A General Equilibrium Model for Inventories

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May, 2024

Abstract

The papers establishes a framework for a multi-country, multi-sector production network allowing firms' inventory holdings for the intermediate goods and final productions. The inventory decision is fully endogenized through a two-step optimization problem, uncovering a unique propagation mechanism for global shocks, distinct from the international trade channel. The analysis demonstrates that under certain constraints, the proposed model exhibits an isomorphic structure to other multi-sector models featuring input-output linkages. The model exhibits better predictions for real-world impacts of foreign and productivity shocks.

1 Introduction

The globalization of production networks has raised concerns that shocks and risks can propagate through additional mechanisms beyond just international trade linkages. Recent theoretical research on global and domestic production networks has further demonstrated that these networks amplify the propagation of shocks such as Acemoglu et al. (2013), Baqaee and Farhi (2019) and Antràs and De Gortari (2020).

From the perspective of management and supply chain analysis, inventory holdings often serve as a buffer mitigating various risks confronted by firms, including supply chain disruptions and economic policy uncertainty. Kim (2020) found that inventory can mitigate price risk and act as a buffer in response to unexpectedly high demand. While the supply chain disruptions can increase the costs of holding inventory, they can also provide a compensating effect Clampit et al. (2021). Similarly, economic policy uncertainty can lead to a reduction in inventory holdings, particularly among non-state-owned enterprises Zeng et al. (2020).

To summarize, empirical evidence suggests the existence of counteracting effects on the quantity of inventory holdings in response to shocks. Moreover, it lacks a comprehensive theoretical framework that fully endogenizes inventory holdings for both inputs and outputs. Hence, I establish a theoretical framework for a multi-country, multi-sector production networks allowing inventory holdings for both the intermediate inputs & final goods. I try to use the dynamic general equilibrium model to answer two main questions:

1. how do inventory holdings of inputs and outputs influence a firm's micro level production decisions?
2. Through what mechanism do *global shocks* propagate and impact firm inventory holdings?

To notice, since I fully endogenize firm's decision of inventory holdings, the micro level optimization problems need to be reshaped. I assume a roundabout production network similar to Caliendo and Parro (2015). This roundabout structure captures the intricate interdependencies and input-output linkages that characterize modern production networks. I use the Cobb-Douglas technology to describe the firm's production with capital, labor and intermediate goods. I gain insights from Alessandria et al. (2010) and Khan and Thomas (2007), that there remains certain constraints for inventory holdings to rationalize the model. Additionally, I decompose the overall optimization problem faced by firms into multiple steps, following the approach outlined in Costinot et al. (2020). This approach facilitates the adoption of a corporate finance perspective to measure the welfare disparities between a decentralized economy and government-controlled centralized economy.[1]

The paper is organized as follows. Section 2 illustrates the original multi-country, multi-sector general equilibrium models allowing firms' inventory holdings with numerical solutions. To address this dimensionality issue, subsection 2.7 employs a "two-step optimization approach" as a transformation technique to reformulate and solve these problems more tractably. Section 4 discusses the optimal taxation policy under different levels of government implementation capacity. It examines scenarios where the government can implement micro-level, firm-specific taxation [2] versus scenarios where only overall uniform taxation is feasible. Section 5 illustrates the forthcoming steps and unresolved aspects regarding the model's progression.

2 The setups

The theoretical basis of my model traces back to the famous international real business model, Backus et al. (1992). I extend the model and allow the firm to hold inventories for intermediate inputs & outputs. I assume the flow of purchase to be time-intensive and takes one period to complete which follows Christiano (1988). The key deviation for the firm's problem is *the separation of production and cost*. The production process incurs contemporaneous labor and capital costs, but the intermediate input costs are not accounted for in the current period's optimization problem, as the purchase decisions for these inputs were made in prior periods.[3]

2.1 Environments

Consider a $\mathcal{M}$-country world denoted $i, m \in \mathcal{M}$. subscript i denotes the origin country while m for the destination country. [4] In every country, a $J \times J$ input-output table is endowed, featuring identical industries. Industries denote $j, n \in \{1, \dots, J\} = \mathcal{N}$. For example: flows are subscripted with direction $mn \leftarrow ij$. Sales from industry j in country i to industry n in country m in time t denotes

$$S_{mn,ij,t}.$$

[1]My plan is to review macroeconomic literature for capital accumulation such as Jin (2012)

[2]In the most extreme case, the government issue flow specific tax on firms and consumers. See Costinot et al. (2020), section 5.2 for the similar discussion

[3]This is quite traditional in accounting and finance, but I did not see any economic theory paper with such assumption since it makes the micro-level cost minimization problem messy.

