The Laws of Probability

Introduction

Some events in life are impossible, other events are quite certain to happen.

For example, it is impossible for a human being to live for 1000 years. We say the probability of a person living for 1000 years is zero.

$$\text{P(a human will live for 1000 years)} = 0$$

On the other hand, it is quite certain that all of us will die someday. We say, the probability that all of us will die is 1

$$\mathrm{P}(\text{a human will die someday}) = 1$$

Many other events in life are neither impossible nor certain. Events such as these are assigned probabilities varying from 0 up to 1.

All probabilities lie in the range 0 to 1

Complementary Events

Consider the following: A lightbulb is tested. Clearly it works or it doesn't work.

Here we have two events

1. The lightbulb works
2. The lightbulb doesn't work

When the bulb is tested, one or other of these events must occur and each event excludes the other.

These two events are **complementary** and so the sum of their probabilities always equals 1.

Calculating theoretical probabilities

Sometimes we have enough information about a set of circumstances to calculate the probability of an event occurring.

e.g. Rolling an unbiased die, the probability of getting a 5:

$$\mathrm{P}(5) = \frac{1}{6}$$

$$= 0.167$$

What about the possibility of obtaining a number other than 5?

$$\mathrm{P}(5') = \frac{5}{6}$$

This event is the complement of obtaining a 5 (written 5’)

So we can see $P(5) + P(5') = \frac{1}{6} + \frac{5}{6} = 1$

When all events are equally likely:

$$\text{P(chosen event)} = \frac{\text{number of ways the event can occur}}{\text{total number of possibilities}}$$

e.g. What is the probability of obtaining a score more than 4?

There are two ways of obtaining a score greater than 4

$$\Rightarrow \text{P}(> 4) = \frac{2}{6}$$

Example 1

There are four Aces in a pack of 52 playing cards. What is the probability that a card selected at random is not an ace?

$$\mathrm{P}(\text{chosen event}) = \frac{\text{number of ways the event can occur}}{\text{total number of possibilities}}$$

$$\mathrm{P}(\mathrm{Ace}') = \frac{48}{52} = 0.923$$

Example 2

Two dice are rolled together and their scores added.
Find the probability that the total score will be

a. 12
b. 0
c. 1
d. 2
e. more than 5

$$P(\text{chosen event}) = \frac{\text{number of ways the event can occur}}{\text{total number of possibilities}}$$

