From Differentiable Manifolds to Gauge Fields

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submitted by

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Von differenzierbaren Mannigfaltigkeiten zu Eichfeldern

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Manifolds

Manifolds are the fundamental spaces used in physics. They provide a framework to describe topological spaces that locally resemble Euclidean spaces, allowing for the application of known methods from calculus and linear algebra.

2.1 Preliminaries from topology

Although this thesis will not focus on introducing topology, a few important results will be given here, which are necessary to understand the definition of manifolds. The following definitions and theorems are taken from [Nak05]

A **topological space** is a set X equipped with a collection of open sets $\mathcal{T} = \{U_i \mid i \in I\}$ such that:

- $\emptyset, X \in \mathcal{T}$
- For any subcollection J of I the Union of corresponding open sets is itself an open set $\bigcup_{j \in J} U_j \in \mathcal{T}$
- For any finite subcollection K of I the intersection of the corresponding open sets is open: $\cap_{k \in K} U_k \in \mathcal{T}$

A family $\{O_i\}$ of (open) subsets of X is called an (open) covering of X if $X = \bigcup_i O_i$.

A subset N is called a **neighborhood** of a point $p \in X$ if there exists at least one open set $U \in \mathcal{T}$ such that $p \in U \subset N$. A topological space is called **Hausdorff** if for any two distinct points $p, q \in X$ there exist neighborhoods N_p, N_q such that $N_p \cap N_q = \emptyset$.

A map $f : X \to Y$ between two topological spaces is called **continuous** if for every open set $V \subset Y$ the preimage $f^{-1}(V)$ is an open set in X. If the inverse $f^{-1} : Y \to X$ is also continuous, then f is called a **homeomorphism**. Two topological spaces are called **homeomorphic** if there exists a homeomorphism between them.

2.2 Differentiable Manifolds

A Hausdorff topological space $(M, \mathcal{T})$ is called a **d-dimensional manifold** if there exists an open covering $\{U_i\}$ and a family of homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^d$. The pair (U_i, φ_i) is called a **chart** and the family $\{(U_i, \varphi_i)\}$ is called an **atlas**[Fre15f].

M is a **differentiable or smooth manifold** if for any U_i and U_j given that $U_i \cap U_j \neq \emptyset$ the transition function $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_i) \to \varphi_i(U_j \cap U_i)$ is infinite differentiable (C^∞)[Nak05]. In this thesis, smoothness will always be assumed, unless stated otherwise.

Let M and N be two differentiable manifolds of dimension m and n equipped with atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ respectively. A map $f : M \to N$ is called a **differentiable map** at a point $p \in M$ if for $p \in U_i$ and $f(p) \in V_j$ the composition $\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$ is infinite differentiable. If f is also a homeomorphism and the inverse $f^{-1} : N \to M$ is differentiable, then f is called a **diffeomorphism**. M and N are called **diffeomorphic** if there exists a diffeomorphism between them. This will be denoted as $M \cong N$.

2.3 Spacetime Manifold M

A trivial but for obvious reasons important example of a differentiable manifold is the spacetime manifold M used in physics, which is defined as follows:

Let $M := \mathbb{R}^4$, the set of ordered 4-tuples $(x^\mu) \in \mathbb{R}^4$. The so called **standard topology** is defined by the open balls around a point $p \in M$ with radius $r > 0$:

$$B_r(p) := \{x \in \mathbb{R}^4 \mid \|x - p\| < r\}$$

with $\|\cdot\|$ the Euclidean norm:

$$\|x\|^2 = \sum_{\mu=0}^{3} (x^\mu)^2$$

This is obviously a Hausdorff[1], and locally Euclidean topological space. The identity map $\phi(p) = p$ covers M globally. Hence, $(\mathcal{M}, B_r(p), \varphi)$ is a smooth manifold.

[1] For two distinct points $p, q \in M$ it is always sufficient to choose $r = \frac{1}{2}\|q - p\|$

Chapter 3

Bundles

The definition of a vector on a Manifold is non-trivial because a vector space structure might not exist globally on the manifold. We can still equip a Manifold with a Vector space structure locally. Thus, tangent spaces are introduced pointwise. Combining these local structures will lead naturally to the definition of fiber bundles.

3.1 Tangent Space T_pM

Let M be an n-dimensional smooth manifold. A tangent vector at a point $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule[Fre15c]:

$$v[fg] = v[f]g(p) + f(p)v[g]$$

A tangent vector at a point $p \in M$ can be constructed as the directional derivatives of an equivalence class of curves through p[Nak05].

Let $\gamma : [-\epsilon, \epsilon] \to M$ be a smooth curve in M with $\gamma(0) = p$. Then $x^\mu(\gamma(t)) \in \mathbb{R}^n$ is called the coordinate representation of γ induces by a chart (U, φ).

Let $f \in C^\infty(M)$ be a smooth function on M. The directional derivative of f along the curve γ at $t = 0$ is given by:

$$\begin{aligned} \frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=0} &= \frac{d}{dt}\left(f \circ \varphi^{-1} \circ \varphi \circ \gamma(t)\right)\Big|_{t=0} \\ &= \frac{\partial f}{\partial x^r}\frac{dx^r}{dt}\Big|_{t=0} \\ &= \frac{dx^r}{dt}\Big|_{t=0}\frac{\partial f}{\partial x^r}\Big|_p \end{aligned}$$

The definition of a tangent vector is now obtained by introducing an equivalence relation on curves. Two curves γ_1 and γ_2 are called equivalent at $\gamma_1(0) = \gamma_2(0) = p$ if their derivatives at $t = 0$ are equal:

$$\frac{dx_1^r}{dt}\Big|_{t=0} = \frac{dx_2^r}{dt}\Big|_{t=0} = v^r$$

A tangent vector is identified with the differential operator given the equivalence class of curves. Once a chart (U, φ) is chosen, with local coordinates $(x^1, \ldots, x^n)$, a tangent vector is represented as a linear combination of partial derivatives with real coefficients.

$$v = v^r \frac{\partial}{\partial x^r}\Big|_p$$

The tangent space at a point $p \in M$ is then defined as the set of all tangent vectors at p and is denoted by T_pM.

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \quad \text{form a basis of } T_pM$$

3.2 The Tangent Bundle as a Fiber Bundle

To introduce the concept of fiber bundles, a detailed examination of a specific example, the tangent bundle, serves as an effective foundation. The tangent bundle of a smooth n-dimensional manifold M is constructed by taking the disjoint union of all tangent spaces T_pM. This construction can be represented as:

$$TM := \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM$$

Here, T_pM denotes the tangent space at a point $p \in M$. The tangent space T_pM is a vector space that consists of all tangent vectors at the point p. It should be noted that this notation emphasizes the pointwise nature of the tangent bundle, where each tangent space is associated with a specific point on the manifold. Since each tangent space is a different object, a certain tangent vector in a tangent space can only belong to that specific tangent space. Therefore all the information is already in the vector itself and the direct product notation $\{p\} \times T_pM$ is only used to clarify[Fre15f].

