From Differentiable Manifolds to Gauge Theories

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submitted by

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Von differenzierbaren Mannigfaltigkeiten zu Eichtheorien

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Chapter 1

Manifolds

Manifolds are the fundamental spaces used in physics. They provide a framework to describe topological spaces that locally resemble Euclidean spaces, allowing for the application of known methods from calculus and linear algebra.

1.1 Preliminaries from topology

Although this thesis will not focus on introducing topology, a few important results will be given here, which are necessary to understand the definition of manifolds. The following definitions and theorems are based on [Nak05]

A **topological space** is a set X equipped with a collection of open sets $\mathcal{T} = \{U_i \mid i \in I\}$ such that:

- $\emptyset, X \in \mathcal{T}$
- For any subcollection J of I the Union of corresponding open sets is itself an open set $\bigcup_{j \in J} U_j \in \mathcal{T}$
- For any finite subcollection K of I the intersection of the corresponding open sets is open: $\cap_{k \in K} U_k \in \mathcal{T}$

A family $\{O_i\}$ of (open) subsets of X is called an (open) covering of X if $X = \bigcup_i O_i$.

A subset N is called a **neighborhood** of a point $p \in X$ if there exists at least one open set $U \in \mathcal{T}$ such that $p \in U \subset N$. A topological space is called **Hausdorff** if for any two distinct points $p, q \in X$ there exist neighborhoods N_p, N_q such that $N_p \cap N_q = \emptyset$.

A map $f : X \to Y$ between two topological spaces is called **continuous** if for every open set $V \subset Y$ the preimage $f^{-1}(V)$ is an open set in X. If the inverse $f^{-1} : Y \to X$ is also continuous, then f is called a **homeomorphism**. Two topological spaces are called **homeomorphic** if there exists a homeomorphism between them.

1.2 Differentiable Manifolds

A Hausdorff topological space $(M, \mathcal{T})$ is called a **d-dimensional manifold** if there exists an open covering $\{U_i\}$ and a family of homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^d$. The pair (U_i, φ_i) is called a **chart** and the family $\{(U_i, \varphi_i)\}$ is called an **atlas**[Fre15e].

M is a **differentiable or smooth manifold** if for any U_i and U_j given that $U_i \cap U_j \neq \emptyset$ the transition function $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_i) \to \varphi_i(U_j \cap U_i)$ is infinite differentiable (C^∞)[Nak05]. In this thesis, smoothness will always be assumed, unless stated otherwise.

Let M and N be two differentiable manifolds of dimension m and n equipped with atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ respectively. A map $f : M \to N$ is called a **differentiable map** at a point $p \in M$ if for $p \in U_i$ and $f(p) \in V_j$ the composition $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$ is infinite differentiable. If f is also a homeomorphism and the inverse $f^{-1} : N \to M$ is differentiable, then f is called a **diffeomorphism**. M and N are called **diffeomorphic** if there exists a diffeomorphism between them. This will be denoted as $M \cong N$.

1.3 Spacetime Manifold M

A trivial but for obvious reasons important example of a differentiable manifold is the space-time manifold M used in physics, which is defined as follows:

Let $M := \mathbb{R}^4$, the set of ordered 4-tuples $(x^\mu) \in \mathbb{R}^4$. The so called **standard topology** is defined by the open balls around a point $p \in M$ with radius $r > 0$:

$$B_r(p) := \{x \in \mathbb{R}^4 \mid \|x - p\| < r\}$$

with $\|\cdot\|$ the Euclidean norm:

$$\|x\|^2 = \sum_{\mu=0}^{3} (x^\mu)^2$$

This is obviously a Hausdorff[1], and locally Euclidean topological space. The identity map $\phi(p) = p$ covers M globally. Hence, $(\mathcal{M}, B_r(p), \varphi)$ is a smooth manifold.

[1] For two distinct points $p, q \in M$ it is always sufficient to choose $r = \frac{1}{2}\|q - p\|$

Chapter 2

Bundles

The definition of a vector on a Manifold is non-trivial because a vector space structure might not exist globally on the manifold. A Manifold may still be equipped with a Vector space structure locally. Thus, tangent spaces are introduced pointwise. Combining these local structures will lead naturally to the definition of fiber bundles.

2.1 Tangent Space T_pM

Let M be an n-dimensional smooth manifold. A tangent vector at a point $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule [Fre15b]:

$$v[fg] = v[f]g(p) + f(p)v[g]$$

A tangent vector at a point $p \in M$ can be constructed as the directional derivatives of an equivalence class of curves through p[Nak05].

Let $\gamma : [-\epsilon, \epsilon] \to M$ be a smooth curve in M with $\gamma(0) = p$. Then $\phi \circ \gamma(t) = x(t) \in \mathbb{R}^n$ is called the coordinate representation of γ induced by a chart (U, φ). Let $f \in C^\infty(M)$ be a smooth function on M. The directional derivative of f along the curve γ at $t = 0$ is given by:

$$\begin{aligned}
\frac{d}{dt}(f \circ \gamma)(t)\bigg|_{t=0} &= \frac{d}{dt}\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma(t)\right)\bigg|_{t=0} \\
&= \frac{\partial f}{\partial x^r}\frac{dx^r}{dt}\bigg|_{t=0} \\
&= \frac{dx^r}{dt}\bigg|_{t=0}\frac{\partial f}{\partial x^r}\bigg|_{p}
\end{aligned}$$

The definition of a tangent vector is now obtained by introducing an equivalence relation on curves. Two curves γ_1 and γ_2 are called equivalent at $\gamma_1(0) = \gamma_2(0) = p$ if their derivatives at $t = 0$ are equal:

$$\frac{dx_1^r}{dt}\bigg|_{t=0} = \frac{dx_2^r}{dt}\bigg|_{t=0} = v^r$$

A tangent vector is identified with the differential operator given the equivalence class of curves. Once a chart (U, φ) is chosen, with local coordinates $(x^1, \ldots, x^n)$, a tangent vector is represented as a linear combination of partial derivatives with real coefficients.

$$v = v^r\,\frac{\partial}{\partial x^r}\bigg|_{p}$$

The tangent space at a point $p \in M$ is then defined as the set of all tangent vectors at p and is denoted by T_pM.

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \quad \text{form a basis of } T_pM$$

2.2 The Tangent Bundle as a Fiber Bundle

To introduce the concept of fiber bundles, a detailed examination of a specific example, the tangent bundle, serves as an effective foundation. The tangent bundle of a smooth n-dimensional manifold M is constructed by taking the disjoint union of all tangent spaces T_pM. This construction can be represented as:

$$TM := \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM$$

Here, T_pM denotes the tangent space at a point $p \in M$. The tangent space T_pM is a vector space that consists of all tangent vectors at the point p. It should be noted that this notation emphasizes the pointwise nature of the disjoint union, where each tangent space is associated with a specific point on the manifold. Since each tangent space is a different object, a certain tangent vector in a tangent space can only belong to that specific tangent space. Therefore all the information is already in the vector itself and the direct product notation $\{p\} \times T_pM$ is only used to clarify [Fre15e].

Each element of the tangent bundle TM can be expressed as a pair (p, v), where p is a point on the manifold M, and v is a tangent vector belonging to the tangent space T_pM at that point. A natural projection map is defined as follows:

$$\pi : TM \to M, \quad (p, v) \mapsto p$$

This projection, π, serves to "forget" the tangent vector v associated with each point, effectively collapsing all the tangent vectors at p to the single point p in the base manifold [Nak05].

The fiber over a point p is denoted as $\pi^{-1}(p) = T_pM$. This represents all the tangent vectors at the point p. Since T_pM is isomorphic to $\mathbb{R}^n$ as a vector space, it is referred to as the model fiber, denoted $F = \mathbb{R}^n$ [Nak05]. Since the fiber of the tangent bundle is a vector space, the tangent bundle is also referred to as a **vector bundle**.

Consider a coordinate chart (U, φ) on M. The chart φ provides a mapping from an open set $U \subseteq M$ to an open subset of $\mathbb{R}^n$. A diffeomorphism on the preimage $\pi^{-1}(U) \subset TM$ can then be defined as:

$$\Psi : \pi^{-1}(U) \to \mathbb{R}^{2n}, \quad (p, v) \mapsto \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n \right)$$

where the coordinates $x^i(p)$ represent the local coordinates of the point p in U, and $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$ describes the tangent vector in terms of its components in the chosen coordinate system [Nak05]. This establishes a local trivialization of the tangent bundle, expressed as:

$$TM|_U \cong U \times \mathbb{R}^n$$

In summary, the construction of the tangent bundle yields a fiber bundle characterized by the following essential components:

- **Total space:** $TM = \bigsqcup_{p \in M} T_pM$, encapsulating all tangent spaces.

- **Base space:** M, a manifold to which additional structure is added.
- **Projection:** $\pi : TM \to M$, mapping each tangent vector to its associated point on the manifold.
- **Model fiber:** $F = \mathbb{R}^n$, serving as the standard fiber structure over each point.
- **Local trivialization:** $TM|_U \cong U \times \mathbb{R}^n$, ensuring that the tangent bundle locally resembles a product structure.

Similar to the way a manifold is commonly perceived as a space that locally resembles $\mathbb{R}^n$, a fiber bundle may be conceptualized as a space that locally resembles the Cartesian product of the base space with a typical fiber structure.

2.3 Definition of a Fiber Bundle

The formal definition of a fiber bundle reads as follows[Fre15e, DG18]. A fiber bundle is a quadruple (E, B, π, F) where:

- E is the total space
- B is the base space
- $\pi : E \to B$ is a surjective map called the projection
- F is the typical fiber

There exists an open cover $\{U_\alpha\}$ of B such that for each α, there is a diffeomorphism

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[d, "\pi"'] & & U \times F \arrow[dll, "\mathrm{proj}_1"] \\
U & &
\end{tikzcd}$$

where $\mathrm{proj}_1 : A \times B \to A$ is the first projection. $(U_\alpha, \varphi_\alpha)$ are called *local trivialization.* The fiber over a point $b \in B$ is:

$$F_b := \pi^{-1}(\{b\}) \cong F$$

Often the notations $E \xrightarrow{\pi} B$ or $\pi : E \to B$ are used to denote a fiber bundle.

