From Differentiable Manifolds to Gauge Fields

Bachelor Thesis of the Faculty of Physics
Ludwig Maximilian University of Munich

submitted by

Jonas von Stein

Supervisor: Prof. Dr. Ivo Sachs
Arnold Sommerfeld Center for Theoretical Physics
Ludwig-Maximilians-Universität Munich

Munich, July 31, 2025

Von differenzierbaren Mannigfaltigkeiten zu Eichfeldern

Bachelorarbeit der Fakultät für Physik
Ludwig-Maximilians-Universität München

vorgelegt von

Jonas von Stein

Betreuer: Prof. Dr. Ivo Sachs
Arnold Sommerfeld Center for Theoretical Physics
Ludwig-Maximilians-Universität Munich

München, den 31. Juli 2025

Contents

Chapter 1

Introduction

Differential geometry plays a central role in the formulation of modern physics. Many physical theories, notably gauge theories and general relativity, are naturally expressed on smooth manifolds equipped with geometric structures such as tensors, connections, and curvature. These mathematical tools provide a coordinate-independent framework in which physical laws can be formulated and understood. In the 20th century Wu and Yang formalized the idea that gauge fields, such as those describing electromagnetism or the strong and weak nuclear forces, can be understood as connections on principal bundles[WY75]. In this context, the curvature of a connection corresponds directly to the physical field strength, while gauge transformations are interpreted as changes of local trivialization. This geometric formulation reveals the profound mathematical structure behind gauge invariance and highlights the important role of symmetry in physics. The goal of this thesis is to elucidate how such geometric structures naturally lead to gauge fields. Starting from the definition of differentiable manifolds, more refined concepts are introduced—vector bundles, principal bundles, and connections—culminating in a complete formulation of classical gauge theory within the framework of differential geometry. Throughout, the focus remains on building the theory in a mathematically consistent manner.

Structure

This thesis develops the geometric foundations of gauge theories by tracing a path from differentiable manifolds to fiber bundles, connections, and curvature. Starting with manifolds, tangent spaces, and bundles, the necessary tools from topology and geometry are introduced. Building on this, principal bundles and their connections are defined, leading to a geometric formulation of gauge fields as connection forms and their curvature as the field strength. Finally, classical gauge theories used in standard model physics are reintroduced within this framework, illustrating the reinterpretation of gauge fields as geometric objects.

Scope and Approach

This thesis is concerned primarily with the classical, differential-geometric formulation of gauge fields. All manifolds and bundles are assumed to be smooth and locally trivial. Quantum aspects, topological phenomena, and the coupling of gauge fields to matter are not treated. The emphasis lies on comprehending the mathematical structure underlying gauge invariance and field strength from a purely geometric perspective.

Objective

The primary objective is to demonstrate how the mathematical concept of a connection on a principal bundle provides a natural and rigorous formulation of gauge fields. This approach underscores that many features of physical theories, such as field strength and gauge invariance, are consequences of the underlying geometric structure.

Chapter 2

Manifolds

Manifolds are the fundamental spaces used in physics. They provide a framework to describe topological spaces that locally resemble Euclidean spaces, allowing for the application of known methods from calculus and linear algebra.

2.1 Preliminaries from topology

Although this thesis will not focus on introducing topology, a few important results will be given here, which are necessary to understand the definition of manifolds. The following definitions and theorems are taken from [Nak05]

A **topological space** is a set X equipped with a collection of open sets $\mathcal{T} = \{U_i \mid i \in I\}$ such that:

- $\emptyset, X \in \mathcal{T}$
- For any subcollection J of I the Union of corresponding open sets is itself an open set $\bigcup_{j \in J} U_j \in \mathcal{T}$
- For any finite subcollection K of I the intersection of the corresponding open sets is open: $\cap_{k \in K} U_k \in \mathcal{T}$

A family $\{O_i\}$ of (open) subsets of X is called an (open) covering of X if $X = \bigcup_i O_i$.

A subset N is called a **neighborhood** of a point $p \in X$ if there exists at least one open set $U \in \mathcal{T}$ such that $p \in U \subset N$. A topological space is called **Hausdorff** if for any two distinct points $p, q \in X$ there exist neighborhoods N_p, N_q such that $N_p \cap N_q = \emptyset$.

A map $f : X \to Y$ between two topological spaces is called **continuous** if for every open set $V \subset Y$ the preimage $f^{-1}(V)$ is an open set in X. If the inverse $f^{-1} : Y \to X$ is also continuous, then f is called a **homeomorphism**. Two topological spaces are called **homeomorphic** if there exists a homeomorphism between them.

2.2 Differentiable Manifolds

A Hausdorff topological space $(M, \mathcal{T})$ is called a **d-dimensional manifold** if there exists an open covering $\{U_i\}$ and a family of homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^d$. The pair (U_i, φ_i) is called a **chart** and the family $\{(U_i, \varphi_i)\}$ is called an **atlas**[Fre15e].

M is a **differentiable or smooth manifold** if for any U_i and U_j given that $U_i \cap U_j \neq \emptyset$ the transition function $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_i) \to \varphi_i(U_j \cap U_i)$ is infinite differentiable (C^∞)[Nak05]. In this thesis, smoothness will always be assumed, unless stated otherwise.

Let M and N be two differentiable manifolds of dimension m and n equipped with atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ respectively. A map $f : M \to N$ is called a **differentiable map** at a point $p \in M$ if for $p \in U_i$ and $f(p) \in V_j$ the composition $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$ is infinite differentiable. If f is also a homeomorphism and the inverse $f^{-1} : N \to M$ is differentiable, then f is called a **diffeomorphism**. M and N are called **diffeomorphic** if there exists a diffeomorphism between them. This will be denoted as $M \cong N$.

2.3 Spacetime Manifold M

A trivial but for obvious reasons important example of a differentiable manifold is the spacetime manifold M used in physics, which is defined as follows:

Let $M := \mathbb{R}^4$, the set of ordered 4-tuples $(x^\mu) \in \mathbb{R}^4$. The so called **standard topology** is defined by the open balls around a point $p \in M$ with radius $r > 0$:

$$B_r(p) := \{x \in \mathbb{R}^4 \mid \|x - p\| < r\}$$

with $\|\cdot\|$ the Euclidean norm:

$$\|x\|^2 = \sum_{\mu=0}^{3} (x^\mu)^2$$

This is obviously a Hausdorff[1], and locally Euclidean topological space. The identity map $\phi(p) = p$ covers M globally. Hence, $(\mathcal{M}, B_r(p), \varphi)$ is a smooth manifold.

[1] For two distinct points $p, q \in M$ it is always sufficient to choose $r = \frac{1}{2}\|q - p\|$

Chapter 3

Bundles

The definition of a vector on a Manifold is non-trivial because a vector space structure might not exist globally on the manifold. A Manifold may still be equipped with a Vector space structure locally. Thus, tangent spaces are introduced pointwise. Combining these local structures will lead naturally to the definition of fiber bundles.

3.1 Tangent Space T_pM

Let M be an n-dimensional smooth manifold. A tangent vector at a point $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule[Fre15b]:

$$v[fg] = v[f]g(p) + f(p)v[g]$$

A tangent vector at a point $p \in M$ can be constructed as the directional derivatives of an equivalence class of curves through p[Nak05].

Let $\gamma : [-\epsilon, \epsilon] \to M$ be a smooth curve in M with $\gamma(0) = p$. Then $x^\mu(\gamma(t)) \in \mathbb{R}^n$ is called the coordinate representation of γ induces by a chart (U, φ).