Each element of the tangent bundle TM can be expressed as a pair (p, v), where p is a point on the manifold M, and v is a tangent vector belonging to the tangent space T_pM at that point.

A natural projection map is defined as follows:

$$\pi : TM \to M, \quad (p, v) \mapsto p$$

This projection, π, serves to "forget" the tangent vector v associated with each point, effectively collapsing all the tangent vectors at p to the single point p in the base manifold[Nak05].

The fiber over a point p is denoted as $\pi^{-1}(p) = T_pM$. This represents all the tangent vectors at the point p. Since T_pM is isomorphic to $\mathbb{R}^n$ as a vector space, it is referred to as the model fiber, denoted $F = \mathbb{R}^n$[Nak05].

Consider a coordinate chart (U, φ) on M. The chart φ provides a mapping from an open set $U \subseteq M$ to an open subset of $\mathbb{R}^n$. A diffeomorphism on the preimage $\pi^{-1}(U) \subset TM$ can then be defined as:

$$\Psi : \pi^{-1}(U) \to \mathbb{R}^{2n}, \quad (p, v) \mapsto \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n \right)$$

where the coordinates $x^i(p)$ represent the local coordinates of the point p in U, and $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$ describes the tangent vector in terms of its components in the chosen coordinate system[Nak05].

This establishes a local trivialization of the tangent bundle, expressed as:

$$TM|_U \cong U \times \mathbb{R}^n$$

In summary, the construction of the tangent bundle yields a fiber bundle characterized by the following essential components:

- **Total space:** $TM = \bigsqcup_{p \in M} T_pM$, encapsulating all tangent spaces.
- **Base space:** M, a manifold to which additional structure is added.
- **Projection:** $\pi : TM \to M$, mapping each tangent vector to its associated point on the manifold.
- **Model fiber:** $F = \mathbb{R}^n$, serving as the standard fiber structure over each point.
- **Local trivialization:** $TM|_U \cong U \times \mathbb{R}^n$, ensuring that the tangent bundle locally resembles a product structure.

Similar to the way a manifold is commonly perceived as a space that locally resembles $\mathbb{R}^n$, a fiber bundle may be conceptualized as a space that locally resembles the Cartesian product of the base space with a typical fiber structure.

Since the fiber of the tangent bundle is a vector space, the tangent bundle is also referred to as a **vector bundle**.

3.3 Definition of a Fiber Bundle

The formal definition of a fiber bundle reads as follows[Fre15f]. A fiber bundle is a quadruple (E, B, π, F) where:

- E is the total space
- B is the base space
- $\pi : E \to B$ is a surjective map called the projection
- F is the typical fiber

There exists an open cover $\{U_\alpha\}$ of B such that for each α, there is a diffeomorphism

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}$$

where $\mathrm{proj}_1 : A \times B \to A$ is the first projection.

$(U_\alpha, \varphi_\alpha)$ are called *local trivialization.*

The fiber over a point $b \in B$ is:

$$F_b := \pi^{-1}(\{b\}) \cong F$$

Often the notations $E \xrightarrow{\pi} B$ or $\pi : E \to B$ are used to denote a fiber bundle.

3.3.1 The Structure Group of a Fiber Bundle

Above a fiber bundle was defined as a quadruple (E, B, π, F) equipped with local trivializations. These local trivializations are established as diffeomorphisms on an open cover $\{U_\alpha\}$ of the base space B. The definition does not impose the requirement that $U_\alpha \cap U_\beta = \emptyset$. For a point $p \in U_\alpha \cap U_\beta$, multiple local trivializations $\varphi_\alpha(p, f) = \varphi_{\alpha,p}(f)$ and $\varphi_\beta(p, f) = \varphi_{\beta,p}(f)$ may be present, defined on U_α and U_β, respectively.

The **structure group** G of a fiber bundle is defined as the Lie group of diffeomorphisms relating these local trivializations. The corresponding transition function is given by[Nak05]:

$$t_{\alpha\beta}(p) \equiv \varphi_{\alpha,p}^{-1} \circ \varphi_{\beta,p} : F \to F$$

This establishes a smooth map $t_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ that satisfies the following properties:

$$\begin{aligned} t_{\alpha\alpha}(p) &= \mathrm{id}_F && \forall\, p \in U_\alpha \\ t_{\alpha\beta}(p) &= t_{\beta\alpha}(p)^{-1} && \forall\, p \in U_\alpha \cap U_\beta \\ t_{\alpha\beta}(p) \circ t_{\beta\gamma}(p) &= t_{\alpha\gamma}(p) && \forall\, p \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

In the case of the tangent bundle, the structure group corresponds to the general linear group $\mathrm{GL}(n, \mathbb{R})$, which consists of all invertible $n \times n$ matrices. A fiber bundle with transition maps identical to the identity map is termed a **trivial bundle**. In this scenario, the total space E is diffeomorphic to the product space $B \times F$.

Generally, a fiber bundle does not possess a unique trivialization. Let $\{\varphi_\alpha\}$ and $\{\tilde{\varphi}_\alpha\}$ denote two local trivializations over the same open covering that describe the same fiber bundle. These trivializations are related by maps $g_\alpha(p) : F \to F \quad \forall\, p \in B$, where each $g_\alpha(p)$ is a homeomorphism within the structure group G. It will be defined in section 4.1 what it means for a Lie group to act on a manifold. The transition function between the two local trivializations is then given by:

$$g_\alpha(p) \equiv \varphi_{\alpha,p}^{-1} \circ \tilde{\varphi}_{\alpha,p}$$

Considering the tangent bundle as an illustrative example[Nak05], let U_i and U_j represent overlapping charts with $p \in U_i \cap U_j$. Utilizing the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ and $\left\{ \frac{\partial}{\partial y^j}\Big|_p \right\}$, a vector $v \in T_pM$ can be expressed in both bases as:

$$v = v^\mu \frac{\partial}{\partial x^\mu} = \tilde{v}^\mu \frac{\partial}{\partial y^\mu}$$

The transition function $t^\nu{}_\mu$ is thus defined as:

$$\tilde{v}^\nu = \frac{\partial y^\nu}{\partial x^\mu}\bigg|_p v^\mu = t^\nu{}_\mu v^\mu$$

3.3.2 Sections

An important definition in the context of fiber bundles is that of a section or cross-section. This concept enables the selection of an element from each fiber over each point in a continuous manner, facilitating the introduction of ideas such as vector fields over spacetime[Nak05].

A **section** of a fiber bundle $\pi : E \to B$ is defined as a continuous map $s : B \to E$ such that

$$\pi \circ s = \mathrm{id}_B.$$

This condition ensures that exactly one point is chosen from each fiber continuously. The set of all (smooth) sections is denoted by:

$$\Gamma(E) := \{ s : M \to E \mid \pi \circ s = \mathrm{id}_M \}.$$

It is also possible to define a section locally on an open set $U \subset B$ as a map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$. In this case, the section is called a **local section**.

