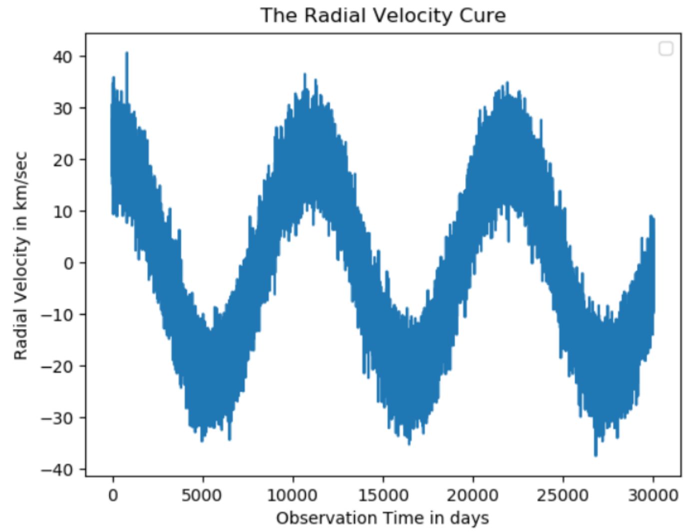

Figure 13: The observation of our suns motion that alien life would probably observe

By using the least square method, we test equation

$$v(t) = v_* \cos(2\pi/P)(t - t_0)$$

for potential values of v_*, P and t_0. We find that

$$v_* = 26.11$$

$$P = 11000$$

$$t_0 = 11000$$

By plotting observed plot, on top of actuall plot, and the calculated by least square plot, we receive figure 14 as the result

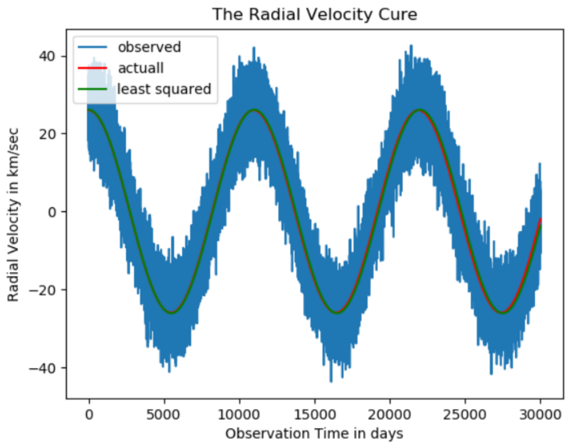

Figure 14: All three curves plottet on top of each other. Looks like our least square method works great!

As for the last part, sadly but our computational fascilities cough on fire this week. We therefore weren't been able to produce significant results of how 3 planets affect suns motion.
Luckily, our observatory did not catch on fire! Our great astronomer have been lucky observing through "Universe Sandbox 2" telescope an extrodonary event. three planets with the mass of earth orbiting around one planet with the mass of jupyter! Sadly we weren't able to produce significant data of this event. However we have made a visual movie of it. (This movie is uploaded together with this report)

IV. DISCUSSION

Figure 6 and 7 simply represents visualisation for the information that we have gotten from table I, in method section. We can confirm visually that the shape of the orbits has elliptical properties.

In figure 8 however, we have used 10 000 timesteps to produce these results. We can clearly see that that the orbit is wrong, because each time it goes a full resolution, it moves either outward or inward, this means that the energy is not conserved. If we change the timestep number to a larger number, we can clearly see from figure 9 that this helps. However, 1 000 000 timesteps is not enough to make a perfect orbit, but we let this to be the limit, and assume that the approximation is close enough to reality. This is because we do not have enough computing power.

Figure 10 shows the comparison between numerical solution for orbit of planet 0 (our home planet), and numerical solution. Here we can see that the path differs slightly at perihelions. We can see that the starting position is different for both solution. This might be because by using analytical approach, we just defined the semi-major axis, but not the starting position. We can see that numerical orbit is shifted a bit to the right. However, the semi-major axis is still approximately the same. It can differ as well because of numerical approximation of the timesteps.

Additionally, we have not included an important factor into our simulation. Aphelion and orbital angle! All the semi-major axis of all planets are parallell in our plots. However, this is not the case in reality, those axis often are rotated to each other. Figure below, illustrates this rotation.

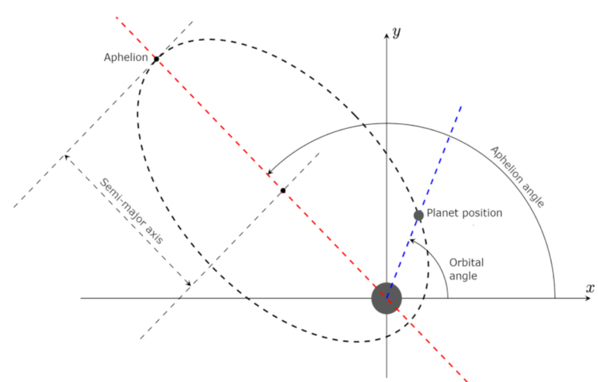

Figure 15: rotation of aphelion from x-axis.

In figure 11 we let the sun out of the fixated point, and let it be affected by the planet. We can observe that it gains a peculiar velocity in to the negative y-direction. We know from theory that both bodies (the sun and the planet) should orbit around the mass center. However, we can see that this mass center moves. This could be a numerical fault. However, we have tested by increasing timesteps. We observe that this does not help. We therefore could think that this could be integration fault.