2.3.1 The Structure Group of a Fiber Bundle

Above a fiber bundle was defined as a quadruple (E, B, π, F) equipped with local trivializations. These local trivializations are established as diffeomorphisms on an open cover $\{U_\alpha\}$ of the base space B. The definition does not impose the requirement that $U_\alpha \cap U_\beta = \emptyset$. For a point $p \in U_\alpha \cap U_\beta$, multiple local trivializations $\varphi_\alpha(p, f) = \varphi_{\alpha,p}(f)$ and $\varphi_\beta(p, f) = \varphi_{\beta,p}(f)$ may be present, defined on U_α and U_β, respectively.

The **structure group** G of a fiber bundle is defined as the Lie group of diffeomorphisms relating these local trivializations. The corresponding transition function is given by[Nak05]:

$$t_{\alpha\beta}(p) \equiv \varphi_{\alpha,p}^{-1} \circ \varphi_{\beta,p} : F \to F$$

This establishes a smooth map $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ that satisfies the following properties:

$$\begin{aligned} t_{\alpha\alpha}(p) &= \mathrm{id}_F && \forall\, p \in U_\alpha \\ t_{\alpha\beta}(p) &= t_{\beta\alpha}(p)^{-1} && \forall\, p \in U_\alpha \cap U_\beta \\ t_{\alpha\beta}(p) \circ t_{\beta\gamma}(p) &= t_{\alpha\gamma}(p) && \forall\, p \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

In the case of the tangent bundle, the structure group corresponds to the general linear group $\mathrm{GL}(n, \mathbb{R})$, which consists of all invertible $n \times n$ matrices. A fiber bundle where all transition maps can be chosen to be the identity map is termed a **trivial bundle**. In this scenario, the total space E is diffeomorphic to the product space $B \times F$.

Generally, a fiber bundle does not possess a unique trivialization. Let $\{\varphi_\alpha\}$ and $\{\tilde{\varphi}_\alpha\}$ denote two local trivializations over the same open covering that describe the same fiber bundle. These trivializations are related by maps $g_\alpha(p) : F \to F \quad \forall\, p \in B$, where each $g_\alpha(p)$ is a homeomorphism within the structure group G. The transition function between the two local trivializations is then given by:

$$g_\alpha(p) \equiv \varphi_{\alpha,p}^{-1} \circ \tilde{\varphi}_{\alpha,p}$$

Considering the tangent bundle as an illustrative example[Nak05], let U_i and U_j represent overlapping charts with $p \in U_i \cap U_j$. Utilizing the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^j}\Big|_p \right\}$, a vector $v \in T_pM$ can be expressed in both bases as:

$$v = v^\mu \frac{\partial}{\partial x^\mu} = \tilde{v}^\mu \frac{\partial}{\partial y^\mu}$$

The transition function $t^\nu{}_\mu$ is thus defined as:

$$\tilde{v}^\nu = \frac{\partial y^\nu}{\partial x^\mu}\bigg|_p v^\mu = t^\nu{}_\mu v^\mu$$

2.3.2 Sections

An important definition in the context of fiber bundles is that of a section or cross-section. This concept enables the selection of an element from each fiber over each point in a continuous manner, facilitating the introduction of ideas such as vector fields over spacetime[Nak05].

A **section** of a fiber bundle $\pi : E \to B$ is defined as a continuous map $s : B \to E$ such that

$$\pi \circ s = \mathrm{id}_B.$$

This condition ensures that exactly one point is chosen from each fiber continuously. The set of all (smooth) sections is denoted by:

$$\Gamma(E) := \{ s : M \to E \mid \pi \circ s = \mathrm{id}_M \}.$$

It is also possible to define a section locally on an open set $U \subset B$ as a map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$. In this case, the section is called a **local section**. For example, a vector field over a manifold M can therefore be understood as a section of the tangent bundle TM over M.

2.4 The cotangent bundle and differential forms

2.4.1 The Cotangent Bundle

In this section the fundamental concepts of the cotangent bundle are introduced, which is essential for the definition of differential forms and the exterior derivative. The cotangent space at a point p on a manifold M is defined as the dual space of the tangent space at that point. Formally, the cotangent space at a point $p \in M$ is the set of all linear maps from the tangent space at that point to the real numbers[Nak05].

$$T_p^*M := \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$$

A covector $\omega \in T_p^*M$ is such a linear function:

$$\omega : T_pM \to \mathbb{R}$$

As an example, consider a function $f \in C^\infty(M)$ and some tangent vector $v \in T_pM$. Then $v[f] \in \mathbb{R}$ by definition. The differential of f at a point p is a covector df_p and therefore $df_p[v] \in \mathbb{R}$ is simply defined as $df_p[v] = v[f]$. Given a coordinate basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ of T_pM the dual basis is:

$$\left\{ dx^i\Big|_p \right\}$$

satisfying the relation:

$$dx^i \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \delta^i_j$$

The example from before can thus be expressed as:

$$df_p = \frac{\partial f}{\partial x^i} dx^i \quad \text{and} \quad df_p[v] = v^i \left.\frac{\partial f}{\partial x^i}\right|_p$$

Analogous to the tangent bundle, the cotangent bundle is defined as the disjoint union of all cotangent spaces at each point in the manifold:

$$T^*M := \bigsqcup_{p \in M} T_p^*M$$

This structure constitutes a vector bundle over M. A section of the cotangent bundle can be defined as:

$$\omega \in \Gamma(T^*M)$$

This section assigns to each $p \in M$ a covector $\omega_p \in T_p^*M$ smoothly. Such a section is referred to as a **1-form**. In a coordinate representation, a 1-form can be expressed as:

$$\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i \quad \text{with} \quad \omega_i \in C^\infty(M)$$

2.4.2 Tensor Fields and the Metric Tensor

Utilizing the fact that the fibers of the cotangent bundle are vector spaces, tensor products of bundles can be defined. For instance:

$$T^*M \otimes T^*M := \bigsqcup_{p \in M} T_p^*M \otimes T_p^*M$$

This forms a bundle whose fibers consist of maps form $TM \otimes TM$ to $\mathbb{R}$. Sections of this bundle are referred to as $(0,2)$-tensor fields. A prominent example of such fields is the metric tensor. The Minkowski metric is defined as:

$$\eta \in \Gamma(T^*M \otimes T^*M)$$

In local coordinates, the Minkowski metric can be expressed as:

$$\eta = \eta_{\mu\nu}\, dx^\mu \otimes dx^\nu \quad \text{with} \quad \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$$

In general, a tensor field of type (i,j) is a section of the bundle [Nak05]:

$$\otimes^i T^*M \otimes^j TM$$

2.4.3 Differential Forms and the Exterior Derivative

A **k-form** or **differential form** of order k is a totally antisymmetric $(k,0)$ tensor. To define a k-form, it is necessary to take the **wedge product** of 1-forms, which is defined by taking the totally antisymmetrized tensor product of 1-forms. This means that all permutations of the tensor product are considered, with even permutations contributing positively and odd permutations contributing negatively. Consider the cotangent space T_p^*M of a manifold M at a point p with basis $\{dx^\mu\}$. The wedge product of two 1-forms dx^μ and dx^ν thus reads:

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

Higher order forms can be constructed analogously. By taking the cotangent bundle T^*M and considering the antisymmetrized tensor product, we can define fields of differential forms of order r as:

$$\Omega^r(M) \equiv \Gamma(\wedge^r(T^*M))$$

The **exterior derivative** is then defined as a map [Nak05]:

$$\begin{aligned} d : \Omega^r(B) &\to \Omega^{r+1}(B) \\ \omega \mapsto d\omega &= \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\ldots\mu_r}\right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \end{aligned}$$

In the following, two relations will be shown that will be used in later proofs [Nak05]. First consider the action of the exterior derivative on a 1-form $\omega = \omega_\mu dx^\mu \in \Omega^1(M)$ on two tangent vector fields $v = v^\mu \frac{\partial}{\partial x^\mu}$ and $w = w^\nu \frac{\partial}{\partial x^\nu}$:

$$\begin{aligned} d\omega(V,W) &= \left(\frac{\partial \omega_\mu}{\partial x^\nu}\, dx^\nu \wedge dx^\mu\right)(V,W) \\ &= \frac{\partial \omega_\mu}{\partial x^\nu}\left(dx^\nu(V)\, dx^\mu(W) - dx^\nu(W)\, dx^\mu(V)\right) \\ &= \frac{\partial \omega_\mu}{\partial x^\nu}\left(V^\nu W^\mu - W^\nu V^\mu\right) \\ &= V^\nu \frac{\partial}{\partial x^\nu}\left(\omega_\mu W^\mu\right) - W^\nu \frac{\partial}{\partial x^\nu}\left(\omega_\mu V^\mu\right) - \omega_\mu\left(V^\nu \frac{\partial W^\mu}{\partial x^\nu} - W^\nu \frac{\partial V^\mu}{\partial x^\nu}\right) \\ &= V[\omega(W)] - W[\omega(V)] - \omega([V,W]) \end{aligned}$$

This gives a coordinate free expression for the action of the exterior derivative of a 1-form. Furthermore, it can easily be shown that the exterior derivative of a exterior derivative is zero, by using the fact that the product of a symmetric and an antisymmetric tensor is zero. By definition the following is obtained:

$$d^2\omega = \frac{1}{r!}\left(\frac{\partial^2}{\partial x^\alpha x^\beta}\omega_{\mu_1\ldots\mu_r}\right) dx^\alpha \wedge dx^\beta \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

From this it follows instantly that $d^2 = 0$ since $\frac{\partial^2}{\partial x^\alpha x^\beta}$ is symmetric and the wedge product is antisymmetric.

Chapter 3

Principal Bundles

A principal bundle is a fiber bundle $P \xrightarrow{\pi} M$ characterized by a fiber that is identical to the structure group. This framework is particularly significant as it facilitates the underlying structure of gauge theories in physics, where the fiber represents the gauge group and the base manifold M represents spacetime. In this chapter, it will first be elaborated how a Lie group acts on a manifold. Then, the definition of a principal bundle will be given, followed by an example of the frame bundle.