3.4 The cotangent bundle and differential forms

3.4.1 The Cotangent Bundle

In this section the fundamental concepts of the cotangent bundle are introduced, which is essential for the definition of differential forms and the exterior derivative. The cotangent space at a point p on a manifold M is defined as the dual space of the tangent space at that point. Formally, the cotangent space at a point $p \in M$ is the set of all linear maps from the tangent space at that point to the real numbers[Nak05].

$$T_p^*M := \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$$

A covector $\omega \in T_p^*M$ is such a linear function:

$$\omega : T_pM \to \mathbb{R}$$

As an example, consider a function $f \in C^\infty(M)$ and some tangent vector $v \in T_pM$. Then $v[f] \in \mathbb{R}$ by definition. The differential of f at a point p is a covector df_p and therefore $df_p[v] \in \mathbb{R}$ is simply defined as $df_p[v] = v[f]$. Given a coordinate basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ of T_pM the dual basis is:

$$\left\{ dx^i\Big|_p \right\}$$

satisfying the relation:

$$dx^i \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta^i_j$$

The example from before can thus be expressed as:

$$df_p = \frac{\partial f}{\partial x^i} dx^i \quad \text{and} \quad df_p[v] = v^i \left. \frac{\partial f}{\partial x^i} \right|_p$$

Analogous to the tangent bundle, the cotangent bundle is defined as the disjoint union of all cotangent spaces at each point in the manifold:

$$T^*M := \bigsqcup_{p \in M} T_p^*M$$

This structure constitutes a vector bundle over M. A section of the cotangent bundle can be defined as:

$$\omega \in \Gamma(T^*M)$$

This section assigns to each $p \in M$ a covector $\omega_p \in T_p^*M$ smoothly. Such a section is referred to as a **1-form**. In a coordinate representation, a 1-form can be expressed as:

$$\omega = \sum_{i=1}^{n} \omega_i(x)\, dx^i \quad \text{with} \quad \omega_i \in C^\infty(M)$$

3.4.2 Tensor Fields and the Metric Tensor

Utilizing the fact that the fibers of the cotangent bundle are vector spaces, tensor products of bundles can be defined. For instance:

$$T^*M \otimes T^*M := \bigsqcup_{p \in M} T_p^*M \otimes T_p^*M$$

This forms a bundle whose fibers consist of maps form $TM \otimes TM$ to $\mathbb{R}$. Sections of this bundle are referred to as $(0,2)$-tensor fields. A prominent example of such fields is the metric tensor. The Minkowski metric is defined as:

$$\eta \in \Gamma(T^*M \otimes T^*M)$$

In local coordinates, the Minkowski metric can be expressed as:

$$\eta = \eta_{\mu\nu}\, dx^\mu \otimes dx^\nu \quad \text{with} \quad \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

In general, a tensor field of type (i, j) is a section of the bundle[Nak05]:

$$\otimes^i T^*M \otimes^j TM$$

3.4.3 Differential Forms and the Exterior Derivative

A **k-form** or **differential form** of order k is a totally antisymmetric $(k, 0)$ tensor. To define a k-form, it is necessary to take the **wedge product** of 1-forms, which is defined by taking the totally antisymmetrized tensor product of 1-forms. This means that all permutations of the tensor product are considered, with even permutations contributing positively and odd permutations contributing negatively. Consider the cotangent space T_p^*M of a manifold M at a point p with basis $\{dx^\mu\}$. The wedge product of two 1-forms dx^μ and dx^ν thus reads:

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

Higher order forms can be constructed analogously. By taking the cotangent bundle T^*M and considering the antisymmetrized tensor product, we can define fields of differential forms of order r as:

$$\Omega^r(M) \equiv \Gamma(\wedge^r(T^*M))$$

The **exterior derivative** is then defined as a map [Nak05]:

$$\begin{aligned} d : \Omega^r(B) &\to \Omega^{r+1}(B) \\ \omega \mapsto d\omega &= \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1 \dots \mu_r}\right) dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \end{aligned}$$

In the following, two relations will be shown that will be used in later proofs[Nak05]. First consider the action of the exterior derivative on a 1-form $\omega = \omega_\mu dx^\mu \in \Omega^1(M)$ on two tangent vector fields $v = v^\mu \frac{\partial}{\partial x^\mu}$ and $w = w^\nu \frac{\partial}{\partial x^\nu}$:

$$\begin{aligned} d\omega(v, w) &= \left(\frac{\partial \omega_\mu}{\partial x^\nu}\, dx^\nu \wedge dx^\mu\right)(v, w) \\ &= \frac{\partial \omega_\mu}{\partial x^\nu}\left(dx^\nu(v)\, dx^\mu(w) - dx^\nu(w)\, dx^\mu(v)\right) \\ &= \frac{\partial \omega_\mu}{\partial x^\nu}\left(v^\nu w^\mu - w^\nu v^\mu\right) \\ &= v^\nu \frac{\partial}{\partial x^\nu}\left(\omega_\mu w^\mu\right) - w^\nu \frac{\partial}{\partial x^\nu}\left(\omega_\mu v^\mu\right) - \omega_\mu\left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu}\right) \\ &= v[\omega(w)] - w[\omega(v)] - \omega([v, w]) \end{aligned}$$

This gives a coordinate free expression for the action of the exterior derivative of a 1-form.Furthermore, it can easily be shown that the exterior derivative of a exterior derivative is zero, by using the fact that the product of a symmetric and an antisymmetric tensor is zero. By definition the following is obtained:

$$d^2\omega = \frac{1}{r!}\left(\frac{\partial^2}{\partial x^\alpha x^\beta}\omega_{\mu_1\dots\mu_r}\right) dx^\alpha \wedge dx^\beta \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

From this it follows instantly that $d^2 = 0$ since $\frac{\partial^2}{\partial x^\alpha x^\beta}$ is symmetric and the wedge product is antisymmetric.

Chapter 4

Principal Bundles

A principal bundle is a fiber bundle $P \xrightarrow{\pi} M$ is a bundle whose fiber is identical to the structure group. This framework is of particular importance, because it allows to understand fibre bundles with fibre F on which G acts. These bundles are called *associated bundles* and are essential to understand Gauge theories in physics.

4.1 Action of a Lie Group on a Manifold

To understand this properly we first need to understand how a Lie group G can act on a manifold M [Fre15e] Let $(G, \cdot)$ be a Lie group, and M a smooth manifold. Then a smooth map

$$\triangleright : G \times M \longrightarrow M$$

satisfying

$$\begin{aligned} e \triangleright p &= p && \forall p \in M, \text{ and } e \text{ being the identity in } G \\ g_2 \triangleright (g_1 \triangleright p) &= (g_2 \cdot g_1) \triangleright p && \forall g_1, g_2 \in G,\, p \in M \end{aligned}$$

is called a **left G-action** on M.

