Finals

Monday, 2 January 2023   22:26

- Week 1

-> Number of elements of the power set $|P(M)| = 2^{|M|}$

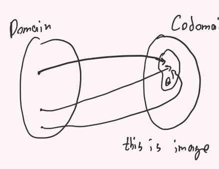
-> De Morgan's laws => $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

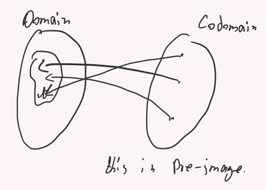
=> $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$

$N \subset N_0 \subset Z \subset Q \subset R$

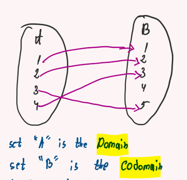
-> $f: X \to Y$ —> Co-domain of the function

Domain of the function





- let $f: X \to Y$ be a function between the sets X and Y and let $A \subset X$
  - -> $f(A) = \{f(x) \in Y : x \in A\}$ is called the image of A under the function f.
  - -> The set $f(x)$ is the range of f.
- let $f: X \to Y$ be a function between the sets X and Y and let $B \subset Y$
  - -> $f^{-1}(B) = \{x \in X : f(x) \in B\}$ is called the pre-image of B un



- The set "A" is the Domain
- The set "B" is the Codomain
- The set of elements that get pointed to in B (The actual values produced by the function) are the Range, also called the Image
  - Domain: $\{1,2,3,4\}$
  - Codomain: $\{1,2,3,4,5\}$
  - Range: $\{1,2,5,3\}$

Injective functions

$f: R \to R$, $f(x) = 2x$

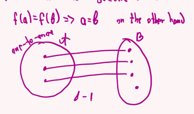
$f: R \to R$, $f(x) = 2x+1$

$f: R \to R$, $f(x) = x^2$ is not injective

$f(-1) = 1$, $f(1) = 1$ here the element from codomain is not injected.

from A to B

$f: A \to B$ is injective (1-1) if $f(a) = f(b) \Rightarrow a = b$; on the other hand if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$



at most one

$f: R \to R$ by $f(x) = 3x+5$

PF Spse $f(a) = f(b)$. Then

$3a+5 = 3b+5$

$3a = 3b$

$\boxed{a = b}$

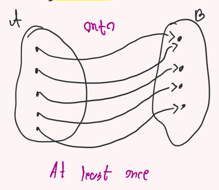
$f: Z \to \{0,1\}$, $f(n) = n \bmod 2$ is surjective

$f: R \to R$, $f(x) = 2x+1$

$A = \{1,-1,2,3\}$, $f: A \to B$, $f(x) = x^2$

$B = \{1,4,9\}$

Surjective functions



At least once

f is called surjective if for every element $y \in Y$ there is at least one element $x \in X$ such that $f(x) = y$

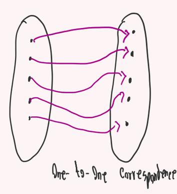
$f: R \to R$, $f(x) = 2x+1$ | $f: R \to R$, $f(x) = x$

$A = \{1,-1,2,3\}$, $f: A \to B$, $f(x) = x^2$

$B = \{1,4,9\}$

Bijective functions

$|\text{Domain}| = |\text{Codomain}|$



One-to-One Correspondence

A function $f: A \to B$ is a bijective function if every element $b \in B$ and every element $a \in A$ such that $f(a) = b$. It is noted that the element "b" is the image of the element "a", and the element "a" is the preimage of the element "b".

$1 \to 4$
$2 \to 3$
$3 \to 2$

$\frac{4!}{4-3}$

if we have 2 sets $A = \{1,2,3\}$ $|A| = n$

$B = \{a,b,c,d\}$ $|B| = m$

how many functions are there from A and B?

$1 \to a,b,c,d$ (4)

$2 \to a,b,c,d$ (4)

$3 \to a,b,c,d$ (4)

$4 \times 4 \times 4 = 4^3 \Rightarrow$ in general m^n

there are 4^3 functions

this is from the general definition of function as it is impossible to x_1 is related to y_1 and at the same time to y_2

All possible functions $A \to B \Rightarrow |B|^{|A|}$

number of injective functions $A \to B \Rightarrow \frac{|B|!}{(|B|-|A|)!}$

number of bijective functions $A \to B \Rightarrow |A| = |B| = |B|!$

number of surjective functions

- how many injective functions are there from these sets?