The total number of possibilities =36

$$\{(6+1),(6+2),(6+3),(6+4),(6+5),(6+6),\}$$

$$\{(5+1),(5+2),(5+3),(5+4),(5+5),(5+6),\}$$

$$\{(4+1),(4+2),(4+3),(4+4),(4+5),(4+6),\}$$

$$\{(3+1),(3+2),(3+3),(3+4),(3+5),(3+6),\}$$

$$\{(2+1),(2+2),(2+3),(2+4),(2+5),(2+6),\}$$

$$\{(1+1),(1+2),(1+3),(1+4),(1+5),(1+6),\}$$

Number of ways the score equals 12

$$\{(6+1),(6+2),(6+3),(6+4),(6+5),\boxed{(6+6),}\}$$

$$\{(5+1),(5+2),(5+3),(5+4),(5+5),(5+6),\}$$

$$\{(4+1),(4+2),(4+3),(4+4),(4+5),(4+6),\}$$

$$\{(3+1),(3+2),(3+3),(3+4),(3+5),(3+6),\}$$

$$\{(2+1),(2+2),(2+3),(2+4),(2+5),(2+6),\}$$

$$\{(1+1),(1+2),(1+3),(1+4),(1+5),(1+6),\}$$

$$\mathrm{P}(12)=\frac{1}{36}=0.028$$

Number of ways the score equals 0

$$\{(6+1),(6+2),(6+3),(6+4),(6+5),(6+6),\}$$

$$\{(5+1),(5+2),(5+3),(5+4),(5+5),(5+6),\}$$

$$\{(4+1),(4+2),(4+3),(4+4),(4+5),(4+6),\}$$

$$\{(3+1),(3+2),(3+3),(3+4),(3+5),(3+6),\}$$

$$\{(2+1),(2+2),(2+3),(2+4),(2+5),(2+6),\}$$

$$\{(1+1),(1+2),(1+3),(1+4),(1+5),(1+6),\}$$

$$\mathrm{P}(0)=\frac{0}{36}=0$$

Number of ways the score equals 1

$$\{(6+1),(6+2),(6+3),(6+4),(6+5),(6+6),\}$$

$$\{(5+1),(5+2),(5+3),(5+4),(5+5),(5+6),\}$$

$$\{(4+1),(4+2),(4+3),(4+4),(4+5),(4+6),\}$$

$$\{(3+1),(3+2),(3+3),(3+4),(3+5),(3+6),\}$$

$$\{(2+1),(2+2),(2+3),(2+4),(2+5),(2+6),\}$$

$$\{(1+1),(1+2),(1+3),(1+4),(1+5),(1+6),\}$$

$$\mathrm{P}(1)=\frac{0}{36}=0$$

Number of ways the score equals 2

$$\{(6+1),(6+2),(6+3),(6+4),(6+5),(6+6),\}$$

$$\{(5+1),(5+2),(5+3),(5+4),(5+5),(5+6),\}$$

$$\{(4+1),(4+2),(4+3),(4+4),(4+5),(4+6),\}$$

$$\{(3+1),(3+2),(3+3),(3+4),(3+5),(3+6),\}$$

$$\{(2+1),(2+2),(2+3),(2+4),(2+5),(2+6),\}$$

$$\{\boxed{(1+1),}(1+2),(1+3),(1+4),(1+5),(1+6),\}$$

$$\mathrm{P}(2)=\frac{1}{36}=0.028$$

Number of ways the score >5

$$\{\boxed{(6+1),}\boxed{(6+2),}\boxed{(6+3)}\boxed{,(6+4)}\boxed{,(6+5),}\boxed{(6+6),}\}$$

$$\{\boxed{(5+1),}\boxed{(5+2),}\boxed{(5+3)}\boxed{,(5+4)}\boxed{,(5+5)}\boxed{,(5+6),}\}$$

$$\{(4+1),\boxed{(4+2),}\boxed{(4+3)}\boxed{,(4+4)}\boxed{,(4+5)}\boxed{,(4+6),}\}$$

$$\{(3+1),(3+2),\boxed{(3+3)}\boxed{,(3+4)}\boxed{,(3+5)}\boxed{,(3+6)},\}$$

$$\{(2+1),(2+2),(2+3)\boxed{,(2+4)}\boxed{,(2+5)}\boxed{,(2+6),}\}$$

$$\{(1+1),(1+2),(1+3),(1+4)\boxed{,(1+5),}\boxed{(1+6),}\}$$

$$\mathrm{P}(>5)=\frac{26}{36}=0.72$$

Calculating experimental probabilities

In many circumstances we do not have sufficient information to calculate a theoretical probability.

In such a case we conduct a (large) number of trials and recording the results.

e.g. we toss a coin that we suspect is biased 100 times. If it lands on tails 65 times then

$$\mathrm{P(tails)} = \frac{65}{100} = 0.65$$

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Such a probability is called an **experimental probability**

We can calculate an experimental probability as follows:

$$\mathrm{P(chosen\ event\ occurs)}$$

$$= \frac{\text{number of times the event occured}}{\text{total number of times the experiment is repeated}}$$

Example 3

A new component is fitted to a washing machine. In a sample of 150 machines tested, seven failed to function correctly. Calculate the probability that a machine fitted with the new component

a. Works correctly
b. Does not work correctly

$$P(\text{chosen event occurs})$$

$$= \frac{No\ of\ times\ event\ occured}{\text{total number of times the experiment is repeated}}$$

$$P(\text{works correctly}) = \frac{143}{150} = 0.95$$

$$P(\text{does not work correctly}) = \frac{7}{150} = 0.05$$

What is the statistical relationship between these two events?

Ans: complementary.

Example 4

In a sample containing 5000 nails manufactured by a company, 5% are too short or too long. A nail is picked at random from the production line, estimate the probability that it is of the right length.

$$\mathrm{P}(\text{right length}) = \frac{95}{100} = 0.95$$

Independent events

If two events are independent then the occurrence of either one in no way affects the occurrence of the other.

e.g. If an unbiased die is thrown twice, the score on the second throw is in no way affected by the score on the first. The two scores are independent.

The **multiplication law** for independent events states

If events A and B are independent, then the probability of obtaining A and B (written $A \wedge B$) is given by

$$\mathrm{P}(\mathrm{A} \ \wedge \ \mathrm{B}) = P(A) \times P(B)$$

Example 5

A die is thrown and a coin is tossed. What is the probability of obtaining a six and a head?

Ans: These events are independent therefore

$$\mathrm{P(six\ and\ a\ head)} = \mathrm{P(six)} \times \mathrm{P(head)}$$

$$= \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

Example 6

A coin is tossed three times. What is the probability of obtaining three heads?