Analogous a **right G-action** $\triangleleft : M \times G \longrightarrow M$ is defined, satisfying:

$$\begin{aligned} p \triangleleft e &= p && \forall p \in M, \text{ and } e \text{ being the identity in } G \\ (p \triangleleft g_1) \triangleleft g_2 &= p \triangleleft (g_1 g_2) && \forall g_1, g_2 \in G,\, p \in M \end{aligned}$$

Given a left action $\triangleright$, we can construct a right action:

$$\begin{aligned} \triangleleft :& M \times G \longrightarrow M, \\ & p \triangleleft g := g^{-1} \triangleright p. \end{aligned}$$

It is trivial to show that this yields a right action.

Let G be a Lie group acting smoothly on a manifold M from the left via

$$\triangleright : G \times M \longrightarrow M.$$

We define an equivalence relation $\sim$ on M by:

$$p \sim \tilde{p} \quad :\Longleftrightarrow \quad \exists g \in G \text{ such that } \tilde{p} = g \triangleright p.$$

This defines an equivalence relation:

- **Reflexivity:** $e \triangleright p = p$, so $p \sim p$.
- **Symmetry:** If $\tilde{p} = g \triangleright p$, then $p = g^{-1} \triangleright \tilde{p}$, so $\tilde{p} \sim p$.
- **Transitivity:** If $\tilde{p} = g_1 \triangleright p$ and $\hat{p} = g_2 \triangleright \tilde{p}$, then
$$\hat{p} = g_2 \triangleright (g_1 \triangleright p) = (g_2 g_1) \triangleright p,$$
so $p \sim \hat{p}$.

The **orbit** of a point $p \in M$ under the group action is then the equivalence class:

$$\mathcal{O}_p := \{\tilde{p} \in M \mid \exists g \in G : \tilde{p} = g \triangleright p\}.$$

We can then define the **quotient space** $M/\sim$, often denoted M/G, by identifyinf points that are in the same orbit.

We define the **stabilizer** of a point $p \in M$ as the set of elements in G that leave p unchanged:

$$S_p \coloneqq \{g \in G \mid g \triangleright p = p\}$$

An action $\triangleright$ is called free if for all $p \in M$ the stabilizer is trivial $S_p = \{e\}$.

4.2 Principal Bundles

We now define a principal fibre bundle as follows [Fre15e]

Let (P, π, M, F) be a fibre bundle. If the following conditions are satisfied, we call it a **principal G-bundle**:

(i) P is equipped with a right G-action $\triangleleft$,
(ii) The action of G is free,
(iii) $\pi : P \to M$ is isomorphic as a bundle to the quotient $\rho : P \to P/G$,

where $\rho(p) \mapsto [p]$ denotes the canonical projection onto the orbit space P/G.

To clarify, two bundles $\pi : E \to M$ and $\pi' : E' \to M'$ are isomorphic if there exist diffeomorphisms $\bar{f} : E \to E'$ and $f : M \to M'$ such that $\pi' \circ \bar{f} = f \circ \pi$.

Since $\triangleleft$ acts freely, each fibre $\rho^{-1}([p])$ is diffeomorphic to G.

The Frame Bundle As an example we consider the frame bundle of a smooth manifold M.

We define a **frame** at a point $p \in M$ with $\dim M = d$ as an ordered basis of the tangent space T_pM and the set of all frames at p as:

$$L_pM \coloneqq \{(e_1, \ldots, e_d) \,|\, \{e_1, \ldots, e_d\} \text{ is a basis of } T_pM\}$$

There exists a natural isomorphism

$$L_pM \cong \mathrm{GL}(d, \mathbb{R}),$$

by identifying each frame $(e_1, \ldots, e_d) \in L_pM$ with the matrix whose columns are the components of the vectors e_i. Concretely, each frame is mapped to a matrix $g \in \mathrm{GL}(d, \mathbb{R})$ such that

$$g^{\mu}{}_{\alpha} = e^{\mu}{}_{\alpha}$$

Analogous to the tangent bundle, we can define the **frame bundle**:

$$LM := \bigsqcup_{p \in M} L_p M$$

Given a chart U_i on M, we can define a local trivialization of the frame bundle [Nak05]. A frame $\epsilon = \{e_1, \dots, e_d\}$ at $p \in M$ is expressed in terms of the natural basis of the tangent space T_pM $\{\partial/\partial x^\mu \mid_p\}$

$$e_\alpha = e^\mu{}_\alpha \, \partial/\partial x^\mu \mid_p \quad \text{where } e^\mu{}_\alpha \in \mathrm{GL}(d, \mathbb{R})$$

The local trivialization is then given by $\varphi_i^{-1}(u) = (p, (e^\mu{}_\alpha))$.

The projection π_L of a frame $\epsilon = \{e_1, \cdot, e_d\}$ at a point $p \in M$ is given by:

$$\begin{aligned} \pi_L \, : \, LM &\longrightarrow M, \\ \epsilon &\mapsto \pi_L(\epsilon) = p \end{aligned}$$

Therefore we have introduced the necessary structure, suck that $(LM, \pi, M, \mathrm{GL}(d, \mathrm{R})$ defines a fiber bundle.

The right action of $\mathrm{GL}(d, \mathbb{R})$ on the frame is defined as we would expect, since it is analogguss to the change of basis in a vector space:

$$\begin{aligned} \triangleleft : LM \times \mathrm{GL}(d, \mathbb{R}) &\longrightarrow LM, \\ (\epsilon, g) &\mapsto \epsilon \triangleleft g = (e_1, \cdots, e_d) \triangleleft g = (e_i g^i{}_1, \cdots, e_i g^i{}_d), \end{aligned}$$

To show that this Bundle equipped with this right action is a principal bundle, we need to check, if the action is free and the bundle is isomorphic to the quotient bundle.

Therefore we need to show that $e_i g^i{}_\alpha = e_\alpha \implies g^i{}_\alpha = \delta^i{}_\alpha \, \forall \alpha$. Since $\{e_1, \dots, e_d\}$ is a basis, they are linearly independent. Therefore the definition already implies that $g^i{}_\alpha = \delta^i{}_\alpha$ and thus the action is free.

To show the second condition, we need to show that the orbit space $LM/\mathrm{GL}(d, \mathbb{R})$ is consists of a single point for each $\epsilon \in \pi^{-1}(p)$. This is indeed the case, since the orbit of a frame at $p \in M$ under the action of $\mathrm{GL}(d, \mathbb{R})$ consists of all frames at p, since the action is transitiv. Thus the quotient space is diffeomorphic to M, with the diffeomorphism being: $f : M \longrightarrow LM/\mathrm{GL}(d, \mathbb{R})$, $p \mapsto [\epsilon]$ where ϵ is a frame at p.

