From Differentiable Manifolds to Gauge Fields

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submitted by

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Von differenzierbaren Mannigfaltigkeiten zu Eichfeldern

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Chapter 1

Introduction

Chapter 2

Manifolds

Manifolds are the fundamental spaces used in physics. They provide a framework to describe topological spaces that locally resemble Euclidean spaces, allowing for the application of known methods from calculus and linear algebra.

2.1 Preliminaries from topology

Although this thesis will not focus on introducing topology, a few important results will be given here, which are necessary to understand the definition of manifolds. The following definitions and theorems are taken from [Nak05]

A **topological space** is a set X equipped with a collection of open sets $\mathcal{T} = \{U_i \mid i \in I\}$ such that:

- $\emptyset, X \in \mathcal{T}$
- For any subcollection J of I the Union of corresponding open sets is itself an open set $\bigcup_{j \in J} U_j \in \mathcal{T}$
- For any finite subcollection K of I the intersection of the corresponding open sets is open: $\cap_{k \in K} U_k \in \mathcal{T}$

A family $\{O_i\}$ of (open) subsets of X is called an (open) covering of X if $X = \bigcup_i O_i$.

A subset N is called a **neighborhood** of a point $p \in X$ if there exists at least one open set $U \in \mathcal{T}$ such that $p \in U \subset N$. A topological space is called **Hausdorff** if for any two distinct points $p, q \in X$ there exist neighborhoods N_p, N_q such that $N_p \cap N_q = \emptyset$.

A map $f : X \to Y$ between two topological spaces is called **continuous** if for every open set $V \subset Y$ the preimage $f^{-1}(V)$ is an open set in X. If the inverse $f^{-1} : Y \to X$ is also continuous, then f is called a **homeomorphism**. Two topological spaces are called **homeomorphic** if there exists a homeomorphism between them.

2.2 Differentiable Manifolds

A Hausdorff topological space $(M, \mathcal{T})$ is called a **d-dimensional manifold** if there exists an open covering $\{U_i\}$ and a family of homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^d$. The pair (U_i, φ_i) is called a **chart** and the family $\{(U_i, \varphi_i)\}$ is called an **atlas**[Fre15e].

M is a **differentiable or smooth manifold** if for any U_i and U_j given that $U_i \cap U_j \neq \emptyset$ the transition function $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_i) \to \varphi_i(U_j \cap U_i)$ is infinite differentiable (C^∞)[Nak05]. In this thesis, smoothness will always be assumed, unless stated otherwise.

Let M and N be two differentiable manifolds of dimension m and n equipped with atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ respectively. A map $f : M \to N$ is called a **differentiable map** at a point $p \in M$ if for $p \in U_i$ and $f(p) \in V_j$ the composition $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$ is infinite differentiable. If f is also a homeomorphism and the inverse $f^{-1} : N \to M$ is differentiable, then f is called a **diffeomorphism**. M and N are called **diffeomorphic** if there exists a diffeomorphism between them. This will be denoted as $M \cong N$.

2.3 Spacetime Manifold M

A trivial but for obvious reasons important example of a differentiable manifold is the spacetime manifold M used in physics, which is defined as follows:

Let $M := \mathbb{R}^4$, the set of ordered 4-tuples $(x^\mu) \in \mathbb{R}^4$. The so called **standard topology** is defined by the open balls around a point $p \in M$ with radius $r > 0$:

$$B_r(p) := \{x \in \mathbb{R}^4 \mid \|x - p\| < r\}$$

with $\| \cdot \|$ the Euclidean norm:

$$\|x\|^2 = \sum_{\mu=0}^{3} (x^\mu)^2$$

This is obviously a Hausdorff[1], and locally Euclidean topological space. The identity map $\phi(p) = p$ covers M globally. Hence, $(\mathcal{M}, B_r(p), \varphi)$ is a smooth manifold.

[1] For two distinct points $p, q \in M$ it is always sufficient to choose $r = \frac{1}{2}\|q - p\|$

Chapter 3

Bundles

The definition of a vector on a Manifold is non-trivial because a vector space structure might not exist globally on the manifold. We can still equip a Manifold with a Vector space structure locally. Thus, tangent spaces are introduced pointwise. Combining these local structures will lead naturally to the definition of fiber bundles.

3.1 Tangent Space $\mathcal{T}_p\mathcal{M}$

Let $\mathcal{M}$ be an n-dimensional smooth manifold. A tangent vector at a point $p \in \mathcal{M}$ is a linear map $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfying the Leibniz rule[1]:

$$v[fg] = v[f]g(p) + f(p)v[g]$$

Such a map is called a *derivation* at p. The set of all such derivations forms a vector space, the *tangent space* at p, denoted $\mathcal{T}_p\mathcal{M}$.

We produce a tangent vector at a point $p \in \mathcal{M}$ as an equivalence class of curves through p.

Let $\gamma_n : [-\epsilon, \epsilon] \to \mathcal{M}$ be a family of smooth curves in $\mathcal{M}$. Then $\varphi(\gamma_n(t)) = x_n^\mu(t) \in \mathbb{R}^n$ is called the coordinate representation of γ_n in a chart (U, φ).

Let $f \in C^\infty(\mathbb{R}^n)$, then we can define

$$\begin{aligned}
\frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=0} &= \frac{d}{dt}\left(f(x^\mu(t))\right)\Big|_{t=0} \\
&= \frac{\partial f}{\partial x^r}\frac{dx^r}{dt}\Big|_{t=0} \\
&= \frac{dx^r}{dt}\Big|_{t=0}\frac{\partial f}{\partial x^r}\Big|_p
\end{aligned}$$

We define an equivalence class. We say two curves are equivalent if

$$\frac{d}{dt}\left(f \circ \gamma_1\right)(t)\Big|_{t=0} = \frac{d}{dt}\left(f \circ \gamma_2\right)(t)\Big|_{t=0}$$

For two equivalent curves γ_1 and γ_2,

$$\frac{dx_1^r}{dt}\Big|_{t=0} = \frac{dx_2^r}{dt}\Big|_{t=0} = v^r$$

[1] $f \in C^\infty(\mathcal{M})$ if for all charts (U, φ) on $\mathcal{M}$, the composition $f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable.

A tangent vector is

$$v = v^r \left.\frac{\partial}{\partial x^r}\right|_p$$

We notice: Once a chart (U, φ) is chosen, with local coordinates $(x^1, \dots, x^n)$, a tangent vector is represented as a linear combination of partial derivatives with real coefficients.