Ans: These events are independent therefore

$$\mathrm{P}(\text{three heads}) = \mathrm{P}(\text{head}) \times \mathrm{P}(\text{head}) \times \mathrm{P}(\text{head})$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Example 7

An urn contains 3 red balls, 2 blue balls and 5 white balls. A ball is selected and its colour noted, then it is replaced. A second ball is selected and its colour noted. Find the probability of:

a. Selecting 2 blue balls
b. Selecting 1 blue ball, then one white ball
c. Selecting one red ball, then one blue ball

Ans: These events are independent (the first ball chosen is replaced) therefore

a. $P(2\ \text{blues}) = P(\text{blue}) \times P(\text{blue})$

$$= \frac{2}{10} \times \frac{2}{10} = \frac{1}{25}$$

b. $P(1\ \text{blue}\ \textit{then}\ 1\ \textit{white}) = P(\text{blue}) \times P(\text{white})$

$$= \frac{2}{10} \times \frac{5}{10} = \frac{1}{10}$$

$$\text{c. } \mathrm{P}(1\ \mathit{red\ then}\ 1\ \mathit{blue}) = \mathrm{P}(\text{red}) \times \mathrm{P}(\text{blue})$$

$$= \frac{3}{10} \times \frac{2}{10} = \frac{3}{50}$$

The **addition law** for mutually exclusive events states

If events A and B are mutually exclusive, then the probability of obtaining A or B (written $A \vee B$) is given by

$$\mathrm{P}(\mathrm{A} \vee \mathrm{B}) = P(A) + P(B)$$

Example 8

A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table.

Number of contamination particles	Proportion of wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

Number of contamination particles	Proportion of wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

a. What is the probability that a wafer selected at random contains no particles?

Ans: P(0 particles)=0.4

b. What is the probability that a wafer selected at random contains three or more particles?

Ans: P(3 or more particles)=0.10+0.05+0.10

=0.10+0.05+0.10

=0.25

Number of contamination particles	Proportion of wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

c. What is the probability that a wafer selected at random contains either 0 or more than 3 particles?

Ans: $P(0 \text{ or } >3 \text{ particles}) = 0.40+0.05+0.10$

$= 0.55$

Looking again at the two dice experiment from example 2. The possibilities are summarised in the table below.

		Second die					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
First die	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

		Second die					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
First die	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

There are six ways of scoring a seven so $P(7) = \frac{6}{36} = \frac{1}{6}$

There are four ways of scoring a nine so $P(9) = \frac{4}{36} = \frac{1}{9}$

		Second die					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
First die	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

The probability that the sum of the scores is either seven or nine

$$P(7 \land 9) = \frac{10}{36} = \frac{5}{18}$$

		Second die					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
First die	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

We can also have arrived at this result by adding the probability of scoring a seven to the probability of scoring a nine

$$P(7 \wedge 9) = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$$

The **addition law** for non-mutually exclusive events states

If events A and B are not mutually exclusive, then the probability of obtaining A or B (written $A \vee B$) is given by

$$P(A \text{ or } B) = P(A) + P(B) - P(A \wedge B)$$

Example 9

A car dealership has the following cars in stock

	SUV	Compact	Mid-sized
European	20	50	20
Asian	65	100	45

If a car is selected at random, find the probability that it is

a. Asian
b. European and Mid-sized
c. Asian or an SUV

	SUV	Compact	Mid-sized
European	20	50	20
Asian	65	100	45

A . $P(Asian) = \frac{210}{300} = 0.7$

b. $P(European\ and\ mid\ sized) = \frac{20}{300} = 0.067$

c. $P(Asian\ or\ SUV) = \frac{65 + 100 + 45}{300} + \frac{20 + 65}{300} - \frac{65}{300} = 0.7667$