3.1 Action of a Lie Group on a Manifold

To understand this concept thoroughly, it is necessary to examine how a Lie group G can act on a manifold M [Fre15d, DG18]. Let $(G, \cdot)$ denote a Lie group, and M a smooth manifold. A smooth map

$$\triangleright : G \times M \longrightarrow M$$

is defined as a **left G-action** on M if it satisfies the following conditions:

$$\begin{aligned} e \triangleright p &= p && \forall p \in M, \text{ where } e \text{ is the identity in } G \\ g_2 \triangleright (g_1 \triangleright p) &= (g_2 \cdot g_1) \triangleright p && \forall g_1, g_2 \in G,\ p \in M. \end{aligned}$$

Analogously, a **right G-action** $\triangleleft$ is defined by the map $\triangleleft : M \times G \longrightarrow M$, which satisfies:

$$\begin{aligned} p \triangleleft e &= p && \forall p \in M, \text{ where } e \text{ is the identity in } G \\ (p \triangleleft g_1) \triangleleft g_2 &= p \triangleleft (g_1 \cdot g_2) && \forall g_1, g_2 \in G,\ p \in M. \end{aligned}$$

Given a left action $\triangleright$, it is possible to construct a right action as follows:

$$\begin{aligned} \triangleleft : & M \times G \longrightarrow M, \\ & p \triangleleft g := g^{-1} \triangleright p. \end{aligned}$$

This transformation yields a valid right action since the inverse of the identity is again the identity and $(g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1}$.

Let G be a Lie group acting smoothly on a manifold M from the left via

$$\triangleright : G \times M \longrightarrow M.$$

An equivalence relation $\sim$ on M is defined by:

$$p \sim \tilde{p} \quad :\Longleftrightarrow \quad \exists g \in G \text{ such that } \tilde{p} = g \triangleright p.$$

This equivalence relation can be verified through the following properties:

- **Reflexivity:** $e \triangleright p = p$, thus $p \sim p$.
- **Symmetry:** If $\tilde{p} = g \triangleright p$, then $p = g^{-1} \triangleright \tilde{p}$, leading to $\tilde{p} \sim p$.
- **Transitivity:** If $\tilde{p} = g_1 \triangleright p$ and $\hat{p} = g_2 \triangleright \tilde{p}$, then

  $$\hat{p} = g_2 \triangleright (g_1 \triangleright p) = (g_2 g_1) \triangleright p,$$

  which implies $p \sim \hat{p}$.

The **orbit** of a point $p \in M$ under the group action constitutes the equivalence class:

$$\mathcal{O}_p := \{\tilde{p} \in M \mid \exists g \in G : \tilde{p} = g \triangleright p\}.$$

Consequently, the **quotient space** $M/\sim$, commonly denoted as M/G, is defined by identifying points within the same orbit.

Additionally, the **stabilizer** of a point $p \in M$ represents the set of elements in G that leave p unchanged:

$$S_p \coloneqq \{g \in G \mid g \triangleright p = p\}.$$

An action $\triangleright$ is termed free if the stabilizer is trivial for all $p \in M$, i.e., $S_p = \{e\}$.

3.2 Principal Bundles

A principal fibre bundle is defined as follows [DG18]:

Let (P, π, M, F) be a fibre bundle. If the following conditions are satisfied, it is classified as a **principal G-bundle**:

(i) P is equipped with a right G-action $\triangleleft$,

(ii) The action of G is free,

(iii) $\pi : P \to M$ is isomorphic as a bundle to the quotient $\rho : P \to P/G$,

where $\rho(p) \mapsto [p]$ denotes the canonical projection onto the orbit space P/G. Since the action $\triangleleft$ is free, each fibre $\rho^{-1}([p])$ is diffeomorphic to G.

For clarification, two bundles $\pi : E \to M$ and $\pi' : E' \to M'$ are considered isomorphic if there exist diffeomorphisms $\bar{f} : E \to E'$ and $f : M \to M'$ such that $\pi' \circ \bar{f} = f \circ \pi$. Thus, (iii) can be reformulated as the existence of a diffeomorphism $f : M \to P/G$ such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\mathrm{id}_P"] \arrow[d, "\pi"'] & P \arrow[d, "\rho"] \\ M \arrow[r, "f"'] & P/G \end{tikzcd}$$

Figure 3.1: Commutative diagram showing the isomorphism between a principal G-bundle $\pi : P \to M$ and its quotient bundle $\rho : P \to P/G$.

3.3 The Frame Bundle

As an example, consider the frame bundle of a smooth manifold M [Nak05].

A **frame** at a point $p \in M$ with $\dim M = d$ is defined as an ordered basis of the tangent space T_pM. The set of all frames at p is given by:

$$L_pM := \{(e_1, \dots, e_d) \,|\, \{e_1, \dots, e_d\} \text{ is a basis of } T_pM\}\,.$$

There exists a natural isomorphism

$$L_pM \cong \mathrm{GL}(d, \mathbb{R}),$$

by identifying each frame $(e_1, \dots, e_d) \in L_pM$ with the matrix whose columns are the vectors e_i. Specifically, each frame is mapped to a matrix $g \in \mathrm{GL}(d, \mathbb{R})$ such that

$$g^\mu{}_\alpha = e^\mu{}_\alpha.$$

Analogous to the tangent bundle, the **frame bundle** can be defined as:

$$LM := \bigsqcup_{p \in M} L_pM.$$

Given a chart U_i on M, a local trivialization of the frame bundle can be defined [Nak05]. A frame $\epsilon = (e_1, \dots, e_d)$ at $p \in M$ can be expressed in terms of the natural basis of the tangent space T_pM as $\{\partial/\partial x^\mu \,|_p\}$:

$$e_\alpha = e^\mu{}_\alpha \, \partial/\partial x^\mu \,|_p, \quad \text{where } e^\mu{}_\alpha \in \mathrm{GL}(d, \mathbb{R}).$$

The local trivialization is then given by $\varphi_i^{-1}(u) = (p, (e^\mu{}_\alpha))$.

The projection π_L of a frame $\epsilon = \{e_1, \cdots, e_d\}$ at a point $p \in M$ is specified by:

$$\begin{aligned} \pi_L \,:\, LM &\longrightarrow M, \\ \epsilon &\mapsto \pi_L(\epsilon) = p. \end{aligned}$$

Thus, the necessary structure has been introduced such that $(LM, \pi_L, M, \mathrm{GL}(d, \mathbb{R}))$ defines a fiber bundle.

The right action of $\mathrm{GL}(d, \mathbb{R})$ on the frame is analogous to a change of basis in a vector space. This action is defined as follows:

$$\begin{aligned} \triangleleft : LM \times \mathrm{GL}(d, \mathbb{R}) &\longrightarrow LM, \\ (\epsilon, g) &\mapsto \epsilon \triangleleft g = (e_1, \cdots, e_d) \triangleleft g = (e_i g^i{}_1, \cdots, e_i g^i{}_d). \end{aligned}$$

To demonstrate that this bundle, equipped with the right action, is a principal bundle, it is necessary to verify that the action is free and that the bundle is isomorphic to the quotient bundle.

First, it must be shown that $e_i g^i{}_\alpha = e_\alpha \implies g^i{}_\alpha = \delta^i{}_\alpha \, \forall \alpha$. Since $\{e_1, \ldots, e_d\}$ forms a basis, the vectors are linearly independent. Therefore, the definition implies that $g^i{}_\alpha = \delta^i{}_\alpha$, confirming that the action is free.

Next, it is necessary to show that the orbit space $LM/\mathrm{GL}(d, \mathbb{R})$ consists of a single point for each $\epsilon \in \pi^{-1}(p)$. This is indeed the case, as the orbit of a frame at $p \in M$ under the action of $\mathrm{GL}(d, \mathbb{R})$ includes all frames at p, since the action is transitive. Hence, the quotient space is diffeomorphic to M, with the diffeomorphism defined by: $f : M \longrightarrow LM/\mathrm{GL}(d, \mathbb{R})$, $p \mapsto [\epsilon]$, where ϵ is a frame at p.

Thus, the frame bundle $(LM, \pi_L, M, \mathrm{GL}(d, \mathbb{R}))$ is indeed a principal $\mathrm{GL}(d, \mathbb{R})$-bundle.

Chapter 4

Connections on Principal Bundles

4.1 General Definition

A Connection is a consistent way to separate the tangent space of a principal bundle into a vertical subspace tangent to the fiber and a horizontal subspace that complements the former. This will be constructed by considering a vector field generated by the right action of the one-parameter subgroup of the structure group G on the principal bundle. As before a vector is defined by its action on a function evaluated along a curve which will be generated by the former mentioned right action.

Let $P \xrightarrow{\pi} M$ be a principal bundle with structure group G. The right action of G on P induces a vector field as follows: For each $A \in \mathfrak{g} \cong T_eG$, the action of the one-parameter subgroup $\exp(tA)$ on an element $p \in P$ yields a curve. Since the group acts within the fiber, it holds that $\pi(p) = \pi(p \triangleleft \exp(tA)) = p$. A vector $X_p^A \in T_pP$ is defined by its action on a function $f \in C^\infty(P)$ [Nak05]:

$$X_p^A f = \frac{d}{dt} f(p \triangleleft \exp(tA)) \,|_{t=0} \,.$$

Furthermore, a vector space isomorphism $i : \mathfrak{g} \longrightarrow \Gamma(TP)$ is defined that assigns to each element $A \in \mathfrak{g}$ the vector field X^A which is generated by A. This vector field is referred to as a **fundamental vector field** on P.

The **Pushforward** [Pus25] of a smooth map $F : M \longrightarrow N$ between smooth manifolds M and N is defined as a map between the tangent spaces:

$$F_* : T_pM \longrightarrow T_{F(p)}N.$$

This is identified using $(F_*v)(f) = v(f \circ F)$ for $v \in T_pM$ and $f \in C^\infty(N)$.

The **Pullback** [Nak05] of a smooth map $F : M \longrightarrow N$ between smooth manifolds M and N is defined as a map between the cotangent spaces:

$$F^* : T^*_{F(p)}N \longrightarrow T^*_pM.$$

This is characterized by the relation

$$(F^*\omega)(v) = \omega(F_*v)$$

for $\omega \in T^*_{F(p)}N$ and $v \in T_pM$, where F_* is the pushforward of F.