Let $f \in C^\infty(M)$ be a smooth function on M. The directional derivative of f along the curve γ at $t = 0$ is given by:

$$\begin{aligned}
\frac{d}{dt}(f \circ \gamma)(t)\bigg|_{t=0} &= \frac{d}{dt}\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma(t)\right)\bigg|_{t=0} \\
&= \frac{\partial f}{\partial x^r}\frac{dx^r}{dt}\bigg|_{t=0} \\
&= \frac{dx^r}{dt}\bigg|_{t=0}\frac{\partial f}{\partial x^r}\bigg|_p
\end{aligned}$$

The definition of a tangent vector is now obtained by introducing an equivalence relation on curves. Two curves γ_1 and γ_2 are called equivalent at $\gamma_1(0) = \gamma_2(0) = p$ if their derivatives at $t = 0$ are equal:

$$\frac{dx_1^r}{dt}\bigg|_{t=0} = \frac{dx_2^r}{dt}\bigg|_{t=0} = v^r$$

A tangent vector is identified with the differential operator given the equivalence class of curves. Once a chart (U, φ) is chosen, with local coordinates $(x^1, \ldots, x^n)$, a tangent vector is represented as a linear combination of partial derivatives with real coefficients.

$$v = v^r\,\frac{\partial}{\partial x^r}\bigg|_p$$

The tangent space at a point $p \in M$ is then defined as the set of all tangent vectors at p and is denoted by T_pM.

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \quad \text{form a basis of } T_pM$$

3.2 The Tangent Bundle as a Fiber Bundle

To introduce the concept of fiber bundles, a detailed examination of a specific example, the tangent bundle, serves as an effective foundation. The tangent bundle of a smooth n-dimensional manifold M is constructed by taking the disjoint union of all tangent spaces T_pM. This construction can be represented as:

$$TM := \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM$$

Here, T_pM denotes the tangent space at a point $p \in M$. The tangent space T_pM is a vector space that consists of all tangent vectors at the point p. It should be noted that this notation emphasizes the pointwise nature of the tangent bundle, where each tangent space is associated with a specific point on the manifold. Since each tangent space is a different object, a certain tangent vector in a tangent space can only belong to that specific tangent space. Therefore all the information is already in the vector itself and the direct product notation $\{p\} \times T_pM$ is only used to clarify[Fre15e].

Each element of the tangent bundle TM can be expressed as a pair (p, v), where p is a point on the manifold M, and v is a tangent vector belonging to the tangent space T_pM at that point.

A natural projection map is defined as follows:

$$\pi : TM \to M, \quad (p, v) \mapsto p$$

This projection, π, serves to "forget" the tangent vector v associated with each point, effectively collapsing all the tangent vectors at p to the single point p in the base manifold[Nak05].

The fiber over a point p is denoted as $\pi^{-1}(p) = T_pM$. This represents all the tangent vectors at the point p. Since T_pM is isomorphic to $\mathbb{R}^n$ as a vector space, it is referred to as the model fiber, denoted $F = \mathbb{R}^n$[Nak05].

Consider a coordinate chart (U, φ) on M. The chart φ provides a mapping from an open set $U \subseteq M$ to an open subset of $\mathbb{R}^n$. A diffeomorphism on the preimage $\pi^{-1}(U) \subset TM$ can then be defined as:

$$\Psi : \pi^{-1}(U) \to \mathbb{R}^{2n}, \quad (p, v) \mapsto \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n \right)$$

where the coordinates $x^i(p)$ represent the local coordinates of the point p in U, and $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$ describes the tangent vector in terms of its components in the chosen coordinate system[Nak05].

This establishes a local trivialization of the tangent bundle, expressed as:

$$TM|_U \cong U \times \mathbb{R}^n$$

In summary, the construction of the tangent bundle yields a fiber bundle characterized by the following essential components:

- **Total space:** $TM = \bigsqcup_{p \in M} T_pM$, encapsulating all tangent spaces.
- **Base space:** M, a manifold to which additional structure is added.
- **Projection:** $\pi : TM \to M$, mapping each tangent vector to its associated point on the manifold.
- **Model fiber:** $F = \mathbb{R}^n$, serving as the standard fiber structure over each point.
- **Local trivialization:** $TM|_U \cong U \times \mathbb{R}^n$, ensuring that the tangent bundle locally resembles a product structure.

Similar to the way a manifold is commonly perceived as a space that locally resembles $\mathbb{R}^n$, a fiber bundle may be conceptualized as a space that locally resembles the Cartesian product of the base space with a typical fiber structure.

Since the fiber of the tangent bundle is a vector space, the tangent bundle is also referred to as a **vector bundle**.

3.3 Definition of a Fiber Bundle

The formal definition of a fiber bundle reads as follows[Fre15e]. A fiber bundle is a quadruple (E, B, π, F) where:

- E is the total space
- B is the base space
- $\pi : E \to B$ is a surjective map called the projection
- F is the typical fiber

There exists an open cover $\{U_\alpha\}$ of B such that for each α, there is a diffeomorphism

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

where $\mathrm{proj}_1 : A \times B \to A$ is the first projection.

$(U_\alpha, \varphi_\alpha)$ are called *local trivialization.*

The fiber over a point $b \in B$ is:

$$F_b := \pi^{-1}(\{b\}) \cong F$$

Often the notations $E \xrightarrow{\pi} B$ or $\pi : E \to B$ are used to denote a fiber bundle.

3.3.1 The Structure Group of a Fiber Bundle

Above a fiber bundle was defined as a quadruple (E, B, π, F) equipped with local trivializations. These local trivializations are established as diffeomorphisms on an open cover $\{U_\alpha\}$ of the base space B. The definition does not impose the requirement that $U_\alpha \cap U_\beta = \emptyset$. For a point $p \in U_\alpha \cap U_\beta$, multiple local trivializations $\varphi_\alpha(p, f) = \varphi_{\alpha,p}(f)$ and $\varphi_\beta(p, f) = \varphi_{\beta,p}(f)$ may be present, defined on U_α and U_β, respectively.

The **structure group** G of a fiber bundle is defined as the Lie group of diffeomorphisms relating these local trivializations. The corresponding transition function is given by[Nak05]:

$$t_{\alpha\beta}(p) \equiv \varphi_{\alpha,p}^{-1} \circ \varphi_{\beta,p} : F \to F$$

This establishes a smooth map $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ that satisfies the following properties:

$$\begin{aligned} t_{\alpha\alpha}(p) &= \mathrm{id}_F && \forall\, p \in U_\alpha \\ t_{\alpha\beta}(p) &= t_{\beta\alpha}(p)^{-1} && \forall\, p \in U_\alpha \cap U_\beta \\ t_{\alpha\beta}(p) \circ t_{\beta\gamma}(p) &= t_{\alpha\gamma}(p) && \forall\, p \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

In the case of the tangent bundle, the structure group corresponds to the general linear group $\mathrm{GL}(n, \mathbb{R})$, which consists of all invertible $n \times n$ matrices. A fiber bundle with transition maps identical to the identity map is termed a **trivial bundle**. In this scenario, the total space E is diffeomorphic to the product space $B \times F$.

Generally, a fiber bundle does not possess a unique trivialization. Let $\{\varphi_\alpha\}$ and $\{\tilde{\varphi}_\alpha\}$ denote two local trivializations over the same open covering that describe the same fiber bundle. These trivializations are related by maps $g_\alpha(p) : F \to F \quad \forall\, p \in B$, where each $g_\alpha(p)$ is a homeomorphism within the structure group G. It will be defined in section 4.1 what it means for a Lie group to act on a manifold. The transition function between the two local trivializations is then given by:

$$g_\alpha(p) \equiv \varphi_{\alpha,p}^{-1} \circ \tilde{\varphi}_{\alpha,p}$$

Considering the tangent bundle as an illustrative example[Nak05], let U_i and U_j represent overlapping charts with $p \in U_i \cap U_j$. Utilizing the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^j}\Big|_p \right\}$, a vector $v \in T_pM$ can be expressed in both bases as:

$$v = v^\mu \frac{\partial}{\partial x^\mu} = \tilde{v}^\mu \frac{\partial}{\partial y^\mu}$$

The transition function $t^\nu{}_\mu$ is thus defined as:

$$\tilde{v}^\nu = \frac{\partial y^\nu}{\partial x^\mu}\bigg|_p v^\mu = t^\nu{}_\mu v^\mu$$

3.3.2 Sections

An important definition in the context of fiber bundles is that of a section or cross-section. This concept enables the selection of an element from each fiber over each point in a continuous manner, facilitating the introduction of ideas such as vector fields over spacetime[Nak05].

