

# Theory of Statistics II (Fall 2016)

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# Preface & Disclaimer

This note is a summary of the lecture Theory of Statistics II (326.522) held at Seoul National University, Fall 2016. Lecturer was B.U.Park, and the note was summarized by J.P.Kim, who is a Ph.D student. There are few textbooks and references in this course. Contents and corresponding references are following.

- Asymptotic Approximations. Reference: *Mathematical Statistics: Basic ideas and selected topics, Vol. I., 2nd edition, P.Bickel & K.Doksum, 2007.*
- Weak Convergence. Reference: *Convergence of Probability Measures, P.Billingsley, 1999.*
- Empirical Processes. Reference: *Empirical Processes in M-estimation, S.A. van de Geer, 2000.*

Lecture notes are available at [stat.snu.ac.kr/theostat](http://stat.snu.ac.kr/theostat). Also I referred to following books when I write this note. The list would be updated continuously.

- *Probability: Theory and Examples, R.Durrett*
- *Mathematical Statistics (in Korean), W.C.Kim*

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# Chapter 1

## Asymptotic Approximations

### 1.1 Consistency

#### 1.1.1 Preliminary for the chapter

**Definition 1.1.1** (Notations). Let  $\Theta$  be a parameter space. Then we consider a ‘random variable’  $X$  on the probability space  $(\Omega, \mathcal{F}, P_\theta)$  which is a function

$$X : (\Omega, \mathcal{F}, P_\theta) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), P_\theta^X),$$

where  $P_\theta^X := P_\theta \circ X^{-1}$ . Note that  $P_\theta$  is a probability measure from the model  $\mathcal{P} := \{P_\theta : \theta \in \Theta\}$ . For the convenience, now we omit the explanation of fundamental setting.

**Definition 1.1.2** (Convergence). Let  $\{X_n\}$  be a sequence of random variables.

1.  $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$  if  $P\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1 \Leftrightarrow P(|X_n - X| > \epsilon \text{ i.o.}) = 0 \forall \epsilon > 0$   
 $\Leftrightarrow \lim_{N \rightarrow \infty} P\left(\bigcup_{n=N}^{\infty} (|X_n - X| > \epsilon)\right) = 0 \forall \epsilon > 0$
2.  $X_n \xrightarrow[n \rightarrow \infty]{P} X$  if  $\forall \epsilon > 0 \ P(|X_n - X| > \epsilon) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proposition 1.1.3.**  $X_n \xrightarrow[n \rightarrow \infty]{P} X$  if and only if for any subsequence  $\{n_k\} \subseteq \{n\}$  there is a further subsequence  $\{n_{k_j}\} \subseteq \{n_k\}$  such that  $X_{n_{k_j}} \xrightarrow[j \rightarrow \infty]{a.s.} X$ .

*Proof.* Durrett, p.65. □

**Definition 1.1.4** (Consistency).  $\hat{q}_n = q_n(X_1, \dots, X_n)$  is consistent estimator of  $q(\theta)$  if

$$\hat{q}_n \xrightarrow[n \rightarrow \infty]{P_\theta} q(\theta)$$

for any  $\theta \in \Theta$ . (We don't know what is the true parameter.)

**Remark 1.1.5.** There are some tools to obtain consistency.

$$1. \text{Var}(Z_n) \rightarrow 0, EZ_n \rightarrow \mu \text{ as } n \rightarrow \infty \Rightarrow Z_n \xrightarrow[n \rightarrow \infty]{P} \mu.$$

$$\begin{aligned} \because P(|Z_n - \mu| > \epsilon) &\leq P(|Z_n - EZ_n| + |EZ_n - \mu| > \epsilon) \\ &\leq P(|Z_n - EZ_n| > \epsilon/2) + \underbrace{P(|EZ_n - \mu| > \epsilon/2)}_{=0 \text{ for sufficiently large } n} \\ &\leq \frac{4}{\epsilon^2} \text{Var}(Z_n) \rightarrow 0 \end{aligned}$$

$$2. \text{WLLN: } X_1, \dots, X_n: \text{i.i.d. and } E|X_1| < \infty \Rightarrow \bar{X}_n \xrightarrow[n \rightarrow \infty]{P} EX_1.$$

3. If  $Z_n \xrightarrow[n \rightarrow \infty]{P} Z$  and  $g$  is continuous on the support of  $Z$ , then  $g(Z_n) \xrightarrow[n \rightarrow \infty]{P} g(Z)$ . Note that uniform convergence of  $g$  implies this directly, and for the general case, we can use Proposition 1.1.3.

4. Followings are the corollary of 3. Or, we can prove it directly. Suppose that  $Y_n \xrightarrow[n \rightarrow \infty]{P} Y$  and  $Z_n \xrightarrow[n \rightarrow \infty]{P} Z$ . Then,

$$(a) Y_n + Z_n \xrightarrow[n \rightarrow \infty]{P} Y + Z.$$

$$(b) Y_n Z_n \xrightarrow[n \rightarrow \infty]{P} YZ.$$

$$(c) Y_n/Z_n \xrightarrow[n \rightarrow \infty]{P} Y/Z \text{ provided that } Z \neq 0 \text{ P-a.s..}$$

*Proof.* (b) Note that  $|Y_n Z_n - YZ| \leq |Y_n||Z_n - Z| + |Z||Y_n - Y| \leq |Y_n - Y||Z_n - Z| + |Y||Z_n - Z| + |Z||Y_n - Y|$ . Now for any  $\eta > 0$  there exists  $M > 0$  such that  $P(|Y| > M) \leq \eta$  and  $P(|Z| > M) \leq \eta$ . Now,

$$\begin{aligned} P(|Y_n Z_n - YZ| > \epsilon) &\leq P(|Y_n||Z_n - Z| > \epsilon/2) + P(|Z||Y_n - Y| > \epsilon/2) \\ &\leq P(|Y_n - Y||Z_n - Z| > \epsilon/4) + P(|Y||Z_n - Z| > \epsilon/4) + P(|Z||Y_n - Y| > \epsilon/2) \end{aligned}$$

and note that  $P(|Y||Z_n - Z| > \epsilon/4) = P(|Y||Z_n - Z| > \epsilon/4, |Y| > M) + P(|Y||Z_n - Z| > \epsilon/4, |Y| \leq M) \leq \eta + P(|Z_n - Z| \geq \epsilon/4M)$ . Thus

$$\limsup_{n \rightarrow \infty} P(|Y||Z_n - Z| > \epsilon/4) \leq \eta$$

and similarly

$$\limsup_{n \rightarrow \infty} P(|Z||Y_n - Y| > \epsilon/2) \leq \eta.$$

Now, since

$$\begin{aligned} P(|Y_n - Y||Z_n - Z| > \epsilon/4) &= P(|Y_n - Y||Z_n - Z| > \epsilon/4, |Y_n - Y| > \sqrt{\epsilon/4}) \\ &\quad + P(|Y_n - Y||Z_n - Z| > \epsilon/4, |Y_n - Y| \leq \sqrt{\epsilon/4}) \\ &\leq P(|Y_n - Y| > \sqrt{\epsilon/4}) + P(|Z_n - Z| \geq \sqrt{\epsilon/4}) \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , we get

$$\limsup_{n \rightarrow \infty} P(|Y_n Z_n - Y Z| > \epsilon) \leq 2\eta.$$

Finally, since  $\eta > 0$  was arbitrary, we get the result.

(c) By (b), it's sufficient to show that  $Z_n^{-1} \xrightarrow[n \rightarrow \infty]{P} Z^{-1}$ . Since  $P(Z = 0) = 0$ , for any  $\eta > 0$  there exists  $\delta > 0$  such that  $P(|Z| \leq \delta) \leq \eta$ . (If not,  $\exists \eta > 0$  such that  $\forall \delta > 0$   $P(|Z| \leq \delta) > \eta$ . Then by continuity of measure,  $P(\bigcup_{\delta > 0} (|Z| \leq \delta)) = P(Z = 0) \geq \eta > 0$ . Contradiction.)

Thus

$$\begin{aligned} P\left(\left|\frac{1}{Z_n} - \frac{1}{Z}\right| > \epsilon\right) &= P\left(\frac{|Z_n - Z|}{|Z_n Z|} > \epsilon\right) \\ &\leq P\left(\frac{|Z_n - Z|}{|Z|(|Z| - |Z_n - Z|)} > \epsilon\right) \\ &\leq \underbrace{P\left(\frac{|Z_n - Z|}{|Z|(|Z| - |Z_n - Z|)} > \epsilon, |Z| > \delta, |Z_n - Z| \leq \delta/2\right)}_{\leq P(|Z_n - Z| > \frac{\delta^2}{2}\epsilon) \xrightarrow[n \rightarrow \infty]{} 0} \\ &\quad + \underbrace{P(|Z| \leq \delta)}_{\leq \eta} + \underbrace{P(|Z_n - Z| > \delta/2)}_{\xrightarrow[n \rightarrow \infty]{} 0} \end{aligned}$$

and hence

$$\limsup_{n \rightarrow \infty} P\left(\left|\frac{1}{Z_n} - \frac{1}{Z}\right| > \epsilon\right) \leq \eta$$

holds. Note that  $\eta > 0$  was arbitrary. □

**Definition 1.1.6** (Probabilistic  $O$ -notation). *Let  $X_n$  be a sequence of r.v.'s.*

1.  $X_n = O_p(1)$  if  $\lim_{c \rightarrow \infty} \sup_n P(|X_n| > c) = 0 \Leftrightarrow \lim_{c \rightarrow \infty} \limsup_{n \rightarrow \infty} P(|X_n| > c) = 0$ . (“Bounded in probability”)

2.  $X_n = o_p(1)$  if  $X_n \xrightarrow[n \rightarrow \infty]{P} 0$ .

3.  $X_n = O_p(a_n)$  if  $X_n/a_n = O_p(1)$ , and  $X_n = o_p(a_n)$  if  $X_n/a_n = o_p(1)$ .

**Proposition 1.1.7.** If  $X_n \xrightarrow[n \rightarrow \infty]{d} X$  for some  $X$ , then  $X_n = O_p(1)$ .

*Proof.* For given  $\epsilon > 0$ , there exists  $c$  such that  $P(|X| > c) < \epsilon/2$ . For such  $c$ ,  $P(|X_n| > c) \rightarrow P(|X| > c)$ , so  $\exists N$  s.t.

$$\sup_{n > N} |P(|X_n| > c) - P(|X| > c)| < \frac{\epsilon}{2}.$$

Thus  $\sup_{n > N} P(|X_n| > c) < \epsilon$ . For  $n = 1, 2, \dots, N$ , there exists  $c_n$  such that  $P(|X_n| > c_n) < \epsilon$ , and letting  $c^* = \max(c_1, \dots, c_N, c)$ , we get  $\sup_n P(|X_n| > c^*) < \epsilon$ .  $\square$

**Example 1.1.8** (Simple Linear Regression). Consider a simple linear regression model  $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ , where  $\epsilon_i \stackrel{i.i.d.}{\sim} (0, \sigma^2)$ . Also assume that  $x_1, \dots, x_n$  are known and not all equal. Note that

$$\hat{\beta}_{1,n} = \frac{\sum_{i=1}^n (x_i - \bar{x}) Y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}.$$

Since  $E(\hat{\beta}_{1,n}) = \beta_1$  and  $Var(\hat{\beta}_{1,n}) = \sigma^2/S_{xx}$ , we obtain consistency

$$\hat{\beta}_{1,n} \xrightarrow[n \rightarrow \infty]{P_{\beta, \sigma^2}} \beta_1$$

provided that  $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2 \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Example 1.1.9** (Sample correlation coefficient). Let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be random sample from the population

$$EX_1 = \mu_1, EY_1 = \mu_2, Var(X_1) = \sigma_1^2 > 0, Var(Y_1) = \sigma_2^2 > 0, \text{ and } Corr(X_1, Y_1) = \rho.$$

Then by WLLN we get

$$(\bar{X}, \bar{Y}, \bar{X}^2, \bar{Y}^2, \bar{XY}) \xrightarrow[n \rightarrow \infty]{P} (EX_1, EY_1, EX_1^2, EY_1^2, EX_1 Y_1).$$

Since the function

$$g(u_1, u_2, u_3, u_4, u_5) = \frac{u_5 - u_1 u_2}{\sqrt{u_3 - u_1^2} \sqrt{u_4 - u_2^2}}$$

is continuous at  $(EX_1, EY_1, EX_1^2, EY_1^2, EX_1Y_1)$ , we get

$$\hat{\rho}_n = \frac{\overline{XY} - \overline{X}\overline{Y}}{\sqrt{\overline{X^2} - \overline{X}^2}\sqrt{\overline{Y^2} - \overline{Y}^2}} \xrightarrow[n \rightarrow \infty]{P} \rho.$$

**Remark 1.1.10.** Note that, if  $X_n \xrightarrow[n \rightarrow \infty]{P} c$  where  $c$  is a constant, then continuity of  $g(x)$  at  $x = c$  is sufficient for consistency  $g(X_n) \xrightarrow[n \rightarrow \infty]{P} g(c)$ . It is trivial from the definition of continuity.

**Example 1.1.11.** Let  $X_1, \dots, X_n$  be a random sample from a population with cdf  $F$ . Then we use an *empirical distribution function*

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n I_{(-\infty, x]}(X_i) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x)$$

for estimation of  $F$ . Then by WLLN, for each  $x$ ,  $\hat{F}_n(x)$  is consistent estimator for  $F(x)$ ,

$$\hat{F}_n(x) \xrightarrow[n \rightarrow \infty]{P} F(x).$$

**Remark 1.1.12.** Note that in this case, we can say more strong result, which is known as *Glivenko-Cantelli theorem*:

$$\sup_x |\hat{F}_n(x) - F(x)| \xrightarrow[n \rightarrow \infty]{P} 0.$$

Sketch of proof is given here. Since  $\hat{F}_n$  and  $F$  are nondecreasing and bounded, we can partition  $[0, 1]$ , which is a range of both functions, into finite number of intervals, and then each interval has a well-defined inverse image which is an interval. For whole proof, see Durrett, p.76.

### 1.1.2 FSE and MLE in Exponential Families

#### FSE

Recall that FSE of  $\nu(F)$  is defined as  $\nu(\hat{F}_n)$ . One example of FSE is MME: to estimate  $EX^j =: \nu_j(F) =: \int x^j dF(x)$ , we use

$$\hat{m}_j = \nu_j(\hat{F}_n) = \int x^j d\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n X_i^j.$$

By WLLN we have  $(\hat{m}_1, \hat{m}_2, \dots, \hat{m}_k)^T \xrightarrow[n \rightarrow \infty]{P} (m_1, m_2, \dots, m_k)^T$  where  $m_j = EX^j$ , so we can obtain consistency of MME easily.

**Proposition 1.1.13.** Let  $q = h(m_1, m_2, \dots, m_k)$  be a parameter of interest where  $m_j$ 's are

population moments. Then for MME

$$\hat{q}_n = h(\hat{m}_1, \dots, \hat{m}_k)$$

based on a random sample  $X_1, \dots, X_n$ ,

$$\hat{q}_n \xrightarrow[n \rightarrow \infty]{P} q$$

holds, provided that  $h$  is continuous at  $(m_1, \dots, m_k)^T$ .

We can do similar work in FSE  $\nu(F)$ . Note that in here,  $\nu$  is a functional, so we may define a continuity of functional. We may use sup norm as a metric in the space of distribution functions.

**Definition 1.1.14.** Let  $\mathcal{F}$  be a space of distribution functions. In this space, we give the norm  $\|\cdot\|$  as a sup norm

$$\|F\| = \sup_x |F(x)|.$$

Then metric is given as

$$\|F - G\| = \sup_x |F(x) - G(x)|.$$

Also, we say that a functional  $\nu : \mathcal{F} \rightarrow \mathbb{R}$  is continuous at  $F$  if for any  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$\|G - F\| < \delta \Rightarrow |\nu(G) - \nu(F)| < \epsilon.$$

**Remark 1.1.15.** Note that since  $\|\hat{F}_n - F\| \rightarrow 0$  as  $n \rightarrow \infty$  from Glivenko-Cantelli theorem, we get consistency of FSE

$$\nu(\hat{F}_n) \xrightarrow[n \rightarrow \infty]{P} \nu(F)$$

provided that  $\nu$  is continuous at  $F$ . In many cases, showing continuity may be difficult problem.

**Example 1.1.16** (Best Linear Predictor). Let  $X_1, \dots, X_n$  be  $k$ -dimensional i.i.d. r.v.'s, and  $Y_1, \dots, Y_n$  be i.i.d. 1-dim random variable. Then we know that

$$BLP(x) = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x - \mu_1),$$

where

$$E \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \text{ and } Var \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$



Thus for sample variance

$$S_{11} = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})^T$$

$$S_{12} = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})^T = S_{21}^T$$

$$S_{22} = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2,$$

we obtain FSE for BLP,

$$\widehat{BLP}^{FSE}(x) = \bar{Y} + S_{21}S_{11}^{-1}(x - \bar{X}).$$

Note that it is same as sample linear regression line. Detail is given in next proposition.

**Proposition 1.1.17.**

(a) *Solution of minimizing problem*

$$(\beta_0^*, \beta_1^*)^T = \arg \min_{\beta_0, \beta_1} E(Y - \beta_0 - \beta_1^T X)^2$$

is

$$BLP(x) := \beta_0^* + \beta_1^{*T} x = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x - \mu_1).$$

(b) For  $\mathbf{Y} = (Y_1, \dots, Y_n)^T$  and design matrix  $\mathbf{X} = (\mathbf{1}, \mathbf{X}_1)$  where  $\mathbf{X}_1 = (X_1, \dots, X_n)^T$ , LSE

is

$$\hat{\beta}_1 = S_{11}^{-1}S_{12} \text{ and } \hat{\beta}_0 = \bar{Y} - \bar{X}^T \hat{\beta}_1.$$

*Proof.* (a) Two approaches are given. First one is direct proof: It is clear because of

$$\begin{aligned} E(Y - \beta_0 - \beta_1^T X)^2 &= E[(Y - \mu_2) - \beta_1^T (X - \mu_1)]^2 + [\mu_2 - \beta_0 - \beta_1^T \mu_1]^2 \\ &= \Sigma_{22} - 2\beta_1^T \Sigma_{12} + \beta_1^T \Sigma_{11} \beta_1 + [\beta_0 - (\mu_2 - \beta_1^T \mu_1)]^2. \end{aligned}$$

Second approach uses projection in  $\mathcal{L}^2$  space. For convenience, suppose  $EX = 0$  and  $EY = 0$ .

Then  $(\beta_0^*, \beta_1^*)^T$  should satisfy

$$\langle \beta_0 + \beta_1^T X, Y - \beta_0^* - \beta_1^{*T} X \rangle = 0 \quad \forall \beta_0, \beta_1.$$

It yields that

$$\beta_0^* = 0, \quad \beta_1^* = (E(XX^T))^{-1} E(XY).$$

(b)  $\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{1}\hat{\beta}_0 + \mathbf{X}_1\hat{\boldsymbol{\beta}}_1$  should satisfy  $\mathbf{1}\hat{\beta}_0 + \mathbf{X}_1\hat{\boldsymbol{\beta}}_1 = \Pi(\mathbf{Y}|\mathcal{C}(\mathbf{X}))$ . For  $\mathcal{X}_1 = \mathbf{X}_1 - \Pi(\mathbf{X}_1|\mathcal{C}(\mathbf{1})) = \mathbf{X}_1 - \mathbf{1}\bar{X}^T$ ,

$$\mathbf{1}\hat{\beta}_0 + \mathbf{X}_1\hat{\beta}_1 = \mathbf{1}\left(\hat{\beta}_0 + \frac{\mathbf{1}^T\mathbf{X}_1}{n}\hat{\beta}_1\right) + \mathcal{X}_1\hat{\beta}_1 = \Pi(\mathbf{Y}|\mathcal{C}(\mathbf{1})) + \Pi(\mathbf{Y}|\mathcal{C}(\mathbf{X}_1))$$

we get

$$\hat{\beta}_0 = \bar{Y} - \bar{X}^T \hat{\beta}_1 \text{ and } \hat{\beta}_1 = (\mathcal{X}_1^T \mathcal{X}_1)^{-1} \mathcal{X}_1^T \mathbf{Y}.$$

Now  $\mathcal{X}_1^T \mathcal{X}_1 = S_{11}$  and  $\mathcal{X}_1^T \mathbf{Y} = S_{12}$  ends the proof.

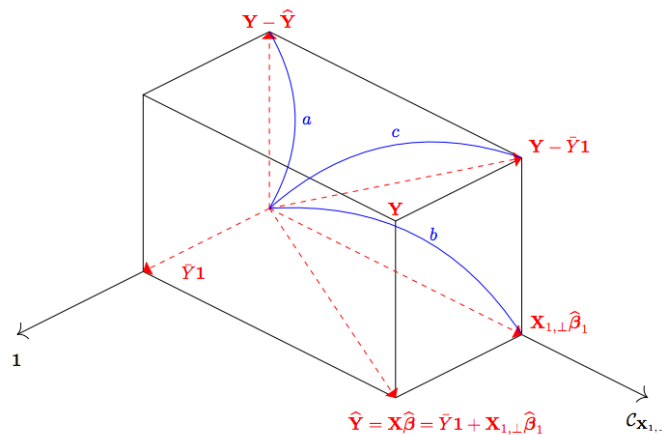


Figure 1.1: Regression with intercept. Image from Lecture Note.

**Example 1.1.18** (Multiple Correlation Coefficient). We define a *multiple correlation coefficient* ( $MCC$ ) as

$$\rho = \max_{\beta_0, \beta_1} \text{Corr}(Y, \beta_0 + \beta_1^T X)$$

and sample MCC is

$$\hat{\rho}_n = \max_{\beta_0, \beta_1} \widehat{\text{Corr}}(Y, \beta_0 + \beta_1^T X).$$

Note that,

$$\begin{aligned}\text{Corr}(Y, \beta_0 + \beta_1^T X) &= \text{Corr}(Y - \mu_2, \beta_1^T (X - \mu_1)) \\ &= \frac{\Sigma_{21} \beta_1}{\sqrt{\Sigma_{22}} \sqrt{\beta_1^T \Sigma_{11} \beta_1}} \\ &= \frac{(\Sigma_{11}^{-1/2} \Sigma_{12})^T (\Sigma_{11}^{1/2} \beta_1)}{\sqrt{\Sigma_{22}} \sqrt{\beta_1^T \Sigma_{11} \beta_1}}\end{aligned}$$

$$\leq \sqrt{\frac{\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}}{\Sigma_{22}}}$$

holds by Cauchy-Schwarz inequality, and equality holds when  $\beta_1 = \Sigma_{11}^{-1}\Sigma_{12}$ . Thus population MCC is obtained as

$$\rho = \sqrt{\frac{\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}}{\Sigma_{22}}}.$$

Meanwhile, sample correlation is obtained as

$$\widehat{\text{Corr}}(\mathbf{Y}, \beta_0 + \beta_1^T \mathbf{X}) = \frac{\langle \mathbf{Y} - \bar{Y}\mathbf{1}, (\mathbf{X} - \mathbf{1}\bar{X}^T)\beta_1 \rangle}{\|\mathbf{Y} - \bar{Y}\mathbf{1}\| \|(\mathbf{X} - \mathbf{1}\bar{X}^T)\beta_1\|}$$

so it is the cosine of the angle between the two rays,  $\mathbf{Y} - \bar{Y}\mathbf{1}$  and  $\mathcal{X}_1\beta_1$ . Its maximal value is attained by  $\mathcal{X}_1\hat{\beta}_1 = \Pi(\mathbf{Y} - \bar{Y}\mathbf{1}|\mathcal{C}(\mathcal{X}_1))$ . Thus,

$$\hat{\rho}^2 = \frac{SSR}{SST} = \frac{\hat{\beta}_1^T \mathcal{X}_1^T \mathcal{X}_1 \hat{\beta}_1}{\|\mathbf{Y} - \bar{Y}\mathbf{1}\|^2} = \frac{S_{21}S_{11}^{-1}S_{12}}{S_{22}}.$$

**Example 1.1.19** (Sample Proportions). Let  $(X_1, \dots, X_k)^T \sim \text{Multi}(n, p)$ , where  $p \in \Theta := \{(p_1, \dots, p_k)^T : \sum_{i=1}^k p_i = 1, p_i \geq 0 \ (i = 1, 2, \dots, k)\}$ . We estimate  $p$  with sample proportion

$$\hat{p}_n = \left( \frac{X_1}{n}, \dots, \frac{X_k}{n} \right)^T.$$

Then,

(a)  $\hat{p}_n$  is consistent estimator of  $p$ , i.e.,

$$\forall \epsilon > 0, \sup_{p \in \Theta} P_p(|\hat{p}_n - p| \geq \epsilon) \xrightarrow{n \rightarrow \infty} 0.$$

(b)  $q(\hat{p}_n)$  is consistent estimator of  $q(p)$  provided that  $q$  is (uniformly) continuous on  $\Theta$ .

*Proof.* (a) Note that there exists a constant  $C > 0$  such that

$$\begin{aligned} \sup_{p \in \Theta} P_p(|\hat{p}_n - p| \geq \epsilon) &\leq \sup_{p \in \Theta} \frac{E|\hat{p}_n - p|^2}{\epsilon^2} \\ &= \sup_{p \in \Theta} \sum_{i=1}^k \frac{p_i(1-p_i)}{n\epsilon^2} \\ &\leq \frac{C}{n\epsilon^2} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

so we get the desired result. Note that first inequality is from Chebyshev's inequality.

(b) Note that  $q$  is uniformly continuous on  $\Theta$ , since  $\Theta$  is closed and bounded. Thus the assertion holds. More precisely, for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$|p' - p| < \delta, p, p' \in \Theta \Rightarrow |q(p') - q(p)| < \epsilon.$$

Therefore, we get

$$\sup_{p \in \Theta} P_p(|q(\hat{p}_n) - q(p)| \geq \epsilon) \leq \sup_{p \in \Theta} P_p(|\hat{p}_n - p| \geq \delta) \xrightarrow{n \rightarrow \infty} 0.$$

□

### MLE in exponential families

Consider a random variable  $X$  with pdf in canonical exponential family

$$q_\eta(x) = h(x) \exp(\eta^T T(x) - A(\eta)) I_{\mathcal{X}}(x), \quad \eta \in \mathcal{E},$$

where  $\mathcal{E}$  is natural parameter space in  $\mathbb{R}^k$ . Our goal is to show consistency of MLE in canonical exponential family.

**Theorem 1.1.20.** *Let*

$$q_\eta(x) = h(x) \exp(\eta^T T(x) - A(\eta)) I_{\mathcal{X}}(x), \quad \eta \in \mathcal{E}$$

*be a canonical exponential family with natural parameter space  $\mathcal{E} \subseteq \mathbb{R}^k$ . Further assume*

*(i)  $\mathcal{E}$  is open.*

*(ii) The family is of rank  $k$ .*

*(iii)  $t_0 := T(x) \in C^0$ , where  $C$  denotes the smallest convex set containing the support of  $T(X)$ , and  $C^0$  be its interior.*

*Then the unique ML estimate  $\hat{\eta}(x)$  exists and is the solution of the likelihood equation*

$$\dot{l}_x(\eta) = T(x) - \dot{A}(\eta) = 0.$$

**Remark 1.1.21.** Note that in (iii),  $x$  is the observation of  $X$ , so  $t_0$  is the observation of  $T(X)$ . It is reasonable to consider  $t_0$  because ML estimate only depends on  $t_0$ . Also, recall that (ii)

means

$$\begin{aligned} & \nexists a \neq 0 \text{ s.t. } [P_\eta(a^T(T(x) - \mu) = 0) = 1 \text{ for some } \eta \in \mathcal{E}] \\ & \Leftrightarrow \nexists a \neq 0 \text{ s.t. } [Var_\eta(a^T T(x)) = 0 \text{ for some } \eta \in \mathcal{E}] \\ & \Leftrightarrow \ddot{A}(\eta) \text{ is positive definite } \forall \eta \in \mathcal{E}. \end{aligned}$$

To prove this, we need some preparation.

**Lemma 1.1.22.**

(a) (“Supporting Hyperplane Theorem”) Let  $C \subseteq \mathbb{R}^k$  be a convex set, and  $C^0$  be its interior. Then for  $t_0 \notin C$  or  $t_0 \in \partial C$ ,

$$\exists a \neq 0 \text{ s.t. } [a^T t \geq a^T t_0 \ \forall t \in C].$$

Conversely, for  $t_0 \in C^0$ ,

$$\nexists a \neq 0 \text{ s.t. } [a^T t \geq a^T t_0 \ \forall t \in C].$$

(b) Let  $P(T \in \mathcal{T}) = 1$  and  $E(\max_i |T_i|) < \infty$ . (i.e.,  $\mathcal{T}$  is support of  $T$ .) Then for a convex hull  $C$  of  $\mathcal{T}$ , we get  $ET \in C^0$ .

(c) Assume the above exponential family model with open  $\mathcal{E}$ . Then the ML estimate exists if the log-likelihood approaches  $-\infty$  on the boundary.

*Proof.* (a) Only second part will be given. (For the first part, see supplementary note.) Let  $t_0 \in C^0$ . Then  $\exists \delta > 0$  such that  $B(t_0, \delta) \subseteq C^0$ , since  $C^0$  is open. Note that for any  $u$  s.t.  $\|u\| = 1$ , we get

$$t_0 - \frac{\delta}{2}u, \ t_0 + \frac{\delta}{2}u \in B(t_0, \delta) \subseteq C.$$

If  $\exists a \neq 0$  such that  $a^T t \geq a^T t_0 \ \forall t \in C$ , then

$$a^T \left( t_0 - \frac{\delta}{2}u \right) \geq a^T t_0, \ a^T \left( t_0 + \frac{\delta}{2}u \right) \geq a^T t_0$$

holds for  $u = a/|a|$ , which yields contradiction. (Note that convexity condition is not used)

(b) Note that since  $C$  is a convex set,  $\mu := ET \in C$  holds. (Convex set contains average of itself) Assume  $\mu \notin C^0$ . Then  $\mu \in \partial C$ . Then by (a),  $\exists a \neq 0$  such that  $a^T t \geq a^T \mu$  for any  $t \in C$ .

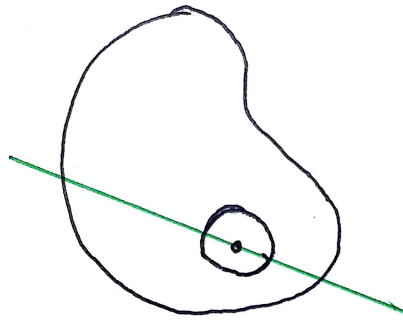


Figure 1.2: Proof of (a)

It implies that,  $\exists a \neq 0$  such that  $P(a^T(T - \mu) \geq 0) = 1$ , since  $\mathcal{T} \subseteq C$ . It implies that

$$P(a^T(T - \mu) = 0) = 1,$$

by the fact that

$$f \geq 0, \int f d\mu = 0 \Rightarrow f = 0 \text{ } \mu - a.e..$$

It is contradictory to (ii), which is full rank condition of the exponential family.