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \quad \text{form a basis of } T_p\mathcal{M}$$

The Tangent Bundle as a Fiber Bundle

To continue, we need to introduce fiber bundles. To do so, we first take a look at a specific example: the tangent bundle.

To construct the tangent bundle of a smooth n-dimensional manifold $\mathcal{M}$, we take the disjoint union of all tangent spaces $T_p\mathcal{M}$:

$$T\mathcal{M} := \bigsqcup_{p\in\mathcal{M}} T_p\mathcal{M} = \bigcup_{p\in\mathcal{M}} \{p\} \times T_p\mathcal{M}$$

Each element of $T\mathcal{M}$ is a pair (p, v), where $p \in \mathcal{M}$ is a point in the base space and $v \in T_p\mathcal{M}$ is a tangent vector at that point.

We define the natural projection:

$$\pi : T\mathcal{M} \to \mathcal{M}, \quad (p, v) \mapsto p$$

which ”forgets” the tangent vector.

We call $\pi^{-1}(p) = T_p\mathcal{M}$ the fiber over p. Since $T_p\mathcal{M} \cong \mathbb{R}^n$ as a vector space, we refer to the model fiber as $F = \mathbb{R}^n$.

The final step is to give $T\mathcal{M}$ a smooth structure of dimension $2n$, we proceed as follows: Given a coordinate chart (U, φ) on $\mathcal{M}$, we define a chart on $\pi^{-1}(U) \subset T\mathcal{M}$ by:

$$\Psi : \pi^{-1}(U) \to \mathbb{R}^{2n}, \quad (p, v) \mapsto \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right)$$

where $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$.

This defines a local trivialization:

$$T\mathcal{M}|_U \cong U \times \mathbb{R}^n$$

Thus, we have constructed a fiber bundle with:

- **Total space:** $T\mathcal{M} = \bigsqcup_{p\in\mathcal{M}} T_p\mathcal{M}$
- **Base space:** $\mathcal{M}$
- **Projection:** $\pi : T\mathcal{M} \to \mathcal{M}$
- **Model fiber:** $F = \mathbb{R}^n$
- **Local trivialization:** $T\mathcal{M}|_U \cong U \times \mathbb{R}^n$

This provides an intuitive picture of fiber bundles: Analogous to a manifold, which we often think of as a space that locally resembles $\mathbb{R}^n$, a fiber bundle can be thought of as a space that locally resembles a product of the base space with a typical fiber[2].

[2] Often the product of two manifolds, but not necessarily so

General Definition of a Fiber Bundle

The formal definition of a fiber bundle reads as follows. A fiber bundle is a quadruple (E, B, π, F) where:

- E is the total space
- B is the base space
- $\pi : E \to B$ is a surjective map called the projection
- F is the typical fiber

There exists an open cover $\{U_\alpha\}$ of B such that for each α, there is a diffeomorphism

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}$$

where $\mathrm{proj}_1 : A \times B \to A$ is the first projection.

$(U_\alpha, \varphi_\alpha)$ are called *local trivialization*.

The fiber over a point $b \in B$ is:

$$F_b := \pi^{-1}(\{b\}) \cong F$$

We often denote a fiber bundle as $E \xrightarrow{\pi} B$

Sections

One important definition in the context of fiber bundles is that of a section or cross-section. This allows us to pick an element from each fiber over each point in a continuous manner, intoducing concepts like vector fields over space time. A *section* of a fiber bundle is a continuous map $s : B \to E$ such that

$$\pi \circ s = \mathrm{id}_B$$

This ensures we choose exactly one point in each fiber continuously. Global sections need not exist, but local sections $s : U \to E$ with $\pi \circ s = \mathrm{id}_U$ often do.

If $(U_\alpha, \varphi_\alpha)$ is a local trivialization, then such local sections always exist.

We denote the set of all (smooth) sections by:

$$\Gamma(E) := \{s : \mathcal{M} \to E \mid \pi \circ s = \mathrm{id}_\mathcal{M}\}$$

3.2 Minkowski Space $\mathbb{M}^4$

The physics of this thesis takes place in Minkowski space. We therefore formally define Minkowski space $\mathbb{M}^4$.

We already defined the spacetime manifold $\mathcal{M} := \mathbb{R}^4$ in section 2.3. We now define the tangent bundle

$$T\mathbb{M}^4 := \bigsqcup_{p \in \mathbb{M}^4} T_p\mathbb{M}^4$$

together with the natural projection:

$$\pi : T\mathbb{M}^4 \to \mathbb{M}^4, \quad (p, v) \mapsto p$$

This is a trivial smooth vector bundle of rank 4 over $\mathbb{M}^4$, i.e.:

$$T\mathbb{M}^4 \cong \mathbb{M}^4 \times \mathbb{R}^4$$

Each fiber is:

$$\pi^{-1}(p) \cong T_p\mathbb{M}^4 \cong \mathbb{R}^4$$

Sections allow us to define vector fields over spacetime:

$$s \in \Gamma(T\mathbb{M}^4)$$

The Cotangent Space

At this point, we introduce the cotangent space. The cotangent space at a point is defined as the dual of the tangent space:

$$T_p^*\mathcal{M} := \mathrm{Hom}_{\mathbb{R}}(T_p\mathcal{M}, \mathbb{R})$$

That is, a covector $\omega \in T_p^*\mathcal{M}$ is a linear functional:

$$\omega : T_p\mathcal{M} \to \mathbb{R}$$

Given a coordinate basis of $T_p\mathcal{M}$:

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}$$

the dual basis is:

$$\left\{ \left. dx^i \right|_p \right\}$$

satisfying the relation:

$$dx^i \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta^i_j$$

This is an example of a **vector bundle**, which is a fiber bundle where the fibers are vector spaces.

The Cotangent Bundle

Analogous to the tangent bundle, the cotangent bundle is:

$$T^*\mathcal{M} := \bigsqcup_{p \in \mathcal{M}} T_p^*\mathcal{M}$$

This forms a vector bundle over $\mathcal{M}$.

We define a section:

$$\omega \in \Gamma(T^*\mathcal{M})$$

which assigns to each $p \in \mathcal{M}$ a covector $\omega_p \in T_p^*\mathcal{M}$ smoothly. Such a section is called a **1-form**.

