Module 2

The Language of Mathematics

Mathematical Language and Symbols

Mathematics in the Modern World

Overview

Mathematics as a language is composed of vocabularies consisting of symbols, words, and grammar as well as a set of rules on the proper usage of the said elements. One needs to learn and understand the mathematical language in order to think, comprehend, and to communicate mathematically. There are two forms that can be used in communicating mathematically: verbal and written, each may be either formal or informal.

Several reasons for using mathematical language are:

a. Statements of mathematics are *clear*;
b. Statements of mathematics are *completely precise*;
c. Statements of mathematics are *concise*;
d. Statements of mathematics are *free from vagueness and ambiguities of ordinary speech*.

This module covers the lessons identified as the most essential under Chapter 2: Mathematical Language and Symbols. Here, you will learn about set operations, propositions, logical connectives, predicate, and quantifiers. Exercises are also provided to help you practice and master the knowledge and skills necessary for your attainment of the learning outcomes. How much you have learned from the said lessons will be determined in the last part of this module. Have fun learning!

Learning Outcomes

After completing the study of this module, you should be able to:

- Illustrate the language, symbols, and conventions of mathematics through examples;
- Explain the nature of mathematics as a language;
- Justify that mathematics is a useful language through citing concrete examples of its applications;
- Perform the operations on sets using proper notation;
- Draw and interpret Venn diagrams of set operations, and use Venn diagrams to solve problems on sets;
- Determine the truth values of propositions at any context; and
- Construct tables to determine the truth values of propositions;

Initial Activity *(Accessing Prior Knowledge)*

Answer the following questions:
- ✓ What is a language?
- ✓ What are the components of a language?
- ✓ Why do you think is mathematics considered a language?

Language is “a systematic means of communicating by the use of sounds or conventional symbols” (Chen,2010). It is the code humans use as a form of expressing themselves and communicating with others. It is a system of words used in a particular discipline and is composed of the following components:

- a vocabulary of symbols or words;
- a grammar consisting of rules on the use of these symbols;
- a community of people who use and understand these symbols; and
- a range of meaning that can be communicated with these symbols.

Since all of the components mentioned above are found in mathematics, we can therefore say that mathematics qualifies as a language. In fact, mathematics as a language, is universal. It is the only one shared by all human being regardless of culture, religion, or gender. Like any language, mathematics has nouns (such as the name of the number 12), pronouns (such as the variables x or y), verbs (such as “equals”), and sentences (such as 3x + 7 = 24). Mathematics has its own vocabulary, grammar, rule, syntax, synonyms, negations, sentence structure, paragraph structure, conventions, and abbreviations. It is designed in such a way that one can write about numbers, sets, functions, and others, as well as the processes undergone by these elements (like adding, multiplying, grouping, evaluating, etc). And like any language, mathematics has its own symbols. There are ten digits (0, 1, 2, …, 9), symbols for operations $(+,\ -,\ \times, \div)$, symbols that represent values $(x, y, z,$ etc.$)$, and many other special symbols $(=, <, \leq, \pi,$ etc.$)$.

A. Set Operations

Set theory is about identifying relationships between things that are grouped together for some reason. The word **set** was first formally used in 1879 by Georg Cantor, a German mathematician, to refer to a well-defined collection of objects. Each object in a set is called an **element** or a member of the set. For instance, if C is the set of all regions in the Philippines, then Region IV B MIMAROPA is an element of C. The symbol $\in$ is used to denote that an object is an element of a set, and the symbol $\notin$ is used to denote that an object is not an element of a set.

Example 1: Let $S = \{a, b, c\}$. It follows that:

$c \in \{a, b, c\}$ $\quad$ $c \in S$ $\quad$ $f \notin \{a, b, c\}$ $\quad$ $f \notin S$

Example 2: Let B be the set of first 10 positive odd integers. Then

$B = \{1,3,5,7,9,11,13,15,17,19\}$

So we can say that $9 \in B$ and $2 \notin B$.

Sets often have relationships with other sets. For example, it could be that you are a member of both the set of college students and the set of students taking college math course. Or you could be in the set of freshmen or in the set of sophomores, but not in both. You might be in the set of students living off campus and the set of students who walk to class. Or maybe you're in the set of students who eat lunch in the canteen and the set of students who think that egg sandwiches are too bland, but not in the set of people who put ketchup in their food. In such cases, a system for displaying and organizing all of these complicated connections between sets will come in handy through studying set theory.