(c) Done in TheoStat I.

*Proof of theorem.* By lemma, it's sufficient to show that:

(1)  $l(\theta)$  diverges to  $-\infty$  at the boundary. (Existence)

(2) Uniqueness

Note that Uniqueness is clear since  $l_x(\eta)$  is  $\mathcal{C}^2$  function and strictly concave from  $\ddot{A}(\eta) > 0$ .

Thus, our claim is

**Claim.**  $l(\theta)$  approaches  $-\infty$  on the boundary  $\partial\mathcal{E}$ .

Let  $\eta^0 \in \partial\mathcal{E}$ . Then there is  $\eta_n \xrightarrow{n \rightarrow \infty} \eta^0$  such that  $\eta_n \in \mathcal{E}$ . Now our claim is, for any such sequence  $\eta_n$ , we get  $l_x(\eta_n) \xrightarrow{n \rightarrow \infty} -\infty$ . Note that  $|\eta_n| \xrightarrow{n \rightarrow \infty} \infty$  or  $\sup |\eta_n| < \infty$ . Also note that, for both cases, from  $l_x(\eta) = \log h(x) + \eta^T T(x) - A(\eta)$  and  $e^{A(\eta)} = \int_{\mathcal{X}} h(x) e^{\eta^T T(x)} d\mu(x)$ , we get

$$-l_x(\eta_n) + \log h(x) = A(\eta_n) - \eta_n^T t_0$$

$$= \log \int_{\mathcal{X}} \exp(\eta_n^T(T(y) - t_0)) h(y) d\mu(y).$$

CASE 1.  $|\eta_n| \rightarrow \infty$ .

Then since

$$\begin{aligned} \int_{\mathcal{X}} e^{\eta_n^T(T(y)-t_0)} h(y) d\mu(y) &\geq \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > \frac{1}{k}} e^{|\eta_n| \cdot \frac{\eta_n^T}{|\eta_n|}(T(y)-t_0)} h(y) d\mu(y) \\ &\geq \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > \frac{1}{k}} e^{|\eta_n|/k} h(y) d\mu(y) \\ &= \exp(|\eta_n|/k) \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > \frac{1}{k}} h(y) d\mu(y), \end{aligned}$$

if we can conclude

$$\inf_n \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > \frac{1}{k}} h(y) d\mu(y) > 0,$$

by the assumption  $|\eta_n| \rightarrow \infty$ , we get  $l_x(\eta_n) \rightarrow -\infty$ . Note that if

$$\inf_{u: \|u\|=1} \int_{u^T(T(y)-t_0) > 0} h(y) d\mu(y) > 0,$$

then

$$\inf_n \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > 0} h(y) d\mu(y) > 0,$$

and from

$$\inf_n \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > \frac{1}{k}} h(y) d\mu(y) \xrightarrow{k \rightarrow \infty} \inf_n \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > 0} h(y) d\mu(y),$$

we get  $\exists \epsilon > 0$  &  $k$  s.t.

$$\inf_n \int_{\frac{\eta_n^T}{|\eta_n|}(T(y)-t_0) > \frac{1}{k}} h(y) d\mu(y) > \epsilon$$

and the assertion holds. So our claim is:

**Claim.**  $\inf_{u: \|u\|=1} \int_{u^T(T(y)-t_0) > 0} h(y) d\mu(y) > 0.$

Assume not. If

$$\inf_{u: \|u\|=1} \int_{u^T(T(y)-t_0) > 0} h(y) d\mu(y) = 0,$$

then since  $\{u : \|u\| = 1\}$  is compact, there exists  $u_0 \in \{u : \|u\| = 1\}$  such that

$$\int_{u_0^T(T(y)-t_0)>0} h(y) d\mu(y) = 0.$$

It implies  $h(y) = 0$  on the set  $\{y : u_0^T(T(y) - t_0) > 0\}$   $\mu$ -a.e., and hence

$$\int_{u_0^T(T(y)-t_0)>0} h(y) e^{\eta^T T(y) - A(\eta)} d\mu(y) = 0,$$

which implies that

$$P_\eta(u_0^T(T(X) - t_0) > 0) = 0.$$

Thus, we get

$$P_\eta(u_0^T(T(X) - t_0) \leq 0) = 1,$$

which is equivalent to

$$u_0^T(t - t_0) \leq 0 \quad \forall t \in \mathcal{T}.$$

Since  $C$  is convex hull of  $\mathcal{T}$ , it implies

$$u_0^T(t - t_0) \leq 0 \quad \forall t \in C,$$

however, this yields contradiction to

$$\nexists a \neq 0 \text{ s.t. } a^T(t - t_0) \leq 0 \quad \forall t \in C,$$

from  $t_0 \in C^0$ .

CASE 2.  $\sup |\eta_n| < \infty$

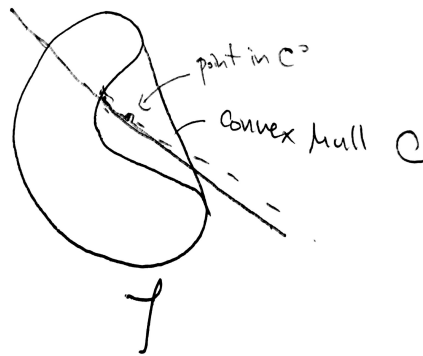
In this case, we get

$$\liminf_{n \rightarrow \infty} \int_{\mathcal{X}} e^{\eta_n^T(T(y)-t_0)} h(y) d\mu(y) \geq \int_{\mathcal{X}} e^{\eta^0 T(y)-t_0} h(y) d\mu(y) \stackrel{(*)}{=} \infty$$

by Fatou's lemma.  $(*)$  holds because  $\mathcal{E}$  is natural parameter space, and  $\eta^0 \in \partial\mathcal{E}$  implies  $\eta^0 \notin \mathcal{E}$ ,

since  $\mathcal{E}$  is open. Thus  $-l_x(\eta_n) \xrightarrow{n \rightarrow \infty} \infty$ .



Figure 1.3: Convex hull of  $\mathcal{T}$ 

□

Now we are ready to prove consistency.

**Theorem 1.1.23.** Let  $X_1, \dots, X_n$  be a random sample from a population with pdf

$$p_\eta(x) = h(x) \exp\{\eta^T T(x) - A(\eta)\} I_{\mathcal{X}}(x), \quad \eta \in \mathcal{E}$$

where  $\mathcal{E}$  is the natural parameter space in  $\mathbb{R}^k$ . Further, assume that

- (i)  $\mathcal{E}$  is open.
- (ii) The family is of rank  $k$ .

Then, the followings hold:

(a)  $P_\eta(\hat{\eta}_n^{MLE} \text{ exists}) \xrightarrow{n \rightarrow \infty} 1$

(b)  $\hat{\eta}_n^{MLE}$  is consistent.

*Proof.* (a) Let  $\bar{T}_n = \frac{1}{n} \sum_{i=1}^n T(X_i)$ . Then by WLLN, we get

$$\lim_{n \rightarrow \infty} P_\eta(|\bar{T}_n - E_\eta T(X_1)| < \epsilon) = 1 \quad \forall \epsilon > 0.$$

Also note that  $E_\eta T(X_1) \in C^0$ , where  $C^0$  is the interior of the convex hull of the support of  $T(X_1)$ . Then since  $C^0$  is open, open ball  $(|\bar{T}_n - E_\eta T(X_1)| < \epsilon)$  is contained in  $C^0$  for sufficiently small  $\epsilon > 0$ , which implies

$$\lim_{n \rightarrow \infty} P_\eta(\bar{T}_n \in C^0) = 1.$$

Now consider  $\bar{T}_n$  instead of  $T(X_1)$  in previous theorems, and note that (convex hull of support of  $\bar{T}_n$ ) = (convex hull of support of  $T(X_1)$ ). Then we can find that

$$(\bar{T}_n \in C^0) \subseteq (\hat{\eta}_n^{MLE} \text{ exists})$$

and therefore

$$\lim_{n \rightarrow \infty} P_\eta(\hat{\eta}_n^{MLE} \text{ exists}) = 1.$$

(b) From  $\ddot{A} > 0$ , we get  $\dot{A}(\eta)$  is one-to-one and continuous for any  $\eta$ . Then we get

$$(\bar{T}_n \in C^0) \subseteq (\hat{\eta}_n^{MLE} \text{ exists}) = (\dot{A}(\hat{\eta}_n^{MLE}) = \bar{T}_n)$$

and hence

$$\lim_{n \rightarrow \infty} P_\eta(\hat{\eta}_n^{MLE} = (\dot{A})^{-1}(\bar{T}_n)) = 1 \quad \forall \eta \in \mathcal{E}. \quad (1.1)$$

Further, by inverse function theorem, and  $C^2$  property of  $A$ , we have that  $(\dot{A})^{-1}$  is continuous. Thus by WLLN and continuous mapping theorem,

$$(\dot{A})^{-1}(\bar{T}_n) \xrightarrow[n \rightarrow \infty]{P_\eta} (\dot{A})^{-1}(E_\eta T(X_1)) = (\dot{A})^{-1}(\dot{A}(\eta)) = \eta$$

and since  $(\dot{A})^{-1}(\bar{T}_n) \approx \hat{\eta}_n^{MLE}$  in the sense of (1.1), we get

$$\lim_{n \rightarrow \infty} P_\eta(|\hat{\eta}_n^{MLE} - \eta| < \epsilon) = 1 \quad \forall \epsilon > 0,$$

$$\text{i.e., } \hat{\eta}_n^{MLE} \xrightarrow[n \rightarrow \infty]{P_\eta} \eta. \quad \square$$

Now let's see some general results. Suppose we have  $\lim_{n \rightarrow \infty} \Psi_n(\theta) = \Psi_0(\theta)$  and

$$\theta_n : \text{solution of } \Psi_n(\theta) = 0, \quad \theta \in C \quad (n = 1, 2, \dots)$$

$$\theta_0 : \text{solution of } \Psi_0(\theta) = 0, \quad \theta \in C.$$

Under what conditions,  $\lim_{n \rightarrow \infty} \theta_n = \theta_0$ ? We need following four conditions:

Uniform convergence of  $\Psi_n$ , Continuity of  $\Psi_0$ , Uniqueness of  $\theta_0$ , and Compactness of  $C$ .

Note that these are sufficient conditions *simultaneously*. Our goal is to obtain similar result for optimization.

**Theorem 1.1.24.** Suppose that we have  $\lim_{n \rightarrow \infty} D_n(\theta) = D_0(\theta)$  and

$$\theta_n = \arg \min_{\theta \in C} D_n(\theta) \quad (n = 1, 2, \dots)$$

$$\theta_0 = \arg \min_{\theta \in C} D_0(\theta)$$

where  $D_n$  and  $D_0$  are deterministic functions. Also assume that

(i)  $D_n$  converges to  $D_0$  uniformly.

(ii)  $D_0$  is continuous on  $C$ .

(iii) Minimizer  $\theta_0$  is unique.

(iv)  $C$  is compact.

Then  $\lim_{n \rightarrow \infty} \theta_n = \theta_0$ .

*Proof.* Assume not. In other words,  $\theta_n \not\rightarrow \theta_0$ . Then  $\exists \epsilon > 0$  such that  $|\theta_n - \theta_0| > \epsilon$  i.o.. It means that there is a subsequence  $\{n'\} \subseteq \{n\}$  s.t.  $|\theta_{n'} - \theta_0| > \epsilon \forall n'$ . Now define

$$\Delta_n = \sup_{\theta \in C} |D_n(\theta) - D_0(\theta)|.$$

Then by **uniform convergence** of  $D_n$ , we get  $\Delta_n \xrightarrow{n \rightarrow \infty} 0$ . Now note that

$$\begin{aligned} \inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) &= \inf_{|\theta - \theta_0| > \epsilon} \{D_0(\theta) - D_{n'}(\theta) + D_{n'}(\theta)\} \\ &\leq \inf_{|\theta - \theta_0| > \epsilon} \{|D_0(\theta) - D_{n'}(\theta)| + D_{n'}(\theta)\} \\ &\leq \Delta_{n'} + \inf_{|\theta - \theta_0| > \epsilon} D_{n'}(\theta) \end{aligned}$$

holds. Because minimization of  $D_{n'}$  is achieved at  $\theta_{n'} \in \{\theta : |\theta - \theta_0| > \epsilon\}$ , we get

$$\begin{aligned} \Delta_{n'} + \inf_{|\theta - \theta_0| > \epsilon} D_{n'}(\theta) &\leq \Delta_{n'} + \inf_{|\theta - \theta_0| \leq \epsilon} D_{n'}(\theta) \\ &\leq \Delta_{n'} + \inf_{|\theta - \theta_0| \leq \epsilon} \{|D_{n'}(\theta) - D_0(\theta)| + D_0(\theta)\} \\ &\leq 2\Delta_{n'} + \inf_{|\theta - \theta_0| \leq \epsilon} D_0(\theta) \\ &= 2\Delta_{n'} + D_0(\theta_0). \end{aligned}$$

The last equality holds from  $\theta_0 = \arg \min D_0(\theta)$  and  $\theta_0 \in \{\theta : |\theta - \theta_0| \leq \epsilon\}$ . Thus

$$\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) \leq 2\Delta_{n'} + D_0(\theta_0)$$

holds, which implies

$$\frac{1}{2} \left( \inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) - D_0(\theta_0) \right) \leq \Delta_{n'}.$$

Letting  $n' \rightarrow \infty$ , we get

$$\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) - D_0(\theta_0) = 0.$$

It is contradictory due to our claim that will be shown:

**Claim.**  $\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) - D_0(\theta_0) > 0.$

Intuitively, since  $\theta_0$  is **unique minimizer**, our claim seems trivial, but we also need continuity and compactness condition to guarantee this. (For this see next remark.)

Note that, by definition of infimum, there is a sequence  $\{\theta_k\} \subseteq \{\theta : |\theta - \theta_0| > \epsilon\} \cap C$  such that

$$\lim_{k \rightarrow \infty} D_0(\theta_k) = \inf_{|\theta - \theta_0| > \epsilon} D_0(\theta).$$

Now, by **compactness of  $C$** , there is a subsequence  $\{k'\} \subseteq \{k\}$  that makes  $\theta_{k'}$  converge to some  $\theta_0^*$  (“Bolzano-Weierstrass”), so with the abuse of notation, let  $\theta_k \rightarrow \theta_0^*$  as  $k \rightarrow \infty$ . Then note that  $\theta_0^*$  should belong to  $\{\theta : |\theta - \theta_0| \geq \epsilon\} \cap C$ , so  $\theta_0^* \neq \theta_0$ . Now, **continuity of  $D_0$**  makes

$$\lim_{k \rightarrow \infty} D_0(\theta_k) = D_0(\theta_0^*),$$

which implies

$$\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) = D_0(\theta_0^*).$$

Therefore, by **uniqueness of minimizer**,  $D_0(\theta_0^*) > D_0(\theta_0)$ , and combining to above result we can obtain

$$\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) > D_0(\theta_0).$$

□

**Remark 1.1.25.** See next figures. Each example tells that we need continuity and compactness, respectively.

**Remark 1.1.26.** For deterministic case, one can give an alternative proof. Suppose  $\theta_n \not\rightarrow \theta_0$ . Then since  $C$  is compact, we can find a subsequence  $\{\theta_{n_k}\}$  such that  $\theta_{n_k} \rightarrow \theta_0^*$ ,  $\theta_0^* \neq \theta_0$ . (If any

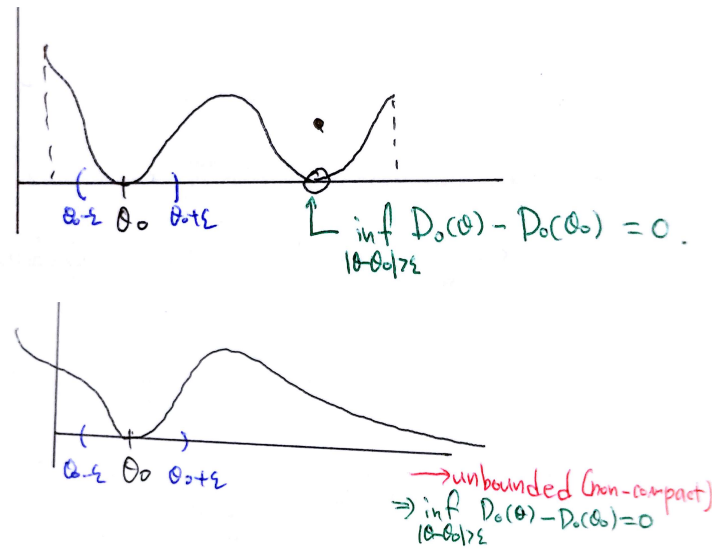


Figure 1.4: Continuity and Compactness are needed.

convergent subsequence converges to  $\theta_0$ , then origin sequence should converge to  $\theta_0$ .) Now for sufficiently large  $n_k$ ,

$$\sup_{\theta \in C} |D_{n_k}(\theta) - D_0(\theta)| < \frac{\epsilon}{3}$$

holds, so

$$\begin{aligned} D_0(\theta_0) &\geq D_{n_k}(\theta_0) - \frac{\epsilon}{3} \quad (\because \text{uniform convergence}) \\ &\geq D_{n_k}(\theta_{n_k}) - \frac{\epsilon}{3} \quad (\because \text{minimizer}) \\ &\geq D_0(\theta_{n_k}) - \frac{2}{3}\epsilon \quad (\because \text{uniform convergence}) \\ &\geq D_0(\theta_0^*) - \epsilon \quad (\because D_0(\theta_{n_k}) \rightarrow D_0(\theta_0^*) \text{ from continuity of } D_0) \end{aligned}$$

and hence taking  $\epsilon \searrow 0$  gives  $D_0(\theta_0) \geq D_0(\theta_0^*)$ , which is contradictory to uniqueness of  $\theta_0$ .

In fact, our real goal was, to get the similar result for *random*  $D_n$ .

**Theorem 1.1.27.** *Let  $D_n$  be a sequence of random functions, and  $D_0$  be deterministic. Similarly, define*

$$\hat{\theta}_n = \arg \min_{\theta \in C} D_n(\theta) \quad (n = 1, 2, \dots)$$

$$\theta_0 = \arg \min_{\theta \in C} D_0(\theta).$$

Now suppose that

(i)  $D_n$  converges in probability to  $D_0$  **uniformly**. It means that,

$$\sup_{\theta \in C} |D_n(\theta) - D_0(\theta)| \xrightarrow[n \rightarrow \infty]{P} 0.$$

(ii)  $D_0$  is continuous on  $C$ .

(iii) Minimizer  $\theta_0$  is unique.

(iv)  $C$  is compact.

Then  $\hat{\theta}_n \xrightarrow[n \rightarrow \infty]{P} \theta_0$ .

*Proof.* Note that in the proof of theorem 1.1.24, we did not used convergence in deriving

$$\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) \leq 2\Delta_{n'} + D_0(\theta_0).$$

Rather, we only used  $|\theta_{n'} - \theta_0| > \epsilon$ . (Convergence is used when deriving  $\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) - D_0(\theta_0)$ )

Thus,

$$|\hat{\theta}_n - \theta_0| > \epsilon \Rightarrow \inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) \leq 2\Delta_{n'} + D_0(\theta_0) \Rightarrow \Delta_n \geq \frac{1}{2} \left( \inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) - D_0(\theta_0) \right)$$

holds. Define

$$\frac{1}{2} \left( \inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) - D_0(\theta_0) \right) =: \delta(\epsilon).$$

Then, we get

$$\left( |\hat{\theta}_n - \theta_0| > \epsilon \right) \subseteq (\Delta_n \geq \delta(\epsilon)),$$

and therefore, by uniform P-convergence,  $\Delta_n \xrightarrow[n \rightarrow \infty]{P} 0$  and hence

$$P(|\hat{\theta}_n - \theta_0| > \epsilon) \leq P(\Delta_n \geq \delta(\epsilon)) \xrightarrow[n \rightarrow \infty]{} 0.$$

□

**Example 1.1.28** (Consistency of MLE when  $\Theta$  is finite). Let  $X_1, \dots, X_n$  be a random sample from a population with pdf  $f_\theta(\cdot)$ ,  $\theta \in \Theta$ . Assume that the parametrization is identifiable and  $\Theta = \{\theta_1, \dots, \theta_k\}$ . Then

$$\hat{\theta}_n^{MLE} \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} \theta_0,$$

provided that

(0) (Identifiability)  $P_{\theta_1} = P_{\theta_2} \Rightarrow \theta_1 = \theta_2$

(1) (Kullback-Leibler divergence)  $E_{\theta_0} \left| \log \frac{f_{\theta}(X_1)}{f_{\theta_0}(X_1)} \right| < \infty$ .

*Proof.* Note that, we defined

$$\hat{\theta}_n^{MLE} = \arg \min_{\theta \in \Theta} D_n(\theta) \text{ for } D_n(\theta) = -\frac{1}{n} \sum_{i=1}^n \log \frac{f_{\theta}(X_i)}{f_{\theta_0}(X_i)},$$

and by Kullback-Leibler divergence,

$$\theta_0 = \arg \min_{\theta \in \Theta} D_0(\theta) \text{ for } D_0(\theta) = -E_{\theta_0} \log \frac{f_{\theta}(X_1)}{f_{\theta_0}(X_1)}.$$

Then,

(i)  $\Theta = \{\theta_1, \dots, \theta_k\}$  is compact.

(ii)  $\theta_0$  is unique minimizer of  $D_0$ . (For this, see next remark.)

(iii) Uniform convergence is achieved from

$$\begin{aligned} P_{\theta_0} \left\{ \max_{1 \leq j \leq k} |D_n(\theta_j) - D_0(\theta_j)| > \epsilon \right\} &= P_{\theta_0} \left\{ \bigcup_{1 \leq j \leq k} (|D_n(\theta_j) - D_0(\theta_j)| > \epsilon) \right\} \\ &\leq \sum_{j=1}^k P_{\theta_0} (|D_n(\theta_j) - D_0(\theta_j)| > \epsilon) \\ &= o(1) \text{ by WLLN.} \end{aligned}$$

so we can derive the result similarly. In precise, it's sufficient to show

$$\inf_{|\theta - \theta_0| > \epsilon} D_0(\theta) - D_0(\theta_0) > 0$$

for  $\epsilon$  s.t.  $|\theta_n - \theta_0| > \epsilon$  i.o.. Uniqueness of  $\theta_0$  implies it clearly, because  $\Theta$  is finite in here. Note that continuity of  $D_0$  is not considered.  $\square$

**Remark 1.1.29.** *Kullback-Leibler divergence.* Since  $1 + \log z \leq z$ , we get

$$\begin{aligned} -E_{\theta_0} \log \frac{f_{\theta}(X_1)}{f_{\theta_0}(X_1)} &= -\int \log \frac{f_{\theta}(X_1)}{f_{\theta_0}(X_1)} dP_{\theta_0} \\ &\geq 1 - \int_{S(\theta_0)} \frac{f_{\theta}(x)}{f_{\theta_0}(x)} f_{\theta_0}(x) d\mu(x) \end{aligned}$$

$$\geq 0,$$

and hence  $D_0(\theta) \geq 0$ . In here  $S(\theta_0) = \{x : f_{\theta_0}(x) > 0\}$  and  $S(\theta) = \{x : f_{\theta}(x) > 0\}$ . Note that  $1 + \log z \leq z \Leftrightarrow z = 1$ . Thus equality of  $D_0(\theta) = 0$  holds if and only if

$$\begin{aligned} \frac{f_{\theta}(x)}{f_{\theta_0}(x)} &= 1 \quad \mu - \text{a.e. on } S(\theta_0) \\ \text{and } \int_{S(\theta_0)} f_{\theta}(x) d\mu(x) &= 1. \end{aligned}$$

Since

$$\begin{aligned} 1 &= \int_{S(\theta)} f_{\theta}(x) d\mu(x) = \int_{S(\theta_0) \cup S(\theta)} f_{\theta}(x) d\mu(x) \\ &= \int_{S(\theta_0)} f_{\theta}(x) d\mu(x) + \int_{S(\theta) \setminus S(\theta_0)} f_{\theta}(x) d\mu(x) \end{aligned}$$

we get

$$\int_{S(\theta_0)} f_{\theta}(x) d\mu(x) = 1 \Leftrightarrow \int_{S(\theta) \setminus S(\theta_0)} f_{\theta}(x) d\mu(x) = 0.$$

However, by definition of the support,  $f_{\theta}(x) > 0$  on  $S(\theta) \setminus S(\theta_0)$ , and hence

$$\int_{S(\theta) \setminus S(\theta_0)} f_{\theta}(x) d\mu(x) = 0 \Leftrightarrow \mu(S(\theta) \setminus S(\theta_0)) = 0.$$

Thus  $D_0(\theta)$  holds if and only if

$$\begin{aligned} f_{\theta}(x) &= f_{\theta_0}(x) \quad \mu - \text{a.e. on } S(\theta_0) \\ \text{and } \mu(S(\theta) \setminus S(\theta_0)) &= 0. \end{aligned}$$

However, note that

$$f_{\theta}(x) = f_{\theta_0}(x) \quad \mu - \text{a.e. on } S(\theta_0) \text{ implies } \mu(S(\theta) \setminus S(\theta_0)) = 0,$$

because

$$\begin{aligned} 1 &= \int_{S(\theta)} f_{\theta}(x) d\mu(x) = \int_{S(\theta_0)} f_{\theta}(x) d\mu(x) + \int_{S(\theta) \setminus S(\theta_0)} f_{\theta}(x) d\mu(x) \\ &= \int_{S(\theta_0)} f_{\theta_0}(x) d\mu(x) + \int_{S(\theta) \setminus S(\theta_0)} f_{\theta}(x) d\mu(x) \end{aligned}$$



$$= 1 + \int_{S(\theta) \setminus S(\theta_0)} f_\theta(x) d\mu(x).$$

Therefore we get,

$$D_0(\theta) = 0 \Leftrightarrow f_\theta(x) = f_{\theta_0}(x) \text{ } \mu - \text{a.e. on } S(\theta_0).$$

Now  $\mu(S(\theta) \setminus S(\theta_0)) = 0$  implies  $f_\theta(x) = f_{\theta_0}(x)$   $\mu - \text{a.e. on } S(\theta) \setminus S(\theta_0)$ , and therefore  $f_\theta(x) = f_{\theta_0}(x)$   $\mu - \text{a.e.}$ , if  $f_\theta(x) = f_{\theta_0}(x)$   $\mu - \text{a.e. on } S(\theta_0)$ . Therefore we get

$$D_0(\theta) = 0 \Leftrightarrow f_\theta(x) = f_{\theta_0}(x) \text{ } \mu - \text{a.e.} \Leftrightarrow \theta = \theta_0 \text{ } (\cdot \text{ identifiability}).$$

It means that  $\theta_0$  is unique minimizer of  $D_0(\theta)$ .

**Example 1.1.30** (Consistency of MCE). Let  $X_1, \dots, X_n$  be a random sample from  $P_\theta$ ,  $\theta \in \Theta \subseteq \mathbb{R}^k$ , and

$$\hat{\theta}_n^{MCE} = \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \rho(X_i, \theta).$$

Assume the following along with  $E_{\theta_0}|\rho(X_1, \theta)| < \infty \forall \theta_0, \theta \in \Theta$ :

For a fixed  $\theta_0 \in \Theta$ ,  $\exists$  a compact set  $K \subseteq \Theta$  containing  $\theta_0$  such that

- (i) (Unique minimizer)  $\theta_0 = \arg \min_{\theta \in K} E_{\theta_0} \rho(X_1, \theta)$ , and  $\theta_0$  is the unique minimizer.
- (ii) (Uniform convergence)  $\sup_{\theta \in K} |\bar{\rho}_n(\theta) - E_{\theta_0} \rho(X_1, \theta)| \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} 0$ .
- (iii) ( $K$  instead of  $\Theta$ )  $P_{\theta_0}(\hat{\theta}_n^{MCE} \in K) \xrightarrow[n \rightarrow \infty]{} 1$ .
- (iv) (Continuous  $D_0$ ) A function  $\theta \mapsto E_{\theta_0} \rho(X_1, \theta)$  is continuous on  $K$ .

In here,

$$\bar{\rho}_n(\theta) = \frac{1}{n} \sum_{i=1}^n \rho(X_i, \theta).$$

Then  $\hat{\theta}_n^{MCE} \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} \theta_0$ .

*Proof.* Note that  $\Theta$  need not be compact. Thus, we may use  $K$  instead of  $\Theta$ . By (the proof of) theorem 1.1.24, we get

$$P_{\theta_0} \left[ |\hat{\theta}_n^{MCE} - \theta_0| > \epsilon, \hat{\theta}_n^{MCE} \in K \right] \xrightarrow[n \rightarrow \infty]{} 0.$$

Thus, we get

$$P_{\theta_0} \left[ |\hat{\theta}_n^{MCE} - \theta_0| > \epsilon \right] \leq P_{\theta_0} \left[ |\hat{\theta}_n^{MCE} - \theta_0| > \epsilon, \hat{\theta}_n^{MCE} \in K \right] + P_{\theta_0} \left[ \hat{\theta}_n^{MCE} \notin K \right] \xrightarrow{n \rightarrow \infty} 0.$$

**Remark 1.1.31.** Indeed, we did not see consistency of MCE yet, but we only verified for fixed  $\theta_0 \in \Theta$ . For the consistency of MCE, we need that *for any  $\theta_0 \in \Theta \exists K \subseteq \Theta$  containing  $\theta_0$  such that the conditions (i)-(iv) are fulfilled.* Suppose that

(a) *for all compact  $K \subseteq \Theta$  and for all  $\theta_0 \in \Theta$ ,*

$$\sup_{\theta \in K} |\bar{\rho}_n(\theta) - E_{\theta_0} \rho(X_1, \theta)| \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} 0.$$

(b) *for any  $\theta_0 \in \Theta$  there exists a compact subset  $K$  of  $\Theta$  containing  $\theta_0$  such that*

$$P_{\theta_0} \left( \inf_{\theta \in K^c} (\bar{\rho}_n(\theta) - \bar{\rho}_n(\theta_0)) > 0 \right) \xrightarrow{n \rightarrow \infty} 1.$$

(c)  $\theta \mapsto E_{\theta_0} \rho(X_1, \theta)$  is continuous on  $K$ .

Then *for any  $\theta_0 \in \Theta$  there exists a compact subset  $K$  of  $\Theta$  containing  $\theta_0$  such that (ii)-(iv) hold.* Note that, (b) implies (iii) with (i) and (c).

Also note that, MLE is a special case for MCE,  $\rho(x, \theta) = -\log f(x, \theta)$ .

**Remark 1.1.32.** In many cases, it's difficult to verify uniform convergence condition. For this, following **convexity lemma** is useful: *If  $K$  is convex,*

$$\bar{\rho}_n(\theta) \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} E_{\theta_0} \rho(X_1, \theta) \quad \forall \theta \in K, \quad (\text{"pointwise convergence"})$$

*and  $\bar{\rho}_n$  is a convex function on  $K$  with  $P_{\theta_0}$ -a.s., then we get "uniform convergence"*

$$\sup_{\theta \in K} |\bar{\rho}_n(\theta) - E_{\theta_0} \rho(X_1, \theta)| \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} 0.$$

See D. Pollard (1991), *Econometric Theory*, 7, 186-199.