In a coordinate representation, a 1-form can be expressed as:

$$\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i \quad \text{with} \quad \omega_i \in C^\infty(\mathcal{M})$$

Tensor Products and the Metric Tensor

Since the fibers are vector spaces, we can define tensor products of bundles.

For example:

$$T^*\mathcal{M} \otimes T^*\mathcal{M} := \bigsqcup_{p\in\mathcal{M}} T_p^*\mathcal{M} \otimes T_p^*\mathcal{M}$$

This forms a bundle whose fibers are bilinear forms on the tangent space.

Sections of this bundle are called $(0,2)$-tensor fields. A prominent example is the metric tensor.

We define the Minkowski metric as:

$$\eta \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$$

In local coordinates:

$$\eta = \eta_{\mu\nu}\, dx^\mu \otimes dx^\nu \quad \text{with} \quad \eta_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$$

Notice that those sections are not called 2-forms. To define a **k-form**, we need to take the exterior product of two 1-forms, which is defined as follows:

Let $E \xrightarrow{\pi} B$ be a vector bundle and $\{e_\alpha\}$ be a basis of the fiber F. We define the **exterior product** as:

$$e_\alpha \wedge e_\beta \equiv e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha$$

We can define a vector bundle $\wedge^r(E)$ of totally antisymmetric r-tensors, called the **exterior power** of the vector bundle E. The fibers of $\wedge^r(E)$ are spanned by elements of the form $\{e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r}\}$

The set of k-forms is defined as:

$$\Omega^r(B) \equiv \Gamma(\wedge^r(T^*B))$$

The **exterior derivative** is then defined as a map [Nak05]:

$$\begin{aligned} d : \Omega^r(B) &\to \Omega^{r+1}(B) \\ \omega \mapsto d\omega &= \left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\ldots\mu_r}\right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \end{aligned}$$

The Structure Group of a Fiber Bundle

Above, we defined a fiber bundle as a quadruple (E, B, π, F) equipped with local trivializations. These local trivializations are given as diffeomorphisms on an open cover of the base space B. Thus, the definition does not require that $U_\alpha \cap U_\beta = \emptyset$. For a point $p \in U_\alpha \cap U_\beta$, we may have multiple local trivializations $\varphi_\alpha(p, f) = \varphi_{\alpha,p}(f)$ and $\varphi_\beta(p, f) = \varphi_{\beta,p}(f)$, defined on U_α and U_β, respectively.

We define the **structure group** G of a fiber bundle as the Lie group of diffeomorphisms relating these local trivializations. The corresponding transition function is given by

$$t_{\alpha\beta}(p) \equiv \varphi_{\alpha,p}^{-1} \circ \varphi_{\beta,p} : F \to F$$

This defines a smooth map $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfying the following properties:

$$\begin{aligned} t_{\alpha\alpha}(p) &= \mathrm{id}_F && \forall\, p \in U_\alpha \\ t_{\alpha\beta}(p) &= t_{\beta\alpha}(p)^{-1} && \forall\, p \in U_\alpha \cap U_\beta \\ t_{\alpha\beta}(p) \circ t_{\beta\gamma}(p) &= t_{\alpha\gamma}(p) && \forall\, p \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

In the case of the tangent bundle, the structure group is the general linear group $\mathrm{GL}(n, \mathbb{R})$, which consists of all invertible $n \times n$ matrices. This means that the transition functions between local trivializations are smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ that preserve linearity and invertibility.

A fiber bundle with transition maps identical to the identity map is called a **trivial bundle**. In this case, the total space E is diffeomorphic to the product space $B \times F$. In general, a fiber bundle has no unique trivialization. Let $\{\varphi_\alpha\}$ and $\{\tilde{\varphi}_\alpha\}$ be two local trivializations over the same open covering describing the same fiber bundle. Then these are related by maps $g_\alpha(p) : F \to F \quad \forall\, p \in B$ where each $g_\alpha(p)$ is a homeomorphism in the structure group G.

$$g_\alpha(p) \equiv \varphi_{\alpha,p}^{-1} \circ \tilde{\varphi}_{\alpha,p}$$

Again we take the tangent bundle as an example. Let U_i and U_j be overlapping chats with $p \in U_i \cap U_j$. With the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^j}\Big|_p \right\}$, a vector $v \in T_p\mathcal{M}$ can be expressed in both bases as:

$$v = v^\mu \frac{\partial}{\partial x^\mu} = \tilde{v}^\mu \frac{\partial}{\partial y^\mu}$$

The transition function $t^\nu{}_\mu$ is then given by:

$$\tilde{v}^\nu = \frac{\partial y^\nu}{\partial x^\mu}\bigg|_p v^\mu = t^\nu{}_\mu v^\mu$$

Chapter 4

Principal Bundles

A principal bundle is a fiber bundle $P \xrightarrow{\pi} M$ is a bundle whose fiber is identical to the structure group. This framework is of particular importance, because it allows to understand fibre bundles with fibre F on which G acts. These bundles are called *associated bundles* and are essential to understand Gauge theories in physics.

4.1 Action of a Lie Group on a Manifold

To understand this properly we first need to understand how a Lie group G can act on a manifold M [Fre15d] Let $(G, \cdot)$ be a Lie group, and M a smooth manifold. Then a smooth map

$$\triangleright : G \times M \longrightarrow M$$

satisfying

$$\begin{aligned} e \triangleright p &= p && \forall p \in M, \text{ and } e \text{ being the identity in } G \\ g_2 \triangleright (g_1 \triangleright p) &= (g_2 \cdot g_1) \triangleright p && \forall g_1, g_2 \in G,\ p \in M \end{aligned}$$

is called a **left G-action** on M.

Analogous a **right G-action** $\triangleleft : M \times G \longrightarrow M$ is defined, satisfying:

$$\begin{aligned} p \triangleleft e &= p && \forall p \in M, \text{ and } e \text{ being the identity in } G \\ (p \triangleleft g_1) \triangleleft g_2 &= p \triangleleft (g_1 g_2) && \forall g_1, g_2 \in G,\ p \in M \end{aligned}$$

Given a left action $\triangleright$, we can construct a right action:

$$\begin{aligned} \triangleleft :& M \times G \longrightarrow M, \\ & p \triangleleft g := g^{-1} \triangleright p. \end{aligned}$$

It is trivial to show that this yields a right action.