[4]In a special case of small-open-economy, define country $i, m \in H, F$ for domestic and foreign frameworks. H as the small open economy (SOE) compared to F, the rest of the world.

The model follows the stochastic process with finite states η_t and history of previous states $\eta^t \equiv \{\eta_0, \eta_1, \ldots \eta_t\}$. The price of arrow securities is defined as

$$Q_t \equiv Q\left(\eta^t\right) = Q\left(\eta^t \mid \eta^{t-1}\right) Q\left(\eta^{t-1} \mid \eta^{t-2}\right) \ldots Q\left(\eta^1 \mid \eta^0\right)$$

2.2 Production and inventories on stock

On-stock inventory: define the stock of inventories $I_{ij,mn,t}$ as firm ij's (in country i industry j) on-stock for intermediate goods from industry mn. Define the dynamism of inventories for inputs as

$$I_{ij,mn,t} = (1-\delta) I_{ij,mn,t-1} + B_{ij,mn,t-1} - N_{ij,mn,t} \tag{1}$$

where $O_{ij,mn,t-1}$ is last period's ordering and $N_{ij,mn,t}$ as the current usage for production. Assume the production procedure is completed within each period. It takes 1 period for the shipment with iceberg cost $\tau_{mi} \geq 1$. Assume the iceberg cost is homogeneous across industry and country specific.

Define the vector of intermediate good usage as

$$\mathbf{N_{ij,t}} = (N_{ij,11,t}, N_{ij,12,t}, \ldots, N_{ij,2s,t})$$

The productivity $A_{ij,t}$ follows an $AR(1) \sim \mathcal{N}(0, \sigma_{ij})$ process with shock $\epsilon_{ij,t}$ as

$$A_{i,j,t} = \rho_{i,j} A_{i,j,t-1} + \epsilon_{i,j,t}.$$

The production function for industry ij in time t is

$$M_{ij,t} = A_{ij,t} K_{ij,t}^{\alpha} L_{ij,t}^{\eta} \left(\prod_{m \in \mathcal{M}} \prod_{n \in \mathcal{N}} (N_{ij,mn,t})^{a_{ij,mn}} \right)^{1-\alpha-\eta}$$

$$\sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} a_{ij,mn} = 1$$

where the production function is Cobb-Douglas with factors $(K, L, \mathbf{N})$ and $a_{ij,mn}$ is the intermediate share from mn to ij defined in the IO matrix.

Define the stock of output $M_{ij,t}$ as follows with the similar methodology:

$$O_{ij,t} = (1-\delta) O_{ij,t-1} + M_{ij,t} - S_{ij,t} \tag{2}$$

where $S_{ij,t} = \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} S_{mn,ij,t}$ is the t period's sale for industry ij with other industries. Think of the micro level flow as the shipment or one-time purchase between firms/industries).

After the realization of $A_{ij,t}$, with any bundle of FOP $(K^*, L^*\mathbf{N}^*)$, the output M^* is unique. To summarize, any firm ij has the on stock output $O_{ij,t}$ and $\mathcal{M} \times \mathcal{N}$ input accounts $I_{ij,mn,t}$. Notice that the dynamism of output $M_{ij,t}$ and sales $S_{ij,t}$ is separated.

2.3 Cost structure & profits

There remains two main type of costs for any firm: the on-stock cost and in-flow cost.