Below are the operations involving sets.

1. ***Union of Sets***

If A and B are any two sets, the union of A and B, denoted by $A \cup B$, is the set containing of all the elements that are found in A or in B or in both A and B. In symbols,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Note: The notation $\{x \mid x \in A \text{ or } x \in B\}$ reads: " the set of all x such that x is in A or x is in B".

Example 3 :

a. Let A = {1, 2, 4} and B = {3, 4, 5, 6}
   Then, $A \cup B = \{1, 2, 3, 4, 5, 6\}$

b. Let D = {German Shepherd, Labrador} and E = {Bulldog, Rottweiler}
   Then, D ∪ E = {German Shepherd, Labrador, Bulldog, Rottweiler}

Note: Elements common to both sets are listed only once in the union.

The inclusive "or" in the definition means that if $x \in A \cup B$, then at least one of these conditions must be true: x can be found in A; or x can be found in B; or x can be found in A and in B.

The figures below illustrate the union of two sets.

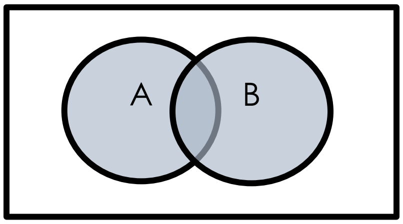


Fig 1a. A and B are overlapping or joint sets. (Some A are B)

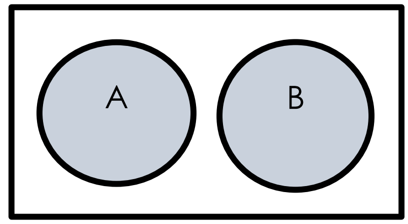


Fig 1b. A and B are disjoint sets. (No A are B.)

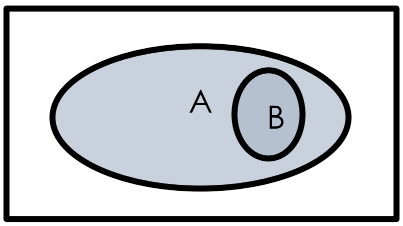


Fig 1c. B is contained in A. (All B are A.)

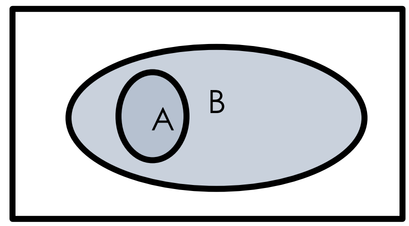


Fig 1d. A is contained in B. (All A are B.)

The illustrations above are called Venn diagrams, named after the English logician John Venn who first devised it, representing the universal set as a rectangle and all of its subsets as circles (or oblongs) within the rectangle. A Venn diagram is pictorial representation for sets and their fundamental operations. The shaded portion of each diagram here represents the union of sets.

For better understanding of the figures above, we introduce some important terms and symbols in set theory.

Definition 1**.** Let A and B be sets. Then A is said to be a subset of B, written $A \subseteq B$, if every element of A is an element of B. If A is not a subset of B, then we write $A \nsubseteq B$.

Example 4: Let $A = \{0,2,4,6,8\}$, $B = \{0,1,2,3,4,5,6,7,8,9,10\}$, and $C = \{1,2,3,4,5\}$. Then we say that:

$$A \subseteq B; \qquad A \nsubseteq C$$

Example 5: {1, 2, 3, 4} ⊆ {1, 2, 3, 4}

Notes:

- ✓ For every set A, we have $A \subseteq A$. (Every set is a subset of itself.)
- ✓ For A to not to be a subset of B, there must be at least one member of A that is not in B.

> Definition 2: Let A and B be sets. Then A is a **proper subset** of B, written $A \subset B$, if A is a subset of B and there exists at least one element in B that is not in A.

Example 6: Let $A = \{a, b\}$ and $B = \{a, b, c\}$. Because every element of A is an element of B, we have $A \subseteq B$. Now $c \in B$ and $c \notin A$. Therefore there exists an element in B that is not in A. It now follows that A is a proper subset of B. That is, $A \subset B$, or in simple terms, A is fully contained in B.

Other examples:

In Figures 1a and 1b, we can say that $A \nsubseteq B$, but in Figures 1c and 1d, we can say that $B \subseteq A$ and $A \subseteq B$, respectively. In fact, we can more appropriately say that $B \subset A$, and $A \subset B$, respectively.