Let G be a Lie group acting smoothly on a manifold M from the left via

$$\triangleright : G \times M \longrightarrow M.$$

We define an equivalence relation $\sim$ on M by:

$$p \sim \tilde{p} \quad :\Longleftrightarrow \quad \exists g \in G \text{ such that } \tilde{p} = g \triangleright p.$$

This defines an equivalence relation:

- **Reflexivity:** $e \triangleright p = p$, so $p \sim p$.
- **Symmetry:** If $\tilde{p} = g \triangleright p$, then $p = g^{-1} \triangleright \tilde{p}$, so $\tilde{p} \sim p$.
- **Transitivity:** If $\tilde{p} = g_1 \triangleright p$ and $\hat{p} = g_2 \triangleright \tilde{p}$, then

$$\hat{p} = g_2 \triangleright (g_1 \triangleright p) = (g_2 g_1) \triangleright p,$$

so $p \sim \hat{p}$.

The **orbit** of a point $p \in M$ under the group action is then the equivalence class:

$$\mathcal{O}_p := \{\tilde{p} \in M \mid \exists g \in G : \tilde{p} = g \triangleright p\}.$$

We can then define the **quotient space** $M/\sim$, often denoted M/G, by identifyinf points that are in the same orbit.

We define the **stabilizer** of a point $p \in M$ as the set of elements in G that leave p unchanged:

$$S_p \coloneqq \{g \in G \mid g \triangleright p = p\}$$

An action $\triangleright$ is called free if for all $p \in M$ the stabilizer is trivial $S_p = \{e\}$.

4.2 Principal Bundles

We now define a principal fibre bundle as follows [Fre15d]

Let (P, π, M, F) be a fibre bundle. If the following conditions are satisfied, we call it a **principal G-bundle**:

(i) P is equipped with a right G-action $\triangleleft$,

(ii) The action of G is free,

(iii) $\pi : P \to M$ is isomorphic as a bundle to the quotient $\rho : P \to P/G$,

where $\rho(p) \mapsto [p]$ denotes the canonical projection onto the orbit space P/G.

To clarify, two bundles $\pi : E \to M$ and $\pi' : E' \to M'$ are isomorphic if there exist diffeomorphisms $\bar{f} : E \to E'$ and $f : M \to M'$ such that $\pi' \circ \bar{f} = f \circ \pi$.

Since $\triangleleft$ acts freely, each fibre $\rho^{-1}([p])$ is diffeomorphic to G.

The Frame Bundle As an example we consider the frame bundle of a smooth manifold M.

We define a **frame** at a point $p \in M$ with $\dim M = d$ as an ordered basis of the tangent space T_pM and the set of all frames at p as:

$$L_pM \coloneqq \{(e_1, \dots, e_d) \mid \{e_1, \dots, e_d\} \text{ is a basis of } T_pM\}$$

There exists a natural isomorphism

$$L_pM \cong \mathrm{GL}(d, \mathbb{R}),$$

by identifying each frame $(e_1, \dots, e_d) \in L_pM$ with the matrix whose columns are the components of the vectors e_i. Concretely, each frame is mapped to a matrix $g \in \mathrm{GL}(d, \mathbb{R})$ such that

$$g^{\mu}{}_{\alpha} = e^{\mu}{}_{\alpha}$$

Analogous to the tangent bundle, we can define the **frame bundle**:

$$LM := \bigsqcup_{p \in M} L_p M$$

Given a chart U_i on M, we can define a local trivialization of the frame bundle [Nak05]. A frame $\epsilon = \{e_1, \dots, e_d\}$ at $p \in M$ is expressed in terms of the natural basis of the tangent space T_pM $\{\partial/\partial x^\mu \mid_p\}$

$$e_\alpha = e^\mu{}_\alpha \, \partial/\partial x^\mu \mid_p \quad \text{where } e^\mu{}_\alpha \in \mathrm{GL}(d, \mathbb{R})$$

The local trivialization is then given by $\varphi_i^{-1}(u) = (p, (e^\mu{}_\alpha))$.

The projection π_L of a frame $\epsilon = \{e_1, \cdot, e_d\}$ at a point $p \in M$ is given by:

$$\begin{aligned} \pi_L \, : \, LM &\longrightarrow M, \\ \epsilon &\mapsto \pi_L(\epsilon) = p \end{aligned}$$

Therefore we have introduced the necessary structure, suck that $(LM, \pi, M, \mathrm{GL}(d, \mathrm{R})$ defines a fiber bundle.

The right action of $\mathrm{GL}(d, \mathbb{R})$ on the frame is defined as we would expect, since it is analogguss to the change of basis in a vector space:

$$\begin{aligned} \triangleleft : LM \times \mathrm{GL}(d, \mathbb{R}) &\longrightarrow LM, \\ (\epsilon, g) &\mapsto \epsilon \triangleleft g = (e_1, \cdots, e_d) \triangleleft g = (e_i g^i{}_1, \cdots, e_i g^i{}_d), \end{aligned}$$

To show that this Bundle equipped with this right action is a principal bundle, we need to check, if the action is free and the bundle is isomorphic to the quotient bundle.

Therefore we need to show that $e_i g^i{}_\alpha = e_\alpha \implies g^i{}_\alpha = \delta^i{}_\alpha \, \forall \alpha$. Since $\{e_1, \dots, e_d\}$ is a basis, they are linearly independent. Therefore the definition already implies that $g^i{}_\alpha = \delta^i{}_\alpha$ and thus the action is free.

To show the second condition, we need to show that the orbit space $LM/\mathrm{GL}(d, \mathbb{R})$ is consists of a single point for each $\epsilon \in \pi^{-1}(p)$. This is indeed the case, since the orbit of a frame at $p \in M$ under the action of $\mathrm{GL}(d, \mathbb{R})$ consists of all frames at p, since the action is transitiv. Thus the quotient space is diffeomorphic to M, with the diffeomorphism being: $f : M \longrightarrow LM/\mathrm{GL}(d, \mathbb{R})$, $p \mapsto [\epsilon]$ where ϵ is a frame at p.

Chapter 5

Connections on Principal Bundles

5.1 General Definition

Let $P \xrightarrow{\pi} M$ be a principal bundle with structure group G. The right action of G on P induces a vector field as follows: For each $A \in \mathfrak{g} \cong T_e G$ the action of the one parameter subgroup $\exp(tA)$ on $p \in P$ yields a curve. Since the group acts within the fiber $\pi(p) = \pi(p \triangleleft \exp(tA)) = p$. We define a vector $X_p^A \in T_p P$ by its action on a function $f \in C^\infty(P)$ [Nak05]:

$$X_p^A f = \frac{d}{dt} f(p \triangleleft \exp(tA)) \mid_{t=0}$$

Futhermore a vector space isomorphism is defined $i : \mathfrak{g} \longrightarrow \Gamma(TP)$ that assigns to each element $A \in \mathfrak{g}$ the vector field X^A. This is called a **fundamental vector field** on P.