$f(1) \to a,b,c,d$ (4)

$f(2) \to a,b,c$ (3)

$f(3) \to a,b$ (2)

$4 \times 3 \times 2 = \frac{4!}{(4-2)!} \Rightarrow$ in general $\frac{m!}{(m-n)!}$

this is from definition of injective function

- how many bijective functions are there from Q and P sets?

Firstly cardinality of $|Q| = |P|$ otherwise there won't be bijective functions

so $|Q| = |P| = m$ and number of bijective functions are $m!$

- how many Surjective funtions are there?

A bit of complicated formula!

Central Exercise week 1

1.1.2

$D = \{2, \frac{1}{4}, 8, \pi\}$

$P(D) \cap D = ? \Rightarrow \{\{2, \frac{1}{4}, 8, \pi\}\} \neq \{2, \frac{1}{4}, 8, \pi\}$, here $P(D) \cap D = \emptyset$

$P(D) = \{\{\emptyset\}, \{2\}, \{\frac{1}{4}\}, \{8\}, \{\pi\}, \{2, \frac{1}{4}\} \ldots \{2, \frac{1}{4}, 8, \pi\}\}$

1.2.1

show by induction that

$T_n = \frac{n(n+1)}{2} \Rightarrow T_n = \frac{n^2+n}{2}$

base case when $n=1$, $T_1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$ ✓

let's assume that this formula for some n is right.

then, for $(n+1)$ this also should be correct

$T_{n+1} = \frac{(n+1)(n+1+1)}{2}$

$T_{n+1} = \frac{(n+1)(n+2)}{2}$

$= \frac{n^2+2n+n+2}{2}$

$= \frac{n^2+n}{2} + \frac{2(n+1)}{2}$

$= \underbrace{\frac{n^2+n}{2}}_{} + (n+1)$

By induction Hypothesis T_n

$T_{n+1} = T_n + (n+1)$

$T_{n+1} = \frac{n(n+1)}{2} + (n+1)$

$T_{n+1} = \frac{(n+1)(n+2)}{2}$

$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

for the sake of base case $n=1$, $1^3 = \frac{1^2(1+1)^2}{4}$

$1 = \frac{2^2}{4} = \frac{4}{4} = 1$ ✓

for some n, let's assume that this function is right.

then for $(n+1)$ this formula should be correct too.

$\underbrace{1^3+2^3+3^3+\ldots+n^3}_{} + (n+1)^3 = \frac{(n+1)^2(n+1+1)^2}{4}$

$\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^2 = a^2$

$n^2(n+1)^2 + 4(n+1)^3 = (n+1)^2(n+2)^2$

$(n+1)^2(n^2+4(n+1)) = (n+1)^2(n+2)^2$

$(n+1)^2(n^2+4n+4) = (n+1)^2(n+2)^2$

$(n+1)^2(n+2)^2 = (n+1)^2(n+2)^2$

$n(n+1)(n+2)$ is divisible by 6.

$n(n+1)(n+2) = 6k$, $k \in Z$

for the sake of base case: $n=0$, $n=1$

$0(0+1)(0+2) = 6k$ | $1(1+1)(1+2) = 6k$

$0 = 6k$ | $2 \cdot 3 = 6k$

$k = 0$, $k \in Z$ | $1 = k$, $k \in Z$

let's assume for some n this formula is right

then the value of the formula when $n = n+1$, should be divisible by 6

$(n+1)(n+1+1)(n+1+2) = 6k$

$(n+1)(n+2)(n+3) = 6k$

$(n+1)(n+2)(n+3) = n(n+1)(n+2) + 3(n+1)(n+2)$

$= 6k + \underbrace{3(n+1)(n+2)}_{}$ -> This is also divisible by 6 since either $(n+1)$ or $(n+2)$ is even.

by induction hypothesis

$3^{2n} - 1$ is divisible by 8

for the sake of base case $n=0$, $3^{2 \cdot 0} - 1 = 1 - 1 = 0$ and zero is divisible by 8 ✓

let's assume for some n $3^{2n} - 1$ is divisibly by 8

then for $n+1$, getting value should be divisible by 8.

