Probability Theory II (Fall 2016)

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Preface & Disclaimer

This note is a summary of the lecture Probability Theory II (326.516) held at Seoul National University, Fall 2016. Lecturer was S.Y.Lee, and the note was summarized by J.P.Kim, who is a Ph.D student. There are few textbooks and references in this course, which are following.

- *Probability: Theory and Examples, R.Durrett*

Also I referred to following books when I write this note. The list would be updated continuously.

- *Probability and Measures, P.Billingsley, 1995.*
- *Convergence in Probability Measures, P.Billingsley, 1999.*
- *Lecture notes on Financial Mathematics I & II (in course), Gerald Trutnau, 2015.*
- *Lecture notes on Topics in Mathematics I (in course), Gerald Trutnau, 2015.*
- *Lecture notes on Introduction to Stochastic Differential Equations (in course), Gerald Trutnau, 2015.*

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Chapter 1

Central Limit Theorems

In this chapter, we prove Central Limit Theorems in various cases, and find sufficient or necessary conditions to CLT be held.

1.1 i.i.d. case

Following lemma is very useful in our story.

Lemma 1.1.1. *Let X be a random variable with $E|X|^n < \infty$ and $\varphi(t) = Ee^{itX}$ be its characteristic function. Then*

$$\left|\varphi(t) - \sum_{k=0}^{n} \frac{(it)^k EX^k}{k!}\right| \le E \min\left(\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\right).$$

Proof. Note that, by Taylor's theorem, there exists ξ between 0 and x such that

$$e^{ix} = \sum_{k=0}^{n} \frac{(ix)^k}{k!} + \frac{(ix)^{n+1}}{(n+1)!} e^{i\xi},$$

so we can obtain that

$$\left|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Similarly, there exists ξ' between 0 and x such that

$$e^{ix} = \sum_{k=0}^{n-1} \frac{(ix)^k}{k!} + \frac{(ix)^n}{n!} e^{i\xi'} = \sum_{k=0}^{n} \frac{(ix)^k}{k!} + \frac{(ix)^n}{n!} e^{i\xi'} - \frac{(ix)^n}{n!} e^{ix},$$

so

$$\left|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}\right| \le \frac{2|x|^n}{n!}$$

holds. Thus, we get

$$\left|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}\right| \le \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\right),$$

and put tX into x then we get

$$\left|e^{itX} - \sum_{k=0}^{n} \frac{(itX)^k}{k!}\right| \le \min\left(\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\right).$$

Therefore, by Jensen $|EX| \le E|X|$ we get

$$\left|\varphi(t) - \sum_{k=0}^{n} \frac{(it)^k EX^k}{k!}\right| \le E\left|e^{itX} - \sum_{k=0}^{n} \frac{(it)^k X^k}{k!}\right| \le E\min\left(\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\right).$$

□

Corollary 1.1.2. *For a random variable such that* $EX = 0$ *and* $EX^2 = \sigma^2$,

$$\varphi(t) = 1 - \frac{t^2\sigma^2}{2} + o(|t|^2)$$

as $t \approx 0$.

Proof. Note that, if $E|X|^n < \infty$, by LDCT,

$$E\min\left(\frac{|t||X|^{n+1}}{(n+1)!}, \frac{2|X|^n}{n!}\right) \xrightarrow[|t|\to 0]{} 0$$

holds, so

$$E\min\left(\frac{|tX|^{n+1}}{(n+1)!}, \frac{2|tX|^n}{n!}\right) = o(|t|^n)$$

and hence

$$\varphi(t) = \sum_{k=0}^{n} \frac{(it)^k EX^k}{k!} + o(|t|^n).$$

Now consider a special case $n = 2$, then

$$\varphi(t) = 1 - \frac{\sigma^2 t^2}{2} + o(|t|^2)$$

is obtained, because $EX = 0$. □

Theorem 1.1.3 (CLT for i.i.d. case)**.** *Let $X_1, \cdots, X_n$ be i.i.d. random variables such that $EX_1 = 0$ and $EX_1^2 = \sigma^2 > 0$. Then, for $S_n = X_1 + X_2 + \cdots + X_n$,*

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{d} N(0,1).$$

Proof. Let $\varphi(t) = Ee^{itX_1}$ be a characteristic function of X_1. Then characteristic function of $\dfrac{S_n}{\sigma\sqrt{n}}$ is

$$\begin{aligned}
\varphi_{S_n/\sigma\sqrt{n}}(t) &= Ee^{it\frac{S_n}{\sigma\sqrt{n}}} \\
&= \left[\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n \\
&= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{\sigma^2 n}\right)\right]^n \\
&= \left[1 - \frac{t^2}{2n} + o(n^{-1})\right]^n.
\end{aligned}$$

Note that in here t is fixed, but $\dfrac{t}{\sigma\sqrt{n}} \approx 0$. Also note that, for a sequence c_n such that $nc_n \xrightarrow[n\to\infty]{} c$,

$$\lim_{n\to\infty} (1 + c_n)^n = e^c$$

holds. Therefore,

$$\varphi_{S_n/\sigma\sqrt{n}}(t) = \left[1 - \frac{t^2}{2n} + o(n^{-1})\right]^n \xrightarrow[n\to\infty]{} e^{-t^2/2},$$

and by Lévy's continuity theorem, we get the conclusion. □

1.2 Double arrays

Definition 1.2.1 (Lindeberg's condition)**.** *Let $\{X_{nk} : k = 1, 2, \cdots, r_n\}$ be a double array of r.v.'s where $r_n \to \infty$ with*

1. *$X_{n1}, X_{n2}, \cdots, X_{nr_n}$ are independent.*
2. *$EX_{nk} = 0$ for $k = 1, 2, \cdots, r_n$.*
3. *$EX_{nk}^2 < \infty$.*

Then $\{X_{nk}\}$ is said to satisfy Lindeberg's condition if

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{r_n} \int_{|X_{nk}|\ge \epsilon s_n} X_{nk}^2 d\mathbb{P} = 0 \ \forall \epsilon > 0$$

where $s_n^2 = \sigma_{n1}^2 + \cdots + \sigma_{nr_n}^2 = Var(X_{n1} + \cdots + X_{nr_n})$ *and* $Var(X_{nk}) = \sigma_{nk}^2$.

Theorem 1.2.2. *Let* $S_n = X_{n1} + \cdots + X_{nr_n}$, *where notations are those of definition 1.2.1. Then under Lindeberg's condition,*

$$\frac{S_n}{s_n} \xrightarrow[n\to\infty]{d} N(0,1).$$

Remark 1.2.3. Note that 2nd assumption in Lindeberg's condition is just for convenience. Also, this theorem and Lindeberg condition say that tail behavior (when $|X_{nk}| \ge \epsilon s_n$) of random variables are important for central convergence. If the distribution of r.v.'s has heavy tail and so X_{nk} can have extreme values, summation may not cancel out extreme effects.

Proof. WLOG we assume $s_n^2 = 1$. Put $\varphi_n(t) = Ee^{itS_n}$ and $\varphi_{nk}(t) = Ee^{itX_{nk}}$, then

$$\varphi_n(t) = \prod_{k=1}^{r_n} \varphi_{nk}(t)$$

holds. Now our goal is to show that:

Claim. $\varphi_n(t) \to e^{-t^2/2}$

Note that for two sequences w_i and z_i of complex numbers, if $|w_i|, |z_i| \le 1$, then

$$\left| \prod_{i=1}^m w_i - \prod_{i=1}^m z_i \right| \le \sum_{i=1}^m |w_i - z_i|$$

by induction on m. Thus,

$$\begin{aligned}
|\varphi_n(t) - e^{-t^2/2}| &\overset{s_n^2=1}{=} \left|\varphi_n(t) - e^{-\frac{t^2}{2}\sum_{k=1}^{r_n}\sigma_{nk}^2}\right| \\
&= \left| \prod_{k=1}^{r_n} \varphi_{nk}(t) - \prod_{k=1}^{r_n} e^{-\frac{t^2}{2}\sigma_{nk}^2} \right| \\
&\le \sum_{k=1}^{r_n} \left| \varphi_{nk}(t) - e^{-\frac{t^2}{2}\sigma_{nk}^2} \right| \\
&\le \underbrace{\sum_{k=1}^{r_n} \left| \varphi_{nk}(t) - \left(1 - \frac{t^2}{2}\sigma_{nk}^2\right) \right|}_{=:A_n} + \underbrace{\sum_{k=1}^{r_n} \left| 1 - \frac{t^2}{2}\sigma_{nk}^2 - e^{-\frac{t^2}{2}\sigma_{nk}^2} \right|}_{=:B_n}
\end{aligned}$$

holds. Now by lemma 1.1.1,

$$\left| \varphi_{nk}(t) - \left(1 - \frac{t^2}{2}\sigma_{nk}^2\right) \right| \le E\min(|tX_{nk}|^3, |tX_{nk}|^2)$$

holds, so

$$\begin{aligned}
A_n &\le \sum_{k=1}^{r_n} E\min\left(|tX_{nk}|^3, |tX_{nk}|^2\right) \\
&= \sum_{k=1}^{r_n} \int \min\left(|tX_{nk}|^3, |tX_{nk}|^2\right) d\mathbb{P} \\
&\underset{(*)}{\le} \sum_{k=1}^{r_n} \int_{|X_{nk}|<\epsilon} |tX_{nk}|^3 d\mathbb{P} + \sum_{k=1}^{r_n} \int_{|X_{nk}|\ge\epsilon} |tX_{nk}|^2 d\mathbb{P} \\
&\le \sum_{k=1}^{r_n} \int |t|^3\epsilon|X_{nk}|^2 d\mathbb{P} + \sum_{k=1}^{r_n} \int_{|X_{nk}|\ge\epsilon s_n} |tX_{nk}|^2 d\mathbb{P} \\
&= \underbrace{\sum_{k=1}^{r_n} |t|^3\epsilon\sigma_{nk}^2}_{=|t|^3\epsilon} + \underbrace{\sum_{k=1}^{r_n} \int_{|X_{nk}|\ge\epsilon s_n} |tX_{nk}|^2 d\mathbb{P}}_{\xrightarrow[n\to\infty]{} 0 \text{ (Lindeberg)}}
\end{aligned}$$

holds for sufficiently small $\epsilon > 0$. Letting $\epsilon \searrow 0$ we get $A_n \xrightarrow[n\to\infty]{} 0$ (For (*), see next remark). Next, note that,

$$\begin{aligned}
\sigma_{nk}^2 &= \int_{|X_{nk}|<\epsilon} X_{nk}^2 d\mathbb{P} + \int_{|X_{nk}|\ge\epsilon} X_{nk}^2 d\mathbb{P} \\
&\le \epsilon^2 + \int_{|X_{nk}|\ge\epsilon} X_{nk}^2 d\mathbb{P}
\end{aligned}$$

so

$$\max_{1\le k\le r_n} \sigma_{nk}^2 \le \epsilon^2 + \underbrace{\sum_{k=1}^{r_n} \int_{|X_{nk}|\ge\epsilon} X_{nk}^2 d\mathbb{P}}_{\xrightarrow[n\to\infty]{} 0}$$

holds. It implies that,

$$\frac{\max_k \sigma_{nk}^2}{s_n^2} \xrightarrow[n\to\infty]{} 0. \tag{1.1}$$

Now note that $\exists K > 0$ such that $|e^x - (1+x)| \le K|x|^2$ if $|x| \le 1$ (For this, see next remark). Thus

$$\begin{aligned}
B_n &\le K \sum_{k=1}^{r_n} \left(\frac{t^2}{2}\sigma_{nk}^2\right)^2 \\
&= K \cdot \frac{t^4}{4} \sum_{k=1}^{r_n} \sigma_{nk}^4 \\
&\le K \cdot \frac{t^4}{4} \max_{1\le k'\le r_n} \sigma_{nk'}^2 \sum_{k=1}^{r_n} \sigma_{nk}^2
\end{aligned}$$

$$= K \cdot \frac{t^4}{4} \max_{1 \le k' \le r_n} \sigma_{nk'}^2 \xrightarrow[n\to\infty]{} 0$$

holds, and it implies the conclusion. □

Remark 1.2.4.

(a) In (*), following fact is used. Note that $\min(|x|^3, |x|^2) = |x|^3$ if $|x| < 1$, and $= |x|^2$ otherwise. Thus if $\epsilon < 1/t$, we get

$$|tx|^3 I(|x| < \epsilon) + |tx|^2 I(|x| \ge \epsilon) \ge \min(|tx|^3, |tx|^2).$$

For this, see figure 1.1.

(b) Note that $\frac{|e^x - (1+x)|}{|x^2|}$ converges as $|x| \to 0$, so

$$\left\{ \frac{|e^x - (1+x)|}{|x^2|} : |x| \le 1 \right\}$$

is a bounded set. Thus there exists $K > 0$ such that $|e^x - (1+x)| \le K|x|^2$ if $|x| \le 1$.

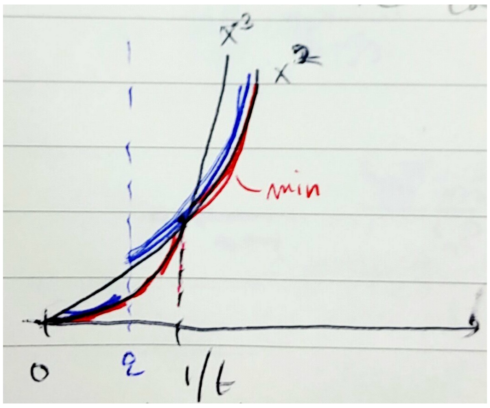

Figure 1.1: The graph of $\min(|tx|^3, |tx|^2)$.

Definition 1.2.5 (Lyapunov's condition)**.** *Let $\{X_{nk}\}$ be a double array such that $X_{n1}, \cdots, X_{nr_n}$ are independent. $\{X_{nk}\}$ satisfies Lyapunov condition if for some $\delta > 0$,*

(a) $EX_{nk} = 0$

(b) $E|X_{nk}|^{2+\delta} < \infty$

(c) $\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{E|X_{nk}|^{2+\delta}}{s_n^{2+\delta}} = 0.$

Proposition 1.2.6. *Lyapunov condition implies Lindeberg condition.*

Proof.

$$\begin{aligned}\sum_{k=1}^{r_n}\frac{1}{s_n^2}\int_{|X_{nk}|\ge\epsilon s_n}1\cdot X_{nk}^2d\mathbb{P} &\le \sum_{k=1}^{r_n}\frac{1}{s_n^2}\int_{|X_{nk}|\ge\epsilon s_n}\left(\frac{|X_{nk}|}{\epsilon s_n}\right)^{\delta}\cdot X_{nk}^2d\mathbb{P}\\ &=\sum_{k=1}^{r_n}\frac{1}{s_n^{2+\delta}}\int_{|X_{nk}|\ge\epsilon s_n}\frac{|X_{nk}|^{2+\delta}}{\epsilon^{\delta}}d\mathbb{P}\\ &\le\sum_{k=1}^{r_n}\frac{E|X_{nk}|^{2+\delta}}{s_n^{2+\delta}}\frac{1}{\epsilon^{\delta}}\xrightarrow[n\to\infty]{\text{Lyapunov}}0.\end{aligned}$$

□

We showed that Lindeberg condition implies CLT. However, next example says that converse does not hold.

Example 1.2.7. Let $\sigma_1^2>0$ be a real number and $\sigma_n^2=\sigma_1^2+\cdots+\sigma_{n-1}^2$ for $n=2,3,\cdots$. Let $X_n\sim N(0,\sigma_n^2)$, and note that $s_n^2=\sigma_1^2+\cdots+\sigma_n^2=2\sigma_n^2$. Then

$$\frac{X_1+\cdots+X_n}{s_n}\sim N(0,1)$$

so CLT holds. But for $Z\sim N(0,1)$,

$$\begin{aligned}\frac{1}{s_n^2}\sum_{k=1}^{n}\int_{|X_k|>\epsilon s_n}X_k^2d\mathbb{P} &\ge \int_{|X_k|>\epsilon s_n}\left(\frac{X_n}{s_n}\right)^2d\mathbb{P}\\ &=\int_{|X_n|/\sigma_n>\sqrt{2}\epsilon}\frac{1}{2}\left(\frac{X_n}{\sigma_n}\right)^2\\ &=\frac{1}{2}E[Z^2I(Z>\sqrt{2}\epsilon)]\end{aligned}$$

so Lindeberg condition does not hold.

Now our interest is: what is an equivalent condition for CLT? Fortunately, following Feller's theorem is well known.

Theorem 1.2.8 (Feller's theorem)**.** *Lindeberg condition* $\Leftrightarrow$ *CLT* $+\left[\frac{\max_{1\le k\le r_n}\sigma_{nk}^2}{s_n^2}\xrightarrow[n\to\infty]{}0\right]$.

Proof. $\Rightarrow$ part was already done. To show $\Leftarrow$ part, WLOG $s_n^2=1$. By the CLT,

$$\prod_{k=1}^{r_n}\varphi_{nk}(t)\xrightarrow[n\to\infty]{}e^{-t^2/2}$$

holds, where $\varphi_{nk}(t) = Ee^{itX_{nk}}$. Recall that: since $EX_{nk} = 0$ and $EX_{nk}^2 = \sigma_{nk}^2$, by lemma 1.1.1,

$$|\varphi_{nk}(t) - 1| \le t^2\sigma_{nk}^2$$

holds, so

$$\max_{1\le k\le r_n} |\varphi_{nk}(t) - 1| \le \max_{1\le k\le r_n} t^2\sigma_{nk}^2 \xrightarrow[n\to\infty]{} 0$$

is obtained. Meanwhile, note that

$$|e^z - 1 - z| \le K|z|^2 \ \forall z \ s.t.\ |z| \le 2$$

holds for some K. Hence, we get

$$\begin{aligned}
\sum_{k=1}^{r_n} \left| e^{\varphi_{nk}(t)-1} - 1 + 1 - \varphi_{nk}(t) \right| &\le K \sum_{k=1}^{r_n} |\varphi_{nk}(t) - 1|^2 \\
&\le K \max_{1\le k\le r_n} |\varphi_{nk}(t) - 1| \underbrace{\sum_{k'=1}^{r_n} |\varphi_{nk'}(t) - 1|}_{\le t^2} \\
&\le Kt^4 \max_{1\le k\le r_n} \sigma_{nk}^2 \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Now since $|e^z| = e^{\mathcal{R}ez} \le e^{|z|}$,

$$\left| e^{\varphi_{nk}(t)-1} \right| \le e^{-1}e^{|\varphi_{nk}(t)|} < 1$$

holds, so by lemma,

$$\left| e^{\sum_{k=1}^{r_n}(\varphi_{nk}(t)-1)} - \prod_{k=1}^{r_n} \varphi_{nk}(t) \right| \le \sum_{k=1}^{r_n} \left| e^{\varphi_{nk}(t)-1} - \varphi_{nk}(t) \right| \xrightarrow[n\to\infty]{} 0$$

is obtained. Thus by CLT, we get

$$e^{\sum_{k=1}^{r_n}(\varphi_{nk}(t)-1)} \xrightarrow[n\to\infty]{} e^{-t^2/2},$$

which implies

$$\left| e^{\sum_{k=1}^{r_n}(\varphi_{nk}(t)-1)} \right| \xrightarrow[n\to\infty]{} \left| e^{-t^2/2} \right| = e^{-t^2/2}.$$

Note that

$$|e^z| = \left| e^{\mathcal{R}e(z) + i\mathcal{I}m(z)} \right| = e^{\mathcal{R}e(z)}$$

holds, so it implies that

$$e^{\mathcal{R}e(\sum_{k=1}^{r_n}(\varphi_{nk}(t)-1))} \xrightarrow[n\to\infty]{} e^{-t^2/2},$$

and hence

$$\mathcal{R}e\left(\sum_{k=1}^{r_n}(\varphi_{nk}(t)-1)\right) \xrightarrow[n\to\infty]{} -\frac{t^2}{2}$$

holds. Thus,

$$\mathcal{R}e\left(\sum_{k=1}^{r_n}(\varphi_{nk}(t)-1)\right)+\frac{t^2}{2}=\sum_{k=1}^{r_n}(E\cos tX_{nk}-1)+\frac{t^2}{2} \xrightarrow[n\to\infty]{} 0.$$

Now, since $EX_{nk}^2=\sigma_{nk}^2$, and by our assumption, it is equivalent to

$$\sum_{k=1}^{r_n} E\left(\cos tX_{nk}-1+\frac{t^2}{2}X_{nk}^2\right) \xrightarrow[n\to\infty]{} 0.$$

Note that for any real number y, $\cos y - 1 + y^2/2 \ge 0$ holds. Therefore,

$$\begin{aligned}
\sum_{k=1}^{r_n} E\underbrace{\left(\cos tX_{nk}-1+\frac{t^2}{2}X_{nk}^2\right)}_{\ge 0} &\ge \sum_{k=1}^{r_n} E\left(\underbrace{\cos tX_{nk}-1}_{\ge -2}+\frac{t^2}{2}X_{nk}^2\right) I(|X_{nk}|\ge\epsilon)\\
&\ge \sum_{k=1}^{r_n} E\left(\frac{t^2}{2}X_{nk}^2 I(|X_{nk}|\ge\epsilon) - \underbrace{2I(|X_{nk}|\ge\epsilon)}_{\le 2X_{nk}^2\epsilon^{-2}I(|X_{nk}|\ge\epsilon)}\right)\\
&\ge \left(\frac{t^2}{2}-\frac{2}{\epsilon^2}\right)\sum_{k=1}^{r_n} EX_{nk}^2 I(|X_{nk}|\ge\epsilon)
\end{aligned}$$

holds for any arbitrarily given $\epsilon > 0$. Letting t such that $\dfrac{t^2}{2}-\dfrac{2}{\epsilon^2}>0$, we get

$$\sum_{k=1}^{r_n} EX_{nk}^2 I(|X_{nk}|\ge\epsilon).$$

□

1.3 Poisson convergence

Theorem 1.3.1. *For each n, X_{nm} are independent r.v.'s with $P(X_{nm}=1)=p_{nm}$ and $P(X_{nm}=0)=1-p_{nm}$. Assume that*

(i) $\sum_{m=1}^{n} p_{nm} \to \lambda \in (0,\infty)$

(ii) $\max_{1\le m\le n} p_{nm} \xrightarrow[n\to\infty]{} 0$

Then $S_n := X_{n1} + \cdots + X_{nm} \xrightarrow[n\to\infty]{d} Poi(\lambda)$.