A **section** of a fiber bundle $\pi : E \to B$ is defined as a continuous map $s : B \to E$ such that

$$\pi \circ s = \mathrm{id}_B.$$

This condition ensures that exactly one point is chosen from each fiber continuously. The set of all (smooth) sections is denoted by:

$$\Gamma(E) := \{ s : M \to E \mid \pi \circ s = \mathrm{id}_M \}.$$

It is also possible to define a section locally on an open set $U \subset B$ as a map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$. In this case, the section is called a **local section**.

3.4 The cotangent bundle and differential forms

3.4.1 The Cotangent Bundle

In this section the fundamental concepts of the cotangent bundle are introduced, which is essential for the definition of differential forms and the exterior derivative. The cotangent space at a point p on a manifold M is defined as the dual space of the tangent space at that point. Formally, the cotangent space at a point $p \in M$ is the set of all linear maps from the tangent space at that point to the real numbers[Nak05].

$$T_p^*M := \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$$

A covector $\omega \in T_p^*M$ is such a linear function:

$$\omega : T_pM \to \mathbb{R}$$

As an example, consider a function $f \in C^\infty(M)$ and some tangent vector $v \in T_pM$. Then $v[f] \in \mathbb{R}$ by definition. The differential of f at a point p is a covector df_p and therefore $df_p[v] \in \mathbb{R}$ is simply defined as $df_p[v] = v[f]$. Given a coordinate basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ of T_pM the dual basis is:

$$\left\{ dx^i \Big|_p \right\}$$

satisfying the relation:

$$dx^i \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \delta^i_j$$

The example from before can thus be expressed as:

$$df_p = \frac{\partial f}{\partial x^i} dx^i \quad \text{and} \quad df_p[v] = v^i \left. \frac{\partial f}{\partial x^i} \right|_p$$

Analogous to the tangent bundle, the cotangent bundle is defined as the disjoint union of all cotangent spaces at each point in the manifold:

$$T^*M := \bigsqcup_{p \in M} T_p^*M$$

This structure constitutes a vector bundle over M. A section of the cotangent bundle can be defined as:

$$\omega \in \Gamma(T^*M)$$

This section assigns to each $p \in M$ a covector $\omega_p \in T_p^*M$ smoothly. Such a section is referred to as a **1-form**. In a coordinate representation, a 1-form can be expressed as:

$$\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i \quad \text{with} \quad \omega_i \in C^\infty(M)$$

3.4.2 Tensor Fields and the Metric Tensor

Utilizing the fact that the fibers of the cotangent bundle are vector spaces, tensor products of bundles can be defined. For instance:

$$T^*M \otimes T^*M := \bigsqcup_{p \in M} T^*_pM \otimes T^*_pM$$

This forms a bundle whose fibers consist of maps form $TM \otimes TM$ to $\mathbb{R}$. Sections of this bundle are referred to as $(0, 2)$-tensor fields. A prominent example of such fields is the metric tensor. The Minkowski metric is defined as:

$$\eta \in \Gamma(T^*M \otimes T^*M)$$

In local coordinates, the Minkowski metric can be expressed as:

$$\eta = \eta_{\mu\nu}\, dx^\mu \otimes dx^\nu \quad \text{with} \quad \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

In general, a tensor field of type (i, j) is a section of the bundle[Nak05]:

$$\otimes^i T^*M \otimes^j TM$$

3.4.3 Differential Forms and the Exterior Derivative

A **k-form** or **differential form** of order k is a totally antisymmetric $(k, 0)$ tensor. To define a k-form, it is necessary to take the **wedge product** of 1-forms, which is defined by taking the totally antisymmetrized tensor product of 1-forms. This means that all permutations of the tensor product are considered, with even permutations contributing positively and odd permutations contributing negatively. Consider the cotangent space T^*_pM of a manifold M at a point p with basis $\{dx^\mu\}$. The wedge product of two 1-forms dx^μ and dx^ν thus reads:

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

Higher order forms can be constructed analogously. By taking the cotangent bundle T^*M and considering the antisymmetrized tensor product, we can define fields of differential forms of order r as:

$$\Omega^r(M) \equiv \Gamma(\wedge^r(T^*M))$$

The **exterior derivative** is then defined as a map [Nak05]:

$$\begin{aligned} d : \Omega^r(B) &\to \Omega^{r+1}(B) \\ \omega \mapsto d\omega &= \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\ldots\mu_r}\right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \end{aligned}$$

In the following, two relations will be shown that will be used in later proofs[Nak05]. First consider the action of the exterior derivative on a 1-form $\omega = \omega_\mu dx^\mu \in \Omega^1(M)$ on two tangent vector fields $v = v^\mu \frac{\partial}{\partial x^\mu}$ and $w = w^\nu \frac{\partial}{\partial x^\nu}$:

$$\begin{aligned} d\omega(v, w) &= \left(\frac{\partial \omega_\mu}{\partial x^\nu}\, dx^\nu \wedge dx^\mu\right)(v, w) \\ &= \frac{\partial \omega_\mu}{\partial x^\nu}\left(dx^\nu(v)\, dx^\mu(w) - dx^\nu(w)\, dx^\mu(v)\right) \\ &= \frac{\partial \omega_\mu}{\partial x^\nu}\left(v^\nu w^\mu - w^\nu v^\mu\right) \\ &= v^\nu \frac{\partial}{\partial x^\nu}\left(\omega_\mu w^\mu\right) - w^\nu \frac{\partial}{\partial x^\nu}\left(\omega_\mu v^\mu\right) - \omega_\mu\left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\right) \\ &= v[\omega(w)] - w[\omega(v)] - \omega([v, w]) \end{aligned}$$

This gives a coordinate free expression for the action of the exterior derivative of a 1-form.Furthermore, it can easily be shown that the exterior derivative of a exterior derivative is zero, by using the fact that the product of a symmetric and an antisymmetric tensor is zero. By definition the following is obtained:

$$d^2\omega = \frac{1}{r!}\left(\frac{\partial^2}{\partial x^\alpha x^\beta}\omega_{\mu_1\dots\mu_r}\right) dx^\alpha \wedge dx^\beta \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

From this it follows instantly that $d^2 = 0$ since $\frac{\partial^2}{\partial x^\alpha x^\beta}$ is symmetric and the wedge product is antisymmetric.

Chapter 4

Principal Bundles

A principal bundle is a fiber bundle $P \xrightarrow{\pi} M$ characterized by a fiber that is identical to the structure group. This framework is particularly significant as it facilitates the underlying structure of gauge theories in physics, where the fiber represents the gauge group and the base manifold M represents spacetime. In this chapter, it will first be elaborated how a Lie group acts on a manifold. Then, the definition of a principal bundle will be given, followed by an example of the frame bundle.

4.1 Action of a Lie Group on a Manifold

To understand this concept thoroughly, it is necessary to examine how a Lie group G can act on a manifold M [Fre15d][DG18]. Let $(G, \cdot)$ denote a Lie group, and M a smooth manifold. A smooth map

$$\triangleright : G \times M \longrightarrow M$$

is defined as a **left G-action** on M if it satisfies the following conditions:

$$\begin{aligned} e \triangleright p &= p && \forall p \in M, \text{ where } e \text{ is the identity in } G \\ g_2 \triangleright (g_1 \triangleright p) &= (g_2 \cdot g_1) \triangleright p && \forall g_1, g_2 \in G,\, p \in M. \end{aligned}$$

Analogously, a **right G-action** $\triangleleft$ is defined by the map $\triangleleft : M \times G \longrightarrow M$, which satisfies:

$$\begin{aligned} p \triangleleft e &= p && \forall p \in M, \text{ where } e \text{ is the identity in } G \\ (p \triangleleft g_1) \triangleleft g_2 &= p \triangleleft (g_1 \cdot g_2) && \forall g_1, g_2 \in G,\, p \in M. \end{aligned}$$

Given a left action $\triangleright$, it is possible to construct a right action as follows:

$$\begin{aligned} \triangleleft : &M \times G \longrightarrow M, \\ &p \triangleleft g := g^{-1} \triangleright p. \end{aligned}$$

This transformation yields a valid right action since the inverse of the identity is again the identity and $(g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1}$.

