# Magnetism as a Consequence of Special Relativity

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## 1 Introduction

Historically, Maxwell's equations, which are derived empirically, paved the way for the discovery of special relativity. Later, it was discovered that relativity is remarkably compatible with Maxwell's equations and classical electromagnetism. However, it is shown here that explaining classical electromagnetism backward from special relativity can lead to more intuitive results, providing clear motivations for counterintuitive concepts such as the Lorentz force law and displacement current. The following is a comprehensive derivation of the classical laws of electromagnetism, which includes the Lorentz force law, Maxwell-Ampère law, and Faraday's law, from a basic set of postulates of special relativity and electrostatics.

## 2 Lorentz transformation

First, a source charge moving with velocity $v$ along the x-axis, and a target particle with charge $q$ moving with velocity $\mathbf{u}$ in an arbitrary direction, is considered. The reference frame where the source charge appears to be stationary is $S'$, and the lab frame is $S$. The goal is to analyze how the electric force exerted on the target charge by the source transforms between $S$ and $S'$. To get started, a Lorentz transformation is performed on the 4-momentum of the particle, which yields:

$$\begin{aligned} E' &= \gamma(E - vp_x) \\ p'_x &= \gamma\left(p_x - \frac{E}{c^2}v\right) \\ p'_y &= p_y \\ p'_z &= p_z \end{aligned} \tag{1}$$

Where $\gamma$ is the Lorentz factor. Differentiating the momentum with respect to $t$ gives:

$$\begin{aligned}
\frac{\mathrm{d}p'_x}{\mathrm{d}t} &= \gamma\Big(F_x - \frac{v}{c^2}\mathbf{F}\cdot\mathbf{u}\Big) \\
\frac{\mathrm{d}p'_y}{\mathrm{d}t} &= F_y \\
\frac{\mathrm{d}p'_z}{\mathrm{d}t} &= F_z
\end{aligned}$$

And since time transforms as:

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \gamma\Big(1 - \frac{vu_x}{c^2}\Big)$$

Using the chain rule on the LHS of the force equation:

$$\begin{aligned}
&F'_x\Big(1 - \frac{vu_x}{c^2}\Big) = F_x - \frac{v}{c^2}\mathbf{F}\cdot\mathbf{u} \\
&\gamma F'_y\Big(1 - \frac{vu_x}{c^2}\Big) = F_y \\
&\gamma F'_z\Big(1 - \frac{vu_x}{c^2}\Big) = F_z
\end{aligned}$$

Expanding out the dot product in the $x$ component and rearranging:

$$\begin{aligned}
F_x &= F'_x + \gamma\frac{v}{c^2}(F'_y u_y + F'_z u_z) \\
F_y &= \gamma F'_y\Big(1 - \frac{vu_x}{c^2}\Big) \\
F_z &= \gamma F'_z\Big(1 - \frac{vu_x}{c^2}\Big)
\end{aligned} \tag{2}$$

This is the relativistic force transformation equation where $\mathbf{F}$ is the force observed from the lab frame, and $\mathbf{F}'$ is the force observed from the source charge's frame.

# 3 Origins of magnetism

The force transformation equation predicts additional forces that are only exerted on moving objects; This is precisely the magnetic force. Thus, $\mathbf{F}$ can be decomposed into two components, one due to the static electric field, and a secondary velocity-dependent magnetic term:

$$\begin{aligned}
\mathbf{F} &= \mathbf{F_E} + \mathbf{F_B} \\
\mathbf{F_E} &= q\mathbf{E} \\
\mathbf{F_B} &\propto \mathbf{u}
\end{aligned} \tag{3}$$

Now because $\mathbf{B}'$ is assumed to be zero in this scenario, $\mathbf{F}'$ only contains the electric component: $\mathbf{F}' = q\mathbf{E}'$. Thus decomposing Eq. (2) into its components yields:

$$\begin{aligned}
F_B x + qE_x &= qE'_x + \gamma\frac{v}{c^2}(qE'_y u_y + qE'_z u_z) \\
F_B y + qE_y &= \gamma q E'_y\Big(1 - \frac{vu_x}{c^2}\Big) \\
F_B z + qE_z &= \gamma q E'_z\Big(1 - \frac{vu_x}{c^2}\Big)
\end{aligned} \tag{4}$$

Since $\mathbf{F_B} = \mathbf{0}$ for $\mathbf{u} = \mathbf{0}$, plugging in $\mathbf{u} = \mathbf{0}$ and canceling gives:

$$\begin{aligned}
E_x &= E'_x \\
E_y &= \gamma E'_y \\
E_z &= \gamma E'_z
\end{aligned} \tag{5}$$

Substituting this back into Eq. (4) and canceling:

$$\begin{aligned}
F_{Bx} &= \gamma\frac{v}{c^2}(qE'_y u_y + qE'_z u_z) \\
F_{By} &= -\gamma\frac{vu_x}{c^2} qE'_y \\
F_{Bz} &= -\gamma\frac{vu_x}{c^2} qE'_z
\end{aligned} \tag{6}$$