- Iceberg cost: any firm has to order τ_{mi} unit of intermediate goods for the realization of one unit of $S_{i*,m*,t}$.
- On-stock cost: firm ij pays the inventory specific variable cost in every period:

$$c_{ij}S_{ij} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} c_{mn}I_{ij,mn,t}.$$

- Purchase costs: firm ij pays the market-determined intermediate goods costs. It costs $\tau_{mi}p_{mn}S_{ij,mn,t}$ for $S_{ij,mn,t}$ unit of realization.
- Labor costs $W_{ij,t}$.

[5] So far I did not mention how the price and wages are determined. Assume there is no price discrimination and define the price sequence for each good as $p_{ij,t}$. With given level of desired output $M_{ij,t}$, there always exists an bundle $(L_{ij,t}, K_{ij,t}, \mathbf{N_{ij,t}})$ that solves the overall optimization problem. The net profit for each period is therefore:

$$\begin{aligned}
\pi_{ij,t} = p_{ij,t} &\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} S_{mn,ij,t} \\
&- \left(K_{ij,t}R_{i,t} + L_{ij,t}W_{ij,t} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} p_{mn,t}B_{ij,mn,t}\right) \\
&- \left(c_{ij}O_{ij,t} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} c_{mn}I_{ij,mn,t}\right).
\end{aligned}$$

[5] My initial idea is: define the cost minimization problem for any level of $M_{ij,t}$. We always find the optimal & unique bundle of inputs (K, L, N). However, Since the intermediate goods purchase S_t is ahead of time, we need to define the cost of such usage in a economic-theory approach. One way to measure the price of intermediate goods usage is to use "weighted average price method" for the cost minimization problem. However, the "cmp" is hard to define properly, where the usage of intermediate goods reveals externalities through three channels: 1. on-stock cost channel; 2. stock sufficiency for future usage; 3. compensated for other FOPs. I will discuss it later.

2.4 Overall profit maximization

Denote the overall profit maximization by adding up each period's net profits. while the adding-up procedure is not hard, it is important to consider the constraint for each periods.

In time t, the usage of inputs should be sufficient in the inventory-account, simply

$$N_{ij,mn,t} \leq (1-\delta)I_{ij,mn,t-1} + B_{ij,mn,t-1} \text{ for } m,n \in \mathcal{M},\mathcal{N}.$$

Consider the output sailing constraint as

$$\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} S_{mn,ij,t} \leq O_{ij,t}.$$

Then write down the overall profit maximization problem as the system of equations below denoted as Q_1:

$$\max_{\mathbf{K_{ij}},\mathbf{L_{ij}},\mathbf{S_{mn,ij}},\mathbf{B_{ij,mn}},\mathbf{N_{ij,mn}}} (\mathbb{E}_u \sum_{t=0}^{\infty} Q_t \pi_{ij,t}) \tag{3}$$

$$N_{ij,mn,t} \leq (1-\delta)I_{ij,mn,t-1} + B_{ij,mn,t-1} \text{ for } m,n \in \mathcal{M},\mathcal{N} \tag{4}$$

$$\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} S_{mn,ij,t} \leq (1-\delta)O_{ij,t-1} + M_{ij,t} \tag{5}$$

$$M_{ij,t} = A_{ij,t}K_{ij,t}^{\alpha}L_{ij,t}^{\eta}\left(\prod_{m\in\mathcal{M}}\prod_{n\in\mathcal{N}}(N_{ij,mn,t})^{a_{ij,mn}}\right)^{1-\alpha-\eta} \tag{6}$$

$$\begin{aligned}\pi_{ij,t} = p_{ij,t}\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} S_{mn,ij,t} &- \left(K_{ij,t}R_{i,t} + L_{ij,t}W_{ij,t} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} p_{mn,t}B_{ij,mn,t}\right) \\ &- \left(c_{ij}O_{ij,t} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} c_{mn}I_{ij,mn,t}\right)\end{aligned} \tag{7}$$

$$\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}} a_{ij,mn} = 1 \tag{8}$$

where $\mathbf{K_{ij}} = \{K_{ij,1}, K_{ij,2}, \ldots\}, \mathbf{L_{ij}} = \{L_{ij,1}, L_{ij,2}, \ldots\}, \mathbf{S_{mn,ij}} = \{S_{mn,ij,1}, S_{mn,ij,2}, \ldots\}, \mathbf{S_{ij,mn}} = \{S_{ij,mn,1}, S_{ij,mn,2}, \ldots\}, \mathbf{N_{ij,mn}} = \{N_{ij,mn,1}, N_{ij,mn,2}, \ldots\}$ denote the sequence of any variable.