**Remark 1.1.33.** The condition (b) in remark 1.1.31 is satisfied if the empirical contrast

$$\rho_n(\theta) = \frac{1}{n} \sum_{i=1}^n \rho(X_i, \theta)$$

is convex on a convex open parameter space  $\Theta \subseteq \mathbb{R}^k$ , and approaches  $+\infty$  on the boundary with probability tending to 1.

## 1.2 The Delta Method

Basic intuition of the Delta Method is Taylor expansion.

**Theorem 1.2.1.** Suppose  $\sqrt{n}(X_n - a) \xrightarrow[n \rightarrow \infty]{d} X$ . Then

$$\sqrt{n}(g(X_n) - g(a)) = \dot{g}(a)\sqrt{n}(X_n - a) + o_P(1)$$

and hence

$$\sqrt{n}(g(X_n) - g(a)) \xrightarrow[n \rightarrow \infty]{d} \dot{g}(a)X,$$

provided  $g$  is differentiable at  $a$ .

*Proof.* By Taylor theorem,  $\exists R(x, a)$  s.t.

$$g(x) = g(a) + (\dot{g}(a) + R(x, a))(x - a)$$

where  $R(x, a) \rightarrow 0$  as  $x \rightarrow a$ . Note that if  $X_n \xrightarrow[n \rightarrow \infty]{P} a$  and  $R(x, a) \rightarrow 0$  as  $x \rightarrow a$  then  $R(X_n, a) \xrightarrow[n \rightarrow \infty]{P} 0$  ( $\because \forall \epsilon > 0 \exists \delta > 0$  s.t.  $|x - a| < \delta \Rightarrow |R(x, a)| < \epsilon$  implies

$$P(|R(X_n, a)| > \epsilon) \leq P(|X_n - a| \geq \delta) \xrightarrow[n \rightarrow \infty]{} 0$$

and then  $R(X_n, a) = o_P(1)$ . Thus

$$g(X_n) = g(a) + (\dot{g}(a) + R(X_n, a))(X_n - a)$$

and hence

$$\sqrt{n}(g(X_n) - g(a)) = \dot{g}(a)\sqrt{n}(X_n - a) + \underbrace{R(X_n, a)}_{=o_P(1)} \underbrace{\sqrt{n}(X_n - a)}_{=O_P(1)} = \dot{g}(a)\sqrt{n}(X_n - a) + o_P(1).$$

In multivariate case, statement becomes  $\dot{g}(a)^\top (X_n - a)$ . □

**Remark 1.2.2.** When  $g$  is a function of several variables, the differentiability means the total differentiability, which is implied by the existence of “continuous partial derivatives.”

**Example 1.2.3.**  $(X_1, Y_1), \dots, (X_n, Y_n) : \text{iid from } (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ . Let

$$\hat{\rho}_n = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2} \sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2}}.$$

(i) As far as the distribution of  $\hat{\rho}_n$  is concerned, we may assume  $\mu_1 = \mu_2 = 0$ ,  $\sigma_1^2 = \sigma_2^2 = 1$ .

Let

$$W_i = (X_i, Y_i, X_i^2, Y_i^2, X_i Y_i)^\top \stackrel{i.i.d.}{\sim} (0, 0, 1, 1, \rho),$$

and  $Z_n = \sqrt{n}(\bar{W}_n - (0, 0, 1, 1, \rho)^\top)$ , i.e.,

$$Z_{n1} = \sqrt{n}\bar{X}, \quad Z_{n2} = \sqrt{n}\bar{Y}, \quad Z_{n3} = \sqrt{n}(\bar{X}^2 - 1), \quad Z_{n4} = \sqrt{n}(\bar{Y}^2 - 1), \quad Z_{n5} = \sqrt{n}(\bar{X}\bar{Y} - \rho).$$

Note that  $Z_n = O_P(1)$ . Then

$$\begin{aligned} \hat{\rho}_n &= \frac{\frac{1}{\sqrt{n}}Z_{n5} + \rho - \left(\frac{1}{\sqrt{n}}Z_{n1}\right)\left(\frac{1}{\sqrt{n}}Z_{n2}\right)}{\sqrt{1 + \frac{1}{\sqrt{n}}Z_{n3} - \left(\frac{1}{\sqrt{n}}Z_{n1}\right)^2} \sqrt{1 + \frac{1}{\sqrt{n}}Z_{n4} - \left(\frac{1}{\sqrt{n}}Z_{n2}\right)^2}} \\ &= \left(\frac{1}{\sqrt{n}}Z_{n5} + \rho - \left(\frac{1}{\sqrt{n}}Z_{n1}\right)\left(\frac{1}{\sqrt{n}}Z_{n2}\right)\right) \\ &\quad \cdot \left(1 + \frac{1}{\sqrt{n}}Z_{n3} - \left(\frac{1}{\sqrt{n}}Z_{n1}\right)^2\right)^{-1/2} \left(1 + \frac{1}{\sqrt{n}}Z_{n4} - \left(\frac{1}{\sqrt{n}}Z_{n2}\right)^2\right)^{-1/2} \\ &= \left(\frac{1}{\sqrt{n}}Z_{n5} + \rho - o_P\left(\frac{1}{\sqrt{n}}\right)\right) \\ &\quad \cdot \left(1 - \frac{1}{2}\left(\frac{1}{\sqrt{n}}Z_{n3} - \left(\frac{1}{\sqrt{n}}Z_{n1}\right)^2\right) + o_P\left(\frac{1}{\sqrt{n}}\right)\right) \left(1 - \frac{1}{2}\left(\frac{1}{\sqrt{n}}Z_{n4} - \left(\frac{1}{\sqrt{n}}Z_{n2}\right)^2\right) + o_P\left(\frac{1}{\sqrt{n}}\right)\right) \\ &= \left(\frac{1}{\sqrt{n}}Z_{n5} + \rho - o_P\left(\frac{1}{\sqrt{n}}\right)\right) \left(1 - \frac{1}{2}\frac{1}{\sqrt{n}}Z_{n3} + o_P\left(\frac{1}{\sqrt{n}}\right)\right) \left(1 - \frac{1}{2}\frac{1}{\sqrt{n}}Z_{n4} + o_P\left(\frac{1}{\sqrt{n}}\right)\right) \\ &= \left(\frac{1}{\sqrt{n}}Z_{n5} + \rho - o_P\left(\frac{1}{\sqrt{n}}\right)\right) \left(1 - \frac{1}{2}\frac{1}{\sqrt{n}}Z_{n3} - \frac{1}{2}\frac{1}{\sqrt{n}}Z_{n4} + o_P\left(\frac{1}{\sqrt{n}}\right)\right) \\ &= \rho + \frac{1}{\sqrt{n}}Z_{n5} - \frac{\rho}{2}\left(\frac{1}{\sqrt{n}}Z_{n3} + \frac{1}{\sqrt{n}}Z_{n4}\right) + o_P\left(\frac{1}{\sqrt{n}}\right) \end{aligned}$$

holds, so we get

$$\sqrt{n}(\hat{\rho}_n - \rho) = Z_{n5} - \frac{\rho}{2}Z_{n3} - \frac{\rho}{2}Z_{n4} + o_P(1)$$

$$\begin{aligned}
&= \sqrt{n} \left( (\overline{XY} - \rho) - \frac{\rho}{2}(\overline{X^2} - 1) - \frac{\rho}{2}(\overline{Y^2} - 1) \right) + o_P(1) \\
&= \sqrt{n} \left( \overline{XY} - \frac{\rho}{2}\overline{X^2} - \frac{\rho}{2}\overline{Y^2} \right) + o_P(1) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( X_i Y_i - \frac{\rho}{2} X_i^2 - \frac{\rho}{2} Y_i^2 \right) + o_P(1) \\
&\xrightarrow[n \rightarrow \infty]{d} N(0, \text{Var} \left( X_1 Y_1 - \frac{\rho}{2} X_1^2 - \frac{\rho}{2} Y_1^2 \right)).
\end{aligned}$$

(ii) Now additionally suppose that  $(X_i, Y_i)$ 's are from bivariate normal distribution. Then  $Y_1 - \rho X_1$  is independent of  $X_1$ . Letting  $Z_1 = Y_1 - \rho X_1$ , we get  $\text{Var}(Z_1) = 1 - \rho^2$ ,  $\text{Var}(Z_1^2) = 2(1 - \rho^2)^2$  and hence

$$\begin{aligned}
\text{Var} \left( X_1 Y_1 - \frac{\rho}{2} X_1^2 - \frac{\rho}{2} Y_1^2 \right) &= \text{Var} \left( (1 - \rho^2) X_1 Z_1 - \frac{\rho}{2} Z_1^2 + \frac{\rho}{2} (1 - \rho^2) X_1^2 \right) \\
&= \text{Var} \left( \frac{\rho}{2} (1 - \rho^2) X_1^2 - \frac{\rho}{2} Z_1^2 \right) + \text{Var} \left( (1 - \rho^2) X_1 Z_1 \right) \\
&\quad + \underbrace{2 \text{Cov} \left( \frac{\rho}{2} (1 - \rho^2) X_1^2 - \frac{\rho}{2} Z_1^2, (1 - \rho^2) X_1 Z_1 \right)}_{=0} \\
&= \frac{\rho^2}{4} \left( (1 - \rho^2)^2 \text{Var}(X_1^2) - 2(1 - \rho^2) \text{Cov}(X_1^2, Z_1^2) + \text{Var}(Z_1^2) \right) \\
&\quad + (1 - \rho^2)^2 \text{Var}(X_1 Z_1) \\
&= \frac{\rho^2}{4} \left( 2(1 - \rho^2)^2 + 2(1 - \rho^2)^2 \right) + (1 - \rho^2)^2 (1 - \rho^2) \\
&= \rho^2 (1 - \rho^2)^2 + (1 - \rho^2)^2 (1 - \rho^2) \\
&= (1 - \rho^2)^2
\end{aligned}$$

holds. It implies that

$$\sqrt{n}(\hat{\rho}_n - \rho) \xrightarrow[n \rightarrow \infty]{d} N(0, (1 - \rho^2)^2).$$

Therefore, if we define  $h(\rho)$  as  $h'(\rho) = (1 - \rho^2)^{-1}$ , i.e.,

$$h(\rho) = \frac{1}{2} \log \frac{1 + \rho}{1 - \rho}, \quad (\text{"Fisher's } z\text{-transform"})$$

then we get

$$\sqrt{n}(h(\hat{\rho}_n) - h(\rho)) \xrightarrow[n \rightarrow \infty]{d} N(0, 1),$$

and with this, we can find a confidence region "with stabilized variance."

We can also expand with higher order terms.

**Theorem 1.2.4** (Higher order stochastic expansion). *Let  $X_1, \dots, X_n$  be a random sample with  $EX_1 = \mu$  and finite  $Var(X_1) = \Sigma$ .*

(a) (1-dim case) *For  $g$  with  $\exists \ddot{g}$ ,*

$$g(\bar{X}_n) = g(\mu) + \frac{\sigma}{\sqrt{n}} \dot{g}(\mu) Z_n + \frac{\sigma^2}{2n} \ddot{g}(\mu) Z_n^2 + o_P\left(\frac{1}{n}\right),$$

where  $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ .

(b) (general case) *For  $g$  with  $\exists \ddot{g}$ ,*

$$g(\bar{X}_n) = g(\mu) + \dot{g}(\mu)^\top (\bar{X}_n - \mu) + \frac{1}{2} (\bar{X}_n - \mu)^\top \ddot{g}(\mu) (\bar{X}_n - \mu) + o_P\left(\frac{1}{n}\right).$$

*Proof.* Again, use Taylor theorem. Only prove (a). Note that

$$g(x) = g(a) + \dot{g}(a)(x - a) + \frac{1}{2} (\ddot{g}(a) + R(x, a)) (x - a)^2$$

for  $R(x, a) \rightarrow 0$  as  $x \rightarrow a$ , so letting  $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ , we get

$$\begin{aligned} g(\bar{X}_n) &= g(\mu) + \dot{g}(\mu)(\bar{X}_n - \mu) + \frac{1}{2} \ddot{g}(\mu)(\bar{X}_n - \mu)^2 + \frac{1}{2} \underbrace{R(\bar{X}_n, \mu)}_{=o_P(1)} \underbrace{(\bar{X}_n - \mu)^2}_{=O_P(1/n)} \\ &= g(\mu) + \frac{\sigma}{\sqrt{n}} \dot{g}(\mu) Z_n + \frac{\sigma^2}{2n} \ddot{g}(\mu) Z_n^2 + o_P(1/n) \end{aligned}$$

which implies the conclusion.

**Remark 1.2.5.** For general case, following notation is also frequently used. For  $(Z_n^i)_{i=1}^d = \sqrt{n}(\bar{X}_n - \mu)$ ,

$$g(\bar{X}_n) = g(\mu) + \frac{1}{\sqrt{n}} g_{/i}(\mu) Z_n^i + \frac{1}{2n} g_{/ij}(\mu) Z_n^i Z_n^j + o_P\left(\frac{1}{n}\right).$$

In here, we omit the “ $\sum$ ,” i.e.,

$$g_{/i}(\mu) Z_n^i := \sum_{i=1}^d g_{/i}(\mu) Z_n^i, \quad g_{/ij}(\mu) Z_n^i Z_n^j = \sum_{i=1}^d \sum_{j=1}^d g_{/ij}(\mu) Z_n^i Z_n^j.$$

**Example 1.2.6** (Estimation of Reliability). Let  $X_1, \dots, X_n$  be a random sample from  $Exp(\lambda)$ , where  $\lambda > 0$  is a rate.

(i) Note that

$$\hat{\eta}_n^{MLE} = e^{-a/\bar{X}}.$$

Let  $Z_n = \sqrt{n}(\lambda\bar{X} - 1)$ . Then  $Z_n \xrightarrow[n \rightarrow \infty]{d} N(0, 1)$  and

$$\bar{X}^{-1} = \lambda \left( \frac{Z_n}{\sqrt{n}} + 1 \right)^{-1}.$$

Thus, we get

$$\begin{aligned} \hat{\eta}_n^{MLE} &= \exp \left( -a\lambda \left( \frac{Z_n}{\sqrt{n}} + 1 \right)^{-1} \right) \\ &= \exp \left( -a\lambda \left( 1 - \frac{Z_n}{\sqrt{n}} + \frac{Z_n^2}{n} + o_P \left( \frac{1}{n} \right) \right) \right) \\ &= e^{-a\lambda} \left( 1 + \left( a\lambda \frac{Z_n}{\sqrt{n}} - a\lambda \frac{Z_n^2}{n} \right) + \frac{1}{2} \left( a\lambda \frac{Z_n}{\sqrt{n}} - a\lambda \frac{Z_n^2}{n} \right)^2 + o_P \left( \frac{1}{n} \right) \right) \\ &= e^{-a\lambda} \left( 1 + a\lambda \frac{Z_n}{\sqrt{n}} + \frac{-a\lambda + (a\lambda)^2/2}{n} Z_n^2 + o_P \left( \frac{1}{n} \right) \right) \\ &= \eta + \frac{a\lambda e^{-a\lambda}}{\sqrt{n}} Z_n + \frac{(-a\lambda + (a\lambda)^2/2)e^{-a\lambda}}{n} Z_n^2 + o_P \left( \frac{1}{n} \right) \end{aligned} \quad (1.2)$$

from  $(1+x)^{-1} = 1 - x + x^2 + o(x^2)$  and  $e^x = 1 + x + x^2/2 + o(x^2)$  when  $x \approx 0$ .

(ii) Now consider

$$\hat{\eta}^{UMVUE} = \left( 1 - \frac{a}{n\bar{X}} \right)^{-1} I \left( \frac{a}{n\bar{X}} < 1 \right).$$

Note that from  $P \left( \frac{a}{n\bar{X}} < 1 \right) \xrightarrow[n \rightarrow \infty]{} 1$ , we can let  $\hat{\eta}^{UMVUE} = \left( 1 - \frac{a}{n\bar{X}} \right)^{-1}$  (See next remark). Then

$$\begin{aligned} \log \hat{\eta}^{UMVUE} &= (n-1) \log \left( 1 - \frac{a}{n\bar{X}} \right) \\ &= (n-1) \log \left( 1 - \frac{a\lambda}{n} \left( \frac{Z_n}{\sqrt{n}} + 1 \right)^{-1} \right) \\ &= (n-1) \log \left( 1 - \frac{a\lambda}{n} \left( 1 - \frac{Z_n}{\sqrt{n}} + \frac{Z_n^2}{n} + o_P \left( \frac{1}{n} \right) \right) \right) \\ &= (n-1) \log \left( 1 - \frac{a\lambda}{n} + \frac{a\lambda}{n\sqrt{n}} Z_n - \frac{a\lambda}{n^2} Z_n^2 + o_P \left( \frac{1}{n^2} \right) \right) \\ &= (n-1) \left\{ \left( -\frac{a\lambda}{n} + \frac{a\lambda}{n\sqrt{n}} Z_n - \frac{a\lambda}{n^2} Z_n^2 \right) - \frac{1}{2} \left( -\frac{a\lambda}{n} + \frac{a\lambda}{n\sqrt{n}} Z_n - \frac{a\lambda}{n^2} Z_n^2 \right)^2 \right\} + o_P \left( \frac{1}{n} \right) \\ &= -a\lambda + \frac{a\lambda}{\sqrt{n}} Z_n - \frac{a\lambda}{n} Z_n^2 - \frac{(a\lambda)^2}{2n} + \frac{a\lambda}{n} + o_P \left( \frac{1}{n} \right) \\ &= -a\lambda + \frac{a\lambda}{\sqrt{n}} Z_n + \frac{-a\lambda Z_n^2 + a\lambda - (a\lambda)^2/2}{n} + o_P \left( \frac{1}{n} \right) \end{aligned}$$

implies

$$\begin{aligned}
\hat{\eta}^{UMVUE} &= \exp \left( -a\lambda + \frac{a\lambda}{\sqrt{n}}Z_n + \frac{-a\lambda Z_n^2 + a\lambda - (a\lambda)^2/2}{n} + o_P \left( \frac{1}{n} \right) \right) \\
&= e^{-a\lambda} \left( 1 + \left( \frac{a\lambda}{\sqrt{n}}Z_n + \frac{-a\lambda Z_n^2 + a\lambda - (a\lambda)^2/2}{n} \right) + \frac{1}{2} \left( \frac{a\lambda}{\sqrt{n}}Z_n + \frac{-a\lambda Z_n^2 + a\lambda - (a\lambda)^2/2}{n} \right)^2 \right) \\
&\quad + o_P \left( \frac{1}{n} \right) \\
&= e^{-a\lambda} \left( 1 + \frac{a\lambda}{\sqrt{n}}Z_n + \left( -a\lambda Z_n^2 + a\lambda - \frac{(a\lambda)^2}{2} + \frac{1}{2}(a\lambda)^2 Z_n^2 \right) \frac{1}{n} \right) + o_P \left( \frac{1}{n} \right) \\
&= \eta + \frac{a\lambda e^{-a\lambda}}{\sqrt{n}}Z_n + \frac{(-a\lambda + (a\lambda)^2/2)e^{-a\lambda}}{n}(Z_n^2 - 1) + o_P \left( \frac{1}{n} \right) \tag{1.3}
\end{aligned}$$

from  $\log(1+x) = x - x^2/2 + o(x^2)$ ,  $x \approx 0$ . Comparing (1.3) to (1.2), we can say that UMVUE is “closer” than MLE to  $\eta$ , since MLE’s leading term has a bias

$$\frac{(-a\lambda + (a\lambda)^2/2)e^{-a\lambda}}{n},$$

while UMVUE’s leading term has no bias. Like this case, if one suggests a new estimator, then in many cases, one compares 2nd order term to judge its asymptotic behavior.

**Remark 1.2.7.** If there is an event that occurring probability converges to 1, then in an asymptotic sense, we may ignore such event, in the sense that:

- (i) If  $P(\mathcal{E}_n) \xrightarrow{n \rightarrow \infty} 1$  and  $P(X_n \leq x, \mathcal{E}_n) \xrightarrow{n \rightarrow \infty} F(x)$ , then  $X_n \xrightarrow[n \rightarrow \infty]{d} F$ .
- (ii) If  $P(\mathcal{E}_n) \xrightarrow{n \rightarrow \infty} 1$  and  $X_n = X + O_P(n^{-\alpha})$  on  $\mathcal{E}_n$ , then  $X_n = X + O_P(n^{-\alpha})$  in general. Convergence rate of  $P(\mathcal{E}_n)$  does not matter!

( $\because P(n^\alpha |X_n - X| \geq C) \leq P(n^\alpha |X_n - X| \geq C, \mathcal{E}_n) + \underbrace{P(\mathcal{E}_n^c)}_{\xrightarrow[n \rightarrow \infty]{} 0}$ , take  $\limsup_n$  and  $\lim_C$  on both sides.)

**Example 1.2.8.** Consider a sample correlation coefficient again. Assume  $EX_1^4 < \infty$  and  $EY_1^4 < \infty$ . Then

$$\begin{aligned}
\hat{\rho}_n &= \left( \frac{1}{\sqrt{n}}Z_{n5} + \rho - \left( \frac{1}{\sqrt{n}}Z_{n1} \right) \left( \frac{1}{\sqrt{n}}Z_{n2} \right) \right) \\
&\quad \cdot \left( 1 + \frac{1}{\sqrt{n}}Z_{n3} - \left( \frac{1}{\sqrt{n}}Z_{n1} \right)^2 \right)^{-1/2} \left( 1 + \frac{1}{\sqrt{n}}Z_{n4} - \left( \frac{1}{\sqrt{n}}Z_{n2} \right)^2 \right)^{-1/2} \\
&= \left( \rho + \frac{1}{\sqrt{n}}Z_{n5} - \frac{1}{n}Z_{n1}Z_{n2} + o_P \left( \frac{1}{n} \right) \right)
\end{aligned}$$



$$\begin{aligned}
& \cdot \left( 1 - \frac{1}{2} \left( \frac{1}{\sqrt{n}} Z_{n3} - \left( \frac{1}{\sqrt{n}} Z_{n1} \right)^2 \right) + \frac{3}{8} \left( \frac{1}{\sqrt{n}} Z_{n3} - \left( \frac{1}{\sqrt{n}} Z_{n1} \right)^2 \right)^2 + o_P \left( \frac{1}{n} \right) \right) \\
& \cdot \left( 1 - \frac{1}{2} \left( \frac{1}{\sqrt{n}} Z_{n4} - \left( \frac{1}{\sqrt{n}} Z_{n2} \right)^2 \right) + \frac{3}{8} \left( \frac{1}{\sqrt{n}} Z_{n4} - \left( \frac{1}{\sqrt{n}} Z_{n2} \right)^2 \right)^2 + o_P \left( \frac{1}{n} \right) \right) \\
& = \left( \rho + \frac{1}{\sqrt{n}} Z_{n5} - \frac{1}{n} Z_{n1} Z_{n2} + o_P \left( \frac{1}{n} \right) \right) \\
& \cdot \left( 1 - \frac{1}{2\sqrt{n}} Z_{n3} + \frac{1}{n} \left( \frac{1}{2} Z_{n1}^2 + \frac{3}{8} Z_{n3}^2 \right) + o_P \left( \frac{1}{n} \right) \right) \\
& \cdot \left( 1 - \frac{1}{2\sqrt{n}} Z_{n4} + \frac{1}{n} \left( \frac{1}{2} Z_{n2}^2 + \frac{3}{8} Z_{n4}^2 \right) + o_P \left( \frac{1}{n} \right) \right) \\
& = \rho + \frac{1}{\sqrt{n}} \left( Z_{n5} - \frac{\rho}{2} Z_{n3} - \frac{\rho}{2} Z_{n4} \right) \\
& + \frac{1}{n} \left( -Z_{n1} Z_{n2} - \frac{1}{2} Z_{n3} Z_{n5} - \frac{1}{2} Z_{n4} Z_{n5} + \rho \left( \frac{1}{4} Z_{n3} Z_{n4} + \frac{1}{2} Z_{n1}^2 + \frac{1}{2} Z_{n2}^2 + \frac{3}{8} Z_{n3}^2 + \frac{3}{8} Z_{n4}^2 \right) \right) + o_P \left( \frac{1}{n} \right)
\end{aligned}$$

holds. The leading term has bias

$$\frac{1}{n} \left\{ \frac{\rho}{4} E X_1^2 Y_1^2 + \frac{3}{8} \rho (E X_1^4 + E Y_1^4) - \frac{1}{2} (E X_1^3 Y_1 + E X_1 Y_1^3) \right\}$$

from

$$\begin{aligned}
E(Z_{3n} Z_{4n}) &= E X_1^2 X_2^2 - 1 \\
E(Z_{3n}^2 + Z_{4n}^2) &= E X_1^4 + E Y_1^4 - 2 \\
E(Z_{1n}^2 + Z_{2n}^2) &= 2 \\
E(Z_{3n} Z_{5n} + Z_{4n} Z_{5n}) &= E X_1^3 Y_1 + E X_1 Y_1^3 - 2\rho \\
E(Z_{1n} Z_{2n}) &= \rho.
\end{aligned}$$

For bivariate normal case, it becomes

$$-\frac{1}{2n} \rho (1 - \rho^2).$$

**Remark 1.2.9.** Note that in using stochastic expansion, we can get the mean, variance, skewness, ... of the leading term and might expect that they become the approximation of the moments of  $g(\bar{X}_n)$ , but this is not true! For example, if  $X_n \sim Ber(1/n)$ , then

$$P(nX_n > \epsilon) = P(X_n = 1) = \frac{1}{n} \xrightarrow[n \rightarrow \infty]{} 0$$

from  $X_n = 0$  or  $1$ , so  $nX_n = o_P(1)$ , but we get  $E(nX_n) = 1 \forall n$ . Thus, we need to check the behavior of the remainder.

From now on, we compare “moments of leading terms” and “approximation of moments.”

**Example 1.2.10.** Recall mgf

$$mgf_X(t) = Ee^{tX} \quad |t| < \epsilon$$

and cgf

$$cgf_X(t) = \log mgf_X(t) = \log Ee^{tX}. \quad |t| < \epsilon$$

For  $m_r = EX^r$ , mgf has a Taylor expansion

$$mgf_X(t) = 1 + m_1t + \frac{m_2}{2!}t^2 + \frac{m_3}{3!}t^3 + \dots,$$

and if  $X$  and  $Y$  are independent,

$$mgf_{X+Y}(t) = mgf_X(t) \cdot mgf_Y(t)$$

and

$$cgf_{aX+b}(t) = cgf_X(at) + bt$$

holds for constants  $a$  and  $b$ . From this we get

$$c_r(aX + b) = a^r c_r(X),$$

where  $c_r$  denotes  $r$ th cumulant. Also recall that, for  $A \approx 0$ ,

$$\log(1 + A) = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \dots,$$

and with this, we can obtain

$$cgf_X(t) = \log \left( 1 + \underbrace{(mgf_X(t) - 1)}_{=A} \right) = c_1t + \frac{c_2}{2!}t^2 + \frac{c_3}{3!}t^3 + \dots$$

where

$$c_1 = m_1, \quad c_2 = m_2 - m_1^2, \quad c_3 = m_3 - 3m_1m_2 + 2m_1^3, \quad c_4 = m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - m_1^4, \dots$$

If observations are normalized, i.e.,  $m_1 = 0$  and  $m_2 = 1$ , then

$$c_1 = 0, \quad c_2 = 1, \quad c_3 = m_3, \quad c_4 = m_4 - 3.$$

**Example 1.2.11.** Let

$$Z_n = \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \underbrace{X_i - \mu}_{=: \tilde{Z}_i \stackrel{d}{=} Z_1}.$$

Then from

$$cgf_{Z_n}(t) = cgf_{n^{-1/2} \sum \tilde{Z}_i}(t) = n \cdot cgf_{Z_1} \left( \frac{t}{\sqrt{n}} \right),$$

we obtain

$$c_r(Z_n) = n \cdot \left( \frac{1}{\sqrt{n}} \right)^r c_r(Z_1) = n^{-\frac{r}{2}+1} c_r(Z_1).$$

From this, we obtain

$$EZ_n^3 = c_3(Z_n) = \frac{1}{\sqrt{n}} c_3(Z_1) \tag{1.4}$$

$$EZ_n^4 = c_4(Z_n) + 3 = \frac{1}{n} c_4(Z_1) + 3. \tag{1.5}$$

**Example 1.2.12.** Now see the multivariate case. Let  $X = (X_1, \dots, X_d)^\top$  and  $t = (t_1, \dots, t_d)^\top$ .

