

Data Sec HW 4

Problem 1. Find the last digit of 9^{123456} .

$$9^1 = 9$$

$$9^2 = 81$$

$$9^3 = 729$$

$$9^4 = 6561$$

...

You can see that there is a pattern. If the exponent is odd, then the last digit will be a 9.

If the exponent is even, the last digit will be 1. In this example, since our exponent is 123456, which is an even number, we know that the last digit is a 1.

We can also say that $123456 \% 2 = 0$. Then using the remainder, 0, we say $9^0 = 1$

If we had an odd exponent, say 123457, we can still do the same operations. $123457 \% 2 = 1$, $9^1 = 9$ as our remainder.

Problem 2. What is $(1! + 2! + 3! + 4! + 5! + 6! + \dots) \bmod 12$?

$$\sum_{i=1}^n i! \bmod 12.$$

$$1 + 2 + 6 + 24 + 120 + 720 + 5040 + \dots \bmod 12$$

Can also look at it as:

$$(1) + (1\ 2) + (1\ 2\ 3) + (1\ 2\ 3\ 4) + (1\ 2\ 3\ 4\ 5) + \dots \bmod 12$$

This can be $(1 \bmod 12) + (2 \bmod 12) + (6 \bmod 12) + (24 \bmod 12) + (120 \bmod 12) + (720 \bmod 12) + (5040 \bmod 12) + (40320 \bmod 12)$

$$1 + 2 + 6 + 0 + 0 + 0 + 0 + 0 + \dots \text{ (infinite \# of zeroes)}$$

That means the remainder for this infinite sum of factorials will be $1 + 2 + 6 = 9$.

Problem 3. In class, we looked at the definition of perfect secrecy, which states that the ciphertext C doesn't carry any information about M and hence it doesn't increase the adversary's probability of guessing C correctly.

$$P[M = m | C = c] = P[M = m]$$

(a) Show that the above definition is equivalent to the following definition.

$$P[M = m, C = c] = P[M = m] \cdot P[C = c]$$

$$P[M = m | C = c] = P[M = m]$$

$$P[M = m] = \frac{P[M = m, C = c]}{P[C = c]} \quad \text{By the definition of conditional probability}$$

$$P[M = m] \cdot P[C = c] = P[M = m, C = c] \quad \text{divide both sides by } P[C = c]$$

$$P[M = m, C = c] = P[M = m] \cdot P[C = c] \quad \text{flip the equation}$$

(b) Consider a simple one-time pad for binary string of length 2 in which the message space, ciphertext space, and key space are defined as $M = C = K = 00, 01, 10, 11$. It is known that the one-time pad is perfectly secret if the key k is chosen uniformly random, i.e., $P[k = 00] = P[k = 01] = P[k = 10] = P[k = 11] = \frac{1}{4}$. This can be proved by showing that, for any m_1, m_2, c , the method satisfies

$$P[Enc(k_1, m_1) = c] = P[Enc(k_2, m_2) = c]. \quad (1)$$

Let's look at the first probability. Given m_1 and c ,

$$P[Enc(k, m_1) = c] = P[k \oplus m_1 = c] = P[k = c \oplus m_1] = \frac{1}{4}.$$

The above means that m_1 will be encrypted into c if the specific key $k = c \oplus m_1$ is chosen. Since in one-time pad all keys are equally likely, m_1 has the probability of $\frac{1}{4}$ to be encrypted into c . Observe that, with the same reasoning, m_2 also has the same probability to be encrypted into c , i.e., $P[Enc(k, m_2) = c] = \frac{1}{4}$.

Suppose that you decided to remove the key $k = 00$ from the key space as it sends the message in clear. When $k = 00$, we have

$$c = Enc(k, m) = 00 \oplus m = m.$$

Prove or disprove whether the modified method is still perfectly secret. In other words, either show that (1) still holds true with $k = 00$ removed or provide a counter example (m_1, m_2, c) for which

$$P[Enc(k_1, m_1) = c] \neq P[Enc(k_2, m_2) = c].$$

Hint: given fixed m and c , there exists a unique k such that $m \oplus k = c$.

If you do the exclusive or of 0 with any bit string, for example

$000000000 \oplus 100011001$, you get the same number: 100011001 . This is one of the properties of exclusive or. In practice, you would not use this key because the plaintext would be the same as the ciphertext. By removing this key value, this would no longer satisfy perfect privacy.