Chapter 5

Connections on Principal Bundles

5.1 General Definition

Let $P \xrightarrow{\pi} M$ be a principal bundle with structure group G. The right action of G on P induces a vector field as follows: For each $A \in \mathfrak{g} \cong T_eG$ the action of the one parameter subgroup $\exp(tA)$ on $p \in P$ yields a curve. Since the group acts within the fiber $\pi(p) = \pi(p \triangleleft \exp(tA)) = p$. We define a vector $X_p^A \in T_pP$ by its action on a function $f \in C^\infty(P)$ [Nak05]:

$$X_p^A f = \frac{d}{dt} f(p \triangleleft \exp(tA)) \mid_{t=0}$$

Futhermore a vector space isomorphism is defined $i : \mathfrak{g} \longrightarrow \Gamma(TP)$ that assigns to each element $A \in \mathfrak{g}$ the vector field X^A. This is called a **fundamental vector field** on P.

We define the **Pushforward** [Pus25] of a smooth map $F : M \longrightarrow N$ of smooth manifolds M and N as the a map between the tangent spaces:

$$F_* : T_pM \longrightarrow T_{F(p)}N$$

By identifying $(F_*v)(f) = v(f \circ F)$ for $v \in T_pM$ and $f \in C^\infty(N)$.

The pushforward of the projection map $\pi_* : TP \longrightarrow TM$ allows the construction of the **vertical subspace** $V_pP \coloneqq \ker(\pi_*)$ at a point $p \in P$, which is a vector subspace of the tangent space of P.

Notice that that each fundamental vector $X_p^A \in V_pP$, since by construction $\pi_*(X_p^A) = 0$.

A **connection** on a principal bundle $P \xrightarrow{\pi} M$ is a separation of the tangent space T_pP into a vertical subspace V_pP and a **horizontal subspace** H_pP, by choosing a complement to the vertical subspace at each point $p \in P$ such that:

(i) $T_pP = H_pP \oplus V_pP$

(ii) $(\triangleleft g)_*(H_pP) = H_{p \triangleleft g}P \quad$ for all $g \in G$

(iii) For every smooth vector field $X \in \Gamma(TP),$ the unique decomposition $X = X^H + X^V$ with $X^H(p) \in H_pP$ and $X^V(p) \in V_pP$ yields smooth vector fields $X^H \in \Gamma(HP),\ X^V \in \Gamma(VP)$.

The condition (ii) ensures that when moving along the fibers by the action of G the horizontal subspace changes in a smooth way, while (iii) guarantees that moving along P the horizontal subspace changes smoothly.

5.2 Connection one-form

The choice of a horizontal subspace at each point $p \in P$ can be achieved by defining a Lie algebra valued one-form. The horizontal subspace is then interpreted as the kernel of this one-form. We define the **connection one-form** $\omega \in \mathfrak{g} \otimes T^*P$ as a $\mathfrak{g}$-valued one-form on P such that:

$$\begin{aligned} &\text{(i)} \quad \omega(X^A) = A \quad \text{for all } A \in \mathfrak{g} \\ &\text{(ii)} \quad (\triangleleft g)^*\omega = \mathrm{Ad}_{g^{-1}*}\omega \quad \text{for all } g \in G \end{aligned}$$

Here, $(\triangleleft g)^*\omega$ denotes the pullback [1] of the connection one-form by the right action of $g \in G$ on P, and $\mathrm{Ad}_{g^{-1}}$ is the adjoint action of g^{-1} on the Lie Group G. That is $\omega_{p\triangleleft g}(X_p(\triangleleft g)_*) = g^{-1} \cdot \omega_p(X_p) \cdot g$

It will be stated without proof that any connection one-form satisfying these properties induces a horizontal subspace H_pP that satisfies the conditions of a connection [Fre15a].

5.3 local connection form

The connection one-form as defined above is a global object on the principal bundle P. However, in practice it is often useful to work with local connection forms. As will be shown in the next chapter, this local connection form is identified with the gauge potential in physical gauge theories.

Consider an open covering $\{U_i\}$ of the base manifold M and local sections $\sigma_i : U_i \to \pi^{-1}(U_i)$. We define a Lie-algebra valued one-form

$$\mathcal{A}_i \equiv \sigma_i^*\omega \in \mathfrak{g} \otimes \Omega^1(U_i)$$

for a global connection one-form ω [Nak05]. Such a local connection form is called a **Yang-Mills field** [Fre15d]

Given a local section $\sigma_i : U_i \to P$, one obtains a local trivialization:

$$\begin{aligned} \psi_i : U_i \times G &\longrightarrow \pi^{-1}(U_i) \subset P \\ (p, g) &\mapsto \sigma_i(p) \triangleleft g \end{aligned}$$

This trivialization introduces a local representation of the global connection one-form ω via its pullback:

$$\begin{aligned} \psi_i^*\omega : T_{(p,g)}(U_i \times G) &\longrightarrow \mathfrak{g} \\ (\psi_i^*\omega)_{(p,g)}(X) &= \omega_{\sigma_i(p)\triangleleft g}\left((\psi_i)_* X\right) \end{aligned}$$

This local representation is related to the Yang-Mills field $\mathcal{A}_i$ by[Fre15d]:

$$(\psi_i^*\omega)_{(p,g)}(X) = \mathrm{Ad}_{g^{-1}*}(\mathcal{A}_i(X)) + \Xi_g(X)$$

[1] The **pullback** of a of a one-form $\omega \in \Gamma(T^*N)$ by a smooth map $\phi : M \to N$ between smoth manifolds is defined as $\phi^*\omega_p(X) = \omega_{\phi(p)}(\phi_{p*}(X))$ for $X \in T_pM$.[Pul24]

The above-used Ξ is the **Maurer–Cartan form** of the Lie group G. This form takes a tangent vector $v \in T_gG$ and maps it to the unique Lie algebra element (i.e., a tangent vector at the identity) that generates v via left translation:

$$\Xi(v) = (g^{-1}\triangleright)_* v.$$

This uses the fact that every tangent vector on G arises as the pushforward of a unique element of the Lie algebra $\mathfrak{g} = T_eG$ under the left action, i.e., for every $v \in T_gG$, there exists a unique $X \in \mathfrak{g}$ such that

$$v = (g\triangleright)_* X.$$

Thus, Ξ identifies the tangent bundle TG with $G \times \mathfrak{g}$ via left translation[Mau25].

5.4 Connection on the Frame Bundle

The Frame Bundle LM is of particular interest, because many groups relevant in physics are subgroups of the general linear group $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. Therefore in the following a local connection form and the Maurer-Cartan form will be derived.

Any choice of a chart (U_i, x) on the base manifold M induces a section on the frame bundle LM by associating to each point $m \in U_i$ the frame given by its coordinates. This section is denoted as:

$$\begin{aligned} \sigma : U_i &\longrightarrow LM \\ m &\mapsto \sigma_i(m) := \left(\left.\frac{\partial}{\partial x^1}\right|_m, \dots, \left.\frac{\partial}{\partial x^{\dim M}}\right|_m \right) \end{aligned}$$

Then the Yang-Mills field $\mathcal{A}_i = \sigma_i^* \omega$ is a one-form on U_i with values in the Lie algebra $\mathfrak{gl}(\dim M, \mathbb{R}) = \{M \mid M \text{ is a } n \times n \text{ matrix with components } M^\alpha_\beta \in \mathbb{R}\}$. The Yang-Mills field can be expressed in its components as:

$$(\mathcal{A}^i)^\alpha{}_{\beta\mu}$$

Where α, β are labels for the Lie algebra components and μ is the index of the base manifold.