We define the **Pushforward** [Pus25] of a smooth map $F : M \longrightarrow N$ of smooth manifolds M and N as the a map between the tangent spaces:

$$F_* : T_p M \longrightarrow T_{F(p)} N$$

By identifying $(F_* v)(f) = v(f \circ F)$ for $v \in T_p M$ and $f \in C^\infty(N)$.

The pushforward of the projection map $\pi_* : TP \longrightarrow TM$ allows the construction of the **vertical subspace** $V_p P \coloneqq \ker(\pi_*)$ at a point $p \in P$, which is a vector subspace of the tangent space of P.

Notice that that each fundamental vector $X_p^A \in V_p P$, since by construction $\pi_*(X_p^A) = 0$.

A **connection** on a principal bundle $P \xrightarrow{\pi} M$ is a separation of the tangent space $T_p P$ into a vertical subspace $V_p P$ and a **horizontal subspace** $H_p P$, by choosing a complement to the vertical subspace at each point $p \in P$ such that:

(i) $T_p P = H_p P \oplus V_p P$

(ii) $(\triangleleft g)_*(H_p P) = H_{p \triangleleft g} P \quad$ for all $g \in G$

(iii) For every smooth vector field $X \in \Gamma(TP)$, the unique decomposition $X = X^H + X^V$ with $X^H(p) \in H_p P$ and $X^V(p) \in V_p P$ yields smooth vector fields $X^H \in \Gamma(HP),\ X^V \in \Gamma(VP)$.

The condition (ii) ensures that when moving along the fibers by the action of G the horizontal subspace changes in a smooth way, while (iii) guarantees that moving along P the horizontal subspace changes smoothly.

5.2 Connection one-form

The choice of a horizontal subspace at each point $p \in P$ can be achieved by defining a Lie algebra valued one-form. The horizontal subspace is then interpreted as the kernel of this one-form. We define the **connection one-form** $\omega \in \mathfrak{g} \otimes T^*P$ as a $\mathfrak{g}$-valued one-form on P such that:

$$\begin{aligned} &\text{(i)} \quad \omega(X^A) = A \quad \text{for all } A \in \mathfrak{g} \\ &\text{(ii)} \quad (\triangleleft g)^*\omega = \mathrm{Ad}_{g^{-1}*}\omega \quad \text{for all } g \in G \end{aligned}$$

Here, $(\triangleleft g)^*\omega$ denotes the pullback [1] of the connection one-form by the right action of $g \in G$ on P, and $\mathrm{Ad}_{g^{-1}}$ is the adjoint action of g^{-1} on the Lie Group G. That is $\omega_{p\triangleleft g}(X_p(\triangleleft g)_*) = g^{-1} \cdot \omega_p(X_p) \cdot g$

It will be stated without proof that any connection one-form satisfying these properties induces a horizontal subspace H_pP that satisfies the conditions of a connection [Fre15a].

5.3 local connection form

The connection one-form as defined above is a global object on the principal bundle P. However, in practice it is often useful to work with local connection forms. As will be shown in the next chapter, this local connection form is identified with the gauge potential in physical gauge theories.

Consider an open covering $\{U_i\}$ of the base manifold M and local sections $\sigma_i : U_i \to \pi^{-1}(U_i)$. We define a Lie-algebra valued one-form

$$\mathcal{A}_i \equiv \sigma_i^*\omega \in \mathfrak{g} \otimes \Omega^1(U_i)$$

for a global connection one-form ω [Nak05]. Such a local connection form is called a **Yang-Mills field** [Fre15c]

Given a local section $\sigma_i : U_i \to P$, one obtains a local trivialization:

$$\begin{aligned} \psi_i : U_i \times G &\longrightarrow \pi^{-1}(U_i) \subset P \\ (p, g) &\mapsto \sigma_i(p) \triangleleft g \end{aligned}$$

This trivialization introduces a local representation of the global connection one-form ω via its pullback:

$$\begin{aligned} \psi_i^*\omega : T_{(p,g)}(U_i \times G) &\longrightarrow \mathfrak{g} \\ (\psi_i^*\omega)_{(p,g)}(X) &= \omega_{\sigma_i(p)\triangleleft g}\left((\psi_i)_* X\right) \end{aligned}$$

This local representation is related to the Yang-Mills field $\mathcal{A}_i$ by[Fre15c]:

$$(\psi_i^*\omega)_{(p,g)}(X) = \mathrm{Ad}_{g^{-1}*}\left(\mathcal{A}_i(X)\right) + \Xi_g(X)$$

[1] The **pullback** of a of a one-form $\omega \in \Gamma(T^*N)$ by a smooth map $\phi : M \to N$ between smoth manifolds is defined as $\phi^*\omega_p(X) = \omega_{\phi(p)}(\phi_{p*}(X))$ for $X \in T_pM$.[Pul24]

The above-used Ξ is the **Maurer–Cartan form** of the Lie group G. This form takes a tangent vector $v \in T_g G$ and maps it to the unique Lie algebra element (i.e., a tangent vector at the identity) that generates v via left translation:

$$\Xi(v) = (g^{-1}\triangleright)_* v.$$

This uses the fact that every tangent vector on G arises as the pushforward of a unique element of the Lie algebra $\mathfrak{g} = T_e G$ under the left action, i.e., for every $v \in T_g G$, there exists a unique $X \in \mathfrak{g}$ such that

$$v = (g\triangleright)_* X.$$

Thus, Ξ identifies the tangent bundle TG with $G \times \mathfrak{g}$ via left translation[Mau25].

5.4 Connection on the Frame Bundle

The Frame Bundle LM is of particular interest, because many groups relevant in physics are subgroups of the general linear group $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. Therefore in the following a local connection form and the Maurer-Cartan form will be derived.