$3^{2n} - 1 = 8k$, where $k \in Z$

$3^{2(n+1)} - 1 = 3^{2n+2} = \underbrace{3^{2n}}_{} \cdot 3^2$

8k, by induction hypothesis

$= 8k \cdot 3 \cdot 3$

so definitely $8k \cdot 9$ is divisible by 8

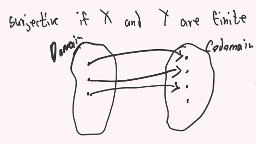
1.3.2

let $f: X \to Y$ be a function between sets X and Y.

1) $B \in Y$ $f^{-1}(B) = \{x \in X : f(x) \in B\}$

2) ?

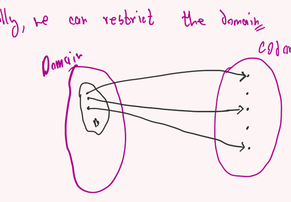
3) Can f be surjective if X and Y are finite and $|X| < |Y|$



NO, it's totally impossible.

4) Can f be injective but not bijective if X and Y are finite and $|X| = |Y|$?

Basically, we can restrict the domain



So, this now is injective function.

In the end, answer is $\boxed{Yes}$

1.3.3

1) Can you find an injective function from N to N?

$f: N \to N$

$f(x) = 2x+1$.

2) Can you find a bijective function from N to Z?

$f: N \to Z$

well it's impossible as |Domain| has to be equal |Codomain| WTF?

Yes, here is an example of a bijective function (also called a bijection) from the set of natural numbers (N) to the set of integers (Z):
h(x) = x - 1
This function is defined for all natural numbers x, and it maps each element in the domain (N) to an element in the range (Z).
For example, h(1) = 0, h(2) = 1, h(3) = 2, and so on. This function is a bijection because it is both injective (no two elements in the domain are mapped to the same element in the range) and surjective (every element in the range is mapped to by at least one element in the domain).
In other words, h is a bijection because it establishes a one-to-one correspondence between the elements in the domain and the range. Every element in the domain is mapped to a unique element in the range, and every element in the range is the image of exactly one element in the domain.

3) Can you find a surjective function from Z to Q

$f: Z \to Q$

$f(x) = y$ for every element from Z, in the codomain there should be at least one element

$f(x) = \frac{x}{2}$

4) Can you find a bijective function from R to $R \times R$?

$f(x) = y$ // $f(x) = (x, x^2)$

$f(x) = (x^3, x^3+1)$

5) How many functions are there from $\{1,2\}$ to $\{3,4,5\}$

$1 \to 3,4,5$
$2 \to 3,4,5$

$3^2 = 9$

6) $f: Z \to Z$ by $f(n) = n^2$.

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

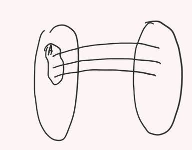
$f(-1) = (-1)^2 = 1$
$f(1) = (1)^2 = 1$

$x_1 \neq x_2$, $f(x_1) = f(x_2)$ It's not injective

$f(x) = y$
$\boxed{f(x) = x^2}$

it's not surjective as not for $\forall y \in Y$ we can find x.

if $y = -1$, there is not such x where $x^2 = -1$.



$f^{-1}(A) = \{x \in X : f(A) \in y\}$

9) Find the pre-image of the set $\{k \in Z : k \text{ is even}\}$ under f from Exercise 1.3.4 (3)?

Week 2.

-> Equivalence relations —> A relation R of a set M is called equivalence relation if it is reflexive, symmetric and transitive.

A relation on a set M is called

-> Reflexive if xRx for all $x \in M$.

-> symmetric if from xRy follows always yRx.

-> Anti-symmetric if from xRy and yRx follows always $x = y$.

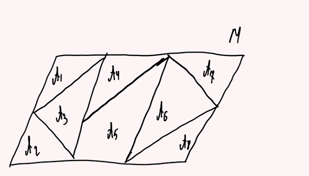
-> Transitive if from xRy and yRz follows always xRz.