The Maurer-Cartan form Ξ can be constructed as follows:

Let $gl \subseteq GL(d, \mathbb{R})$ be an open subset of the general linear group containing the identity. Coordinates are introduced by:

$$\begin{aligned} x : gl &\longrightarrow \mathbb{R} \\ g &\mapsto x(g)^a{}_b := g^a{}_b \end{aligned}$$

Consider a left-invariant vector field L^A generated by the Lie algebra element $A \in \mathfrak{gl}(d, \mathbb{R})$. Since it is a vector field on the group, it acts on the coordinate functions:

$$\begin{aligned} \left(L^A x^a{}_b\right)_g &= x^a{}_b \frac{d}{dt} \left(g \cdot \exp(tA)\right) \Big|_{t=0} \\ &= \frac{d}{dt} \left(g^a{}_c \exp(tA)^c{}_b\right) \Big|_{t=0} \\ &= g^a{}_c A^c{}_b \end{aligned}$$

Therefore the components of the vector field are given by $L_g^A = g^a{}_b\, A^b{}_c\, \frac{\partial}{\partial x^a{}_c}$[Fre15d]

The Maurer-Cartan form Ξ then is defined as the one-form that maps the left-invariant vector field L^A to the Lie algebra element A:

$$(\Xi_g)^a_{\ b} = (g^{-1})^a_{\ c}(dx^c{}_b)$$

It can be easily checked that this expression satisfies the properties of a Maurer-Cartan form:

$$\begin{aligned}
\Xi_g(L_g^A) &= (g^{-1})^a_{\ c}\,(dx)^c_{\ b}\left(g^p_{\ r}\,A^r_{\ q}\,\frac{\partial}{\partial x^p_{\ q}}\right)\\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\left((dx)^c_{\ b}\,\frac{\partial}{\partial x^p_{\ q}}\right)\\
&= (g^{-1})^a_{\ c}\,g^p_{\ r}\,A^r_{\ q}\,\delta^c_p\,\delta^q_b\\
&= (g^{-1})^a_{\ p}\,g^p_{\ r}\,A^r_{\ b}\\
&= A^a_{\ b}
\end{aligned}$$

5.5 Compatibility condition for local connection forms

It was stated before, that the local connection forms $\mathcal{A}_i$ relate to a unique global connection one-form ω. For this to be true, the local connection forms must satisfy a compatibility condition. This condition is given by the requirement that the local connection forms on overlapping charts $U_i \cap U_j \neq \emptyset$ are related by a gauge transformation[Nak05]. Specifically, let σ_i and σ_j be sections respectively defining Yang-Mills fields $\mathcal{A}_i$ and $\mathcal{A}_j$ on the overlapping region $U_i \cap U_j$. Introduce a gauge map

$$\Omega : U_i \cap U_j \longrightarrow G$$

defined by the relation

$$\sigma_j(m) = \sigma_i(m) \triangleleft \Omega(m) \quad \forall m \in U_i \cap U_j$$

Then the local connection forms are related as follows:

$$\mathcal{A}_j = \mathrm{Ad}_{\Omega^{-1}(m)*}\mathcal{A}_i + \Omega^*\Xi_m$$

In this will be shown for the case of the frame bundle LM. First, we calculate the latter expression. Notice that $\Omega^*\Xi_m$ is a map from the tangent space of the intersection on the base manifold $U_i \cap U_j$ to the Lie algebra $\mathfrak{gl}(d, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Therefore, to find the explicit form, it is calculated how this map acts on a vector in the tangent space:

$$\begin{aligned}
(\Omega^*\Xi)_p\left(\frac{\partial}{\partial x^\mu}\right)_p &= \Xi_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)\\
&= (\Omega^{-1}(p))^i_{\ k}(dx^k_{\ j})_{\Omega(p)}\left(\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\right)\\
&= \Omega^{-1}(p)^i_{\ k}\left(\Omega_*\left(\frac{\partial}{\partial x^\mu}\right)_p\right)_{\Omega(p)}\left(x^k_{\ j}\right)\\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\left(x^k_{\ j}\circ\Omega\right)_p\\
&= \Omega^{-1}(p)^i_{\ k}\left(\frac{\partial}{\partial x^\mu}\right)_p\Omega(p)^k_{\ j}
\end{aligned}$$

Therefore, the components of the pullback of the Maurer–Cartan form are given by[Fre15d]:

$$((\Omega^*\Xi)_p)^i_{\ j} = \Omega^{-1}(p)^i_{\ k} \left(\frac{\partial}{\partial x^\mu}\right)_p \Omega(p)^k_{\ j}\, dx^\mu := \mathbf{\Omega}^{-1} d\mathbf{\Omega}$$

Futhermore, the pushforward of the adjoint action on the Yang-Mills field is easily obtained by definition of the adjoint action:

$$\begin{aligned} \mathrm{Ad}_g &: G \longrightarrow G\,, \quad h \mapsto ghg^{-1} \\ \mathrm{Ad}_{g*} &: T_eG \longrightarrow T_eG\,, \quad A \mapsto \mathbf{gAg}^{-1} \end{aligned}$$

Here the notation $\mathbf{g}$ is used to denote the matrix product, since the adjoint action is defined on the group G not the Lie algebra $\mathfrak{g}$.

Altogether the transition between two Yang-Mills fields on the intersection of two charts is given by:

$$\begin{aligned} \mathcal{A}_j &= \mathbf{\Omega}^{-1}\, \mathcal{A}_i\, \mathbf{\Omega} + \mathbf{\Omega}^{-1}\, d\mathbf{\Omega} \\ (\mathcal{A}_j)^i_{\ r\mu} &= \left(\Omega^{-1}(p)\right)^i_{\ k} (\mathcal{A}_i)^k_{\ l\mu}\, \Omega(p)^l_{\ r} + \left(\Omega^{-1}(p)\right)^i_{\ k} \partial_\mu \Omega(p)^k_{\ r} \end{aligned}$$

This is simply the **gauge transformation** as known form gauge theories[Nak05].

As an example, consider the case of a $U(1)$ principal bundle. The transition function Ω is a smooth function $U_i \cap U_j \longrightarrow U(1)$, which can be expressed as $\Omega(m) = \exp[i\Lambda(m)]$ for some real-valued function $\Lambda : U_i \cap U_j \longrightarrow \mathbb{R}$. Since $U(1)$ is a subgroup of $GL(d, \mathbb{C})$, two local connection forms $\mathcal{A}_i$ and $\mathcal{A}_j$ on the intersection $U_i \cap U_j$ are then related by:

$$\begin{aligned} \mathcal{A}_j &= \Omega^{-1}\, \mathcal{A}_i\, \Omega + \Omega^{-1}\, d\Omega \\ &= \mathcal{A}_i + e^{-i\Lambda(m)}\, d\left(e^{i\Lambda(m)}\right) \\ &= \mathcal{A}_i + e^{-i\Lambda(m)} \cdot ie^{i\Lambda(m)}\, d\Lambda \\ &= \mathcal{A}_i + i\, d\Lambda \end{aligned}$$

Which in components reads:

$$\mathcal{A}_{j\,\mu} = \mathcal{A}_{i\,\mu} + i\partial_\mu \Lambda$$

This is the familiar form of the gauge transformation in electromagnetism[Nak05].