Proof. Let $\varphi_{nm}(t) = Ee^{itX_{nm}} = (1-p_{nm}) + p_{nm}e^{it}$. Then

$$Ee^{itS_n} = \prod_{m=1}^{n}((1-p_{nm}) + p_{nm}e^{it}).$$

Note that

$$\left|e^{p_{nm}(e^{it}-1)}\right| = e^{\mathcal{R}e(p_{nm}(e^{it}-1)} = e^{p_{nm}(\cos t-1)} \le 1$$

and

$$\left|(1-p_{nm}) + p_{nm}e^{it}\right| \le (1-p_{nm}) + p_{nm}|e^{it}| = 1,$$

so we get

$$\begin{aligned}
\left|e^{\sum_{m=1}^n p_{nm}(e^{it}-1)} - \prod_{m=1}^{n}\left((1-p_{nm}) + p_{nm}e^{it}\right)\right| &\le \sum_{m=1}^{n}\left|e^{p_{nm}(e^{it}-1)} - \left((1-p_{nm}) + p_{nm}e^{it}\right)\right| \\
&\underset{(*)}{\le} K\sum_{m=1}^{n}\left(p_{nm}\underbrace{|e^{it}-1|}_{\le 2}\right)^2 \\
&\le 4K\sum_{m=1}^{n}p_{nm}^2 \\
&\le 4K\underbrace{\max_{1\le m'\le n} p_{nm'}}_{\xrightarrow[n\to\infty]{}0}\underbrace{\sum_{m=1}^{n}p_{nm}}_{\xrightarrow[n\to\infty]{}\lambda} \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

In (*), we used $|e^z - 1 - z| \le K|z|^2$ ($\because p_{nm}|e^{it}-1| \le 2p_{nm} \le 2$). Note that

$$e^{\sum_{m=1}^n p_{nm}(e^{it}-1)} \xrightarrow[n\to\infty]{} e^{\lambda(e^{it}-1)} = \varphi_Z(t),$$

where $\varphi_Z(t)$ is ch.f of $Poi(\lambda)$, and therefore

$$Ee^{itS_n} = \prod_{m=1}^{n}\left((1-p_{nm}) + p_{nm}e^{it}\right) \xrightarrow[n\to\infty]{} \varphi_Z(t),$$

and Lévy continuity theorem ends the proof. $\square$

Corollary 1.3.2. *Let X_{nm} be independent nonnegative integer valued random variables for $1 \le m \le n$, with*

$$P(X_{nm} = 1) = p_{nm},\ P(X_{nm} \ge 2) = \epsilon_{nm}.$$

Assume that

(i) $\sum_{m=1}^{n} p_{nm} \to \lambda \in (0, \infty)$

(ii) $\max_{1 \le m \le n} p_{nm} \xrightarrow[n\to\infty]{} 0$

(iii) $\sum_{m=1}^{n} \epsilon_{nm} \xrightarrow[n\to\infty]{} 0$

Then $S_n := X_{n1} + \cdots + X_{nm} \xrightarrow[n\to\infty]{d} Poi(\lambda)$.

Proof. Let $X'_{nm} = I(X_{nm} = 1)$ and $S'_n = X'_{n1} + \cdots + X'_{nn}$. Then since $P(X'_{nm} = 1) = p_{nm}$, by previous theorem,

$$S'_n \xrightarrow[n\to\infty]{d} Poi(\lambda)$$

holds. Now, note that

$$\begin{aligned} P(S_n \ne S'_n) &\le P\left(\bigcup_{m=1}^{n} (X_{nm} \ne X'_{nm})\right) \\ &\le \sum_{m=1}^{n} P(X_{nm} \ne X'_{nm}) \\ &= \sum_{m=1}^{n} P(X_{nm} \ge 2) \\ &= \sum_{m=1}^{n} \epsilon_{nm} \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

With this, we get

$$P(\underbrace{|S_n - S'_n|}_{\text{integer}} \ge \epsilon) \le P(S_n \ne S'_n) \xrightarrow[n\to\infty]{} 0$$

so $S_n - S'_n \xrightarrow[n\to\infty]{P} 0$. Therefore, the assertion holds. □

Chapter 2

Martingales

2.1 Hilbert space

Recall that Hilbert space is a "complete inner product space."

Definition 2.1.1. *Let E be a $\mathbb{C}$-vector space. Inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ is a function satisfies followings.*

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

(iv) $\langle x, x \rangle \geq 0, \ \langle x, x \rangle \Leftrightarrow x = 0$

Definition 2.1.2. *Let* $\|x\| = \sqrt{\langle x, x \rangle}$ *be the norm.*

Proposition 2.1.3. *Followings hold.*

(a) $\|x + y\| \leq \|x\| + \|y\|$

(b) $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

(c) $2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$

Theorem 2.1.4 (Projection)**.** *Suppose that M is a closed convex subset of Hilbert space E. Then $\forall y \in E$, $\exists! w \in M$ such that*

$$\|y - w\| = d(y, M) := \inf\{\|y - z\| : z \in M\}.$$

We may denote it as $\mathcal{P}_M y = w$.

Proof. Let $d := \inf\{\|y-z\| : z \in M\}$. For $n \geq 1$, $\exists z_n \in M$ such that

$$d \leq \|y - z_n\| < d + \frac{1}{n}.$$

Then, since

$$2(\|y+z_n\|^2 + \|y-z_n\|^2) = \|2y - z_n - z_m\|^2 + \|z_n - z_m\|^2,$$

we get

$$\begin{aligned}\|z_n - z_m\|^2 &= 2\|y-z_n\|^2 + 2\|y+z_n\|^2 - 4\left\|y - \frac{z_n+z_m}{2}\right\|^2 \\ &\leq 2\|y-z_n\|^2 + 2\|y+z_n\|^2 - 4d^2 \ (\because M \text{ is convex, and } d \text{ is minimum distance}) \\ &\xrightarrow[m,n\to\infty]{} 0 \ (\because \|y-z_n\|, \|y-z_m\| \to d)\end{aligned}$$

and hence $\{z_n\}$ is Cauchy sequence. Since M is Hilbert, $\exists w = \lim_n z_n \in M$, which makes $\|y-w\| = d$. For uniqueness, let $\exists z \in M$ such that $\|y-z\| = d$. Then

$$d^2 \leq \left\|y - \frac{z+w}{2}\right\|^2 = 2\left\|\frac{y-z}{2}\right\|^2 + 2\left\|\frac{y-w}{2}\right\|^2 - \left\|\frac{z-w}{2}\right\|^2 = d^2 - \frac{\|z-w\|^2}{4} \leq d^2$$

and therefore we get $z = w$. □

Theorem 2.1.5. *Let $M \subseteq E$ be a closed subspace. Then $\forall y \in E$, $\exists! w \in M$ and $v \in M^\perp$ such that $y = w + v$, where $M^\perp = \{u : \langle u, v\rangle = 0 \ \forall v \in M\}$.*

Proof. By previous theorem, there exists $w \in M$ such that $\|y - w\| = d(y, M) =: d$. Let $z \in M, z \neq 0$. Then for any $\lambda \in \mathbb{C}$,

$$d^2 \leq \|y - (w + \lambda z)\|^2 = \|(y-w) - \lambda z\|^2$$

holds. Using

$$\|x+y\|^2 = \|x\|^2 + 2\mathcal{R}e\langle x, y\rangle + \|y\|^2,$$

we obtain

$$d^2 \leq \|(y-w) - \lambda z\|^2 = \|y-w\|^2 - 2\mathcal{R}e\bar{\lambda}\langle y-w, z\rangle + |\lambda|^2\|z\|^2$$

and hence

$$2\mathcal{R}e\bar{\lambda}\langle y-w, z\rangle \leq |\lambda|^2\|z\|^2$$

is obtained. Especially take $\bar{\lambda} = r\overline{\langle y - w, z\rangle}$ for $r \in \mathbb{R}$, and then

$$2r|\langle y - w, z\rangle|^2 \le r^2|\langle y - w, z\rangle|^2\|z\|^2$$

holds, which implies $\langle y - w, z\rangle = 0$. (To show this, assume not, and yield contradiction.) Since z was arbitrary, $y - w \in M^\perp$, and then $y = w + (y - w)$ is the desired decomposition. For uniqueness, let $y = w + v, w' + v'$ such that $w, w' \in M$ and $v, v' \in M^\perp$. Then

$$w - w' = v' - v$$

holds. Note that $w - w' \in M$ and $v' - v \in M^\perp$, and since $M \cap M^\perp = \{0\}$, we obtain $w = w'$ and $v = v'$. □

2.2 Conditional Expectation

Now let's go back to the space of random variables.

Theorem 2.2.1. *Let $\mathcal{L}^2 = \{X : EX^2 < \infty\}$. Then $\mathcal{L}^2$ is a Hilbert space with inner product $\langle X, Y\rangle = EXY$.*

Proof. It's enough to show completeness. First we need a lemma.

Lemma 2.2.2. *If $\{X_n\} \subseteq \mathcal{L}^2$ and $\|X_n - X_{n+1}\| \le 2^{-n}$ for any $n = 1, 2, \cdots$, then $\exists X \in \mathcal{L}^2$ such that*

(1) $P(X_n \to X \text{ as } n \to \infty) = 1$.

(2) $\|X_n - X\| \xrightarrow[n\to\infty]{} 0$.

Proof of lemma. Put $X_0 \equiv 0$. Note

$$\begin{aligned} E\Big(\sum_{j=1}^\infty |X_j - X_{j+1}|\Big) &\underset{\text{MCT}}{=} \sum_{j=1}^\infty E|X_{j+1} - X_j| \\ &\le \sum_{j=1}^\infty \left(E|X_{j+1} - X_j|^2\right)^{1/2} \\ &\le \sum_{j=1}^\infty 2^{-j} < \infty. \end{aligned}$$

Thus $\sum_{j=1}^{\infty}|X_{j+1}-X_j|<\infty$ (Note that $E|X|<\infty \Rightarrow |X|<\infty$ a.s.), and hence $\sum_{j=1}^{\infty}(X_{j+1}-X_j)$ converges P-a.s.. Let

$$X := X_1 + \sum_{j=1}^{\infty}(X_{j+1}-X_j) = \sum_{j=0}^{\infty}(X_{j+1}-X_j).$$

Then $\lim_n X_n = X$ P-a.s. and because

$$\|X\| \le \sum_{j=0}^{\infty}\|X_{j+1}-X_j\| < \infty$$

we get $X \in \mathcal{L}^2$. Therefore

$$\|X_n - X\| = \left\|\sum_{j=n}^{\infty}(X_{j+1}-X_j)\right\| \le \sum_{j=n}^{\infty}\|X_{j+1}-X_j\| \xrightarrow[n\to\infty]{} 0.$$

$\square$ (Lemma)

Now suppose that $\{X_n\} \subseteq \mathcal{L}^2$ is a Cauchy sequence. Then for any $\epsilon > 0$ there is $N(\epsilon)$ such that

$$n, m \ge N(\epsilon) \Rightarrow \|X_n - X_m\| < \epsilon.$$

Put $k_n = \max(N(2^{-1}), N(2^{-2}), \cdots, N(2^{-n})) + 1$. Then $k_n \le k_{n+1}$ for any n, and $k_n, k_{n+1} \ge N(2^{-n})$ so

$$\|X_{k_{n+1}} - X_{k_n}\| \le \frac{1}{2^n}.$$

Thus by lemma, there exists $X \in \mathcal{L}^2$ such that $X = \lim_{n\to\infty} X_{k_n}$. To show for general n, note that

$$\|X_n - X\| \le \underbrace{\|X_n - X_{k_n}\|}_{\to 0 \text{ (Cauchy)}} + \|X_{k_n} - X\| \xrightarrow[n\to\infty]{} 0.$$

$\square$

Theorem 2.2.3. *Let $X \in \mathcal{L}^2$ and let*

$$\mathcal{L}^2(X) = \{h(X) : h : \mathbb{R} \to \mathbb{R} \textit{ is a Borel function and } E[h(X)]^2 < \infty\}.$$

Then $\mathcal{L}^2(X)$ is a closed subspace.

Proof. Since subspace is trivial (show $(\alpha h + \beta\tilde{h})(X) \in \mathcal{L}^2(X)$), so closedness is left. Let $\{h_n(X)\} \subseteq \mathcal{L}^2(X)$ be a convergent sequence. Then since it is Cauchy, there is a subsequence $\{k_n\}$ such that

$\|h_{k_n}(X) - h_{k_{n+1}}(X)\| \le 2^{-n}$, so by previous lemma, there exists Y such that

$$Y = \lim_{n\to\infty} h_{k_n}(X).$$

Note that $\|Y - h_{k_n}(X)\| \xrightarrow[n\to\infty]{} 0$. ("converge" means that $\|Y - h_n(X)\| \xrightarrow[n\to\infty]{} 0$.) Letting

$$M = \{x : -\infty < \liminf_{n\to\infty} h_{k_n}(x) = \limsup_{n\to\infty} h_{k_n}(x) < \infty\}$$

and

$$h(x) := \limsup_{n\to\infty} h_{k_n}(x) I_M(x),$$

we obtain $Y = h(X)$ P-a.s.. Therefore $Y = h(X) \in \mathcal{L}^2(X)$. □

Note that since $\mathcal{L}^2(X)$ is closed subspace (subspace is convex!) of $\mathcal{L}^2$, there exists a "projection" of $Y \in \mathcal{L}^2$ on $\mathcal{L}^2(X)$, and if we define

$$E(Y|X) = \mathcal{P}_{\mathcal{L}^2(X)} Y,$$

it will satisfy

$$\|Y - E(Y|X)\| = \inf_{h(X)\in\mathcal{L}^2(X)} \|Y - h(X)\|.$$

Furthermore, since $Y - E(Y|X)$ is orthogonal to $h(X)$, $E(Y|X)$ should satisfy

$$E\left[(Y - E(Y|X))h(X)\right] = 0 \ \forall h(X) \in \mathcal{L}^2(X).$$

Also note that such $E(Y|X)$ is unique by previous theorems.

Definition 2.2.4 (*Temporary* definition)**.** *Let $X, Y \in \mathcal{L}^2$. Then $E(Y|X)$ is defined as the only function of X satisfying*

$$E\left[(Y - E(Y|X))h(X)\right] = 0 \ \forall h(X) \in \mathcal{L}^2(X).$$

Proposition 2.2.5. *Followings hold.*

(a) $E(c|X) = c$ for a constant c.

(b) $E(\alpha Y + \beta Z|X) = \alpha E(Y|X) + \beta E(Z|X)$.

(c) If $EXY = EXEY$, $E(Y|X) = EY$.

(d) If g is bounded, $E[g(X)Y|X] = g(X)E[Y|X]$.

(e) $EE(Y|X) = EY$.

Proof. Trivial from the definition. Note that in (d), to be well-defined, $g(X)Y$ should be in $\mathcal{L}^2$. Verifying this may be difficult for general g. If g is bounded, it is easily checked. (e) can be proved with definition, considering the case $h(X) \equiv 1$. $\square$

Note that, in particular we choose $h(X) = I(X \in A)$ for a Borel set A, then definition becomes

$$E(YI(X \in A)) = E(E(Y|X)I(X \in A)),$$

i.e.,

$$\int_{(X\in A)} Y d\mathbb{P} = \int_{(X\in A)} E(Y|X)d\mathbb{P}.$$

Note that since $\sigma(X) = \{(X \in A) : A \in \mathcal{B}(\mathbb{R})\}$, if Z is a $\sigma(X)$-measurable r.v. such that

$$\int_{(X\in A)} Z d\mathbb{P} = \int_{(X\in A)} E(Y|X)d\mathbb{P},$$

then $Z = E(Y|X)$ $\mathbb{P}$-a.s.. (Note that $\int_B f d\mu = \int_B g d\mu\ \forall B \Rightarrow f = g\ \mu-$a.e.) Thus if we define conditional expectation using this property, we can omit the assumption that $E(Y|X)$ is in $\mathcal{L}^2$. In other words, we can *extend* the definition.

We can also interpret the conditional expectation as Radon-Nikodym derivative.

Theorem 2.2.6 (Radon-Nikodym theorem)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space and let μ, ν be σ-finite measures with $\nu << \mu$. (It means that $\mu(A) = 0 \Rightarrow \nu(A) = 0$) Then there exists a ($\mu$-a.e.) nonnegative $\mathcal{F}$-measurable function f such that*

$$\nu(A) = \int_A f d\mu\ \forall A \in \mathcal{F}$$

*and denote it as $f = \dfrac{d\nu}{d\mu}$. f is called **Radon-Nikodym derivative**.*

Now we are ready to define a conditional expectation.

Theorem 2.2.7. *Let $(\Omega, \mathcal{F}_0, P)$ be a probability space and $\mathcal{F} \subseteq \mathcal{F}_0$ be a sub-σ-field. Consider $X \in \mathcal{L}^1$. Then there exists a unique r.v. Y satisfying*

(i) Y is $\mathcal{F}$-measurable.

(ii) For any $A \in \mathcal{F}$, $\int_A XdP = \int_A YdP$.

Proof. (Existence) Let $X = X^+ - X^-$. Letting

$$Q^+(A) = \int_A X^+dP \text{ and } Q^-(A) = \int_A X^-dP$$

for any $A \in \mathcal{F}$, by Radon-Nikodym theorem, there are $\mathcal{F}$-measurable random variables

$$\frac{dQ^+}{dP} \text{ and } \frac{dQ^-}{dP} \text{ satisfying } Q^+(A) = \int_A \frac{dQ^+}{dP}dP,\ Q^-(A) = \int_A \frac{dQ^-}{dP}dP\ \forall A \in \mathcal{F}.$$

Note that

$$\frac{dQ^+}{dP} \text{ and } \frac{dQ^-}{dP} \text{ are integrable because } Q^+(\Omega) = \int_\Omega \frac{dQ^+}{dP}dP < \infty \text{ and similar for } \frac{dQ^-}{dP}.$$

Therefore, we get

$$\int_A XdP = \int_A (X^+ - X^-)dP = \int_A \left(\frac{dQ^+}{dP} - \frac{dQ^-}{dP}\right)dP\ \forall A \in \mathcal{F}.$$

(Uniqueness) If Y' also satisfies (i) and (ii), then

$$\int_A YdP = \int_A Y'dP\ \forall A \in \mathcal{F}.$$

Taking $A = \{Y - Y' \geq \epsilon\}$ for $\epsilon > 0$, and then

$$0 = \int_A (Y - Y')dP \geq \int_A \epsilon dP = \epsilon P(A)$$

holds, hence $P(A) = 0$. Since $\epsilon > 0$ was arbitrary, we get $Y \leq Y'$ P-a.s., and by symmetry, we get $Y = Y'$ P-a.s.. □

Definition 2.2.8. *Such Y is called a* ***conditional expectation*** *of X, and denoted as $Y = E(X|\mathcal{F})$. Also, if $\mathcal{F} = \sigma(X)$, we denote*

$$E(Y|\sigma(X)) = E(Y|X)$$

for integrable r.v.'s X, Y.

Remark 2.2.9. Note that $E(X|\mathcal{F})$ is also $\mathcal{L}^1$. To show this, letting $A = (E(X|\mathcal{F}) > 0) \in \mathcal{F}$,

we get

$$0 \le \int_A E(X|\mathcal{F})dP = \int_A XdP \le \int_A |X|dP$$

and

$$0 \le \int_{A^c} -E(X|\mathcal{F})dP = \int_{A^c} -XdP \le \int_{A^c} |X|dP$$

so we have $E|E(X|\mathcal{F})| \le E|X|$.

Definition 2.2.10. *We define*

$$P(A|\mathcal{F}) := E[I_A|\mathcal{F}]$$

for any $A \in \mathcal{F}_0$.

Proposition 2.2.11. *Followings hold. In here, $X \in \mathcal{L}^1$. Also, for convenience, I omitted "P-a.s."*

(a) $E(c|\mathcal{F}) = c$.

(b) For $Y \in \mathcal{L}^1$, and constants a, b, $E(aX + bY|\mathcal{F}) = aE(X|\mathcal{F}) + bE(Y|\mathcal{F})$.

(c) For Borel function $\varphi : \mathbb{R} \to \mathbb{R}$, if $E|\varphi(X)| < \infty$, then $E[\varphi(X)|X] = \varphi(X)$.

(d) If $\mathcal{F} = \{\phi, \Omega\}$, then $E(X|\mathcal{F}) = EX$. ("trivial σ-field")

(e) If $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ for $\Omega_i \cap \Omega_j = \phi$ $\forall i \ne j$, and

$$\mathcal{F} = \sigma(\Omega_i : i \in \mathbb{N}) = \left\{ \bigcup_{i \in I} \Omega_i : I \subseteq \mathbb{N} \right\},$$

then

$$E(X|\mathcal{F}) = \sum_{i=1}^{\infty} \frac{E[XI_{\Omega_i}]}{P(\Omega_i)} I_{\Omega_i}.$$

(f) If $E|Y| < \infty$ and $E|XY| < \infty$, and X is $\mathcal{F}$-mb, then

$$E(XY|\mathcal{F}) = X \cdot E(Y|\mathcal{F}).$$

(g) (Tower property) If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_0$, then

$$E\left[E[X|\mathcal{F}_1]|\,\mathcal{F}_2\right] = E\left[E[X|\mathcal{F}_2]|\,\mathcal{F}_1\right] = E[X|\mathcal{F}_1].$$

Specifically, $EE(X|\mathcal{F}) = EX$.

(h) $|E(X|\mathcal{F})| \le E\left[|X||\mathcal{F}\right]$

(i) *(Markov)* $P(|X| \ge c|\mathcal{F}) \le c^{-1}E[|X||\mathcal{F}]$ *for* $c > 0$.

(j) *(MCT) If* $X_n \ge 0,\ X_n \nearrow X$, *then* $E(X_n|\mathcal{F}) \nearrow E(X|\mathcal{F})$.

(k) *(DCT) If* $X_n \xrightarrow[n\to\infty]{a.s} X$ *and* $|X_n| \le Y$ *for* $E|Y| < \infty$, *then* $E(X_n|\mathcal{F}) \xrightarrow[n\to\infty]{a.s} E(X|\mathcal{F})$.

(l) *(Continuity) Let* $B_n \nearrow B$ *be events. Then* $P(B_n|\mathcal{F}) \nearrow P(B|\mathcal{F})$.