Eg. Equality relation / area equality of plane triangles

-> Properties of Equivalence classes:

If R (also denoted by $\sim$) is an equivalence relation on set A, then

- Every element $a \in A$ is a member of the equivalence class $[a]$
  $\forall a \in A, a \in [a]$
- Two elements $a, b \in A$ are equivalent if and only if they belong to the same equivalence class: $\forall a,b \in A, a \sim b$ iff $[a] = [b]$
- Every two equivalence classes $[a]$ and $[b]$ are either equal or disjoint: $\forall a,b \in A, [a] = [b]$ or $[a] \cap [b] = \emptyset$



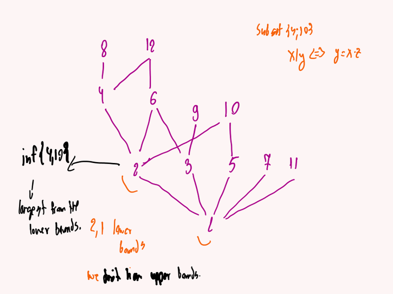
- Partial order relations
  - -> A relation R on a set M is called partial order relation if R is reflexive, anti-symmetric and transitive.
  - $(R, \leq)$ less or equal relations
  - $(N, |)$ divisibility
  - $(P(M), \subseteq)$ set inclusion

Infimum

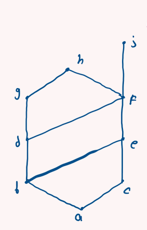
(so in other words, when we have lower bound and I want to find largest lower bound (infimum) I should try to find such lower bound which all other lower bound would be related to it.

Supremum

respectively to smallest upper bound, we should find upper bound which would be related to all other upper bounds.



Ex. Find the lower and upper bounds of the subsets $\{a,b,c\}$, $\{j,h\}$, $\{a,c,d,f\}$ in the poset with the Hasse diagram



lower bound of $\{a,b,c\}$ is a

upper bound of $\{a,b,c\}$ is e, f, j, h

lower bound $\{j,h\}$ is f, e, c, a, b, d

upper bound of $\{j,h\}$ is no upper bound. $\emptyset$

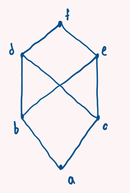
lower bound of $\{a,c,d,f\}$ is a,

upper bound of $\{a,c,d,f\}$ is f, j, h

- **Lattice**

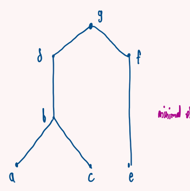
A partially ordered set $(M, \leq)$ is called lattice if $\inf\{a,b\}$ and $\sup\{a,b\}$ for each two elements $a, b \in M$ always exist.

- poset can have more than one minimal element



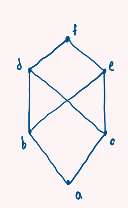
minimal element is a since there is no element which is related to a

b is not minimal be $aRb \to a$ is related to b



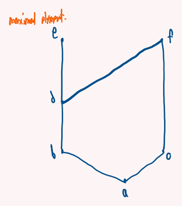
minimal elements are a, c, e.

- A poset can have more than one maximal element.

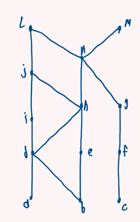


f is maximal element since f is not related to other vertexes

d is not maximal element as it is related to f.



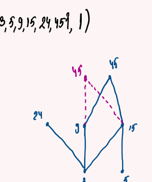
e and f are maximal element since they are not related to other vertexes.



Maximal elements l and m

Minimal elements: a, d, c

- What are the maximal and minimal elements for the poset $(\{3,5,9,15,24,45\}, |)$



maximal elements: 24, 45;

minimal elements: 3, 5;

- 3 and 5 are incomparable and no element divides 3 and 5 so, let's put them in the bottom most place

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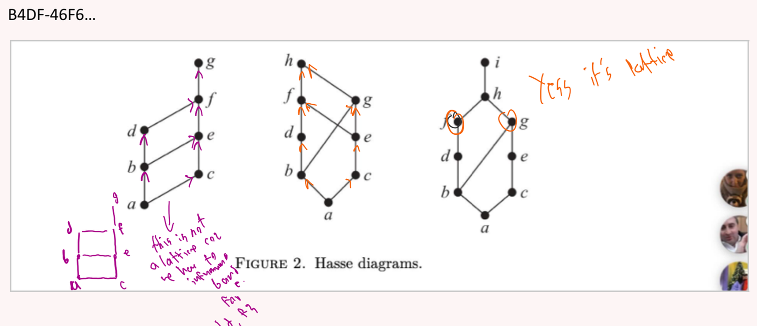

FIGURE 2. Hasse diagrams.