A counter example would be if $m_1 = 00$, $m_2 = 11$ and $c = 11$

$$k \oplus m_1 = 11, \text{ which means } k = 11$$

$$k \oplus m_2 = 11, \text{ which means } k = 00$$

Since there are 3 keys in the keyspace $\{01, 10, 11\}$, the probability of choosing the first example ($k = 11$) is $\frac{1}{3}$ since the probability distribution is it's uniformly distributed. However, in our second example, $k = 00$ is not in our keyspace, so the probability of choosing $k = 00$ is 0.

$$P[Enc(k, m_1) = c] \neq P[Enc(k, m_2) = c]$$

$$\frac{1}{3} \neq 0$$

Problem 4. Consider the following cryptosystem in which

- the message space (or plaintext space) is $M = \{a, b, c\}$,
- the keyspace is $K = \{k_1, k_2, k_3\}$, and
- the ciphertext space is $C = \{1, 2, 3\}$.

The following table describes how the encryption is done, i.e., $Enc(k, m)$.

K/M	a	b	c
k_1	2	3	1
k_2	3	1	2
k_3	1	2	3

For example, $Enc(k_1, b) = 3$. This means the plaintext 'b' is mapped to the ciphertext 3 if the key k_2 is chosen. Suppose that messages occur with probability

- $P(M = a) = \frac{1}{4}$,
- $P(M = b) = \frac{1}{2}$, and
- $P(M = c) = \frac{1}{4}$.

Let's further assume the probability distribution over keys is given by

- $P(K = k_1) = \frac{1}{2}$,
- $P(K = k_2) = \frac{1}{4}$, and
- $P(K = k_3) = \frac{1}{4}$.

(a) Compute the probability of each ciphertext, $P(C = 1)$, $P(C = 2)$, and $P(C = 3)$.

$$P[C = 1] = \sum_{m \in M} P[C = 1 | M = m] * P[M = m]$$

=

$$P[K = k_1] * P[M = c] = \frac{1}{2} * \frac{1}{4} = \frac{1}{8}$$

+

$$P[K = k_2] * P[M = b] = \frac{1}{4} * \frac{1}{2} = \frac{1}{8}$$

+

$$P[K = k_3] * P[M = a] = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{16} = \frac{5}{16}$$

$$P[C = 2]$$

=

$$P[K = k_1] * P[M = a] = \frac{1}{2} * \frac{1}{4} = \frac{1}{8}$$

$$P[K = k_2] * P[M = c] = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

$$P[K = k_3] * P[M = b] = \frac{1}{4} * \frac{1}{2} = \frac{1}{8}$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{16} = \frac{5}{16}$$

$$P[C = 3]$$

=

$$P[K = k_1] * P[M = b] = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

$$P[K = k_2] * P[M = a] = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

$$P[K = k_3] * P[M = c] = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

$$= \frac{1}{4} + \frac{1}{16} + \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$$

(b) Compute $P[M = a|C = 1]$.

$$P[M = a|C = 1] = \frac{P[C=1|M=a]*P[M=a]}{P[C=1|M=a]*P[M=a] + P[C=1|M=b]*P[M=b] + P[C=1|M=c]*P[M=c]}$$

$$= \frac{P[K=k_3]*P[M=a]}{P[C=1]}$$

$$= \frac{\frac{1}{4} * \frac{1}{4}}{\frac{6}{16}} = \frac{\frac{1}{16}}{\frac{6}{16}} = \frac{1}{6} * \frac{16}{6} = \frac{16}{80} = \frac{1}{5}$$

$P[C = 1]$ is already calculated above in (a)

Problem 5. In class, we learned a simple symmetric shift cipher (a.k.a. Caesar cipher). The secret key K is an integer in $\{0, 1, \dots, 25\}$. As always, we map each alphabet letter $x \in \{A, B, \dots, Z\}$ to an integer $\{0, 1, \dots, 25\}$. The encryption and decryption are defined by

$$Enc(x, k) = (x + k) \bmod 26, Dec(x, k) = (x - k) \bmod 26.$$

1. (a) As a warm-up, encrypt "CSCI IS COOL" using a Caesar cipher with $k = 'F'$.

$F = 5$

A	B	C	D	E	F	G	H	I	J	K
F	G	H	I	J	K	L	M	N	O	P
L	M	N	O	P	Q	R	S	T	U	V
Q	R	S	T	U	V	W	X	Y	Z	A
W	X	Y	Z							
B	C	D	E							

The encrypted message would be: "HXHN NX HTTP" using the shifted Caesar's cipher mapping above.