(m) $P\left(\bigcup_{n=1}^{\infty} C_n \middle| \mathcal{F}\right) = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} C_k \middle| \mathcal{F}\right) = \lim_{n\to\infty}\sum_{k=1}^{n} P(C_k|\mathcal{F})$ *holds. Last equality holds provided that* C_k*'s are disjoint.*

(n) *(Jensen) If* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a convex function, and* $E|\varphi(X)| < \infty$, *then* $E[\varphi(X)|\mathcal{F}] \le \varphi\left(E[X|\mathcal{F}]\right)$.

Proof. (a), (b), (c), (d). By definition.

(e) Note that if g is $\mathcal{F}$-mb function, then $g = \sum_{i=1}^{\infty} a_i I_{\Omega_i}$ for some a_i. Then we get

$$E(X|\mathcal{F}) = \sum_{i=1}^{\infty} a_i I_{\Omega_i}.$$

Taking $\int_{\Omega_i}$ on both sides, we get

$$P(\Omega_i)a_i = \int_{\Omega_i} X dP$$

and the assertion holds.

(f) Standard machine. If $X = I_B$ for $B \in \mathcal{F}$, for any $A \in \mathcal{F}$, we get

$$\int_A E(XY|\mathcal{F})dP = \int_A XY dP = \int_{A\cap B} Y dP = \int_{A\cap B} E(Y|\mathcal{F})dP = \int_A X \cdot E(Y|\mathcal{F})dP$$

from $A \cap B \in \mathcal{F}$. If X is simple, i.e.,

$$X = \sum_{i=1}^{m} a_i I_{B_i} \text{ for } B_i \in \mathcal{F},\ a_i \in \mathbb{R},$$

then

$$E(XY|\mathcal{F}) = E\left[\sum_{i=1}^{m} a_i I_{B_i} Y \middle| \mathcal{F}\right] = \sum_{i=1}^{m} a_i E(I_{B_i}Y|\mathcal{F}) = \sum_{i=1}^{m} a_i I_{B_i} E(Y|\mathcal{F}) = X \cdot E(Y|\mathcal{F})$$

holds. If $X \ge 0$, there is a sequence of simple r.v.'s such that $X_n \nearrow X$, so $|X_nY| \le |XY|$ holds.

Thus by DCT ((k)),

$$E[X_nY|\mathcal{F}] \xrightarrow[n\to\infty]{} E[XY|\mathcal{F}],$$

and from $E[X_nY|\mathcal{F}] = X_nE[Y|\mathcal{F}] \xrightarrow[n\to\infty]{} X \cdot E[Y|\mathcal{F}]$, we get the desired result. Finally, for general X, decomposition $X = X^+ - X^-$ gives the conclusion. (For $X \geq 0$ case, we can also prove it directly. For any $A \in \mathcal{F}$, we get

$$\int_A E[XY|\mathcal{F}]dP = \int_A XYdP \overset{DCT}{=} \lim_{n\to\infty} \int_A X_nYdP = \lim_{n\to\infty} \int_A E[X_nY|\mathcal{F}]dP \overset{DCT}{=} \int_A \lim_{n\to\infty} X_nE[Y|\mathcal{F}]dP$$

and hence

$$\int_A E[XY|\mathcal{F}]dP = \int_A XE[Y|\mathcal{F}]dP.)$$

(g) First, since $E[X|\mathcal{F}_1]$ is $\mathcal{F}_1$-mb, it is also $\mathcal{F}_2$-mb, and hence by (f), $E\left[E[X|\mathcal{F}_1]|\mathcal{F}_2\right] = E[X|\mathcal{F}_1]$.

Second, for any $A \in \mathcal{F}_1$,

$$\int_A E[X|\mathcal{F}_2]dP \overset{A\in\mathcal{F}_2}{=} \int_A XdP \overset{A\in\mathcal{F}_1}{=} \int_A E[X|\mathcal{F}_1]dP$$

holds, and therefore $E\left[E[X|\mathcal{F}_2]|\mathcal{F}_1\right] = E[X|\mathcal{F}_1]$.

(h) $-|X| \leq X \leq |X|$.

(i) Clear.

(j) Since $E(X_n|\mathcal{F})$ is monotone, we can define $\lim_{n\to\infty} E(X_n|\mathcal{F})$. Thus, for any $A \in \mathcal{F}$,

$$\begin{aligned}
\int_A \lim_{n\to\infty} E(X_n|\mathcal{F})dP &\underset{MCT}{=} \lim_{n\to\infty} \int_A E(X_n|\mathcal{F})dP \\
&= \lim_{n\to\infty} \int_A X_ndP \\
&\underset{MCT}{=} \int_A \lim_{n\to\infty} X_ndP \\
&= \int_A XdP = \int_A E(X|\mathcal{F})dP.
\end{aligned}$$

Also, $\lim_{n\to\infty} E(X_n|\mathcal{F})$ is $\mathcal{F}$-mb.

(k) Let

$$Y_n := \sup_{k\geq n} |X_k - X|.$$

Then Y_n is monotone, $Y_n \xrightarrow[n\to\infty]{a.s} 0$, and $Y_n \leq 2Y$. Then $EY_n \xrightarrow[n\to\infty]{} 0$ by DCT. Note that since

$E(Y_n|\mathcal{F})$ is monotone, $\exists Z \geq 0$ such that $E(Y_n|\mathcal{F}) \searrow Z$. Then by Fatou's lemma,

$$0 \leq EZ \leq \liminf_{n\to\infty} EE(Y_n|\mathcal{F}) = \liminf_{n\to\infty} EY_n = 0,$$

and hence

$$|E(X_n|\mathcal{F}) - E(X|\mathcal{F})| \leq E\left(|X_n - X||\mathcal{F}\right) \leq E(Y_n|\mathcal{F}) \xrightarrow[n\to\infty]{} 0.$$

(l) Clear by (k).

(m) Clear by (k) and (l).

(n) Note that

$$\varphi(x) = \sup\{ax + b : (a, b) \in S\}$$

where

$$S = \{(a, b) : a, b \in \mathbb{R},\ ax + b \leq \varphi(x)\ \forall x\}.$$

(By definition of S, $\varphi(x) \geq \sup\{ax + b : (a, b) \in S\}$. Also, for any x, there is a and b such that $\varphi(x) = ax+b$ and $\varphi(y) \geq ay+b\ \forall y$, so because of supremum, we get $\varphi(x) \leq \sup\{ax+b : (a, b) \in S\}$.) Therefore, from

$$E(\varphi(X)|\mathcal{F}) \geq a \cdot E(X|\mathcal{F}) + b,$$

we get

$$E(\varphi(X)|\mathcal{F}) \geq \sup_{a,b\in S} a \cdot E(X|\mathcal{F}) + b = \varphi\left(E(X|\mathcal{F})\right).$$

Proposition 2.2.12. *Let X, Y be integrable independent random variables with $E|\varphi(X, Y)|\infty$, where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable. Also, define*

$$g(x) = E[\varphi(x, Y)].$$

Then

$$E[\varphi(X, Y)|X] = g(X).$$

Proof. By proof of Fubini theorem, g is Borel measurable, so $g(X)$ is $\sigma(X)$-mb. Thus we may show

$$\int_A \varphi(X, Y)dP = \int_A g(X)dP\ \forall A \in \sigma(X).$$

Note that for $A \in \sigma(X)$, $\exists C \in \mathcal{B}$ such that $A = (X \in C)$. Also note that from independence, we get $P^{(X,Y)} = P^X \otimes P^Y$. Therefore,

$$\begin{aligned}\int_A \varphi(X,Y)dP &= E\left[\varphi(X,Y)I_C(X)\right]\\ &= \int\int \varphi(x,y)I_C(x)P^{(X,Y)}(dxdy)\\ &= \int\left(\int \varphi(x,y)P^Y(dy)\right)I_C(x)P^X(dx) \ (\because \text{Fubini})\\ &= \int E[\varphi(x,Y)]I_C(x)P^X(dx)\\ &= \int g(x)I_C(x)P^X(dx) = \int_A g(X)dP.\end{aligned}$$

□

Note that conditional expectation can be interpreted as a *projection* in $\mathcal{L}^2$. In other words, our definition is concident to the *temporary* definition in definition 2.2.4.

Theorem 2.2.13. *Suppose that X is r.v. with $EX^2 < \infty$. Define*

$$\mathcal{C} := \{Y : Y \in \mathcal{F} \ \& \ EY^2 < \infty\}.$$

In here, $Y \in \mathcal{F}$ means that Y is $\mathcal{F}$-mb. Then,

$$E\left((X - E[X|\mathcal{F}])^2\right) = \inf_{Y\in\mathcal{C}} E(X-Y)^2.$$

Proof. If $Y \in \mathcal{C}$,

$$E(X-Y)^2 = E[(X-E(X|\mathcal{F}))^2] + E[(E(X|\mathcal{F})-Y)^2] + 2E[(X-E(X|\mathcal{F}))(E(X|\mathcal{F})-Y)]$$

and

$$\begin{aligned}E[(X-E(X|\mathcal{F}))(E(X|\mathcal{F})-Y)] &= EE[(X-E(X|\mathcal{F}))(E(X|\mathcal{F})-Y)|\mathcal{F}]\\ &= E\left[(E(X|\mathcal{F})-Y)\underbrace{E[(X-E(X|\mathcal{F}))|\mathcal{F}]}_{=0}\right] = 0\end{aligned}$$

ends the proof. □

Remark 2.2.14. Note that $E(X|\mathcal{F})$ is also $\mathcal{L}^2$, by Cauchy-Schwarz inequality,

$$[E(X|\mathcal{F})]^2 \le E[X^2|\mathcal{F}].$$

Thus we can say that

$$E(X|\mathcal{F}) = \arg\min_{Y\in\mathcal{C}} E(X-Y)^2.$$

2.3 Martingales and Stopping Times

2.3.1 Definitions and Basic Theory

Fix a probability space $(\Omega, \mathcal{F}, P)$.

Definition 2.3.1. *Let $\{\mathcal{F}_n\}$ be a sequence of sub σ-fileds of $\mathcal{F}$ Then $\{\mathcal{F}_n\}_{n=0}^{\infty}$ is called a **filtration** if $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ $\forall n$.*

Definition 2.3.2. *Let $\{\mathcal{F}_n\}_n$ be a filtration. A sequence of r.v. $\{X_n\}_n$ is called $\mathcal{F}_n$-**adapted** if $X_n \in \mathcal{F}_n$ for any n.*

Definition 2.3.3. *Let $\{\mathcal{F}_n\}$ be a filtration and $\{X_n\}$ be $\mathcal{F}_n$-adapted integrable r.v.'s. Then $\{X_n\}$ or $(X_n, \mathcal{F}_n)$ is called*

martingale *if $E[X_n\mathcal{F}_{n-1}] = X_{n-1}$ $\forall n \ge 1$.*

submartingale *if $E[X_n\mathcal{F}_{n-1}] \ge X_{n-1}$ $\forall n \ge 1$.*

supermartingale *if $E[X_n\mathcal{F}_{n-1}] \le X_{n-1}$ $\forall n \ge 1$.*

Example 2.3.4. Let $\xi_1, \xi_2, \cdots \overset{i.i.d.}{\sim} (0, \sigma^2)$, and let

$$\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n),\ X_n = \xi_1 + \cdots + \xi_n = X_{n-1} + \xi_n.$$

Then $\{\mathcal{F}_n\}$ is filtration $\{X_n\}$ is $\mathcal{F}_n$-adapted, and $\{X_n\}$ is a martinagle.

Example 2.3.5. Let $\eta_1, \eta_2, \cdots \overset{i.i.d.}{\sim} (0, \sigma^2)$, and let

$$X_{n+1} = X_n + h_n(X_1, \cdots, X_n)\eta_{n+1},\ X_1 = \eta_1,$$

where $h_n : \mathbb{R}^n \to \mathbb{R}$ is Borel. Assume that X_n's are integrable. Then letting $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$, we get $\{X_n\}$ is martingale.

Following is clear by Jensen.

Proposition 2.3.6. *Let $\{\mathcal{F}_n\}$ be a filtration, and $\{X_n\}$ be $\mathcal{F}_n$-adapted integrable random variables.*

(a) If $\{X_n\}$ is a martinagle and $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function satisfying $E|\varphi(X_n)| < \infty\ \forall n$, then $\{\varphi(X_n)\}$ is a submartingale.

(b) If $\{X_n\}$ is a submartinagle and $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing, convex function satisfying $E|\varphi(X_n)| < \infty\ \forall n$, then $\{\varphi(X_n)\}$ is a submartingale.

(c) If $\{X_n\}$ is a supermartinagle and $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing, concave function satisfying $E|\varphi(X_n)| < \infty\ \forall n$, then $\{\varphi(X_n)\}$ is a supermartingale.

Remark 2.3.7. Consequence of previous proposition that we will use frequently is $\varphi(x) = |x|,\ x^+, |x|^p\ (p \ge 1), |x-a|,\ (x-a)^+, \cdots$.

Definition 2.3.8. *Let $\{\mathcal{F}_n\}$ be a filtration. Then $\{H_n\}$ is called* ***predictable*** *if $H_n \in \mathcal{F}_{n-1}\ \forall n \ge 1$. It means that, $E(H_n|\mathcal{F}_{n-1}) = H_n$.*

Definition 2.3.9 (Martingale Transform)**.** *Let X_n be a $(\mathcal{F}_n)$-martingale (sub- or super-), and H_n be predictable process, i.e., $H_n \in \mathcal{F}_{n-1}$. Then $\forall n \ge 1$,*

$$(H \cdot X)_n := \sum_{m=1}^{n} H_m(X_m - X_{m-1}).$$

Theorem 2.3.10. *Let H_n be predictable process, and suppose that each H_n is bounded. Then*

(a) If X_n is $(\mathcal{F}_n)$-martingale, then $(H \cdot X)_n$ is $(\mathcal{F}_n)$-martingale.

(b) If X_n is $(\mathcal{F}_n)$-submartingale, then $(H \cdot X)_n$ is $(\mathcal{F}_n)$-submartingale, "provided that $H_n \ge 0$."

(c) If X_n is $(\mathcal{F}_n)$-supermartingale, then $(H \cdot X)_n$ is $(\mathcal{F}_n)$-supermartingale, "provided that $H_n \ge 0$."

Proof. Note that

$$\begin{aligned}
E\left[(H \cdot X)_{n+1}|\mathcal{F}_n\right] &= E\left[\left.\sum_{m=1}^{n+1} H_m(X_m - X_{m-1})\right|\mathcal{F}_n\right] \\
&= \sum_{m=1}^{n} E\left[H_m(X_m - X_{m-1})|\mathcal{F}_n\right] + E\left[H_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n\right] \\
&= \sum_{m=1}^{n} H_m(X_m - X_{m-1}) + H_{n+1}E\left[X_{n+1} - X_n|\mathcal{F}_n\right]
\end{aligned}$$

$$= (H \cdot X)_n + \underbrace{H_{n+1} E\left[X_{n+1} - X_n | \mathcal{F}_n\right]}_{(*)}. \tag{2.1}$$

If X_n is martingale, (*) is equal to 0, so (2.1) becomes $(H \cdot X)_n$. If X_n is submartingale, $(*) \geq 0$, which implies (2.1) $\geq (H \cdot X)_n$. □

Now it's time to introduce a stopping time.

Definition 2.3.11 (Stopping Time)**.** *Let N be a r.v. taking values of nonnegative integers (& ∞). N is called a* ***stopping time*** *if*

$$\forall n \geq 0, \ (N = n) \in \mathcal{F}_n.$$

Note that if N is a stopping time, then $(N \leq n) \in \mathcal{F}_n$ and $(N > n) \in \mathcal{F}_n$ also hold.

Example 2.3.12 (Stopped process)**.** Let X_n be a (sub-/super-) martingale, and N be a stopping time. Letting $H_m = I(N \geq m)$, it becomes predictable ($H_m \in \mathcal{F}_{m-1}$). Thus,

$$\begin{aligned}
(H \cdot X)_n &= \sum_{m=1}^{n} I(N \geq m)(X_m - X_{m-1}) \\
&= \sum_{m=1}^{\infty} I(m \leq n) I(N \geq m)(X_m - X_{m-1}) \\
&= \sum_{m=1}^{\infty} I(m \leq N \wedge n)(X_m - X_{m-1}) \\
&= \sum_{m=1}^{N \wedge n} (X_m - X_{m-1}) \\
&= X_{N \wedge n} - X_0
\end{aligned}$$

holds. It implies that a "stopped process" $(X_{N \wedge n})_{n \geq 0}$ is $(\mathcal{F}_n)$-(sub-/super-) martingale.

Following "upcrossing process" is set-up for convergence theorem.

Example 2.3.13. Let X_n be $(\mathcal{F}_n)$-submartingale, and $a < b$. Define

$$N_1 = \inf\{m \geq 0 : X_m \leq a\}$$

$$N_2 = \inf\{m > N_1 : X_m \geq b\}$$

$$N_3 = \inf\{m > N_2 : X_m \leq a\}$$

$$N_4 = \inf\{m > N_3 : X_m \geq b\}$$

$$\vdots$$

See figure 2.1.

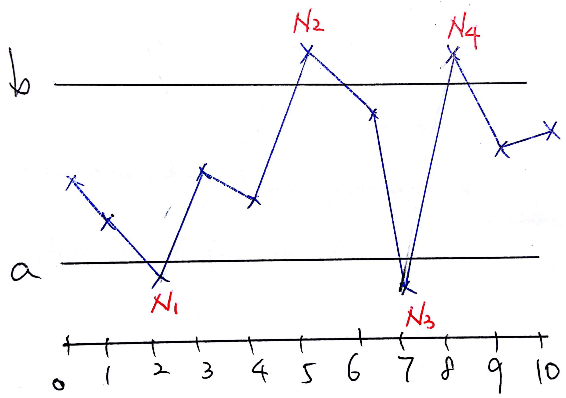

Figure 2.1: X_n and N_n's. For example, $N_4 = 8$.

Then N_k's become a stopping time. First, N_1 is a stopping time, because

$$(N_1 = n) = (X_m > a\ \forall m \leq n-1,\ X_n \leq a) = \bigcap_{m=0}^{n-1} (X_m > a) \cap (X_n \leq a) \in \mathcal{F}_n.$$

Next, N_2 is also a stopping time from

$$(N_2 = n) = \bigcup_{m=0}^{n-1} (N_1 = m) \cap (X_l < b\ \forall l \text{ s.t. } m < l \leq n-1) \cap (X_n \geq b) \in \mathcal{F}_n.$$

Then N_3 is a stopping time, ..., and by induction, we get N_k is a stopping time.

Now define an "upcrossing process,"

$$U_n := \sup\{k : N_{2k} \leq n\} \text{ for } n \geq 1.$$

Then U_n is "the number of upcrossings (from a to b) completely by time n." Note that $U_n \leq n$. Also note that, $N_{2U_n} \leq n$. See figure 2.2.

Now our assertion is:

Theorem 2.3.14 (Upcrossing inequality). $(b-a)EU_n \leq E(X_n - a)^+ - E(X_0 - a)^+$.

Proof. Let $Y_n = (X_n - a)^+ + a = X_n \vee a$ (See figure 2.3). Then by Jensen's inequality, Y_n is $(\mathcal{F}_n)$-submartingale, and the numbers of upcrossings of X_n and Y_n are the same. Thus, we may

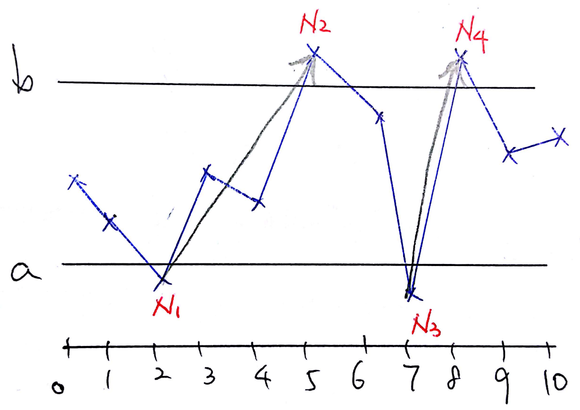

Figure 2.2: Upcrossing process. For example, in this figure, $U_{10} = 2$.

consider Y_n instead of X_n without loss of generality.

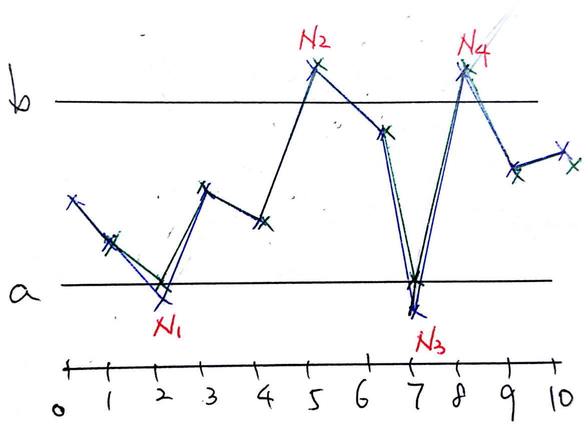

Figure 2.3: Upcrossing process and Y_n.

Note that from $Y_{N_{2k}} - Y_{N_{2k-1}} \geq b - a$, we get

$$(b-a)U_n \leq \sum_{k=1}^{U_n} (Y_{N_{2k}} - Y_{N_{2k-1}}).$$

Now letting $J_k = \{N_{2k-1}+1, \cdots, N_{2k}\} = \{m : N_{2k-1} < m \leq N_{2k}\}$ and $J = \bigcup_{k=1}^{U_n} J_k$, we get

$$\begin{aligned}(b-a)U_n &\leq \sum_{k=1}^{U_n} (Y_{N_{2k}} - Y_{N_{2k-1}}) \\ &= \sum_{k=1}^{U_n} \sum_{i \in J_k} (Y_i - Y_{i-1}) \\ &= \sum_{m \in J} (Y_m - Y_{m-1}).\end{aligned}$$

Now define a predictable process

$$H_m = I(m \in J) = I(N_{2k-1} < m \le N_{2k} \text{ for some } k = 1, 2, \cdots, n).$$

(Note that $N_{2U_n} \le n$) Then

$$\sum_{m \in J} (Y_m - Y_{m-1}) = \sum_{m=1}^{n} H_m (Y_m - Y_{m-1}) = (H \cdot Y)_n$$

becomes a martingale transform. (H_m is predictable from $(N_{2k-1} < m \le N_{2k}) = (N_{2k-1} \le m-1) \cap (N_{2k} \le m-1)^c \in \mathcal{F}_{m-1}$.) Hence, $(H \cdot Y)_n$ is submartingale. Now, define $\tilde{H}_m = 1 - H_m$. Then $(\tilde{H} \cdot Y)_n$ also becomes submartingale and

$$Y_n - Y_0 = \sum_{m=1}^{n} (H_m + \tilde{H}_m)(Y_m - Y_{m-1}) = (H \cdot Y)_n + (\tilde{H} \cdot Y)_n,$$

so we get $E(\tilde{H} \cdot Y)_n \ge E(\tilde{H} \cdot Y)_1 \ge 0$ and hence

$$Y_n - Y_0 = (H \cdot Y)_n + (\tilde{H} \cdot Y)_n \ge (H \cdot Y)_n,$$

i.e.,

$$E(Y_n - Y_0) \ge E(H \cdot Y)_n.$$

Recall that $Y_n = (X_n - a)^+ + a$. Therefore, we get

$$(b-a)EU_n \le E(H \cdot Y)_n \le E(Y_n - Y_0) = E(X_n - a)^+ - E(X_0 - a)^+.$$

□

Remark 2.3.15. The key fact is that $E(\tilde{H} \cdot Y)_n \ge 0$, that is, *no matter how hard you try, you can't lose money betting on a submartingale.* (Note that $(\tilde{H} \cdot Y)_n$ is "total profit resulted in downcrossing.")