The pushforward of the projection map $\pi_* : TP \longrightarrow TM$ facilitates the construction of the **vertical subspace** $V_pP \coloneqq \ker(\pi_*)$ at a point $p \in P$, which serves as a vector subspace of the tangent space of P. It is noteworthy that each fundamental vector $X_p^A \in V_pP$, since $\pi_*(X_p^A) = 0$ by construction.

A **connection** on a principal bundle $P \xrightarrow{\pi} M$ is defined as a decomposition of the tangent space T_pP into a vertical subspace V_pP and a **horizontal subspace** H_pP. This is achieved by choosing a complement to the vertical subspace at each point $p \in P$ such that [DG18]:

(i) $T_pP = H_pP \oplus V_pP$

(ii) $(\triangleleft g)_*(H_pP) = H_{p\triangleleft g}P \quad \text{for all } g \in G$

(iii) For every smooth vector field $X \in \Gamma(TP)$, the unique decomposition $X = X^H + X^V$ with $X^H(p) \in H_pP$ and $X^V(p) \in V_pP$ produces smooth vector fields $X^H \in \Gamma(HP)$, $X^V \in \Gamma(VP)$.

Condition (ii) ensures that when moving along the fibers via the action of G, the horizontal subspace changes smoothly, while condition (iii) guarantees that the horizontal subspace varies smoothly when traversing the manifold P.

4.2 Connection One-Form

The choice of a horizontal subspace at each point $p \in P$ can be achieved by defining a Lie algebra-valued one-form. The horizontal subspace is then interpreted as the kernel of this one-form. The **connection one-form** $\omega \in \mathfrak{g} \otimes T^*P$ is defined as a $\mathfrak{g}$-valued one-form on P such that:

$$\begin{aligned} &\text{(i)} \quad \omega(X^A) = A \quad \text{for all } A \in \mathfrak{g} \\ &\text{(ii)} \quad (\triangleleft g)^*\omega = \mathrm{Ad}_{g^{-1}*}\omega \quad \text{for all } g \in G \end{aligned}$$

Here, $(\triangleleft g)^*\omega$ denotes the pullback of the connection one-form by the right action of $g \in G$ on P, and $\mathrm{Ad}_{g^{-1}}$ is the adjoint action of g^{-1} on the Lie Group G. This implies that $\omega_{p\triangleleft g}(X_p(\triangleleft g)_*) = g^{-1} \cdot_* \omega_p(X_p) \cdot_* g$. The horizontal subspace H_pP is then defined as the kernel of the connection one-form [Nak05]

$$H_pP \equiv \{X \in T_pP \mid \omega(X) = 0\}$$

This is consistent with the general definition of a connection. The smoothness of the decomposition is guaranteed by the fact that the connection one-form is a section, which is smooth by definition. It is therefore sufficient to show that the horizontal subspace is invariant under the right action of G. To show this, consider a vector X at a point $p \in P$ such that $X \in H_pP$. By definition $\omega(X) = 0$. For any element $g \in G$ the pushforward of the right action on X can be acted on by ω [Nak05]:

$$\omega((\triangleleft g)_*X) = (\triangleleft g)^*\omega(X) = g^{-1} \cdot \omega(X) \cdot g = 0$$

Therefore $(\triangleleft g)^*X$ is again a horizental vector at the point $p \triangleleft g$. Futhermore, any vector $\tilde{X} \in H_{p\triangleleft g}P$ can thus be obtained by the right action on some vector $X \in H_pP$. Therefore $(\triangleleft g)_*(H_pP) = H_{p\triangleleft g}P$

4.3 Local Connection Form

The connection one-form, as defined above, is a global object on the principal bundle P. However, in practice, it is often useful to work with local connection forms, which will be identified with the gauge potential in physical gauge theories.

Consider an open covering $\{U_i\}$ of the base manifold M and local sections $\sigma_i : U_i \to \pi^{-1}(U_i)$. A Lie-algebra valued one-form $\mathcal{A}_i \equiv \sigma_i^*\omega \in \mathfrak{g} \otimes \Omega^1(U_i)$ is defined for a global connection one-form ω [Nak05]. This local connection form is termed a **Yang-Mills field** [Fre15c].

Given a local section $\sigma_i : U_i \to P$, a local trivialization is established:

$$\begin{aligned} \psi_i : U_i \times G &\longrightarrow \pi^{-1}(U_i) \subset P \\ (p, g) &\mapsto \sigma_i(p) \triangleleft g \end{aligned}$$

This trivialization introduces a local representation of the global connection one-form ω via its pullback:

$$\begin{aligned} \psi_i^*\omega : T_{(p,g)}(U_i \times G) &\longrightarrow \mathfrak{g} \\ (\psi_i^*\omega)_{(p,g)}(X) &= \omega_{\sigma_i(p)\triangleleft g}\left((\psi_i)_* X\right) \end{aligned}$$

The relations of the above maps are illustrated in the following diagram:

$$\begin{tikzcd}
\psi_i^*\omega \arrow[dr, dashed] & & \\
& U_i \times G \arrow[r, "\psi_i"] \arrow[d, "\mathrm{proj}_1"'] & \pi^{-1}(U_i) \subset P \arrow[d, "\pi"] \\
& U_i \arrow[ur, "\sigma_i"] \arrow[r, "\mathrm{id}"'] & M \\
\sigma_i^*\omega \arrow[ur, dashed] & &
\end{tikzcd}$$

Figure 4.1: Local trivialization ψ_i and local connection forms $\psi_i^*\omega$ and $\sigma_i^*\omega$ associated to $U_i \times G$ and U_i, respectively.

This local representation is related to the Yang-Mills field $\mathcal{A}_i$ by [Fre15c]:

$$(\psi_i^*\omega)_{(p,g)}(X) = \mathrm{Ad}_{g^{-1}*}\left(\mathcal{A}_i(X)\right) + \Xi_g(X)$$

Here, Ξ denotes the **Maurer–Cartan form** of the Lie group G. This form takes a tangent vector $v \in T_gG$ and maps it to the unique Lie algebra element (i.e., a tangent vector at the identity) that generates v via left translation:

$$\Xi(v) = (g^{-1}\triangleright)_* v \in T_eG \cong \mathfrak{g}$$

This formulation exploits the fact that every tangent vector on G arises as the pushforward of a unique element of the Lie algebra $\mathfrak{g} = T_eG$ under left action [Rag25]. Thus, for every $v \in T_gG$, there exists a unique $X \in \mathfrak{g}$ such that

$$v = (g\triangleright)_* X.$$

Consequently, Ξ identifies the tangent bundle TG with $G \times \mathfrak{g}$ via left translation [Mau25].

4.4 Connection on the Frame Bundle

The Frame Bundle LM is of particular interest, because many groups relevant in physics are subgroups of the general linear group $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. Therefore in the following a local connection form and the Maurer-Cartan form will be derived.

Any choice of a chart (U_i, x) on the base manifold M induces a section on the frame bundle LM by associating to each point $m \in U_i$ the frame given by its coordinates. This section is denoted as:

$$\begin{aligned} \sigma : U_i &\longrightarrow LM \\ m &\mapsto \sigma_i(m) := \left(\frac{\partial}{\partial x^1}\bigg|_m, \dots, \frac{\partial}{\partial x^{\dim M}}\bigg|_m \right) \end{aligned}$$

Then the Yang-Mills field $\mathcal{A}_i = \sigma_i^* \omega$ is a one-form on U_i with values in the Lie algebra $\mathfrak{gl}(\dim M, \mathbb{R}) = \{M \mid M \text{ is a } n \times n \text{ matrix with components } M^\alpha_\beta \in \mathbb{R}\}$. The Yang-Mills field can be expressed in its components as:

$$(\mathcal{A}^i)^\alpha{}_{\beta\mu}$$

Where α, β are labels for the Lie algebra components and μ is the index of the base manifold. The Maurer-Cartan form Ξ can be constructed as follows:

Let $gl \subseteq GL(d,\mathbb{R})$ be an open subset of the general linear group containing the identity. Coordinates are introduced by:

$$\begin{aligned} x : gl &\longrightarrow \mathbb{R} \\ g &\mapsto x(g)^a{}_b := g^a{}_b \end{aligned}$$

Consider a left-invariant vector field L^A generated by the Lie algebra element $A \in \mathfrak{gl}(d,\mathbb{R})$. Since it is a vector field on the group, it acts on the coordinate functions:

$$\begin{aligned} \left(L^A x^a{}_b\right)_g &= x^a{}_b \frac{d}{dt} \left(g \cdot \exp(tA)\right)\bigg|_{t=0} \\ &= \frac{d}{dt} \left(g^a{}_c \exp(tA)^c{}_b\right)\bigg|_{t=0} \\ &= g^a{}_c A^c{}_b \end{aligned}$$

Therefore the components of the vector field are given by $L^A_g = g^a{}_b\, A^b{}_c \frac{\partial}{\partial x^a{}_c}$ [Fre15c]

The Maurer-Cartan form Ξ then is defined as the one-form that maps the left-invariant vector field L^A to the Lie algebra element A:

$$(\Xi_g)^a{}_b = (g^{-1})^a{}_c (dx^c{}_b)$$

It can be easily checked that this expression satisfies the properties of a Maurer-Cartan form:

$$\begin{aligned}
\Xi_g(L_g^A) &= (g^{-1})^a_{\ c}\,(dx)^c_{\ b}\left(g^p_{\ r}\,A^r_{\ q}\,\frac{\partial}{\partial x^p_q}\right) \\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\left((dx)^c_{\ b}\,\frac{\partial}{\partial x^p_q}\right) \\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\,\delta^c_p\,\delta^q_b \\
&= (g^{-1})^a_{\ p}\,g^p_{\ r}\,A^r_{\ b} \\
&= A^a_{\ b}
\end{aligned}$$