Central exercises

$xk = y$

1.1

1) Find all x such that $x | 10$ $x = \{1,2,5,10\}$

2) Is it true that $x | x$ for all $x \in Z$? in most cases $x/x = 1$ but in case of $x = 0$, $0 \cdot k = 0$ so for $\forall k \in Z$ this is correct.

3) Is it true that if $x | y$ then $y | x$ for all $x, y \in Z$? well, no it's not true $\frac{4}{2} \in Z$ but $\frac{2}{4} \notin Z$.

4) Is it true that if $x | y$ and $y | z$ then $x | z$ $\forall x,y,z \in Z$? well, hell yeah!! it's called fucking transitivity.

1.2.1 (x,y) $xy = 0$

reflexive? xRx $x \cdot x = 0$ well, it's reflexive

symmetric? xRy then yRx $xy = 0$ then $yx = 0$ well, it's symmetric

Anti-symmetric? xRy and yRx then $x = y$ well, not really $4 \cdot 0 = 0$, $0 \cdot 4 = 0$ but $4 \neq 0$

Transitive? xRy and yRz then xRz

$5 \cdot 0 = 0$, $0 \cdot 4 = 0$

but $5 \cdot 4 \neq 0$ so, no it's not transitive

1.2.2 Give an example of a relation on a set that is neither symmetric nor antisymmetric

having a set $A = \{1,2,3\}$

and we have some relation $R = \{(1,2), \{1,3\}, \{3,1\}\}$

1.2.3.

1) All ordered pairs of real numbers (x,y) such that $x-y$ is rational

reflexive xRx $x - x = 0$ it's rational ✓

symmetric xRy then yRx $x - y = k$, $y - x = -k$ ✓

antisymmetric xRy and yRx then $x = y$ $5-4 = 1$, $4-5 = -1$ but $5 \neq 4$ NO

transitive yes

3) All ordered pairs of real numbers (x,y) such that $xy > 0$

reflexive $x^2 > 0$ well, if $x = 0$ NO

symmetric xRy, yRx $5 \cdot 4 > 0$ $4 \cdot 5 > 0$ ✓

anti-symmetric xRy, yRx then $x = y$ $5 \cdot 4 > 0$, $4 \cdot 5 > 0$, $5 \neq 4$ ✗

transitive xRy and yRz then xRz ✓

$2 \cdot 3 > 0$ $3 \cdot 5 > 0$ $2 \cdot 5 > 0$

1.2.4 give an example of a relation on a set that is both symmetric and antisymmetric

$A = \{1,2,3\}$

$R = \{(1,1), (2,2), (3,3)\}$

1.4

$\{(3,1), (0,1), (0,2), (2,0), (4,0), (4,2), (1,2), (3,4)\}$

$\{(1,3), (1,0), (0,4), (2,4), (2,1), (4,3)\}$

1.5

$\{(0,0), (1,1), (2,0), (2,2), (2,3), (3,3)\}$ on $\{0,1,2,3\}$ a partial ordering?

- Reflexive +
- Anti-symmetric +
- Transitive +

Yes

2,0 0,0 2,0

2,3 3 3 2,3

2,2

1.5.2. Denote by J the set of closed intervals in the real numbers, and define a relation S on J by $[a,b] \, S \, [c,d]$ If and only if $[a,b] = [c,d]$ or $b < c$. Show that S is a partial ordering but not a linear ordering on J.

1.5.3 $\{0,1,2,3\}$

ref
Anti-sym
Trans

1) $\{(0,0), (2,2), (3,3)\}$ NO

2) $\{(0,0), (1,1), (1,2), (2,2), (3,1), (3,3)\}$ NO.

1,2 2,2 1,2

3,1 1,1 3,1

3,1 1→2 3,2 well, it's not transitive

3) $\{(0,0), (1,1), (1,2), (1,3), (2,0), (2,2), (2,3), (3,0), (3,3)\}$

ref ✓
Anti-sym ✓
Trans NO

1→2 2→0
1→0

1.6 Lattices

1) $(\{-1, 2, 3, 7\}, \leq)$ Yes

2) $(N, \leq)$ NO

3) (Z, Z) NO

4) $(P(S), \supseteq)$ Yes

1.6.2 which of these posets are lattices?

1) $(\{1,3,6,9,12\}, |)$

NO we have inf but not sup

2) $(\{1,5,25,125\}, |)$

Well, yes having both of the bolls.