In example 4, we can say that $A \subset B$.

The set of all men is a proper subset of the set of all people.

{1, 3} ⊂ {1, 2, 3, 4}

The set of all even integers is a proper subset of the set of all integers.

As a side trip to the set of complex numbers, in connection with subsets and proper subsets, we present the diagram of the subsets of Complex Numbers.

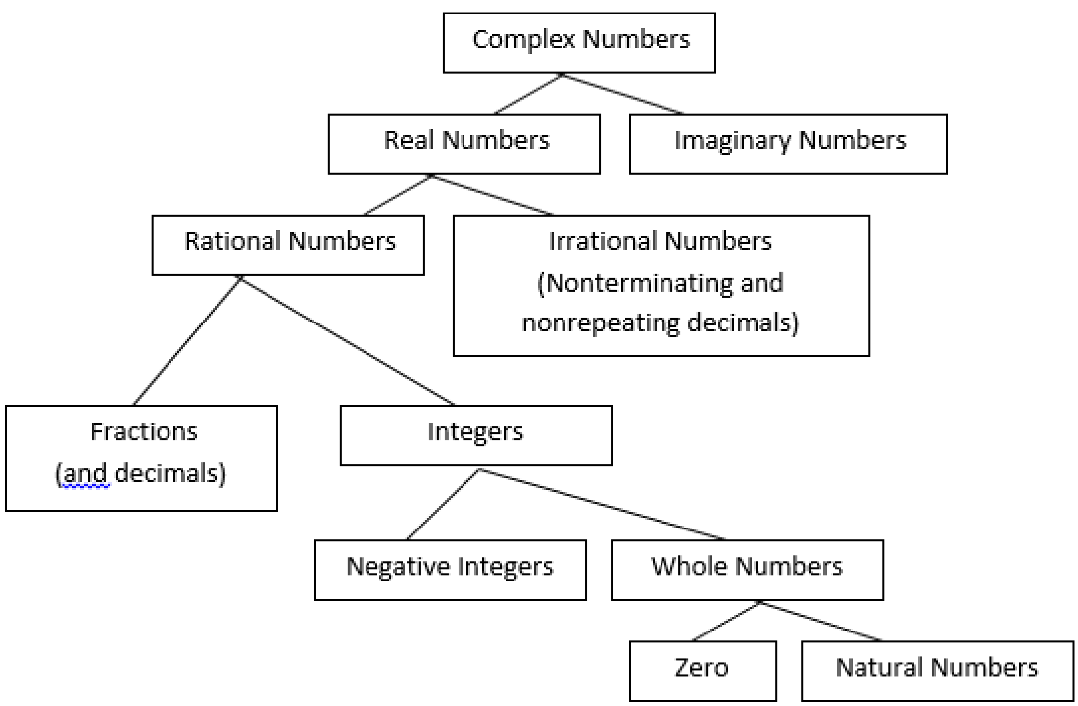


Figure 2 The Subset of Complex Numbers

The symbols that are conventionally used to denote these various sets of numbers are the following:

$\mathbb{N}$ The set of all natural numbers (i.e., all positive integers)
$\mathbb{Z}$ The set of all integers
$\mathbb{Z}^*$ The set of all nonzero integers
$\mathbb{E}$ The set of all even numbers
$\mathbb{Q}$ The set of all rational numbers
$\mathbb{Q}^*$ The set of all nonzero rational numbers
$\mathbb{Q}^+$ The set of all positive rational numbers
$\mathbb{R}$ The set of all real numbers
$\mathbb{R}^*$ The set of all nonzero real numbers
$\mathbb{R}^+$ The set of all positive real numbers
$\mathbb{C}$ The set of all complex numbers
$\mathbb{C}^*$ The set of all nonzero complex numbers

From the concept of proper subsets that we learned earlier, and using the diagram of the subset of the complex numbers, we can verify that: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Other useful terms:

- ✓ Joint sets- are sets that have common elements
- ✓ Disjoint sets- are sets that do not have common elements
- ✓ Equal sets- sets that contain exactly the same elements
- ✓ Equivalent sets- sets that have the same number of elements

Try this!

Three experimental medications are being evaluated for safety. Each has a list of side effects that has been reported by at least 1% of the people trying the medication. This is a blind trial, so the medications are simply labeled A, B, and C. The side effects for each are listed below.