Let G be a Lie group acting smoothly on a manifold M from the left via

$$\triangleright : G \times M \longrightarrow M.$$

An equivalence relation $\sim$ on M is defined by:

$$p \sim \tilde{p} \quad :\Longleftrightarrow \quad \exists g \in G \text{ such that } \tilde{p} = g \triangleright p.$$

This equivalence relation can be verified through the following properties:

- **Reflexivity:** $e \triangleright p = p$, thus $p \sim p$.
- **Symmetry:** If $\tilde{p} = g \triangleright p$, then $p = g^{-1} \triangleright \tilde{p}$, leading to $\tilde{p} \sim p$.
- **Transitivity:** If $\tilde{p} = g_1 \triangleright p$ and $\hat{p} = g_2 \triangleright \tilde{p}$, then

$$\hat{p} = g_2 \triangleright (g_1 \triangleright p) = (g_2 g_1) \triangleright p,$$

  which implies $p \sim \hat{p}$.

The **orbit** of a point $p \in M$ under the group action constitutes the equivalence class:

$$\mathcal{O}_p := \{\tilde{p} \in M \mid \exists g \in G : \tilde{p} = g \triangleright p\}.$$

Consequently, the **quotient space** $M/\sim$, commonly denoted as M/G, is defined by identifying points within the same orbit.

Additionally, the **stabilizer** of a point $p \in M$ represents the set of elements in G that leave p unchanged:

$$S_p := \{g \in G \mid g \triangleright p = p\}.$$

An action $\triangleright$ is termed free if the stabilizer is trivial for all $p \in M$, i.e., $S_p = \{e\}$.

4.2 Principal Bundles

A principal fibre bundle is defined as follows [DG18]:

Let (P, π, M, F) be a fibre bundle. If the following conditions are satisfied, it is classified as a **principal G-bundle**:

(i) P is equipped with a right G-action $\triangleleft$,

(ii) The action of G is free,

(iii) $\pi : P \to M$ is isomorphic as a bundle to the quotient $\rho : P \to P/G$,

where $\rho(p) \mapsto [p]$ denotes the canonical projection onto the orbit space P/G. Since the action $\triangleleft$ is free, each fibre $\rho^{-1}([p])$ is diffeomorphic to G.

For clarification, two bundles $\pi : E \to M$ and $\pi' : E' \to M'$ are considered isomorphic if there exist diffeomorphisms $\bar{f} : E \to E'$ and $f : M \to M'$ such that $\pi' \circ \bar{f} = f \circ \pi$. Thus, (iii) can be reformulated as the existence of a diffeomorphism $f : M \to P/G$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\mathrm{id}_P"] \arrow[d, "\pi"'] & P \arrow[d, "\rho"] \\ M \arrow[r, "f"] & P/G \end{tikzcd}$$

Figure 4.1: Commutative diagram showing the isomorphism between a principal G-bundle $\pi : P \to M$ and its quotient bundle $\rho : P \to P/G$.

4.3 The Frame Bundle

As an example, consider the frame bundle of a smooth manifold M.

A **frame** at a point $p \in M$ with $\dim M = d$ is defined as an ordered basis of the tangent space T_pM. The set of all frames at p is given by:

$$L_pM \coloneqq \{(e_1, \ldots, e_d) \,|\, \{e_1, \ldots, e_d\} \text{ is a basis of } T_pM\}\,.$$

There exists a natural isomorphism

$$L_pM \cong \mathrm{GL}(d, \mathbb{R}),$$

by identifying each frame $(e_1, \ldots, e_d) \in L_pM$ with the matrix whose columns are the vectors e_i. Specifically, each frame is mapped to a matrix $g \in \mathrm{GL}(d, \mathbb{R})$ such that

$$g^\mu{}_\alpha = e^\mu{}_\alpha.$$

Analogous to the tangent bundle, the **frame bundle** can be defined as:

$$LM := \bigsqcup_{p \in M} L_pM.$$

Given a chart U_i on M, a local trivialization of the frame bundle can be defined [Nak05]. A frame $\epsilon = (e_1, \ldots, e_d)$ at $p \in M$ can be expressed in terms of the natural basis of the tangent space T_pM as $\{\partial/\partial x^\mu \,|_p\}$:

$$e_\alpha = e^\mu{}_\alpha \, \partial/\partial x^\mu \,|_p, \quad \text{where } e^\mu{}_\alpha \in \mathrm{GL}(d, \mathbb{R}).$$

The local trivialization is then given by $\varphi_i^{-1}(u) = (p, (e^\mu{}_\alpha))$.

The projection π_L of a frame $\epsilon = \{e_1, \cdots, e_d\}$ at a point $p \in M$ is specified by:

$$\begin{aligned} \pi_L \,:\, LM &\longrightarrow M, \\ \epsilon &\mapsto \pi_L(\epsilon) = p. \end{aligned}$$

Thus, the necessary structure has been introduced such that $(LM, \pi_L, M, \mathrm{GL}(d, \mathbb{R}))$ defines a fiber bundle.

The right action of $\mathrm{GL}(d, \mathbb{R})$ on the frame is analogous to a change of basis in a vector space. This action is defined as follows:

$$\begin{aligned} \triangleleft : LM \times \mathrm{GL}(d, \mathbb{R}) &\longrightarrow LM, \\ (\epsilon, g) &\mapsto \epsilon \triangleleft g = (e_1, \cdots, e_d) \triangleleft g = (e_i g^i{}_1, \cdots, e_i g^i{}_d). \end{aligned}$$

To demonstrate that this bundle, equipped with the right action, is a principal bundle, it is necessary to verify that the action is free and that the bundle is isomorphic to the quotient bundle.

First, it must be shown that $e_i {g^i}_\alpha = e_\alpha \implies {g^i}_\alpha = {\delta^i}_\alpha \, \forall \alpha$. Since $\{e_1, \ldots, e_d\}$ forms a basis, the vectors are linearly independent. Therefore, the definition implies that ${g^i}_\alpha = {\delta^i}_\alpha$, confirming that the action is free.

Next, it is necessary to show that the orbit space $LM/\mathrm{GL}(d, \mathbb{R})$ consists of a single point for each $\epsilon \in \pi^{-1}(p)$. This is indeed the case, as the orbit of a frame at $p \in M$ under the action of $\mathrm{GL}(d, \mathbb{R})$ includes all frames at p, since the action is transitive. Hence, the quotient space is diffeomorphic to M, with the diffeomorphism defined by: $f : M \longrightarrow LM/\mathrm{GL}(d, \mathbb{R})$, $p \mapsto [\epsilon]$, where ϵ is a frame at p.

Thus, the frame bundle $(LM, \pi_L, M, \mathrm{GL}(d, \mathbb{R}))$ is indeed a principal $\mathrm{GL}(d, \mathbb{R})$-bundle.

Chapter 5

Connections on Principal Bundles

5.1 General Definition

A Connection is a consistent way to separate the tangent space of a principal bundle into a vertical subspace tangent to the fiber and a horizontal subspace that complements the former. This will be constructed by considering a vector field generated by the right action of the one-parameter subgroup of the structure group G on the principal bundle. As before a vector is defined by its action on a function evaluated along a curve which will be generated by the former mentioned right action.

Let $P \xrightarrow{\pi} M$ be a principal bundle with structure group G. The right action of G on P induces a vector field as follows: For each $A \in \mathfrak{g} \cong T_eG$, the action of the one-parameter subgroup $\exp(tA)$ on an element $p \in P$ yields a curve. Since the group acts within the fiber, it holds that $\pi(p) = \pi(p \triangleleft \exp(tA)) = p$. A vector $X_p^A \in T_pP$ is defined by its action on a function $f \in C^\infty(P)$ [Nak05]:

$$X_p^A f = \frac{d}{dt} f(p \triangleleft \exp(tA)) \mid_{t=0} .$$

Furthermore, a vector space isomorphism $i : \mathfrak{g} \longrightarrow \Gamma(TP)$ is defined that assigns to each element $A \in \mathfrak{g}$ the vector field X^A which is generated by A. This vector field is referred to as a **fundamental vector field** on P.