Any choice of a chart (U_i, x) on the base manifold M induces a section on the frame bundle LM by associating to each point $m \in U_i$ the frame given by its coordinates. This section is denoted as:

$$\begin{aligned} \sigma : U_i &\longrightarrow LM \\ m &\mapsto \sigma_i(m) \coloneqq \left(\left.\frac{\partial}{\partial x^1}\right|_m, \ldots, \left.\frac{\partial}{\partial x^{\dim M}}\right|_m \right) \end{aligned}$$

Then the Yang-Mills field $\mathcal{A}_i = \sigma_i^* \omega$ is a one-form on U_i with values in the Lie algebra $\mathfrak{gl}(\dim M, \mathbb{R}) = \{M \mid M \text{ is a } n \times n \text{ matrix with components } M^\alpha_\beta \in \mathbb{R}\}$. The Yang-Mills field can be expressed in its components as:

$$(\mathcal{A}^i)^\alpha{}_{\beta\mu}$$

Where α, β are labels for the Lie algebra components and μ is the index of the base manifold.

The Maurer-Cartan form Ξ can be constructed as follows:

Let $gl \subseteq GL(d, \mathbb{R})$ be an open subset of the general linear group containing the identity. Coordinates are introduced by:

$$\begin{aligned} x : gl &\longrightarrow \mathbb{R} \\ g &\mapsto x(g)^a{}_b \coloneqq g^a{}_b \end{aligned}$$

Consider a left-invariant vector field L^A generated by the Lie algebra element $A \in \mathfrak{gl}(d, \mathbb{R})$. Since it is a vector field on the group, it acts on the coordinate functions:

$$\begin{aligned} \left(L^A x^a{}_b\right)_g &= x^a{}_b \frac{d}{dt} \left(g \cdot \exp(tA)\right) \Big|_{t=0} \\ &= \frac{d}{dt} \left(g^a{}_c \exp(tA)^c{}_b\right) \Big|_{t=0} \\ &= g^a{}_c A^c{}_b \end{aligned}$$

Therefore the components of the vector field are given by $L_g^A = g^a{}_b\, A^b{}_c\, \frac{\partial}{\partial x^a{}_c}$[Fre15c]

The Maurer-Cartan form Ξ then is defined as the one-form that maps the left-invariant vector field L^A to the Lie algebra element A:

$$(\Xi_g)^a_{\ b} = (g^{-1})^a{}_c(dx^c{}_b)$$

It can be easily checked that this expression satisfies the properties of a Maurer-Cartan form:

$$\begin{aligned}
\Xi_g(L_g^A) &= (g^{-1})^a_{\ c}\,(dx)^c_{\ b}\left(g^p_{\ r}\,A^r_{\ q}\,\frac{\partial}{\partial x^p_{\ q}}\right)\\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\left((dx)^c_{\ b}\,\frac{\partial}{\partial x^p_{\ q}}\right)\\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\,\delta^c_p\,\delta^q_b\\
&= (g^{-1})^a_{\ p}\,g^p_{\ r}\,A^r_{\ b}\\
&= A^a_{\ b}
\end{aligned}$$

5.5 Compatibility condition for local connection forms

It was stated before, that the local connection forms $\mathcal{A}_i$ relate to a unique global connection one-form ω. For this to be true, the local connection forms must satisfy a compatibility condition. This condition is given by the requirement that the local connection forms on overlapping charts $U_i \cap U_j \neq \emptyset$ are related by a gauge transformation[Nak05]. Specifically, let σ_i and σ_j be sections respectively defining Yang-Mills fields $\mathcal{A}_i$ and $\mathcal{A}_j$ on the overlapping region $U_i \cap U_j$. Introduce a gauge map

$$\Omega : U_i \cap U_j \longrightarrow G$$

defined by the relation

$$\sigma_j(m) = \sigma_i(m) \triangleleft \Omega(m) \quad \forall m \in U_i \cap U_j$$

Then the local connection forms are related as follows:

$$\mathcal{A}_j = \mathrm{Ad}_{\Omega^{-1}(m)*}\mathcal{A}_i + \Omega^*\Xi_m$$

In this will be shown for the case of the frame bundle LM. First, we calculate the latter expression. Notice that $\Omega^*\Xi_m$ is a map from the tangent space of the intersection on the base manifold $U_i \cap U_j$ to the Lie algebra $\mathfrak{gl}(d, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Therefore, to find the explicit form, it is calculated how this map acts on a vector in the tangent space:

$$\begin{aligned}
(\Omega^*\Xi)_p\left(\frac{\partial}{\partial x^\mu}\right)_p &= \Xi_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)\\
&= (\Omega^{-1}(p))^i_{\ k}(dx^k_{\ j})_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)\\
&= \Omega^{-1}(p)^i_{\ k}\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\left(x^k_{\ j}\right)\\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\left(x^k_{\ j}\circ\Omega\right)_p\\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\Omega(p)^k_{\ j}
\end{aligned}$$

Therefore, the components of the pullback of the Maurer–Cartan form are given by[Fre15c]:

$$((\Omega^*\Xi)_p)^i_{\ j} = \Omega^{-1}(p)^i_{\ k} \left(\frac{\partial}{\partial x^\mu}\right)_p \Omega(p)^k_{\ j}\, dx^\mu := \mathbf{\Omega}^{-1} d\mathbf{\Omega}$$

Futhermore, the pushforward of the adjoint action on the Yang-Mills field is easily obtained by definition of the adjoint action:

$$\begin{aligned} \mathrm{Ad}_g &: G \longrightarrow G\,, \quad h \mapsto ghg^{-1} \\ \mathrm{Ad}_{g*} &: T_eG \longrightarrow T_eG\,, \quad A \mapsto \mathbf{gAg}^{-1} \end{aligned}$$

Here the notation $\mathbf{g}$ is used to denote the matrix product, since the adjoint action is defined on the group G not the Lie algebra $\mathfrak{g}$.

Altogether the transition between two Yang-Mills fields on the intersection of two charts is given by:

$$\begin{aligned} \mathcal{A}_j &= \mathbf{\Omega}^{-1}\, \mathcal{A}_i\, \mathbf{\Omega} + \mathbf{\Omega}^{-1}\, d\mathbf{\Omega} \\ (\mathcal{A}_j)^i_{\ r\mu} &= \left(\Omega^{-1}(p)\right)^i_{\ k} (\mathcal{A}_i)^k_{\ l\mu}\, \Omega(p)^l_{\ r} + \left(\Omega^{-1}(p)\right)^i_{\ k} \partial_\mu \Omega(p)^k_{\ r} \end{aligned}$$

This is simply the **gauge transformation** as known form gauge theories[Nak05].