Then

$$mgf_X(t) = Ee^{t^\top X} = Ee^{t_1 X_1 + \dots + t_d X_d}$$

and

$$\begin{aligned} m_1 &= \left[ \frac{\partial}{\partial t_i} mgf_X(t) \Big|_{t=0} \right]_i \\ m_2 &= \left[ \frac{\partial^2}{\partial t_i \partial t_j} mgf_X(t) \Big|_{t=0} \right]_{i,j} \\ m_3 &= \left[ \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} mgf_X(t) \Big|_{t=0} \right]_{i,j,k} \\ &\vdots \end{aligned}$$

and we get

$$mgf_X(t) = 1 + \sum_i m_1(i) t_i + \frac{1}{2!} \sum_{i,j} m_2(i,j) t_i t_j + \frac{1}{3!} \sum_{i,j,k} m_3(i,j,k) t_i t_j t_k + \dots$$

If data is centered, i.e.,  $EX = 0$ , then  $m_1 = 0$  and so

$$cgf_X(t) = \frac{1}{2!} \sum_{i,j} m_2(i,j) t_i t_j + \frac{1}{3!} \sum_{i,j,k} m_3(i,j,k) t_i t_j t_k + \frac{1}{4!} \sum_{i,j,k,l} (m_4(i,j,k,l) - 3m_2(i,j)m_2(k,l)) t_i t_j t_k t_l + \dots$$

Now let  $Z_n = (Z_n^1, \dots, Z_n^d)^\top = \sqrt{n}(\bar{X}_n - \mu)$ . Then

$$cgf_{Z_n}(t) = n \cdot cgf_{Z_1}(t/\sqrt{n})$$

implies

$$EZ_n^i Z_n^j =: \sigma^{i,j}, \quad (\sigma^{i,j})_{i,j} = \text{Var}(X_1)$$

$$EZ_n^i Z_n^j Z_n^k = \frac{1}{\sqrt{n}} c_3(i, j, k)$$

$$EZ_n^i Z_n^j Z_n^k Z_n^l = \frac{1}{n} c_4(i, j, k, l) + 3\sigma^{ij}\sigma^{kl}$$

where

$$c_3(i, j, k) = c_3(Z_1)(i, j, k) = EZ_1^i Z_1^j Z_1^k$$

$$c_4(i, j, k, l) = c_4(Z_1)(i, j, k, l) = EZ_1^i Z_1^j Z_1^k Z_1^l - 3(EZ_1^i Z_1^j)(EZ_1^k Z_1^l).$$

**Proposition 1.2.13** (Moments of the leading terms). *Let  $X_1, \dots, X_n$  be i.i.d. with  $E|X_1|^4 < \infty$ <sup>1</sup>.*

(a) (Univariate case) For  $g$  with  $\exists \ddot{g}$  and  $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ ,

$$g(\bar{X}_n) = g(\mu) + \underbrace{\frac{\sigma}{\sqrt{n}} \dot{g}(\mu) Z_n + \frac{\sigma^2}{2n} \ddot{g}(\mu) Z_n^2}_{=: W_n} + o_P(n^{-1}),$$

and for the leading term  $W_n$ ,

$$E(W_n) = g(\mu) + \frac{\sigma^2}{2n} \ddot{g}(\mu)$$

$$\text{Var}(W_n) = \frac{\sigma^2}{n} (\dot{g}(\mu))^2 + \frac{1}{n^2} \left( \sigma^3 c_3(Z_1) \dot{g}(\mu) \ddot{g}(\mu) + \frac{1}{2} \sigma^4 (\ddot{g}(\mu))^2 \right) + O(n^{-3})$$

$$E(W_n - EW_n)^3 = \frac{1}{n^2} \left( \sigma^3 c_3(Z_1) (\dot{g}(\mu))^3 + 3\sigma^4 (\dot{g}(\mu))^2 (\ddot{g}(\mu)) \right) + O(n^{-3}).$$

---

<sup>1</sup>In fact, stronger condition is needed: mgf of  $X_1$  exists

(b) (Multivariate case) For  $g$  with  $\exists \ddot{g}$  and  $(Z_n^i) = \sqrt{n}(\bar{X}_n - \mu)$ ,

$$g(\bar{X}_n) = g(\mu) + \underbrace{\frac{1}{\sqrt{n}}g_{/i}(\mu)Z_n^i + \frac{1}{2n}g_{/ij}(\mu)Z_n^iZ_n^j}_{=:W_n} + o_P(n^{-1}),$$

and for the leading term  $W_n$ ,

$$\begin{aligned} E(W_n) &= g(\mu) + \frac{1}{2n}g_{/ij}(\mu)\sigma^{ij} \\ \text{Var}(W_n) &= \frac{1}{n}g_{/i}(\mu)g_{/j}(\mu)\sigma^{ij} + O(n^{-2}) \end{aligned}$$

where  $(\sigma^{ij}) = \text{Var}(X_1)$ .

*Proof.* (a) Nothing but tedious calculation. First,

$$E(W_n) = g(\mu) + \frac{\sigma^2}{2n}\ddot{g}(\mu)$$

is easily obtained. Next, note that

$$\begin{aligned} \text{Var}(W_n) &= \text{Var}\left(\frac{\sigma}{\sqrt{n}}\dot{g}(\mu)Z_n + \frac{\sigma^2}{2n}\ddot{g}(\mu)Z_n^2\right) \\ &= \frac{\sigma^2}{n}(\dot{g}(\mu))^2\text{Var}(Z_n) + \frac{\sigma^3}{n\sqrt{n}}\dot{g}(\mu)\ddot{g}(\mu)\text{Cov}(Z_n, Z_n^2) + \frac{\sigma^4}{4n^2}(\ddot{g}(\mu))^2\text{Var}(Z_n^2). \end{aligned}$$

First,  $\text{Var}(Z_n) = 1$ . Also,  $\text{Var}(Z_n^2) = E(Z_n^4) - [E(Z_n^2)]^2 = 2 + n^{-1}c_4(Z_1)$  from (1.5). Finally,  $\text{Cov}(Z_n, Z_n^2) = EZ_n^3 - EZ_n \cdot EZ_n^2 = n^{-1/2}c_3(Z_1)$  from (1.4). Now we get

$$\begin{aligned} \text{Var}(W_n) &= \frac{\sigma^2}{n}(\dot{g}(\mu))^2 + \frac{\sigma^3}{n^2}\dot{g}(\mu)\ddot{g}(\mu)c_3(Z_1) + \frac{\sigma^4}{2n^2}(\ddot{g}(\mu))^2 + \frac{\sigma^4}{4n^3}(\ddot{g}(\mu))^2c_4(Z_1) \\ &= \frac{\sigma^2}{n}(\dot{g}(\mu))^2 + \frac{1}{n^2}\left(\sigma^3c_3(Z_1)\dot{g}(\mu)\ddot{g}(\mu) + \frac{1}{2}\sigma^4(\ddot{g}(\mu))^2\right) + O(n^{-3}). \end{aligned}$$

For  $E(W_n - EW_n)^3$ , note that  $EZ_n^5 = O(n^{-1/2})$  and  $EZ_n^6 = O(1)$ . (Check!)<sup>2</sup> Then

$$\begin{aligned} E(W_n - EW_n)^3 &= E\left[\frac{\sigma}{\sqrt{n}}\dot{g}(\mu)Z_n + \frac{\sigma^2}{2n}\ddot{g}(\mu)(Z_n^2 - 1)\right]^3 \\ &= \frac{\sigma^3}{n\sqrt{n}}\dot{g}(\mu)^3 \underbrace{EZ_n^3}_{=n^{-1/2}c_3(Z_1)} + \frac{3\sigma^4}{2n^2}\dot{g}(\mu)^2\ddot{g}(\mu) \underbrace{E[Z_n^2(Z_n^2 - 1)]}_{=n^{-1}c_4(Z_1)+2} \end{aligned}$$

---

<sup>2</sup>For  $EZ_n^5$ , check directly, and for  $EZ_n^6$ , you can check directly, or use  $Z_n^6 = O_P(1)$  from  $Z_n = O_P(1)$ .

$$\begin{aligned}
& + \underbrace{\frac{\sigma^5}{4n^2\sqrt{n}}\dot{g}(\mu)\ddot{g}(\mu)^2 E[Z_n(Z_n^2 - 1)^2] + \frac{\sigma^6}{8n^3}\ddot{g}(\mu)^3 E[(Z_n^2 - 1)^3]}_{=O(n^{-3})} \\
& = \frac{1}{n^2} \left( \sigma^3 c_3(Z_1) (\dot{g}(\mu))^3 + 3\sigma^4 (\dot{g}(\mu))^2 (\ddot{g}(\mu)) \right) + O(n^{-3})
\end{aligned}$$

by (1.4) and (1.5).

(b) Note that

$$W_n = g(\mu) + \frac{1}{\sqrt{n}}g_{/i}(\mu)Z_n^i + \frac{1}{2n}g_{/ij}(\mu)Z_n^iZ_n^j = g(\mu) + \frac{1}{\sqrt{n}}\sum_{i=1}^d g_{/i}(\mu)Z_n^i + \frac{1}{2n}\sum_{i,j=1}^d g_{/ij}(\mu)Z_n^iZ_n^j$$

so

$$\begin{aligned}
Var(W_n) &= \frac{1}{n}\sum_{i=1}^d\sum_{j=1}^d g_{/i}(\mu)g_{/j}(\mu)Cov(Z_n^i, Z_n^j) + \frac{1}{n\sqrt{n}}\sum_{i=1}^d\sum_{k,l=1}^d g_{/i}(\mu)g_{/kl}(\mu)Cov(Z_n^i, Z_n^kZ_n^l) \\
&+ \frac{1}{4n^2}\sum_{i,j=1}^d\sum_{k,l=1}^d g_{/ij}(\mu)g_{/kl}(\mu)Cov(Z_n^iZ_n^j, Z_n^kZ_n^l) \\
&= \frac{1}{n}g_{/i}(\mu)g_{/j}(\mu)\underbrace{Cov(Z_n^i, Z_n^j)}_{=\sigma^{ij}} + \frac{1}{n\sqrt{n}}g_{/i}(\mu)g_{/kl}(\mu)\underbrace{Cov(Z_n^i, Z_n^kZ_n^l)}_{=EZ_n^iZ_n^kZ_n^l} \\
&+ \frac{1}{4n^2}g_{/ij}(\mu)g_{/kl}(\mu)\underbrace{Cov(Z_n^iZ_n^j, Z_n^kZ_n^l)}_{EZ_n^iZ_n^jZ_n^kZ_n^l - (EZ_n^iZ_n^j)(EZ_n^kZ_n^l)} \\
&= \frac{1}{n}g_{/i}(\mu)g_{/j}(\mu)\sigma^{ij} + \frac{1}{n^2}g_{/i}(\mu)g_{/kl}(\mu)c_3(i, j, k) + \frac{1}{4n^2}g_{/ij}(\mu)g_{/kl}(\mu)\left(\frac{1}{n}c_4(i, j, k, l) + 2\sigma^{ij}\sigma^{kl}\right) \\
&= \frac{1}{n}g_{/i}(\mu)g_{/j}(\mu)\sigma^{ij} + O(n^{-2}).
\end{aligned}$$

□

**Proposition 1.2.14** (Approximation of moments). *Let  $X_1, \dots, X_n$  be i.i.d. with  $E|X_1|^4 < \infty$ .*

(a) (Univariate case) For  $g$  with bounded  $g^{(r)}$  ( $r = 0, 1, \dots, 4$ ),

$$\begin{aligned}
E(g(\bar{X}_n)) &= g(\mu) + \frac{\sigma^2}{2n}g^{(2)}(\mu) + O(n^{-2}) \\
Var(g(\bar{X}_n)) &= \frac{\sigma^2}{n}g^{(1)}(\mu)^2 + O(n^{-2})
\end{aligned}$$

where  $\mu = EX_1$ ,  $\sigma^2 = Var(X_1)$ .

(b) (Multivariate case) For  $g$  with bounded and continuous  $g_{/I}$  ( $|I| = 0, 1, \dots, 4$ ),

$$\begin{aligned} E(g(\bar{X}_n)) &= g(\mu) + \frac{1}{2n} g_{/ij}(\mu) \sigma^{ij} + O(n^{-2}) \\ \text{Var}(g(\bar{X}_n)) &= \frac{1}{n} g_{/i}(\mu) g_{/j}(\mu) \sigma^{ij} + O(n^{-2}) \end{aligned}$$

where  $\mu = EX_1$ ,  $(\sigma^{ij}) = \text{Var}(X_1)$ .

*Proof.* (a) Note that,

$$g(\bar{X}_n) = g(\mu) + \frac{\sigma}{\sqrt{n}} g^{(1)}(\mu) Z_n + \frac{\sigma^2}{2n} g^{(2)}(\mu) Z_n^2 + \frac{1}{3!} \frac{\sigma^3}{n\sqrt{n}} g^{(3)}(\mu) Z_n^3 + R_n,$$

where

$$R_n = \frac{1}{4!} \frac{\sigma^4}{n^2} g^{(4)}(\xi_n) Z_n^4, \quad \xi_n : \text{ a number between } \mu \text{ and } \bar{X}_n.$$

Note that

$$E|R_n| \leq \frac{1}{4!} \frac{\sigma^4}{n^2} \sup_x |g^{(4)}(x)| \cdot EZ_n^4 = O(n^{-2})$$

from “boundedness of  $g^{(4)}$ ” and  $EZ_n^4 = 3 + n^{-1}c_4(Z_1)$ . Also note that

$$\frac{\sigma^3}{n\sqrt{n}} EZ_n^3 = \frac{\sigma^3}{n\sqrt{n}} \frac{1}{\sqrt{n}} c_3(Z_1) = O(n^{-2}).$$

From these, we obtain

$$Eg(\bar{X}_n) = g(\mu) + \frac{\sigma^2}{2n} g^{(2)}(\mu).$$

Next, for the variance, we get

$$\begin{aligned} \text{Var}g(\bar{X}_n) &= \frac{\sigma^2}{n} g^{(1)}(\mu)^2 \text{Var}(Z_n) + \frac{\sigma^2}{4n^2} g^{(2)}(\mu)^2 \text{Var}(Z_n^2) \\ &\quad + O(n^{-3}) \text{Var}(Z_n^3) + \text{Var}(R_n) \\ &\quad + O(n^{-3/2}) \text{Cov}(Z_n, Z_n^2) + O(n^{-2}) \text{Cov}(Z_n, Z_n^3) + O(n^{-5/2}) \text{Cov}(Z_n^2, Z_n^3) \\ &\quad + O(n^{-1/2}) \text{Cov}(Z_n, R_n) + O(n^{-1}) \text{Cov}(Z_n^2, R_n) + O(n^{-3/2}) \text{Cov}(Z_n^3, R_n). \end{aligned}$$

Note that,

$$\text{Var}(Z_n) = 1, \quad \text{Var}(Z_n^2) = 2 + \frac{1}{n} c_4(Z_1) = O(1), \quad \text{Var}(Z_n^3) = EZ_n^6 - (EZ_n^3)^2 \leq O(1),$$

$$\text{Var}(R_n) \leq ER_n^2 = O(n^{-4}) \cdot EZ_n^8 \leq O(n^{-4}),$$

$$|Cov(Z_n, R_n)| \leq \sqrt{Var Z_n} \sqrt{Var R_n} \leq O(n^{-2}),$$

$$|Cov(Z_n^2, R_n)| \leq \sqrt{Var Z_n^2} \sqrt{Var R_n} \leq O(n^{-2}),$$

$$|Cov(Z_n^3, R_n)| \leq \sqrt{Var Z_n^3} \sqrt{Var R_n} \leq O(n^{-2}),$$

$$Cov(Z_n, Z_n^2) = EZ_n^3 - (EZ_n)(EZ_n^2) = O(n^{-1/2}),$$

$$|Cov(Z_n, Z_n^3)| \leq \sqrt{Var Z_n} \sqrt{Var Z_n^3} \leq O(1),$$

$$|Cov(Z_n^2, Z_n^3)| \leq \sqrt{Var Z_n^2} \sqrt{Var Z_n^3} \leq O(1).$$

From these, we get

$$\begin{aligned} Var g(\bar{X}_n) &= \frac{\sigma^2}{n} g^{(1)}(\mu)^2 \underbrace{Var(Z_n)}_{=1} + \frac{\sigma^2}{4n^2} g^{(2)}(\mu)^2 \underbrace{Var(Z_n^2)}_{=O(1)} \\ &\quad + O(n^{-3}) \underbrace{Var(Z_n^3)}_{=O(1)} + \underbrace{Var(R_n)}_{\leq O(n^{-4})} \\ &\quad + O(n^{-3/2}) \underbrace{Cov(Z_n, Z_n^2)}_{=O(n^{-1/2})} + O(n^{-2}) \underbrace{Cov(Z_n, Z_n^3)}_{\leq O(1)} + O(n^{-5/2}) \underbrace{Cov(Z_n^2, Z_n^3)}_{\leq O(1)} \\ &\quad + O(n^{-1/2}) \underbrace{Cov(Z_n, R_n)}_{\leq O(n^{-2})} + O(n^{-1}) \underbrace{Cov(Z_n^2, R_n)}_{\leq O(n^{-2})} + O(n^{-3/2}) \underbrace{Cov(Z_n^3, R_n)}_{\leq O(n^{-2})} \\ &= \frac{\sigma^2}{n} g^{(1)}(\mu)^2 + O(n^{-2}). \end{aligned}$$

(b) Recall the Taylor theorem for multivariate function: for  $(k+1)$ -time continuously differentiable function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$f(\mathbf{x}) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(\mathbf{a})}{\alpha!} (\mathbf{x} - \mathbf{a})^\alpha + \sum_{|\beta| = k+1} R_\beta(\mathbf{x}) (\mathbf{x} - \mathbf{a})^\beta,$$

where for  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  we define  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $\alpha! = \alpha_1! \dots \alpha_n!$ , and  $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ ,

$$R_\beta(\mathbf{x}) = \frac{|\beta|}{\beta!} \int_0^1 (1-t)^{|\beta|-1} D^\beta f(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) dt.$$

In here,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$



Using this, we obtain

$$g(\bar{X}_n) = g(\mu) + \frac{1}{\sqrt{n}}g_{/i}(\mu)Z_n^i + \frac{1}{2!}\frac{1}{n}g_{/ij}Z_n^iZ_n^j + \frac{1}{3!}\frac{1}{n\sqrt{n}}g_{/ijk}Z_n^iZ_n^jZ_n^k + R_n,$$

where

$$|R_n| \leq \frac{1}{6n^2} \left| \int_0^1 (1-u)^3 g_{/ijkl} \left( \mu + \frac{1}{\sqrt{n}}uZ_n \right) Z_n^i Z_n^j Z_n^k Z_n^l du \right|.$$

By boundedness of  $g_{/I}$  and

$$E|Z_n^i Z_n^j Z_n^k Z_n^l| \leq \frac{1}{4}E((Z_n^i)^4 + (Z_n^j)^4 + (Z_n^k)^4 + (Z_n^l)^4) = O(1), \quad (\text{"AM-GM"})$$

we obtain the desired result with similar procedure as univariate case.  $\square$

**Remark 1.2.15** (Open Questions).

(i) In (a), we may not use boundedness of  $g^{(r)}$  ( $r = 0, 1, 2, 3$ ), but only use that of  $g^{(4)}$ . Why boundedness condition is given?

(ii) Expand  $g(\bar{X}_n)$  by quadratic terms

$$g(\bar{X}_n) = g(\mu) + \frac{\sigma}{\sqrt{n}}g^{(1)}(\mu)Z_n + \frac{\sigma^2}{2n}g^{(2)}(\mu)Z_n^2 + R_n,$$

$$R_n = \frac{1}{3!}\frac{\sigma^3}{n\sqrt{n}}g^{(3)}(\xi_n)Z_n^3, \quad \xi_n : \text{ a number between } \mu \text{ and } \bar{X}_n.$$

Then

$$E|R_n| \leq O(n^{-3/2})E|Z_n^3| = O(n^{-2})$$

from boundedness of  $g^{(3)}$ . Then with similar procedure, we can get the same result. Why we should consider 3rd order term?

(Sol. We cannot guarantee  $E|Z_n^3| = O(n^{-1/2})$  from  $EZ_n^3 = O(n^{-1/2})!$ )

(iii) I think there is a typo in the lecture note; in the note, it is written that

$$R_n = \frac{1}{6n^2} \int_0^1 (1-u)^3 g_{/ijkl} \left( \frac{1}{\sqrt{n}}uZ_n \right) Z_n^i Z_n^j Z_n^k Z_n^l du.$$

**Example 1.2.16** (Estimation of reliability). Let  $X_1, \dots, X_n$  be a random sample from  $Exp(\lambda)$ ,  $\lambda > 0$ . Define

$$\eta = P_\lambda(X_1 > a) = e^{-a\lambda}.$$

Then

$$\hat{\eta}_n^{MLE} = \exp(-a/\bar{X}).$$

Here,  $g(x) = \exp(-a/x)$  is infinitely differentiable with bounded derivatives. Thus

$$E(\hat{\eta}_n^{MLE}) = \eta + \frac{(-a\lambda + (a\lambda)^2/2)e^{-a\lambda}}{n} + O(n^{-2})$$

$$Var(\hat{\eta}_n^{MLE}) = \frac{(a\lambda)^2 e^{-2a\lambda}}{n} + O(n^{-2}),$$

the leading terms of which agree with the mean and variance of the leading term in its stochastic expansion. On the other hand, for

$$\hat{\eta}_n^{UMVUE} = \left(1 - \frac{a}{n\bar{x}}\right)^{n-1} I\left(\frac{a}{n\bar{X}} < 1\right),$$

we cannot apply the result for the approximation of moments. But, it can be shown that its variance is also approximated by the same approximation formula.

**Remark 1.2.17.** The boundedness of the derivatives for the approximation of moments is rather stronger than needed. Whenever the approximation can be proved, the formulae agree with the moments of the leading term of its stochastic expansion. So only the validity of the order of the remainder needs to be proved. For example, in the bivariate normal case, the mean and variance of the sample correlation coefficient can be approximated as follows;

$$E(\hat{\rho}_n) = \frac{-\rho(1-\rho^2)}{2n} + O(n^{-2})$$

$$Var(\hat{\rho}_n) = \frac{(1-\rho^2)^2}{n} + O(n^{-2}).$$

## 1.3 Asymptotic Theory

### 1.3.1 MLE in Exponential Family

**Proposition 1.3.1.** *Let  $X_1, \dots, X_n$  be a random sample from a population with pdf*

$$p_\eta(x) = h(x) \exp(\eta^\top T(x) - A(\eta)) I_{\mathcal{X}}(x), \quad \eta \in \mathcal{E},$$

where  $\mathcal{E}$  is a natural parameter space in  $\mathbb{R}^k$ . Further, assume that

(i)  $\mathcal{E}$  is open.

(ii) The family is of rank  $k$ .

Then

$$(a) \hat{\eta}_n^{MLE} = \eta + (\ddot{A}(\eta))^{-1}(\bar{T}_n - \dot{A}(\eta)) + o_{P_\eta}(n^{-1/2}) = \eta + (-\ddot{l}(\eta))^{-1}\dot{l}(\eta) + o_{P_\eta}(n^{-1/2}).$$

$$(b) \sqrt{n}(\hat{\eta}_n^{MLE} - \eta) \xrightarrow[n \rightarrow \infty]{d} N\left(0, (\ddot{A}(\eta))^{-1}\right) = N(0, I_1^{-1}(\eta)).$$

*Proof.* Recall that, under the same assumptions,

$$(\bar{T}_n \in C^0) \subseteq (\dot{A}(\hat{\eta}_n^{MLE}) = \bar{T}_n) \subseteq (\hat{\eta}_n^{MLE} = \left(\dot{A}\right)^{-1}(\bar{T}_n)),$$

with the probabilities of these events tending to 1 (See theorem 1.1.20). Also note that, by full rank condition,  $\dot{A}$  is one-to-one and differentiable on  $\mathcal{E}$  with  $\ddot{A} > 0$ .

Now let  $t = \dot{A}(\eta)$ . Then by **Inverse Function Theorem** (see next Remark),  $\dot{A}^{-1}$  is also differentiable and

$$D(\dot{A}^{-1})(t) = \frac{\partial}{\partial t} \dot{A}^{-1}(t) = \left(\ddot{A}(\eta)\right)^{-1}.$$

CLT implies

$$\sqrt{n}(\bar{T}_n - E_\eta T(X_1)) \xrightarrow[n \rightarrow \infty]{d} N(0, \text{Var}_\eta T(X_1)),$$

and recall that  $E_\eta T(X_1) = \dot{A}(\eta)$ ,  $\text{Var}_\eta T(X_1) = \ddot{A}(\eta)$ . Therefore,  $\Delta$ -method implies

$$\begin{aligned} \hat{\eta}_n^{MLE} &= \dot{A}^{-1}(\bar{T}_n) \\ &= \dot{A}^{-1}(t) + \left(\ddot{A}(\eta)\right)^{-1}(\bar{T}_n - t) + o(|\bar{T}_n - t|) \\ &= \eta + \left(\ddot{A}(\eta)\right)^{-1}(\bar{T}_n - t) + o_{P_\eta}(n^{-1/2}), \end{aligned}$$

and hence

$$\begin{aligned} \sqrt{n}(\hat{\eta}_n^{MLE} - \eta) &= \left(\ddot{A}(\eta)\right)^{-1} \sqrt{n}(\bar{T}_n - t) + o_{P_\eta}(1) \\ &\xrightarrow[n \rightarrow \infty]{d} N(0, \text{Var}_\eta T(X_1)^{-1}). \end{aligned}$$

Rest part is obtained from  $\bar{T}_n - \dot{A}(\eta) = \dot{l}_n(\eta)/n$  and  $-\ddot{l}_n(\eta) = n\ddot{A}(\eta)$ . □

**Remark 1.3.2** (Inverse Function Theorem). Let  $F : \mathbb{U} \rightarrow \mathbb{R}^d$ , where  $U \subseteq \mathbb{R}^d$  is an open set. If

(i)  $F$  is one-to-one

(ii)  $F$  is Fréchet differentiable near  $x_0 \in U$

(iii)  $DF_{x_0} := \left[ \frac{\partial F}{\partial x_j} \Big|_{x=x_0} \right]_{i,j}$  is invertible.

Then  $F^{-1} : F(U) \rightarrow U$  is also Fréchet differentiable at  $y_0 = F(x_0)$ , and it satisfies  $D(F^{-1})(y_0) = (DF_{x_0})^{-1}$ .

**Example 1.3.3** (Multinomial case). Let  $X_1, \dots, X_n$  be a random sample from  $\text{Multi}(1, p(\theta))$ , where  $p(\theta) = (p_1(\theta), \dots, p_k(\theta))^\top$ , and  $\theta \in \Theta \subseteq \mathbb{R}$ . For example, consider Hardy-Weinberg proportions  $p(\theta) = (\theta^2, 2\theta(1-\theta), (1-\theta)^2)^\top$ ,  $0 < \theta < 1$ . Assume that

- (i)  $\Theta$  is open and  $0 < p_i(\theta) < 1$ ,  $\sum_{i=1}^k p_i(\theta) = 1$ .
- (ii)  $p(\theta) = (p_1(\theta), \dots, p_k(\theta))^\top$  is twice (totally) differentiable.

Then we can derive the asymptotic distribution of estimator of  $\theta$ . Let  $\theta = h(p(\theta))$  for any  $\theta \in \Theta$  for some differentiable function  $h$ . Let

$$\hat{p}(\theta) = \frac{1}{n} \sum_{i=1}^n X_i = \left( \frac{N_1}{n}, \dots, \frac{N_k}{n} \right)^\top,$$

where  $N_j = \sum_{i=1}^n I(X_{ij} = 1)$ . Then we get

$$E_\theta \hat{p}(\theta) = p(\theta)$$

and hence

$$h(\bar{X}_n) = h(p(\theta)) + \dot{h}(p(\theta))^\top (\bar{X}_n - p(\theta)) + o(|\bar{X}_n - p(\theta)|).$$

Note that  $Z_n := \sqrt{n}(\bar{X}_n - p(\theta)) = O_P(1)$ , and it has an asymptotic distribution

$$Z_n \xrightarrow[n \rightarrow \infty]{d} N(0, \Sigma(\theta)), \quad \Sigma(\theta) := \text{diag}(p(\theta)) - p(\theta)p(\theta)^\top,$$

and therefore we get

$$\sqrt{n}(h(\bar{X}_n) - h(p(\theta))) = \dot{h}(p(\theta))^\top Z_n + o_P(1),$$

which implies

$$\sqrt{n}(h(\bar{X}_n) - h(p(\theta))) \xrightarrow[n \rightarrow \infty]{d} N(0, \sigma_h^2(\theta)),$$

where

$$\sigma_h^2(\theta) = \dot{h}(p(\theta))^\top \Sigma(\theta) \dot{h}(p(\theta)).$$

Furthermore, we can obtain that

$$\sigma_h^2(\theta) \geq I_1^{-1}(\theta),$$

where equality holds iff

$$\dot{h}(p(\theta))^\top (X_1 - p(\theta)) = I_1^{-1}(\theta) \dot{l}_1(\theta).$$

It can be shown as following. First, note that

$$\dot{h}(p(\theta))^\top \dot{p}(\theta) = 1.$$

It implies that

$$\begin{aligned} 1 &= \dot{h}(p(\theta))^\top \frac{\partial}{\partial \theta} E_\theta X_1 \\ &= \dot{h}(p(\theta))^\top \int \frac{\partial}{\partial \theta} x f(x : \theta) d\mu(x) \\ &= \dot{h}(p(\theta))^\top \int x \frac{\partial}{\partial \theta} f(x : \theta) d\mu(x) \\ &= \dot{h}(p(\theta))^\top \text{Cov}_\theta \left( X_1, \frac{\partial}{\partial \theta} l_1(\theta) \right) \quad (\because E_\theta \frac{\partial}{\partial \theta} f(X_1 : \theta) = E_\theta \frac{\partial}{\partial \theta} l_1(\theta) = 0) \\ &= \text{Cov}_\theta \left( \dot{h}(p(\theta))^\top X_1, \frac{\partial}{\partial \theta} l_1(\theta) \right) \\ &\leq \sqrt{\text{Var}_\theta \left( \dot{h}(p(\theta))^\top X_1 \right) \text{Var}_\theta \left( \frac{\partial}{\partial \theta} l_1(\theta) \right)} \\ &= \sqrt{\dot{h}(p(\theta))^\top \Sigma(\theta) \dot{h}(p(\theta)) \cdot I_1(\theta)} \end{aligned}$$

holds. In here, “=” holds when  $\dot{h}(p(\theta))^\top X_1$  and  $\partial l_1(\theta)/\partial \theta$  has a linear relationship, i.e.,

$$\dot{h}(p(\theta))^\top (X_1 - p(\theta)) = I_1^{-1}(\theta) \dot{l}_1(\theta).$$

**Remark 1.3.4.** Actually, previous example shows *how to deal with asymptotic distribution of FSE*, more generally.

### 1.3.2 Asymptotic Normality of MCE

Our real goal of this section is right here.

**Theorem 1.3.5** (Asymptotic Normality of MCE). *Let  $X_1, \dots, X_n$  be a random sample from*

$P_\theta$ , where  $\theta \in \Theta$  and parameter space  $\Theta$  is open in  $\mathbb{R}^k$ . Let

$$\hat{\theta}_n^{MCE} = \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \rho(X_i, \theta)$$

$$\theta_0 = \arg \min_{\theta \in \Theta} E_{\theta_0} \rho(X_1, \theta).$$

Under the assumption of their existence, let

$$\begin{aligned} \Psi_1(\theta) &= \Psi(X_1, \theta) = \frac{\partial}{\partial \theta} \rho(X_1, \theta), & \bar{\Psi}_n(\theta) &= \frac{1}{n} \sum_{i=1}^n \Psi(X_i, \theta) \\ \dot{\Psi}_1(\theta) &= \frac{\partial \Psi_1(\theta)}{\partial \theta}, & \dot{\bar{\Psi}}_n(\theta) &= \frac{\partial \bar{\Psi}_n(\theta)}{\partial \theta} \\ \ddot{\Psi}_1(\theta) &= \frac{\partial^2 \Psi_1(\theta)}{\partial \theta^2}, & \ddot{\bar{\Psi}}_n(\theta) &= \frac{\partial^2 \bar{\Psi}_n(\theta)}{\partial \theta^2} \end{aligned}$$

Assume that

$$(A0) \quad P_{\theta_0} \left( \bar{\Psi}_n(\hat{\theta}_n^{MCE}) = 0 \right) \xrightarrow{n \rightarrow \infty} 1.$$

$$(A1) \quad E_{\theta_0} \Psi_1(\theta_0) = 0.$$

$$(A2) \quad \text{Var}_{\theta_0}(\Psi_1(\theta_0)) \text{ exists.}$$

$$(A3) \quad E_{\theta_0}(\dot{\Psi}_1(\theta_0)) \text{ exists and is nonsingular.}$$

$$(A4) \quad \exists \delta > 0 \text{ and } \exists M(X_1) = M_{\theta_0, \delta}(X_1) \text{ s.t.}$$

$$\max_{\substack{|\theta - \theta_0| \leq \delta \\ \theta \in \Theta}} |\ddot{\Psi}_1(\theta)| \leq M(X_1), \text{ where } E_{\theta_0} M(X_1) < \infty.$$

$$(A5) \quad \hat{\theta}_n^{MCE} \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} \theta_0 \text{ as } n \rightarrow \infty.$$

Then

$$\hat{\theta}_n^{MCE} = \theta_0 + \left[ -E_{\theta_0} \dot{\Psi}_1(\theta_0) \right]^{-1} \bar{\Psi}_n(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}),$$

so that

$$\sqrt{n}(\hat{\theta}_n^{MCE} - \theta_0) \xrightarrow[n \rightarrow \infty]{d} N \left( 0, \left[ -E_{\theta_0} \dot{\Psi}_1(\theta_0) \right]^{-1} \text{Var}_{\theta_0}(\Psi_1(\theta_0)) \left[ -E_{\theta_0} \dot{\Psi}_1(\theta_0) \right]^{-1} \right) \text{ under } P_{\theta_0}.$$

**Remark 1.3.6** (Gradient of vector map). Let  $F : \mathbb{R}^k \rightarrow \mathbb{R}^d$  be a smooth function, where

$$F(x) = (F_1(x), \dots, F_d(x))^\top.$$

(1) (1st-order gradient)

$$\frac{\partial F}{\partial x}(x_0) := DF(x_0) = \left( \frac{\partial F}{\partial x_1}(x_0), \frac{\partial F}{\partial x_2}(x_0), \dots, \frac{\partial F}{\partial x_k}(x_0) \right) \in \mathbb{R}^{d \times k},$$

where  $\frac{\partial F}{\partial x_j}(x_0) = \left( \frac{\partial F_1}{\partial x_j}(x_0), \dots, \frac{\partial F_d}{\partial x_j}(x_0) \right)^\top$  is a column vector. It can be interpreted as “a linear map.”