The **Pushforward** [Pus25] of a smooth map $F : M \longrightarrow N$ between smooth manifolds M and N is defined as a map between the tangent spaces:

$$F_* : T_pM \longrightarrow T_{F(p)}N.$$

This is identified using $(F_*v)(f) = v(f \circ F)$ for $v \in T_pM$ and $f \in C^\infty(N)$.

The pushforward of the projection map $\pi_* : TP \longrightarrow TM$ facilitates the construction of the **vertical subspace** $V_pP \coloneqq \ker(\pi_*)$ at a point $p \in P$, which serves as a vector subspace of the tangent space of P. It is noteworthy that each fundamental vector $X_p^A \in V_pP$, given that $\pi_*(X_p^A) = 0$ by construction.

A **connection** on a principal bundle $P \xrightarrow{\pi} M$ is defined as a decomposition of the tangent space T_pP into a vertical subspace V_pP and a **horizontal subspace** H_pP. This is achieved by choosing a complement to the vertical subspace at each point $p \in P$ such that[DG18]:

(i) $T_pP = H_pP \oplus V_pP$

(ii) $(\triangleleft g)_*(H_pP) = H_{p\triangleleft g}P \quad \text{for all } g \in G$

(iii) For every smooth vector field $X \in \Gamma(TP)$, the unique decomposition $X = X^H + X^V$ with $X^H(p) \in H_pP$ and $X^V(p) \in V_pP$ produces smooth vector fields $X^H \in \Gamma(HP)$, $X^V \in \Gamma(VP)$.

Condition (ii) ensures that when moving along the fibers via the action of G, the horizontal subspace changes smoothly, while condition (iii) guarantees that the horizontal subspace varies smoothly when traversing the manifold P.

5.2 Connection One-Form

The choice of a horizontal subspace at each point $p \in P$ can be achieved by defining a Lie algebra-valued one-form. The horizontal subspace is then interpreted as the kernel of this one-form. The **connection one-form** $\omega \in \mathfrak{g} \otimes T^*P$ is defined as a $\mathfrak{g}$-valued one-form on P such that:

$$\begin{aligned}
&\text{(i)} \quad \omega(X^A) = A \quad \text{for all } A \in \mathfrak{g} \\
&\text{(ii)} \quad (\triangleleft g)^*\omega = \mathrm{Ad}_{g^{-1}*}\omega \quad \text{for all } g \in G
\end{aligned}$$

Here, $(\triangleleft g)^*\omega$ denotes the pullback of the connection one-form by the right action of $g \in G$ on P, and $\mathrm{Ad}_{g^{-1}}$ is the adjoint action of g^{-1} on the Lie Group G. This implies that $\omega_{p\triangleleft g}(X_p(\triangleleft g)_*) = g^{-1} \cdot_* \omega_p(X_p) \cdot_* g$. The horizontal subspace H_pP is then defined as the kernel of the connection one-form[Nak05]

$$H_pP \equiv \{X \in T_pP \mid \omega(X) = 0\}$$

This is consistent with the general definition of a connection. The smoothness of the decomposition is guaranteed by the fact that the connection one-form is a section, which is smooth by definition. It is therefore sufficient to show that the horizontal subspace is invariant under the right action of G. To show this, consider a vector X at a point $p \in P$ such that $X \in H_pP$. By definition $\omega(X) = 0$. For any element $g \in G$ the pushforward of the right action on X can be acted on by ω[Nak05]:

$$\omega((\triangleleft g)_*X) = (\triangleleft g)^*\omega(X) = g^{-1} \cdot_* \omega(X) \cdot_* g = 0$$

Therefore $(\triangleleft g)^*X$ is again a horizental vector at the point $p \triangleleft g$. Futhermore, any vector $\tilde{X} \in H_{p\triangleleft g}P$ can thus be obtained by the right action on some vector $X \in H_pP$. Therefore $(\triangleleft g)_*(H_pP) = H_{p\triangleleft g}P$

5.3 Local Connection Form

The connection one-form, as defined above, is a global object on the principal bundle P. However, in practice, it is often useful to work with local connection forms, which will be identified with the gauge potential in physical gauge theories.

Consider an open covering $\{U_i\}$ of the base manifold M and local sections $\sigma_i : U_i \to \pi^{-1}(U_i)$. A Lie-algebra valued one-form $\mathcal{A}_i \equiv \sigma_i^*\omega \in \mathfrak{g} \otimes \Omega^1(U_i)$ is defined for a global connection one-form ω [Nak05]. This local connection form is termed a **Yang-Mills field** [Fre15c].

Given a local section $\sigma_i : U_i \to P$, a local trivialization is established:

$$\begin{aligned} \psi_i : U_i \times G &\longrightarrow \pi^{-1}(U_i) \subset P \\ (p, g) &\mapsto \sigma_i(p) \triangleleft g \end{aligned}$$

This trivialization introduces a local representation of the global connection one-form ω via its pullback:

$$\begin{aligned} \psi_i^*\omega : T_{(p,g)}(U_i \times G) &\longrightarrow \mathfrak{g} \\ (\psi_i^*\omega)_{(p,g)}(X) &= \omega_{\sigma_i(p)\triangleleft g}\left((\psi_i)_* X\right) \end{aligned}$$

The relations of the above maps are illustrated in the following diagram:

$$\begin{tikzcd}
\psi_i^*\omega \arrow[dr, dashed] & & \\
& U_i \times G \arrow[r, "\psi_i"] \arrow[d, "\mathrm{proj}_1"'] & \pi^{-1}(U_i) \subset P \arrow[d, "\pi"] \\
& U_i \arrow[r, "\mathrm{id}"'] \arrow[ur, "\sigma_i"] & M \\
\sigma_i^*\omega \arrow[ur, dashed] & &
\end{tikzcd}$$

Figure 5.1: Local trivialization ψ_i and local connection forms $\psi_i^*\omega$ and $\sigma_i^*\omega$ associated to $U_i \times G$ and U_i, respectively.

This local representation is related to the Yang-Mills field $\mathcal{A}_i$ by [Fre15c]:

$$(\psi_i^*\omega)_{(p,g)}(X) = \mathrm{Ad}_{g^{-1}*}\left(\mathcal{A}_i(X)\right) + \Xi_g(X)$$

Here, Ξ denotes the **Maurer–Cartan form** of the Lie group G. This form takes a tangent vector $v \in T_gG$ and maps it to the unique Lie algebra element (i.e., a tangent vector at the identity) that generates v via left translation:

$$\Xi(v) = (g^{-1}\triangleright)_* v \in T_eG \cong \mathfrak{g}$$

This formulation exploits the fact that every tangent vector on G arises as the pushforward of a unique element of the Lie algebra $\mathfrak{g} = T_eG$ under left action[Rag25]. Thus, for every $v \in T_gG$, there exists a unique $X \in \mathfrak{g}$ such that

$$v = (g\triangleright)_* X.$$

Consequently, Ξ identifies the tangent bundle TG with $G \times \mathfrak{g}$ via left translation [Mau25].

5.4 Connection on the Frame Bundle

The Frame Bundle LM is of particular interest, because many groups relevant in physics are subgroups of the general linear group $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. Therefore in the following a local connection form and the Maurer-Cartan form will be derived.