Indeed, our goal was following **Martingale convergence theorem**.

Theorem 2.3.16 (Martingale convergence theorem)**.** *If X_n is a $((\mathcal{F}_n)$-)submartingale with $\sup_n EX_n^+ < \infty$, then as $n \to \infty$, X_n converges a.s. to a limit X with $E|X| < \infty$.*

Proof. Note that $(x-a)^+ \le x^+ + |a|$ (See figure 2.4). Then we get

$$EU_n \le \frac{E(X_n-a)^+ - E(X_0-a)^+}{b-a} \le \frac{E(X_n-a)^+}{b-a} \le \frac{EX_n^+ + |a|}{b-a} \le \frac{\sup_n EX_n^+ + |a|}{b-a}.$$

Note that U_n is monotone, so $\exists U$ s.t. $U_n \nearrow U$. Then from MCT (proposition 2.2.11) $EU_n \nearrow EU$ and hence

$$EU \le \frac{\sup_n EX_n^+ + |a|}{b-a} < \infty.$$

From this we get $EU < \infty$, which implies $U < \infty$ a.s.. As U means "the number of whole upcrossings," from $U < \infty$, we get

$$P\left(\liminf_{n\to\infty} X_n < a < b < \limsup_{n\to\infty} X_n\right) = 0.$$

(The number of whole upcrossing should not be infinite) Since it holds for any $a, b \in \mathbb{Q}$ s.t. $a < b$, we get

$$P\left(\bigcup_{\substack{a,b\in\mathbb{Q}\\ a<b}} \left\{\liminf_{n\to\infty} X_n < a < b < \limsup_{n\to\infty} X_n\right\}\right) = 0,$$

i.e., $\liminf X_n = \limsup X_n$ P-a.s., which implies $\exists \lim X_n =: X$ P-a.s.. (For well-definedness, let $X = 0$ if $\liminf X_n \ne \limsup X_n$) Now by Fatou's lemma,

$$EX^+ \le \liminf_{n\to\infty} EX_n^+ < \infty$$

holds, so $EX^+ < \infty$ and $X < \infty$ P-a.s.. Since X_n is submartingale, $EX_n \ge EX_0$, so

$$EX_n^- = EX_n^+ - EX_n \le EX_n^+ - EX_0$$

holds, and by Fatou again, we get

$$EX^- \le \liminf_{n\to\infty} EX_n^- \le \sup_n EX_n^+ - EX_0 < \infty.$$

Therefore, $EX^- < \infty$, which implies that (with $EX^+ < \infty$) X is finite almost surely, and integrable (i.e., $E|X| < \infty$).

□

Corollary 2.3.17. *If $X_n \ge 0$ is a $((\mathcal{F}_n)$-$)$supermartingale, then as $n \to \infty$, $\exists X$ s.t. $X_n \to X$*

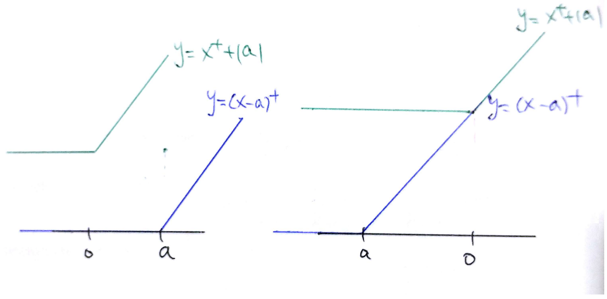

Figure 2.4: $y = (x - a)^+$ and $y = x^+ + |a|$.

a.s. and $EX \le EX_0 < \infty$.

Proof. $Y_n = -X_n \le 0$ is a submartingale with $EY_n^+ = 0$. Thus by previous theorem, Y_n has a limit Y, and $X_n \xrightarrow[n\to\infty]{a.s} -Y =: X$. As X_n is a supermartingale, we get $EX_0 \ge EX_n$, and with Fatou's lemma, we obtain $EX \le EX_0$.

Example 2.3.18. Let $\xi_1, \xi_2, \cdots,$ be i.i.d. r.v.'s with $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. Also define

$$S_0 = 1,\ S_n = S_{n-1} + \xi_n,\ n \ge 1,$$

and $\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$. Then S_n is $(\mathcal{F}_n)$-martingale. Let $N = \inf\{n : S_n = 0\}$. Then from $S_0 = 1$, $N > 0$. Also note that N becomes a stopping time. Let

$$X_n = S_{N\wedge n}.$$

Then by example 2.3.12, X_n is also a martingale. Now, note that by definition of N, and from $S_0 = 1$,

$$m \le N \Rightarrow S_m \ge 0,$$

which implies $X_n \ge 0$. Note that on $(N = \infty)$, $X_n = S_n$ holds $(\star)$. Also, it's known that

$$\limsup_{n\to\infty} \frac{S_n}{n^{1/2}(\log\log n)^{1/2}} = \sigma\sqrt{2},$$

and with this we can obtain that

$$\limsup_{n\to\infty} S_n = \infty,\ \liminf_{n\to\infty} S_n = -\infty\ P-a.s..$$

Thus,

$$P(N=\infty)=P\left(N=\infty,\ \limsup_{n\to\infty} S_n=\infty,\ \liminf_{n\to\infty} S_n=-\infty\right)\le P\left(\limsup_{n\to\infty} X_n=\infty,\ \liminf_{n\to\infty} X_n=-\infty\right)$$

holds from ($\star$). Note that by previous corollary, since X_n is martingale, it converges to some X almost surely, which implies that

$$P\left(\limsup_{n\to\infty} X_n=\infty,\ \liminf_{n\to\infty} X_n=-\infty\right)=0.$$

This implies that $N<\infty$ a.s.. Therefore,

$$\lim_{n\to\infty} X_n=\lim_{n\to\infty} S_{N\wedge n}=S_N=0.$$

However, it means that $X_n \xrightarrow[n\to\infty]{a.s} 0$, while $EX_n=EX_0=1$ for any n. Therefore, even if X_n converges almost surely, we cannot say that X_n also converges in $\mathcal{L}^1$. □

Example 2.3.19. If X_n is $(\mathcal{F}_n)_{n\ge0}$-submartingale s.t. $X_n\le 0$, then we can define

$$X_\infty=\lim_{n\to\infty} X_n,\ \mathcal{F}_\infty=\sigma\left(\bigcup_{n=0}^{\infty}\mathcal{F}_n\right)$$

and it can be obtained that

$$(X_n)_{0\le n\le\infty} \text{ is } (\mathcal{F}_n)_{0\le n\le\infty}\text{-submartingale,}$$

i.e.,

$$E(X_\infty|\mathcal{F}_n)\ge X_n\ P-a.s.\ \forall n\ge 0.$$

In this situation, we say that X_n is "closable." To show this, we need *Fatou's lemma* in conditional context.

Lemma 2.3.20 (Conditional Fatou lemma)**.** *Suppose that $X_n\ge 0$, $X_n \xrightarrow[n\to\infty]{a.s} X$, and $E|X|<\infty$. Then for sub σ-field $\mathcal{F}$,*

$$E(X|\mathcal{F})\le \liminf_{n\to\infty} E(X_n|\mathcal{F}).$$

Proof. Let $M>0$ be a constant. Then by DCT (proposition 2.2.11),

$$E(X\wedge M|\mathcal{F})=\lim_{n\to\infty} E(X_n\wedge M|\mathcal{F})$$

holds. $X_n \wedge M \le X_n$ implies that $\lim_{n\to\infty} E(X_n \wedge M|\mathcal{F}) \le \liminf_{n\to\infty} E(X_n|\mathcal{F})$, so we get

$$E(X \wedge M|\mathcal{F}) \le \liminf_{n\to\infty} E(X_n|\mathcal{F}) \ \forall M > 0.$$

Letting $M \to \infty$, we get $E(X \wedge M|\mathcal{F}) \xrightarrow[n\to\infty]{} E(X|\mathcal{F})$ by MCT (proposition 2.2.11), and hence

$$E(X|\mathcal{F}) \le \liminf_{n\to\infty} E(X_n|\mathcal{F}).$$

□

Now come back to our example. By martingale convergence theorem, $\exists X_\infty = \lim_{n\to\infty} X_n \in \mathcal{F}_\infty$, and $X_\infty \le 0$, by negativity of X_n. By conditional Fatou,

$$E(-X_\infty|\mathcal{F}_n) \le \liminf_{m\to\infty} E(-X_m|\mathcal{F}_n) \le (-X_n)$$

for arbitrary given n. The last inequality holds because $(-X_n)$ is supermartingale. Therefore, we get

$$E(X_\infty|\mathcal{F}_n) \ge X_n \ P-a.s..$$

Following theorem is very useful in martingale theory.

Theorem 2.3.21 (Doob decomposition theorem)**.** *Any submartingale X_n can be expressed uniquely as $X_n = M_n + A_n$, where M_n is a martingale, and A_n is a predictable increasing sequence with $A_0 = 0$.*

Proof. (Motivation: if it holds, $E(X_n|\mathcal{F}_{n-1}) = E(M_n|\mathcal{F}_{n-1}) + E(A_n|\mathcal{F}_{n-1}) = M_{n-1} + A_n = X_{n-1} - A_{n-1} + A_n$.)

Let

$$A_n = A_{n-1} + E(X_n|\mathcal{F}_{n-1}) - X_{n-1}.$$

Then since X_n is submartingale, $E(X_n|\mathcal{F}_{n-1}) - X_{n-1} \ge 0$, and hence A_n is increasing. Further, by induction, A_n is predictable. Define

$$M_n = X_n - A_n,$$

and then we obtain

$$E(M_n|\mathcal{F}_{n-1}) = E(X_n - A_n|\mathcal{F}_{n-1}) = E(X_{n-1} - A_{n-1}|\mathcal{F}_{n-1}) = X_{n-1} - A_{n-1} = M_{n-1},$$

which implies that M_n is a martingale. In here, the second equality holds from the definition of A_n and predictability, while the third one comes from $X_{n-1} \in \mathcal{F}_{n-1}$.

Now for uniqueness, suppose that we have two decompositions,

$$X_n = M_n + A_n = M'_n + A'_n.$$

Then from

$$M_n - M'_n = A'_n - A_n,$$

$M_n - M'_n$ is predictable martingale, which implies that $M_n - M'_n = M_0 - M'_0$. Since $A_0 = A'_0$, it yields that $M_n = M'_n$. □

Note that Doob decomposition implies that, if X_n is a martingale, X_n^2 is a submartingale, and therefore, there exists a unique predictable increasing process $\langle X \rangle_n$ such that $X_n^2 - \langle X \rangle_n$ becomes a martingale. $\langle X \rangle$ is called a "quadratic variation."

Remark 2.3.22 (Annotation by compiler)**.** In 1953, Doob published previous theorem, and conjectured a continuous time version of the theorem. In 1962 and 1963, Paul-André Meyer proved such a theorem, which became known as the *Doob-Meyer decomposition*. It implies following: For filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$ and any right-continuous square-integrable $(\mathcal{F}_t)$-adapted martingale $(X_t)_{t\geq 0}$, there exists a unique continuous increasing predictable process $\langle X \rangle$, $\langle X \rangle_0 = 0$ and such that $X^2 - \langle X \rangle$ is a martingale. For example, if $(B_t)_{t\geq 0}$ is a standard Brownian motion, then $\langle B \rangle_t = t$.

One important application of Doob-Meyer decomposition in statistics is for survival analysis. Let $N(t)$ be a counting process, which is defined as a stochastic process with the properties that $N(0) = 0,\ P(N(t) < \infty) = 1$, and the sample paths of $N(t)$ are right-continuous, piecewise constant with jumps of size $+1$. In survival analysis, $N(t)$ often denotes "the number of event occurs," i.e., the number of dead people at time t. Then there is a smooth predictable process $\Lambda(t)$ which makes $M(t) := N(t) - \Lambda(t)$ a martingale. $M(t)$ is called a counting process martingale. Now, for quadratic variation $\langle M \rangle$ of M^2, we have $Var(dM(t)|\mathcal{F}_{t-}) = d\langle M \rangle(t)$. Using this, we can construct a *stochastic integrals* of the basic martingale. For example, let $Y(t)$ be "at risk process," which denotes the number of individuals at risk at a given time. Then $Y(t)$ becomes predictable, so we can define a stochastic integral

$$\int_0^t Y(s)dM(s),$$

which also becomes a martingale (Indeed, it is "generalization of martingale transform"), and quadratic variation becomes

$$\left\langle \int_0^t Y(s)dM(s) \right\rangle = \int_0^t Y^2(s)d\langle M\rangle(s).$$

2.3.2 Examples

Bounded increments

Proposition 2.3.23 (Bounded increments). *Let X_n be a martingale with $|X_{n+1}-X_n| \le M < \infty$ for any n, and define*

$$C = \{\lim X_n \text{ exists and is finite}\}$$

$$D = \{\limsup X_n = \infty \text{ and } \liminf X_n = -\infty\}.$$

Then $P(C \cup D) = 1$.

Proof. WLOG $X_0 = 0$. (Why "WLOG"? Let $\tilde{X}_n = X_n - X_0$. Then $\tilde{X}_n$ is also a martingale, and it has bounded increments, i.e., $|\tilde{X_{n+1}} - \tilde{X}_n| \le M$. Further, for

$$\tilde{C} = \{\lim \tilde{X}_n \text{ exists and is finite}\}$$

$$\tilde{D} = \{\limsup X_n = \infty \text{ and } \liminf X_n = -\infty\},$$

$\tilde{C} = C$ and $\tilde{D} = D$ holds.) For any $K > 0$, define

$$N_K = \inf\{n \ge 1 : X_n \le -K\}.$$

Then

$$(N_K = n) = (\forall m < n\ X_m > -K,\ X_n \le -K) \in \mathcal{F}_n$$

for any n, so N_K is a stopping time, and hence $\{X_{n\wedge N_K} : n \ge 0\}$ is a martingale. Note that on $(N_K < \infty)$,

$$X_k > -K \text{ for } k = 1, 2, \cdots, N_K - 1,$$

and thus

$$X_{N_K} = X_{N_K-1} + \underbrace{(X_{N_K} - X_{N_K-1})}_{\ge -M} \ge -K - M,$$

and on $(N_K = \infty)$, $X_n > -K > -K - M$, so for any cases $X_{n\wedge N_K} + K + M \ge 0$. Thus

($X_{n\wedge N_K} + K + M$ is a nonnegative (super)martingale) by martingale convergence theorem, ($X_{n\wedge N_K}+K+M$, and consequently,) $X_{n\wedge N_K}$ converges almost surely to some integrable random variable. In particular, X_n converges (P-)a.s. "on $(N_K = \infty)$." (It means that, $\exists E \subseteq (N_K = \infty)$ s.t. $P\left((N_K = \infty)\backslash E\right) = 0$ and X_n converges pointwisely on E.) Since $K > 0$ was arbitrary, so X_n converges P-a.s. on $\bigcup_{K=1}^{\infty}(N_K = \infty)$. Now, from

$$(\liminf X_n > -\infty) \subseteq \bigcup_{K=1}^{\infty}(N_K = \infty),$$

($\because$ if $\forall K$ $(N_K < \infty)$, then for any K we can find n s.t. $X_n < -K$, i.e., $\liminf X_n = -\infty$) we can obtain that X_n converges P-a.s. on $(\liminf X_n > -\infty)$. Applying such procedure to $-X_n$ repeatedly, we can obtain that

$$-X_n \text{ converges on } (\liminf(-X_n) > -\infty) = (\limsup X_n < \infty).$$

Therefore, $\underbrace{X_n \text{ converges } P\text{-a.s.}}_{=C}$ on $\underbrace{(-\infty < \liminf X_n) \cup (\limsup X_n < \infty)}_{=D^c}$, i.e., $C \supseteq D^c$ (except probability zero set). It implies that $P(C \cup D) = 1$. $\square$

With this, we can find similar argument as "Borel-Cantelli Lemma" in filtered probability space. It can be also called as "conditional Borel-Cantelli lemma."

Theorem 2.3.24 (Second Borel Cantelli Lemma, "conditional")**.** *Let $\mathcal{F}_0 = \{\phi, \Omega\}$ and $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. If $A_n \in \mathcal{F}_n$ $\forall n \geq 1$, then*

$$(A_n \ i.o.) = \left(\sum_{n=1}^{\infty} P(A_n|\mathcal{F}_{n-1}) = \infty\right) \quad P-a.s.$$

Remark 2.3.25. If A_n's are independent set, letting $\mathcal{F}_n = \sigma(A_1, \cdots, A_n)$, we get $P(A_n|\mathcal{F}_{n-1}) = P(A_n)$, and hence

$$(A_n \text{ i.o.}) = \left(\sum_{n=1}^{\infty} P(A_n) = \infty\right),$$

i.e.,

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Leftrightarrow A_n \text{ i.o.}$$

In other words, "conditional" version of Borel-Cantelli lemma contains ordinary one.

Proof. Let $X_0 = 0$, and define

$$X_n = \sum_{m=1}^{n} \underbrace{\{I_{A_m} - P(A_m|\mathcal{F}_{m-1})\}}_{\in\mathcal{F}_m}.$$

Then X_n is a martingale, because

$$E(X_{n+1}|\mathcal{F}_n) = X_n + E\left(I_{A_{n+1}} - P(A_{n+1}|\mathcal{F}_n)|\mathcal{F}_n\right) = X_n.$$

Also, note that

$$|X_{n+1} - X_n| = |I_{A_{n+1}} - P(A_{n+1}|\mathcal{F}_n)| \le 1,$$

i.e., $\{X_n\}$ has bounded increments. Now on C, $X_n = \sum_{m=1}^{n} (I_{A_m} - P(A_m|\mathcal{F}_{m-1}))$ converges and is finite, so

$$\sum_{m=1}^{\infty} I_{A_m} = \infty \Leftrightarrow \sum_{m=1}^{\infty} P(A_m|\mathcal{F}_{m-1}) = \infty.$$

Note that $\sum_{m=1}^{\infty} I_{A_m} = \infty$ means A_n occurs infinitely often. On the other hand, on D, from $\sum_{m=1}^{n} I_{A_m} \ge X_n$, we get

$$\sum_{m=1}^{\infty} I_{A_m} \ge \limsup X_n \underset{D}{=} \infty,$$

and from

$$\sum_{m=1}^{n} P(A_m|\mathcal{F}_{m-1}) = \sum_{m=1}^{n} I_{A_m} - X_n \ge -X_n,$$

we get

$$\sum_{m=1}^{\infty} P(A_m|\mathcal{F}_{m-1}) \ge \limsup(-X_n) = -\liminf X_n = \infty.$$

Therefore, on D,

$$\sum_{m=1}^{\infty} I_{A_m} = \infty \text{ and } \sum_{m=1}^{\infty} P(A_m|\mathcal{F}_{m-1}) = \infty \text{ simultaneously.}$$

Now previous proposition ($P(C \cup D) = 1$) ends the proof. More precisely, from

$$\left(\sum_{m=1}^{\infty} I_{A_m} = \infty\right) \cap C = \left(\sum_{m=1}^{\infty} P(A_m|\mathcal{F}_{m-1}) = \infty\right) \cap C$$

and

$$\left(\sum_{m=1}^{\infty} I_{A_m} = \infty\right) \cap D = \left(\sum_{m=1}^{\infty} P(A_m|\mathcal{F}_{m-1}) = \infty\right) \cap D,$$

we get

$$\left(\sum_{m=1}^{\infty} I_{A_m} = \infty\right) \cap (C \cup D) = \left(\sum_{m=1}^{\infty} P(A_m|\mathcal{F}_{m-1}) = \infty\right) \cap (C \cup D),$$

i.e.,

$$\left(\sum_{m=1}^{\infty} I_{A_m} = \infty\right) = \left(\sum_{m=1}^{\infty} P(A_m|\mathcal{F}_{m-1}) = \infty\right) \;\; P - \text{a.s.}$$

$\square$

Branching Process

Definition 2.3.26 (Branching process)**.** *Let ξ_i^n, $i, n \geq 0$ be i.i.d. nonnegative integer valued random variables, and $Z_0 = 1$. Now define*

$$Z_{n+1} = \begin{cases} \sum_{k=1}^{Z_n} \xi_k^{n+1} & \text{if} \;\; Z_n > 0 \\ 0 & \text{if} \;\; Z_n = 0 \end{cases}.$$

$(Z_n)_{n\geq 0}$ is called a branching process.

Remark 2.3.27. In here, Z_n can be interpreted as "a population in generation n." In nth generation, each (Z_n) individual produces some random number of individuals in $(n+1)$th generation. If Z_n becomes 0, it denotes "extinction." In this model, our interest is "the probability of ultimate extinction." It is known that for $\mu = E\xi_i^n$, if $\mu < 1$, then population ultimately extincts with probability 1, and if $\mu > 1$, then the probability of ultimate extinction is less than 1 (but not necessarily zero). In this lecture, we will see the case $\mu < 1$.