(2) (2nd-order gradient)

$$\frac{\partial^2 F}{\partial x^2}(x_0) := D^2F(x_0) = \left( \frac{\partial}{\partial x_1} \frac{\partial F}{\partial x}(x_0), \dots, \frac{\partial}{\partial x_k} \frac{\partial F}{\partial x}(x_0) \right) \in \mathbb{R}^{d \times k \times k},$$

where  $\frac{\partial}{\partial x_i} \frac{\partial F}{\partial x}(x_0)$  is  $d \times k$  matrix with  $\frac{\partial^2 F}{\partial x_i x_j}(x_0)$  as the  $j$ th column vector. Note that it can be interpreted as “a bi-linear map.”

(3) (Taylor expansion of vector-valued map)

$$\begin{aligned} F(x) &\approx F(x_0) + \sum_{j=1}^k \frac{\partial F}{\partial x_j}(x_0)(x_j - x_{0j}) + \frac{1}{2} \sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j}(x_0)(x_i - x_{0i})(x_j - x_{0j}) \\ &= F(x_0) + \underbrace{\frac{\partial F}{\partial x_j}(x_0)}_{\text{matrix}} \underbrace{(x - x_0)}_{\text{vector}} + \frac{1}{2} (x - x_0)^\top \underbrace{\frac{\partial^2 F}{\partial x_i \partial x_j}(x_0)}_{\text{3-array}} (x - x_0). \end{aligned}$$

In here, “matrix  $DF(x_0) \times \text{vector}$ ” becomes a vector, and “quadratic form with 3-array  $D^2F(x_0)$ ” becomes vector-valued. In this view,  $DF(x_0)$  and  $D^2F(x_0)$  can be interpreted as a linear and bi-linear map, respectively.

*Proof.* By Taylor’s theorem,  $\exists \theta_n^*$  in  $\text{line}(\theta_0, \hat{\theta}_n)$  such that  $\|\theta_n^* - \theta_0\| \leq \|\hat{\theta}_n - \theta_0\|$  and

$$\bar{\Psi}_n(\hat{\theta}_n) = \bar{\Psi}_n(\theta_0) + \dot{\bar{\Psi}}_n(\theta_0)(\hat{\theta}_n - \theta_0) + \frac{1}{2}(\hat{\theta}_n - \theta_0)^\top \ddot{\bar{\Psi}}_n(\theta_n^*)(\hat{\theta}_n - \theta_0).$$

Also note that, from (A4) and (A5),

$$\lim_{K \rightarrow \infty} \sup_n P_{\theta_0} \left( \ddot{\bar{\Psi}}_n(\theta_n^*) \geq K \right) = 0$$

holds, and hence

$$\ddot{\bar{\Psi}}_n(\theta_n^*) = O_{P_{\theta_0}}(1),$$

i.e.,

$$\ddot{\Psi}_n(\theta_n^*)(\hat{\theta}_n - \theta_0) = o_{P_{\theta_0}}(1).$$

( $\because$  From

$$\begin{aligned} P_{\theta_0} \left( \left| \ddot{\Psi}_n(\theta_n^*) \right| > K \right) &= P_{\theta_0} \left( \left| \ddot{\Psi}_n(\theta_n^*) \right| > K, |\hat{\theta}_n - \theta_0| \leq \delta \right) + P_{\theta_0} \left( \left| \ddot{\Psi}_n(\theta_n^*) \right| > K, |\hat{\theta}_n - \theta_0| > \delta \right) \\ &\leq P_{\theta_0} \left( \left| \ddot{\Psi}_n(\theta_n^*) \right| > K, |\hat{\theta}_n - \theta_0| \leq \delta \right) + P_{\theta_0} \left( |\hat{\theta}_n - \theta_0| > \delta \right) \\ &\stackrel{(A4)}{\leq} P_{\theta_0} (M(X_1) > K) + \underbrace{P_{\theta_0} \left( |\hat{\theta}_n - \theta_0| > \delta \right)}_{\xrightarrow[n \rightarrow \infty]{(A5)} 0}, \end{aligned}$$

for any  $\epsilon > 0$  we get for large  $N$

$$\sup_{n > N} P_{\theta_0} \left( \left| \ddot{\Psi}_n(\theta_n^*) \right| > K \right) \leq \frac{1}{K} E_{\theta_0} M(X_1) + \epsilon,$$

which implies

$$\lim_{K \rightarrow \infty} \sup_{n > N} P_{\theta_0} \left( \left| \ddot{\Psi}_n(\theta_n^*) \right| > K \right) = 0.$$

(Let  $N$  more large and take  $\epsilon \searrow 0$ ) Thus, we get

$$\begin{aligned} \bar{\Psi}_n(\hat{\theta}_n) &= \bar{\Psi}_n(\theta_0) + \dot{\bar{\Psi}}_n(\theta_0)(\hat{\theta}_n - \theta_0) + \frac{1}{2}(\hat{\theta}_n - \theta_0)^\top \ddot{\bar{\Psi}}_n(\theta_n^*)(\hat{\theta}_n - \theta_0) \\ &= \bar{\Psi}_n(\theta_0) + \left( \dot{\bar{\Psi}}_n(\theta_0) + \frac{1}{2}(\hat{\theta}_n - \theta_0)^\top \ddot{\bar{\Psi}}_n(\theta_n^*) \right) (\hat{\theta}_n - \theta_0) \\ &= \bar{\Psi}_n(\theta_0) + \left( \dot{\bar{\Psi}}_n(\theta_0) + o_{P_{\theta_0}}(1) \right) (\hat{\theta}_n - \theta_0) \\ &= \bar{\Psi}_n(\theta_0) + \left( E_{\theta_0} \dot{\Psi}_1(\theta_0) + o_{P_{\theta_0}}(1) \right) (\hat{\theta}_n - \theta_0). \end{aligned}$$

Now, note that

$$(1) \ P_{\theta_0}(\bar{\Psi}_n(\hat{\theta}_n) = 0) \xrightarrow[n \rightarrow \infty]{} 1.$$

$$(2) \ E_{\theta_0} \dot{\Psi}_1(\theta_0) + o_{P_{\theta_0}}(1) \text{ is nonsingular with probability 1 (See remark 1.3.7)}$$

$$(3) \ \text{If } X_n = Y_n + O_P(a_n) \text{ on } \mathcal{E}_n \text{ with } P(\mathcal{E}_n) \xrightarrow[n \rightarrow \infty]{} 1, \text{ then } X_n = Y_n + O_P(a_n) \text{ (on whole space),}$$

and the same holds for  $O_P$  (See remark 1.2.7).

Thus, on the set with probability tending to 1,

$$\hat{\theta}_n - \theta_0 = \left( -E_{\theta_0} \dot{\Psi}_1(\theta_0) + o_{P_{\theta_0}}(1) \right)^{-1} \bar{\Psi}_n(\theta_0)$$



$$= \left[ \left( -E_{\theta_0} \dot{\Psi}_1(\theta_0) \right)^{-1} + o_{P_{\theta_0}}(1) \right] \bar{\Psi}_n(\theta_0)$$

holds, which yields

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \left( -E_{\theta_0} \dot{\Psi}_1(\theta_0) \right)^{-1} \underbrace{\sqrt{n} \bar{\Psi}_n(\theta_0)}_{O_{P_{\theta_0}}(1)} + o_{P_{\theta_0}}(1)$$

and therefore

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow[n \rightarrow \infty]{d} N \left( 0, \left( -E_{\theta_0} \dot{\Psi}_1(\theta_0) \right)^{-1} \text{Var}_{\theta_0} \Psi_1(\theta_0) \left( -E_{\theta_0} \dot{\Psi}_1(\theta_0) \right)^{-1} \right).$$

□

**Remark 1.3.7.** If  $A \in \mathbb{R}^{d \times d}$  is symmetric positive definite matrix, then for small perbutation  $\Delta$  s.t.  $\|\Delta\|_2 < \sigma_{\min}(A)$ ,  $\text{rank}(A + \Delta) = d$ . Note that  $\text{rank}(A) = d$ . In here,  $\sigma_{\min}(A)$  denotes the smallest eigenvalue of  $A$ , and  $\|\cdot\|_p$  is a matrix norm induced by corresponding  $\mathcal{L}^p$  vector norm, i.e.,

$$\|\Delta\|_p = \sup_{x: \|x\|_p=1} \|\Delta x\|_p.$$

*Proof.* (Motivation: for  $c \approx 0$  and  $x \neq 0$ ,

$$\frac{1}{x+c} = \frac{x}{c} - \frac{x^2}{c^2} + \frac{x^3}{c^3} - \dots$$

exists, or for small vectors  $u, v$ ,  $A + uv^\top$  is nonsingular from

$$(A + uv^\top)^{-1} = A^{-1} - \frac{A^{-1}uv^\top A^{-1}}{1 + u^\top A^{-1}v},$$

if  $A$  is invertible) Let  $\text{rank}(A + \Delta) < d$ . Then  $\exists x_0 \neq 0$  s.t.  $(A + \Delta)x_0 = 0$  and  $\|x_0\|_2 = 1$ . Then by definition of matrix norm,

$$\|\Delta\|_2 \geq \|\Delta x_0\|_2 = \|Ax_0\|_2 \geq \sigma_{\min}(A)$$

holds. The last inequality is from spectral theorem. It is contradictory to our assumption that  $\Delta$  is small. □

### 1.3.3 Asymptotic Normality and Efficiency of MLE

Note that MLE is just a special case of MCE.

**Theorem 1.3.8.** *Let  $X_1, \dots, X_n$  be a random sample from  $P_\theta$ , where  $\theta \in \Theta$  and parameter space  $\Theta$  is open in  $\mathbb{R}^k$ . Recall that MLE is an MCE with*

$$\rho(x, \theta) = -\log p_\theta(x), \quad p_\theta : \text{pdf of } P_\theta.$$

*Under the assumption of their existence, denote*

$$\begin{aligned} \dot{l}_n(\theta) &= \sum_{i=1}^n \frac{\partial}{\partial \theta} \log p_\theta(X_i), & \bar{\dot{l}}_n(\theta) &= \frac{1}{n} \dot{l}_n(\theta) \\ \ddot{l}_n(\theta) &= \frac{\partial^2}{\partial \theta^2} \log p_\theta(X_i), & \bar{\ddot{l}}_n(\theta) &= \frac{1}{n} \ddot{l}_n(\theta) \\ \dddot{l}_n(\theta) &= \frac{\partial^3}{\partial \theta^3} \log p_\theta(X_i), & \bar{\dddot{l}}_n(\theta) &= \frac{1}{n} \dddot{l}_n(\theta). \end{aligned}$$

*Also assume that*

$$(M0) \quad P_{\theta_0} \left( \dot{l}_n(\hat{\theta}_n^{MLE}) = 0 \right) \xrightarrow{n \rightarrow \infty} 1.$$

$$(M1) \quad E_{\theta_0} \dot{l}_1(\theta_0) = 0.$$

$$(M2) \quad I(\theta_0) = \text{Var}_{\theta_0}(\dot{l}_1(\theta_0)) \text{ exists.}$$

$$(M3) \quad E_{\theta_0}(\ddot{l}_1(\theta_0)) \text{ exists and is nonsingular.}$$

$$(M4) \quad \exists \delta_0 > 0 \text{ and } \exists M(X_1) = M_{\theta_0, \delta}(X_1) \text{ s.t.}$$

$$\max_{\substack{|\theta - \theta_0| \leq \delta_0 \\ \theta \in \Theta}} |\ddot{l}_1(\theta)| \leq M(X_1), \text{ where } E_{\theta_0} M(X_1) < \infty.$$

$$(M5) \quad \hat{\theta}_n^{MLE} \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} \theta_0 \text{ as } n \rightarrow \infty.$$

$$(M6) \quad I(\theta_0) = E_{\theta_0}(-\ddot{l}_1(\theta_0)).$$

*Under (M0)  $\sim$  (M6),*

$$\hat{\theta}_n^{MLE} = \theta_0 + I(\theta_0)^{-1} \bar{\dot{l}}_n(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}),$$

*so that*

$$\sqrt{n}(\hat{\theta}_n^{MLE} - \theta_0) \xrightarrow[n \rightarrow \infty]{d} N(0, I(\theta_0)^{-1}).$$

Even though MCE is more general version of MLE, we frequently use MLE to estimate the parameter. Following theorem says that, “MLE is more (asymptotically) efficient than MCE,” i.e., log contrast function makes MCE the most efficient.

**Theorem 1.3.9** (Asymptotic efficiency of MLE). *Assume  $(A0) \sim (A6)$ , and  $(M0) \sim (M6)$  hold, where*

$$(A6) : E_{\theta_0} \dot{\Psi}_1(\theta_0) = -E_{\theta_0} \dot{l}_1(\theta_0) \Psi_1^\top(\theta_0).$$

*Then*

$$\begin{aligned} \sqrt{n}(\hat{\theta}_n^{MLE} - \theta_0) &\xrightarrow[n \rightarrow \infty]{d} N(0, I(\theta_0)^{-1}) \\ \sqrt{n}(\hat{\theta}_n^{MCE} - \theta_0) &\xrightarrow[n \rightarrow \infty]{d} N(0, \Sigma_\Psi(\theta_0)) \end{aligned}$$

*with  $\Sigma_\Psi(\theta_0) - I(\theta_0)^{-1}$  being nonnegative definite (i.e., “ $\Sigma_\Psi(\theta_0) \geq I(\theta_0)^{-1}$ ”), and “=” holds if and only if*

$$\Psi_1(\theta_0) = (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) I_1(\theta_0)^{-1} \dot{l}_1(\theta_0). \quad (\text{“Determining contrast function”})$$

*Proof.* By Cauchy-Schwarz inequality,

$$\begin{aligned} \text{Corr}(\lambda^\top \dot{l}_1(\theta_0), \gamma^\top \Psi_1(\theta_0)) &= \frac{\lambda^\top (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) \gamma}{\sqrt{\lambda^\top I_1(\theta_0) \lambda} \sqrt{\gamma^\top \text{Var}_{\theta_0} \Psi_1(\theta_0) \gamma}} \\ &= \frac{\lambda^\top I_1^{1/2}(\theta_0) I_1^{-1/2}(\theta_0) (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) \gamma}{\sqrt{\lambda^\top I_1(\theta_0) \lambda} \sqrt{\gamma^\top \text{Var}_{\theta_0} \Psi_1(\theta_0) \gamma}} \\ &\leq \frac{\sqrt{\lambda^\top I_1(\theta_0) \lambda} \sqrt{\gamma^\top (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) I_1^{-1}(\theta_0) (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) \gamma}}{\sqrt{\lambda^\top I_1(\theta_0) \lambda} \sqrt{\gamma^\top \text{Var}_{\theta_0} \Psi_1(\theta_0) \gamma}} \\ &= \frac{\sqrt{\gamma^\top (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) I_1^{-1}(\theta_0) (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) \gamma}}{\sqrt{\gamma^\top \text{Var}_{\theta_0} \Psi_1(\theta_0) \gamma}} \end{aligned}$$

holds, and hence

$$\max_{\lambda \neq 0} \text{Corr}(\lambda^\top \dot{l}_1(\theta_0), \gamma^\top \Psi_1(\theta_0)) = \frac{\sqrt{\gamma^\top (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) I_1^{-1}(\theta_0) (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) \gamma}}{\sqrt{\gamma^\top \text{Var}_{\theta_0} \Psi_1(\theta_0) \gamma}}$$

is obtained (we can find  $\lambda$  that achieves maximum). Since correlation coefficient is less than 1, we get

$$\gamma^\top (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) I_1^{-1}(\theta_0) (-E_{\theta_0} \dot{\Psi}_1(\theta_0)) \gamma \leq \gamma^\top \text{Var}_{\theta_0} \Psi_1(\theta_0) \gamma$$

for any  $\gamma$ . Using  $(-E_{\theta_0} \dot{\Psi}_1(\theta_0))^{-1} \gamma$  instead of  $\gamma$ , we obtain that

$$\gamma^\top I_1^{-1}(\theta_0) \gamma \leq \gamma^\top (-E_{\theta_0} \dot{\Psi}_1(\theta_0))^{-1} \text{Var}_{\theta_0} \Psi_1(\theta_0) (-E_{\theta_0} \dot{\Psi}_1(\theta_0))^{-1} \gamma,$$

i.e.,

$$I_1^{-1}(\theta_0) \leq (-E_{\theta_0} \dot{\Psi}_1(\theta_0))^{-1} \text{Var}_{\theta_0} \Psi_1(\theta_0) (-E_{\theta_0} \dot{\Psi}_1(\theta_0))^{-1} = \Sigma_\Psi(\theta_0).$$

Equality holds iff

$$\gamma^\top (-E_{\theta_0} \dot{\Psi}_1(\theta_0))^{-1} I_1^{-1}(\theta_0) \dot{l}_1(\theta_0) = \gamma^\top \Psi_1(\theta_0),$$

i.e.,

$$\Psi_1(\theta_0) = (-E_{\theta_0} \dot{\Psi}_1(\theta_0))^{-1} I_1^{-1}(\theta_0) \dot{l}_1(\theta_0).$$

□

**Remark 1.3.10.** The claim that the MLE has smaller variance than other asymptotically normal estimators was known as *Fisher's conjecture*. This is true for a certain class of estimators in a *regular* parametric model. Essential property for such a comparison is the “*uniform*” convergence to the normal distribution as it can be seen in the following example.

**Example 1.3.11** (Hodge's example: “Superefficient estimator”). Let  $\bar{X}$  be the sample mean in  $N(\theta, 1)$ . Let

$$\hat{\theta}_n^s = \bar{X} I(|\bar{X}| > n^{-1/4}).$$

(Actually, not only  $1/4$ , but any positive number less than  $1/2$  is OK.)

If  $\theta \neq 0$ , then

$$\begin{aligned} P_\theta(\hat{\theta}_n^s = \bar{X}) &= 1 - P_\theta(|\bar{X}| \leq n^{-1/4}) \\ &= 1 - P_\theta(-n^{-1/4} \leq \bar{X} \leq n^{-1/4}) \\ &= 1 - P_\theta(-\sqrt{n}\theta - n^{1/4} \leq \sqrt{n}(\bar{X} - \theta) \leq -\sqrt{n}\theta + n^{1/4}) \\ &= 1 - \underbrace{\Phi(-\sqrt{n}\theta + n^{1/4})}_{\xrightarrow{n \rightarrow \infty} 0 \text{ if } \theta > 0; 1 \text{ if } \theta < 0} + \underbrace{\Phi(-\sqrt{n}\theta - n^{1/4})}_{\xrightarrow{n \rightarrow \infty} 0 \text{ if } \theta > 0; 1 \text{ if } \theta < 0} \\ &\xrightarrow{n \rightarrow \infty} 1 \end{aligned}$$

holds. Intuitively, if  $\theta \neq 0$ ,  $\bar{X}$  converges to nonzero value with  $\sqrt{n}$ -rate, so  $I(\bar{X} > n^{-1/4})$  becomes

1. Thus,

$$\begin{aligned}
P_\theta \left( \sqrt{n}(\hat{\theta}_n^s - \theta) \leq x \right) &= P_\theta \left( \sqrt{n}(\hat{\theta}_n^s - \theta) \leq x, \hat{\theta}_n^s = \bar{X} \right) + P_\theta \left( \sqrt{n}(\hat{\theta}_n^s - \theta) \leq x, \hat{\theta}_n^s \neq \bar{X} \right) \\
&= P_\theta \left( \sqrt{n}(\bar{X} - \theta) \leq x, \hat{\theta}_n^s = \bar{X} \right) + P_\theta \left( \sqrt{n}(\hat{\theta}_n^s - \theta) \leq x, \hat{\theta}_n^s \neq \bar{X} \right) \\
&\xrightarrow[n \rightarrow \infty]{} \Phi(x)
\end{aligned}$$

holds, i.e.,

$$\sqrt{n}(\hat{\theta}_n^s - \theta) \xrightarrow[n \rightarrow \infty]{d} N(0, 1) \text{ under } P_\theta.$$

Now, assume that  $\theta = 0$ . Then  $\sqrt{n}\bar{X} \sim N(0, 1)$ , so

$$P_\theta(\hat{\theta}_n^s = 0) = P_\theta(|\bar{X}| \leq n^{-1/4}) = \Phi(n^{-1/4}) - \Phi(-n^{-1/4}) \xrightarrow[n \rightarrow \infty]{} 1.$$

Then we obtain  $\lim_{n \rightarrow \infty} P_\theta(\hat{\theta}_n^s = 0) = 1$ , and therefore,

$$\begin{aligned}
\lim_{n \rightarrow \infty} P_\theta \left( \sqrt{n}(\hat{\theta}_n^s - \theta) \leq x \right) &= \lim_{n \rightarrow \infty} P_\theta(0 \leq x) \\
&= \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \\
&= I_{[0, \infty)}(x),
\end{aligned}$$

which yields

$$\sqrt{n}(\hat{\theta}_n^s - \theta) \xrightarrow[n \rightarrow \infty]{d} N(0, 0) \text{ under } P_\theta.$$

Thus, we get

$$\sqrt{n}(\hat{\theta}_n^s - \theta) \xrightarrow[n \rightarrow \infty]{d} N(0, \sigma_s^2(\theta)) \text{ under } P_\theta,$$

where

$$\sigma_s^2(\theta) = \begin{cases} 1 & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0 \end{cases}.$$

It implies that, there is a “superefficient” estimator than “optimal” one with variance  $I(\theta)^{-1}$ , i.e., *Fisher’s conjecture* is wrong!

**Remark 1.3.12.** Note that Fisher’s conjecture is wrong in general, but it is “partially correct” in “regular” model. It means “uniform convergence” of CLT. First note that

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n \rightarrow \infty]{d (P_\theta)} N(0, \nu(\theta))$$

only means “pointwise” convergence *for each*  $\theta$ . It is equivalent to, for some metric  $d$ , and law (distribution function)  $L_\theta$  under  $P_\theta$ ,

$$d\left(L_\theta(\sqrt{n}(\hat{\theta}_n - \theta)), \Phi\left(\frac{\cdot}{\nu(\theta)}\right)\right) \xrightarrow{n \rightarrow \infty} 0.$$

“Regular estimator” means, on the other hand, that

$$\sup_{\theta: |\theta - \theta_0| < Mn^{-1/2}} d\left(L_\theta(\sqrt{n}(\hat{\theta}_n - \theta)), \Phi\left(\frac{\cdot}{\nu(\theta_0)}\right)\right) \xrightarrow{n \rightarrow \infty} 0$$

holds *for any*  $\theta_0 \in \Theta$ . Essentially it means that *for any sequence*  $\{\theta_n\}$  s.t.  $\sqrt{n}|\theta_n - \theta| \leq M$ , we get

$$P_{\theta_n}\left(\sqrt{n}(\hat{\theta}_n - \theta_n) \leq x\right) \xrightarrow{n \rightarrow \infty} \Phi\left(\frac{x}{\nu(\theta)}\right).$$

In Hodge’s example,  $\hat{\theta}_n^s$  is not regular, because cases ‘ $\theta = 0$ ’ and ‘ $\theta \approx 0$ ’ are different. Take  $\{\theta_n\}$  s.t.  $\theta_n \rightarrow 0$  at  $\sqrt{n}$ -rate, e.g.,

$$\theta_n = \frac{a_0}{\sqrt{n}}.$$

Then

$$\sqrt{n}\left(\hat{\theta}_n^s - \frac{a_0}{\sqrt{n}}\right) \xrightarrow[n \rightarrow \infty]{d(P_{a_0/\sqrt{n}})} -a_0,$$

and “limiting distribution depends on  $a_0$ ,” i.e., “the sequence  $\{\theta_n\}$ .”

**Example 1.3.13** (Linear model with stochastic covariates). Consider the model

$$Y|Z = z \sim N(\alpha + z^\top \beta, \sigma^2), \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}^k, \quad \sigma^2 > 0,$$

where  $E(Z) = 0$  and  $Var(Z)$  is non-singular. Assume that there are  $n$  independent copies of  $Y$  and  $Z$ , and denote  $Z_{(n)} = (Z_1, \dots, Z_n)^\top$ . Then as long as the distribution of  $Z$  does not depend on  $\theta = (\alpha, \beta, \sigma^2)$ , we get MLE of  $\beta$  is equivalent to least square procedure, from

$$l_n(\theta) = -\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \alpha - z_i^\top \beta)^2 - \frac{n}{2} \log 2\pi\sigma^2 + \sum_{i=1}^n \log pdf_Z(z_i).$$

In other words,

$$\begin{aligned} \hat{\beta}^{MLE} &= (\tilde{Z}_{(n)}^\top \tilde{Z}_{(n)})^{-1} \tilde{Z}_{(n)}^\top Y, \quad \tilde{Z}_{(n)} = (Z_1 - \bar{Z}, \dots, Z_n - \bar{Z})^\top \\ \hat{\alpha}^{MLE} &= \bar{Y} - \bar{Z}^\top \hat{\beta}^{MLE} \end{aligned}$$

$$\hat{\sigma}^{2MLE} = \frac{1}{n} \|Y - (\mathbf{1}\hat{\alpha}^{MLE} + Z_{(n)}\hat{\beta}^{MLE})\|^2.$$

It also says that,

$$l_1(\theta) = -\frac{1}{2\sigma^2}(Y_1 - \alpha - z_1^\top \beta)^2 - \frac{1}{2} \log 2\pi\sigma^2 + \log pdf_Z(z_1)$$

and hence

$$\dot{l}_1(\theta) = \left( \frac{\epsilon_1}{\sigma^2}, z_1^\top \frac{\epsilon_1}{\sigma^2}, \frac{\epsilon_1^2}{2\sigma^4} - \frac{1}{2\sigma^2} \right)^\top, \text{ where } \epsilon_1 = Y_1 - \alpha - z_1^\top \beta.$$

Thus

$$I(\theta) = \begin{pmatrix} \frac{1}{\sigma^2} & & \\ & \frac{1}{\sigma^2} E(Z_1 Z_1^\top) & \\ & & \frac{1}{2\sigma^4} \end{pmatrix}.$$

(It can be easily obtained from  $\epsilon_1|z_1 \sim N(0, \sigma^2)$ . Note that

$$Var(\epsilon_1) = EVar(\epsilon_1|z_1) + VarE(\epsilon_1|z_1) = \sigma^2,$$

$$Var(z_1 \epsilon_1) = EVar(z_1 \epsilon_1|z_1) + VarE(z_1 \epsilon_1|z_1) = E(z_1 \sigma^2 z_1^\top) + Var z_1 \cdot 0,$$

and similarly we can obtain  $Var(\epsilon_1^2)$ .) It implies that,

$$\sqrt{n}(\hat{\theta}_n^{MLE} - \theta) \xrightarrow[n \rightarrow \infty]{d} N(0, \Sigma),$$

where

$$\Sigma = I(\theta)^{-1} = \begin{pmatrix} \sigma^2 & & \\ & \sigma^2 (E(Z_1 Z_1^\top))^{-1} & \\ & & 2\sigma^4 \end{pmatrix},$$

provided that  $E(Z_1 Z_1^\top)$ , or equivalently  $Var(Z_1)$ , is non-singular.