A = {nausea, night sweats, nervousness, dry mouth, swollen feet}
B = {weight gain, nausea, nervousness, blurry vision, fever, trouble sleeping}
C = {dry mouth, nausea, blurry vision, fever, weight loss, eczema}

Find each of the following sets.

1. $A \cup B$
2. $A \cup C$
3. $A \cup B \cup C$

Solution: Remember that to get the union of sets, we just collect altogether the elements of all sets involved, but we list common elements just once, and never forget the braces to indicate that the list is a set.

1. A ∪ B= { nausea, night sweats, nervousness, dry mouth, swollen feet, weight gain, blurry vision, fever, trouble sleeping}

2. A ∪ C= {nausea, night sweats, nervousness, dry mouth, swollen feet, blurry vision, fever, weight loss, eczema}

3. A ∪ B ∪ C= { nausea, night sweats, nervousness, dry mouth, swollen feet, weight gain, blurry vision, fever, trouble sleeping, weight loss, eczema }

2. ***Intersection of Sets***

The intersection of two sets A and B, denoted by A ∩ B, is the set of all elements common to both A and B. In symbols,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Example 7:

a. Let A = {1, 2, 4} and B = {3, 4, 5, 6}
   Then, A ∩ B = {4}
b. Let D = {German Shepherd, Labrador} and E = {Bulldog, Rottweiler}
   Then, D ∩ E = { }

The following figures illustrate the intersection of two sets.

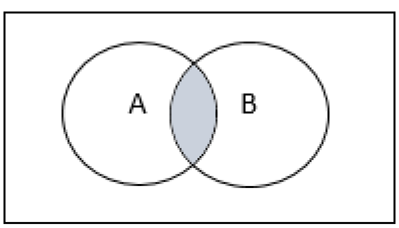


Figure 3a. A and B are overlapping sets. (Some A are B.)

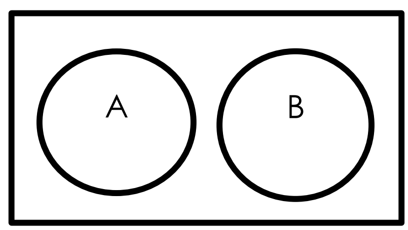


Figure 3b. A and B are disjoint sets. (No A are B.)

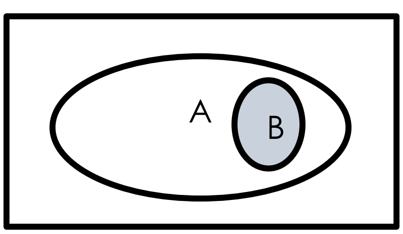


Figure 3c. B is contained in A. (All B are A.)

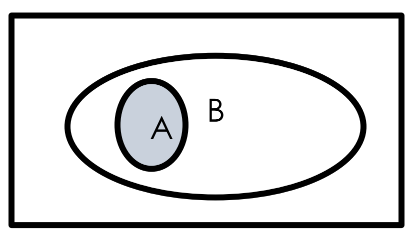


Figure 3d. A is contained in B. All B are A.)

Notice that there is no shaded portion in Figure 3b, because A and B are disjoint sets.

Definition 3: Two sets A and B are said to be **disjoint** if $\mathrm{A} \cap \mathrm{B} = \emptyset$.
(The symbol $\emptyset$ is called the null or empty set, which is a set that does not have any element at all.)

Example 8: Let $= \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \mathrm{e}\}$; $\mathrm{Y} = \{\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}, \mathrm{h}\}$; and $\mathrm{Z} = \{\mathrm{h}, \mathrm{p}, \mathrm{q}, \mathrm{r}\}$.
Then $\mathrm{X} \cap \mathrm{Y} = \{\mathrm{c}, \mathrm{d}, \mathrm{e}\}$
and $\mathrm{X} \cap \mathrm{Z} = \emptyset$

Note: You can also represent an empty set by an empty braces. So $\mathrm{X} \cap \mathrm{Z} = \emptyset$ can be written as $\mathrm{X} \cap \mathrm{Z} = \{\ \}$.

Example 9: If W = { 1, 2, 3 }and S = { a, b, c } then $\mathrm{W} \cap \mathrm{S} = \emptyset$ or $\mathrm{W} \cap \mathrm{S} = \{\ \}$.