This is the precise formula for the magnetic component of the force. Now, to generalize this to an arbitrary boost, first $\mathbf{E}'$ is decomposed into components parallel and perpendicular to $\mathbf{v}$:

$$\begin{aligned}
\hat{\mathbf{v}} &= \frac{\mathbf{v}}{||\mathbf{v}||} \\
\mathbf{E}'_{\parallel} &= (\mathbf{E}' \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \\
\mathbf{E}'_{\perp} &= \mathbf{E}' - \mathbf{E}'_{\parallel}
\end{aligned} \tag{7}$$

Decomposing $\mathbf{F_B}$ following from Eq. (6) and (7):

$$\begin{aligned}
\mathbf{F}_{\mathbf{B}\parallel} &= \gamma\frac{q}{c^2}(\mathbf{E}'_{\perp} \cdot \mathbf{u})\mathbf{v} \\
\mathbf{F}_{\mathbf{B}\perp} &= -\gamma\frac{q}{c^2}(\mathbf{v} \cdot \mathbf{u})\mathbf{E}'_{\perp}
\end{aligned} \tag{8}$$

Adding back the components:

$$\begin{aligned}
\mathbf{F_B} &= \mathbf{F}_{\mathbf{B}\parallel} + \mathbf{F}_{\mathbf{B}\perp} \\
&= \gamma\frac{q}{c^2}[(\mathbf{E}'_{\perp} \cdot \mathbf{u})\mathbf{v} - (\mathbf{v} \cdot \mathbf{u})\mathbf{E}'_{\perp}] \\
&= \gamma\frac{q}{c^2}\mathbf{u} \times (\mathbf{v} \times \mathbf{E}'_{\perp}) \\
&= q\mathbf{u} \times (\gamma\frac{1}{c^2}\mathbf{v} \times \mathbf{E}')
\end{aligned}$$

$\mathbf{E}'_\perp$ and $\mathbf{E}'$ are interchangeable here since the cross product eliminates the parallel component of $\mathbf{E}'$. At this point, it feels reasonable to define a new vector quantity $\mathbf{B} = \gamma\frac{1}{c^2}\mathbf{v}\times\mathbf{E}'$ to make the calculations simpler. One can see that the $\mathbf{B}$ field is essentially a bookkeeping device used to simplify the transformation of the $\mathbf{E}$ field. Now, formulating force transformation in terms of $\mathbf{B}$ gives:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B}) \tag{9}$$

$$\mathbf{B} = \gamma\frac{1}{c^2}\mathbf{v}\times\mathbf{E}' \tag{10}$$

$$\mathbf{E} = \gamma\mathbf{E}'_\perp + \mathbf{E}'_\parallel \tag{11}$$

This is the exact Lorentz force law with the standard relativistic field transformation equations for $\mathbf{B}' = 0$. Note that equation (11) is a generalization of Eq. (5) to an arbitrary boost. Since $\gamma \approx 1$ for $v$ much smaller than the speed of light, the Lorentz factor can be ignored for low velocity approximations. And because $\mathbf{E}'$ is defined by Coulomb's law in $S'$, ignoring the electric field transformation and plugging in Coulomb's expression for the $\mathbf{E}$ field into Eq. (10) gives:

$$\begin{aligned}\mathbf{B} &= \frac{\mathbf{v}}{c^2}\times\left(\frac{1}{4\pi\epsilon_0}\frac{q\hat{\mathbf{r}}}{r^2}\right)\\ &= \frac{q\mathbf{v}}{4\pi\epsilon_0 c^2}\times\frac{\hat{\mathbf{r}}}{r^2}\end{aligned}$$

plugging in $\mu_0\epsilon_0 = \frac{1}{c^2}$:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\times\hat{\mathbf{r}}}{r^2} \tag{12}$$

Yields the Biot-Savart law for point charges. One can see that the Biot-Savart law is essentially a low velocity approximation for $\mathbf{B}$ where $\mathbf{E}' \approx \mathbf{E}$. In the next section, this is generalized to the Maxwell-Ampère law, which holds for any source velocity, including accelerating sources.

# 4 Maxwell's equations

The field transformation equations derived in Section 3 break down if the source velocity varies, since Lorentz transformations are only valid for inertial frames. This issue can be mitigated by considering the motion of individual electric field disturbances instead of the source. Since the update speed of the electric field is limited to the speed of light, one can think of every charge continuously emitting electric field signals that propagate radially outward. To analyze this, first, since it is no longer necessary to consider $\mathbf{E}'$, it is eliminated from eq. (10)

$$\mathbf{B} = \gamma\frac{1}{c^2}\mathbf{v}\times\left(\frac{\mathbf{E}_\perp}{\gamma}\right) = \frac{1}{c^2}\mathbf{v}\times\mathbf{E} \tag{13}$$

Since a Lorentz transform to a frame moving at the speed of light isn't valid, instead, a limit of the Eq. (13) as $\mathbf{v}$ approaches $c$ is taken:

$$\lim_{u\to c^-}\frac{1}{c^2}\mathbf{u}\times\mathbf{E} = \frac{1}{c}\,\hat{\mathbf{n}}\times\mathbf{E},$$