### 1.3.4 Asymptotic Null distribution of LRT

Consider a random sample  $X_1, \dots, X_n$  from  $P_\theta$ , where  $\theta \in \Theta \subseteq \mathbb{R}^k$ . Denote  $\theta = (\xi^\top, \eta^\top)^\top$ , where  $\eta^\top \in \mathbb{R}^{k_0}$ . We wish to test

$$H_0 : \xi = \xi_0 \text{ vs } H_1 : \xi \neq \xi_0.$$

(Note that it is composite null!) Let  $\Theta$  be a  $k$ -dimensional open set, and

$$\Theta_0 = \{(\xi_0^\top, \eta^\top)^\top : (\xi_0^\top, \eta^\top)^\top \in \Theta\}$$

be a  $k_0$ -dimensional open set. Now denote

$$i(\theta) = \begin{pmatrix} i_1(\theta) \\ i_2(\theta) \end{pmatrix}, \quad I(\theta) = \begin{pmatrix} I_{11}(\theta) & I_{12}(\theta) \\ I_{21}(\theta) & I_{22}(\theta) \end{pmatrix},$$

where

$$i_1(\theta) = \frac{\partial}{\partial \xi} l(\theta), \quad i_2(\theta) = \frac{\partial}{\partial \eta} l(\theta).$$

**Theorem 1.3.14** (Asymptotic null distribution of LRT). *Assume  $(M0) \sim (M6)$ . Then under  $H_0 : \xi = \xi_0$*

$$2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) \xrightarrow[n \rightarrow \infty]{d} \chi^2(k - k_0)$$

where

$$\hat{\theta}_n = \arg \max_{\theta \in \Theta} l(\theta), \quad \hat{\theta}_n^0 = \arg \max_{\theta \in \Theta_0} l(\theta).$$

*Proof.* Recall that,

$$l(\theta_0) = l(\hat{\theta}_n) + \frac{1}{2} \sqrt{n}(\hat{\theta}_n - \theta_0)^\top \left[ \frac{1}{n} \ddot{l}(\theta_0) + o_{P_{\theta_0}}(1) \right] \sqrt{n}(\hat{\theta}_n - \theta_0)$$

holds. Since  $\sqrt{n}(\hat{\theta}_n - \theta_0) = O_{P_{\theta_0}}(1)$ , and  $O_{P_{\theta_0}}(1) \cdot o_{P_{\theta_0}}(1) = o_{P_{\theta_0}}(1)$ , we get

$$2(l(\hat{\theta}_n) - l(\theta_0)) = \sqrt{n}(\hat{\theta}_n - \theta_0)^\top I(\theta_0) \sqrt{n}(\hat{\theta}_n - \theta_0) + o_{P_{\theta_0}}(1),$$

from

$$-\frac{1}{n} \ddot{l}(\theta_0) \xrightarrow[n \rightarrow \infty]{P_{\theta_0}} I(\theta_0).$$

Also recall that

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = I(\theta_0)^{-1} \sqrt{n} \bar{\dot{l}}(\theta_0) + o_{P_{\theta_0}}(1).$$

It implies that

$$2(l(\hat{\theta}_n) - l(\theta_0)) = \sqrt{n} \bar{\dot{l}}(\theta_0)^\top I(\theta_0)^{-1} \sqrt{n} \bar{\dot{l}}(\theta_0) + o_{P_{\theta_0}}(1).$$



Similarly, we get

$$2(l(\hat{\theta}_n^0) - l(\theta_0)) = \sqrt{n}(\hat{\theta}_n^0 - \theta_0)^\top I_{22}(\theta_0) \sqrt{n}(\hat{\theta}_n^0 - \theta_0) + o_{P_{\theta_0}}(1)$$

and

$$\sqrt{n}(\hat{\theta}_n^0 - \theta_0) = I_{22}(\theta_0)^{-1} \sqrt{n} \bar{l}_2(\theta_0) + o_{P_{\theta_0}}(1)$$

under  $H_0$ , so we get

$$2(l(\hat{\theta}_n^0) - l(\theta_0)) = \sqrt{n} \bar{l}_2(\theta_0)^\top I_{22}(\theta_0)^{-1} \sqrt{n} \bar{l}_2(\theta_0) + o_{P_{\theta_0}}(1)$$

under  $H_0$ . Thus, we get

$$2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) = \sqrt{n} \bar{l}(\theta_0)^\top I(\theta_0)^{-1} \sqrt{n} \bar{l}(\theta_0) - \sqrt{n} \bar{l}_2(\theta_0)^\top I_{22}(\theta_0)^{-1} \sqrt{n} \bar{l}_2(\theta_0) + o_{P_{\theta_0}}(1)$$

under  $H_0$ . Now note that

$$\begin{aligned} I(\theta_0)^{-1} &= \begin{pmatrix} I_{11}(\theta_0) & I_{12}(\theta_0) \\ I_{21}(\theta_0) & I_{22}(\theta_0) \end{pmatrix}^{-1} \\ &= \begin{pmatrix} J_1 & 0 \\ -I_{22}^{-1}(\theta_0)I_{21}(\theta_0) & J_2 \end{pmatrix} \begin{pmatrix} I_{11.2}^{-1}(\theta_0) & 0 \\ 0 & I_{22}^{-1}(\theta_0) \end{pmatrix} \begin{pmatrix} J_1 & -I_{12}(\theta_0)I_{22}^{-1}(\theta_0) \\ 0 & J_2 \end{pmatrix} \end{aligned}$$

holds, for identity matrices  $J_1$  and  $J_2$  with suitable sizes. Then

$$\begin{aligned} \bar{l}(\theta_0)^\top I(\theta_0)^{-1} \bar{l}(\theta_0) &= \bar{l}(\theta_0)^\top \begin{pmatrix} J_1 & 0 \\ -I_{22}^{-1}(\theta_0)I_{21}(\theta_0) & J_2 \end{pmatrix} \begin{pmatrix} I_{11.2}^{-1}(\theta_0) & 0 \\ 0 & I_{22}^{-1}(\theta_0) \end{pmatrix} \begin{pmatrix} J_1 & -I_{12}(\theta_0)I_{22}^{-1}(\theta_0) \\ 0 & J_2 \end{pmatrix} \bar{l}(\theta_0) \\ &= \begin{pmatrix} \bar{l}_1(\theta_0) - I_{21}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) \\ \bar{l}_2(\theta_0) \end{pmatrix}^\top \begin{pmatrix} I_{11.2}^{-1}(\theta_0) & 0 \\ 0 & I_{22}^{-1}(\theta_0) \end{pmatrix} \begin{pmatrix} \bar{l}_1(\theta_0) - I_{21}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) \\ \bar{l}_2(\theta_0) \end{pmatrix} \end{aligned}$$

holds, so we get

$$\begin{aligned} 2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) &= \sqrt{n} \left( \bar{l}_1(\theta_0) - I_{21}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) \right)^\top I_{11.2}^{-1}(\theta_0) \sqrt{n} \left( \bar{l}_1(\theta_0) - I_{21}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) \right) \\ &\quad + o_{P_{\theta_0}}(1). \end{aligned} \tag{1.6}$$

Now by CLT,

$$\begin{aligned}\sqrt{n}(\bar{l}_1(\theta_0) - I_{21}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0)) &\xrightarrow[n \rightarrow \infty]{d} \begin{pmatrix} J_1 & -I_{12}(\theta_0)I_{22}^{-1}(\theta_0) \end{pmatrix} N(0, I(\theta_0)) \\ &= N(0, I_{11}(\theta_0) - I_{12}(\theta_0)I_{22}^{-1}(\theta_0)I_{21}(\theta_0)) \\ &= N(0, I_{11 \cdot 2}(\theta_0)),\end{aligned}$$

and hence,

$$2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) \xrightarrow[n \rightarrow \infty]{d} \chi^2(k - k_0).$$

□

**Remark 1.3.15.** (i) The extension to a null hypothesis

$$H_0 : g_1(\theta) = 0, \dots, g_{k_1}(\theta) = 0$$

is rather trivial by considering a smooth reparametrization

$$\xi = g_1(\theta), \dots, \xi_{k_1}(\theta) = g_{k_1}(\theta), \eta_1 = g_{k_1+1}(\theta), \dots, \eta_{k_0} = g_k(\theta)$$

whenever such  $g_j(\theta)$  ( $j = k_1 + 1, \dots, k$ ) can be found.

(ii) Large sample confidence set based on the maximum likelihood is obtained by duality.

**Remark 1.3.16.** Note that to perform LRT we should find MLE on  $\Theta$  and  $\Theta_0$ , however, it may not be so easy. Thus our interest is to find *asymptotic equivalent tests* of LRT.

### 1.3.5 Asymptotic Equivalents of LRT

Let  $X_1, \dots, X_n$  be a random sample from  $P_\theta$ , where  $\theta \in \Theta \subseteq \mathbb{R}^k$ . For  $\theta = (\xi^\top, \eta^\top)^\top \in \Theta$ , we again wish to test

$$H_0 : \xi = \xi_0 \text{ vs } H_1 : \xi \neq \xi_0.$$

We assume the regularity conditions, and in addition, that  $I(\theta)$  is continuous. Also, we denote the true parameter under  $H_0$  as  $\theta_0 = (\xi_0^\top, \eta_0^\top)^\top$ .

**Lemma 1.3.17.** (a) From

$$\sqrt{n}(\hat{\theta}_n^{MLE} - \theta_0) = I(\theta_0)^{-1}\sqrt{n}\bar{l}(\theta_0) + o_{P_{\theta_0}}(1),$$

we get

$$I_{11}(\theta_0)(\hat{\xi}_n - \xi_0) + I_{12}(\theta_0)(\hat{\eta}_n - \eta_0) = \bar{l}_1(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}),$$

$$I_{21}(\theta_0)(\hat{\xi}_n - \xi_0) + I_{22}(\theta_0)(\hat{\eta}_n - \eta_0) = \bar{l}_2(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}).$$

(b)

$$\sqrt{n}(\hat{\eta}_n^0 - \eta_0) = I_{22}^{-1}(\theta_0)\sqrt{n}\bar{l}_2(\theta_0) + o_{P_{\theta_0}}(1)$$

$$\hat{\eta}_n^0 - \eta_0 = I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) + o_{P_{\theta_0}}(n^{-1/2})$$

$$(c) \ 2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) = \sqrt{n}(\bar{l}_1(\theta_0) - I_{21}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0))^\top I_{11 \cdot 2}^{-1}(\theta_0)\sqrt{n}(\bar{l}_1(\theta_0) - I_{21}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0)) + o_{P_{\theta_0}}(1).$$

**Theorem 1.3.18** (Wald's test). *Let*

$$W_n = \sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^0)^\top I(\hat{\theta}_n^0)\sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^0).$$

Then,

$$\begin{aligned} W_n &= \sqrt{n}(\hat{\xi}_n - \xi_0)^\top I_{11 \cdot 2}(\theta_0)\sqrt{n}(\hat{\xi}_n - \xi_0) + o_{P_{\theta_0}}(1) \\ &= 2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) + o_{P_{\theta_0}}(1). \end{aligned}$$

*Proof.* First note that, by continuity of  $I$ ,

$$\begin{aligned} W_n &= n(\hat{\theta}_n - \hat{\theta}_n^0)^\top \left( I(\theta_0) + o_{P_{\theta_0}}(1) \right) (\hat{\theta}_n - \hat{\theta}_n^0) \\ &= n(\hat{\theta}_n - \hat{\theta}_n^0)^\top I(\theta_0)(\hat{\theta}_n - \hat{\theta}_n^0) + o_{P_{\theta_0}}(1). \end{aligned}$$

Then, we get

$$\begin{aligned} W_n &= n(\hat{\theta}_n - \hat{\theta}_n^0)^\top I(\theta_0)(\hat{\theta}_n - \hat{\theta}_n^0) + o_{P_{\theta_0}}(1) \\ &= n \left[ (\hat{\xi}_n - \xi_0)^\top I_{11}(\theta_0)(\hat{\xi}_n - \xi_0) + 2(\hat{\xi}_n - \xi_0)^\top I_{12}(\theta_0)(\hat{\eta}_n - \hat{\eta}_n^0) \right. \\ &\quad \left. + (\hat{\eta}_n - \hat{\eta}_n^0)^\top I_{22}(\theta_0)(\hat{\eta}_n - \hat{\eta}_n^0) \right] + o_{P_{\theta_0}}(1) \\ &= n \left[ (\hat{\xi}_n - \xi_0)^\top \left( I_{11}(\theta_0)(\hat{\xi}_n - \xi_0) + I_{12}(\theta_0)(\hat{\eta}_n - \hat{\eta}_n^0) \right) \right. \\ &\quad \left. + \left( (\hat{\xi}_n - \xi_0)^\top I_{12}(\theta_0) + (\hat{\eta}_n - \hat{\eta}_n^0)^\top I_{22}(\theta_0) \right) (\hat{\eta}_n - \hat{\eta}_n^0) \right] + o_{P_{\theta_0}}(1), \end{aligned}$$

and from

$$\begin{aligned} I_{11}(\theta_0)(\hat{\xi}_n - \xi_0) + I_{12}(\theta_0)(\hat{\eta}_n - \hat{\eta}_n^0) &= I_{11}(\theta_0)(\hat{\xi}_n - \xi_0) + I_{12}(\theta_0)(\hat{\eta}_n - \eta_0) - I_{12}(\theta_0)(\hat{\eta}_n^0 - \eta_0) \\ &= \bar{l}_1(\theta_0) - I_{12}(\theta_0)(\hat{\eta}_n^0 - \eta_0) + o_{P_{\theta_0}}(n^{-1/2}) \\ &= \bar{l}_1(\theta_0) - I_{12}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}) \end{aligned}$$

and

$$\begin{aligned} (\hat{\xi}_n - \xi_0)^\top I_{12}(\theta_0) + (\hat{\eta}_n - \hat{\eta}_n^0)^\top I_{22}(\theta_0) &= (\hat{\xi}_n - \xi_0)^\top I_{12}(\theta_0) + (\hat{\eta}_n - \eta_0)^\top I_{22}(\theta_0) - (\hat{\eta}_n^0 - \eta_0)^\top I_{22}(\theta_0) \\ &= \bar{l}_2(\theta_0) - I_{22}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}) \end{aligned}$$

we get

$$\begin{aligned} W_n &= n \left[ \underbrace{(\hat{\xi}_n - \xi_0)^\top}_{=o_{P_{\theta_0}}(n^{-1/2})} \left( \bar{l}_1(\theta_0) - I_{12}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}) \right) \right. \\ &\quad \left. + \left( \underbrace{\bar{l}_2(\theta_0) - I_{22}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0)}_{=0} + o_{P_{\theta_0}}(n^{-1/2}) \right) \underbrace{(\hat{\eta}_n - \hat{\eta}_n^0)}_{=o_{P_{\theta_0}}(n^{-1/2})} \right] + o_{P_{\theta_0}}(1) \\ &= n(\hat{\xi}_n - \xi_0)^\top \left( \bar{l}_1(\theta_0) - I_{12}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) \right) + o_{P_{\theta_0}}(1). \end{aligned}$$

Now by (a) of lemma, we get

$$\hat{\xi}_n - \xi_0 = I_{11.2}^{-1}(\theta_0) \left( \bar{l}_1(\theta_0) - I_{12}(\theta_0)I_{22}^{-1}(\theta_0)\bar{l}_2(\theta_0) \right) + o_{P_{\theta_0}}(n^{-1/2}),$$

and hence,

$$W_n = n(\hat{\xi}_n - \xi_0)^\top I_{11.2}(\theta_0)(\hat{\xi}_n - \xi_0) + o_{P_{\theta_0}}(1).$$

Use (a) part again, and then we can obtain

$$W_n = 2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) + o_{P_{\theta_0}}(1)$$

by (1.6). □

**Remark 1.3.19.** The leading term in previous theorem depends on unknown  $\eta_0$  through

$I_{11.2}(\theta_0)$ , so in practice, it should be estimated as

$$n(\hat{\xi}_n - \xi_0)^\top I_{11.2}(\hat{\theta}_n^0)(\hat{\xi}_n - \xi_0), \quad (\text{"estimated Wald statistics"})$$

which is also the leading term of LRT.

**Theorem 1.3.20** (Rao's test; score test statistic). *Let*

$$R_n = n\bar{l}(\hat{\theta}_n^0)^\top \widehat{I(\theta^0)}^{-1} \bar{l}(\hat{\theta}_n^0),$$

where

$$\widehat{I(\theta^0)} = -\bar{\ddot{l}}(\hat{\theta}_n^0)/n \text{ or } I(\hat{\theta}_n^0).$$

i.e., "observed information." Define

$$\widehat{I_{11.2}(\theta^0)} = \widehat{I_{11}(\theta^0)} - \widehat{I_{12}(\theta^0)} \widehat{I_{22}(\theta^0)}^{-1} \widehat{I_{21}(\theta^0)},$$

just as

$$I_{11.2}(\theta^0) = I_{11}(\theta^0) - I_{12}(\theta^0) I_{22}(\theta^0)^{-1} I_{21}(\theta^0).$$

Then,

$$\begin{aligned} R_n &= n\bar{l}_1(\hat{\theta}_n^0)^\top \widehat{I_{11.2}(\theta^0)}^{-1} \bar{l}_1(\hat{\theta}_n^0) \\ &= 2(l(\hat{\theta}_n) - l(\hat{\theta}_n^0)) + o_{P_{\theta_0}}(1). \end{aligned}$$

**Remark 1.3.21.** Note that, unlike Wald statistics or LRT statistics, to obtain  $R_n$  we only need MLE on the null,  $\hat{\theta}_n^0$ .

*Proof.* Note that, under  $H_0$ ,  $\dot{l}_2(\hat{\theta}_n^0) = 0$  holds,

$$\dot{l}(\hat{\theta}_n^0) = \begin{pmatrix} \dot{l}_1(\hat{\theta}_n^0) \\ 0 \end{pmatrix}$$

so we get

$$R_n = n\bar{l}_1(\hat{\theta}_n^0)^\top \widehat{I_{11.2}(\theta^0)}^{-1} \bar{l}_1(\hat{\theta}_n^0)$$

directly. Now with Taylor expansion,  $\exists \hat{\theta}_n^{0,*} \in \text{line} \left( \begin{pmatrix} \xi_0 \\ \hat{\eta}_n^0 \end{pmatrix}, \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} \right)$  s.t.

$$\bar{l}_1(\hat{\theta}_n^0) - \bar{l}_1(\theta_0) = \frac{\partial}{\partial \eta} \bar{l}_1(\theta) \Big|_{\theta=\hat{\theta}_n^{0,*}} (\hat{\eta}_n^0 - \eta_0)$$

(by chain rule, and  $\xi_0 - \xi_0 = 0$ ) Thus by continuity of  $I$ ,

$$\bar{l}_1(\hat{\theta}_n^0) - \bar{l}_1(\theta_0) = \left( -I_{12}(\theta_0) + o_{P_{\theta_0}}(1) \right) (\hat{\eta}_n^0 - \eta_0)$$

holds. In the same way, we get

$$\bar{l}_2(\hat{\theta}_n^0) - \bar{l}_2(\theta_0) = \left( -I_{22}(\theta_0) + o_{P_{\theta_0}}(1) \right) (\hat{\eta}_n^0 - \eta_0),$$

but from  $\bar{l}_2(\hat{\theta}_n^0) = 0$ ,

$$\hat{\eta}_n^0 - \eta_0 = \left( I_{22}(\theta_0) + o_{P_{\theta_0}}(1) \right)^{-1} \bar{l}_2(\theta_0) = \left[ I_{22}^{-1}(\theta_0) + o_{P_{\theta_0}}(1) \right] \bar{l}_2(\theta_0)$$

is obtained. Therefore,

$$\begin{aligned} \bar{l}_1(\hat{\theta}_n^0) &= \bar{l}_1(\theta_0) - \left[ I_{12}(\theta_0) + o_{P_{\theta_0}}(1) \right] (\hat{\eta}_n^0 - \eta_0) \\ &= \bar{l}_1(\theta_0) - \left[ I_{12}(\theta_0) + o_{P_{\theta_0}}(1) \right] \left[ I_{22}^{-1}(\theta_0) + o_{P_{\theta_0}}(1) \right] \bar{l}_2(\theta_0) \\ &= \bar{l}_1(\theta_0) - I_{12}(\theta_0) I_{22}^{-1}(\theta_0) \bar{l}_2(\theta_0) + o_{P_{\theta_0}}(n^{-1/2}) \end{aligned}$$

holds, and the proof completes if one uses

$$\widehat{I_{11.2}(\theta^0)} = I_{11.2}(\theta_0) + o_{P_{\theta_0}}(1)$$

and (1.6). □

**Remark 1.3.22.** Note that, for a simple null  $H_0 : \theta = \theta_0$ ,

$$W_n = n(\hat{\theta}_n - \theta_0)I(\theta_0)(\hat{\theta}_n - \theta_0)$$

$$R_n = n\bar{l}_1(\theta_0)I(\theta_0)^{-1}\bar{l}_2(\theta_0).$$

In this case,  $k_0 = 0$ , so  $W_n$  and  $R_n$  converges (in distribution) to  $\chi^2(k)$  under  $H_0$ . For the

composite,  $\theta_0$  is estimated by  $\hat{\theta}_n^0$ .

**Example 1.3.23** (GoF test in a multinomial model). Let  $X_1, \dots, X_n$  be a random sample from  $\text{Multi}(1, (p_1, \dots, p_r)^\top)$ . Let  $\theta = (p_1, \dots, p_k)^\top$ , where  $k = r - 1$ . Recall that

$$\Sigma(\theta) = \text{diag}(\theta_i) - \theta\theta^\top$$

$$I(\theta) = \Sigma(\theta)^{-1} = \text{diag}(\theta_i^{-1}) + \theta_r^{-1} \mathbf{1}\mathbf{1}^\top$$

$$\hat{\theta}^{MLE} = (\hat{p}_1, \dots, \hat{p}_k)^\top, \quad \hat{p}_i = \frac{1}{n} \sum_{j=1}^n X_{ji} = \frac{O_i}{n}$$

hold.

(a) Under simple null  $H_0 : p = p_0$ , we get (for convenience, denote  $\hat{\theta}^{MLE}$  as  $\hat{\theta}$ )

$$\begin{aligned} W_n &= n(\hat{\theta} - \theta_0)^\top I(\theta_0)(\hat{\theta} - \theta_0) \\ &= n(\hat{\theta} - \theta_0)^\top \left( \text{diag}(p_{0i}^{-1}) + p_{0r}^{-1} \mathbf{1}\mathbf{1}^\top \right) (\hat{\theta} - \theta_0) \\ &= n \left( \sum_{i=1}^k \frac{(\hat{p}_i - p_{0i})^2}{p_{0i}} + \frac{1}{p_{0r}} \left( \sum_{i=1}^k \hat{p}_i - \sum_{i=1}^k p_{0i} \right)^2 \right) \\ &= n \sum_{i=1}^r \frac{(\hat{p}_i - p_{0i})^2}{p_{0i}}, \end{aligned}$$

if we define

$$\hat{p}_r = 1 - \hat{\theta}_1 - \dots - \hat{\theta}_k.$$

Then letting  $n\hat{p}_i = \sum_{j=1}^n X_{ji} = O_i$ , and  $E_i^0 = np_{0i}$  be “expected occurrence” under the null, we get

$$W_n = \sum_{i=1}^r \frac{(O_i - E_i^0)^2}{E_i^0},$$

and  $W_n \xrightarrow[n \rightarrow \infty]{d} \chi_k^2 = \chi_{r-1}^2$  under  $H_0$ .

(b) Under the composite null  $H_0 : p \in \Theta_0$ , let  $\Theta_0 = \{p(\eta) : \eta \in \mathcal{E}_0\}$ , where  $p(\cdot)$  is known and  $\mathcal{E}_0$  is a  $k_0$ -dimensional space. Then  $\hat{\theta}_0 = p(\hat{\eta}^0)$ , and so similarly as (a) we get

$$W_n = \sum_{i=1}^r \frac{(O_i - \hat{E}_i^0)^2}{\hat{E}_i^0},$$

where  $\hat{E}_i^0 = np_i(\hat{\eta}^0)$ , and

$$W_n \xrightarrow[n \rightarrow \infty]{d} \chi_{k-k_0}^2$$

under  $H_0$ .

**Example 1.3.24** (Testing independence in a contingency table). Consider the model

$$O_{ij} \sim \text{Multi}(n, (p_{ij})), \quad i = 1, \dots, r, \quad j = 1, \dots, c.$$

In here, we wish to test

$$H_0 : p_{ij} = p_{i\cdot} \times p_{\cdot j}. \quad (\text{"Independence"})$$

Let

$i \backslash j$	1	2	$\dots$	$c$	
1	$p_{11}$	$p_{12}$	$\dots$	$p_{1c}$	$p_{1\cdot}$
2	$p_{21}$	$p_{22}$	$\dots$	$p_{2c}$	$p_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$r$	$p_{r1}$	$p_{r2}$	$\dots$	$p_{rc}$	$p_{r\cdot}$
	$p_{\cdot 1}$	$p_{\cdot 2}$	$\dots$	$p_{\cdot c}$	1

Table 1.1: Contingency table.

$$\theta = (p_{11}, p_{12}, \dots, p_{1c}, p_{21}, \dots, p_{2c}, \dots, p_{r1}, \dots, p_{r,c-1})^\top$$

be a  $rc - 1$  dimensional parameter, and

$$\Theta = \{(p_{ij}) : p_{\cdot\cdot} = 1, p_{ij} > 0\}$$

be a parameter space. From

$$l(\theta) = \sum_{i=1}^r \sum_{j=1}^c O_{ij} \log p_{ij} + C,$$

and

$$p_{rc} = 1 - \sum_{(i,j) \neq (r,c)} p_{ij},$$

we get

$$\frac{\partial}{\partial \theta} l(\theta) = \begin{bmatrix} \vdots \\ \frac{O_{ij}}{p_{ij}} - \frac{O_{rc}}{p_{rc}} \\ \vdots \end{bmatrix}.$$

Note that it is  $rc - 1$  dimensional vector. Thus likelihood equation becomes

$$\frac{O_{ij}}{p_{ij}} = \frac{O_{rc}}{p_{rc}} \quad \forall i, j,$$



and hence

$$\hat{p}_{ij} = \frac{O_{ij}}{n}.$$

Further, under  $H_0$ , parameter space is

$$\Theta_0 = \{(p_{i\cdot} p_{\cdot j}) : p_{1\cdot} + \cdots + p_{r\cdot} = 1, p_{i\cdot} > 0, p_{\cdot 1} + \cdots + p_{\cdot c} = 1, p_{\cdot j} > 0\},$$

and likelihood becomes

$$l(\theta) = \sum_{i=1}^r \sum_{j=1}^c O_{ij} \log p_{i\cdot} p_{\cdot j} + C$$

so

$$\hat{p}_{ij}^0 = \hat{p}_{i\cdot}^0 \hat{p}_{\cdot j}^0 = \frac{O_{i\cdot} O_{\cdot j}}{n^2}.$$

Further,  $(i, j)$ -th diagonal term of  $\partial^2 l(\theta) / \partial \theta^2$  is

$$-\frac{O_{ij}}{p_{ij}^2} - \frac{O_{rc}}{p_{rc}^2},$$

and other elements are  $-O_{rc}/p_{rc}^2$ , and so

$$I(\theta) = \frac{1}{n} E_{\theta} \left[ -\frac{\partial^2}{\partial \theta^2} l(\theta) \right] = \text{diag}(\theta_i^{-1}) + \frac{1}{p_{rc}} \mathbf{1} \mathbf{1}^{\top}.$$

Thus, we get

$$\begin{aligned} W_n &= n(\hat{\theta}_n - \hat{\theta}_n^0)^{\top} I(\hat{\theta}_n^0)(\hat{\theta}_n - \hat{\theta}_n^0) \\ &= n(\hat{\theta}_n - \hat{\theta}_n^0)^{\top} \left( \text{diag}((\hat{\theta}_i^0)^{-1}) + \frac{1}{\hat{p}_{rc}^0} \mathbb{K} \mathbb{K}^{\top} \right) (\hat{\theta}_n - \hat{\theta}_n^0) \\ &= n \left\{ \sum_{(i,j) \neq (r,c)} \frac{(\hat{p}_{ij} - \hat{p}_{ij}^0)^2}{\hat{p}_{ij}^0} + \frac{1}{\hat{p}_{rc}^0} \left( \underbrace{\sum_{(i,j) \neq (r,c)} (\hat{p}_{ij} - \hat{p}_{ij}^0)}_{=\hat{p}_{rc}^0 - \hat{p}_{rc}} \right)^2 \right\} \\ &= n \sum_{i=1}^r \sum_{j=1}^c \frac{(\hat{p}_{ij} - \hat{p}_{ij}^0)^2}{\hat{p}_{ij}^0} \\ &= \sum_{i=1}^r \sum_{j=1}^c \frac{(O_{ij} - \hat{E}_{ij}^0)^2}{\hat{E}_{ij}^0}, \end{aligned}$$

where  $\hat{E}_{ij}^0 = n\hat{p}_{ij}^0$ . Then

$$\text{dimension of } \Theta =: k = rc - 1$$

dimension of  $\Theta_0 =: k_0 = (r - 1) + (c - 1)$

so

$$k - k_0 = (r - 1)(c - 1),$$

and therefore,

$$W_n = \sum_{i=1}^r \sum_{j=1}^c \frac{(O_{ij} - \hat{E}_{ij}^0)^2}{\hat{E}_{ij}^0} \xrightarrow[n \rightarrow \infty]{d} \chi^2((r - 1)(c - 1))$$

under  $H_0$ .

**Example 1.3.25** (Homogeneity of contingency table). Now consider the model

$$O_i = (O_{i1}, \dots, O_{ic})^\top \stackrel{\text{indep}}{\sim} \text{Multi}(n_i, p_i)$$

for  $i = 1, 2, \dots, r$  in the contingency table. We wish to test

$$H_0 : p_1 = p_2 = \dots = p_r, \quad (\text{“Homogeneity”})$$

i.e.,

$$H_0 : \begin{pmatrix} p_{11} \\ p_{12} \\ \vdots \\ p_{1c} \end{pmatrix} = \begin{pmatrix} p_{21} \\ p_{22} \\ \vdots \\ p_{2c} \end{pmatrix} = \dots = \begin{pmatrix} p_{r1} \\ p_{r2} \\ \vdots \\ p_{rc} \end{pmatrix}.$$

Let  $\theta = (p_{11}, p_{12}, \dots, p_{1,c-1}, p_{21}, \dots, p_{2,c-1}, \dots, p_{r1}, \dots, p_{r,c-1})^\top$ . Then dimension of the parameter space

$$\Theta = \{(p_{ij}) : p_{i\cdot} = 1, p_{ij} > 0\}$$

is

$$k := r(c - 1),$$

and that of space under null

$$\Theta_0 = \{(p_{ij}) : p_{1j} = p_{2j} = \dots = p_{rj}, p_{i\cdot} = 1, p_{ij} > 0\}$$

is

$$k_0 := c - 1.$$

Now, note that log likelihood is obtained as

$$l(\theta) = \sum_{i=1}^r \sum_{j=1}^c O_{ij} \log p_{ij},$$

so

$$\frac{\partial}{\partial \theta} l(\theta) = \begin{bmatrix} \vdots \\ \frac{O_{ij}}{p_{ij}} - \frac{O_{ic}}{p_{ic}} \\ \vdots \end{bmatrix}.$$

Hence likelihood equation is obtained as

$$\frac{O_{ij}}{p_{ij}} = \frac{O_{ic}}{p_{ic}},$$

and therefore

$$\hat{p}_{ij} = \frac{1}{n_i} O_{ij}.$$

Meanwhile, under the null, likelihood is

$$l(\theta) = \sum_{i=1}^r \sum_{j=1}^c O_{ij} \log p_{1j} + C = \sum_{j=1}^c O_{.j} \log p_{1j} + C,$$

and hence

$$\hat{p}_{ij}^0 = \hat{p}_{1j}^0 = \frac{1}{n} O_{.j}.$$

Now we obtain the information. Note that  $\partial^2 l(\theta) / \partial \theta^2$  has diagonal term

$$-\frac{O_{ij}}{p_{ij}^2} - \frac{O_{ic}}{p_{ic}^2}$$

and other elements of  $\ddot{l}$  are

$$-\frac{O_{ic}}{p_{ic}^2},$$

and hence  $I(\theta)$  has the form

$$I(\theta) = \begin{bmatrix} I_1 & & & \\ & I_2 & & \\ & & \ddots & \\ & & & I_r \end{bmatrix},$$

where

$$I_i = \begin{bmatrix} \frac{n_i}{p_{i1}} & & \\ & \ddots & \\ & & \frac{n_i}{p_{i,c-1}} \end{bmatrix} + \begin{bmatrix} \frac{n_i}{p_{ic}} & \frac{n_i}{p_{ic}} & \dots \\ & \vdots & \\ & & \ddots \end{bmatrix} = n_i \left( \text{diag}(\theta_i^{-1}) + \frac{1}{p_{ic}} \mathbf{1}_{c-1} \mathbf{1}_{c-1}^\top \right),$$

$$\theta_i = (p_{i1}, \dots, p_{i,c-1})^\top.$$

Note that there are repetition of independent trials. Thus, by additivity, Wald's statistic which is given as

$$\begin{aligned} W_n &= (\hat{\theta}_n - \hat{\theta}_n^0)^\top I(\hat{\theta}_n^0) (\hat{\theta}_n - \hat{\theta}_n^0) \\ &= (\hat{\theta}_n - \hat{\theta}_n^0)^\top n_i \left( \text{diag}(\theta_i^{-1}) + \frac{1}{p_{ic}} \mathbf{1}_{c-1} \mathbf{1}_{c-1}^\top \right) (\hat{\theta}_n - \hat{\theta}_n^0) \\ &= \sum_{i=1}^r \sum_{j=1}^{c-1} \frac{n_i (\hat{p}_{ij} - \hat{p}_{ij}^0)^2}{\hat{p}_{ij}^0} + \sum_{i=1}^r \frac{n_i}{\hat{p}_{ic}^0} \left( \underbrace{\sum_{j=1}^{c-1} (\hat{p}_{ij} - \hat{p}_{ij}^0)}_{=\hat{p}_{ic}^0 - \hat{p}_{ic}} \right)^2 \\ &= \sum_{i=1}^r \sum_{j=1}^c \frac{n_i (\hat{p}_{ij} - \hat{p}_{ij}^0)^2}{\hat{p}_{ij}^0} \end{aligned}$$

converges to  $\chi^2(k - k_0)$  distribution under  $H_0$ . Finally, note that

$$O_{ij} = n_i \hat{p}_{ij} \text{ and } \hat{E}_{ij}^0 = n_i \hat{p}_{ij}^0,$$

and hence  $W_n$  can be represented as

$$W_n = \sum_{i=1}^r \sum_{j=1}^c \frac{(O_{ij} - \hat{E}_{ij}^0)^2}{\hat{E}_{ij}^0}.$$

Therefore, our final result is that

$$W_n = \sum_{i=1}^r \sum_{j=1}^c \frac{(O_{ij} - \hat{E}_{ij}^0)^2}{\hat{E}_{ij}^0} \xrightarrow[n \rightarrow \infty]{d} \chi^2((r-1)(c-1))$$

holds under  $H_0$ .