Lemma 2.3.28. *Let $\mathcal{F}_n = \sigma(\xi_i^m : i \geq 1, \; 1 \leq m \leq n)$. ($m$ denotes "generation") Then under the assumption $0 < \mu < \infty$,*

$$\frac{Z_n}{\mu^n} \;\; \textit{is } (\mathcal{F}_n) - \textit{martingale.}$$

Proof. First, it is clear that $Z_n \in \mathcal{F}_n$. Next,

$$\begin{aligned} E\left[\left.\frac{Z_{n+1}}{\mu^{n+1}}\right| \mathcal{F}_n\right] &= \sum_{k=0}^{\infty} \frac{1}{\mu^{n+1}} E\left[Z_{n+1} I(Z_n = k)\right| \mathcal{F}_n] \;\; \text{(conditional MCT)} \\ &= \sum_{k=0}^{\infty} \frac{1}{\mu^{n+1}} E\left[\left. \underbrace{(\xi_i^{n+1} + \cdots + \xi_k^{n+1})}_{\text{independent of } \mathcal{F}_n} \underbrace{I(Z_n = k)}_{\in \mathcal{F}_n} \right| \mathcal{F}_n \right] \\ &= \sum_{k=0}^{\infty} I(Z_n = k) \cdot \frac{E(\xi_i^{n+1} + \cdots + \xi_k^{n+1})}{\mu^{n+1}} \end{aligned}$$

$$= \frac{1}{\mu^n} \sum_{k=0}^{n} I(Z_n = k)k$$
$$= \frac{1}{\mu^n} \sum_{k=0}^{n} I(Z_n = k)Z_n$$
$$= \frac{Z_n}{\mu^n}$$

holds. □

Theorem 2.3.29. *If $0 < \mu < 1$, then $Z_n = 0$ $\forall$ large n, P-a.s.*

Proof. Since Z_n is integer, $P(Z_n > 0) = P(Z_n \geq 1) \leq E(Z_n I(Z_n \geq 1)) = E(Z_n I(Z_n > 0))$, and so

$$P(Z_n > 0) \leq E(Z_n I(Z_n > 0)) = E(Z_n I(Z_n > 0) + Z_n I(Z_n = 0)) = EZ_n = \mu^n$$

holds. The last equality is from $E(\mu^{-n} Z_n) = E(\mu^{-0} Z_0) = 1$ ($\because \mathcal{F}_0 = \{\phi, \Omega\}$). Thus we get $P(Z_n > 0) \leq \mu^n$, and therefore, by Borel-Cantelli lemma, $Z_n = 0$ holds for all but finite n. □

It also implies that,

$$\frac{Z_n}{\mu^n} \xrightarrow[n\to\infty]{a.s} 0.$$

2.3.3 Doob's inequality

Proposition 2.3.30. *If $\{X_n\}$ is a submartingale, and N is a stopping time with $P(N \leq k) = 1$ for some $k \geq 0$, then*

$$EX_0 \leq EX_N \leq EX_k.$$

Proof. Note that $X_{n\wedge N}$ is a submartingale. Thus,

$$EX_0 = EX_{0\wedge N} \leq EX_{k\wedge N} = EX_N$$

holds. Thus our claim is:

Claim. $EX_N \leq EX_k$.

Let $K_n = I(N \leq n-1)$. Then $K_n \in \mathcal{F}_{n-1}$ so it is predictable, and hence we can define $(K \cdot X)_n$. Then since

$$I(N < m \leq n) = I(N \wedge n < m \leq n) = I(N \wedge n + 1 \leq m \leq n),$$

we get

$$\begin{aligned}
(K \cdot X)_n &= \sum_{m=1}^{n} I(N \le m-1)(X_m - X_{m-1}) \\
&= \sum_{m=1}^{n} I(N < m \le n)(X_m - X_{m-1}) \\
&= \sum_{m=1}^{n} I(N \wedge n + 1 \le m \le n)(X_m - X_{m-1}) \\
&= \sum_{N \wedge n+1}^{n} (X_m - X_{m-1}) \\
&= X_n - X_{N \wedge n}.
\end{aligned}$$

Note that $(K \cdot X)_n$ is also a submartingale; hence we get

$$E(K \cdot X)_k = EX_k - EX_{N \wedge k} \ge E(K \cdot X)_1 = E[I(N=0)(X_1 - X_0)] = E\left[I(N=0) \underbrace{E(X_1 - X_0|\mathcal{F}_0)}_{\ge 0} \right] \ge 0,$$

i.e.,

$$EX_k \ge EX_{N \wedge k}.$$

However, $N \wedge k = N$, so we get the conclusion. $\square$

Theorem 2.3.31 (Submartingale Ineqaulity)**.** *Let X_n be a submartingale. Then for $\tilde{X}_n = \max_{0 \le m \le n} X_m$ and $\lambda > 0$,*

$$\lambda P(\tilde{X}_n \ge \lambda) \le EX_n I(\tilde{X}_n \ge \lambda) \le EX_n^+ I(\tilde{X}_n \ge \lambda) \le EX_n^+.$$

Proof. Let $A = (\tilde{X}_n \ge \lambda)$, and $N = \inf\{m \le n : X_m \ge \lambda\} \wedge n$. Then N is a stopping time less than n. Note that

$$X_N I_A \ge \lambda I_A$$

holds (On A. $\exists m \le n$ s.t. $X_m \ge \lambda$, so $X_N \ge \lambda$. On A^c, both sides are all zero). Therefore, we get

$$\lambda P(A) \le EX_N I_A = EX_N - EX_N I_{A^c} \underset{(*)}{\le} EX_n - EX_n I_{A^c} = EX_n I_A.$$

$(*)$ is obtained from $EX_N \le EX_n$ (previous proposition), and on A^c, $N = n$, i.e., $X_N = X_n$. $\square$

If X_n is a submartingale, then $-X_n$ is a supermartingale; thus, we can also get a similar result for a supermartingale.

Corollary 2.3.32 (Supermartingale inequality). *Let X_n be a supermartingale. Then*

$$\lambda P(\tilde{X}_n \geq \lambda) \leq EX_0 - EX_n I_{A^c} \leq EX_0 + EX_n^-.$$

Proof. Let $N = \inf\{m \leq n : X_m \geq \lambda\} \wedge n$. Then

$$EX_0 \geq EX_N = EX_N I_A + EX_N I_{A^c} \underset{(*)}{\geq} \lambda P(A) + EX_n I_{A^c}$$

holds. $(*)$ holds from: On A^c, $N = n$, and on A. $X_N \geq \lambda$. ☐

Following is called **Doob's inequality**, or **Doob's maximal inequality**, which is very important result in martingale theory.

Theorem 2.3.33 (Doob's maximal inequality). *If X_n is a nonnegative submartingale, then for $1 < p < \infty$,*

$$E \max_{1 \leq m \leq n} X_m^p \leq \left(\frac{p}{p-1}\right)^p EX_n^p.$$

We often use the case $p = 2$, i.e.,

$$E \max_{1 \leq m \leq n} X_m^2 \leq 4EX_n^2.$$

Proof. If $E\tilde{X}_n^p = 0$, then $X_n = 0$ almost surely for any n, so there is nothing to show. So we may assume that $E\tilde{X}_n^p > 0$. Let $M > 0$. Then

$$\begin{aligned}
E(\tilde{X}_n \wedge M)^p &= \int_0^\infty P(\tilde{X}_n \wedge M > y) p y^{p-1} dy \ (\because \text{Fubini}) \\
&= \int_0^M p y^{p-1} P(\tilde{X}_n \wedge M > y) dy \\
&\leq \int_0^M p y^{p-1} P(\tilde{X}_n > y) dy \\
&\leq \int_0^M p y^{p-1} \cdot \frac{1}{y} E X_n I(\tilde{X}_n \geq y) dy \ (\because \lambda P(A) \leq EX_n I_A) \\
&= \int_0^M \int X_n I(\tilde{X}_n \geq y) d\mathbb{P} p y^{p-2} dy \\
&= \int X_n \left(\int_0^M I(\tilde{X}_n \geq y) p y^{p-2} dy \right) d\mathbb{P} \\
&= \int X_n \int_0^{M \wedge \tilde{X}_n} p y^{p-2} dy d\mathbb{P}
\end{aligned}$$

$$
\begin{aligned}
&= E\left[X_n \cdot \frac{p}{p-1}\left(\tilde{X}_n \wedge M\right)^{p-1}\right] \\
&= \frac{p}{p-1} E\left[X_n \cdot \left(\tilde{X}_n \wedge M\right)^{p-1}\right]
\end{aligned}
$$

holds. Now let q be a Hölder conjugate of p, i.e., $q = \dfrac{p}{p-1}$. Then by Hölder inequality,

$$
\begin{aligned}
E\left[X_n \cdot \left(\tilde{X}_n \wedge M\right)^{p-1}\right] &\le (E(X_n^p))^{\frac{1}{p}} \left(E\left[(\tilde{X}_n \wedge M)^{p-1}\right]^q\right)^{\frac{1}{q}} \\
&= (E(X_n^p))^{\frac{1}{p}} \left(E(\tilde{X}_n \wedge M)^p\right)^{\frac{1}{q}}
\end{aligned}
$$

is obtained, and hence, we get

$$
E(\tilde{X}_n \wedge M)^p \le \frac{p}{p-1} (E(X_n^p))^{\frac{1}{p}} \left(E(\tilde{X}_n \wedge M)^p\right)^{\frac{1}{q}}.
$$

It is equivalent to

$$
\left(E(\tilde{X}_n \wedge M)^p\right)^{\frac{1}{p}} \le \frac{p}{p-1} (E(X_n^p))^{\frac{1}{p}},
$$

and therefore

$$
E(\tilde{X}_n \wedge M)^p \le \left(\frac{p}{p-1}\right)^p E(X_n^p).
$$

As it holds for any $M > 0$, letting $M \to \infty$, we get

$$
E\tilde{X}_n^p \le \left(\frac{p}{p-1}\right)^p E(X_n^p)
$$

with MCT. □

2.3.4 Stopping time and filtration

Definition 2.3.34. *Let $(\Omega, (\mathcal{F}_n)_{n\ge 0}, P)$ be a filtered probability space, and τ be a stopping time. Then $\mathcal{F}_\tau$ is defined as*

$$
\mathcal{F}_\tau := \{A : A \cap (\tau = n) \in \mathcal{F}_n \ \forall n\}.
$$

Remark 2.3.35. Note that $\mathcal{F}_\tau$ is a σ-field.

(i) $\phi \in \mathcal{F}_\tau$, because for any n, $\phi \cap (\tau = n) = \phi \in \mathcal{F}_n$.

(ii) If $A \in \mathcal{F}_\tau$, for any n, $A^c \cap (\tau = n) = (\tau = n) \cap \{A \cap (\tau = n)\}^c \in \mathcal{F}_n$, so $A^c \in \mathcal{F}_\tau$.

(iii) If $A_k \in \mathcal{F}_\tau$ for $k = 1, 2, \cdots$, then $(\cup_k A_k) \cap (\tau = n) = \cup_k (A_k \cap (\tau = n)) \in \mathcal{F}_n$, so $\cup_k A_k \in \mathcal{F}_\tau$.

Also, τ is $\mathcal{F}_\tau$-measurable, because for any k we get $(\tau = k) \in \mathcal{F}_\tau$, from

$$(\tau = k) \cap (\tau = n) = \begin{cases} (\tau = n) & n = k \\ \phi & n \neq k \end{cases} \in \mathcal{F}_n.$$

Following theorem is one version of **optional sampling theorem**, which is very important result. In here, we only see for bounded stopping times. We will deal with the general one later.

Theorem 2.3.36 ((Bounded) Optional Sampling Theorem)**.** *Let X_n be a submartingale and $\sigma \le \tau$ be bounded stopping times. Then,*

$$E(X_\tau | \mathcal{F}_\sigma) \ge X_\sigma \ P-a.s.$$

Especially, if X_n is a martingale, then

$$E(X_\tau | \mathcal{F}_\sigma) = X_\sigma \ P-a.s.$$

This statement seems very intuitive, by the definition of (sub)martingale.

Proof. From boundedness, we can find $B > 0$ s.t. $\sigma \le \tau \le B \in \mathbb{N}$. First, our claim is:

Claim. $E(X_\tau|\mathcal{F}_\sigma)I(\sigma = n) = E(X_\tau|\mathcal{F}_n)I(\sigma = n)$ P-a.s.

Proof of Claim.) For any $a \in \mathbb{R}$ and $k = 0, 1, 2, \cdots$, we get

$$\begin{aligned}(E(X_\tau|\mathcal{F}_n)I(\sigma = n) \le a) \cap (\sigma = k) &= \{(E(X_\tau|\mathcal{F}_n) \le a) \cap (\sigma = n) \cap (\sigma = k)\} \\ &\quad \cup \{(0 \le a) \cap (\sigma \neq n) \cap (\sigma = k)\} \\ &= \begin{cases} (E(X_\tau|\mathcal{F}_n) \le a) \cap (\sigma = n) & n = k \\ (0 \le a) \cap (\sigma = k) & n \neq k \end{cases} \in \mathcal{F}_k\end{aligned}$$

and hence $(E(X_\tau|\mathcal{F}_n)I(\sigma = n) \le a) \in \mathcal{F}_\sigma$ $\forall a$. It implies $E(X_\tau|\mathcal{F}_n)I(\sigma = n) \in \mathcal{F}_\sigma$ $(\star)$. Thus, for any $A \in \mathcal{F}_\sigma$,

$$\begin{aligned}\int_A E(X_\tau|\mathcal{F}_\sigma)I(\sigma = n)dP &= \underbrace{\int_{A\cap(\sigma=n)}}_{\in \mathcal{F}_\sigma} E(X_\tau|\mathcal{F}_\sigma)dP \\ &= \underbrace{\int_{A\cap(\sigma=n)}}_{\in \mathcal{F}_n} X_\tau dP \text{ (def. of conditional expectation)}\end{aligned}$$

$$= \int_{A\cap(\sigma=n)} E(X_\tau|\mathcal{F}_n)dP$$
$$= \int_A E(X_\tau|\mathcal{F}_n)I(\sigma=n)dP$$

holds, which implies

$$\int_A \underbrace{(E(X_\tau|\mathcal{F}_\sigma)I(\sigma=n) - E(X_\tau|\mathcal{F}_n)I(\sigma=n))}_{\in\mathcal{F}_\sigma\ (\because(\star))} dP = 0\ \forall A\in\mathcal{F}_\sigma.$$

Hence we get

$$E(X_\tau|\mathcal{F}_\sigma)I(\sigma=n) - E(X_\tau|\mathcal{F}_n)I(\sigma=n) = 0.$$

(Recall that: if f is $\mathcal{G}$-mb, and $\int_A f = 0$ for any $A\in\mathcal{G}$, then $f=0$ a.e.: Take $A=(f>0)$ and $A=(f<0)$!)

□ (Claim)

Back to our main theorem. To show $E(X_\tau|\mathcal{F}_\sigma)\ge X_\sigma$, it is sufficient to show that:

$$E(X_\tau|\mathcal{F}_\sigma)I(\sigma=n)\ge X_\sigma I(\sigma=n)\ \forall n=0,1,\cdots,B.$$

From $X_\sigma I(\sigma=n) = X_n I(\sigma=n)$ and $E(X_\tau|\mathcal{F}_\sigma)I(\sigma=n) = E(X_\tau|\mathcal{F}_n)I(\sigma=n)$ (Claim), for any $A\in\mathcal{F}_n$, we get

$$\begin{aligned}
\int_A X_\sigma I(\sigma=n)dP - \int_A E(X_\tau|\mathcal{F}_\sigma)I(\sigma=n)dP &= \int_A X_n I(\sigma=n)dP - \int_A E(X_\tau|\mathcal{F}_n)I(\sigma=n)dP\\
&= \int_{A\cap(\sigma=n)} X_n dP - \int_{\underbrace{A\cap(\sigma=n)}_{\in\mathcal{F}_n}} E(X_\tau|X_n)dP\\
&= \int_{A\cap(\sigma=n)} X_n dP - \int_{A\cap(\sigma=n)} X_\tau dP\\
&= \int_{A\cap(\sigma=n)} (X_n - X_\tau)dP\\
&\underset{\sigma\le\tau}{=} \int_{A\cap(\sigma=n)\cap(\tau\ge n)} (X_n - X_\tau)dP\\
&= \underbrace{\int_{A\cap(\sigma=n)\cap(\tau=n)} (X_n - X_\tau)dP}_{=0\ (\because X_n=X_\tau\text{ on }(\tau=n))}\\
&\quad + \int_{A\cap(\sigma=n)\cap(\tau\ge n+1)} (X_n - X_\tau)dP\\
&= \int_{A\cap(\sigma=n)\cap(\tau\ge n+1)} (X_n - X_\tau)dP
\end{aligned}$$

$$
\begin{aligned}
&\leq \underbrace{\int_{A\cap(\sigma=n)\cap(\tau\geq n+1)}}_{\in\mathcal{F}_n\ (\because(\tau\geq n+1)=(\tau\leq n)^c)} (E(X_{n+1}|\mathcal{F}_n)-X_\tau)dP \\
&= \int_{A\cap(\sigma=n)\cap(\tau\geq n+1)} (X_{n+1}-X_\tau)dP \\
&\quad \text{(def. of conditional expectation)} \\
&\leq \int_{A\cap(\sigma=n)\cap(\tau\geq n+2)} (X_{n+2}-X_\tau)dP \text{ (Same way)} \\
&\leq \cdots \\
&\leq \int_{A\cap(\sigma=n)\cap(\tau\geq B)} (X_B-X_\tau)dP \\
&= \int_{A\cap(\sigma=n)\cap(\tau=B)} (X_B-X_\tau)dP\ (\because \tau\leq B) \\
&= 0,
\end{aligned}
$$

i.e.,

$$\int_A \underbrace{(X_\sigma I(\sigma=n)-E(X_\tau|\mathcal{F}_\sigma)I(\sigma=n))}_{\in\mathcal{F}_n} dP \leq 0\ \forall A\in\mathcal{F}_n.$$

Therefore, we get

$$X_\sigma I(\sigma=n)\leq E(X_\tau|\mathcal{F}_\sigma)I(\sigma=n)\ P-\text{a.s.},$$

which ends the proof. Also recall that: if $f\in\mathcal{G}$ and $\int_A f\geq 0\ \forall A\in\mathcal{G}$, with taking $A=(f<0)$, we get $f\geq 0$. □

2.4 Uniform Integrability

Definition 2.4.1. *The family $\{X_t : t\in T\}$ of random variables is said to be* ***uniformly integrable*** *if*

$$\lim_{a\to\infty}\sup_{t\in T}\int_{|X_t|\geq a}|X_t|dP=0.$$

Example 2.4.2. If $\exists X\in\mathcal{L}^1$ s.t. $|X_t|\leq X\ \forall t\in T$, then

$$\int_{|X_t|\geq a}|X_t|dP\leq\int_{|X_t|\geq a}|X|dP\leq\int_{|X|\geq a}|X|dP=aP(|X|\geq a)\xrightarrow[a\to\infty]{}0,$$

so $\{X_t : t\in T\}$ is uniformly integrable. Especially, *the set of finite number of intebrable r.v.'s* is uniformly integrable.

Following proposition shows equivalent condition of uniform integrability. Such equivalence is very useful.

Proposition 2.4.3. $\{X_t : t \in T\}$ *is uniformly integrable if and only if*

(a) $\sup_{t\in T} E|X_t| < \infty$.

(b) $\forall \epsilon > 0$ $\delta > 0$ *s.t.*

$$A \in \mathcal{F},\ P(A) < \delta \Rightarrow \sup_{t\in T} \int_A |X_t| dP < \epsilon.$$

Proof. $\Rightarrow$) (a) Let a be s.t. $\sup_{t\in T} E[|X_t| I(|X_t| \ge a)] \le 1$ (Such a exists because it converges to 0 as $a \to \infty$). Then for any $t \in T$

$$E|X_t| = \underbrace{E|X_t| I(|X_t| < a)}_{\le a} + \underbrace{E|X_t| I(|X_t| \ge a)}_{\le 1} \le a + 1$$

holds, and hence,

$$\sup_{t\in T} E|X_t| \le a + 1 < \infty.$$

(b) Let $A \in \mathcal{F}$ and $a > 0$. Now note that

$$\begin{aligned}
\int_A |X_t| dP &= \int_{A\cap(|X_t|\ge a)} |X_t| dP + \int_{A\cap(|X_t|<a)} |X_t| dP \\
&\le \int_{|X_t|\ge a} |X_t| dP + \int_{|X_t|<a} a I_A dP \\
&\le \int_{|X_t|\ge a} |X_t| dP + aP(A)
\end{aligned}$$

holds. Thus we get

$$\sup_{t\in T} \int_A |X_t| dP \le \sup_{t\in T} \int_{|X_t|\ge a} |X_t| dP + aP(A).$$

Now choose a_0 s.t.

$$\sup_{t\in T} \int_{|X_t|\ge a_0} |X_t| dP < \frac{\epsilon}{2}$$

and let $\delta = \epsilon/2a_0$. Then for measurable set A s.t. $P(A) < \delta$,

$$\sup_{t\in T} \int_A |X_t| dP \le \sup_{t\in T} \int_{|X_t|\ge a} |X_t| dP + aP(A) \le \frac{\epsilon}{2} + a_0\delta = \epsilon$$

holds.

$\Leftarrow$) Let $\epsilon > 0$ be arbitrarily given, and $\delta > 0$ be the real number satisfying (b). Now put

$$M = \sup_{t\in T} E|X_t| \underset{\text{(a)}}{<} \infty$$

and let $a_0 = M/\delta$. Then

$$P(|X_t| \ge a_0) \le \frac{E|X_t|}{a_0} \le \frac{M}{a_0} = \delta,$$

so by (b),

$$\sup_{s\in T} \int_{|X_t|\ge a_0} |X_s|dP < \epsilon$$

holds for any $t \in T$. It implies that

$$\sup_{t\in T} \int_{|X_t|\ge a_0} |X_t|dP \le \sup_{t\in T} \sup_{s\in T} \int_{|X_t|\ge a_0} |X_s|dP < \epsilon.$$

Now for any $a \ge a_0$,

$$\sup_{t\in T} \int_{|X_t|\ge a} |X_t|dP < \epsilon$$

holds, i.e.,

$$\sup_{t\in T} \int_{|X_t|\ge a} |X_t|dP \xrightarrow[a\to\infty]{} 0.$$

□

Recall that, even if $X_n \xrightarrow[n\to\infty]{a.s} X$, we cannot guarantee that $X_n \xrightarrow[n\to\infty]{\mathcal{L}^1} X$, or, $EX_n \not\to EX$. (See example 2.3.18) However, with uniform integrability, we can say that convergence in probability is equivalent to $\mathcal{L}^1$-convergence.