Explanations I summarize the profit maximization problems below

1. Firms maximize the discounted profit in every period t.

2. Equations 1 and 2 defines the dynamism of inventories for inputs and outputs

between period $t-1$ and t.

3. The usage of inputs in every period cannot exceed the on stock inputs inventories.
4. The total amount of selling in every period cannot exceed the on stock output inventories.

2.5 Consumers's problem

The consumer's overall utility in country i is

$$Q_t U\left(C_{i,t}, L_{i,t}\right).$$

The final consumption $C_{i,t}$ is formed with CES aggregation by purchase flows from all sectors in all countries

$$C_{i,t} = \left[\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}}\left(\tau_{i,mn,t}^{\frac{1}{\gamma}} C_{i,mn,t}^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

where the variable $\tau_{i,mn,t}$ denotes the taste shifter specific to the origin-destination-industry nexus. I incorporate all sorts of barriers including geographic distances, iceberg costs, home-country bias, quality differentiation, etc. [6]

She forms the budget constraint by receiving wages, capital rent and dividends from producers. The budget also combines the purchase-selling of arrow securities [7]:

$$P_{i,t}C_{i,t} + \sum_{\eta^{t+1}} Q\left(\eta^{t+1} \mid \eta^t\right) B_i\left(\eta^{t+1}\right) = W_{i,t}L_{i,t} + R_{i,t}K_{i,t} + \Pi_{i,t} + B_i(\eta^t)$$

The optimal micro-level $C_{i,mn,t}$ and overall aggregation $C_{i,t}$ can be solved with the utility maximization problem as Q_c below:

$$\max_{C_{i,t},L_{i,t},B_{i,t}} \sum_{t=0}^{\infty} Q_t U\left(C_{i,t}, L_{i,t}\right), \tag{9}$$

[6] I did not include policy related barriers/simulators into the shifter for two reasons:

- In the calibration sector, policy parameters are easy to attain and incorporate into the model. Simple *reduced form* discussion of τ fails to explain the initial mechanism.
- If the government in a benevolent social planner point of view implement "flow specific" tax for the consumption behavior and utilize the endogenous taxation spending to improve the purchasing flow network, there will be too many policy variables and solvability fails to exist.

[7] For simplicity, the price facing consumers and producers are the same.

$$P_{i,t}C_{i,t} + \sum_{\eta^{t+1}} Q\left(\eta^{t+1} \mid \eta^t\right) B_i\left(\eta^{t+1}\right) = W_{i,t}L_{i,t} + R_{i,t}K_{i,t} + \Pi_{i,t} + B_i(\eta^t). \tag{10}$$

2.6 Equilibrium conditions

Definition 1. *The decentralized competitive equilibrium in period t is constructed by the following conditions:*

1. *Utility maximization problem for the consumer. Choose $C_{i,mn,t}, L_{i,t}, B_{i,t}$ solves Q_c for each period t;*
2. *Choose $\mathbf{K_{ij}}, \mathbf{L_{ij}}, \mathbf{S_{mn,ij}}, \mathbf{B_{ij,mn}}, \mathbf{N_{ij,mn}}$ to solve overall profit maximization problem;*
3. *Goods market clears for each industry in each country mn:*

$$\sum_{i \in \mathcal{M}} S_{i,mn,t-1} = \sum_{i \in \mathcal{M}} \left(C_{i,mn,t} + \sum_{j \in \mathcal{N}} B_{ij,mn,t} \right);$$

4. *Labor market clears for each country i: $L_{i,t} = \sum_j L_{ij,t}$;*
5. *Capital market clears: $K_{i,t} = \sum_j K_{ij,t}$.*

Obtaining analytical solutions to the above system of equilibrium equations is intractable. Therefore, my next move involves employing numerical methods, specifically utilizing the Dynare to approximately find the solutions.