Note: Do not use the symbols $\emptyset$ and { } together like this $\{\emptyset\}$ if you want to represent an empty set because that would no longer be an empty set. This notation means that the set contains the element $\emptyset$, which makes it "not empty".

Try this!

Three experimental medications are being evaluated for safety. Each has a list of side effects that has been reported by at least 1% of the people trying the medication. This is a blind trial, so the medications are simply labeled A, B, and C. The side effects for each are listed below.

A = {nausea, night sweats, nervousness, dry mouth, swollen feet}
B = {weight gain, nausea, nervousness, blurry vision, fever, trouble sleeping}
C = {dry mouth, nausea, blurry vision, fever, weight loss, eczema}

Find:
1. A ∩ B
2. B ∩ C
3. A ∩ B ∩ C

Solution: Remember that in taking the intersection of two sets, we are going to collect only the elements that are common in both or all the sets involved.

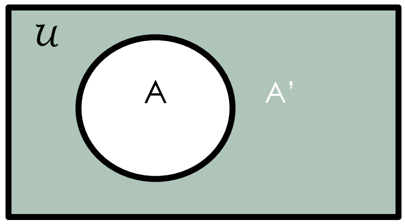
1. A ∩ B = { nausea, nervousness}
2. B ∩ C = {nausea, blurry vision, fever}
3. A ∩ B ∩ C = {nausea}

3. ***Set Complement (or Absolute Complement)***

The complement of a set is the set whose elements are found in the universal set but are not found in the given set. The *universal set* (U) is the set which contains the elements of all sets being considered. If A is the given set, then its complement A' would have elements in U that are not in A. In symbols,

$$A' = \{x \mid x \in U \text{ and } x \notin A\} \text{ or } A' = \{x \mid x \in (U - A)\}.$$

A' is represented by the shaded region in the figure below.



Example 10:

a. Let U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} and A = {1, 3, 5, 7}
   A' = {0, 2, 4, 6, 8}

b. Let U = {PSU college students} and A = {PSU engineering students}
   A' = {PSU college students who are not taking up engineering}

c. Let $U = \{a, b, c, d, e, f, g, h\}$ and $A = \{a, c, g, h\}$. Then $A' = \{b, d, e, f\}$.

d. Let U = { x | x is a letter in the alphabet } and A = { x |x is a consonant }, then
   A' = { x | x is not a consonant } or
   A' = { x | x is a vowel }

d. Let U=$\mathbb{Z}$. then $\mathbb{E}'$= the set of odd integers

Note: The set of vowel letters and consonant letters of the alphabet are complementary sets. Same thing with the set of even and odd integers. Complementary sets are two sets whose union makes the Universal set.

Try this!

Three experimental medications are being evaluated for safety. Each has a list of side effects that has been reported by at least 1% of the people trying the medication. This is a blind trial, so the medications are simply labeled A, B, and C. The side effects for each are listed below.

A = {nausea, night sweats, nervousness, dry mouth, swollen feet}
B = {weight gain, nausea, nervousness, blurry vision, fever, trouble sleeping}
C = {dry mouth, nausea, blurry vision, fever, weight loss, eczema}

The universal set below is the set of all side effects reported by ANY user.
U = {nausea, night sweats, nervousness, dry mouth, swollen feet, weight gain, blurry vision, fever, trouble sleeping, weight loss, eczema, motor mouth, darting eyes, uncontrollable falling down}

Use this to find the following sets.
1. A’ ∩ C’ 2. (A ∩ B)’ ∩ C 3. B’ ∪ (A ∩ C’)

Solutions: Remember that to take the absolute complement of a set, we collect all elements of the universal set that are not in the given set. In expressions containing ∪, ∩, ′, and grouping symbols, we first evaluate those inside the grouping symbols. If there is more than one pair of grouping symbols, we start from the innermost. Then we evaluate ′. In the absence of grouping symbols, union (∪) and intersection (∩) usually have the same level of precedence.

1. A’ ∩ C’ = { weight gain, blurry vision, fever, trouble sleeping, weight loss, eczema, motor mouth, darting eyes, uncontrollable falling down} ∩ {night sweats, nervousness, swollen feet, weight gain, trouble sleeping, motor mouth, darting eyes, uncontrollable falling down}
= {weight gain, trouble sleeping, motor mouth, darting eyes, uncontrollable falling down}

Note: The first step of the solution shows the complement of A and C. The second step shows the intersection (∩) of the two complements.