4.5 Compatibility condition for local connection forms

It was stated before, that the local connection forms $\mathcal{A}_i$ relate to a unique global connection one-form ω. For this to be true, the local connection forms must satisfy a compatibility condition. This condition is given by the requirement that the local connection forms on overlapping charts $U_i \cap U_j \neq \emptyset$ are related by a gauge transformation [Nak05]. Specifically, let σ_i and σ_j be sections respectively defining Yang-Mills fields $\mathcal{A}_i$ and $\mathcal{A}_j$ on the overlapping region $U_i \cap U_j$. Introduce a gauge map

$$\Omega : U_i \cap U_j \longrightarrow G$$

defined by the relation

$$\sigma_j(m) = \sigma_i(m) \triangleleft \Omega(m) \quad \forall m \in U_i \cap U_j$$

Then the local connection forms are related as follows:

$$\mathcal{A}_j = \mathrm{Ad}_{\Omega^{-1}(m)*}\mathcal{A}_i + \Omega^*\Xi_m$$

Again this will be shown for the case of the frame bundle LM. First, we calculate the latter expression. Notice that $\Omega^*\Xi_m$ is a map from the tangent space of the intersection on the base manifold $U_i \cap U_j$ to the Lie algebra $\mathfrak{gl}(d, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Therefore, to find the explicit form, it is calculated how this map acts on a vector in the tangent space at a point $p \in U_i \cap U_j$:

$$\begin{aligned}
(\Omega^*\Xi)_p\left(\frac{\partial}{\partial x^\mu}\right)_p &= \Xi_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right) \\
&= (\Omega^{-1}(p))^i_{\ k}(dx^k_j)_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right) \\
&= \Omega^{-1}(p)^i_{\ k}\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\left(x^k_j\right) \\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\left(x^k_j \circ \Omega\right)_p \\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\Omega(p)^k_{\ j}
\end{aligned}$$

Therefore, the components of the pullback of the Maurer–Cartan form are given by [Fre15c]:

$$((\Omega^*\Xi)_p)^i_{\ j} = \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\Omega(p)^k_{\ j}\,dx^\mu := \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}$$

Futhermore, the pushforward of the adjoint action on the Yang-Mills field is easily obtained by definition of the adjoint action:

$$\begin{aligned} \mathrm{Ad}_g : G \longrightarrow G\,, &\quad h \mapsto ghg^{-1} \\ \mathrm{Ad}_{g*} : T_eG \longrightarrow T_eG\,, &\quad A \mapsto \mathbf{g}\mathbf{A}\mathbf{g}^{-1} \end{aligned}$$

Here the notation $\mathbf{g}$ is used to denote the matrix product, since the adjoint action is defined on the group G not the Lie algebra $\mathfrak{g}$.

Altogether the transition between two Yang-Mills fields on the intersection of two charts is given by:

$$\begin{aligned} \mathcal{A}_j &= \mathbf{\Omega}^{-1}\,\mathcal{A}_i\,\mathbf{\Omega} + \mathbf{\Omega}^{-1}\,d\mathbf{\Omega} \\ (\mathcal{A}_j)^i{}_{r\mu} &= \left(\Omega^{-1}(p)\right)^i{}_k\,(\mathcal{A}_i)^k_{l\mu}\,\Omega(p)^l{}_r + \left(\Omega^{-1}(p)\right)^i{}_k\,\partial_\mu\Omega(p)^k_r \end{aligned}$$

This is simply the **gauge transformation** as known form gauge theories [Nak05].

As an example, consider the case of a $U(1)$ principal bundle. The transition function Ω is a smooth function $U_i \cap U_j \longrightarrow U(1)$, which can be expressed as $\Omega(m) = \exp[i\Lambda(m)]$ for some real-valued function $\Lambda : U_i \cap U_j \longrightarrow \mathbb{R}$. Since $U(1)$ is a subgroup of $GL(d, \mathbb{C})$, two local connection forms $\mathcal{A}_i$ and $\mathcal{A}_j$ on the intersection $U_i \cap U_j$ are then related by:

$$\begin{aligned} \mathcal{A}_j &= \Omega^{-1}\,\mathcal{A}_i\,\Omega + \Omega^{-1}\,d\Omega \\ &= \mathcal{A}_i + e^{-i\Lambda(m)}\,d\left(e^{i\Lambda(m)}\right) \\ &= \mathcal{A}_i + e^{-i\Lambda(m)} \cdot ie^{i\Lambda(m)}\,d\Lambda \\ &= \mathcal{A}_i + i\,d\Lambda \end{aligned}$$

Which in components reads:

$$\mathcal{A}_{j\,\mu} = \mathcal{A}_{i\,\mu} + i\partial_\mu\Lambda$$

This is the familiar form of the gauge transformation in electromagnetism [Nak05].

Chapter 5

Curvature and Field Strength

5.1 Curvature

Let P be a principal G-bundle with a connection one form ω and let $\phi \in \Omega^k(P) \otimes V$ be a V valued k-form on P, where V is some k-dimensional vector space with basis $\{e_i\}$. The connection one form ω allows for the separation of the tangent space of P into horizontal and vertical components. Then the map:

$$\begin{aligned} &D\phi : \Gamma(T_u^{k+1}P) \to V, \\ &(X_1, \dots X_{k+1}) \mapsto D\phi(X_1, \dots X_k) := d\phi(X_1^H, \dots X_{k+1}^H) \end{aligned}$$

is called the **exterior covariant derivative** of ϕ. Here $d\phi \equiv d\phi^i \otimes e_i$ is the exterior derivative.

This introduces the **curvature two-form** Ω as the exterior covariant derivative of the connection one-form ω:

$$\Omega \equiv D\omega \in \Omega^2(P) \otimes \mathfrak{g}$$

First it will be shown, that Ω takes the following form:

$$\Omega = d\omega + \omega \wedge_{\mathfrak{g}} \omega$$

Where $\wedge_{\mathfrak{g}}$ denotes the wedge product in the Lie algebra $\mathfrak{g}$ of G defined by its action on $\Gamma(T^2P)$: $(\omega \wedge_{\mathfrak{g}} \omega)(X, Y) := [\omega(X), \omega(Y)]_{\mathfrak{g}}$

Note that if G is a matrix group, then the above can be written in terms of its components as:

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

This will be proven by considering three separate cases [Fre15a]:

a) $X, Y \in \Gamma(TP)$ are vertical vector fields

$\Rightarrow \quad \exists A, B \in T_eG : X = X^A, \quad Y = X^B$

Left-hand side:

$$\Omega(X^A, X^B) = D\omega(X^A, X^B) = d\omega\left((X^A)^H, (X^B)^H\right)$$
$$= d\omega(0,0) = 0$$

Right-hand side:

$$d\omega(X^A, X^B) + (\omega \wedge_{\mathfrak{g}} \omega)(X^A, X^B)$$
$$= X^A(\omega(X^B)) - X^B(\omega(X^A)) - \omega([X^A, X^B]) + \left[\omega(X^A), \omega(X^B)\right]_{\mathfrak{g}}$$
$$= X^A(B) - X^B(A) - \omega(X^{[A,B]_{\mathfrak{g}}}) + [A,B]_{\mathfrak{g}}$$
$$= 0 - 0 - [A,B]_{\mathfrak{g}} + [A,B]_{\mathfrak{g}}$$
$$= 0$$

b) $X, Y \in \Gamma(TP)$ are horizontal vector fields

Left-hand side:

$$\Omega(X, Y) = D\omega(X, Y) = d\omega(X^H, Y^H)$$
$$= d\omega(X, Y)$$

Right-hand side:

$$d\omega(X, Y) + (\omega \wedge_{\mathfrak{g}} \omega)(X, Y)$$
$$= d\omega(X^H, Y^H) + \left[\omega(X), \omega(Y)\right]_{\mathfrak{g}}$$
$$= d\omega(X, Y) + \left[0, 0\right]_{\mathfrak{g}}$$
$$= d\omega(X, Y)$$

c) $X \in \Gamma(TP)$ is horizontal and $Y = X^A \in \Gamma(TP)$ is vertical

Left-hand side:

$$\Omega(X, X^A) = D\omega(X, X^A) = d\omega(X^H, (X^A)^H)$$
$$= d\omega(X, 0)$$
$$= 0$$

Right-hand side:

$$d\omega(X, X^A) + (\omega \wedge_{\mathfrak{g}} \omega)(X, X^A)$$
$$= d\omega(X, X^A) + \left[\omega(X), \omega(X^A)\right]_{\mathfrak{g}}$$
$$= X(\omega(X^A)) - X^A(\omega(X)) - \omega([X, X^A]) + \left[\omega(X), \omega(X^A)\right]_{\mathfrak{g}}$$
$$= X(A) - X^A(0) - \omega([X, X^A]) + \left[0, A\right]_{\mathfrak{g}}$$
$$= 0$$

Where in the last step the fact that the comutator of a horizontal and a vertical vector field is again a horizontal vector field was used [Nak05].

5.2 Local from of the curvature and Yang-Mills field strength

As the connection one-form ω can be expressed locally as the pullback by a section $\mathcal{A}_i = \sigma^*\omega$, the local from of the curvature two-form Ω is defined analogous [Nak05]:

$$\mathscr{F} \equiv \sigma^*\Omega \in \Omega^2(M) \otimes \mathfrak{g}$$

In terms of the local connection one-form $\mathcal{A}$, the curvature two-form can be expressed as:

$$\begin{aligned}
\mathscr{F} &= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\
&= \sigma^*(d\omega) + \sigma^*(\omega \wedge_{\mathfrak{g}} \omega) \\
&= \sigma^*(d\omega) + \sigma^*(\omega) \wedge_{\mathfrak{g}} \sigma^*(\omega) \\
&= d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_j
\end{aligned}$$

Let x^μ be the coordinates on the open set U_i where the section σ is defined. Then the Yang-Mills field is given by $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. We therefore get the following expression:

$$\begin{aligned}
\mathscr{F} &= d(\mathcal{A}_\mu dx^\mu) + (\mathcal{A}_\mu dx^\mu \wedge_{\mathfrak{g}} \mathcal{A}_\nu dx^\nu) \\
&= \frac{1}{2}\left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]_{\mathfrak{g}}\right) dx^\mu \wedge dx^\nu \\
&:= \frac{1}{2}\mathscr{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu
\end{aligned}$$

In physics, the local curvature two-form $\mathscr{F}$ is identified with the **Yang-Mills field strength**.