Theorem 2.4.4 (Vitalli's Lemma)**.** *Suppose that $X_n \xrightarrow[n\to\infty]{P} X$, and $X_n \in \mathcal{L}^r$ for $r \ge 1$. Then TFAE.*

(i) $\{|X_n|^r : n \ge 1\}$ *is uniformly integrable.*

(ii) $X_n \xrightarrow[n\to\infty]{\mathcal{L}^r} X$*, i.e.,* $E|X_n - X|^r \xrightarrow[n\to\infty]{} 0$.

(iii) $E|X_n|^r \xrightarrow[n\to\infty]{} E|X|^r$.

To show this, we need some basic properties of uniform integrable sequences.

Lemma 2.4.5. *(a) If $\{X_n\}$ and $\{Y_n\}$ are both uniformly integrable, then so is $\{X_n + Y_n\}$.*

(b) If $\{X_n\}$ is uniformly integrable and $|Y_n| \le |X_n|$, then $\{Y_n\}$ is also uniformly integrable.

Proof of lemma. (a) We get the result from

$$\sup_n \int_{|X_n+Y_n|\ge a} |X_n + Y_n|dP \le \sup_n \int_{|X_n|+|Y_n|\ge a} |X_n + Y_n|dP$$

$$\leq \sup_n \left(\int_{\substack{|X_n|+|Y_n|\geq a \\ |X_n|\geq |Y_n|}} (|X_n|+|Y_n|)dP + \int_{\substack{|X_n|+|Y_n|\geq a \\ |X_n|<|Y_n|}} (|X_n|+|Y_n|)dP \right)$$
$$\leq \sup_n \left(\int_{2|X_n|\geq a} 2|X_n|dP + \int_{2|Y_n|\geq a} 2|Y_n|dP \right)$$
$$\leq \sup_n \int_{|X_n|\geq a/2} 2|X_n|dP + \sup_n \int_{|Y_n|\geq a/2} 2|Y_n|dP$$
$$\xrightarrow[a\to\infty]{} 0.$$

(b) Clear. □

Proof. (i)⇒(ii): Since $X_n \xrightarrow[n\to\infty]{P} X$, $\exists \{n'\} \subseteq \{n\}$ s.t. $X_{n'} \xrightarrow[n'\to\infty]{a.s.} X$. Then by Fatou's lemma,

$$E|X|^r \leq \liminf_{n'\to\infty} E|X_{n'}|^r \leq \sup_n E|X_n|^r < \infty,$$

so $X \in \mathcal{L}^r$. Now from

$$|X_n - X|^r \leq 2^r(|X_n|^r + |X|^r),$$

($\because$ $|a+b|^r \leq 2^r|a|^r$ if $|a| \geq |b|$, and $|a+b|^r \leq 2^r|b|^r$otherwise, so $|a+b|^r \leq 2^r(|a|^r+|b|^r)$) $\{|X_n - X|^r : n \geq 1\}$ is uniformly integrable. Thus, $\forall \epsilon > 0$ $\exists \delta > 0$ s.t.

$$A \in \mathcal{F},\ P(A) < \delta \Rightarrow \int_A |X_n - X|^r dP < \epsilon.$$

Now by assumption, $|X_n - X|^r \xrightarrow[n\to\infty]{P} 0$, and hence $\exists N$ s.t. $P(|X_n - X|^r > \epsilon) \leq \delta$ for any $n > N$. Now

$$n \geq N \Rightarrow E|X_n - X|^r = \underbrace{E|X_n - X|^r I(|X_n - X|^r > \epsilon)}_{\leq \epsilon\ (\because \text{U.I.})} + \underbrace{E|X_n - X|^r I(|X_n - X|^r \leq \epsilon)}_{\leq \epsilon} \leq 2\epsilon$$

holds, i.e.,

$$E|X_n - X|^r \xrightarrow[n\to\infty]{} 0.$$

(ii)⇒(iii): Let $\|X\|_r = (E|X|^r)^{1/r}$. Then by Minköwski inequality,

$$|\|X\|_r - \|X_n\|_r| \leq \|X - X_n\|_r \xrightarrow[n\to\infty]{} 0$$

holds, i.e., $\|X_n\|_r \to \|X\|_r$. It implies $E|X_n|^r \xrightarrow[n\to\infty]{} E|X|^r$.

(iii)⇒(i): We can find infinitely many $a > 0$ s.t. $P(|X|^r = a) = 0$. Since $X_n \xrightarrow[n\to\infty]{P} X$, if one can show

$$I(|X_n|^r \le a) \xrightarrow[n\to\infty]{P} I(|X|^r \le a),$$

then we get

$$|X_n|^r I(|X_n|^r \le a) \xrightarrow[n\to\infty]{P} |X|^r I(|X|^r \le a).$$

Claim. $I(|X_n|^r \le a) \xrightarrow[n\to\infty]{P} I(|X|^r \le a)$.

Let $a_n = P(|I(|X_n|^r \le a) \xrightarrow[n\to\infty]{P} I(|X|^r \le a)| > \epsilon)$. Then for small ϵ,

$$\begin{aligned}
a_n &= P(|I(|X_n|^r \le a) \xrightarrow[n\to\infty]{P} I(|X|^r \le a)| > \epsilon) \\
&= P(|X_n|^r \le a,\ |X|^r > a) + P(|X_n|^r < a,\ |X|^r \le a) \\
&= P(|X_n|^r \le a,\ |X|^r > a+\delta) + P(|X_n|^r \le a,\ a < |X|^r \le a+\delta) \\
&+ P(|X_n|^r > a,\ |X|^r \le a-\delta) + P(|X_n|^r > a,\ a-\delta < |X|^r \le a) \\
&\le P\left(||X_n|^r - |X|^r| > \delta\right) + P(a < |X|^r \le a+\delta) \\
&+ P\left(||X_n|^r - |X|^r| > \delta\right) + P(a-\delta < |X|^r \le a) \\
&= P\left(||X_n|^r - |X|^r| > \delta\right) + P(a-\delta < |X|^r \le a+\delta)
\end{aligned}$$

holds, for arbitrary $\delta > 0$. Thus we get

$$0 \le \limsup_{n\to\infty} a_n \le P(a-\delta < |X|^r \le a+\delta),$$

and letting $\delta \searrow 0$, we get

$$0 \le \limsup_{n\to\infty} a_n \le P(|X|^r = a) = 0,$$

i.e., $a_n \xrightarrow[n\to\infty]{} 0$. □ (Claim) Now,

i) $\{|X_n|^r I(|X_n|^r \le a) : n \ge 1\}$ is the collection of bounded random variables, so it is uniformly integrable.

ii) $|X_n|^r I(|X_n|^r \le a) \xrightarrow[n\to\infty]{P} |X|^r I(|X|^r \le a)$.

So by (i)⇒(iii) of this theorem, we get

$$E|X_n|^r I(|X_n|^r \le a) \xrightarrow[n\to\infty]{} E|X|^r I(|X|^r \le a)$$

holds. The assumption says $E|X_n|^r \to E|X|^r$, so

$$E|X_n|^r I(|X_n|^r > a) \xrightarrow[n\to\infty]{} E|X|^r I(|X|^r > a)$$

holds. Since such a is uncountably many, for any $\epsilon > 0$, we can choose a_0 s.t. $E|X|^r I(|X|^r > a_0) < \epsilon/2$, and then we can find $n > N$ s.t.

$$a \geq a_0 \Rightarrow E|X_n|^r I(|X_n|^r > a) \leq E|X_n|^r I(|X_n|^r > a_0) \leq \epsilon \ \forall n > N.$$

Now let $a_1, \cdots, a_N$ be s.t.

$$E|X_n|^r I(|X_n|^r > a_n 0 \leq \epsilon \text{ for } n = 1, 2, \cdots, N,$$

and $a^* = \max(a_0, a_1, \cdots, a_N)$. Then,

$$a \geq a^* \Rightarrow E|X_n|^r I(|X_n|^r > a) \leq E|X_n|^r I(|X_n|^r > a^*) \leq \epsilon$$

holds for any $n \geq 1$, which implies

$$\sup_n E|X_n|^r I(|X_n|^r > a) \leq \epsilon \ \forall a \geq a^*.$$

Since $\epsilon > 0$ was arbitrary, we get

$$\sup_n E|X_n|^r I(|X_n|^r > a) \xrightarrow[a\to\infty]{} 0.$$

□

Corollary 2.4.6. *Let $X_n \xrightarrow[n\to\infty]{d} X$ and $\{X_n : n \geq 1\}$ be uniformly integrable. Then $E|X_n| \to E|X|$ and $EX_n \to EX$ as $n \to \infty$.*

Proof. By Skorohod theorem, we can find a probability space $(\Omega', \mathcal{F}', P')$ and r.v.'s on this new probability space X'_n and X', such that

$$X'_n \stackrel{d}{\equiv} X_n, X' \stackrel{d}{\equiv} X, \text{ and } X'_n \xrightarrow[n\to\infty]{} X' \ P' - \text{a.s.}.$$

Then

$$\sup_n E'|X'_n| I(|X'_n| \geq a) = \sup_n E|X_n| I(|X_n| \geq a)$$

holds, so $\{X'_n : n \geq 1\}$ is uniformly integrable. Then we get

$$E'|X'_n| \to E'|X'|$$

and

$$E'|X'_n - X'| \xrightarrow[n\to\infty]{} 0,$$

which implies

$$E|X_n| \to E|X| \text{ and } EX_n \to EX \text{ as } n \to \infty.$$

□

2.4.1 Uniform integrable martingales

Now back to the martingale theory.

Definition 2.4.7. *(1) A martingale $(X_n, \mathcal{F}_n)_{n\geq 0}$ is said to be **regular** if $\exists X \in \mathcal{L}^1$ s.t. $X_n = E(X|\mathcal{F}_n)$ (P-a.s.).*

*(2) A martingale $(X_n, (\mathcal{F}_n))_{n\geq 0}$ is said to be **closable** if $\exists X_\infty \in \mathcal{L}^1$ s.t. X_∞ is $\mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n)$-mb and $(X_n, \mathcal{F}_n)_{0\leq n\leq\infty}$ becomes a martingale, i.e.,*

$$E(X_\infty|\mathcal{F}_n) = X_n \ \forall n \geq 0.$$

Note that, closable martingale is obviously regular. However, regular martingale may not be closable, because such X need not be $\mathcal{F}_\infty$-measurable. Nevertheless, under uniform integrability, we get equivalence of both conditions.

Theorem 2.4.8. *Let $\{X_n\}$ be a martingale. Then TFAE.*

(i) $\{X_n\}$ is regular.

(ii) $\{X_n\}$ is uniformly integrable, and it converges a.s. to some X.

(iii) X_n converges in $\mathcal{L}^1$, i.e., $E|X_n - X| \to 0$.

(iv) $\{X_n\}$ is closable martingale, i.e., $E(X_\infty|\mathcal{F}_n) = X_n$ where $X_\infty = \lim X_n$ a.s.

Proof. (ii) ⇒ (iii) : Vitali's lemma.

(iv) ⇒ (i) : Definition.

(i) ⇒ (ii) : Since X_n is regular, we can write $X_n = E(X|\mathcal{F}_n)$ for some $X \in \mathcal{L}^1$. First, from

$$|X_n| = |E(X|\mathcal{F}_n)| \leq E\left(|X||\mathcal{F}_n\right),$$

we get

$$E|X_n| \le E|X|,$$

and hence

$$\sup_n E|X_n| < \infty.$$

Next, since $(|X_n| \ge a) \in \mathcal{F}_n$, by the definition of conditional expectation,

$$\begin{aligned}
\int_{|X_n|\ge a} |X_n| dP &\le \int_{|X_n|\ge a} E(|X||\mathcal{F}_n) dP \\
&= \int_{|X_n|\ge a} |X| dP \\
&= \int_{|X_n|\ge a, |X|\le b} |X| dP + \int_{|X_n|\ge a, |X|>b} |X| dP \\
&\le bP(|X_n| \ge a) + \int_{|X|>b} |X| dP \\
&\le \frac{b}{a} E|X_n| + \int_{|X|>b} |X| dP \\
&\le \frac{b}{a} E|X| + \int_{|X|>b} |X| dP
\end{aligned}$$

holds for any $b > 0$, and hence

$$\sup_n \int_{|X_n|\ge a} |X_n| dP \le \frac{b}{a} E|X| + \int_{|X|>b} |X| dP$$

also holds. Letting $a \to \infty$, we get

$$\limsup_{a\to\infty} \sup_n \int_{|X_n|\ge a} |X_n| dP \le \int_{|X|>b} |X| dP.$$

Since $b > 0$ was arbitrary, letting $b \to \infty$, by integrability of X, we get

$$\limsup_{a\to\infty} \sup_n \int_{|X_n|\ge a} |X_n| dP = 0.$$

Therefore $\{X_n\}$ is uniformly integrable. An a.s. convergence comes from martingale convergence theorem, since $\sup_n E|X_n| < \infty$.

(iii) $\Rightarrow$ (iv) : Suppose that $E|X_n - X| \to 0$ as $n \to \infty$, for some X. It means that

$$\forall \epsilon > 0,\ \exists N \text{ s.t. } n \ge N \Rightarrow E|X_n - X| \le \epsilon,$$

and hence

$$E|X_n| \le E|X| + \epsilon,$$

i.d., $\sup_n E|X_n| < \infty$. Then by martingale convergence theorem, $\exists X_\infty$ s.t.

$$X_n \xrightarrow[n\to\infty]{a.s} X_\infty.$$

Now,
i) $X_n \xrightarrow[n\to\infty]{P} X_\infty$.
ii)$X_n \xrightarrow[n\to\infty]{\mathcal{L}^1} X$ implies $X_n \xrightarrow[n\to\infty]{P} X$.

Two things imply that $X_\infty = X$ a.s., and thus

$$E|X_n - X_\infty| \to 0.$$

Now, for any $m \ge n$,

$$\begin{aligned} E\left|E(X_\infty|\mathcal{F}_n) - X_n\right| &= E\left|E(X_\infty|\mathcal{F}_n) - E(X_m|\mathcal{F}_n)\right| \\ &= E\left|E(X_\infty - X_m|\mathcal{F}_n)\right| \\ &\le EE(|X_\infty - X_m||\mathcal{F}_n) \\ &= E|X_\infty - X_m| \end{aligned}$$

holds, and letting $m \to \infty$, we get

$$E\left|E(X_\infty|\mathcal{F}_n) - X_n\right| \le \lim_{m\to\infty} E|X_\infty - X_m| = 0.$$

The last equality is from $\mathcal{L}^1$-convergence. Therefore,

$$E\left|E(X_\infty|\mathcal{F}_n) - X_n\right| = 0,$$

which implies

$$E(X_\infty|\mathcal{F}_n) = X_n \ P-\text{a.s.}.$$

Note that $X_\infty \in \mathcal{F}_\infty$ comes from a.s.-convergence. $\square$

Corollary 2.4.9 (Lévy)**.** *If $X \in \mathcal{L}^1$ and $(\mathcal{F}_n)_{n\geq 0}$ is a filtration, then*

$$E(X|\mathcal{F}_n) \xrightarrow[n\to\infty]{} E(X|\mathcal{F}_\infty)\ P-a.s.,\ \textit{and in}\ \mathcal{L}^1,$$

where

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n\right).$$

Remark 2.4.10. Now we will denote

$$\bigvee_{n=0}^{\infty} \mathcal{F}_n := \sigma\left(\bigcup_{n=0}^{\infty} \mathcal{F}_n\right).$$

Proof. Let $X_n = E(X|\mathcal{F}_n)$. Then $\{X_n\}$ is regular, and so by theorem 2.4.8, $\exists X_\infty$ s.t.

$$X_n \xrightarrow[n\to\infty]{} X_\infty\ P-\text{a.s., and in } \mathcal{L}^1,$$

and $X_n = E(X_\infty|\mathcal{F}_n)$ almost surely. Now, for any $A \in \mathcal{F}_n$,

$$\int_A X_\infty dP = \int_A E(X_\infty|\mathcal{F}_n)dP = \int_A X_n dP = \int_A E(X|\mathcal{F}_n)dP = \int_A XdP$$

holds, for arbitrarily given n. Thus, we get

$$\underbrace{\bigcup_{n=0}^{\infty} \mathcal{F}_n}_{\pi-\text{sys}} \subseteq \underbrace{\left\{A : \int_A X_\infty dP = \int_A XdP\right\}}_{\lambda-\text{sys}}.$$

Note that

$$\bigcup_{n=0}^{\infty} \mathcal{F}_n$$

is a π-system, and using

$$EX = EX_\infty,\ \text{i.e., } \Omega \in \left\{A : \int_A X_\infty dP = \int_A XdP\right\},$$

we can easily get that

$$\left\{A : \int_A X_\infty dP = \int_A XdP\right\}$$

is a λ-system. Thus by Dynkin's theorem,

$$\bigvee_{n=0}^{\infty} \mathcal{F}_n \subseteq \left\{ A : \int_A X_\infty dP = \int_A X dP \right\},$$

so

$$\forall A \in \mathcal{F}_\infty \quad \int_A X_\infty dP = \int_A X dP.$$

Now, by $X_\infty \in \mathcal{F}_\infty$, we get

$$X_\infty = E(X|\mathcal{F}_\infty),$$

by definition of conditional expectation. □

Following is the another verstion of "dominated convergence theorem."

Theorem 2.4.11. *If $Y_n \xrightarrow[n\to\infty]{a.s} Y$, and $\exists Z \in \mathcal{L}^1$ s.t. $|Y_n| \le Z$ $\forall n$, then*

$$E(Y_n|\mathcal{F}_n) \xrightarrow[n\to\infty]{a.s} E(Y|\mathcal{F}_\infty).$$

Proof. Let $W_n = \sup_{k,l\ge n} |Y_k - Y_l|$. Then,
i) $0 \le W_n \le 2Z$.
ii) W_n "monotonely" (sup) "converges to 0" ($\{Y_n\}$ is pathwise Cauchy)
Thus $W_n \searrow 0$ as $n \nearrow \infty$. Now note that,

$$|Y_n - Y| \le |Y_n - Y_m| + |Y_m - Y| \le W_m + |Y_m - Y|$$

for any $m \le n$, and letting $n \to \infty$,

$$\limsup_{n\to\infty} E(|Y_n - Y||\mathcal{F}_n) \le \lim_{n\to\infty} E(W_m|\mathcal{F}_n) + \lim_{n\to\infty} E(|Y_m - Y||\mathcal{F}_n) \overset{\text{Lévy}}{=} E(W_m|\mathcal{F}_\infty) + E(|Y_m - Y||\mathcal{F}_\infty)$$

for any m. Note that,

$$0 \le E(W_m|\mathcal{F}_\infty) + E(|Y_m - Y||\mathcal{F}_\infty) \le 4E(Z|\mathcal{F}_\infty),$$

and $E(Z|\mathcal{F}_\infty)$ is integrable. Therefore, by DCT, we get

$$\lim_{m\to\infty} (E(W_m|\mathcal{F}_\infty) + E(|Y_m - Y||\mathcal{F}_\infty)) = 0,$$

i.e.,

$$\limsup_{n\to\infty} E(|Y_n - Y||\mathcal{F}_n) = 0.$$

It implies that

$$E(Y_n - Y|\mathcal{F}_n) \xrightarrow[n\to\infty]{a.s} 0,$$

i.e.,

$$\lim_{n\to\infty} E(Y_n|\mathcal{F}_n) = \lim_{n\to\infty} E(Y|\mathcal{F}_n) \overset{\text{Lévy}}{=} E(Y|\mathcal{F}_\infty),$$

which is the desired result. $\square$

2.4.2 Riesz Decomposition

Definition 2.4.12. *A nonnegative supermartingale X_n is* ***potential*** *if $EX_n \to 0$.*

Remark 2.4.13. (i) A potential supermartingale (X_n), indeed, converges to 0 a.s.. By martingale convergence theorem,

$$X_n \xrightarrow[n\to\infty]{a.s} X_\infty$$

for some X_∞, and then by Fatou's lemma,

$$EX_\infty \le \liminf_{n\to\infty} EX_n = 0$$

holds. Nonnegativity yields $X_\infty = 0$ a.s.

(ii) Further, $\{X_n\}$ is uniformly integrable. By potentiality, $\forall \epsilon > 0$, $\exists N$ s.t.

$$n > N \Rightarrow EX_n \le \epsilon.$$

Since N is finite, $\exists a_0$ s.t.

$$a \ge a_0 \Rightarrow \sup_{n\le N} EX_n I(|X_n| \ge a) \le \epsilon,$$

and

$$\sup_{n>N} EX_n I(|X_n| \ge a) \le \sup_{n>N} EX_n \le \epsilon.$$

Therefore, we get

$$\sup_n EX_n I(|X_n| \ge a) \le \epsilon \ \forall a \ge a_0,$$

which yields

$$\sup_n EX_nI(|X_n| \geq a) \xrightarrow[a\to\infty]{} 0.$$

Following theorem shows "Doob-like" decomposition for nonnegative supermartingales, which is called **Riesz decomposition**.

Theorem 2.4.14 (Riesz Decomposition). *Let X_n be a nonnegative supermartingale. Then, $\exists$a "unique" decomposition*

$$X_n = M_n + V_n,$$

where

i) M_n is uniformly integrable martingale.

ii) V_n is a nonnegative supermartingale satisfying $V_n \xrightarrow[n\to\infty]{a.s} 0$.

Proof. (Existence) Note that $\exists X_\infty$ s.t. $X_n \xrightarrow[n\to\infty]{a.s} X_\infty$. Put

$$M_n = E(X_\infty|\mathcal{F}_n) \text{ and } V_n = X_n - M_n.$$

Then M_n is a regular martingale, and hence by theorem 2.4.8, $\{M_n\}$ is uniformly integrable. Furthermore,

i) V_n is a supermartingale from

$$E(V_{n+1}|\mathcal{F}_n) = E(X_{n+1}|\mathcal{F}_n) - E(M_{n+1}|\mathcal{F}_n) \leq X_n - M_n = V_n.$$

ii) $V_n = X_n - E(X_\infty|\mathcal{F}_n)$ is nonnegative from

$$E(X_\infty|\mathcal{F}_n) \leq \liminf_{m\to\infty} E(X_m|\mathcal{F}_n) \leq X_n.$$

First inequality is from conditional Fatou (lemma 2.3.20), and second one is from that X_n is supermartingale.

iii) By Lévy's theorem,

$$\lim_{n\to\infty} V_n = X_\infty - E(X_\infty|\mathcal{F}_\infty) = 0.$$

Thus the assertion holds.