As an example, consider the case of a $U(1)$ principal bundle. The transition function Ω is a smooth function $U_i \cap U_j \longrightarrow U(1)$, which can be expressed as $\Omega(m) = \exp[i\Lambda(m)]$ for some real-valued function $\Lambda : U_i \cap U_j \longrightarrow \mathbb{R}$. Since $U(1)$ is a subgroup of $GL(d, \mathbb{C})$, two local connection forms $\mathcal{A}_i$ and $\mathcal{A}_j$ on the intersection $U_i \cap U_j$ are then related by:

$$\begin{aligned} \mathcal{A}_j &= \Omega^{-1}\, \mathcal{A}_i\, \Omega + \Omega^{-1}\, d\Omega \\ &= \mathcal{A}_i + e^{-i\Lambda(m)}\, d\left(e^{i\Lambda(m)}\right) \\ &= \mathcal{A}_i + e^{-i\Lambda(m)} \cdot ie^{i\Lambda(m)}\, d\Lambda \\ &= \mathcal{A}_i + i\, d\Lambda \end{aligned}$$

Which in components reads:

$$\mathcal{A}_{j\,\mu} = \mathcal{A}_{i\,\mu} + i\partial_\mu \Lambda$$

This is the familiar form of the gauge transformation in electromagnetism[Nak05].

Chapter 6

Curvature and Field Strength

6.1 horizontal lift and parallel transport

Consider a principal G bundle wiht total space P, base manifold M and structure group G. Let $\gamma : [0,1] \to M$ be a curve in M. A curve $\gamma^{\uparrow} : [0,1] \to P$ is called a **horizontal lift** of γ if it satisfies the following conditions:

$$\begin{aligned} (i) \quad & \pi \circ \gamma^{\uparrow} = \gamma \\ (ii) \quad & X^V(\gamma^{\uparrow}(t)) = 0 \quad \forall t \in [0,1] \\ (iii) \quad & \pi_*(X_{\gamma^{\uparrow}(t)}) = X_{\gamma(t)} \quad \forall t \in [0,1] \end{aligned}$$

where $X^V \in V_{\gamma^{\uparrow}(t)}P$ is the vertical vector field.

6.2 Curvature

Let P be a principal G-bundle with a connection one form ω and let $\phi \in \Omega^k(P) \otimes V$ be a V valued k-form on P, where V is some k-dimensional vector space with basis $\{e_i\}$. The connection one form ω allows for the separation of the tangent space of P into horizontal and vertical components. Then the map:

$$\begin{aligned} D\phi : \Gamma(T_u^{k+1}P) \to V, \\ (X_1, \dots X_{k+1}) \mapsto D\phi(X_1, \dots X_k) \coloneqq d\phi(X_1^H, \dots X_{k+1}^H) \end{aligned}$$

is called the **covariant derivative** of ϕ. Here $d\phi \equiv d\phi^i \otimes e_i$ is the exterior derivative.

This introduces the **curvature two-form** Ω as the covariant derivative of the connection one-form ω:

$$\Omega \equiv D\omega \in \Omega^2(P) \otimes \mathfrak{g}$$

First it will be shown, that Ω takes the following form:

$$\Omega = d\omega + \omega \wedge_{\mathfrak{g}} \omega$$

Where $\wedge_{\mathfrak{g}}$ denotes the wedge product in the Lie algebra $\mathfrak{g}$ of G defined by its action on $\Gamma(T^2P)$: $(\omega \wedge_{\mathfrak{g}} \omega)(X,Y) \coloneqq [\omega(X), \omega(Y)]_{\mathfrak{g}}$

Note that if G is a matrix group, then the above can be written in terms of its components as:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

We proof this by considering three separate cases[Fre15b]:

a) $X, Y \in \Gamma(TP)$ are vertical vector fields

$\Rightarrow$ $\exists A, B \in T_eG : X = X^A, \quad Y = X^B$

Left-hand side:

$$\Omega(X^A, X^B) = D\omega(X^A, X^B) = d\omega\left((X^A)^H, (X^B)^H\right)$$
$$= d\omega(0,0) = 0$$

Right-hand side:

$$d\omega(X^A, X^B) + (\omega \wedge_{\mathfrak{g}} \omega)(X^A, X^B)$$
$$= X^A(\omega(X^B)) - X^B(\omega(X^A)) - \omega([X^A, X^B]) + \big[\omega(X^A), \omega(X^B)\big]_{\mathfrak{g}}$$
$$= X^A(B) - X^B(A) - \omega(X^{[A,B]_{\mathfrak{g}}}) + [A, B]_{\mathfrak{g}}$$
$$= 0 - 0 - [A, B]_{\mathfrak{g}} + [A, B]_{\mathfrak{g}}$$
$$= 0$$

b) $X, Y \in \Gamma(TP)$ are horizontal vector fields

Left-hand side:

$$\Omega(X, Y) = D\omega(X, Y) = d\omega(X^H, Y^H)$$
$$= d\omega(X, Y)$$

Right-hand side:

$$d\omega(X, Y) + (\omega \wedge_{\mathfrak{g}} \omega)(X, Y)$$
$$= d\omega(X^H, Y^H) + \big[\omega(X), \omega(Y)\big]_{\mathfrak{g}}$$
$$= d\omega(X, Y) + \big[0, 0\big]_{\mathfrak{g}}$$
$$= d\omega(X, Y)$$

c) $X \in \Gamma(TP)$ is horizontal and $Y = X^A \in \Gamma(TP)$ is vertical

Left-hand side:

$$\Omega(X, X^A) = D\omega(X, X^A) = d\omega(X^H, (X^A)^H)$$
$$= d\omega(X, 0)$$
$$= 0$$

Right-hand side:

$$d\omega(X, X^A) + (\omega \wedge_{\mathfrak{g}} \omega)(X, X^A)$$
$$= d\omega(X, X^A) + \big[\omega(X), \omega(X^A)\big]_{\mathfrak{g}}$$
$$= X(\omega(X^A)) - X^A(\omega(X)) - \omega([X, X^A]) + \big[\omega(X), \omega(X^A)\big]_{\mathfrak{g}}$$
$$= X(A) - X^A(0) - \omega([X, X^A]) + \big[0, A\big]_{\mathfrak{g}}$$
$$= 0$$

Where in the last step the fact that the comutator of a horizontal and a vertical vector field is again a horizontal vector field was used[Nak05].