Any choice of a chart (U_i, x) on the base manifold M induces a section on the frame bundle LM by associating to each point $m \in U_i$ the frame given by its coordinates. This section is denoted as:

$$\begin{aligned} \sigma : U_i &\longrightarrow LM \\ m &\mapsto \sigma_i(m) := \left(\frac{\partial}{\partial x^1}\bigg|_m, \ldots, \frac{\partial}{\partial x^{\dim M}}\bigg|_m \right) \end{aligned}$$

Then the Yang-Mills field $\mathcal{A}_i = \sigma_i^* \omega$ is a one-form on U_i with values in the Lie algebra $\mathfrak{gl}(\dim M, \mathbb{R}) = \{M \mid M \text{ is a } n \times n \text{ matrix with components } M^\alpha_\beta \in \mathbb{R}\}$. The Yang-Mills field can be expressed in its components as:

$$(\mathcal{A}^i)^\alpha{}_{\beta\mu}$$

Where α, β are labels for the Lie algebra components and μ is the index of the base manifold. The Maurer-Cartan form Ξ can be constructed as follows:

Let $gl \subseteq GL(d,\mathbb{R})$ be an open subset of the general linear group containing the identity. Coordinates are introduced by:

$$\begin{aligned} x : gl &\longrightarrow \mathbb{R} \\ g &\mapsto x(g)^a{}_b := g^a{}_b \end{aligned}$$

Consider a left-invariant vector field L^A generated by the Lie algebra element $A \in \mathfrak{gl}(d,\mathbb{R})$. Since it is a vector field on the group, it acts on the coordinate functions:

$$\begin{aligned} \left(L^A x^a{}_b\right)_g &= x^a{}_b \frac{d}{dt} \left(g \cdot \exp(tA)\right)\bigg|_{t=0} \\ &= \frac{d}{dt} \left(g^a{}_c \exp(tA)^c{}_b\right)\bigg|_{t=0} \\ &= g^a{}_c A^c{}_b \end{aligned}$$

Therefore the components of the vector field are given by $L^A_g = g^a{}_b \, A^b{}_c \, \frac{\partial}{\partial x^a{}_c}$[Fre15c]

The Maurer-Cartan form Ξ then is defined as the one-form that maps the left-invariant vector field L^A to the Lie algebra element A:

$$(\Xi_g)^a{}_b = (g^{-1})^a{}_c (dx^c{}_b)$$

It can be easily checked that this expression satisfies the properties of a Maurer-Cartan form:

$$
\begin{aligned}
\Xi_g(L_g^A) &= (g^{-1})^a_{\ c}\,(dx)^c_{\ b}\left(g^p_{\ r}\,A^r_{\ q}\,\frac{\partial}{\partial x^p_q}\right)\\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\left((dx)^c_{\ b}\,\frac{\partial}{\partial x^p_q}\right)\\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\,\delta^c_p\,\delta^q_b\\
&= (g^{-1})^a_{\ p}\,g^p_{\ r}\,A^r_{\ b}\\
&= A^a_{\ b}
\end{aligned}
$$

5.5 Compatibility condition for local connection forms

It was stated before, that the local connection forms $\mathcal{A}_i$ relate to a unique global connection one-form ω. For this to be true, the local connection forms must satisfy a compatibility condition. This condition is given by the requirement that the local connection forms on overlapping charts $U_i \cap U_j \neq \emptyset$ are related by a gauge transformation[Nak05]. Specifically, let σ_i and σ_j be sections respectively defining Yang-Mills fields $\mathcal{A}_i$ and $\mathcal{A}_j$ on the overlapping region $U_i \cap U_j$. Introduce a gauge map

$$\Omega : U_i \cap U_j \longrightarrow G$$

defined by the relation

$$\sigma_j(m) = \sigma_i(m) \triangleleft \Omega(m) \quad \forall m \in U_i \cap U_j$$

Then the local connection forms are related as follows:

$$\mathcal{A}_j = \mathrm{Ad}_{\Omega^{-1}(m)*}\mathcal{A}_i + \Omega^*\Xi_m$$

In this will be shown for the case of the frame bundle LM. First, we calculate the latter expression. Notice that $\Omega^*\Xi_m$ is a map from the tangent space of the intersection on the base manifold $U_i \cap U_j$ to the Lie algebra $\mathfrak{gl}(d, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Therefore, to find the explicit form, it is calculated how this map acts on a vector in the tangent space:

$$
\begin{aligned}
(\Omega^*\Xi)_p\left(\frac{\partial}{\partial x^\mu}\right)_p &= \Xi_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)\\
&= (\Omega^{-1}(p))^i_{\ k}(dx^k_j)_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)\\
&= \Omega^{-1}(p)^i_{\ k}\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\left(x^k_j\right)\\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\left(x^k_j \circ \Omega\right)_p\\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\Omega(p)^k_{\ j}
\end{aligned}
$$

Therefore, the components of the pullback of the Maurer–Cartan form are given by[Fre15c]:

$$((\Omega^*\Xi)_p)^i_{\ j} = \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\Omega(p)^k_{\ j}\,dx^\mu := \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}$$

Futhermore, the pushforward of the adjoint action on the Yang-Mills field is easily obtained by definition of the adjoint action:

$$\begin{aligned}\mathrm{Ad}_g : G \longrightarrow G\,, \quad h \mapsto ghg^{-1}\\ \mathrm{Ad}_{g*} : T_eG \longrightarrow T_eG\,, \quad A \mapsto \mathbf{gAg}^{-1}\end{aligned}$$

Here the notation $\mathbf{g}$ is used to denote the matrix product, since the adjoint action is defined on the group G not the Lie algebra $\mathfrak{g}$.

Altogether the transition between two Yang-Mills fields on the intersection of two charts is given by:

$$\begin{aligned}\mathcal{A}_j &= \mathbf{\Omega}^{-1}\,\mathcal{A}_i\,\mathbf{\Omega} + \mathbf{\Omega}^{-1}\,d\mathbf{\Omega}\\ (\mathcal{A}_j)^i{}_{r\mu} &= \left(\Omega^{-1}(p)\right)^i{}_k\,(\mathcal{A}_i)^k_{l\mu}\,\Omega(p)^l{}_r + \left(\Omega^{-1}(p)\right)^i{}_k\,\partial_\mu\Omega(p)^k_r\end{aligned}$$

This is simply the **gauge transformation** as known form gauge theories[Nak05].

As an example, consider the case of a $U(1)$ principal bundle. The transition function Ω is a smooth function $U_i \cap U_j \longrightarrow U(1)$, which can be expressed as $\Omega(m) = \exp[i\Lambda(m)]$ for some real-valued function $\Lambda : U_i \cap U_j \longrightarrow \mathbb{R}$. Since $U(1)$ is a subgroup of $GL(d, \mathbb{C})$, two local connection forms $\mathcal{A}_i$ and $\mathcal{A}_j$ on the intersection $U_i \cap U_j$ are then related by:

$$\begin{aligned}\mathcal{A}_j &= \Omega^{-1}\,\mathcal{A}_i\,\Omega + \Omega^{-1}\,d\Omega\\ &= \mathcal{A}_i + e^{-i\Lambda(m)}\,d\left(e^{i\Lambda(m)}\right)\\ &= \mathcal{A}_i + e^{-i\Lambda(m)}\cdot ie^{i\Lambda(m)}\,d\Lambda\\ &= \mathcal{A}_i + i\,d\Lambda\end{aligned}$$

Which in components reads:

$$\mathcal{A}_{j\,\mu} = \mathcal{A}_{i\,\mu} + i\partial_\mu\Lambda$$

This is the familiar form of the gauge transformation in electromagnetism[Nak05].

Chapter 6

Curvature and Field Strength

6.1 Curvature

Let P be a principal G-bundle with a connection one form ω and let $\phi \in \Omega^k(P) \otimes V$ be a V valued k-form on P, where V is some k-dimensional vector space with basis $\{e_i\}$. The connection one form ω allows for the separation of the tangent space of P into horizontal and vertical components. Then the map:

$$\begin{aligned} &D\phi : \Gamma(T_u^{k+1} P) \to V, \\ &(X_1, \dots X_{k+1}) \mapsto D\phi(X_1, \dots X_k) \coloneqq d\phi(X_1^H, \dots X_{k+1}^H) \end{aligned}$$

is called the **covariant derivative** of ϕ. Here $d\phi \equiv d\phi^i \otimes e_i$ is the exterior derivative.