(Uniqueness) Let

$$X_n = M_n + V_n = M_n' + V_n'.$$

Then since M_n is uniformly integrable converging to X_∞, by theorem 2.4.8, it is regular, i.e.,

$\exists \eta, \eta'$ s.t.

$$M_n = E(\eta|\mathcal{F}_n),\ M_n' = E(\eta'|\mathcal{F}_n).$$

Now since $V_n - V_n' \xrightarrow[n\to\infty]{a.s} 0$,

$$M_n' - M_n = V_n - V_n' = E(\eta' - \eta|\mathcal{F}_n) \xrightarrow[n\to\infty]{\text{Lévy}} E(\eta' - \eta|\mathcal{F}_\infty) = 0,$$

and hence $E(\eta|\mathcal{F}_\infty) = E(\eta'|\mathcal{F}_\infty)$, i.e.,

$$M_n = E(\eta|\mathcal{F}_n) = E\left(E(\eta|\mathcal{F}_\infty)|\mathcal{F}_n\right) = E\left(E(\eta'|\mathcal{F}_\infty)|\mathcal{F}_n\right) = E(\eta'|\mathcal{F}_n) = M_n'$$

holds. □

2.4.3 Optional Sampling Theorem

Theorem 2.4.15. *If $\{X_n\}$ is uniformly integrable submartingale, and N is a stopping time, then $\{X_{N\wedge n}\}$ is also a uniformly integrable submartingale.*

Proof. $(X_{N\wedge n)}$ is submartingale from example 2.3.12, and hence uniform integrability is left. Proof will be given step by step.

i) From uniform integrability, we get $\sup_n E|X_n| < \infty$, and so by martingale convergence theorem, $\exists X_\infty$ s.t.

$$X_n \xrightarrow[n\to\infty]{a.s} X_\infty.$$

ii) Since (X_n) is a submartingale, we get

$$(X_n^+)\text{: submartingale, } (X_n^-)\text{: supermartingale.}$$

Therefore, so are $(X_{N\wedge n}^+)$ and $(X_{N\wedge n}^-)$, respectively, and so by martingale convergence theorem,

$$\sup_n EX_{N\wedge n}^+ \le \sup_n EX_n^+ < \infty$$

$$\sup_n EX_{N\wedge n}^- \le EX_0^- < \infty.$$

iii) $X_{N\wedge n} \xrightarrow[n\to\infty]{a.s} X_N$. It comes from:

$$\text{On } (N<\infty),\ X_{N\wedge n} \xrightarrow[n\to\infty]{} X_N.$$

$$\text{On } (N=\infty),\ X_{N\wedge n} \xrightarrow[n\to\infty]{} X_\infty = X_N.$$

iv) Then by Fatou's lemma,

$$EX_N^+ \le \liminf_{n\to\infty} EX_{N\wedge n}^+ \le \sup_{n\to\infty} EX_{N\wedge n}^+ < \infty$$

$$EX_N^- \le \liminf_{n\to\infty} EX_{N\wedge n}^- \le \sup_{n\to\infty} EX_{N\wedge n}^- < \infty$$

holds, and hence

$$E|X_N| = EX_N^+ + EX_N^- < \infty,$$

i.e., X_N is integrable.

v) Therefore, we get uniform integrability, from

$$\begin{aligned} E|X_{N\wedge n}|I(|X_{N\wedge n}|\ge a) &= E|X_{N\wedge n}|I(|X_{N\wedge n}|\ge a, N\le n) + E|X_{N\wedge n}|I(|X_{N\wedge n}|\ge a, N>n) \\ &= E|X_N|I(|X_N|\ge a, N\le n) + E|X_n|I(|X_n|\ge a, N>n) \\ &\le E|X_N|I(|X_N|\ge a) + E|X_n|I(|X_n|\ge a) \end{aligned}$$

and consequently

$$\sup_n E|X_{N\wedge n}|I(|X_{N\wedge n}|\ge a) \le E|X_N|I(|X_N|\ge a) + \sup_n E|X_n|I(|X_n|\ge a) \xrightarrow[a\to\infty]{} 0.$$

□

Theorem 2.4.16. *If X_n is a uniformly integrable submartingale, then for any stopping time N,*

$$EX_0 \le EX_N \le EX_\infty,$$

where

$$X_\infty = \lim_{n\to\infty} X_n \ a.s..$$

Remark 2.4.17. Note that, since X_n is uniformly integrable, it satisfies $\sup_n E|X_n| < \infty$, and hence by martingale convergence theorem, we can define X_∞.

Proof. We know that $X_{N\wedge n}$ is uniformly integrable submartingale. so $X_{N\wedge n}$ converges to X_N;

$$X_{N\wedge n} \xrightarrow[n\to\infty]{a.s} X_N \text{ if } N<\infty$$

$$X_{N\wedge n} \xrightarrow[n\to\infty]{a.s} X_\infty \text{ if } N = \infty,$$

Thus, $X_{N\wedge n}$ converges P-a.s. to X_N. Note that $X_{N\wedge n}$ converges to X_N in $\mathcal{L}^1$. Since $N \wedge n$ is bouned stopping time, we get

$$EX_0 \le EX_{N\wedge n} \le EX_n.$$

By Vitali lemma, we get

$$EX_{N\wedge n} \xrightarrow[n\to\infty]{} EX_N, \ EX_n \xrightarrow[n\to\infty]{} EX_\infty$$

($\because |EX_{N\wedge n} - EX_N| \le E|X_{N\wedge n} - X_N| \to 0$) and therefore

$$EX_0 \le EX_N \le EX_\infty.$$

□

Now we reach to our goal.

Theorem 2.4.18 (Optional Sampling Theorem)**.** *If $L \le M$ are stopping times and $Y_{M\wedge n}$ is a uniformly integrable submartingale (assume that Y_∞ is well defined), then $EY_L \le EY_M$, and further,*

$$Y_L \le E(Y_M|\mathcal{F}_L) \ P\text{-}a.s.$$

Remark 2.4.19. Note that if Y_n is uniformly integrable submartingale, then $Y_{M\wedge n}$ is also a uniformly integrable submartingale, so we can apply this theorem.

Proof. Let $X_n = Y_{M\wedge n}$ be a submartingale. Then by previous theorem, we get

$$EX_L \le EX_\infty.$$

Note that $X_L = Y_{M\wedge L} = Y_L$ and $X_\infty = Y_M$, and hence

$$EY_L \le EY_M. \tag{2.2}$$

Now, fix $A \in \mathcal{F}_L$, and let

$$N = \begin{cases} L & \text{on } A \\ M & \text{on } A^c. \end{cases}$$

Then $N = LI_A + MI_{A^c}$ is a stopping time ($\because$ $(N=n) = ((L=n)\cap A) \cup \underbrace{((M=n)\cap A^c)}_{=(M=n)\cap(L\le n)\cap A^c} \in \mathcal{F}_n$ from $(L\le n)\cap A^c \in \mathcal{F}_n$), and $L \le N \le M$ holds. Thus we get

$$EY_N \le EY_M$$

by (2.2), and it implies

$$E[Y_N] = E[Y_L I_A + Y_M I_{A^c}] \le E[Y_M] = E[Y_M I_A + Y_M I_{A^c}],$$

i.e.,

$$EY_L I_A \le EY_M I_A.$$

Since it holds for any $A \in \mathcal{F}_L$, we get

$$\int_A E[Y_M|\mathcal{F}_L]dP = \int_A Y_M dP \ge \int_A Y_L dP \ \forall A \in \mathcal{F}_L,$$

i.e.,

$$E[Y_M|\mathcal{F}_L] \ge Y_L \text{ a.s..}$$

□

Optional sampling theorem has many applications. In here we see some corollaries, and one example, which is related to random walk.

Corollary 2.4.20. *Suppose that X_n is a submartingale and $E[|X_{n+1}-X_n||\mathcal{F}_n] \le B$ P-a.s.. Then if $EN<\infty$, $X_{N\wedge n}$ is uniformly integrable and $EX_N \ge EX_0$.*

Proof. Recall that

$$X_{N\wedge n} = X_0 + \sum_{m=1}^n (X_m - X_{m-1}) I(N\ge m).$$

Thus we get

$$|X_{N\wedge n}| \le |X_0| + \sum_{m=1}^n |X_m - X_{m-1}| I(N \ge m) =: Z.$$

Note that

$$EZ \le E|X_0| + E\sum_{m=1}^\infty |X_m - X_{m-1}| I(N\ge m)$$

$$\begin{aligned}
&= E|X_0| + \sum_{m=1}^{\infty} E|X_m - X_{m-1}|I(N \geq m) \quad \text{(MCT)} \\
&= E|X_0| + \sum_{m=1}^{\infty} EE\left[|X_m - X_{m-1}|I(N \geq m)|\mathcal{F}_{m-1}\right] \\
&= E|X_0| + \sum_{m=1}^{\infty} EE\left[|X_m - X_{m-1}||\mathcal{F}_{m-1}\right] I(N \geq m) \\
&(\because I(N \geq m) = 1 - I(N \leq m-1) \in \mathcal{F}_{m-1}) \\
&\leq E|X_0| + \sum_{m=1}^{\infty} BP(N \geq m) \\
&= E|X_0| + BEN \quad (\because EN < \infty)
\end{aligned}$$

holds, so $EZ < \infty$, i.e., $\{|X_{N\wedge n}| : n \geq 1\}$ is dominated by integrable r.v. Z. Therefore, we get $\{X_{N\wedge n}\}$ is uniformly integrable. Now optional sampling theorem gives $EX_N \geq EX_0$. □

Corollary 2.4.21. *If $X_n \geq 0$ is nonnegative supermartingale and N is a stopping time, then $EX_0 \geq EX_N$.*

Remark 2.4.22. Note that, by martingale convergence theorem, $\exists X_\infty \stackrel{a.s.}{=} \lim_n X_n$.

Proof. By bounded optional sampling theorem, we get

$$EX_0 \geq EX_{N\wedge n}.$$

Now using Fatou's lemma, we obtain

$$EX_N \leq \liminf_{n\to\infty} EX_{N\wedge n} \leq EX_0.$$

□

Example 2.4.23 (Asymmetric simple random walk.)**.** Let $\xi_1, \xi_2, \cdots$ be i.i.d. random variables s.t.

$$P(\xi_i = 1) = p,\ P(\xi = -1) = q = 1 - p.$$

Define

$$S_n = \xi_1 + \cdots + \xi_n,\ S_0 = 0$$

and

$$\mathcal{F}_n = \sigma(\xi_1, \xi_2, \cdots, \xi_n),\ \mathcal{F}_0 = \{\phi, \Omega\}.$$

(a) If $0 < p < 1$, then for $\varphi(x) = \left(\dfrac{1-p}{p}\right)^x$, $\varphi(S_n)$ is a martingale.

(b) Let $T_x = \inf\{n : S_n = x\}$ be "the first time touching x." ($x \in \mathbb{Z}$) Then for $a < 0 < b$,

$$P(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}.$$

(c) Now further assume that $1/2 < p < 1$. If $a < 0 < b$, then $T_b < \infty$ P-a.s., and $P(\inf_n S_n \le a) = P(T_a < \infty) = \left(\dfrac{1-p}{p}\right)^{-a}$.

(d) $ET_b = \dfrac{b}{2p-1}$.

Proof. (a) It comes from

$$\begin{aligned}
E[\varphi(S_{n+1})|\mathcal{F}_n] &= E\left[\left(\frac{1-p}{p}\right)^{S_n}\left(\frac{1-p}{p}\right)^{\xi_{n+1}}\middle|\mathcal{F}_n\right]\\
&= \left(\frac{1-p}{p}\right)^{S_n} E\left[\left(\frac{1-p}{p}\right)^{\xi_{n+1}}\middle|\mathcal{F}_n\right]\\
&= \left(\frac{1-p}{p}\right)^{S_n} E\left[\left(\frac{1-p}{p}\right)^{\xi_{n+1}}\right]\\
&= \left(\frac{1-p}{p}\right)^{S_n}\left[\left(\frac{1-p}{p}\right)^{-1}(1-p) + \left(\frac{1-p}{p}\right)^{p}\right]\\
&= \left(\frac{1-p}{p}\right)^{S_n} = \varphi(S_n).
\end{aligned}$$

(b) Let $N = T_a \wedge T_b$. For any $x \in (a, b)$, we get

$$P(x + S_{b-a} \notin (a,b)) \ge p^{b-a},$$

because $b-a$ steps of size $+1$ in a row will take us out of the interval. Similarly

$$P(x + S_{b-a} \notin (a,b)) \ge q^{b-a}.$$

Now, note that $N = \inf\{n : S_n \notin (a,b)\}$. Thus we get

$$\begin{aligned}
P(N > n(b-a)) &= P(S_{b-a} \in (a,b))P(S_{b-a} + (S_{2(b-a)} - S_{b-a}) \in (a,b))\cdots\\
&\ge (1-p^{b-a})(1-p^{b-a})\cdots(1-p^{b-a})\\
&= (1-p^{b-a})^n
\end{aligned}$$

and hence $EN < \infty$, i.e., $N < \infty$ a.s.. (Or, you can use the approximation

$$S_n \approx n(p-q) \pm \sigma\sqrt{2n\log\log n}$$

from

$$\limsup_{n\to\infty} \frac{S_n - n(p-q)}{\sigma\sqrt{2n\log\log n}} = 1$$

and

$$\liminf_{n\to\infty} \frac{S_n - n(p-q)}{\sigma\sqrt{2n\log\log n}} = -1,$$

where $\sigma^2 = Var(\xi_1)$. Note that $\lim S_n = \infty$ if $p > q$, and $\lim S_n = -\infty$ if $p < q$, so for each case, $T_b < \infty$ and $T_a > \infty$ respectively.) Note that $S_{N\wedge n}$ is between a and b, and so $\varphi(S_{N\wedge n})$ is bounded martingale. Thus, $\varphi(S_{N\wedge n})$ is uniformly integrable, and hence it is closable. Now also note that

$$\varphi(S_{N\wedge n}) \xrightarrow[n\to\infty]{a.s} \varphi(S_N).$$

Note that $S_N = a$ or b, and $T_a = T_b$ is impossible. Thus we get

$$S_N = NI(T_a < T_b) + NI(T_a > T_b) = aI(T_a < T_b) + bI(T_a > T_b).$$

It implies that

$$\begin{aligned}
1 = \varphi(0) = E\varphi(S_0) &= E\varphi(S_N) \\
&= E[\varphi(a)I(T_a < T_b)] + E[\varphi(b)I(T_a > T_b)] \\
&= \varphi(a)P(T_a < T_b) + \varphi(b)P(T_a > T_b) \\
&= (\varphi(a) - \varphi(b))P(T_a < T_b) + \varphi(b)
\end{aligned}$$

and hence

$$P(T_a < T_b) = \frac{\varphi(b) - \varphi(0)}{\varphi(b) - \varphi(a)}.$$

(c) We have already shown that

$$T_b < \infty \ P\text{-a.s.}.$$

Furthermore, since

$$\forall b < b' \Rightarrow T_b < T_{b'},$$

we get

$$\lim_{b\to\infty} T_b = \infty,$$

and hence,

$$P(T_a < \infty) = \lim_{b\to\infty} P(T_a < T_b) = \left(\frac{1-p}{p}\right)^{-a}.$$

Note that we used $1/2 < p < 1$ to get $\varphi(b) \to 0$ as $b \to \infty$.

(d) Note that for any $a \in \mathbb{Z}$, we get

$$\left(\inf_n S_n \le a\right) = (T_a < \infty).$$

Thus, we get

$$P\left(\inf_n S_n \le a\right) = \begin{cases} \left(\dfrac{1-p}{p}\right)^{-a} & a < 0 \\ 1 & a \ge 0 \end{cases}$$

and thus

$$\begin{aligned} E\left|\inf_n S_n\right| &= \sum_{a=-\infty}^{\infty} P\left(\inf_n S_n = a\right)|a| \\ &= \sum_{a=-\infty}^{0}\left[\left(\frac{1-p}{p}\right)^{-a} - \left(\frac{1-p}{p}\right)^{-(a-1)}\right]|a| \\ &= \sum_{a=-\infty}^{0}\left(\underbrace{\frac{1-p}{p}}_{<1}\right)^{-a}\left(1-\frac{1-p}{p}\right)|a| < \infty \end{aligned}$$

holds ("power series"). In other words, $\inf_n S_n$ is integrable. Now put

$$X_n = S_n - (p-q)n,$$

and then X_n is a martingale ($\because E\xi_1 = p-q$). Thus so is $X_{T_b\wedge n}$, and hence by optional sampling theorem,

$$EX_0 = EX_{T_b\wedge n} = E\left(S_{T_b\wedge n} - (p-q)(T_b\wedge n)\right).$$

Now note that

$$\inf_n S_n \le S_{T_b\wedge n} \le b,$$

so by DCT, we get

$$ES_{T_b\wedge n} \xrightarrow[n\to\infty]{} S_{T_b}$$

from

$$S_{T_b \wedge n} \xrightarrow[n\to\infty]{a.s} S_{T_b} \ (\because T_b < \infty).$$

However, by $T_b < \infty$ again, we get

$$S_{T_b} = b \text{ and } T_b \wedge n \xrightarrow[n\to\infty]{a.s} T_b,$$

and therefore,

$$0 = EX_0 = EX_{T_b \wedge n} ES_{T_b \wedge n} - (p-q)E(T_b \wedge n) \xrightarrow[n\to\infty]{} b - (p-q)ET_b$$

by MCT, which implies that

$$0 = b - (p-q)ET_b,$$

i.e.,

$$ET_b = \frac{b}{p-q}.$$

□

2.5 Square integrable Martingales

In this section, we see some special properties of square integrable martingales. In this section, let X_n be a martingale with $X_0 = 0$, and $EX_n^2 < \infty$ for all n. Put

$$A_0 \equiv 0, \ A_n = \sum_{m=1}^{n} \left\{ E(X_m^2 | \mathcal{F}_{m-1}) - X_{m-1}^2 \right\}.$$

Then since $E(X_m^2 | \mathcal{F}_{m-1}) - X_{m-1} \geq 0$, A_n is increasing, and it is also predictable. Now let

$$M_n = X_n^2 - A_n.$$

Then

$$\begin{aligned} E(M_n | \mathcal{F}_{n-1}) &= E(X_n^2 | \mathcal{F}_{n-1}) - A_n \\ &= E(X_n^2 | \mathcal{F}_{n-1}) - (A_{n-1} + E(X_n^2 | \mathcal{F}_{n-1}) - X_{n-1}^2) \\ &= X_{n-1}^2 - A_{n-1} \\ &= M_{n-1}, \end{aligned}$$

so M_n is a martingale. It gives a Doob's decomposition of X_n^2,

$$X_n^2 = M_n + A_n.$$

Also note that

$$A_n = \sum_{m=1}^{n} E\left[(X_m - X_{m-1})^2 | \mathcal{F}_{m-1}\right].$$

Now let $A_\infty = \lim_{n\to\infty} A_n$ (Note that it is well-defined, because A_n is increasing). Then we get an upper bound for $E\sup_m |X_m|^2$:

Theorem 2.5.1. $E\sup_m |X_m|^2 \le 4EA_\infty$.

Proof. By Doob's maximal inequality, we get

$$E\left(\sup_{0\le m\le n} |X_m|^2\right) \le 4E|X_n|^2 = 4EA_n \ (\because EM_n = EM_0 = 0),$$

and therefore, by MCT,

$$E\left(\sup_m |X_m|^2\right) \le 4EA_\infty.$$

□

Actually, we can obtain more sharp bound.

Theorem 2.5.2. $E\sup_m |X_m|^2 \le 3EA_\infty^{1/2}$.

Proof. Let

$$N_a = \inf\left\{n : A_{n+1} > a^2\right\}$$

be a stopping time ($(N=k) = (A_1 \le a^2) \cap \cdots \cap (A_k \le a^2) \cap (A_{k+1} > a^2) \in \mathcal{F}_k$). Then

$$\begin{aligned}
P\left(\sup_{0\le m\le n} |X_m| > a\right) &\le P(N<\infty) + P\left(\sup_{0\le m\le n} |X_{N\wedge m}| > a,\ N=\infty\right) \\
&\le P(A_\infty > a^2) + P\left(\sup_{0\le m} |X_{N\wedge m}| > a\right) \\
&(\because (N<\infty) = (\exists n \text{ s.t. } A_{n+1} > a^2) \subseteq (A_\infty > a^2)) \\
&\le P(A_\infty > a^2) + \lim_{n\to\infty} \underbrace{P\left(\sup_{0\le m\le n} |X_{N\wedge m}| > a\right)}_{=P\left(\sup_{0\le m\le n}|X_{N\wedge m}|^2 > a^2\right)} \\
&\le P(A_\infty > a^2) + \lim_{n\to\infty} a^{-2} EX_{N\wedge n}^2 \ (\because \text{ submtg ineq})
\end{aligned}$$

$$\leq P(A_\infty > a^2) + \lim_{n\to\infty} a^{-2} E A_{N\wedge n} \ (\because EM_{N\wedge n} = 0)$$

$$\leq P(A_\infty > a^2) + a^{-2} E(A_\infty \wedge a^2) \ (\because A_{n\wedge N} \leq A_\infty,\ A_{n\wedge N} \leq a^2)$$

holds. (Check where we used submartingale inequality, optional sampling theorem, and Doob's decomposition!) Therefore, we get

$$\begin{aligned}
E \sup_{0\leq m\leq n} |X_m|^2 &= \int_0^\infty P\left(\sup_{0\leq m\leq n} |X_m| > a\right) da \\
&\leq \int_0^\infty \underbrace{P(A_\infty > a^2)}_{=P(A_\infty^{1/2} > a)} da + \int_0^\infty [a^{-2} E(A_\infty \wedge a^2) da \\
&= EA_\infty^{1/2} + \int_0^\infty a^{-2} \left(\int_0^\infty P(A_\infty \wedge a^2 > b) db\right) da \\
&= EA_\infty^{1/2} + \int_0^\infty a^{-2} \int_0^{a^2} P(A_\infty > b) db da \\
&\overset{\text{Fubini}}{=} EA_\infty^{1/2} + \int_0^\infty \int_{\sqrt{b}}^\infty a^{-2} P(A_\infty > b) da db \\
&= EA_\infty^{1/2} + \int_0^\infty b^{-1/2} P(A_\infty > b) db \\
&\underset{(*)}{=} EA_\infty^{1/2} + 2EA_\infty^{1/2} = 3EA_\infty^{1/2}.
\end{aligned}$$

Note that in $(*)$, we used

$$EX^p = \int_0^\infty py^{p-1} P(X > y) dy$$

when $X \geq 0$. Now MCT gives the conclusion,

$$E \sup_m |X_m|^2 \leq 3EA_\infty^{1/2}.$$

□

Theorem 2.5.3. $\lim_{n\to\infty} X_n$ *exists, and it is finite* P*-a.s. on* $(A_\infty < \infty)$.