6.3 Local from of the curvature and Yang-Mills field strength

As the connection one-form ω can be expressed locally as the pullback by a section $\mathcal{A}_i = \sigma^*\omega$, the local from of the curvature two-form Ω is defined analogous[Nak05]:

$$\mathscr{F} \equiv \sigma^*\Omega \in \Omega^2(M) \otimes \mathfrak{g}$$

In terms of the local connection one-form $\mathcal{A}$, the curvature two-form can be expressed as:

$$\begin{aligned}\mathscr{F} &= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\ &= \sigma^*(d\omega) + \sigma^*(\omega \wedge_{\mathfrak{g}} \omega) \\ &= \sigma^*(d\omega) + \sigma^*(\omega) \wedge_{\mathfrak{g}} \sigma^*(\omega) \\ &= d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_j\end{aligned}$$

Let x^μ be the coordinates on the open set U_i where the section σ is defined. Then the Yang-Mills field is given by $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. We therefore get the following expression:

$$\begin{aligned}\mathscr{F} &= d(\mathcal{A}_\mu dx^\mu) + (\mathcal{A}_\mu dx^\mu \wedge_{\mathfrak{g}} \mathcal{A}_\nu dx^\nu) \\ &= \frac{1}{2}\left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]_{\mathfrak{g}}\right) dx^\mu \wedge dx^\nu \\ &:= \frac{1}{2} \mathscr{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu\end{aligned}$$

In physics, the local curvature two-form $\mathscr{F}$ is identified with the **Yang-Mills field strength**.

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First, compute the exterior derivative:

$$\begin{aligned}&d\left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \\ &= -\boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} \boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\mathcal{A}_i\,\boldsymbol{\Omega} \\ &\quad - \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\boldsymbol{\Omega} - \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}^{-1} \wedge_{\mathfrak{g}} d\boldsymbol{\Omega}\end{aligned}$$

Then, compute the wedge product:

$$\begin{aligned}&\left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \wedge_{\mathfrak{g}} \left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \\ &= \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i\,\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\boldsymbol{\Omega} \\ &\quad + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} \mathcal{A}_i\,\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} d\boldsymbol{\Omega}\end{aligned}$$

Combining both contributions, we obtain:

$$\begin{aligned}\mathscr{F}_j &= \boldsymbol{\Omega}^{-1}\left(d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i\right)\boldsymbol{\Omega} \\ &= \boldsymbol{\Omega}^{-1}\,\mathscr{F}_i\,\boldsymbol{\Omega}\end{aligned}$$

6.4 The Bianchi identity

The Bianchi identity states that the covariant derivative of the curvature two-form vanishes. To show this, the exterior derivative of the curvature two-form is computed:

$$d\Omega = d(d\omega) + d(\omega \wedge_{\mathfrak{g}} \omega) = d\omega \wedge_{\mathfrak{g}} \omega - \omega \wedge_{\mathfrak{g}} d\omega$$

Since for any $X \in H_pP$ the connection one-form vanishes, the following holds:

$$D\Omega(X,Y,Z) = d\omega(X^H, Y^H, Z^H) = 0$$

Therefore, the **Bianchi identity** is $D\Omega = 0$

Localy the Bianchi identity is given by:

$$\begin{aligned}
\sigma^* d\Omega &= d(\sigma^*\Omega) = d\mathscr{F} \\
&= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\
&= d\sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega + \sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega \\
&= d\mathcal{A} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} d\mathcal{A} \\
&= \mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}
\end{aligned}$$

Thus the Bianchi identity in local coordinates is given by:

$$D\mathscr{F} = d\mathscr{F} - (\mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}) = d\mathscr{F} + [\mathcal{A}, \mathscr{F}]_{\mathfrak{g}} = 0$$

Chapter 7

Gauge Theories

In physical gauge theories like electromagnetism, Yang-Mills theories or general relativity, the laws of nature they describe are not just differential equations that happen to describe nature, but they are deeply connected to the geometry of the underlying symmetries. In the following, the above developed mathematical framework is applied to recover Maxwell's equations, Yang-Mills theories.

7.1 Maxwell theory

Consider a $U(1)$ principal bundle P over the four dimensional Minkowski spacteime manifold M equipped with the Minkowski metric $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$. The principal bundle is trivial $P = M \times U(1)$, and the projection map is given by $\pi : P \to M$, $\pi(x, e^{i\Lambda}) = x$. The Yang-Mills field is given by:

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu \in \Omega^1(M) \otimes \mathfrak{u}(1)$$

And the field strength is given by the curvature two-form:

$$\mathscr{F} = d\mathcal{A}$$

We identify the **gauge potential** A by $\mathcal{A}_\mu = iA_\mu$ and the field strength tensor F by $\mathscr{F}_{\mu\nu} = iF_{\mu\nu}$, where i is the factor associated with the Lie algebra. Therefore, the curvature two-form can be written in components as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Bianchi identity is given by:

$$\begin{aligned} D\mathscr{F} &= d\mathscr{F} \\ &= \frac{1}{2} \partial_\mu \mathscr{F}_{\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho = 0 \\ \Rightarrow \quad & \partial_\mu \mathscr{F}_{\nu\rho} + \partial_\nu \mathscr{F}_{\rho\mu} + \partial_\rho \mathscr{F}_{\mu\nu} = 0 \end{aligned}$$

When identifying the electric and magnetic fields with the components of the field strength tensor, we have:

$$\begin{aligned} E_i &= F_{0i} \\ B_i &= \frac{1}{2} \epsilon_{ijk} F_{jk} \end{aligned}$$

The Bianchi identity yields the **homogeneous Maxwell equations**:

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

First, choosing indices $\mu = 0$, $\nu = i$, $\rho = j$ and using antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, we obtain:

$$\begin{aligned} &\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0 \\ \Rightarrow \quad &(\nabla \times \mathbf{E})_k + \partial_t B_k = 0 \end{aligned}$$

Now, choose $\mu = i$, $\nu = j$, $\rho = k$, all spatial indices. Contracting with the Levi-Civita tensor ϵ^{ijk} gives:

$$\begin{aligned} &\epsilon^{ijk} \partial_i F_{jk} = 0 \\ \Rightarrow \quad &\nabla \cdot \mathbf{B} = 0 \end{aligned}$$

Together, these two identities form the homogeneous Maxwell equations:

$$\begin{aligned} \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

7.2 Yang-Mills theory

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