2. (A ∩ B)’ ∩ C = {nausea, nervousness}’ ∩ {dry mouth, nausea, blurry vision, fever, weight loss, eczema}
= {night sweats, dry mouth, swollen feet, weight gain, blurry vision, fever, trouble sleeping, weight loss, eczema, motor mouth, darting eyes, uncontrollable falling down} ∩ {dry mouth, nausea, blurry vision, fever, weight loss, eczema}
= {dry mouth, blurry vision, fever, weight loss, eczema}

Note: Step 1 shows the intersection of A and B, as well as the elements of set C.
Step 2 shows the complement of the intersection of A and B, and we just copied set C.
Step 3 finally shows the intersection of the sets found in step 2.

3. B’ ∪ (A ∩ C’)= {night sweats, dry mouth, swollen feet, weight loss, eczema, motor mouth, darting eyes, uncontrollable falling down} ∪ ({nausea, night sweats, nervousness, dry mouth, swollen feet} ∩ {night sweats, nervousness, swollen feet, weight gain, trouble sleeping, motor mouth, darting eyes, uncontrollable falling down})
   = {night sweats, dry mouth, swollen feet, weight loss, eczema, motor mouth, darting eyes, uncontrollable falling down} ∪ {night sweats, nervousness, swollen feet}
   = {night sweats, dry mouth, swollen feet, weight loss, eczema, motor mouth, darting eyes, uncontrollable falling down, nervousness}

Note: Step 1: Take B’, copy A and take C’. Remember to enclose A and C’ in one group to avoid error.

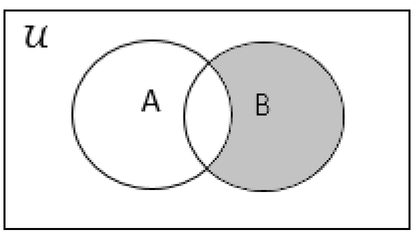
Step 2: Copy B’ and take the intersection of A and C’.

Step 3: Finally take the union of the sets found in step 2.

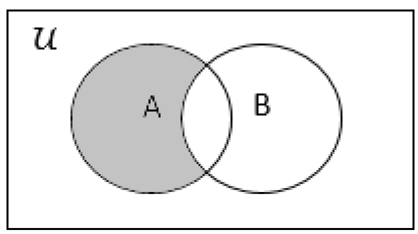
4. ***Set Difference (or Relative Complement)***

The difference of set A from set B, denoted by B – A, is the set of all elements of set B that are not in set A. The difference of set B from set A, denoted by A – B, is the set of all elements of set A that are not in set B. Some books use the symbol A\B to represent the relative complement of A with respect to B.

The illustrations below show set difference.



B - A



A - B

Example 11:

Let A = {1, 2, 4} and B = {3, 4, 5, 6}, then
B – A = {3, 5, 6} and A – B = {1, 2}

Try this!

Three experimental medications are being evaluated for safety. Each has a list of side effects that has been reported by at least 1% of the people trying the medication. This is a blind trial, so the medications are simply labeled A, B, and C. The side effects for each are listed below.

A = {nausea, night sweats, nervousness, dry mouth, swollen feet}
B = {weight gain, nausea, nervousness, blurry vision, fever, trouble sleeping}
C = {dry mouth, nausea, blurry vision, fever, weight loss, eczema}

Find:
1. A – B
2. B – C
3. (A – B) – C

Solution: Remember that subtraction is not commutative, so we have to be careful with taking A- B and B- A.

1. A- B= {night sweats, dry mouth, swollen feet}
2. B- A= {weight gain, blurry vision, fever, trouble sleeping}
3. (A- B)- C= {night sweats, dry mouth, swollen feet}- {dry mouth, nausea, blurry vision, fever, weight loss, eczema}
   ={ night sweats, swollen feet }

Definition 4: Let S be a finite set with n distinct elements, where $n \geq 0$. Then we write $n(S) = n$ and say that the cardinality of S is n. The **cardinality or cardinal number** of a set is the number of elements in the set.