1. (b) Here's the ciphertext generated by the shift cipher. Find the corresponding plaintext (please explain the approach/strategy you used to find the key).

IWXHFJTHIXDCXHTPHN

The approach to use in this case is brute force since there are only 26 possible keys in the keyspace. I implemented a quick program to decipher this.

```
def caesar_decrypt(ciphertext, shift):
    result = ""
    for char in ciphertext:
        if char.isalpha():
            base = ord('A')
            result += chr((ord(char.upper()) - base - shift) % 26 + base)
        else:
            result += char
    return result

ciphertext = "IWXHFJTHIXDCXHTPHN"

for k in range(1, 26):
    decrypted = caesar_decrypt(ciphertext, k)
    print(f"k = {k}, {decrypted}")
```

The output is as shown (notice $k = 15$):

```

(base) johnhughes@johns:~$ cat c
k = 1, HVWGEISGHWCBWGSOGM
k = 2, GUVFDHRFGVBAVFRNFL
k = 3, FTUECGQEFUAZUEQMEK
k = 4, ESTDBFPDETZYTDPLDJ
k = 5, DRSCAE0CDSYXSCOKCI
k = 6, CQRBZDNBCRXWRBNJBH
k = 7, BPQAYCMABQWVQAMIAG
k = 8, AOPZXBLZAPVUPZLHZF
k = 9, ZNOYWAKYZOUTOYKGYE
k = 10, YMNXVZJXYNTSNXJFXD
k = 11, XLMWUYIWXMSRMWIEWC
k = 12, WKLVTXHVWLRQLVHDVB
k = 13, VJKUSWGUVKQPKUGCUA
k = 14, UIJTRVFTUJPOJTFTBTZ
k = 15, THISQUESTIONISEASY
k = 16, SGHRPTDRSHNMHRDZRX
k = 17, RFGQ0SCQRGMLGQCYQW
k = 18, QEFPNRBPQFLKFPBXPV
k = 19, PDEOMQAOPEKJEOAWOU
k = 20, OCDNLPZNODJIDNZVNT
k = 21, NBCMKOYMNCIHCYUMS
k = 22, MABLJNXLMBHGBLXTLR
k = 23, LZAKIMWKLAFKWSKQ
k = 24, KYZJHLVJKZFEZJVRJP
k = 25, JXYIGKUIJYEDYIUQIO

```

The table was taking too long. The Decrypted Ciphertext reads: THISQUESTIONISEASY

Keys	Plaintext
k = 1	JXYIGKUIJYEDYIUQIO
k = 2	KYZJHLVJKZFEZJVRJP
k = 3	
k = 4	
k = 5	
k = 6	

Problem 6. In class, we learned that one-time pad is not secure if the same key is repeatedly used for encrypting messages. Suppose that your classmate Alice is encrypting two messages, denoted by m_1 and m_2 , each of which is an n -bit string, using the one-time pad. She knew that it's not safe to re-use the key k , so she decided to encrypt as follows:

- $c_1 = m_1 \oplus k$ and
 - $c_2 = m_2 \oplus k'$,
- where $k' = 0 \oplus 1 \oplus k$.

Is it safe to release c_1 and c_2 ? Explain your answer with justification.

If it's not safe, explain what information (about m_1 and m_2) can be leaked from c_1 and c_2 .

If you plug in $k' = 0 \oplus 1 \oplus k$, you get

$$c_2 = m_2 \oplus (0 \oplus 1 \oplus k)$$

If you try to XOR c_1 and c_2 , you get

$$c_1 \oplus c_2$$

or

$$(m_1 \oplus k) \oplus (m_2 \oplus (0 \oplus 1 \oplus k))$$

You can get rid of the parenthesis because you're doing exclusive OR.


```
t__t_e__te_tt_t__e__te__t_e__t_t_t__t__t__e__e__e__e_e_t
_e_e__e__e_e_et_t__e__ette__e_t__t_e__t_t_t__e__t_et__e__
__e__t__e__e__ette__t_t_e__e_t_e_e__e__e__e__e_e__e_te__
_t_e_e__e__e_te_e_t__et_e_e__e_e__e__
```

Now you notice how there are a lot of "t_e" in the resulting text.

