Graphs & Algo cheat-sheet

Check bipartite: Look for odd cycles. G is bipartite iff G has no odd cycles. (2-colour, P)
Independent Set: $I \subseteq V$ is independent iff $\neg\exists$ edge between vertices in I. NP

- Max degree d, then $\max$ independent set $|I| \geq \frac{n}{d+1}$.

Vertex Cover: Set $S \subseteq V$ such that every edge has one vertex in S. NP

- S is a vertex cover iff $V \setminus S$ is independent.

Matching: Set $M \subseteq E$ such that no vertex V has more than one edge in E. P
(Also called *independent edge set*.)
Perfect Matching: Every vertex is exactly incident to one edge in M.
X-saturating matching: Match all vertices of X in a bipartite graph (X, Y).

- $\max$ Matching $\leq \min$ Vertex Cover. (Equality holds for bipartite graphs.)
- Maximal matching of M? Then $\min$ vertex cover $|C|$ is: $|M| \leq |C| \leq |2M|$
- **Hall's Theorem**: G has an X-saturating matching iff $\forall S \subseteq X$: $|N(S)| \geq |S|$
- **Tutte-Berge theorem:** $\max M = \min_U \frac{1}{2}(n + U - o(G \setminus U))$
- **Tutte's matching theorem:** G has a perfect matching iff $\forall U \subseteq V$: $o(G \setminus U) \leq |U|$

Dominating set: Set $S \subseteq V$ such that each vertex is either in S or has a neighbour in S.

Probability theory

Useful inequalities:

- $\binom{n}{k} \leq \frac{n^k}{k!}$ and $\binom{n}{k} \leq (\frac{en}{k})^k$ and $\binom{n}{k} \leq 2^n$ and finally for $p \in [0,1] : (1-p)^n \leq e^{-pn}$.
- $n! \approx \binom{n}{e}$ for $O^*(n)$ algorithms.

Markov's inequality: $\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$ (X non-negative and $a > 0$.)

Lovasz Local Lemma (LLL): Consider n bad events $A_1, \ldots, A_n$.

- It also holds that $\forall_{1 \leq i \leq n} : \Pr[A_i] \leq p$ and each A_i depends on at most d other events A_j.
- Then $ep(d+1) \leq 1 \implies \Pr[\bigcap_i \bar{A}_i] > 0$.

Probabilistic idea is to randomly assign (or assign with probability p) stuff:

1. $\exists i : n_i \geq \mathbb{E}[X]$ and $\exists_j : n_j \leq \mathbb{E}[X]$. This can be useful for expected sizes of certain sets.
2. Consider event $A(X)$. If $\Pr[A(X)] > 0$, then there must exist a value of X where A occurs. Similar vice-versa.

Algorithms