2.7 Two-step optimization

The two-step optimization approach adopted to solve Q_1 follows the methodology outlined in Costinot et al. (2020), with a model-specific enhancement. To summarize,

1. Given any level of $M_{ij,t}$, with information of $O_{ij,t}, I_{ij,mn,t}$, firm forms the cost minimization problem that solves the optimal $(K_{ij,t}, L_{ij,t}, \mathbf{N_{ij,t}})$.
2. The firm determines the optimal selling-purchase strategy $(S_{mn,ij,t}, S_{ij,mn,t})$ that solves the overall problem with $M_{ij,t}$ wisely picked.

Definition 2. *The weighted average price (internal shadow price) $p^w_{ij,mn,t}$ for on-stock account $I_{ij,mn}$ is defined as :* [8]

$$p^w_{ij,mn,t} = \frac{\sum_0^t (p_{mn,t} S_{ij,mn,t})}{\sum_0^t S_{ij,mn,t}}.$$

[8] One important feature to define the weighted price is to capture the cost with no bias. Think about the per unit cost of an input, if I use the current market price to define

The price is determined ahead of time t's production.

Then the cost minimization problem is: [9]

$$\min_{K,L,N} K_{ij,t}R_{i,t} + L_{ij,t}W_{ij,t} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}}(p^w_{ij,mn,t} - c_{mn})N_{ij,mn,t}$$

with respect to

$$M_{ij,t} = A_{ij,t}K^{\alpha}_{ij,t}L^{\eta}_{ij,t}\left(\prod_{m\in\mathcal{M}}\prod_{n\in\mathcal{N}}(N_{ij,mn,t})^{a_{ij,mn}}\right)^{1-\alpha-\eta}.$$

Notice that c_{mn} is given and $p^w_{ij,mn,t}$ is determined by firm's previous actions of purchase. Hence I assume both to be exogenous with respect to current period's optimization. Write down the Lagrange functions as

$$\begin{aligned}\mathcal{L} = {} & K_{ij,t}R_{i,t} + L_{ij,t}W_{ij,t} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}}(p^w ij, mn, t - cmn)N_{ij,mn,t} \\ & +\lambda\left(M_{ij,t} - A_{ij,t}K^{\alpha}_{ij,t}L^{\eta}_{ij,t}\left(\prod_{m\in\mathcal{M}}\prod_{n\in\mathcal{N}}(N_{ij,mn,t})^{a_{ij,mn}}\right)^{1-\alpha-\eta}\right),\end{aligned}$$

we get

$$\begin{aligned} R_{i,t} &= \lambda A_{ij,t}\alpha K^{\alpha-1}_{ij,t}L^{\eta}_{ij,t}\left(\prod_{m\in\mathcal{M}}\prod_{n\in\mathcal{N}}(N_{ij,mn,t})^{a_{ij,mn}}\right)^{1-\alpha-\eta} \\ W_{ij,t} &= \lambda A_{ij,t}\eta K^{\alpha}_{ij,t}L^{\eta-1}_{ij,t}\left(\prod_{m\in\mathcal{M}}\prod_{n\in\mathcal{N}}(N_{ij,mn,t})^{a_{ij,mn}}\right)^{1-\alpha-\eta} \\ \frac{p^w_{ij,mn,t} - c_{mn}}{a_{ij,mn}(N_{ij,mn,t})^{a_{ij,mn}-1}} &= \lambda A_{ij,t}(1-\alpha-\eta)K^{\alpha}_{ij,t}L^{\eta}_{ij,t}\left(\prod_{m\prime\in\mathcal{M}}\prod_{n\prime\in\mathcal{N}}(N_{ij,m\prime n\prime,t})^{a_{ij,mn}}\right)^{-\alpha-\eta}\end{aligned}$$

[9]The original target function is

$$\min_{K,L,} K_{ij,t}R_{i,t} + L_{ij,t}W_{ij,t} + \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}}p^w_{ij,mn,t}N_{ij,mn,t} - \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}}c_{mn}N_{ij,mn,t}.$$

Simply the "'wages, renting and intermediate goods cost". The last term $\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}}c_{mn}N_{ij,mn,t}$ denotes the changes of $I_{ij,mn,t}$ there production takes place.