$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{n}} \times \mathbf{E} \tag{14}$$

The limit surprisingly converges to the Lienard-Wiechert solution for $\mathbf{B}$, where $\hat{\mathbf{n}}$ is the unit vector pointing from the position where the signal originates to the field point. This relation holds for sources with any arbitrary motion since the fields propagate at a constant velocity, free of the source's motion. Now working backwards, Maxwell-Ampère law can be derived from equation (14). To get started, Reynolds' transport theorem (RTT) Eq. (15), along with a secondary transport equation for vector fields Eq. (16), is used to describe the electric field signals.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho dV = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right] dV \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)} \mathbf{E}\cdot d\mathbf{S} = \int_{S(t)} \left[\frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{E})\right]\cdot d\mathbf{S} \tag{16}$$

Equation (16), similar to RTT, describes the rate of change of electric flux through a moving surface. Much like RTT, the RHS comprises two terms: the first term represents the variation of flux due to the internal variation of the field, which is given by the surface integral of the derivative of the field, and the second term accounts for the variation due to the movement of the surface. The second term arises when considering an infinitesimal area swept out by the boundary of the surface, which is:

$$\Delta\mathbf{A} = \oint \mathbf{v}dt \times d\mathbf{l}$$

The corresponding change in flux is:

$$\Delta\phi = \int_{\Delta A} \mathbf{E}\cdot d\mathbf{A} = \oint \mathbf{E}\cdot(\mathbf{v}dt \times d\mathbf{l}) = -\oint (\mathbf{v}dt\times\mathbf{E})\cdot d\mathbf{l}$$

Differentiating and applying Stokes' theorem:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\oint (\mathbf{v}\times\mathbf{E})\cdot d\mathbf{l} = -\int_{S(t)} \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{E})\cdot d\mathbf{S}$$

This essentially quantifies the flux flowing in and out of the boundaries. Now, if a charge is moving with velocity $\mathbf{v_q}$, applying RTT to an arbitrary volume moving with the charge gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho dV = \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v_q})\right] dV = 0$$

where $\rho$ is the charge density. This is zero since the charge enclosed is constant. Simplifying gives the standard current density continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v_q}) = 0 \tag{17}$$

Now doing the same for the surface transport theorem, where the surface is moving with the electric field signals:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)} \mathbf{E}\cdot d\mathbf{S} = \int_{S(t)} \left[\frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times(\hat{\mathbf{n}}c\times\mathbf{E})\right]\cdot d\mathbf{S}$$

Applying the divergence theorem and canceling the volume integrals:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\nabla}\cdot\mathbf{E} = \boldsymbol{\nabla}\cdot\left[\frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times(\hat{\mathbf{n}}c\times\mathbf{E})\right]$$

Plugging in Maxwell's first equation:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \tag{18}$$

$$\frac{1}{\epsilon_0}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \boldsymbol{\nabla}\cdot\left[\frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times(\hat{\mathbf{n}}c\times\mathbf{E})\right]$$

Plugging in equation (17):

$$\begin{aligned}
-\frac{1}{\epsilon_0}\boldsymbol{\nabla}\cdot(\rho\mathbf{v}_{\mathbf{q}}) &= \boldsymbol{\nabla}\cdot\left[\frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times(\hat{\mathbf{n}}c\times\mathbf{E})\right]\\
-\frac{1}{\epsilon_0}\rho\mathbf{v}_{\mathbf{q}} &= \frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla}\times(\hat{\mathbf{n}}c\times\mathbf{E})\\
\boldsymbol{\nabla}\times\left(\frac{1}{c}\hat{\mathbf{n}}\times\mathbf{E}\right) &= \frac{1}{c^2}\left(\frac{1}{\epsilon_0}\rho\mathbf{v} + \frac{\partial \mathbf{E}}{\partial t}\right)
\end{aligned}$$

Plugging in equation (14), $\epsilon_0\mu_0 = \frac{1}{c^2}$, and $\mathbf{J} = \rho\mathbf{v}_{\mathbf{q}}$

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)$$

Yields the exact Maxwell-Ampère law. The same procedure can be used to derive Faraday's law from:

$$\mathbf{E} = -\hat{\mathbf{n}}c\times\mathbf{B} \tag{19}$$

This follows from the Lorentz force law. Applying RTT and surface transport yields:

$$\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{20}$$

Notice there is no magnetic current term since:

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{21}$$

One can see that Maxwell's third and fourth equations are essentially advection PDEs for the electromagnetic field, which explains why the wave equation follows naturally from them.

## 5 Conclusion

The fundamental equations of magnetism were shown to follow from the basic postulates of electrostatics and special relativity. The **B** field was shown to arise naturally from the Lorentz transformation of the **E** field, and Faraday's law and Maxwell-Ampère law were able to be derived from field transformations, along with Maxwell's first and second equations alone. Thus, one can conclude that magnetism is a derived phenomenon rather than a fundamental one. This provides a new framework for viewing magnetism, explaining displacement currents as a byproduct of field propagation and describing Maxwell's third and fourth equations as advection equations for electromagnetic field updates.