The compatibility condition of the field strengths can be derived by substituting the transformation of the connection one-form into the expression for the curvature two-form.

First, compute the exterior derivative:

$$\begin{aligned}
&d\left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \\
&= -\boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} \boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\mathcal{A}_i\,\boldsymbol{\Omega} \\
&\quad - \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\boldsymbol{\Omega} - \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}^{-1} \wedge_{\mathfrak{g}} d\boldsymbol{\Omega}
\end{aligned}$$

Then, compute the wedge product:

$$\begin{aligned}
&\left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \wedge_{\mathfrak{g}} \left(\boldsymbol{\Omega}^{-1}\mathcal{A}_i\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega}\right) \\
&= \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i\,\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\boldsymbol{\Omega} \\
&\quad + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} \mathcal{A}_i\,\boldsymbol{\Omega} + \boldsymbol{\Omega}^{-1}d\boldsymbol{\Omega} \wedge_{\mathfrak{g}} d\boldsymbol{\Omega}
\end{aligned}$$

Combining both contributions, we obtain:

$$\begin{aligned}
\mathscr{F}_j &= \boldsymbol{\Omega}^{-1}\left(d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i\right)\boldsymbol{\Omega} \\
&= \boldsymbol{\Omega}^{-1}\,\mathscr{F}_i\,\boldsymbol{\Omega}
\end{aligned}$$

5.3 The Bianchi identity

The Bianchi identity states that the covariant derivative of the curvature two-form vanishes. To show this, the exterior derivative of the curvature two-form is computed:

$$d\Omega = d(d\omega) + d(\omega \wedge_{\mathfrak{g}} \omega) = d\omega \wedge_{\mathfrak{g}} \omega - \omega \wedge_{\mathfrak{g}} d\omega$$

Since for any $X \in H_pP$ the connection one-form vanishes, the following holds:

$$D\Omega(X, Y, Z) = d\omega(X^H, Y^H, Z^H) = 0$$

Therefore, the **Bianchi identity** is $D\Omega = 0$

Localy the Bianchi identity is given by:

$$\begin{aligned}
\sigma^* d\Omega &= d(\sigma^*\Omega) = d\mathscr{F} \\
&= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\
&= d\sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega + \sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega \\
&= d\mathcal{A} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} d\mathcal{A} \\
&= \mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}
\end{aligned}$$

Thus the Bianchi identity in local coordinates is given by:

$$D\mathscr{F} = d\mathscr{F} - (\mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}) = d\mathscr{F} + [\mathcal{A}, \mathscr{F}]_{\mathfrak{g}} = 0$$

Chapter 6

Gauge Theories

In physical gauge theories like electromagnetism, Yang-Mills theories or general relativity, the laws of nature they describe are not just differential equations that happen to describe nature, but they are deeply connected to the geometry of the underlying symmetries. In the following, the above developed mathematical framework is applied to recover Maxwell's equations, Yang-Mills theories.

6.1 Maxwell theory

Consider a $U(1)$ principal bundle P over the four dimensional Minkowski spacteime manifold M equipped with the Minkowski metric $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$. The principal bundle is trivial $P = M \times U(1)$, and the projection map is given by $\pi : P \to M$, $\pi(x, e^{i\Lambda}) = x$. The Yang-Mills field is given by:

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu \in \Omega^1(M) \otimes \mathfrak{u}(1)$$

And the field strength is given by the curvature two-form:

$$\mathscr{F} = d\mathcal{A}$$

We identify the **gauge potential** A by $\mathcal{A}_\mu = iA_\mu$ and the field strength tensor F by $\mathscr{F}_{\mu\nu} = iF_{\mu\nu}$, where i is the factor associated with the Lie algebra. Therefore, the curvature two-form can be written in components as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Bianchi identity is given by:

$$\begin{aligned} D\mathscr{F} &= d\mathscr{F} \\ &= \frac{1}{2} \partial_\mu \mathscr{F}_{\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho = 0 \\ \Rightarrow \quad & \partial_\mu \mathscr{F}_{\nu\rho} + \partial_\nu \mathscr{F}_{\rho\mu} + \partial_\rho \mathscr{F}_{\mu\nu} = 0 \end{aligned}$$

When identifying the electric and magnetic fields with the components of the field strength tensor, we have:

$$\begin{aligned} E_i &= F_{0i} \\ B_i &= \frac{1}{2} \epsilon_{ijk} F_{jk} \end{aligned}$$

The Bianchi identity yields the **homogeneous Maxwell equations**:

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

First, choosing indices $\mu = 0$, $\nu = i$, $\rho = j$ and using antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, we obtain:

$$\begin{aligned} \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} &= 0 \\ \Rightarrow \quad (\nabla \times \mathbf{E})_k + \partial_t B_k &= 0 \end{aligned}$$

Now, choose $\mu = i$, $\nu = j$, $\rho = k$, all spatial indices. Contracting with the Levi-Civita tensor ϵ^{ijk} gives:

$$\begin{aligned} \epsilon^{ijk} \partial_i F_{jk} &= 0 \\ \Rightarrow \quad \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

Together, these two identities form the homogeneous Maxwell equations:

$$\begin{aligned} \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

6.2 Yang-Mills theory

The same construction applies to non-Abelian gauge theories with structure group $G = SU(N)$. We consider a trivial principal bundle over the spaceteime manifold $P = M \times SU(N)$, with projection map $\pi : P \to M$, $\pi(x, g) = x$. The Yang–Mills field is given by a $\mathfrak{su}(N)$-valued one-form:

$$\mathcal{A} = A^a_\mu T^a dx^\mu,$$

where $T^a \in \mathfrak{su}(N)$ are the generators of the Lie algebra of $SU(N)$, and A^a_μ are the gauge potentials. The corresponding field strength (curvature two-form) is given by:

$$\mathscr{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

In components, this reads:

$$\begin{aligned} \mathscr{F}_{\mu\nu} &= \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a + A^b_\mu A^c_\nu [T^b, T^c] \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \right) T^a =: F^a_{\mu\nu} T^a, \end{aligned}$$

where f^{abc} are the structure constants of $\mathfrak{su}(N)$, defined via $[T^b, T^c] = f^{abc} T^a$.

The Bianchi identity holds:

$$D\mathscr{F} = d\mathscr{F} + [\mathcal{A}, \mathscr{F}] = 0,$$

which in components becomes:

$$\partial_\lambda F^a_{\mu\nu} + f^{abc} A^b_\lambda F^c_{\mu\nu} = 0.$$

6.3 Gauge sector of the standard model of particle physics

The above elaboration of gauge theories can be applied to the gauge sector of the standard model of particle physics. It is described by a principal bundle with gauge group $SU(3) \times$

$SU(2) \times U(1)$, and a corresponding connection one-form $\mathcal{A} \in \Omega^1(M, \mathfrak{g})$, with curvature $\mathscr{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \in \Omega^2(M, \mathfrak{g})$. The total gauge action takes the form:

$$S_{\text{gauge}} = -\frac{1}{2} \int_M \mathrm{Tr}(\mathscr{F}_{\mu\nu} \mathscr{F}^{\mu\nu})\, \mathrm{d}^4 x,$$

where the trace is taken over the Lie algebra representation [Nak05]. Here, the normalization $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta_{ab}$ was used, which is common in physics literature. This ensures that the action written in components has the proper normalization.

$$S_{\text{gauge}} = \int_M \mathrm{d}^4 x \left(-\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \right),$$

where $G^a_{\mu\nu}$, $W^i_{\mu\nu}$, and $B_{\mu\nu}$ are the field strength components derived from the respective gauge subgroups. These arise from the decomposition of the total curvature two-form $\mathscr{F} \in \mathfrak{g} \otimes \Omega^2(M)$, with $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

The above construction provides the geometric foundation for Yang–Mills theory as applied in the standard model.

Chapter 7

Associated Bundles

Associated bundles are a way to construct vector bundles that are associated in a precise manner to a given principal bundle. They play a crucial role in gauge theories, as they provide the mathematical framework for describing matter fields that transform under the action of a gauge group.

7.1 Definition of Associated Bundles

Given a G-principal bundle $\pi : P \to M$ and a smooth manifold F with a left G-action $\triangleright : G \times F \to F$, an associated bundle $\pi_F : P_F \to M$ is defined as follows:

- We define the equivalence relation:

$$(p, f) \sim_G (p \triangleleft g, g^{-1} \triangleright f) \quad \forall p \in P, f \in F, g \in G.$$

- The associated bundle P_F is the quotient space:

$$(P \times F)/ \sim_G =: P_F$$

- The projection map $\pi_F : P_F \to M$ is defined by:

$$\pi_F([p, f]) = \pi(p),$$

  where $[p, f]$ denotes the equivalence class of (p, f) under the defined relation.

It is instructive to recall that the fiber of the principal bundle P is the group G itself. Therefore taking the modulo of P by the group action effectively reproduces the base Manifold. Now roughly speaking, attaching to it a different fiber F with a group action of G results in a new bundle with fiber F over the same base manifold M.

The projection is well-defined, since $\pi_F([p, f]) = \pi(p) = \pi(p \triangleleft g) = \pi_F([p \triangleleft g, g^{-1} \triangleright f])$ due to the properties of the principal bundle.