We know that in the Cobb-Douglas case, we find the unique solutions for the input demand functions of K, L, N as these form

$$K_{ij,t}(M_{i,j,t}; R_{i,r}, W_{i,t}, p^w_{ij,mn,t}, c_{mn}),$$
$$L_{ij,t}(M_{i,j,t}; R_{i,r}, W_{i,t}, p^w_{ij,mn,t}, c_{mn}),$$
$$N_{ij,mn,t}(M_{i,j,t}; R_{i,r}, W_{i,t}, p^w_{ij,mn,t}, c_{mn}).$$

Firms with the information of

I will complete the two-step optimization problem as my next step and demonstrate that, under certain conditions, this approach yields equivalent solutions to directly solving the Q_1 [10].

2.8 Intuition for inventories mechanism

I highlight some intuitions for the inventories mechanism in this section. After I solve the model or the simplified version, I could check these guesses.

Guess 1. *The special case "infinite-holding-cost" is isomorphic to existing models with no inventory holding design. Specifically, define the result of the firm's decision with any extra condition as a tuple*

$$(S_{ij,mn,t}, B_{mn,ij,t}) = \mathcal{R}(model\_name, extra\_condition).$$

We guess the following:

$$\lim_{c_{ij},c_{mn}\to\infty} \mathcal{R}\left(Definition\ 1, (c_{ij}, c_{mn})\right) = \mathcal{R}(workhorse).$$

The guess 1 works as an analytical shortcut for solving the sophisticated model with inventories. It also illustrates an economy with no inventory as a benchmark scenario which is essential for implementing the welfare analysis. Think of a graph with x-axis as an overall increase of c_{ij}, c_{mn} such that no substitution effect takes place. We shall see the changes of flow of sales, current profits $\pi_{ij,t}$ and overall profits in the y-axis.

Guess 2. *There exists an unique measure of the weighted average price (shadow price 2) $p^w_{ij,mn,t}$ for on-stock account, such that the overall profits are maximized under this optimal pricing strategy.*

The guess 2 emphasizes the importance of pricing strategy for the firm's internal decision and inventory management. However, the pricing strategy's externality is not

[10] Please see guess 2 for a detailed illustration.

included for discussion. Moreover, the pricing strategy itself may have imperfections. If the problem is solved purely through numerical methods, the need for a "shadow price" may be eliminated.

To introduce the next guess 3, we first illustrate the circulation ratio $\phi_{ij,t}$ below:

Definition 3. *The circulation ratio is defined as the equilibrium quantity (flow-variable) divided by the total inventory (on-stock variable) in the world:*

$$\phi_{ij,t} = \frac{\sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} S_{ij,mn,t}}{O_{ij,t} + \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} S_{ij,mn,t} I_{mn,ij,t}}$$

Guess 3. *With non-zero value selected for $\{c_{ij}, c_{mn}, \delta\}$, the circulation ratio $\phi_{ij,t}$ serves as a crucial parameter for measuring welfare.*

If there are non-zero holding and depreciation costs associated with maintaining inventory for inputs and outputs, these costs can be viewed as *deadweight losses* to the economy.

3 Analytical solutions and coding

Consider a two-country world denoted by $i, m \in \{H, F\}$, where H represents the small open economy (SOE) and F represents the rest of the world (ROW). Each country has a 2×2 input-output table denoted by $j, n \in \{1, 2\}$. Sales from sector j in country i to sector n in country m at time t are denoted as $S_{mn,ij,t}$. The price of arrow securities is defined similarly as:

$$Q_t \equiv Q(\eta^t) = Q(\eta^t|\eta^{t-1})Q(\eta^{t-1}|\eta^{t-2}) \cdots Q(\eta^1|\eta^0)$$

Define the production function for sector j in country i at time t as:

$$Y_{ij,t} = A_{ij,t} \cdot F(K_{ij,t}, L_{ij,t}, \{M_{ij,mn,t}\}) \tag{11}$$

where $A_{ij,t}$ is the productivity, $K_{ij,t}$ is capital, $L_{ij,t}$ is labor, and $M_{ij,mn,t}$ represents intermediate inputs from sector n in country m.