## Chapter 2

# Weak Convergence

### 2.1 Weak Convergence in $\mathbb{R}^k$

**Definition 2.1.1.** Let  $X_n$  and  $X$  be random vectors taking values in  $\mathbb{R}^k$ . Define  $F_n(x) = P(X_n \leq x)$  and  $F(x) = P(X \leq x)$ , where inequality is defined coordinatewisely, i.e.,  $P(X \leq x) = P(X_1 \leq x_1, \dots, X_k \leq x_k)$ . Then  $X_n$  converges weakly to a random vector  $X$ , and write  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ , if  $P(X_n \in A) \xrightarrow[n \rightarrow \infty]{} P(X \in A)$  for any Borel set  $A \subseteq \mathbb{R}^k$  with  $P(X \in \partial A) = 0$ .

**Remark 2.1.2.** Note that in weak convergence, we only see the “distribution” of random vectors, so it’s allowed to be defined on different probability spaces, unlike almost surely convergence. However, just for convenience, we fix probability space.

**Proposition 2.1.3.** Let  $C_f = \{x \in \mathbb{R}^k : f \text{ is continuous at } x\}$ . Then followings are equivalent.

(i)  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ .

(ii)  $F_n(x) \xrightarrow[n \rightarrow \infty]{} F(x)$  for any  $x \in C_F$ .

**Theorem 2.1.4** (Skorokhod representation theorem). Suppose that  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ . Then there exists  $X^*$  and  $\{X_n^*\}$  defined on “the same probability space”  $(\Omega^*, \mathcal{F}^*, P^*)$  s.t.

$$X_n^* \stackrel{d}{=} X_n, \quad X^* \stackrel{d}{=} X \quad \text{and} \quad X_n^* \xrightarrow[n \rightarrow \infty]{a.s.} X^*.$$

Note that,  $X_n$  need not be defined on the same space, but  $X_n^*$  should be, because we will say its “almost surely convergence.”

**Theorem 2.1.5** (Continuous mapping theorem). If  $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$  and  $P(X \in C_f) = 1$  for a real valued function  $f$ , then  $f(X_n) \xrightarrow[n \rightarrow \infty]{a.s.} f(X)$ .

*Proof.* It's clear from

$$P(\lim f(X_n) = f(X)) = P(X \in C_f, \lim f(X_n) = f(X)) \geq P(X \in C_f, \lim X_n = X) = 1.$$

□

**Proposition 2.1.6.** *If  $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$ , then  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ .*

*Proof.* (Proof in the lecture note is wrong! It is reported by Shin Jun-ho, and we are searching for alternative proof. Using Portmanteau lemma, it becomes trivial, but I think it breaks our flow.)

However, we cannot say that  $X_n \xrightarrow[n \rightarrow \infty]{d} X$  implies  $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$ . Following theorem shows some equivalent conditions of weak convergence in  $\mathbb{R}^k$ .

**Theorem 2.1.7** (Portmanteau lemma). *TFAE.*

- (i)  $F_n(x) \rightarrow F(x)$  for all  $x \in C_F$ .
- (ii)  $Ef(X_n) \rightarrow Ef(X)$  for all bounded  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  s.t.  $P(X \in C_f) = 1$ .
- (iii)  $Ef(X_n) \rightarrow Ef(X)$  for all bounded continuous  $f : \mathbb{R}^k \rightarrow \mathbb{R}$ .
- (iv)  $Ef(X_n) \rightarrow Ef(X)$  for all bounded and uniformly continuous  $f : \mathbb{R}^k \rightarrow \mathbb{R}$ .

*Proof.* Enough to show (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i).

(i)  $\Rightarrow$  (ii): Let  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  be a bounded function with  $P(X \in C_f) = 1$ , i.e., continuous on the support of  $X$ . By Skorokhod representation theorem, we can find  $X_n^* \stackrel{d}{=} X_n$  and  $X^* \stackrel{d}{=} X$  satisfying

$$X_n^* \xrightarrow[n \rightarrow \infty]{a.s.} X^*.$$

Then by continuous mapping theorem,

$$f(X_n^*) \xrightarrow[n \rightarrow \infty]{a.s.} f(X^*)$$

holds, and thus by bounded convergence theorem,

$$Ef(X_n^*) \xrightarrow[n \rightarrow \infty]{} Ef(X^*).$$

Note that since  $X_n^*$  and  $X^*$  have the same distribution as  $X_n$  and  $X$ , expectation also will be the same, so we get

$$Ef(X_n) \xrightarrow{n \rightarrow \infty} Ef(X).$$

**Remark 2.1.8.** Note that, if  $X_n^*$  and  $X^*$  are defined on  $(\Omega^*, \mathcal{F}^*, P^*)$ , then for expectation  $E^*(\cdot)$  under  $P^*$ , we should write

$$E^*f(X_n^*) \xrightarrow{n \rightarrow \infty} E^*f(X^*),$$

but just for convenience, we abuse the notation.

(iv)  $\Rightarrow$  (i): Fix  $x \in C_F$ , and define

$$f_{m,x}^+ : \mathbb{R}^k \rightarrow \mathbb{R} \text{ and } f_{m,x}^- : \mathbb{R}^k \rightarrow \mathbb{R}$$

as

$$f_{m,x}^+(u) = \begin{cases} 1 & \text{if } u \leq x \\ \text{linear} & \text{if } x \leq u \leq x + 1/m \\ 0 & \text{if } u \geq x + 1/m \end{cases}$$

and

$$f_{m,x}^-(u) = \begin{cases} 1 & \text{if } u \leq x - 1/m \\ \text{linear} & \text{if } x - 1/m \leq u \leq x \\ 0 & \text{if } u \geq x. \end{cases}$$

Precisely, they become

$$f_{m,x}^+(u) = \begin{cases} 1 & \text{if } u \leq x \\ \frac{m}{k} \mathbf{1}^\top \left( x + \frac{1}{m} - u \right) & \text{if } x \leq u \leq x + \frac{1}{m} \\ 0 & \text{if } u \geq x + \frac{1}{m} \end{cases}$$

and

$$f_{m,x}^-(u) = \begin{cases} 1 & \text{if } u \leq x - \frac{1}{m} \\ \frac{m}{k} \mathbf{1}^\top (x - u) & \text{if } x - \frac{1}{m} \leq u \leq x \\ 0 & \text{if } u \geq x. \end{cases}$$

Both functions are bounded and uniformly continuous, so we get

$$Ef_{m,x}^+(X_n) \rightarrow Ef_{m,x}^+(X) \text{ and } Ef_{m,x}^-(X_n) \rightarrow Ef_{m,x}^-(X).$$

Further,  $0 \leq f_{m,x}^- \leq f \leq f_{m,x}^+ \leq 1$ , where  $f = I_{(-\infty, x]}(\cdot)$  holds, so we get

$$F_n(x) = Ef(X_n) \leq Ef_{m,x}^+(X_n),$$

and hence

$$\limsup_{n \rightarrow \infty} F_n(x) \leq \limsup_{n \rightarrow \infty} Ef_{m,x}^+(X_n) = Ef_{m,x}^+(X).$$

Similarly, we get

$$\liminf_{n \rightarrow \infty} F_n(x) \geq \liminf_{n \rightarrow \infty} Ef_{m,x}^-(X_n) = Ef_{m,x}^-(X).$$

Note that

$$\lim_{m \rightarrow \infty} f_{m,x}^+(u) = I_{(-\infty, x]}(u) \text{ and } \lim_{m \rightarrow \infty} f_{m,x}^-(u) = I_{(-\infty, x)}(u).$$

Therefore, we get

$$\lim_{m \rightarrow \infty} Ef_{m,x}^-(X) \leq \liminf_{n \rightarrow \infty} F_n(x) \leq \limsup_{n \rightarrow \infty} F_n(x) \leq \lim_{m \rightarrow \infty} Ef_{m,x}^+(X),$$

i.e.,

$$F(x-) \leq \liminf_{n \rightarrow \infty} F_n(x) \leq \limsup_{n \rightarrow \infty} F_n(x) \leq F(x),$$

which yields

$$\lim_{n \rightarrow \infty} F_n(x) = F(x)$$

from the assumption  $x \in C_F$ . □

**Remark 2.1.9.** Indeed, some more equivalent conditions are known, which is called “*portman-teau lemma*.” Equivalent conditions are given as following:

- (i)  $F_n(x) \rightarrow F(x)$  for all  $x \in C_F$ .
- (ii)  $Ef(X_n) \rightarrow Ef(X)$  for all bounded and uniformly continuous  $f : \mathbb{R}^k \rightarrow \mathbb{R}$ .
- (iii)  $\limsup_{n \rightarrow \infty} P(X_n \in F) \leq P(X \in F)$  for any closed set  $F$ .
- (iv)  $\liminf_{n \rightarrow \infty} P(X_n \in G) \geq P(X \in G)$  for any open set  $G$ .
- (v)  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ , i.e.,  $P(X_n \in A) \rightarrow P(X \in A)$  for any  $X$ -continuity sets  $A$ .



## 2.2 Weak Convergence in Metric Spaces

Let  $(\mathbb{S}, \mathcal{S})$  be a metric space, where  $\mathcal{S} = \mathcal{B}(\mathbb{S})$  denotes the Borel  $\sigma$ -field of  $\mathbb{S}$ . Let  $X_n, X$  be *random elements* taking values in  $\mathbb{S}$ , i.e.,  $X_n$  and  $X$  are “measurable” mappings from  $\Omega$  to  $\mathbb{S}$ . Recall that  $X : \Omega \rightarrow \mathbb{S}$  is  $(\mathcal{F} \setminus \mathcal{S})$ -measurable if

$$X^{-1}(A) \in \mathcal{F} \quad \forall A \in \mathcal{S}.$$

**Definition 2.2.1.** A sequence  $\{X_n\}$  of random elements **converges weakly** to a random element  $X$ , which is written as  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ , if

$$P(X_n \in A) \xrightarrow[n \rightarrow \infty]{} P(X \in A) \quad \forall A \in \mathcal{S} \text{ (i.e., for any Borel set } A, \text{) s.t. } P(X \in \partial A) = 0.$$

**Remark 2.2.2.**  $X_n \xrightarrow[n \rightarrow \infty]{d} X$  if and only if  $Ef(X_n) \rightarrow Ef(X)$  for any bounded and uniformly continuous real-valued function  $f$ .

**Example 2.2.3.** In here, we see an example of a metric space  $(\mathbb{S}, \mathcal{S})$  which is not Euclidean. Consider  $\mathbb{S} = \mathbb{R}^\infty$ , i.e.,  $\mathbb{S}$  is the set of real-valued sequences. Give the metric

$$d(x, y) = \sup_{1 \leq i} |x_i - y_i|$$

in this space. Then one can observe that  $X_n \xrightarrow[n \rightarrow \infty]{d} X$  if *every finite-dimensional distribution of  $X_n$  converges weakly to corresponding finite-dimensional distribution of  $X$* , i.e.,

$$(X_{n,1}, \dots, X_{n,k})^\top \xrightarrow[n \rightarrow \infty]{d} (X_1, \dots, X_k)^\top \quad \forall k \geq 1.$$

It can be also written as

$$P_n \pi_k^{-1} \xrightarrow[n \rightarrow \infty]{w} P \pi_k^{-1},$$

where  $P_n$  and  $P$  are distribution measure of  $X_n$  and  $X$  respectively, and  $\pi_k : \mathbb{R}^\infty \rightarrow \mathbb{R}^k$  is a natural projection s.t.  $\pi_k(x) = (x_1, \dots, x_k)^\top$ .

Unfortunately, statement given in previous example is not true in general. Especially, we can find a counter-example when  $\mathbb{S}$  is a function space, which is introduced in following example.

**Example 2.2.4.** Let  $\mathbb{S} = \mathbb{C}[0, 1]$  be the space of continuous functions defined on  $[0, 1]$  with

sup-metric  $d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|$ . Define

$$X_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq 1/n \\ 2 - nt & \text{if } 1/n \leq t \leq 2/n \\ 0 & \text{if } 2/n \leq t \leq 1. \end{cases}$$

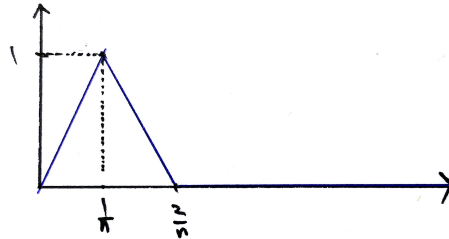


Figure 2.1: ( $P$ -a.s. unique) sample path of  $X_n$ .

(Note that  $X_n$  is a random element on  $\mathbb{C}[0, 1]$  which has only one value; it has only one sample path.) Then for  $X(t) \equiv 0$ , every finite-dimensional distribution of  $X_n$  converges weakly to the corresponding finite-dimensional distribution of  $X$ . It follows from, for any  $0 \leq t_1 < t_2 < \dots < t_k \leq 1$ , (if  $0 = t_1$ , consider  $t_2$  instead of  $t_1$ , so WLOG  $0 < t_1$ )

$$(X_n(t_1), X_n(t_2), \dots, X_n(t_k)) \stackrel{d}{\equiv} (X(t_1), X(t_2), \dots, X(t_k))$$

if  $t_1 > 2/n$ , i.e., for sufficiently large  $n$ . However,  $X_n$  does not converge weakly to  $X$ . To see this, let  $A = B(0, 1/2) = \{y : d(y, 0) \leq 1/2\}$ . Then clearly  $A$  is a Borel set, and from

$$d(X_n, X) = 1 \text{ and } d(X, 0) = 0,$$

we get

$$P(X \in \partial A) = P(d(X, 0) = 1/2) = 0$$

and

$$P(X_n \in A) = P(d(X_n, 0) \leq 1/2) = 0$$

but

$$P(X \in A) = P(d(X, 0) \leq 1/2) = 1,$$

i.e.,

$P(X_n \in A)$  does not converges to  $P(X \in A)$ , even if  $P(X \in \partial A) = 0$ .

⊠

**Remark 2.2.5.** For the natural projection  $\pi_{t_1, \dots, t_k}(x) = (x(t_1), \dots, x(t_k))^\top$ , we get even if

$$P_n \pi_{t_1, \dots, t_k}^{-1} \xrightarrow[n \rightarrow \infty]{w} P \pi_{t_1, \dots, t_k}^{-1} \quad \forall k \geq 1, \quad \forall t_i$$

holds, we cannot say that

$$P_n \xrightarrow[n \rightarrow \infty]{w} P.$$

Therefore, our main task is to find a plausible set of sufficient conditions that ensures  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ . First, we may introduce following proposition, which is very useful tool.

**Proposition 2.2.6.**  $X_n \xrightarrow[n \rightarrow \infty]{d} X$  if and only if

$$\forall \{n'\} \subseteq \{n\} \quad \exists \{n''\} \subseteq \{n'\} \quad \text{s.t.} \quad X_{n''} \xrightarrow[n \rightarrow \infty]{d} X.$$

*Proof.* Only  $\Leftarrow$  part is non-trivial. Suppose that  $X_n$  does not converge to  $X$  weakly. Then by Portmanteau lemma,  $\exists$  bounded, uniformly continuous function  $f : \mathbb{S} \rightarrow \mathbb{R}$  s.t.

$$Ef(X_n) \not\rightarrow Ef(X).$$

(Note that  $Ef(X_n)$  is a real sequence) Then  $\exists \epsilon > 0$  and  $\exists \{n'\} \subseteq \{n\}$  s.t.

$$|Ef(X_{n'}) - Ef(X)| > \epsilon \quad \forall n'. \quad (2.1)$$

However, by the assumption,  $\exists \{n''\} \subseteq \{n'\}$  s.t.

$$X_{n''} \xrightarrow[n'' \rightarrow \infty]{d} X,$$

i.e.,

$$Ef(X_{n''}) \rightarrow Ef(X) \quad \text{as } n'' \rightarrow \infty$$

by Portmanteau lemma, which is contradictory to (2.1). □

Recall that for real-valued random elements,

$$X_n \xrightarrow[n \rightarrow \infty]{d} X \Rightarrow X_n = O_P(1)$$

holds. We already know that converse does not hold; if

$$X_n = \begin{cases} Y_n & n = 2k \\ Z_n & n = 2k - 1 \end{cases}$$

for  $Y_n$  and  $Z_n$  s.t.  $Y_n \xrightarrow[n \rightarrow \infty]{d} Y$ ,  $Z_n \xrightarrow[n \rightarrow \infty]{d} Z$ , then clearly  $X_n = O_P(1)$ , but  $X_n$  does not converge weakly. Similarly, we will think a general notion of “ $O_P(1)$ ,” which is called as *relative compactness*, and then we will find a sufficient condition for weakly convergence, *additional to relative compactness*.

**Definition 2.2.7.** Let  $\{X_n\}$  be a sequence of random elements.  $\{X_n\}$  is called **relatively compact** if  $\forall \{n'\} \subseteq \{n\}$ ,  $\exists \{n''\} \subseteq \{n'\}$  s.t.  $X_{n''}$  converges weakly to some random element.

Note that in this case, the limit may depend on the choice of subsequence  $\{n'\}$ ! Thus, relative compactness cannot guarantee weak convergence in alone.

**Proposition 2.2.8** (Continuous mapping theorem). Let  $(\mathbb{S}, \mathcal{S})$  and  $(\mathbb{S}', \mathcal{S}')$  be metric spaces, and  $h : \mathbb{S} \rightarrow \mathbb{S}'$  be a measurable function s.t.  $P(X \in D_h) = 0$ , where  $D_h$  denotes the set of discontinuities of  $h$ . Then  $X_n \xrightarrow[n \rightarrow \infty]{d} X$  implies  $h(X_n) \xrightarrow[n \rightarrow \infty]{d} h(X)$ .

*Proof.* Skorokhod representation theorem and continuous mapping theorem introduced in previous section. □

With these, we get the first sufficient condition of weak convergence.

**Theorem 2.2.9.** Let  $\mathcal{H}$  be “a” collection of measurable continuous functions from  $(\mathbb{S}, \mathcal{S})$  to  $(\mathbb{S}', \mathcal{S}')$ . If

(i)  $\mathcal{H}^{-1}(\mathcal{S}')$  generates  $\mathcal{S}$ , where  $\mathcal{H}^{-1}(\mathcal{S})$  is defined as

$$\mathcal{H}^{-1}(\mathcal{S}') := \bigcup_{h \in \mathcal{H}} h^{-1}(\mathcal{S}') = \{h^{-1}(A) : A \in \mathcal{S}', h \in \mathcal{H}\}.$$

(ii)  $\{X_n\}$  is relatively compact.

(iii)  $h(X_n)$  converges weakly to  $X$  for any  $h \in \mathcal{H}$ .

Then  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ .

**Remark 2.2.10.** (a) Condition (iii) can be relaxed to:

“ $h(X_n)$  converges weakly to some random element  $Y_h$  for any  $h \in \mathcal{H}$ .”

Then we can further show that there exists  $X$  s.t.  $Y_h = h(X)$  for any  $h \in \mathcal{H}$ . Define  $X$  to satisfy

$$P(X \in h^{-1}(A')) = P(Y_h \in A') \quad \forall A' \in \mathcal{S}', \quad \forall h \in \mathcal{H}.$$

Then we can “define”  $X$ ; because the collection of  $h^{-1}(A')$  generates Borel  $\sigma$ -field  $\mathcal{S}$ . Next, it is “well-defined,” i.e., our definition is “consistent”; for another function  $\tilde{h} \in \mathcal{H}$  and corresponding limit  $Y_{\tilde{h}}$ , if  $h^{-1}(A) = \tilde{h}^{-1}(B)$ , then

$$\begin{aligned} P(Y_h \in A) &= \lim P(h(X_n) \in A) = \lim P(X_n \in h^{-1}(A)) \\ &= \lim P(X_n \in \tilde{h}^{-1}(B)) = \lim P(\tilde{h}(X_n) \in B) = P(Y_{\tilde{h}} \in B). \end{aligned}$$

Finally,  $X$  is uniquely determined, and therefore  $Y_h = h(X)$ .

(b) Indeed,  $\mathcal{H}$  is related to finite-dimensional distribution. For example, on  $\mathcal{C}[0, 1]$ , if  $x : [0, 1] \rightarrow \mathbb{S}$ , then for  $h(x) = (x(t_1), \dots, x(t_k))$ ,  $h(X)$  is finite-dimensional distribution of  $X$ .

*Proof.* By relative compactness, it's enough to show that weak limit of  $X_{n''}$  does not depend on the choice of  $\{n'\}$ . Let  $X_{n''} \xrightarrow[n'' \rightarrow \infty]{d} Y$ . Then by continuous mapping theorem,  $h(X_{n''}) \xrightarrow[n'' \rightarrow \infty]{d} h(Y)$ . However, by the assumption,  $h(X_{n''}) \xrightarrow[n'' \rightarrow \infty]{d} h(X)$ , so we get

$$h(X) \stackrel{d}{=} h(Y).$$

(Yet  $Y$  depends on  $\{n'\}$ ) It implies that

$$P(X \in A) = P(Y \in A) \quad \forall A \in \mathcal{H}^{-1}(\mathcal{S}'),$$

and by Dynkin's  $\pi$ - $\lambda$  theorem (note that  $\mathcal{H}^{-1}(\mathcal{S}')$  is  $\pi$ -system; the collection of such  $A$ 's is  $\lambda$ -system), we get

$$P(X \in A) = P(Y \in A) \quad \forall A \in \mathcal{S},$$

with the assumption that

$$\sigma(\mathcal{H}^{-1}(\mathcal{S}')) = \mathcal{S}.$$

Therefore, we get

$$X \stackrel{d}{=} Y,$$

i.e., weak limit  $Y$  does not depend on the choice of subsequence ( $\because X$  is predetermined).  $\square$

However, relative compactness is hard to show. Thus, we introduce another notion that implies relative compactness; which is *tightness*.

**Definition 2.2.11.** (a)  $A$  is **totally bounded** if  $\forall \epsilon > 0$ ,  $A$  is covered by “finitely” many open balls of radius  $\epsilon$  in  $\mathbb{S}$ . (Note that center of each ball need not be contained in  $A$ )

(b)  $\mathbb{S}$  is **complete** if every Cauchy sequence has a limit in  $\mathbb{S}$ . A set  $S$  in  $\mathbb{S}$  is **complete** if every Cauchy sequence in  $S$  converges to a value in  $S$ .

(c) A set  $K$  in  $(\mathbb{S}, \mathcal{S})$  is **compact** if every open cover of  $K$  has finite subcover.

**Proposition 2.2.12.** If  $A$  is totally bounded and complete, then  $A$  is compact.

**Remark 2.2.13.** Note that, in complete metric space  $(\mathbb{S}, \mathcal{S})$ , if  $A$  is totally bounded but not complete, then think closure of  $A$ , denoted as  $\bar{A}$ , and then we get that  $\bar{A}$  is compact. ( $\because \bar{A}$  is complete, and it is totally bounded; just cover with closed ball instead of open balls!)

**Definition 2.2.14.** A (single) random element  $X$  is **tight** if  $\forall \epsilon > 0$ ,  $\exists$  compact set  $K$  s.t.  $P(X \in K) > 1 - \epsilon$ .

**Remark 2.2.15.** Note that, in Euclidean space  $\mathbb{R}^k$ , or sequence space  $\mathbb{R}^\infty$ , single random element  $\{X\}$  is always tight; in general, it may not be. Hence, we should find a sufficient condition for tightness.

**Definition 2.2.16.** (a)  $D$  is a **dense** subset of  $S$  if  $\forall x \in S \forall \epsilon > 0 \exists y \in D$  s.t.  $y \in B(x, \epsilon)$ , or equivalently,  $d(x, y) < \epsilon$ .

(b)  $\mathbb{S}$  is **separable** space if it contains countable dense subset, i.e.,

$$\exists \text{countable subset } D \text{ s.t. } \forall \epsilon > 0 \forall x \in \mathbb{S} \exists y \in D \text{ s.t. } d(x, y) < \epsilon,$$

or equivalently,

$$\exists \text{countable subset } D \text{ s.t. } \forall \epsilon > 0 \bigcup_{y \in D} B(y, \epsilon) \supseteq \mathbb{S}.$$

(c) A topological space is called a **Polish space** if it is (i) separable, and (ii) metrizable in such a way that it becomes complete.

Following theorem says that, in *separable and complete metric space*, our convention works.

**Theorem 2.2.17.** *If  $\mathbb{S}$  is separable and complete, then each random element in  $(\mathbb{S}, \mathcal{S})$  is tight.*

*Proof.* Given  $\epsilon > 0$ . We want to show that:

$$\exists \text{compact set } K \text{ s.t. } P(X \in K) > 1 - \epsilon.$$

Since  $\mathbb{S}$  is separable,  $\exists$  countable number of balls  $B_{k,1}, B_{k,2}, \dots$  with radius  $1/k$  that covers  $\mathbb{S} (\cdot \cup B(y_k, 1/k) \supseteq \mathbb{S} \text{ for some } \{y_k : k \geq 1\})$ . It implies that

$$P \left( X \in \bigcup_{j=1}^{\infty} B_{k,j} \right) = 1.$$

By continuity of probability measure, we get

$$P \left( X \in \bigcup_{j=1}^J B_{k,j} \right) \xrightarrow{J \rightarrow \infty} 1,$$

and therefore,  $\exists J = J(k)$  s.t.

$$P \left( X \in \bigcup_{j=1}^{J(k)} B_{k,j} \right) > 1 - \frac{\epsilon}{2^k}.$$

Now define

$$B = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{J(k)} B_{k,j},$$

and then  $B$  is totally bounded, because  $\bigcup_{j=1}^{J(k)} B_{k,j}$  is already totally bounded set, including  $B$ .

Then closure  $\bar{B}$  of  $B$  is complete, and hence,  $\bar{B}$  is compact (cf. remark 2.2.13). Then we get

$$P(X \in \bar{B}^c) \leq P(X \in B^c) \leq \sum_{k=1}^{\infty} P \left( X \in \bigcap_{j=1}^{J(k)} B_{k,j}^c \right) < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} < \epsilon,$$

and therefore,

$$P(X \in \bar{B}) > 1 - \epsilon$$

holds. Let  $K = \bar{B}$ . □

Note that, we directly constructed a compact set with probability “near to 1.”

## 2.3 Weak Convergence in $\mathbb{C}[0, 1]$

Here we consider weak convergence in  $\mathbb{C} \equiv \mathbb{C}[0, 1]$ , *the space of real valued continuous functions defined on the interval  $[0, 1]$* . (Do not confuse  $\mathbb{C}$  with the set of complex numbers!) Let  $\mathcal{C}$  be the Borel  $\sigma$ -field of  $\mathbb{C}$ , i.e.,  $\mathcal{C} = \mathcal{B}(\mathbb{C})$ . We endow  $\mathbb{C}$  with the uniform metric

$$d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|.$$

It is well known that  $\mathbb{C}$  is separable and complete. Separability and completeness facilitate derivation of a plausible set of sufficient conditions for weak convergence.