Chapter 6

Curvature and Field Strength

6.1 horizontal lift and parallel transport

Consider a principal G bundle wiht total space P, base manifold M and structure group G. Let $\gamma : [0,1] \to M$ be a curve in M. A curve $\gamma^{\uparrow} : [0,1] \to P$ is called a **horizontal lift** of γ if it satisfies the following conditions:

$$\begin{aligned} (i) &\quad \pi \circ \gamma^{\uparrow} = \gamma \\ (ii) &\quad X^V(\gamma^{\uparrow}(t)) = 0 \quad \forall t \in [0,1] \\ (iii) &\quad \pi_*(X_{\gamma^{\uparrow}(t)}) = X_{\gamma(t)} \quad \forall t \in [0,1] \end{aligned}$$

where $X^V \in V_{\gamma^{\uparrow}(t)}P$ is the vertical vector field.

6.2 Curvature

Let P be a principal G-bundle with a connection one form ω and let $\phi \in \Omega^k(P) \otimes V$ be a V valued k-form on P, where V is some k-dimensional vector space with basis $\{e_i\}$. The connection one form ω allows for the separation of the tangent space of P into horizontal and vertical components. Then the map:

$$\begin{aligned} &D\phi : \Gamma(T_u^{k+1}P) \to V, \\ &(X_1, \dots X_{k+1}) \mapsto D\phi(X_1, \dots X_k) \coloneqq d\phi(X_1^H, \dots X_{k+1}^H) \end{aligned}$$

is called the **covariant derivative** of ϕ. Here $d\phi \equiv d\phi^i \otimes e_i$ is the exterior derivative.

This introduces the **curvature two-form** Ω as the covariant derivative of the connection one-form ω:

$$\Omega \equiv D\omega \in \Omega^2(P) \otimes \mathfrak{g}$$

First it will be shown, that Ω takes the following form:

$$\Omega = d\omega + \omega \wedge_{\mathfrak{g}} \omega$$

Where $\wedge_{\mathfrak{g}}$ denotes the wedge product in the Lie algebra $\mathfrak{g}$ of G defined by its action on $\Gamma(T^2P)$: $(\omega \wedge_{\mathfrak{g}} \omega)(X,Y) \coloneqq [\omega(X), \omega(Y)]_{\mathfrak{g}}$

Note that if G is a matrix group, then the above can be written in terms of its components as:

$${\Omega^i}_j = d{\omega^i}_j + {\omega^i}_k \wedge {\omega^k}_j$$

We proof this by considering three separate cases[Fre15b]:

a) $X, Y \in \Gamma(TP)$ are vertical vector fields

$\Rightarrow$ $\exists A, B \in T_eG : X = X^A, \quad Y = X^B$

Left-hand side:

$$\Omega(X^A, X^B) = D\omega(X^A, X^B) = d\omega\left((X^A)^H, (X^B)^H\right)$$
$$= d\omega(0,0) = 0$$

Right-hand side:

$$d\omega(X^A, X^B) + (\omega \wedge_{\mathfrak{g}} \omega)(X^A, X^B)$$
$$= X^A(\omega(X^B)) - X^B(\omega(X^A)) - \omega([X^A, X^B]) + \left[\omega(X^A), \omega(X^B)\right]_{\mathfrak{g}}$$
$$= X^A(B) - X^B(A) - \omega(X^{[A,B]_{\mathfrak{g}}}) + [A, B]_{\mathfrak{g}}$$
$$= 0 - 0 - [A, B]_{\mathfrak{g}} + [A, B]_{\mathfrak{g}}$$
$$= 0$$

b) $X, Y \in \Gamma(TP)$ are horizontal vector fields

Left-hand side:

$$\Omega(X, Y) = D\omega(X, Y) = d\omega(X^H, Y^H)$$
$$= d\omega(X, Y)$$

Right-hand side:

$$d\omega(X, Y) + (\omega \wedge_{\mathfrak{g}} \omega)(X, Y)$$
$$= d\omega(X^H, Y^H) + \left[\omega(X), \omega(Y)\right]_{\mathfrak{g}}$$
$$= d\omega(X, Y) + \left[0, 0\right]_{\mathfrak{g}}$$
$$= d\omega(X, Y)$$

c) $X \in \Gamma(TP)$ is horizontal and $Y = X^A \in \Gamma(TP)$ is vertical

Left-hand side:

$$\Omega(X, X^A) = D\omega(X, X^A) = d\omega(X^H, (X^A)^H)$$
$$= d\omega(X, 0)$$
$$= 0$$

Right-hand side:

$$d\omega(X, X^A) + (\omega \wedge_{\mathfrak{g}} \omega)(X, X^A)$$
$$= d\omega(X, X^A) + \left[\omega(X), \omega(X^A)\right]_{\mathfrak{g}}$$
$$= X(\omega(X^A)) - X^A(\omega(X)) - \omega([X, X^A]) + \left[\omega(X), \omega(X^A)\right]_{\mathfrak{g}}$$
$$= X(A) - X^A(0) - \omega([X, X^A]) + \left[0, A\right]_{\mathfrak{g}}$$
$$= 0$$

Where in the last step the fact that the comutator of a horizontal and a vertical vector field is again a horizontal vector field was used[Nak05].

6.3 Local from of the curvature and Yang-Mills field strength

As the connection one-form ω can be expressed locally as the pullback by a section $\mathcal{A}_i = \sigma^*\omega$, the local from of the curvature two-form Ω is defined analogous[Nak05]:

$$\mathscr{F} \equiv \sigma^*\Omega \in \Omega^2(M) \otimes \mathfrak{g}$$

In terms of the local connection one-form $\mathcal{A}$, the curvature two-form can be expressed as:

$$\begin{aligned}\mathscr{F} &= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\ &= \sigma^*(d\omega) + \sigma^*(\omega \wedge_{\mathfrak{g}} \omega) \\ &= \sigma^*(d\omega) + \sigma^*(\omega) \wedge_{\mathfrak{g}} \sigma^*(\omega) \\ &= d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_j\end{aligned}$$

Let x^μ be the coordinates on the open set U_i where the section σ is defined. Then the Yang-Mills field is given by $\mathcal{A} = \mathcal{A}_\mu dx^\mu$. We therefore get the following expression:

$$\begin{aligned}\mathscr{F} &= d(\mathcal{A}_\mu dx^\mu) + (\mathcal{A}_\mu dx^\mu \wedge_{\mathfrak{g}} \mathcal{A}_\nu dx^\nu) \\ &= \frac{1}{2}\left(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]_{\mathfrak{g}}\right) dx^\mu \wedge dx^\nu \\ &:= \frac{1}{2} \mathscr{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu\end{aligned}$$

In physics, the local curvature two-form $\mathscr{F}$ is identified with the **Yang-Mills field strength**.