The most common trigram in the English alphabet is "the", so it's safe to say that "t_e" -> "the"

The first occurrence of "t_e" is in the 7th character, which means our 'h' should be in the 8th character. If you look at the ciphertext, 's' is in our 8th char spot, and you can see how g is on the left of s, and v is on the right of s, the two chars we replaced, so let's try replacing s with h. This is the resulting text:

```
th__the__te_tth_t__e__te__the__t_t_t__t__t__h__e__e__e__he_e_t
he_e__e__e_e_eth_t__h__e_ette__e_t__h__the__t_t_t__he__thet__e__
__e__t__e__e__ette__t_the__e_the_e__e__e__he_e__e_te__
_the_e__e__e_te_e_t__ethe_e__he_e__e__
```

I chose "the" because this is the most common trigram in the english alphabet. Now let's look at the second most common trigram: "and". In our ngram program, it says the second most common trigram is "rmt". Let's try replacing "rmt" with "and". Here is the resulting text:

```
tha_a_the__ante_tth_t__en__te__andthe__tat_ta_n_n__nd__t__ta_n__h__e__e__e__a_he_e_t
he_e__de_n_a_e__ae_eth_t__n__h__e_ette__n_e_t_n__and__nh__the__a__tat_ta_n_a__he__thet_a__n_e__
__e__and__t_n_e__e__ette__ant_the__e_the_e__anand_ne__a__a__he_ea__ente_an
dthe_e__and_e__enten_e_t__ethe_e__he_e__ae__
```

However, this doesn't work because if you look at the first few letters, there are very little to no words that start with 'tha' and don't also have a 't', 'a', 'h', 'e', 'n', or 'd'. The only ones I can think of that could fit here, is "that", but it doesn't fit these conditions still. So let's get rid of "and" as rmt. Let's keep trying different bigrams. So we know that the is already in the resulting text. We know that 'g' maps to 't', 's' maps to 'h' and 'v' maps to 'e'. The next most common bigram in our ciphertext is 'vi'. Since we know that v = e, let's find the most common bigram in English alphabet that starts with e: "er". Let's try replacing 'i' with 'r'. This is the resulting text:

```
th__the__te_tth_t__e__r__te__the__t_t_t__r__t__t__h__e__e__e__here_t
he_e__e__e_e_eth_t__h__e_etter__ert__h__the__t_t_t__her__r__thetr__err
_r_r__e__r__t__e__e__etter__t_the__e_there__e__r__here__e_ter__
_the_e__e__e_te_e_t__ethe_e__here__er__
```

You can see how you can read a couple of "there"s and "here"s. I think this is correct now that we have a couple words other than "the". Now if we look at the most bigrams again, we get the table:

sv	gs	vi	rm
13	12	11	9

We know that "the" is "gsv", so we can essentially figure out:

he	th	e'i'	"rm"
13	12	11	9

We know that "in" is the next most frequently found bigram, so let's substitute "rm" with "in" because we know that the first letter of "vi" does not start with an 'i', but with an 'e'. This is the resulting text:

```
thi_i_the__inte_tth_t__en_r__te__in_the__tit_ti_n_n__n_r_t__ti_n__h__e__e__e__e_i_here_t
he_e__e__e_n_i_e_ie_eth_t__n__h__e_etter_n_ert_n_in__nh__the__i__tit_ti_n_i__her__r__thetri__n_err
```

```
_r_r_e____rin__t_n_e_e_etter____int_the____e_there__inin_ne__i____i____r_herei__enterin
_the_e_nin__e__enten_e_t____ethe_e_i_here__ier
```

Now we're getting somewhere. We have a lot of words we can make out now: "there", "here", "the", "in," and even "enter". Let's go back to unigrams. The most next unused most common letter is 'h' = 28. The most common letters in the english alphabet are: 'E', 'T', 'O', 'A', 'I', 'N', 'S', 'H', ... according to the slides. However, we already used 'E', 'T', 'N', 'I', and 'H' so we're left with 'O', 'A', 'S'. Let's plug each one of them for 'h' to see if anything else makes sense:

o: here we have words like "theo"

```
thioiothe__inte_tth_t_oen_r_te__oin_theo__otit_ti_n_n__n_r_t__ti_no__h_eo__eoo____e_i_here_t
he_eoo__e_n_i_e_ie_eth_t__n_h_e_etter_n_ert_n_in_nh__the__oi_o__otit_ti_n_i_her__r_othetri__n_err
_r_r_eoo__o__rin__t_n_e_e_ettero____int_the____eothere__inin_neo_i____o__i____r_herei__enterin
_the_e_nin__eoooenten_eot____ethe_e_i_here_oier
```