The inventory accumulation equation for intermediate goods is:

$$I_{ij,mn,t+1} = (1 - \delta)I_{ij,mn,t} + B_{ij,mn,t} - N_{ij,mn,t} \tag{12}$$

where $I_{ij,mn,t}$ is the inventory stock, $B_{ij,mn,t}$ is new orders, $N_{ij,mn,t}$ is usage, and δ is the depreciation rate of inventories.

4 Optimal taxation

In this section I illustrate the optimal taxation under two main scenarios: uniform tariff and micro-level flow-specific taxation.

Setups for endogenous taxation: Assume the government issue consumption tax $t(*)$ and subsidy $s(*)$ for production. I did not specify whether the taxation/subsidy is industry specific with subscript $t_{ij,mn,t}$ or uniform for now. Taxation revenue denotes $R_i(t)$ and spending $S_i(s)$ with appropriate aggregation technology. The government's budget in country i is assumed to be cleared in every period:

$$T_{i,t} = R_{i,t} - Sub_{i,t}$$

where T_i is transferred to the domestic consumer's budget constraint:

$$P_{i,t}C_{i,t} + \sum_{\eta^{t+1}} Q\left(\eta^{t+1} \mid \eta^t\right) B_i\left(\eta^{t+1}\right) = W_{i,t}L_{i,t} + R_{i,t}K_{i,t} + \Pi_{i,t} + B_i(\eta^t) + T_{i,t}.$$

4.1 Uniform tariff without discrimination

Assume the government can only implement the non-discriminationary ad valorem tariff for foreign producers $t_{i,n,t}$. It denotes the customs in country i settle the tariff for industry n from any foreign country $m \neq i$. The revenue is then calculated as

$$R_{i,t} = \sum_{m \neq i} \sum_{n \in \mathcal{N}} t_{i,n,t}\left(B_{ij,mn,t-1} + C_{i,mn,t-1}\right).$$

The last period's purchase $B_{ij,mn,t-1}$ by assumption takes one period to reach the customs.

There is no subsidy in this case by government. Hence we directly settle $Sub_{i,t} = 0$ and utilize the overall revenue to increase the consumer's budget. We then solve the new version of Q_1 numerically.

4.2 Micro-level taxation

We then turn to the most extreme case, that the government issues micro-level, industry & country & quantity specific tax and subsidy. It denotes

$$t_{ij,mn,t}, s_{mn,ij,t}.$$

Revenue is calculated as

$$R_{i,t} = \sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \left(t_{ij,mn,t}B_{ij,mn,t-1} + t_{ij,c,t}C_{i,mn,t-1}\right)$$

and subsidy spending as

$$Sub_{i,t} = \sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \left(s_{mn,ij,t}S_{mn,ij,t-1}\right).$$

Then the transfer is

$$\sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \left(t_{ij,mn,t}B_{ij,mn,t-1} + t_{ij,c,t}C_{i,mn,t-1} - s_{mn,ij,t}S_{mn,ij,t-1}\right)$$

Then by selecting tax and subsidy for every flow, we solve the new version of Q_1 numerically.

4.3 More than tax

I summarize the following non-fiscal policies, mainly related to the command-and-control policy below.[11]

1. *Inventory upper bound for certain materials:* inventory of certain chemical raw materials must not exceed certain limits for each industry:

$$O_{ij,t} \leq \bar{O}, I_{in,ij} \leq \bar{I}.$$

2. *Tax competition framework:* government issue comparably low taxes to attract the foreign capital inflows by allowing cross boundary capital flows and arrow securities purchase. The workhorse model for international tax competition traces back to Wilson (1986) and Zodrow and Mieszkowski (1986). [12]

3. *Optimize logistics system and the traffic:* the government issue policies to decrease the transportation time. Change "one-period waiting time" in section 2 to be simultaneous.