To prove that this construction indeed yields a fiber bundle with typical fiber F, consider a local trivialization of the principal bundle. Let $\{U_i\}$ be an open cover of M over which P admits smooth local sections $s_i : U_i \to P$, and let

$$\phi_i : \pi^{-1}(U_i) \to U_i \times G, \qquad p \mapsto (\pi(p), g_i(p)),$$

be the corresponding local trivializations, where $p = s_i(\pi(p)) \triangleleft g_i(p)$ is unique. Using this, define a map

$$\psi_i : U_i \times F \longrightarrow \pi_F^{-1}(U_i), \qquad \psi_i(x, f) := [\, s_i(x),\ f\,].$$

This map is smooth and surjective. To show that it is a trivialization, define its inverse

$$\psi_i^{-1}([p, f]) := (\pi(p),\, g_i(p)^{-1} \triangleright f).$$

This is well-defined: if $(p, f) \sim_G (p \triangleleft g, g^{-1} \triangleright f)$, then

$$p = s_i(\pi(p)) \triangleleft g_i(p) \quad\Longrightarrow\quad p \triangleleft g = s_i(\pi(p)) \triangleleft (g_i(p)g),$$

so the G–component transforms as $g_i(p) \mapsto g_i(p)g$, and therefore

$$(g_i(p)g)^{-1} \triangleright (g^{-1} \triangleright f) = g^{-1} g_i(p)^{-1} \triangleright g^{-1} \triangleright f = g_i(p)^{-1} \triangleright f,$$

so the expression is independent of the chosen representative. Hence ψ_i is a smooth bijection with smooth inverse, and

$$\pi_F \circ \psi_i(x, f) = x,$$

so (U_i, ψ_i) is a local trivialization of P_F. In particular,

$$\pi_F^{-1}(x) \cong F \qquad \text{for all } x \in M.$$

Thus $\pi_F : P_F \to M$ is a smooth fiber bundle with typical fiber F.

7.1.1 The tangent bundle as an associated bundle

First, consider the frame bundle LM of a smooth manifold M. The frame bundle is a principal $GL(d, \mathbb{R})$-bundle over M, where $d = \dim M$. The fiber at each point $x \in M$ consists of all ordered bases (frames) of the tangent space T_xM. The right action of $GL(d, \mathbb{R})$ on the frame bundle was defined as:

$$\begin{aligned} \triangleleft : GL(d, \mathbb{R}) \times LM &\to LM \\ (e_1, \dots, e_d) \triangleleft g &:= (g_1^m e_m, g_2^m e_m, \cdots, g_d^m e_m) \end{aligned}$$

Now construct a associated fiber bundle with typical fiber $\mathbb{R}^d$ and the standard representation of $GL(d, \mathbb{R})$ on $\mathbb{R}^d$. The left action of $GL(d, \mathbb{R})$ on $\mathbb{R}^d$ can be defined as:

$$\triangleright : GL(d, \mathbb{R}) \times F \to F; \quad (g \triangleright f)^a := g_b^a f^b$$

Notice that this coincides with the usual definition of how vectors and there components transform under a change of basis. In the discussion of the tangent bundle in chapter 2, this was a condition that was a conclusion. Notice that from the perspective of associated bundles, this is a choice for definition of a left action. As we will see in following sections, this choice is not unique, and different choices lead to different associated bundles.

For the definition given above, the associated bundle is $\pi_{\mathbb{R}^d} : LM_{\mathbb{R}^d} \to M$. There exists a natural isomorphism $u : LM_{\mathbb{R}^d} \to TM$

$$\begin{aligned} u : LM_{\mathbb{R}^d} \equiv (LM \times \mathbb{R}^d)/{\sim_G} &\longrightarrow TM \\ [e, f] &\longmapsto f^a e_a \end{aligned}$$

This map is invertible, since for any $X \in TM$ any choice of frame $e \in LM$ at $\pi_{TM}(X)$ gives a unique $f \in \mathbb{R}^d$ such that $X = f^a e_a$. The map is well-defined, since for any other representative $(e \triangleleft g, g^{-1} \triangleright f)$ reproduces the equivalence class $[e, f]$.

$$
\begin{tikzcd}
LM_{\mathbb{R}^d} \arrow[r, "u"] \arrow[d, "\pi_{\mathbb{R}^d}"'] & TM \arrow[d, "\pi_{TM}"] \\
M \arrow[r, "\mathrm{id}_M"'] & M
\end{tikzcd}
$$

Figure 7.1: Commutative diagram exhibiting the tangent bundle TM as the associated bundle $LM_{\mathbb{R}^d}$ of the frame bundle.

7.1.2 Tensor bundles as associated bundles

Equivalently to the tangent bundle, one can simply extend this construction to bundles associated to the frame bundle LM with typical fiber

$$F = \left(\mathbb{R}^d\right)^p \times \left(\mathbb{R}^{d^*}\right)^q$$

define a left-actoin $\quad \triangleright : GL(1,1)) \times F \to F$ by

$$(g \triangleright f)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} := g^{i_1}{}_{k_1} \cdots g^{i_p}{}_{k_p} (g^{-1})^{l_1}{}_{j_1} \cdots (g^{-1})^{l_q}{}_{j_q} \, f^{k_1 \cdots k_p}{}_{l_1 \cdots l_q}.$$

As above this bundle $\pi_F : LM_F \to M$ is isomorphic to the (p,q)-tensor bundle $\pi : T^p_q M \to M$. This definition formalizes, what in physics is often taken as given. Intuitively, the coordinate transformations is a choice of a different frame in the principal bundle, and each associated bundle must follow the transformation rules defined by the left action of the group on the typical fiber.

7.1.3 Tensor densities as associated bundles

Consider an associated bundle to the frame bundle LM with the same typical fiber as the tensor bundle above

$$F = \left(\mathbb{R}^d\right)^p \times \left(\mathbb{R}^{d^*}\right)^q$$

But now define a different left action $\quad \triangleright : GL(1,1)) \times F \to F$ by

$$(g \triangleright f)^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} := \left(\det g^{-1}\right)^{\omega} g^{i_1}{}_{k_1} \cdots g^{i_p}{}_{k_p} (g^{-1})^{l_1}{}_{j_1} \cdots (g^{-1})^{l_q}{}_{j_q} \, f^{k_1 \cdots k_p}{}_{l_1 \cdots l_q}.$$

for some $\omega \in \mathbb{Z}$. Then the bundle $\pi_F : LM_F \to M$ is called the bundle of (p,q)-tensor densities of weight ω over M. Choosing $F = \mathbb{R}$ and $\omega = 1$ yields the bundle of scalar densities of weight 1. The significance of tensor densities becomes clear when considering integration on manifolds. Ordinary tensor fields do not transform in a way that makes their coordinate expressions compatible with a well-defined notion of integration. What *is* integrable on a manifold are *densities*, i.e. sections of associated bundles whose transition functions include a power of the determinant of the Jacobian. The weight ω in the $GL(d)$–action above precisely controls how such objects respond to a change of frame. For instance, scalar densities of weight 1 transform as

$$\rho \;\mapsto\; (\det g^{-1})\, \rho,$$

ensuring that the combination $\rho \, \mathrm{d}^d x$ is invariantly defined and can therefore be integrated over M.

This is directly visible in General Relativity. Under a coordinate transformation with Jacobian matrix

$$g^i{}_j = \frac{\partial x^i}{\partial x'^j},$$

the metric tensor γ transforms as

$$\gamma'_{ij} = g^k{}_i \, g^l{}_j \, \gamma_{kl},$$

and therefore its determinant transforms as

$$\det \gamma' = (\det g^k{}_i)^2 \, \det \gamma = (\det g^{-1})^2 \det \gamma.$$

Thus $\det \gamma$ is a *scalar density of weight* 2, and consequently $\sqrt{|\det \gamma|}$ is a scalar density of weight 1. This is exactly the object needed to compensate for the transformation of the coordinate measure, so that

$$\sqrt{-\gamma} \, \mathrm{d}^4 x$$

is invariantly defined. Hence the Einstein–Hilbert action is written as

$$\int_M \sqrt{-\gamma} \, R \, \mathrm{d}^4 x.$$

which is independent of any coordinate choice.

The introduction of tensor densities is therefore not merely formal: it captures the geometric fact that integration on manifolds is only well defined for objects that transform as densities. This also explains why the *Lagrangian density* in field theory is called a density: it is a section of a weight-1 scalar density bundle, ensuring that the action integral is well defined. The relation to differential forms and volume elements will be discussed further in the context of integration on manifolds.

7.2 Associated bundle maps

Above associated bundles were constructed as fiber bundles which carry additional structure induced by a principal bundle. Therefore we now define maps between associated bundles which preserve this additional structure. To define these maps consider two associated bundles, with the same fiber F, but associated to arbitrary principal bundles $\pi : P \to M$ and $\pi' : P' \to M'$ with the same structure group G. An associated bundle map $(\tilde{u}, \tilde{h}$ is a bundle map which can be constructed a principal bundle map between the underlying principal bundles (u, h).

$$\begin{tikzcd}
P \arrow[r, "u"] & P' \\
P \arrow[u, "\triangleleft G"] \arrow[r, "u"] \arrow[d, "\pi"'] & P' \arrow[u, "\triangleleft' G"'] \arrow[d, "\pi'"] \\
M \arrow[r, "h"] & M'
\end{tikzcd}$$

$$\tilde{u}(p \triangleleft g) = \tilde{u}(p) \triangleleft' g, \qquad \pi' \circ \tilde{u} = h \circ \pi.$$

Figure 7.2: A morphism of principal G-bundles.

Then the associated bundle map $(\tilde{u}, \tilde{h})$ is defined as:

$$\begin{aligned}
\tilde{u} : P_F &\longrightarrow P'_F & \qquad \tilde{h} : M &\longrightarrow M' \\
[p, f] &\longmapsto \tilde{u}([p, f]) := [u(p), f] & m &\longmapsto h(m)
\end{aligned}$$

This definition is indeed necessary, since two F–fiber bundles may be isomorphic as bundles but still fail to be isomorphic as associated bundles. This situation occurs when the underlying principal bundles are not isomorphic as *principal* G–bundles: even though there may exist a bundle isomorphism between the total spaces, it might not respect the G–action. In that case, the formula

$$[p, f] \longmapsto [u(p), f]$$

is not well defined on the associated bundles. For $(p, f) \sim (p \triangleleft g,\, g^{-1} \triangleright f)$ we obtain

$$[u(p), f] \qquad \text{and} \qquad [u(p \triangleleft g),\, g^{-1} \triangleright f],$$

and these coincide precisely when

$$u(p \triangleleft g) = u(p) \triangleleft' g,$$

i.e. when u is G–equivariant. Thus G–equivariance is exactly the condition ensuring that the induced map between associated bundles is well defined.

7.3 Sections on associated bundles

This section discusses the relation between sections of associated bundles and equivariant maps on the underlying principal bundle. We will proof the following theorem:

Given a principal G-bundle $\pi : P \to M$ and an associated bundle $\pi_F : P_F \to M$ with typical fiber F. There exists a bijective correspondence between sections $\sigma_F \in \Gamma(P_F)$ of the associated bundle and G-equivariant functions $\phi : P \to F$ satisfying $\phi(p \triangleleft g) = g^{-1} \triangleright \phi(p)$.