Example 12: Let $A = \{a, b, c, d, e\}$. Then $n(A) = 5$

Example 13: What is the cardinal number of the set of all positive integers, $\mathbb{Z}^+$?
Answer: $n(\mathbb{Z}^+) = \infty$

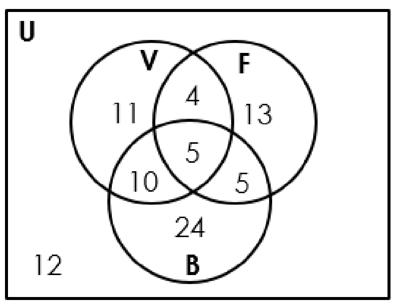
Application Problem: A PE instructor at Palawan State University conducted a survey to determine the favorite sports of 84 students in her 2 freshmen classes. The individual interviews showed the following: 44 liked basketball, 30 liked volleyball, 27 liked football, 15 liked both basketball and volleyball, 9 liked both volleyball and football, 10 liked both football and basketball, 5 liked all three sports. How many of the students did not like any of the three sports? How many of the students liked basketball only? How many of the students liked football but not volleyball?

Solution:

This kind of problem is best approached by drawing a Venn diagram. The first thing to consider is the number of sets involved and which of these sets are joint and which are disjoint. Then values that fall within each region are found by working on the problem/ information backwards. Let us list down the things that we know about the problem.

- ✓ There were 84 freshmen students interviewed. (Universal set)
- ✓ Forty- four (44) students liked basketball. (Set B)
- ✓ Thirty (30) students liked volleyball. (Set V)
- ✓ Twenty- seven (27) liked football. (Set F)
- ✓ Fifteen (15) liked both basketball and volleyball. ($B \cap V$)
- ✓ Nine (9) liked both volleyball and football. ($V \cap F$)
- ✓ Ten (10) liked both football and basketball. ($F \cap B$)
- ✓ Five (5) liked all three sports. ($B \cap V \cap F$)

Based on these information, we know that there are three sets involved and they are all joint sets, so our Venn diagram will contain three overlapping circles. Some of you might be wondering why it seems that the total number of data is more than the cardinality of the universal set, which is 84. That is because you have double- counted some elements in the intersections. So, it is important that we work backwards in solving this problem.



This is the Venn diagram for the problem. To know how the numbers were derived, and how double counting was avoided, let us list down the cardinality of the sets, beginning from the intersection of all three sets, ($B \cap V \cap F$).

- ✓ $n(B \cap V \cap F) = 5$ (because 5 students liked all three sports)
- ✓ $n(F \cap B) = 10 - 5 = 5$ (because 5 students have already been counted in $B \cap V \cap F$)
- ✓ $n(V \cap F) = 9 - 5 = 4$ (because 5 students have already been counted in $B \cap V \cap F$)
- ✓ $n(B \cap V) = 15 - 5 = 10$ (because 5 students have already been counted in $B \cap V \cap F$)
- ✓ $n(F) = 27 - 5 - 5 - 4 = 13$ (because 5 students have already been counted in $B \cap V \cap F$, 5 in $F \cap B$, and 4 in $V \cap F$)
- ✓ $n(V) = 30 - 5 - 10 - 4 = 11$ (because 5 students have already been counted in $B \cap V \cap F$, 10 in $B \cap V$, and 4 in $V \cap F$)
- ✓ $n(B) = 44 - 5 - 10 - 5 = 24$ (because 5 students have already been counted in $B \cap V \cap F$, 10 in $B \cap V$, and 5 in $F \cap B$)

If we add these values, we will get 72, which is not equal to the cardinality of the Universal set. This means that there were students in the interview who answered that neither of the three sports (basketball, volleyball, football) is their favorite. We put these students, 84- 72= 12, outside the circles but still inside the rectangle.

Now that we have the Venn diagram, all we have to do is locate which region is covered by each of the operations in the problem. It is important that we know how to translate the values that we see in the diagram into words. It is equally important that we know how to translate verbal expressions into symbols. For instance, the questions in this problem can be translated as:

How many of the students did not like any of the three sports?	$n(B \cup F \cup V)'$
How many of the students liked basketball only?	$n(B - (F \cup V))'$
How many of the students liked football but not volleyball?	$n(F - V)$

By translating the sentences into symbols, it made us easier to find the answer by referring to the Venn diagram. Study the diagram and verify that:

1. $n(B \cup F \cup V)' = 12$. That is, 12 of the students did not like any of the three sports.
2. $n(B - (F \cup V)) = 24$. That is, 24 of students liked basketball only.
3. $n(F - V) = 18$. That is, 18 students liked football but not volleyball.