a: here we have words like "eat"

```
thiaiathe__inte_tth_t_aen_r_te__ain_thea__atit_ti_n_n__n_r_t__ti_na__h_ea__eaa____e_i_here_t
he_eaa__e_n_i_e_ie_eth_t__n_h_e_etter_n_ert_n_in_nh__the__ai_a__atit_ti_n_i_her__r_athetri__n_err
_r_r_eaa__a__rin__t_n_e_e_ettera____int_the____eathere__inin_nea_i____a__i____r_herei__enterin
_the_e_nin__eaaaenten_eat____ethe_e_i_here_aier
```

s: here we have words like "this", "is", "sentenc_e", "s__stit_tio_n"

```
thisisthe__inte_tth_t_sen_r_te__sin_thes__stit_ti_n_n__n_r_t__ti_ns__h_es__ess____e_i_here_t
he_ess__e_n_i_e_ie_eth_t__n_h_e_etter_n_ert_n_in_nh__the__si_s__stit_ti_n_i_her__r_sthetri__n_err
_r_r_ess__s__rin__t_n_e_e_etters____int_the____esthere__inin_nes_i____s__i____r_herei__enterin
_the_e_nin__esssnten_est____ethe_e_i_here_sier
```

We can see that the letter that makes the most sense is most likely 's'.

We can make out a few words and infer that senten_e is likely sentence, and s__stit_ti_n is most likely substitution.

Let's make this happen by replacing the respective letters: x -> c, f -> u, y -> b. This is the resulting text:

```
thisisthe__inte_tth_t_sencr__te_usin_thesubstituti_n_n_c_n_r_tu__ti_ns__uh_esuccess_u____eci_here_t
he_ess__e_n_ib_e_ie_eth_t__un_h_ebetterun_ert_n_in_nh__theb_sicsubstituti_nci_her__r_sthetri__n_err
_r_r_ess__sb_rin__butnce_e_etters____int_the____cesthere__inin_nes_i____s__uic____r_herei__enterin
_the_e_nin__esssntencest____ethe_eci_here_sier
```

we can vaguely make out the words "substitution" "cipher", so we can infer that l -> o and k -> p. Let's try this and see what we can as the resulting text:

```
thisisthep__inte_tth_t_sencr_pte_usin_thesubstitution_n_con_r_tu__tions_ouh_esuccess_u____eciphert
he_ess__e_n_ib_e_ie_eth_t_ouno_h_ebetterun_ert_n_in_onho_theb_sicsubstitutionncipher_or_sthetri__n_err
orprocess__sborin__butonce_e_etters____intothe__cesthere__inin_ones_i____oso_uic____ro_herei__enterin
_the_e_nin__esssntencesto____ethe_eciphertsier
```

we can infer that "b_sic" is "basic", so let's replace the corresponding letter (z -> a). This is the resulting text:

```
thisisthep_a__inte_tthat__asencr_pte_usin_thesubstitutionan_con_ratu__ations_ouha_esuccess_u____eciphert
he_essa_ean_ib_e_ie_ethat_ouno_ha_ebetterun_ertan_in_onho_thebasicsubstitutionncipher_or_sthetria__n_err
orprocess__asborin__butoncea_e_etters_a__intothe__acesthere__ainin_ones_i____oso_uic____ro_hereia__enterin
_the_eanin__esssntencesto_a__ethe_eciphertasier
```

We can see "encrpye" goes to encrypted. We can vaguely make out "con_ratu__ations" to congratulations. Therefore, b -> y, t -> g, and o -> l. This is the resulting text:

thisistheplainte_tthat_asencrypte_usingthesubstitutionan_congratulationsyouha_esuccess_ully_eciphere_t
 he_essagean_ibelie_ethatyouno_ha_ebetterun_ertan_ingonho_thebasicsubstitutioncipher_or_sthetrialan_err
 orprocess_asboringbutoncea_e_letters_allintotheplacesthere_ainingones_ill_oso_uic_ly_ro_hereia_enterin
 gthe_eaninglesssentencesto_a_ethe_eciphereasier

You can make out a lot from this text.

thisistheplaintexttthatwasencryptedusingthesubstitutionandcongratulationsyouhavesuccessfullydecipheredt
 hemessageandibelievethatyounowhavebetterundertandingonhowthebasicsubstitutioncipherworksthetrialanderr
 orprocesswasboringbutonceafewlettersfallintotheplacestheremainingoneswilldosomuchmorequicklyfromhereiamenterin
 gthmeaninglesssentences tomakethedeciphereasier

The message is certainly true. The more letters you figure out, the easier it becomes to decipher the message.