HW2 Joohyun Lee

1. Let $\rho_j^n = \xi_k^n e^{ikj\Delta x}$.

Then, we have

$$\xi_k^{n+1} e^{ikj\Delta x} - \xi_k^{n-1} e^{ikj\Delta x} = -v_0 \frac{\Delta t}{\Delta x}\left(\xi_k^n e^{ik(j+1)\Delta x} - \xi_k^n e^{ik(j-1)\Delta x}\right)$$

$$\Rightarrow \xi_k - \xi_k^{-1} = -v_0 \frac{\Delta t}{\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$= -v_0 \frac{\Delta t}{\Delta x} 2i \sin(k\Delta x)$$

$$\Rightarrow \xi_k^2 + 2\frac{v_0 \Delta t}{\Delta x} i \sin(k\Delta x)\, \xi_k - 1 = 0$$

$$\Rightarrow \xi_k = -\frac{v_0 \Delta t}{\Delta x} i \sin(k\Delta x) \pm \sqrt{-\left(\frac{v_0\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) + 1}$$

$$\Rightarrow |\xi_k|^2 = \begin{cases} 1 & \left(\frac{v_0 \Delta t}{\Delta x} < 1\right) \\ 1 \quad \text{or} \quad \pm \frac{2 v_0 \Delta t}{\Delta x} \sin(k\Delta x) \sqrt{\left(\frac{v_0\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) - 1} & \left(\frac{v_0\Delta t}{\Delta x} > 1\right) \end{cases}$$

If $\frac{v_0 \Delta t}{\Delta x} \leq 1$, it is stable.

If not, because $\max(\sin(k\Delta x)) = 1$

and we know that $2X\sqrt{X^2-1} < 1 \left(X = \frac{v_0 \Delta t}{\Delta x}\right)$

when $X < \sqrt{\frac{1+\sqrt{2}}{2}}$.

$$\left(\begin{array}{l} 4X^2(X^2-1) < 1 \\ 4X^4 - 4X^2 - 1 < 0 \\ X^2 < \frac{1+\sqrt{2}}{2} \end{array}\right)$$

$\therefore$ The leapfrog sheme is stable

if $\Delta t \leq \frac{\Delta x}{v_0}$ $(X \leq 1)$

and also if $\Delta t \leq \sqrt{\frac{1+\sqrt{2}}{2}}\frac{\Delta x}{v_0}$ $\left(X \leq \sqrt{\frac{1+\sqrt{2}}{2}}\right)$

2. The answers are in a IPython Notebook file.