The addition law for non-mutually exclusive events

$$P(A\ or\ B) = P(A) + P(B) - P(A\ \wedge\ B)$$

Reduces to

$$P(A\ or\ B) = P(A) + P(B)$$

For mutually exclusive events because $P(A\ \wedge\ B) = 0$

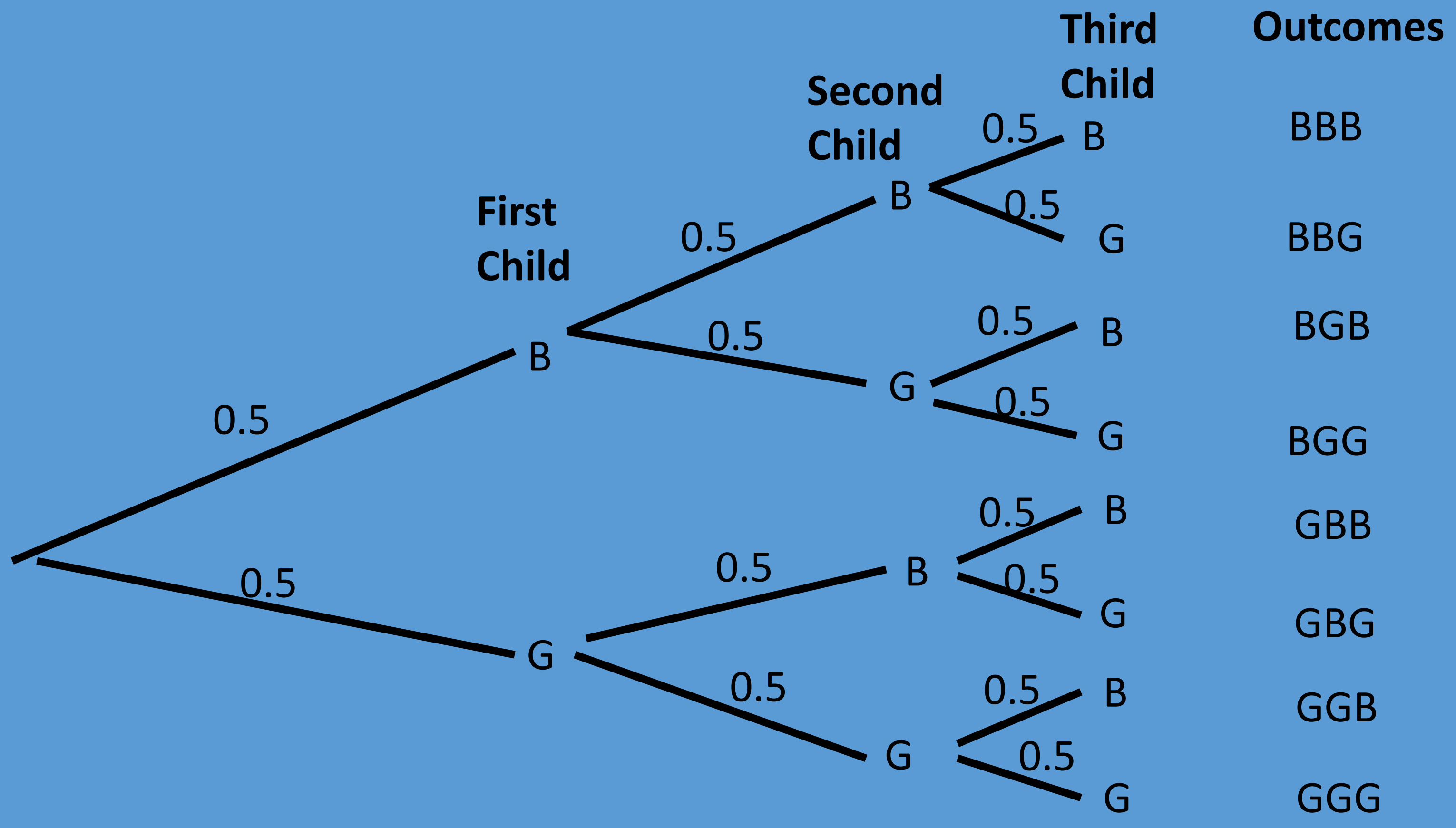
Applications of the laws of probability

Example 10

A tree diagram consists of line segments coming from a starting point and also from an outcome point used to determine all possible outcomes of a probability experiment

Use a tree diagram to find the sample space (all possibilities) for the gender of three children in a family.

Find the probability of having two girls and a boy in any order.



$$P(2G \text{ and } 1\ B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

Example 11

In the past, two building contractors, A and B, have competed for 20 building contracts of which 10 were awarded to A and six were awarded to B. The remaining four contracts were not awarded to A or B. Three contracts for buildings of the kind in which they both specialise have been offered for tender.

Assuming that the market has not changed, find the probability that:

a. A will obtain all three contracts
b. B will obtain at least one contract
c. A contract will be awarded to A or B.

Ans:

The probability that A gets one contract $P(A) = \frac{10}{20} = \frac{1}{2}$

The probability that B gets one contract $P(B) = \frac{6}{20} = \frac{3}{10}$

a. The probability that A will obtain all three contracts

$$P(\text{A gets all three}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

b. The probability that B will obtain at least one contract

$$P(\text{B gets at least one})$$

$$= 1 - P(B\ gets\ none)$$

$$=1-\left(\frac{7}{10} \times \frac{7}{10} \times \frac{7}{10}\right)$$

$$= \frac{657}{1000}$$

c. The probability that a contract will be awarded to A or B

$$=P(\mathrm{A}\ \vee B)=\frac{10}{20}+\frac{6}{20}=\frac{4}{5}$$