MAT-MEK4270: Mandatory assignment 1

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October 2023

1.2.3 Exact solution

We would like to show that

$$u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}, \tag{1}$$

satisfies the wave equation in two dimensions, i.e

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \tag{2}$$

To do this we compute each side of the equation separately, starting with the RHS

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u$$

and LHS

$$c^2 \nabla^2 u = -c^2 (k_x^2 + k_y^2) u$$

and we see that (1) is a solution to (2), with the following criteria on the parameters

$$\omega = c\sqrt{k_x^2 + k_y^2}$$

which is the dispersion relation that holds true for any given wave.

1.2.4 Dispersion coefficient

We assume that $k_x = k_y = k$, meaning we can re-write (1) in a discrete version as follows

$$u_{ij}^n = e^{\mathbf{i}(kh(i+j) - \tilde{\omega} n \Delta t)} \tag{3}$$

where $\tilde{\omega}$ is some numerical approximation to the exact ω. Inserted into the discretized wave equation (eq. (1.3) in assigment), yields for the LHS:

$$\rightarrow \frac{u_{ij}^n(e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t})}{\Delta t^2}$$

and for the RHS:

$$\rightarrow \frac{c^2 u_{ij}^n(e^{ikh} - 2 + e^{-ikh} + e^{ikh} - 2 + e^{-ikh})}{h^2}$$

Now using that the CFL number is given as $C = \frac{1}{\sqrt{2}} = \frac{c\Delta t}{h}$, we can simplify the LHS = RHS expression greatly

$$e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} = 2C^2(e^{ikh} - 2 + e^{-ikh}) = e^{ikh} - 2 + e^{-ikh}$$

from this we must have that

$$\tilde{\omega}\Delta t = kh \rightarrow \tilde{\omega} = c\sqrt{2}k$$

where we've used $\frac{c\Delta t}{h} = \frac{1}{\sqrt{2}} \rightarrow h = c\Delta t\sqrt{2}$. Furthermore, by definition, the wave number $k = \sqrt{k_x^2 + k_y^2} = \sqrt{k^2 + k^2} = k\sqrt{2}$. Which means, that

$$\tilde{\omega} = c\sqrt{2}k = \omega$$

and we've shown what we wanted to show. ■