This introduces the **curvature two-form** Ω as the covariant derivative of the connection one-form ω:

$$\Omega \equiv D\omega \in \Omega^2(P) \otimes \mathfrak{g}$$

First it will be shown, that Ω takes the following form:

$$\Omega = d\omega + \omega \wedge_{\mathfrak{g}} \omega$$

Where $\wedge_{\mathfrak{g}}$ denotes the wedge product in the Lie algebra $\mathfrak{g}$ of G defined by its action on $\Gamma(T^2P)$: $(\omega \wedge_{\mathfrak{g}} \omega)(X, Y) \coloneqq [\omega(X), \omega(Y)]_{\mathfrak{g}}$

Note that if G is a matrix group, then the above can be written in terms of its components as:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

We proof this by considering three separate cases[Fre15a]:

a) $X, Y \in \Gamma(TP)$ are vertical vector fields

$\Rightarrow \quad \exists A, B \in T_e G : X = X^A, \quad Y = X^B$

Left-hand side:

$$\Omega(X^A, X^B) = D\omega(X^A, X^B) = d\omega\left((X^A)^H, (X^B)^H\right)$$
$$= d\omega(0, 0) = 0$$

Right-hand side:

$$d\omega(X^A, X^B) + (\omega \wedge_{\mathfrak{g}} \omega)(X^A, X^B)$$
$$= X^A(\omega(X^B)) - X^B(\omega(X^A)) - \omega([X^A, X^B]) + \left[\omega(X^A), \omega(X^B)\right]_{\mathfrak{g}}$$
$$= X^A(B) - X^B(A) - \omega(X^{[A,B]_{\mathfrak{g}}}) + [A, B]_{\mathfrak{g}}$$
$$= 0 - 0 - [A, B]_{\mathfrak{g}} + [A, B]_{\mathfrak{g}}$$
$$= 0$$

b) $X, Y \in \Gamma(TP)$ are horizontal vector fields

Left-hand side:

$$\Omega(X, Y) = D\omega(X, Y) = d\omega(X^H, Y^H)$$
$$= d\omega(X, Y)$$

Right-hand side:

$$d\omega(X, Y) + (\omega \wedge_{\mathfrak{g}} \omega)(X, Y)$$
$$= d\omega(X^H, Y^H) + \left[\omega(X), \omega(Y)\right]_{\mathfrak{g}}$$
$$= d\omega(X, Y) + \left[0, 0\right]_{\mathfrak{g}}$$
$$= d\omega(X, Y)$$

c) $X \in \Gamma(TP)$ is horizontal and $Y = X^A \in \Gamma(TP)$ is vertical

Left-hand side:

$$\Omega(X, X^A) = D\omega(X, X^A) = d\omega(X^H, (X^A)^H)$$
$$= d\omega(X, 0)$$
$$= 0$$

Right-hand side:

$$d\omega(X, X^A) + (\omega \wedge_{\mathfrak{g}} \omega)(X, X^A)$$
$$= d\omega(X, X^A) + \left[\omega(X), \omega(X^A)\right]_{\mathfrak{g}}$$
$$= X(\omega(X^A)) - X^A(\omega(X)) - \omega([X, X^A]) + \left[\omega(X), \omega(X^A)\right]_{\mathfrak{g}}$$
$$= X(A) - X^A(0) - \omega([X, X^A]) + \left[0, A\right]_{\mathfrak{g}}$$
$$= 0$$

Where in the last step the fact that the comutator of a horizontal and a vertical vector field is again a horizontal vector field was used[Nak05].

6.2 Local from of the curvature and Yang-Mills field strength

As the connection one-form ω can be expressed locally as the pullback by a section $\mathcal{A}_i = \sigma^*\omega$, the local from of the curvature two-form Ω is defined analogous[Nak05]:

$$\mathscr{F} \equiv \sigma^*\Omega \in \Omega^2(M) \otimes \mathfrak{g}$$

In terms of the local connection one-form $\mathcal{A}$, the curvature two-form can be expressed as:

$$\begin{aligned}
\mathscr{F} &= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\
&= \sigma^*(d\omega) + \sigma^*(\omega \wedge_{\mathfrak{g}} \omega) \\
&= \sigma^*(d\omega) + \sigma^*(\omega) \wedge_{\mathfrak{g}} \sigma^*(\omega) \\
&= d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_j
\end{aligned}$$

Let x^μ be the coordinates on the open set U_i where the section σ is defined. Then the Yang-Mills field is given by $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. We therefore get the following expression:

$$\begin{aligned}
\mathscr{F} &= d(\mathcal{A}_\mu dx^\mu) + (\mathcal{A}_\mu dx^\mu \wedge_{\mathfrak{g}} \mathcal{A}_\nu dx^\nu) \\
&= \frac{1}{2} \left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]_{\mathfrak{g}} \right) dx^\mu \wedge dx^\nu \\
&:= \frac{1}{2} \mathscr{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu
\end{aligned}$$

In physics, the local curvature two-form $\mathscr{F}$ is identified with the **Yang-Mills field strength**.

The co

First, compute the exterior derivative:

$$\begin{aligned}
& d\left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \\
&= -\boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} \boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\mathcal{A}_i\,\boldsymbol{\Omega} \\
&\quad - \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\boldsymbol{\Omega} - \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}^{-1} \wedge_{\mathfrak{g}} d\boldsymbol{\Omega}
\end{aligned}$$

Then, compute the wedge product:

$$\begin{aligned}
& \left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \wedge_{\mathfrak{g}} \left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \\
&= \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i\,\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\boldsymbol{\Omega} \\
&\quad + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} \mathcal{A}_i\,\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} d\boldsymbol{\Omega}
\end{aligned}$$

Combining both contributions, we obtain:

$$\begin{aligned}
\mathscr{F}_j &= \boldsymbol{\Omega}^{-1} \left(d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i \right) \boldsymbol{\Omega} \\
&= \boldsymbol{\Omega}^{-1}\, \mathscr{F}_i\, \boldsymbol{\Omega}
\end{aligned}$$

6.3 The Bianchi identity

The Bianchi identity states that the covariant derivative of the curvature two-form vanishes. To show this, the exterior derivative of the curvature two-form is computed:

$$d\Omega = d(d\omega) + d(\omega \wedge_{\mathfrak{g}} \omega) = d\omega \wedge_{\mathfrak{g}} \omega - \omega \wedge_{\mathfrak{g}} d\omega$$

Since for any $X \in H_pP$ the connection one-form vanishes, the following holds:

$$D\Omega(X, Y, Z) = d\omega(X^H, Y^H, Z^H) = 0$$

Therefore, the **Bianchi identity** is $D\Omega = 0$

Localy the Bianchi identity is given by:

$$\begin{aligned}
\sigma^* d\Omega &= d(\sigma^*\Omega) = d\mathscr{F} \\
&= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\
&= d\sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega + \sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega \\
&= d\mathcal{A} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} d\mathcal{A} \\
&= \mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}
\end{aligned}$$

Thus the Bianchi identity in local coordinates is given by:

$$D\mathscr{F} = d\mathscr{F} - (\mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}) = d\mathscr{F} + [\mathcal{A}, \mathscr{F}]_{\mathfrak{g}} = 0$$

Chapter 7

Gauge Theories

In physical gauge theories like electromagnetism, Yang-Mills theories or general relativity, the laws of nature they describe are not just differential equations that happen to describe nature, but they are deeply connected to the geometry of the underlying symmetries. In the following, the above developed mathematical framework is applied to recover Maxwell's equations, Yang-Mills theories.

7.1 Maxwell theory

Consider a $U(1)$ principal bundle P over the four dimensional Minkowski spacteime manifold M equipped with the Minkowski metric $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$. The principal bundle is trivial $P = M \times U(1)$, and the projection map is given by $\pi : P \to M$, $\pi(x, e^{i\Lambda}) = x$. The Yang-Mills field is given by:

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu \in \Omega^1(M) \otimes \mathfrak{u}(1)$$

And the field strength is given by the curvature two-form:

$$\mathscr{F} = d\mathcal{A}$$

We identify the **gauge potential** A by $\mathcal{A}_\mu = iA_\mu$ and the field strength tensor F by $\mathscr{F}_{\mu\nu} = iF_{\mu\nu}$, where i is the factor associated with the Lie algebra. Therefore, the curvature two-form can be written in components as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Bianchi identity is given by:

$$\begin{aligned} D\mathscr{F} &= d\mathscr{F} \\ &= \frac{1}{2}\partial_\mu \mathscr{F}_{\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho = 0 \\ \Rightarrow \quad & \partial_\mu \mathscr{F}_{\nu\rho} + \partial_\nu \mathscr{F}_{\rho\mu} + \partial_\rho \mathscr{F}_{\mu\nu} = 0 \end{aligned}$$

When identifying the electric and magnetic fields with the components of the field strength tensor, we have:

$$\begin{aligned} E_i &= F_{0i} \\ B_i &= \frac{1}{2}\epsilon_{ijk} F_{jk} \end{aligned}$$