The co

First, compute the exterior derivative:

$$\begin{aligned}&d\left(\mathbf{\Omega}^{-1}\mathcal{A}_i\mathbf{\Omega} + \mathbf{\Omega}^{-1}d\mathbf{\Omega}\right) \\ &= -\mathbf{\Omega}^{-1}d\mathbf{\Omega} \wedge_{\mathfrak{g}} \mathbf{\Omega}^{-1}\mathcal{A}_i\mathbf{\Omega} + \mathbf{\Omega}^{-1}d\mathcal{A}_i\,\mathbf{\Omega} \\ &\quad - \mathbf{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\mathbf{\Omega} - \mathbf{\Omega}^{-1}d\mathbf{\Omega}\cdot\mathbf{\Omega}^{-1} \wedge_{\mathfrak{g}} d\mathbf{\Omega}\end{aligned}$$

Then, compute the wedge product:

$$\begin{aligned}&\left(\mathbf{\Omega}^{-1}\mathcal{A}_i\mathbf{\Omega} + \mathbf{\Omega}^{-1}d\mathbf{\Omega}\right) \wedge_{\mathfrak{g}} \left(\mathbf{\Omega}^{-1}\mathcal{A}_i\mathbf{\Omega} + \mathbf{\Omega}^{-1}d\mathbf{\Omega}\right) \\ &= \mathbf{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i\,\mathbf{\Omega} + \mathbf{\Omega}^{-1}\mathcal{A}_i \wedge_{\mathfrak{g}} d\mathbf{\Omega} \\ &\quad + \mathbf{\Omega}^{-1}d\mathbf{\Omega} \wedge_{\mathfrak{g}} \mathcal{A}_i\,\mathbf{\Omega} + \mathbf{\Omega}^{-1}d\mathbf{\Omega} \wedge_{\mathfrak{g}} d\mathbf{\Omega}\end{aligned}$$

Combining both contributions, we obtain:

$$\begin{aligned}\mathscr{F}_j &= \mathbf{\Omega}^{-1}\left(d\mathcal{A}_i + \mathcal{A}_i \wedge_{\mathfrak{g}} \mathcal{A}_i\right)\mathbf{\Omega} \\ &= \mathbf{\Omega}^{-1}\,\mathscr{F}_i\,\mathbf{\Omega}\end{aligned}$$

6.4 The Bianchi identity

The Bianchi identity states that the covariant derivative of the curvature two-form vanishes. To show this, the exterior derivative of the curvature two-form is computed:

$$d\Omega = d(d\omega) + d(\omega \wedge_{\mathfrak{g}} \omega) = d\omega \wedge_{\mathfrak{g}} \omega - \omega \wedge_{\mathfrak{g}} d\omega$$

Since for any $X \in H_pP$ the connection one-form vanishes, the following holds:

$$D\Omega(X,Y,Z) = d\omega(X^H, Y^H, Z^H) = 0$$

Therefore, the **Bianchi identity** is $D\Omega = 0$

Localy the Bianchi identity is given by:

$$\begin{aligned}
\sigma^* d\Omega &= d(\sigma^*\Omega) = d\mathscr{F} \\
&= \sigma^*(d\omega + \omega \wedge_{\mathfrak{g}} \omega) \\
&= d\sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega + \sigma^*\omega \wedge_{\mathfrak{g}} \sigma^*\omega \\
&= d\mathcal{A} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} d\mathcal{A} \\
&= \mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}
\end{aligned}$$

Thus the Bianchi identity in local coordinates is given by:

$$D\mathscr{F} = d\mathscr{F} - (\mathscr{F} \wedge_{\mathfrak{g}} \mathcal{A} - \mathcal{A} \wedge_{\mathfrak{g}} \mathscr{F}) = d\mathscr{F} + [\mathcal{A}, \mathscr{F}]_{\mathfrak{g}} = 0$$

Chapter 7

Gauge Theories

In physical gauge theories like electromagnetism, Yang-Mills theories or general relativity, the laws of nature they describe are not just differential equations that happen to describe nature, but they are deeply connected to the geometry of the underlying symmetries. In the following, the above developed mathematical framework is applied to recover Maxwell's equations, Yang-Mills theories.

7.1 Maxwell theory

Consider a $U(1)$ principal bundle P over the four dimensional Minkowski spacteime manifold M equipped with the Minkowski metric $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$. The principal bundle is trivial $P = M \times U(1)$, and the projection map is given by $\pi : P \to M$, $\pi(x, e^{i\Lambda}) = x$. The Yang-Mills field is given by:

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu \in \Omega^1(M) \otimes \mathfrak{u}(1)$$

And the field strength is given by the curvature two-form:

$$\mathscr{F} = d\mathcal{A}$$

We identify the **gauge potential** A by $\mathcal{A}_\mu = iA_\mu$ and the field strength tensor F by $\mathscr{F}_{\mu\nu} = iF_{\mu\nu}$, where i is the factor associated with the Lie algebra. Therefore, the curvature two-form can be written in components as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Bianchi identity is given by:

$$\begin{aligned} D\mathscr{F} &= d\mathscr{F} \\ &= \frac{1}{2} \partial_\mu \mathscr{F}_{\nu\rho} \, dx^\mu \wedge dx^\nu \wedge dx^\rho = 0 \\ \Rightarrow \quad & \partial_\mu \mathscr{F}_{\nu\rho} + \partial_\nu \mathscr{F}_{\rho\mu} + \partial_\rho \mathscr{F}_{\mu\nu} = 0 \end{aligned}$$

When identifying the electric and magnetic fields with the components of the field strength tensor, we have:

$$\begin{aligned} E_i &= F_{0i} \\ B_i &= \frac{1}{2} \epsilon_{ijk} F_{jk} \end{aligned}$$

The Bianchi identity yields the **homogeneous Maxwell equations**:

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

First, choosing indices $\mu = 0$, $\nu = i$, $\rho = j$ and using antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, we obtain:

$$\begin{aligned} \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} &= 0 \\ \Rightarrow \quad (\nabla \times \mathbf{E})_k + \partial_t B_k &= 0 \end{aligned}$$

Now, choose $\mu = i$, $\nu = j$, $\rho = k$, all spatial indices. Contracting with the Levi-Civita tensor ϵ^{ijk} gives:

$$\begin{aligned} \epsilon^{ijk} \partial_i F_{jk} &= 0 \\ \Rightarrow \quad \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

Together, these two identities form the homogeneous Maxwell equations:

$$\begin{aligned} \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

7.2 Yang-Mills theory

Chapter 8

Conclusion

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