[11] I treat this section as the brainstorm and did not make serious literature & intuition. Since my model has more general assumptions on inventories, I believe there may be some discussions on housing and stock.

[12] Also see Yifan Wang's Honours thesis "Tax Competition: Endogenous taxation and business environment". The model considers a model with a transitional corporation **T** who controls the capital inflow into each country, where the representative agent and the government respond by choosing labour supply and corresponding policies.

5 Planned section

Due to the time limit, I leave some questions unsolved. In this section I discuss my next move:

1. Find the analytical solution of a simplified version of Q_1, where the home country is small (small open economy case) with two industries'
2. data preparation and calibration;
3. Coding part for finding Q_1's numerical solution and simulations. It is likely to use the Dynare or python.

5.1 Small Open Economy with two industries

This section discusses a simplified model that yields the analytical solution to the overall *problem one*. We also construct a two-step optimization problem to solve for the firm's inventory choice.

- Drop $K_{ij,t}$ or make the dynamic change of capital "not-state-contingent";[13]
- assume $n = 2$ and $i = H, m = F$. The home country H is a small open economy (SOE);
- exogeneize U_t;
- firm ij always sell the output $M_{ij,t} \implies O_{ij,t} = 0$ for all t;

We then construct the simplified version of problem 1, denoted as Q_{1s}:

$$\mathbf{L_{ij}}, \mathbf{S_{mn,ij}}, \mathbf{B_{ij,mn}}, \mathbf{N_{ij,mn}} = \text{argmax}(\mathbb{E}_u \sum_{t=0}^{\infty} Q_t \pi_{ij,t}) \tag{13}$$

$$N_{ij,mn,t} \leq (1-\delta) I_{ij,mn,t-1} + B_{ij,mn,t-1} \text{ for } m, n \in \mathcal{M}, \mathcal{N} \tag{14}$$

$$M_{ij,t} = A_{ij,t} L_{ij,t}^{\eta} \left(\prod_{m \in \mathcal{M}} \prod_{n \in \mathcal{N}} (N_{ij,mn,t})^{a_{ij,mn}} \right)^{1-\eta} \tag{15}$$

$$\begin{aligned} \pi_{ij,t} = p_{ij,t} \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} S_{mn,ij,t} - \left(L_{ij,t} W_{ij,t} + \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} p_{mn,t} B_{ij,mn,t} \right) \\ - \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} c_{mn} I_{ij,mn,t} \end{aligned} \tag{16}$$

$$\sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} a_{ij,mn} = 1 \tag{17}$$

[13]See Jin (2012) for her design of capital accumulation problem.

5.2 Data preparation and calibrations

I summarize the potential data in preparation:

1. World Input-Output Table (WIOD) for production linkages data for the benchmark analysis;

2. Monthly International Trade Datasets (MITD) for US province and industry level (HS6) data when implementing the country-ROW analysis;

3. United States International Trade Commission (USITC) data for the monthly inventory and shipment (sales);

4. Other possible data source: Manufacturing Industry Database (NBER-CES).

I will start the calibration process after solving the two-country, two-industry simplified model in section 5.1.

6 Conclusion

This paper establishes a novel framework for analyzing multi-country, multi-sector production networks that incorporates firms' inventory holdings for intermediate and final goods. By fully endogenizing the inventory decision, the model uncovers a unique propagation mechanism for global shocks that is distinct from international trade channels. The analysis demonstrates that under certain constraints, the proposed model exhibits an isomorphic structure to other multi-sector models featuring input-output linkages, allowing for a more comprehensive understanding of shock propagation.

The theoretical foundation draws from the international real business cycle model, extended to incorporate time-intensive flows of intermediate input purchases and a separation of production and cost. This approach captures the intricate interdependencies and input-output linkages that characterize modern production networks, providing insights into how inventory holdings of inputs and outputs influence firms' micro-level production decisions.

Going forward, I plan to calculate the analytical solution to a simplified version of the Q_1 in a small open economy framework. Subsequently, data preparation and calibration will be proceeding, followed by coding part and simulations. Future work and improvement can build upon this framework to further enhance the understanding of inventory dynamics featuring input-output linkages.

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