The Bianchi identity yields the **homogeneous Maxwell equations**:

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

First, choosing indices $\mu = 0$, $\nu = i$, $\rho = j$ and using antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, we obtain:

$$\begin{aligned} \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} &= 0 \\ \Rightarrow \quad (\nabla \times \mathbf{E})_k + \partial_t B_k &= 0 \end{aligned}$$

Now, choose $\mu = i$, $\nu = j$, $\rho = k$, all spatial indices. Contracting with the Levi-Civita tensor ϵ^{ijk} gives:

$$\begin{aligned} \epsilon^{ijk} \partial_i F_{jk} &= 0 \\ \Rightarrow \quad \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

Together, these two identities form the homogeneous Maxwell equations:

$$\begin{aligned} \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

7.2 Yang-Mills theory

The same construction applies to non-Abelian gauge theories with structure group $G = SU(N)$. We consider a trivial principal bundle over the spaceime manifold $P = M \times SU(N)$, with projection map $\pi : P \to M$, $\pi(x, g) = x$. The Yang–Mills field is given by a $\mathfrak{su}(N)$-valued one-form:

$$\mathcal{A} = A^a_\mu T^a dx^\mu,$$

where $T^a \in \mathfrak{su}(N)$ are the generators of the Lie algebra of $SU(N)$, and A^a_μ are the gauge potentials. The corresponding field strength (curvature two-form) is given by:

$$\mathscr{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

In components, this reads:

$$\begin{aligned} \mathscr{F}_{\mu\nu} &= \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a + A^b_\mu A^c_\nu [T^b, T^c] \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \right) T^a =: F^a_{\mu\nu} T^a, \end{aligned}$$

where f^{abc} are the structure constants of $\mathfrak{su}(N)$, defined via $[T^b, T^c] = f^{abc} T^a$.

The Bianchi identity holds:

$$D\mathscr{F} = d\mathscr{F} + [\mathcal{A}, \mathscr{F}] = 0,$$

which in components becomes:

$$D_\lambda F^a_{\mu\nu} = \partial_\lambda F^a_{\mu\nu} + f^{abc} A^b_\lambda F^c_{\mu\nu} = 0.$$

7.3 Gauge sector of the standard model of particle physics

The above elaboration of gauge theories can be applied to the gauge sector of the standard model of particle physics. It is described by a principal bundle with the gauge group $SU(3) \times SU(2) \times U(1)$. Combining the above results, the action describing the dynamics of the gauge fields may be written as:

$$S_{\text{gauge}} = \int_M d^4x - \frac{1}{4} \left(G^a_{\mu\nu} G^{a\,\mu\nu} + W^i_{\mu\nu} W^{i\,\mu\nu} + B_{\mu\nu} B^{\mu\nu} \right) \mathrm{d}^4x$$

where $G^a_{\mu\nu}$, $W^i_{\mu\nu}$, and $B_{\mu\nu}$ denote the field strength components of the curvature two-forms corresponding to $SU(3)$, $SU(2)$, and $U(1)$, respectively. These arise from the decomposition of the total curvature $\mathscr{F} \in \mathfrak{g} \otimes \Omega^2(M)$.

Chapter 8

Conclusion

The above chapters established the foundations of modern physics in a geometrical framework. Chapter 2 presented preliminary concepts from topology and introduced the basic definitions of differentiable manifolds. Chapter 3 discussed fiber bundles, beginning with the tangent bundle as a motivating example. The general definition of fiber bundles was elaborated, including structure groups, sections, and local trivializations. The cotangent bundle and differential forms were examined to establish the necessary tools for field theories. Chapter 4 focused on principal bundles, which utilized a Lie group as the typical fiber. The frame bundle was explored as a key example, and the connection to symmetry groups was elucidated. Chapter 5 introduced connections on principal bundles. The global connection one-form and the local connection form (interpreted as a gauge potential) were defined, and horizontal and vertical subspaces were introduced. The transformation behavior under gauge transformations was derived. Chapter 6 defined the curvature of a connection and demonstrated how it led to the field strength in physical gauge theories. The Bianchi identity, a fundamental identity satisfied by the curvature, was derived both globally and locally. Chapter 7 applied the developed framework to classical gauge theories, beginning with Maxwell theory as a $U(1)$ gauge theory and extending the discussion to the general structure of Yang–Mills theories. This journey—from manifolds to fiber bundles and curvature—has shown how the language of differential geometry offers not just a reformulation, but a profound reinterpretation of gauge theories as geometric objects.

This thesis focused on the mathematical foundations of gauge theories, which are an important aspect in physics, especially in the context of quantum field theories. Another essential subtopic in the field of differential geometry in physics is the study of associated bundles, which was not covered in this thesis. Associated bundles are bundles that transform under a representation of gauge group. This is of interest, particularly in relation to the standard model of particle physics, where associated bundles are used to describe matter fields.

Furthermore, this introduced framework can be applied to explore the Higgs mechanism, which is a crucial aspect of the standard model of particle physics. The Higgs mechanism involves spontaneous symmetry breaking and the generation of masses. In the context of differential geometry, this can be understood as a reduction of the symmetry group associated with the principal bundle. This reduction can be analyzed using the tools developed in this thesis, providing a deeper understanding of how gauge theories can incorporate mass generation through geometric structures.

The geometric approach not only clarifies the structure of gauge theories, but also reveals profound mathematical coherence—suggesting that the language of geometry is the natural framework for modern physics.

Bibliography

[DG18] DUDEK, Marta ; GARECKI, Janusz: *Ehreshmann Theory of Connection in a Principal Bundle - Compendium for Physicists*. http://dx.doi.org/10.48550/arXiv.1810.03447. Version: Oktober 2018

[Fre15a] FREDERIC SCHULLER: *Curvature and Torsion on Principal Bundles - Lec 24 - Frederic Schuller*. https://www.youtube.com/watch?v=j36o4DLLK2k. Version: September 2015

[Fre15b] FREDERIC SCHULLER: *Differential Structures: The Pivotal Concept of Tangent Vector Spaces - Lec 09 - Frederic Schuller*. https://www.youtube.com/watch?v=UPGoXBfm6Js. Version: September 2015

[Fre15c] FREDERIC SCHULLER: *Local Representations of a Connection on the Base Manifold: Yang-Mills Fields - Lec 22*. https://www.youtube.com/watch?v=KhagmmNvOvQ. Version: September 2015

[Fre15d] FREDERIC SCHULLER: *Principal Fibre Bundles - Lec 19 - Frederic Schuller*. https://www.youtube.com/watch?v=vYAXjTGr_eM. Version: September 2015

[Fre15e] FREDERIC SCHULLER: *Topological Manifolds and Manifold Bundles- Lec 06 - Frederic Schuller*. https://www.youtube.com/watch?v=uGEV0Wk0eIk. Version: September 2015

[Mau25] *Maurer–Cartan Form*. https://en.wikipedia.org/w/index.php?title=Maurer%E2%80%93Cartan_form&oldid=1292760293. Version: Mai 2025

[Nak05] NAKAHARA, Mikio: *Geometry, Topology, and Physics*. 2. ed., repr. Bristol : Inst. of Physics Publishing, 2005 (Graduate Student Series in Physics). – ISBN 978–0–7503–0606–5

[Pus25] *Pushforward*. https://de.wikipedia.org/w/index.php?title=Pushforward&oldid=257543341. Version: Juli 2025

[Rag25] RAGHUNATHAN, M. S.: *Lie Groups and Lie Algebras*. Singapore : Springer Nature, 2025 (Texts and Readings in Mathematics). http://dx.doi.org/10.1007/978-981-97-1832-0. http://dx.doi.org/10.1007/978-981-97-1832-0. – ISBN 978–981–97–1832–0

[WY75] WU, Tai T. ; YANG, Chen N.: Concept of Nonintegrable Phase Factors and Global Formulation of Gauge Fields. In: *Physical Review D* 12 (1975), Dezember, Nr. 12, 3845–3857. http://dx.doi.org/10.1103/PhysRevD.12.3845. – DOI 10.1103/PhysRevD.12.3845