Proof. (a) Given a G–equivariant map $\phi : P \to F$, define a section

$$\sigma_\phi : M \longrightarrow P_F, \qquad \sigma_\phi(x) := [p,\, \phi(p)],$$

where p is any element of the fiber $\pi^{-1}(x)$.

To see that this is well defined, let $p' \in \pi^{-1}(x)$. Since P is a principal G–bundle, there exists a unique $g \in G$ such that $p' = p \triangleleft g$. Then

$$[p', \phi(p')] = [p \triangleleft g,\, \phi(p \triangleleft g)] = [p \triangleleft g,\, g^{-1} \triangleright \phi(p)] = [p, \phi(p)],$$

where the last equality uses the G–equivariance of ϕ. Thus σ_ϕ is independent of the choice of p and therefore well defined.

(b) Conversely, given a section $\sigma : M \to P_F$, we construct a map $\phi_\sigma : P \to F$. For each $p \in P$, the value $\sigma(\pi(p))$ lies in the fiber $\pi_F^{-1}(\pi(p)) \subset P_F$. Define

$$\phi_\sigma(p) := (i_p)^{-1}\big(\sigma(\pi(p))\big),$$

where

$$i_p : F \longrightarrow \pi_F^{-1}(\pi(p)), \qquad i_p(f) := [p, f].$$

The map i_p is a bijection: every element of $\pi_F^{-1}(\pi(p))$ has the form $[p, f]$ for a unique $f \in F$. Furthermore, for any $g \in G$,

$$i_p(f) = [p, f] = [p \triangleleft g,\; g^{-1} \triangleright f] = i_{p \triangleleft g}(g^{-1} \triangleright f),$$

which describes how i_p transforms when p is replaced by another point in the same principal fiber.

We now show that ϕ_σ is G–equivariant. For any $p \in P$ and $g \in G$,

$$\begin{aligned}\phi_\sigma(p \triangleleft g) &= (i_{p\triangleleft g})^{-1}\big(\sigma(\pi(p \triangleleft g))\big) \\ &= (i_{p\triangleleft g})^{-1}\big(\sigma(\pi(p))\big) \\ &= (i_{p\triangleleft g})^{-1}\big(i_p(\phi_\sigma(p))\big) \\ &= (i_{p\triangleleft g})^{-1}\big(i_{p\triangleleft g}(g^{-1} \triangleright \phi_\sigma(p))\big) \\ &= g^{-1} \triangleright \phi_\sigma(p),\end{aligned}$$

using the transformation rule above. Thus ϕ_σ is G–equivariant.

(c) Finally, we show that the two constructions are inverse to one another.

(1) Recovering σ from ϕ_σ. Let $\sigma : M \to P_F$ be a section. For $x \in M$ choose any $p \in \pi^{-1}(x)$. Then

$$\begin{aligned}\sigma_{\phi_\sigma}(x) &= [p, \phi_\sigma(p)] \\ &= [p, (i_p)^{-1}(\sigma(\pi(p)))] \\ &= \sigma(\pi(p)) \\ &= \sigma(x),\end{aligned}$$

because $\pi(p) = x$ and i_p is a bijection onto the fiber over x.

(2) Recovering ϕ from σ_ϕ. Let $\phi : P \to F$ be G–equivariant. Then for any $p \in P$,

$$\begin{aligned}\phi_{\sigma_\phi}(p) &= (i_p)^{-1}\big(\sigma_\phi(\pi(p))\big) \\ &= (i_p)^{-1}\big([p, \phi(p)]\big).\end{aligned}$$

Since $i_p(f) = [p, f]$, the only f satisfying $[p, f] = [p, \phi(p)]$ is $f = \phi(p)$. Therefore

$$\phi_{\sigma_\phi}(p) = \phi(p),$$

so $\phi_{\sigma_\phi} = \phi$.

We conclude that

$$\phi \longmapsto \sigma_\phi, \qquad \sigma \longmapsto \phi_\sigma$$

are mutually inverse bijections between G–equivariant maps $P \to F$ and sections of the associated bundle P_F. $\square$

Chapter 8

Covariant Derivatives

Covariant derivatives are a fundamental concept in differential geometry and gauge theories, providing a way to differentiate sections of associated vector bundles while respecting the underlying geometric structure. In this chapter, the notion of covariant derivatives will be introduced, and their relationship with connections on principal bundles will be explored.

Recall that sections of bundles associated to a G-principal bundle $\pi : P \to M$ are equivalent to G-eqivariant functions $\phi : P \to F$, where F is the typical fiber of the associated bundle. For the definition of the covariant derivative, we first identify the equivarianz condition for finite dimensional linear left actions. This imideatly restricts us to the discussion of associated vector bundles, i.e. F is a finite dimensional vector space.

For any $g \in G$ close to the identity, we can write the equivariant condition as:

$$\phi(p \triangleleft \exp(At)) = \exp(-At) \triangleright \phi(p)$$

for $A \in T_eG$ and $t \in \mathbb{R}$ small enough. Differentiating both sides with respect to t at $t = 0$ yields:

$$\begin{aligned} \frac{d}{dt}\, \phi(p \triangleleft \exp(At))\Big|_{t=0} &= \frac{d}{dt}\left(\exp(-At) \, \triangleright \, \phi(p)\right)\Big|_{t=0} \\ d\phi(p)\left(X_p^A\right) &= -\, A \triangleright \phi(p) \\ &= -\omega(X^A) \triangleright \phi(p) \end{aligned}$$

Therefore the G-equivariance condition for linear left actions becomes:

$$d\phi\left(X^A\right) + \omega(X^A) \triangleright \phi(p) = 0$$

8.1 Definition of Covariant Derivative

Consider a G-principal bundle $\pi : P \to M$ with connection form $\omega \in \Omega^1(P, \mathfrak{g})$ and an associated vector bundle $\pi_F : P_F \to M$ with a section $\sigma \in \Gamma(P_F)$. We now construct a covariant derivative:

$$\nabla_T \sigma \in \Gamma(P_F) \quad \text{for}\, T \in T_xM$$

such that:

$$\text{(i) } \nabla_{fT+S}\, \sigma = f\, \nabla_T \sigma + \nabla_S \sigma, \qquad f \in C^\infty(M),\;\; T, S \in T_xM,$$

$$\text{(ii) } \nabla_T(\sigma + \tau) = \nabla_T \sigma + \nabla_T \tau,$$

$$\text{(iii) } \nabla_T(f\sigma) = (Tf)\, \sigma + f\, \nabla_T \sigma.$$

At this point, most physics literature would derive from these properties the expression for the covariant derivative in a local coordinate patch. However, rather than postulating this expression, we derive it from the underlying geometry. In particular, we show that the covariant derivative emerges naturally from the additional structure that an associated bundle inherits from its principal bundle.

Let ϕ be the induced G-equivariant function corresponding to the section σ. Since it is a function, it is a F-valued zero-form on P, i.e. $\phi \in \Omega^0(P, F)$. Recall the definition of the exterior covariant derivative D for a F-valued k-form $\alpha \in \Omega^k(P, F)$ on P, as the exterior derivative d acting only on the horizontal components of vector fields. Thus we define the covariant derivative:

$$D\phi := d\phi \circ \mathrm{hor}$$

This will take the form:

$$D\phi(X) = d\phi(X) + \omega(X) \wedge \phi \quad \text{for } X \in T_pP$$

Since ϕ is a zero-form, the wedge product reduces to ordinary multiplication $\omega(X) \wedge \phi = \omega(X) \triangleright \phi$. To proof this, we consider the cases where X is either horizontal or vertical:

Proof. (a) Suppose first that X is vertical. Then $X = X^A$ for some $A \in \mathfrak{g}$, i.e. X is the fundamental vector field generated by A. Using the definition of $D\phi$ and the decomposition of X into horizontal and vertical components, we have

$$D\phi(X) = d\phi(\mathrm{hor}(X)) = d\phi(0) = 0.$$

On the other hand, by G–equivariance of ϕ,

$$d\phi(X^A) + \omega(X^A) \triangleright \phi = 0.$$

Since $\mathrm{hor}(X) = 0$ for vertical X.

(b) Now let X be horizontal. Then $\omega(X) = 0$, so we compute

$$D\phi(X) = d\phi(\mathrm{hor}(X)) = d\phi(X).$$

$\square$

We denote this by $D_X\phi := D\phi(X)$. This defines a differential operator but on the total space P. We obtain the covariant derivative $\nabla_X\sigma$ by introducing a local section $\varphi_i : U_i \subset M \to P$ of the principal bundle $\pi : P \to M$ over an open subset U of M. Then we can pull back $D_X\phi$ to U_i via φ_i:

$$\begin{aligned}
\phi : P \to F \qquad &\longrightarrow \qquad \phi^*\phi \ = \ \phi \circ \varphi := s, \\
\omega \in \Omega^1(P) \otimes T_eG \qquad &\longrightarrow \qquad \varphi^*\omega := \mathcal{A}_i \ \in \ \Omega^1(M) \otimes T_eG, \\
D\phi \in \Omega^1(P) \otimes F \qquad &\longrightarrow \qquad \varphi^*(D\phi) \ \in \ \Omega^1(M) \otimes F.
\end{aligned}$$

Then we find:

$$\begin{aligned}
\nabla_T s := (\varphi^* D\phi)(T) &:= \varphi^*\big(d\phi + \omega \triangleright \phi\big)(T) \\
&= \varphi^*(d\phi)(T) \ + \ \varphi^*(\omega \triangleright \phi)(T) \\
&= d(\varphi^*\phi)(T) \ + \ (\varphi^*\omega)(T) \triangleright (\varphi^*\phi) \\
&= ds(T) \ + \ \mathcal{A}_i(T) \triangleright s,
\end{aligned}$$

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Selbständigkeitserklärung

Ich versichere hiermit, die vorliegende Arbeit mit dem Titel **From Differentiable Manifolds to Gauge Theories** selbständig verfasst zu haben und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben.

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Jonas von Stein