Proof. Let $a > 0$ and

$$N_a = \inf\{n : A_{n+1} > a^2\}$$

be a stopping time. Then $X_{N_a\wedge n}$ is a martingale, so by previous theorem, we get

$$E\left[\sup_n E|X_{N_a\wedge n}|^2\right] \leq 4E \lim_{n\to\infty} A_{N_a\wedge n} \leq 4a^2,$$

from $A_{N_a\wedge n} \leq a^2$. (Check: $X_{N_a\wedge n}^2 = M_{N_a\wedge n} + A_{N_a\wedge n}$ is also a Doob's decomposition for

$X^2_{N_a\wedge n}$) Thus by (sub)martingale convergence theorem, $X_{N_a\wedge n}$ converges P-a.s., and by DCT, $EX^2_{N_a\wedge n} \to EX^2_{N_a}$. Now, for $k = 1, 2, \cdots$, let

$$C_k = \left(X_{N_k\wedge n} \text{ converges}\right),$$

and

$$C = \bigcap_{k=1}^{\infty} C_k.$$

Then clearly $P(C_k) = 1$, and hence, $P(C) = 1$. Now, if $\omega \in C \cap (A_\infty < \infty)$, then from $A_\infty(\omega) < \infty$, $\exists k$ s.t. $N_k(\omega) = \infty$. Thus, for such k,

$$\left(X_{N_k\wedge n}\right)(\omega) = X_{N_k(\omega)\wedge n}(\omega) = X_n(\omega)$$

holds, and for almost all $\omega \in C$, it should converges. Therefore,

$$X_n(\omega) \text{ converges for almost all } \omega \in C \cap (A_\infty < \infty),$$

i.e.,

$$X_n \text{ converges } P\text{-a.s. on } (A_\infty < \infty).$$

□

Theorem 2.5.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be increasing function with $f \geq 1$ and $\int_0^\infty f(t)^{-2}dt < \infty$. Then*

$$\frac{X_n}{f(A_n)} \xrightarrow[n\to\infty]{a.s} 0 \ \text{on} \ (A_\infty = \infty).$$

Proof. Let $H_n := 1/f(A_n) \in \mathcal{F}_{n-1}$. Then H_n is nonnegative bounded ($\because 1/f \leq 1$) predictable process, so

$$Y_n := (H \cdot X)_n = \sum_{m=1}^{n} \frac{X_m - X_{m-1}}{f(A_m)}$$

is a martingale, and Y_n is also square integrable. Thus Y_n^2 is a submartingale, and hence we can find its Doob decomposition

$$Y_n^2 = M_n^y + B_n.$$

Note that

$$B_0 \equiv 0 \text{ and } B_n = \sum_{m=1}^{n} E\left[(Y_m - Y_{m-1})^2 | \mathcal{F}_{m-1}\right]$$

and hence $B_{n+1} - B_n = E\left[(Y_{n+1} - Y_n)^2|\mathcal{F}_n\right]$. Since B_n is increasing, we can define $B_\infty := \lim_{n\to\infty} B_n$ and it is obtained as

$$\begin{aligned}
B_\infty &= \sum_{n=0}^{\infty}(B_{n+1} - B_n) \\
&= \sum_{n=0}^{\infty} E\left[\left.\frac{(X_{n+1} - X_n)^2}{f(A_{n+1})^2}\right| \mathcal{F}_n\right] \\
&= \sum_{n=0}^{\infty} \frac{E[(X_{n+1} - X_n)^2|\mathcal{F}_n]}{f(A_{n+1})^2} \\
&\overset{(*)}{=} \sum_{n=0}^{\infty} \frac{A_{n+1} - A_n}{f(A_{n+1})^2} \\
&= \sum_{n=0}^{\infty} \int_{A_n}^{A_{n+1}} \frac{1}{f(A_{n+1})^2} dt \\
&\le \sum_{n=0}^{\infty} \int_{A_n}^{A_{n+1}} \frac{1}{f(t)^2} dt \\
&= \int_0^\infty \frac{1}{f(t)^2} dt < \infty,
\end{aligned}$$

so $B_\infty < \infty$ a.s.. ($(*)$ is obtained from $E[(X_{n+1} - X_n)^2|\mathcal{F}_n] = E(X_{n+1}^2|\mathcal{F}_n) - X_n^2 = A_{n+1} - A_n$) By previous theorem, Y_n converges and is finite on $(B_\infty < \infty)$, so we get

$$Y_n \xrightarrow[n\to\infty]{a.s} Y_\infty < \infty.$$

Therefore, on $(A_\infty = \infty)$, $f(A_n) \nearrow \infty$, so by Kronecker's lemma, we get

$$\frac{1}{f(A_n)} \sum_{m=1}^{n} (X_m - X_{m-1}) = \frac{X_n}{f(A_n)} \xrightarrow[n\to\infty]{a.s} 0.$$

$\square$

With this, we can further extend second Borel-Cantelli lemma.

Corollary 2.5.5 (Second Borel Cantelli Lemma). *Let $B_n \in \mathcal{F}_n$ be adapted events and $p_n = P(B_n|\mathcal{F}_{n-1})$ for $n \ge 1$. Then*

$$\frac{\sum_{m=1}^{n} I_{B_m}}{\sum_{m=1}^{n} p_m} \xrightarrow[n\to\infty]{a.s} 1 \text{ on } \left(\sum_{m=1}^{\infty} p_m = \infty\right).$$

Proof. Let $X_0 = 0$ and $X_n - X_{n-1} = I_{B_n} - P(B_n|\mathcal{F}_{n-1})$, i.e.,

$$X_n = \sum_{m=1}^{n}(I_{B_m} - P(B_m|\mathcal{F}_{m-1})).$$

Then X_n becomes a square-integrable martingale. Note that

$$\frac{\sum_{m=1}^{n} I_{B_m}}{\sum_{m=1}^{n} p_m} - 1 = \frac{\sum_{m=1}^{n}(I_{B_m} - p_m)}{\sum_{m=1}^{n} p_m} = \frac{\sum_{m=1}^{n}(I_{B_m} - P(B_m|\mathcal{F}_{m-1}))}{\sum_{m=1}^{n} p_m} = \frac{X_n}{\sum_{m=1}^{n} p_m}.$$

Let $X_n^2 = M_n + A_n$ be a Doob decomposition of X_n^2. Then we get

$$A_n - A_{n-1} = E[(X_n - X_{n-1})^2|\mathcal{F}_{n-1}] = E[(I_{B_n} - p_n)^2|\mathcal{F}_{n-1}] = p_n - p_n^2 \le p_n,$$

and hence

$$A_n \le \sum_{m=1}^{n} p_m.$$

On $(A_\infty < \infty)$, X_n converges to a finite limit X_∞, and so

$$\text{On } (A_\infty < \infty) \cap \left(\sum_{m=1}^{\infty} p_m = \infty\right), \ \frac{X_n}{\sum_{m=1}^{n} p_m} \xrightarrow[n\to\infty]{a.s} 0.$$

Meanwhile, on $(A_\infty = \infty)$, $A_n - A_{n-1} \le p_n$ implies $\sum_{n=1}^{\infty} p_n = \infty$, i.e.,

$$(A_\infty = \infty) = (A_\infty = \infty) \cap \left(\sum_{m=1}^{\infty} p_m = \infty\right),$$

and applying previous theorem to $f(t) = t \vee 1$, we get

$$\text{On } (A_\infty = \infty) \cap \left(\sum_{m=1}^{\infty} p_m = \infty\right), \ \frac{X_n}{\sum_{m=1}^{n} p_m} \underset{(\star)}{\le} \frac{X_n}{f(A_n)} \xrightarrow[n\to\infty]{a.s} 0.$$

($(\star)$ holds for sufficiently large n from $A_n \to \infty$) Therefore, we get the conclusion. $\square$

2.6 Further Topics

2.6.1 Square Function Inequalities

Our first goal is to show following *Burkholder's inequality*:

Theorem 2.6.1. *If S_i, $i = 1, 2, \cdots, n$ is a martingale, then for $1 < p < \infty$, $\exists C_1$ and C_2, depending only upon p, such that*

$$C_1 \cdot E\left|\sum_{i=1}^n X_i^2\right|^{p/2} \le E|S_n|^p \le C_2 \cdot E\left|\sum_{i=1}^n X_i^2\right|^{p/2},$$

where $X_i = S_i - S_{i-1}$ and $S_0 \equiv 0$.

Remark 2.6.2. In here, X_t is called "martingale difference." With this inequality, we can handle martingale with squared sum of martingale difference sequences. For the proof, we need some lemmas.

Lemma 2.6.3. *Suppose that S_i, $i = 1, 2, \cdots, n$ is a martingale or nonnegative submartingale. Then for $\lambda > 0$, defining the stopping time τ by*

$$\tau = \inf\{i : 1 \le i \le n, |S_i| > \lambda\} \wedge (n+1),$$

we get

$$E\left(\sum_{i=1}^{\tau-1} X_i^2\right) + E(S_{\tau-1}^2) \le 2\lambda E|S_n|.$$

In here, we defined as $X_i = S_i - S_{i-1}$, $S_0 \equiv 0$, $S_{n+1} = S_n$, and $X_{n+1} = 0$.

Proof. For any $m = 1, 2, \cdots n+1$ we get

$$S_{m-1}^2 = (X_1 + \cdots + X_{m-1})^2 = \sum_{i=1}^{m-1} X_i^2 + 2\sum_{1 \le i < j \le m-1} X_i X_j$$

and hence

$$\begin{aligned}\sum_{i=1}^{m-1} X_i^2 + S_{m-1}^2 &= 2S_{m-1}^2 - 2\sum_{1 \le i < j \le m-1} X_i X_j \\ &= 2S_{m-1}^2 - 2\sum_{j=2}^{m-1}\sum_{i=1}^{j-1} X_i X_j\end{aligned}$$

$$\begin{aligned}
&= 2S_{m-1}^2 - 2\sum_{j=2}^{m-1} S_{j-1}X_j \\
&= 2(S_m - X_m)S_{m-1} - 2\sum_{j=2}^{m-1} S_{j-1}X_j \\
&= 2S_mS_{m-1} - 2\sum_{j=2}^{m} S_{j-1}X_j
\end{aligned}$$

holds, which implies

$$\sum_{i=1}^{\tau-1} X_i^2 + S_{\tau-1}^2 = 2S_\tau S_{\tau-1} - 2\sum_{j=2}^{\tau} S_{j-1}X_j.$$

Note that, letting $\mathcal{F}_{n+1} = \mathcal{F}_n$, $(S_i, \mathcal{F}_i)_{i=1,2,\cdots,n+1}$ also becomes a martingale or nonnegative submartingale. Thus we get

$$\begin{aligned}
E\left(\sum_{i=2}^{\tau} S_{i-1}X_i\right) &= E\left(\sum_{i=2}^{n+1} I(\tau \ge i)S_{i-1}X_i\right) \\
&= \sum_{i=2}^{n+1} ES_{i-1}X_iI(\tau \ge i) \\
&= \sum_{i=2}^{n+1} E\left[E(S_{i-1}X_iI(\tau \ge i)|\mathcal{F}_{i-1})\right] \\
&= \sum_{i=2}^{n+1} E\left[S_{i-1}E(X_i|\mathcal{F}_{i-1})I(\tau \ge i)\right] \\
&\ge 0.
\end{aligned}$$

The last inequality holds from: if S_i is a martingale, then $E(X_i|\mathcal{F}_{i-1}) = 0$, and if S_i is a nonnegative submartingale, then $S_{i-1} \ge 0$, $E(X_i|\mathcal{F}_{i-1}) \ge 0$. From this, we obtain

$$E\left(\sum_{i=1}^{\tau-1} X_i^2\right) + E(S_{\tau-1}^2) = E(2S_\tau S_{\tau-1}) - 2\underbrace{E\left(\sum_{j=2}^{\tau} S_{j-1}X_j\right)}_{\ge 0} \le 2E(S_\tau S_{\tau-1}).$$

Note that, by definition, $|S_{\tau-1}| \le \lambda$. This yields that

$$E(S_\tau S_{\tau-1}) \le E|S_\tau S_{\tau-1}| \le \lambda E|S_\tau|.$$

Note that $|S_n|$ is a submartingale in any case (S_n is martingale or nonnegative submartingale), and by optional sampling theorem, $E|S_\tau| \le E|S_{n+1}| = E|S_n|$ (Recall that $S_{n+1} = S_n$). Therefore, we get

$$E\left(\sum_{i=1}^{\tau-1} X_i^2\right) + E(S_{\tau-1}^2) \le 2E(S_\tau S_{\tau-1}) \le 2\lambda E|S_n|.$$

□

Lemma 2.6.4. *Let S_i, $i = 1, 2, \cdots, n$ be a nonnegative submartingale and define*

$$Y = \max\left(\theta\left(\sum_{i=1}^{n} X_i^2\right)^{1/2}, \max_{1\le i\le n} S_i\right) \text{ for } \theta > 0.$$

Then $\forall\lambda > 0$,

$$\lambda P(Y > \beta\lambda) \le 3ES_n I(Y > \lambda), \tag{2.3}$$

where $\beta = (1 + 2\theta^2)^{1/2}$, and for each $1 < p < \infty$,

$$\left\|\left(\sum_{i=1}^{n} X_i^2\right)^{1/2}\right\|_p \le 9p^{1/2}q\|S_n\|_p, \tag{2.4}$$

where q is the Hölder conjugate of p.

Proof. First note that

$$\begin{aligned}\lambda P(Y > \beta\lambda) &= \lambda P\left(\max_{1\le i\le n} S_i > \lambda, Y > \beta\lambda\right) + \lambda P\left(\max_{1\le i\le n} S_i \le \lambda, Y > \beta\lambda\right)\\ &\le \underbrace{\lambda P\left(\max_{1\le i\le n} S_i > \lambda\right)}_{:=F_n} + \underbrace{\lambda P\left(\max_{1\le i\le n} S_i \le \lambda, Y > \beta\lambda\right)}_{:=G_n}.\end{aligned}$$

Letting $A_i = (S_1 \le \lambda, \cdots, S_{i-1} \le \lambda, S_i > \lambda)$, we get

$$\left(\max_{1\le i\le n} S_i > \lambda\right) = \bigcup_{i=1}^{n} A_i,$$

and so

$$\begin{aligned}F_n &= \lambda P\left(\bigcup_{i=1}^{n} A_i\right)\\ &= \lambda\sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{n} E\lambda I_{A_i}\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{n} ES_iI_{A_i} \ (\because \text{on } A_i,\ \lambda < S_i) \\
&\leq \sum_{i=1}^{n} E\left(E(S_n|\mathcal{F}_i)I_{A_i}\right) \ (\because S_i \leq E(S_n|\mathcal{F}_i)) \\
&= \sum_{i=1}^{n} E\left(E(S_nI_{A_i}|\mathcal{F}_i)\right) \\
&= \sum_{i=1}^{n} ES_nI_{A_i} = ES_n\sum_{i=1}^{n} I_{A_i} \\
&= ES_nI\left(\max_{1\leq i\leq n} S_i > \lambda\right) \\
&\leq ES_nI(Y > \lambda) \ (\because \max S_i \leq Y).
\end{aligned}$$

On the other hand, let's see G_n. If $Y > \beta\lambda$, from $\beta > 1$, $Y > \lambda$ holds, and by definition of Y, if further $\max_{1\leq i\leq n} S_i \leq \lambda$, we get $Y = \theta(\sum X_i^2)^{1/2} > \beta\lambda$. Thus,

$$G_n = \lambda P\left(\max_{1\leq i\leq n} S_i \leq \lambda, Y > \beta\lambda\right) \leq \lambda P\left(\max_{1\leq i\leq n} S_i \leq \lambda,\ \theta\left(\sum_{i=1}^{n} X_i^2\right)^{1/2} > \beta\lambda\right).$$

Our goal is to show that $G_n \leq 2ES_nI(Y > \lambda)$, so that (2.3) is proven. Let T_m be a "truncated S_m," which is defined as

$$T_m = S_mI\left(\theta\left(\sum_{i=1}^{m} X_i^2\right)^{1/2} > \lambda\right).$$

Then from

$$\begin{aligned}
E(T_m|\mathcal{F}_{m-1}) &\geq E\left(S_m\ I\left(\theta\left(\sum_{i=1}^{m-1} X_i^2\right)^{1/2} > \lambda\right)\middle|\mathcal{F}_{m-1}\right) \\
&= E(S_m|\mathcal{F}_{m-1})I\left(\theta\left(\sum_{i=1}^{m-1} X_i^2\right)^{1/2} > \lambda\right) \\
&= S_{m-1}I\left(\theta\left(\sum_{i=1}^{m-1} X_i^2\right)^{1/2} > \lambda\right) \\
&= T_{m-1}
\end{aligned}$$

T_m is a nonnegative submartingale. Now define $Y_1 = T_1$, $Y_2 = T_2 - T_1$, $\cdots$, $Y_n = T_n - T_{n-1}$. Put

$$\mathcal{E}_1 = \left\{\theta\left(\sum_{i=1}^{n} X_i^2\right)^{1/2} > \beta\lambda,\ \max_{1\leq i\leq n} S_i \leq \lambda\right\}$$

and

$$\mathcal{E}_2 = \left\{ \left(\sum_{i=1}^n Y_i^2 \right)^{1/2} > \lambda,\ \max_{1 \le i \le n} T_i \le \lambda \right\}.$$

We will show that $\mathcal{E}_1 \subseteq \mathcal{E}_2$. Let

$$\nu = \inf \left\{ i : 1 \le i \le n, \theta \left(\sum_{j=1}^i X_j^2 \right)^{1/2} > \lambda \right\} \wedge n$$

be a stopping time. On $\mathcal{E}_1$, from the definition of T_i, $T_i \le S_i$ holds, which implies $\max_i T_i \le \max_i S_i \le \lambda$. Further, on $\mathcal{E}_1$, we get

$$\begin{aligned}
\beta^2\lambda^2 &< \theta^2 \sum_{i=1}^n X_i^2 \\
&= \underbrace{\theta^2 \sum_{i=1}^{\nu-1} X_i^2}_{\le \lambda^2 \text{ (def of } \nu)} + \theta^2 \underbrace{X_\nu^2}_{\le \lambda^2 (*)} + \theta^2 \underbrace{\sum_{i=\nu+1}^n X_i^2}_{= \sum Y_i^2 (**)} \\
&\le \lambda^2 + \theta^2\lambda^2 + \theta^2 \sum_{i=\nu+1}^n Y_i^2 \\
&\le \lambda^2 + \theta^2\lambda^2 + \theta^2 \sum_{i=1}^n Y_i^2
\end{aligned}$$

holds ($(*)$: since S_n is nonnegative, $|X_\nu| = |S_\nu - S_{\nu-1}| \le \max(S_{\nu-1}, S_\nu) \le \lambda$ on $\mathcal{E}_1$; $(**)$: if $i \ge \nu$, $S_i = T_i$, so $X_i = Y_i$ if $i \ge \nu + 1$). Thus we get

$$\lambda^2 + \theta^2\lambda^2 + \theta^2 \sum_{i=1}^n Y_i^2 > \beta^2\lambda^2,$$

i.e.,

$$(1 + 2\theta^2)\lambda^2 + \theta^2 \sum_{i=1}^n Y_i^2 > \beta^2\lambda^2 + \theta^2\lambda^2,$$

which implies

$$\sum_{i=1}^n Y_i^2 > \lambda^2.$$

Therefore $\mathcal{E} \subseteq \mathcal{E}_2$. Now, under this, we get

$$G_n \le \lambda P(\mathcal{E}_1) \le \lambda P(\mathcal{E}_2)$$

$$
\begin{aligned}
&= \lambda P\left(\left(\sum_{i=1}^n Y_i^2\right)^{1/2} > \lambda, \max_{1\le i\le n} T_i \le \lambda\right)\\
&= \lambda P\left(\left(\sum_{i=1}^n Y_i^2\right)^{1/2} I\left(\max_{1\le i\le n} T_i \le \lambda\right) > \lambda\right)\\
&\le \lambda^{-1} E\left[\left(\sum_{i=1}^n Y_i^2\right) I\left(\max_{1\le i\le n} T_i \le \lambda\right)\right]\\
&\le \lambda^{-1} E\left[\left(\sum_{i=1}^n Y_i^2\right) I\left(\max_{1\le j\le i} T_j \le \lambda\right)\right].
\end{aligned}
$$

Now, let

$$\tau = \inf\{i : 1 \le i \le n, |T_i| > \lambda\} \wedge (n+1)$$

be a stopping time. Then if $\tau \le n$, then $i \ge \tau \Rightarrow I(\max_{1\le j\le i} T_j \le \lambda) = 0$, and if $\tau = n+1$, then $\tau - 1 = n$, so we get

$$E\left[\left(\sum_{i=1}^n Y_i^2\right) I\left(\max_{1\le j\le i} T_j \le \lambda\right)\right] = E\left[\left(\sum_{i=1}^{\tau-1} Y_i^2\right) I\left(\max_{1\le j\le i} T_j \le \lambda\right)\right].$$

Consequently, we get

$$
\begin{aligned}
G_n &\le \lambda^{-1} E\left[\left(\sum_{i=1}^n Y_i^2\right) I\left(\max_{1\le j\le i} T_j \le \lambda\right)\right]\\
&\le \lambda^{-1} E\left[\left(\sum_{i=1}^{\tau-1} Y_i^2\right) I\left(\max_{1\le j\le i} T_j \le \lambda\right)\right]\\
&\le \lambda^{-1} E\left[\left(\sum_{i=1}^{\tau-1} Y_i^2\right)\right]\\
&\underset{(\star)}{\le} 2ET_n \le 2ES_n I(Y > \lambda),
\end{aligned}
$$

and hence get (2.3). Note that $(\star)$ comes from lemma 2.6.3;

$$E\left[\left(\sum_{i=1}^{\tau-1} Y_i^2\right)\right] \le E\left[\left(\sum_{i=1}^{\tau-1} Y_i^2\right)\right] + E(T_{\tau-1}^2) \le 2\lambda E|T_n|.$$

For (2.4),