**Theorem 2.3.1** (Stone-Weierstrass theorem). *Any continuous function can be approximated by a polynomial function. In other words,  $\mathbb{C}$  is **separable**.*

*Proof.* There is elementary proof introduced by S.N.Bernstein. For this, see K.A.Ross, pp.217-220. □

**Theorem 2.3.2** (Completeness).  *$\mathbb{C}$  is **complete**.*

*Proof.* Royden, pp.193-194. □

### 2.3.1 Projection from $\mathbb{C}$ to $\mathbb{R}^k$

**Definition 2.3.3.** *For a set of points  $\{t_1, t_2, \dots, t_k\} \subseteq [0, 1]$ , let*

$$\pi_{t_1, \dots, t_k} : (\mathbb{C}, \mathcal{C}) \rightarrow (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$$

*be a **projection** that carries a point(=function)  $x$  of  $\mathbb{C}$  to the points  $(x(t_1), \dots, x(t_k)) \in \mathbb{R}^k$ .*

The following theorem says that, *the collection of all projections satisfies the conditions of  $\mathcal{H}$  in theorem 2.2.9.*

**Theorem 2.3.4.** *The projection  $\pi_{t_1, \dots, t_k}$  is measurable and continuous for all  $t_1, \dots, t_k \in [0, 1]$  and  $k \geq 1$ . Furthermore, all sets of the form*

$$\pi_{t_1, \dots, t_k}^{-1}(B) \text{ for some } B \in \mathcal{B}(\mathbb{R}^k), t_1, \dots, t_k \in [0, 1] \text{ and } k \geq 1$$

*form a field that generates  $\mathcal{C}$ .*



*Proof.* Continuity is obvious: Letting  $\delta = \epsilon/k$ , we get

$$d(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)| < \delta \Rightarrow |\pi_{t_1, \dots, t_k}(x) - \pi_{t_1, \dots, t_k}(y)| = \sqrt{(x(t_1) - y(t_1))^2 + \dots + (x(t_k) - y(t_k))^2} < \epsilon.$$

Now let

$$\mathcal{C}_0 = \{\pi_{t_1, \dots, t_k}^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^k), t_1, \dots, t_k \in [0, 1] \text{ and } k \geq 1\}.$$

Then  $\mathcal{C}_0$  is a field from

$$[\pi_{t_1, \dots, t_k}^{-1}(B)]^c = \pi_{t_1, \dots, t_k}^{-1}(B^c),$$

$$\begin{aligned} \pi_{t_1, \dots, t_k}^{-1}(B_1) \bigcup \pi_{s_1, \dots, s_l}^{-1}(B_2) &= \pi_{t_1, \dots, t_k, s_1, \dots, s_l}^{-1}(B_1 \times \mathbb{R}^l) \bigcup \pi_{t_1, \dots, t_k, s_1, \dots, s_l}^{-1}(\mathbb{R}^k \times B_2) \\ &= \pi_{t_1, \dots, t_k, s_1, \dots, s_l}^{-1}\left((B_1 \times \mathbb{R}^l) \cup (\mathbb{R}^k \times B_2)\right). \end{aligned}$$

Since Borel  $\sigma$ -field  $\mathcal{C}$  is generated by closed balls, it's sufficient to show that arbitrary closed ball is contained in  $\sigma(\mathcal{C}_0)$ . Note that if  $d(x, y) \leq \epsilon$ , then for any  $t \in [0, 1]$  we get  $|x(t) - y(t)| \leq \epsilon$ , and hence, we get

$$\forall n \geq 1, 1 \leq i \leq n \Rightarrow \left| y\left(\frac{i}{n}\right) - x\left(\frac{i}{n}\right) \right| \leq \epsilon.$$

Thus we get

$$\bar{B}(x, \epsilon) \subseteq \bigcap_{n=1}^{\infty} \bigcap_{i=1}^n \left\{ y : \left| y\left(\frac{i}{n}\right) - x\left(\frac{i}{n}\right) \right| \leq \epsilon \right\}.$$

Now our claim is:

**Claim.**  $\bar{B}(x, \epsilon) = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^n \{y : |y(\frac{i}{n}) - x(\frac{i}{n})| \leq \epsilon\}$ . To show this, we should prove “ $\supseteq$ ” part. Suppose that

$$y \in \bigcap_{n=1}^{\infty} \bigcap_{i=1}^n \left\{ y : \left| y\left(\frac{i}{n}\right) - x\left(\frac{i}{n}\right) \right| \leq \epsilon \right\}.$$

By max-min theorem,  $\exists t_0 \in [0, 1]$  s.t.

$$\sup_{t \in [0,1]} |x(t) - y(t)| = |x(t_0) - y(t_0)|.$$

Since  $x$  and  $y$  are continuous,  $\forall \delta > 0$ ,  $\exists t_\delta \in \{\frac{i}{n} : 1 \leq i \leq n, n \geq 1\}$  s.t.

$$|x(t_\delta) - x(t_0)| \leq \frac{\delta}{2}, |y(t_\delta) - y(t_0)| \leq \frac{\delta}{2}.$$

Now we implement “3- $\epsilon$  argument”; by construction, we get  $|x(t_\delta) - y(t_\delta)| \leq \epsilon$ , and so

$$\sup_{t \in [0,1]} |x(t) - y(t)| = |x(t_0) - y(t_0)| \leq |x(t_0) - x(t_\delta)| + |x(t_\delta) - y(t_\delta)| + |y(t_\delta) - y(t_0)| \leq \epsilon + \delta$$

holds. Since  $\delta > 0$  was arbitrary, we get

$$\sup_{t \in [0,1]} |x(t) - y(t)| \leq \epsilon,$$

i.e.,  $d(x, y) \leq \epsilon$ . Therefore, we get

$$y \in \bar{B}(x, \epsilon),$$

which yields

$$\bar{B}(x, \epsilon) \supseteq \bigcap_{n=1}^{\infty} \bigcap_{i=1}^n \left\{ y : \left| y\left(\frac{i}{n}\right) - x\left(\frac{i}{n}\right) \right| \leq \epsilon \right\}.$$

□

Now we get the conclusion.

**Theorem 2.3.5.** *In  $\mathcal{C}$ , if  $\{X_n\}$  is tight, and every finite-dimensional distribution of  $X_n$  converges to that of  $X$  weakly, then  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ .*

*Proof.* Note that,

$$\mathcal{H} := \{\pi_{t_1, \dots, t_k}^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^k), t_1, \dots, t_k \in [0, 1] \text{ and } k \geq 1\}$$

satisfies the condition in theorem 2.2.9, by previous theorem. Thus by theorem 2.2.9, we get the conclusion. □

As we have shown, if  $\{X_n\}$  is tight, then we can obtain weak convergence of  $X_n$  via only showing for that of finite-dimensional distributions. Thus, from now on, our interests are sufficient conditions of tightness of  $\{X_n\}$ .

### 2.3.2 Conditions for tightness in $\mathbb{C}$

Our first result is a “probabilistic version” of Arzelá-Ascoli argument.

**Theorem 2.3.6** (Arzelá-Ascoli). *A set  $A \subseteq \mathbb{C}$  has compact closure (i.e.,  $\bar{A}$  is compact) if and only if*

$$(i) \sup_{x \in A} |x(0)| < \infty$$

(ii)  $\lim_{\delta \rightarrow 0} \sup_{x \in A} w_x(\delta) = 0$ , where

$$w_x(\delta) = \sup_{(s,t): |s-t| < \delta} |x(s) - x(t)|$$

is the “modulus of continuity” of  $x$  in  $\mathbb{C}$ .

**Remark 2.3.7.** In fact, these conditions are necessary and sufficient for  $A$  to be “totally bounded.” Therefore, with completeness of  $\mathbb{C}$ , we get that  $\bar{A}$  is totally bounded and complete, and hence compact.

**Theorem 2.3.8.** A sequence  $\{X_n\}$  is tight if and only if

(i)  $\{X_n(0)\}$  is tight in  $\mathbb{R}$ .

(ii) (“asymptotic equicontinuity”) For any  $\epsilon > 0$ ,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P \left( \sup_{(s,t): |s-t| < \delta} |X_n(s) - X_n(t)| \geq \epsilon \right) = 0.$$

**Remark 2.3.9.** Recall that:

(i)  $f$  is continuous at  $x$  if

$$\sup_{y: |x-y| < \delta} |f(y) - f(x)| \xrightarrow{\delta \rightarrow 0} 0.$$

(ii)  $f$  is uniformly continuous if

$$\sup_{(x,y): |x-y| < \delta} |f(y) - f(x)| \xrightarrow{\delta \rightarrow 0} 0.$$

(iii) A sequence  $\{f_n\}$  of functions is equicontinuous if

$$\sup_{(x,y): |x-y| < \delta} \sup_n |f_n(y) - f_n(x)| \xrightarrow{\delta \rightarrow 0} 0.$$

As we need equicontinuity using Arzelá-Ascoli argument, we need condition (ii), which is stochastic version of equi-continuity condition, to use similar analogy.

*Proof.*  $\Rightarrow$ ) Assume that  $\{X_n\}$  is tight. Then for any given  $\epsilon > 0$ , there is a compact set  $K$  such that

$$\inf_n P(X \in K) > 1 - \epsilon.$$

Since  $K$  is compact, we apply Arzelá-Ascoli theorem, and then we get

$$\sup_{x \in K} |x(0)| < \infty, \quad \lim_{\delta \rightarrow 0} \sup_{x \in K} \sup_{(s,t): |s-t| < \delta} |x(s) - x(t)| = 0,$$

i.e.,

$$\exists C_0 \text{ and } \delta_0 \text{ s.t. } \sup_{x \in K} |x(0)| < C_0, \quad \sup_{x \in K} \sup_{(s,t): |s-t| < \delta_0} |x(s) - x(t)| < \epsilon.$$

(Remark: Always think “sup” as “proposition”!)  $\sup_{x \in K} |x(0)| < C_0$  means that

$$x \in K \Rightarrow |x(0)| < C_0,$$

i.e.,

$$\{x : x \in K\} \subseteq \{x : |x(0)| < C_0\},$$

so we get

$$\inf_n P(|X_n(0)| < C_0) \geq \inf_n P(X_n \in K) > 1 - \epsilon.$$

Thus  $\{X_n(0)\}$  is tight in  $\mathbb{R}$ . Similarly, we get

$$x \in K \Rightarrow \sup_{(s,t): |s-t| < \delta_0} |x(s) - x(t)| < \epsilon,$$

and hence

$$\inf_n P \left( \sup_{(s,t): |s-t| < \delta_0} |X_n(s) - X_n(t)| < \epsilon \right) \geq \inf_n P(X_n \in K) > 1 - \epsilon,$$

so that

$$\sup_n P \left( \sup_{(s,t): |s-t| < \delta_0} |X_n(s) - X_n(t)| \geq \epsilon \right) < \epsilon \quad (\text{“diagonal probability”})$$

(To get rid of “diagonal problem,” we find “very small”  $\epsilon$  that is even smaller than  $\epsilon_0$ ) Thus we get

$$\lim_{\delta \rightarrow 0} \sup_n P \left( \sup_{(s,t): |s-t| < \delta_0} |X_n(s) - X_n(t)| \geq \epsilon \right) \leq \epsilon,$$

and hence for any  $\epsilon_0 > 0$ , (we can always find  $\epsilon < \epsilon_0$  and so)

$$\lim_{\delta \rightarrow 0} \sup_n P \left( \sup_{(s,t): |s-t| < \delta_0} |X_n(s) - X_n(t)| \geq \epsilon_0 \right) \leq \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \sup_n P \left( \sup_{(s,t): |s-t| < \delta_0} |X_n(s) - X_n(t)| \geq \epsilon \right) \leq 0,$$

and therefore

$$\lim_{\delta \rightarrow 0} \sup_n P \left( \sup_{(s,t): |s-t| < \delta_0} |X_n(s) - X_n(t)| \geq \epsilon_0 \right) = 0.$$

$\Leftarrow$ ) Now assume that (i) and (ii) hold. We construct a totally bounded set  $K$  such that  $\sup_n P(X_n \in K^c) < \epsilon$ . By (i), we can find  $C_0$  such that

$$\sup_n P(|X_n(0)| > C_0) < \frac{\epsilon}{2}.$$

Also, by (ii), we can choose  $\delta_j > 0$ ,  $j \geq 1$  such that

$$\lim_{n \rightarrow \infty} \sup P \left( \sup_{(s,t): |s-t| < \delta_j} |X_n(s) - X_n(t)| \geq \frac{1}{j} \right) < \frac{\epsilon}{2^j}.$$

Note that, since each single random element  $X_k$  is tight, so by “only if” part that we have shown, we get

$$\lim_{\delta \rightarrow 0} P \left( \sup_{(s,t): |s-t| < \delta_j} |X_n(s) - X_n(t)| \geq \frac{1}{j} \right) = 0,$$

and consequently, we can choose “even smaller”  $\delta_j > 0$  such that

$$\sup_n P \left( \sup_{(s,t): |s-t| < \delta_j} |X_n(s) - X_n(t)| \geq \frac{1}{j} \right) < \frac{\epsilon}{2^j}.$$

Now take

$$K = \{x : |x(0)| \leq C_0\} \cap \bigcap_{j=1}^{\infty} \left\{ x : \sup_{(s,t): |s-t| < \delta_j} |x(s) - x(t)| < \frac{1}{j} \right\}.$$

Then by Arzelà-Ascoli theorem,  $K$  is totally bounded, and hence  $\bar{K}$  is compact. Therefore, we get

$$\sup_n P(X_n \in \bar{K}^c) \leq \underbrace{\sup_n P(|X_n(0)| > C_0)}_{< \epsilon/2} + \sum_{j=1}^{\infty} \underbrace{\sup_n P \left( \sup_{(s,t): |s-t| < \delta_j} |X_n(s) - X_n(t)| \geq \frac{1}{j} \right)}_{< \epsilon/2^j} < \epsilon$$

from

$$K^c = \{x : |x(0)| > C_0\} \cup \bigcup_{j=1}^{\infty} \left\{ x : \sup_{(s,t): |s-t| < \delta_j} |x(s) - x(t)| \geq \frac{1}{j} \right\},$$

which is the desired result.  $\square$

Now we get the very important proposition: By theorem 2.3.5, if  $\{X_n\}$  is tight, then weak convergence of “every finite dimensional distributions” implies weak convergence “of  $X_n$ .” However,

in previous theorem (theorem 2.3.8), we have a sufficient condition for tightness. Consequently, we have following corollary. Note that tightness of  $\{X_n(0)\}$  comes from the weak convergence of “1-dimensional distribution.”

**Corollary 2.3.10.** *Let  $X_n$  and  $X$  be a random element in  $\mathbb{C}$ . If all finite-dimensional distributions of  $X_n$  converge weakly to those of  $X$ , and if  $\forall \epsilon > 0 \exists n_0, \delta > 0$  such that*

$$\sup_{n \geq n_0} P \left( \sup_{(s,t): |s-t| < \delta} |X_n(s) - X_n(t)| \geq \epsilon \right) \leq \epsilon, \quad (2.2)$$

then  $X_n \xrightarrow[n \rightarrow \infty]{d} X$ .

*Proof.* It is just re-statement of

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P \left( \sup_{(s,t): |s-t| < \delta} |X_n(s) - X_n(t)| \geq \epsilon \right) = 0.$$

□

However, showing (2.2) is not an easy task. Thus, we rather show alternative condition, which is given in (2.3).

**Theorem 2.3.11.** (2.2) holds if

$$\sup_{n \geq n_0} \sup_{0 \leq t \leq 1} P \left( \sup_{s \in [t, t+\delta]} |X_n(s) - X_n(t)| \geq \frac{\epsilon}{3} \right) \leq \delta \epsilon. \quad (2.3)$$

**Remark 2.3.12.** Recall the proof of Glivenko-Cantelli theorem: Give a partition (grid), and find an interval that  $x$  belongs to. It is a common technique to handle “sup”, and we employee it in here similarly.

*Proof.* WLOG  $t < s$  (otherwise, just change the role), and  $1/\delta$  is an integer (otherwise, just get smaller  $\delta$  to obtain (2.2); we just need to show existence!). Let  $t_i = \delta i$  for  $i = 0, 1, \dots, \delta^{-1} =: I_n$ . Assume that  $|s - t| < \delta$ . Then (we supposed  $t < s$ )  $\exists$  grid point  $t_i$  such that

(i)  $t_{i-1} \leq t \leq t_i \leq s \leq t_{i+1}$  ( $s$  and  $t$  are located along 2 intervals), or

(ii)  $t_i \leq t < s \leq t_{i+1}$  (both  $s$  and  $t$  are in same interval).

(Check: we used  $|s - t| < \delta$ .)

CASE 1.  $t_{i-1} \leq t \leq t_i \leq s \leq t_{i+1}$

In this case,  $(t \in [t_{i-1}, t_i], s \in [t_i, t_{i+1}])$  from

$$|X_n(t) - X_n(t_i)| \leq \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)|$$

and

$$|X_n(s) - X_n(t_{i+1})| \leq \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)|,$$

we get

$$\begin{aligned} |X_n(t) - X_n(s)| &\leq |X_n(t) - X_n(t_i)| + |X_n(t_i) - X_n(t_{i+1})| + |X_n(t_{i+1}) - X_n(s)| \\ &\leq 3 \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)|. \end{aligned}$$

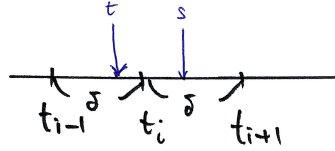


Figure 2.2: Case 1.

CASE 2.  $t_i \leq t < s \leq t_{i+1}$

In this case,  $(t, s \in [t_i, t_{i+1}])$  we get

$$\begin{aligned} |X_n(t) - X_n(s)| &\leq |X_n(t) - X_n(t_i)| + |X_n(t_i) - X_n(s)| \\ &\leq 2 \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)| \\ &< 3 \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)|. \end{aligned}$$

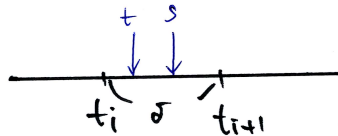


Figure 2.3: Case 2.

Thus, we get

$$|s - t| < \delta \Rightarrow |X_n(t) - X_n(s)| \leq 3 \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)|.$$

(Recall: “sup” is always regarded as a proposition!) Hence,

$$\sup_{(s,t):|s-t|<\delta} |X_n(t) - X_n(s)| \leq 3 \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)|$$

is obtained, and it implies

$$\begin{aligned} P \left( \sup_{(s,t):|s-t|<\delta} |X_n(t) - X_n(s)| \geq \epsilon \right) &\leq P \left( \max_{0 \leq i \leq I_n-1} \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)| \geq \frac{\epsilon}{3} \right) \\ &\leq \sum_{i=0}^{I_n-1} P \left( \sup_{t_i \leq t \leq t_{i+1}} |X_n(t) - X_n(t_i)| \geq \frac{\epsilon}{3} \right) \\ &\leq \sum_{i=0}^{I_n-1} \delta \epsilon \text{ (Assumption)} \\ &= I_n \delta \epsilon = \epsilon. \end{aligned}$$

□

### 2.3.3 Donsker's Theorem

Consider a random element in  $\mathbb{C}$  with the form

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i + (nt - \lfloor nt \rfloor) \frac{1}{\sqrt{n}} \xi_{\lfloor nt \rfloor + 1},$$

where  $\xi_i$  are i.i.d. random variables with  $E\xi_1 = 0$ ,  $Var\xi_1 = 1$ . Note that it is a “continuous embedding” of elements (See figure 2.4)

$$\tilde{X}_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \xi_i.$$

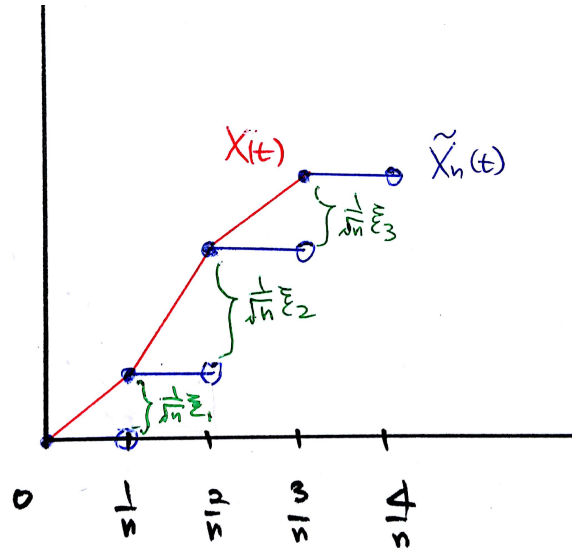
Note that each sample path of  $X_n(t)$  is continuous in  $[0, 1]$ , so it is a random element on  $\mathbb{C}[0, 1]$  (In the interval  $[i/n, (i+1)/n)$ ,  $\tilde{X}_n(t)$  is a constant, and hence

$$X_n(t) = \tilde{X}_n(t) + (nt - i)\sqrt{n}^{-1}\xi_{i+1}$$

becomes a linear function; i.e.,  $X_n(t)$  is piecewise linear interpolation). Donsker's theorem states that:

**Theorem 2.3.13** (Donsker). *The “partial sum process”  $X_n(t)$  converges weakly to the standard*



Figure 2.4:  $X_n(t)$  and  $\tilde{X}_n(t)$ 

Brownian motion  $W$ , i.e.,

$$X_n \xrightarrow[n \rightarrow \infty]{d} W.$$

Before prove Donsker's theorem, we first have to define what the limit process  $W$  is.

**Definition 2.3.14.** A random element  $W$  on  $\mathbb{C}[0, 1]$  is called a (standard) **Brownian motion** or **Wiener process** if

- (i) For each  $0 \leq t \leq 1$ ,  $W(t) \sim N(0, t)$  ("marginally normal").
- (ii)  $W(t_k) - W(t_{k-1}), \dots, W(t_2) - W(t_1)$  are independent for all  $0 \leq t_1 \leq \dots \leq t_k \leq 1$  ("independent increments").

Note that by definition,

- (i)  $W(0) = 0$ , and
- (ii) Each sample path of  $W$  is continuous.

*Proof of theorem 2.3.13.* Note that finite-dimensional distributions of  $X_n$  converge weakly to those of  $W$  by classical CLT. For example, in 1-dimensional case, for given  $t \in [0, 1]$ ,

$$(nt - [nt]) \frac{1}{\sqrt{n}} \xi_{[nt]+1} \xrightarrow[n \rightarrow \infty]{P} 0,$$

and

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i = \sqrt{\frac{[nt]}{n}} \frac{1}{\sqrt{[nt]}} \sum_{i=1}^{[nt]} \xi_i \xrightarrow[n \rightarrow \infty]{d} \sqrt{t} \cdot N(0, 1) = N(0, t)$$

holds by CLT, and

$$W(t) \sim N(0, t).$$

In 2-dimensional case, if  $0 \leq t_1 < t_2 \leq 1$  are given, then we can show that

$$\frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{i=1}^{\lfloor nt_1 \rfloor} \xi_i \\ \sum_{i=1}^{\lfloor nt_2 \rfloor} \xi_i \end{bmatrix} \xrightarrow[n \rightarrow \infty]{d} N(0, \Sigma)$$

where

$$\Sigma = \begin{pmatrix} t_1 & t_1 \\ t_1 & t_2 \end{pmatrix},$$

so from

$$\begin{bmatrix} W(t_1) \\ W(t_2) \end{bmatrix} \sim N(0, \Sigma),$$

we get the desired result. For the proof, see Kim (2012), p. . Thus, we will show that  $\forall \epsilon > 0$ ,  $\exists n_0, \delta > 0$  such that (2.3) holds. In here, to show (2.3) we will further assume that  $E\xi_1^4 < \infty$  (For the proof of general case, i.e.,  $E\xi_1^2 < \infty$ , see Billingsley (1999), p.89-91). First following lemma is introduced without proof, which is given in Billingsley (1999).

**Lemma 2.3.15.** *Let  $\xi_1, \dots, \xi_m$  be random variables. Let  $S_k = \xi_1 + \dots + \xi_k$  for  $k \geq 1$  and put  $S_0 = 0$ . If*

$$E|S_j - S_i|^\gamma \leq (u_{i+1} + \dots + u_j)^\alpha$$

*for some  $\gamma \geq 0$ ,  $\alpha > 1$  and  $u_1, \dots, u_m \geq 0$ , then*

$$P\left(\max_{1 \leq k \leq m} |S_k| \geq \lambda\right) \leq \frac{C_{\gamma, \alpha}}{\lambda^\gamma} (u_1 + \dots + u_m)^\alpha,$$

*where  $C_{\gamma, \alpha}$  is a constant that depends only on  $\gamma$  and  $\alpha$ .*

Now back to the proof. Fix  $t \in [0, 1]$ . Then for any  $n$ ,  $\exists j$  such that

$$t \in \left[\frac{j}{n}, \frac{j+1}{n}\right).$$

(We “partitioned”  $[0, 1]$  with  $n$  subintervals) Let

$$\frac{k}{n} < t + \delta \leq \frac{k+1}{n}$$

for some  $j \leq k \leq n-1$ , and for  $t < s \leq t + \delta$ , let

$$\frac{i}{n} < s \leq \frac{i+1}{n}$$

for  $j \leq i \leq k$ . See figure 2.5.

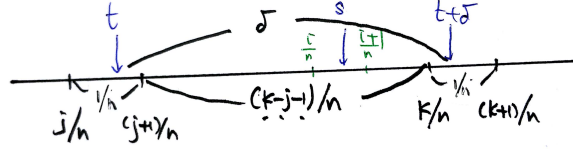


Figure 2.5: Proof of Donsker's theorem

Note that

$$\frac{k-j-1}{n} \leq \delta. \quad (*)$$

Then by triangle inequality and “polygonal character” of  $X_n$  (cf. figure 2.4), we get

$$\begin{aligned} |X_n(s) - X_n(t)| &\leq \left| X_n(s) - X_n\left(\frac{i}{n}\right) \right| + \left| X_n\left(\frac{i}{n}\right) - X_n\left(\frac{j}{n}\right) \right| + \left| X_n\left(\frac{j}{n}\right) - X_n(t) \right| \\ &= \left| X_n(s) - X_n\left(\frac{i}{n}\right) \right| + \left| X_n\left(\frac{i}{n}\right) - X_n\left(\frac{j}{n}\right) \right| + \left| X_n(t) - X_n\left(\frac{j}{n}\right) \right| \\ &\leq \left| X_n\left(\frac{i+1}{n}\right) - X_n\left(\frac{i}{n}\right) \right| + \left| X_n\left(\frac{i}{n}\right) - X_n\left(\frac{j}{n}\right) \right| + \left| X_n\left(\frac{j+1}{n}\right) - X_n\left(\frac{j}{n}\right) \right|. \end{aligned}$$

Now note that, since  $j \leq i \leq k \leq j+1+n\delta$  from (\*), we get  $j \leq i \leq (j+1+n\delta) \wedge (n-1)$ .

Let

$$I(\delta, j) = (j+1+n\delta) \wedge (n-1).$$

Then we have

$$\begin{aligned} |X_n(s) - X_n(t)| &\leq \max_{j \leq i \leq I(\delta, j)} \left| X_n\left(\frac{i}{n}\right) - X_n\left(\frac{j}{n}\right) \right| + 2 \max_{j \leq i \leq I(\delta, j)} \left| X_n\left(\frac{i+1}{n}\right) - X_n\left(\frac{i}{n}\right) \right| \\ &= \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_{j+1} + \cdots + \xi_i}{\sqrt{n}} \right| + 2 \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_i}{\sqrt{n}} \right|. \end{aligned}$$

(Note that

$$X_n\left(\frac{i}{n}\right) = \frac{1}{\sqrt{n}}(\xi_1 + \cdots + \xi_i)$$

by definition) Thus for  $S_k = \xi_1 + \cdots + \xi_k$ , we get

$$\begin{aligned} E(S_{i'} - S_i)^4 &= E(\xi_{i+1} + \cdots + \xi_{i'})^4 \\ &\leq C_1 \left( \sum_{k=i+1}^{i'} \underbrace{E\xi_k^2}_{=1} \right)^2 + \sum_{k=i+1}^{i'} \underbrace{E\xi_k^4}_{\leq (E\xi_k^2)^2=1} \\ &\leq (C_1 + 1)(i' - i)^2 \quad (\cdot: i' - i: \text{integer}) \end{aligned}$$

for  $i' > i > j$  ( $\because E(\sum_k \xi_k)^4 = \sum_k E\xi_k^4 + \sum_{k \neq k'} (E\xi_k^2)(E\xi_{k'}^2) \leq \sum_k E\xi_k^4 + \sum_{k,k'} (E\xi_k^2)(E\xi_{k'}^2)$ ), and hence by lemma ( $\gamma = 4, \alpha = 2, u_k \equiv \sqrt{C_1 + 1}$ ), we get

$$\begin{aligned} P \left( \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_{j+1} + \cdots + \xi_i}{\sqrt{n}} \right| \geq \frac{\sqrt{n}\epsilon}{6} \right) &\leq \frac{C_{4,2}}{(\epsilon/6)^4} (I(\delta, j) - j)^2 (C_1 + 1) \\ &\leq \frac{C_2(n\delta + 1)^2}{n^2\epsilon^4} \quad (\because I(\delta, j) - j \leq n\delta + 1) \\ &\leq \frac{4C_2\delta^2}{\epsilon^4} \end{aligned}$$

for sufficiently large  $n$  (precisely,  $n \geq 1/\delta$ ), and choosing

$$\delta \leq \frac{\epsilon^5}{8C_2},$$

we get

$$P \left( \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_{j+1} + \cdots + \xi_i}{\sqrt{n}} \right| \geq \frac{\epsilon}{6} \right) \leq \frac{4C_2\delta^2}{\epsilon^4} \leq \frac{\delta\epsilon}{2}.$$

Meanwhile, we get

$$\begin{aligned} P \left( \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_i}{\sqrt{n}} \right| \geq \frac{\epsilon}{12} \right) &\leq P \left( \bigcup_{i=j}^{I(\delta, j)} \left( \left| \frac{\xi_i}{\sqrt{n}} \right| \geq \frac{\epsilon}{12} \right) \right) \\ &\leq \sum_{i=j}^{I(\delta, j)} P \left( \left| \frac{\xi_i}{\sqrt{n}} \right| \geq \frac{\epsilon}{12} \right) \\ &\leq \sum_{i=j}^{I(\delta, j)} \frac{C_{4,2}}{(\sqrt{n}\epsilon/12)^4} (C_1 + 1) \\ &= \underbrace{(I(\delta, j) - j - 1)}_{\leq n\delta} \frac{C_{4,2}}{(\sqrt{n}\epsilon/12)^4} (C_1 + 1) \\ &\leq \frac{C_4\delta}{n\epsilon^4} \end{aligned}$$

$$\leq \frac{\delta\epsilon}{2}$$

for sufficiently large  $n$ , by the lemma again. Therefore, from

$$\forall t \in [0, 1], \forall (s, t) : |s - t| < \delta,$$

$$|X_n(s) - X_n(t)| \leq \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_{j+1} + \cdots + \xi_i}{\sqrt{n}} \right| + 2 \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_i}{\sqrt{n}} \right|,$$

we get

$$\sup_{(s, t) : |s - t| < \delta} |X_n(s) - X_n(t)| \leq \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_{j+1} + \cdots + \xi_i}{\sqrt{n}} \right| + 2 \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_i}{\sqrt{n}} \right|,$$

and hence

$$\begin{aligned} P \left( \sup_{(s, t) : |s - t| < \delta} |X_n(s) - X_n(t)| \geq \frac{\epsilon}{3} \right) &\leq P \left( \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_{j+1} + \cdots + \xi_i}{\sqrt{n}} \right| \geq \frac{\epsilon}{6} \right) + P \left( 2 \max_{j \leq i \leq I(\delta, j)} \left| \frac{\xi_i}{\sqrt{n}} \right| \geq \frac{\epsilon}{6} \right) \\ &\leq \frac{\delta\epsilon}{2} + \frac{\delta\epsilon}{2} \\ &= \delta\epsilon \end{aligned}$$

for large  $n$ , i.e.,

$$\sup_{n \geq n_0} \sup_{0 \leq t \leq 1} P \left( \sup_{(s, t) : |s - t| < \delta} |X_n(s) - X_n(t)| \geq \frac{\epsilon}{3} \right) \leq \delta\epsilon.$$

Note that, “sufficiently large”  $n_0$  only depends on  $\delta$  and  $\epsilon$ , because  $C_1, C_2, C_3$  and  $C_4$  are absolutely constant.  $\square$

**Remark 2.3.16** (Annotation by compiler). In fact, Donsker’s theorem says more general thing: For empirical cdf  $F_n(x)$ ,

$$\sqrt{n}(F_n(t) - F(t)) \xrightarrow[n \rightarrow \infty]{d} B(F(t)),$$

where weak convergence holds in Skorokhod topology of  $\mathbb{D}(-\infty, \infty)$  space. In here,  $B$  denotes Brownian bridge. However, empirical cdf is not continuous, so in here, we saw “continuous embedding” of partial sum process. Non-separability of  $\mathbb{D}$  space made Donsker’s formulation (for empirical cdf) wrong, and implementing Skorokhod metric, Skorokhod and Kolmogorov completed the proof (Wikipedia).