In figure 12 we can see the velocity fluxiation towards x-direction. We can aproximetely tell that the amplitude of this cosinosoidal fluxation is aproximetely 0.0016 AU/yr, which is aproximetely the same as 27 km/h. This is a very small value, and in comparison to other astronomical bodies, this value is too small. However, we have choosen a planet which is aproximetely 0.5 AU away from the sun. And its mass is only 6.2e-5 the mass of the sun. This means that planet is pretty far away, and has to little mass in order to affect the suns motion on a large scale! We could therefore also tried with different planets and see how it would affect the sun.

From figure 14 we can see that the least square method was successful. We can see that the calculated curved lies on top of the actual curve, but differ a bit later. We can actually see that his begins to happen after one period. Therefore we could guess that the value of a period was calculated wrongly. In order to solve this problem we could have more iteration for our least square method (more values to guess for). However, the results are reasonably close to real values. In addition, more guesses would take more time to compute.

Even tho we weren't been able to produce any computational results of how three planets affects sun motion. We were able to capture a movie of the event that similar to our computation goal. We can see that the planet in the middle begins to "wobble" as we have seen with one planet. In the right bottom corner we can see the fluxiation of the absolute value of the velocity. However, interesting enought we can also see that the planet begins to move in more in one of the directions! This is not expected, as we would expect that all of these planets would orbit the mass center. This might be because each planets "throws" eachother further and further, and this is not a perfect elliptical orbits, making this simply a 4-body-problem. With 4-body-problem we mean that the orbits are unstable and chaotic, and at any moment one of the bodies would be thrown away or colliding, making it stable 2-body-problem at the end.

V. CONCLUSION

As for the conclution, we have sucsessfully plottet orbits of the planets around the sun. In addition to that, we have managed to simulate those orbits as well. However, we cannot trust those orbits 100 percent. We have seen that for example the choise of timesteps has an affect for the results. In addition, we haven't consideret adding the aphelion angle.

We were not able to properly implement the two of Keplers law that we mentioned in the method section, however we managed to check that our simulations were consistent with our analytical plots

We also helped out our astronomers curiosity by analysing how our planets affects their star and evaluated how intelligent alien lifeforms would observe the motion of our sun (Assume they have the same instrument and advancements as us).
Additionaly, we were extremely lucky to observe an extraordinary event of 4 planets orbiting each other.

ACKNOWLEDGMENTS

I would like thank the team who developed Universe Sandbox Simulator.

REFERENCES

- Ast2000tools v1.1.7, Lars Frogner
- AST2000 Lecture Notes Part 1B, Celestial Mechanics: calculating the orbit
- Properties of the ellipse

Appendix A: Derivation of Keplers $2^{nd} law$

We will attempt to show how the infitesimal area spun out by the position vector as it follows an elliptic motion around the center of mass equals

$$dA = \frac{1}{2}r^2 d\theta$$

When we do this we will approximate the area dA with a triangle with one side of length r (length of position vector), and, as the change in angle becomes infitisimally small dθ the change in distance d$\mathbf{r}$ will equal the arc length of the circle spun by the change in angle dθ.
Thus we can express

$$d\mathbf{r} = r d\theta$$

Now we can compute the area dA

$$\begin{aligned} dA &= \frac{1}{2} r d\mathbf{r} \\ dA &= \frac{1}{2}r^2 d\theta \end{aligned}$$

since the area of a triangle is the two catheti multiplied and divided by 2.
We will also prove Keplers 2nd law by:

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}r^2\frac{d\theta}{dt} \\ &= \frac{1}{2}r(r\omega) \end{aligned}$$

hvor ω er vinkelhastighet. Vi vet videre at $r\omega = v_\theta$ er banefarten i sirkelbanen vi approksimerer. Videre er også banefarten $v_\theta = \frac{1}{r}|\mathbf{r}\times\mathbf{v}|$ fartskomponenten som står vinkelrett på posisjonsvektoren
Kan dermed omskrive

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}r(r\omega) \\ &= \frac{1}{2}rv_\theta = \frac{1}{2}|\mathbf{r}\times\mathbf{v}| \\ \frac{dA}{dt} &= \frac{1}{2}h \end{aligned}$$

hvor $h = |\mathbf{r}\times\mathbf{v}|$.

Appendix B: Derivation of Keplers orbital period

We want to derive, and show that, the orbital period for the elliptic motion of a planet in orbit expressed as

$$P = 2\pi\frac{ab}{h}$$

where πab = area of an ellipse.
To do this we will integrate the expression we have for $\frac{dA}{dt}$

$$\begin{aligned} \int_0^P \frac{dA}{dt}dt &= \int_0^P \frac{1}{2}h dt \\ &= \frac{1}{2}hP \end{aligned}$$

This integral will be to total area under the curve that is our ellipse orbit, and we have the area for an ellipse given by πab.
Therefore

$$\begin{aligned} \int_0^P \frac{dA}{dt}dt &= \pi ab \\ \frac{1}{2}Ph &= \pi ab \\ P &= 2\pi\frac{ab}{h} \end{aligned}$$

■

DERIVATION OF KEPLERS 3RD LAW (NEWTONIAN VERSION)

We want to show how to find the Newtonian expression for Keplers third law

$$P^2 = \frac{4\pi^2}{G(m_1+m_2}a^3$$

We will use our previously found expression for P

$$P = 2\pi\frac{ab}{h}$$

We raise this to the power of 2

$$P^2 = 4\pi^2\frac{a^2b^2}{h^2}$$

And we know that $b = a\sqrt{1-e^2}$ and $h = \sqrt{mp}$ where $m = G(m_1+m_2)$ (reduced mass)
Thus we can find the expression

$$\begin{aligned} P^2 &= 4\pi^2\frac{a^3(1-e^2)}{mp} \\ &= \frac{4\pi^2}{G(m_1+m_2)}a^3 \end{aligned}$$

since $p = (1-e^2)$ for an ellipse and we have found the newtonian version of Keplers $3^{rd} law$. ■