There is another set operation considered in this section. Forming this new set involves a much different process than forming the union, intersection, or complement of sets.

Given elements *a* and *b*, the symbol *(a, b)* denotes the **ordered pair** consisting of *a* and *b* together with the specification that *a* is the first element of the pair and *b* is the second element.
Given sets *A* and *B*, the **Cartesian product of A and B**, denoted *A x B* and read "A cross B", is the set of all ordered pairs *(a, b)*, where *a* is in *A* and *b* is in B. Symbolically:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

Example 14: Let $X = \{1,2\}$, $Y = \{3,4\}$.
Then $X \times Y = \{(1,3), (1,4), (2,3), (2,4)\}$ and $Y \times X = \{(3,1), (3,2), (4,1), (4,2)\}$

Example 15. Let A = {1, 2, 3} and B = {u, v}.

a. Find A x B.
   A x B = {(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)}

b. Find B x A.
   B x A = {(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)}

c. Find B x B.
   B x B = {(u, u), (u, v), (v, u), (v, v)}

e. How many elements are in A x B, B x A, and B x B?
A x B has six elements. Note that this is the number of elements in A times the number of elements in B. B x A has six elements, the number of elements in B times the number of elements in A. B x B has four elements in B times the number of elements in B.

Note: If $A \neq B$, then $A \times B \neq B \times A$. Also, $n(A \times B) = n(A) \cdot n(B)$.

Bonus Concept: Let U = the set of real numbers. The coordinates axes are formed by two intersecting perpendicular lines in order to graph U x U.

Try this!

Let Y = {a, b, c} and Z = {1, 2}.

1. Find Y x Z.
2. Find Z x Y.
3. Y x Y.
4. How many elements are Y x Z, Z x Y, and Y x Y?

Solutions: Remember that in cross- product, the order of elements is important: the first coordinate must come from the first set, and the second coordinate must come from the second set

1. Y x Z= {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}
2. Z x Y= {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}
3. Y x Y= {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)}
4. Y x Z has 3 x 2= 6 elements;
   Z x Y has 2 x 3= 6 elements; and
   Y x Y has 3 x 3= 9 elements.

Learning check

Activity # 1

A. Given: U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
   A = {1, 2, 3, 4, 5}
   B = {5, 6, 7, 8, 9, 10}

C = {2, 4, 6, 8, 10}
D = {1, 3, 5, 7, 9}

Find the following:

1. $A \cup B$
2. $A' \cap B$
3. $(B' \cup C) \cup D$
4. $D - C'$
5. $(A \cup D')'$

B. Workers are grouped into various categories. Set E is the set of workers under 20 years old, Set F is the set of foreign workers, and Set M is the set of male workers. Describe the following sets in words:

a. E' b. $F \cap M$ c. $E \cup M$ d. $E \cap F \cup M$

C. Solve the following problems using formulas and Venn diagram.

1. A survey of 100 college students revealed the following information about their enrolment in Mathematics, Political Science, and Sociology courses. Twenty-six take Mathematics; 65 take Political Science; 65 take Sociology; 14 take Mathematics and Political Science; 13 take Mathematics and Sociology; 40 take Political Science and Sociology; 8 take Mathematics, Political Science and Sociology.

   a. How many students take Mathematics as their only course?
   b. How many students did not take any of the three courses?
   c. How many students were taking Mathematics and Sociology, but not Political Science?
   d. How many students did not take either Mathematics or Political Science?

2. After a genetic experiment, the number of pea plants having certain characteristics was tallied, with the results as follows: 22 were tall; 25 had green peas; 39 had smooth peas; 9 were tall and had green peas; 17 were tall and had smooth peas; 20 had green peas and smooth peas; 6 had all three characteristics; and 4 had none of the characteristics mentioned.

   a. Find the total number of plants counted.
   b. How many plants were tall and had peas that were neither smooth nor green?
   c. How many plants were not tall but had peas that were smooth and green?

D. Let S = {2, 4, 6} and T = {1, 3, 5}. Use the roster notation to write each of the following sets, and indicate the number of elements that are in each set:

a. S x T b. T x S c. S x S d. T x T

B. Propositions

A **proposition** is a declarative sentence that is either true or false but not both. As a declarative sentence, it expresses a complete thought with a definite meaning. A proposition's truth value is denoted by T or 1 if it is true, and F or 0 if it is false. Some books use the term "statement" to equally mean proposition.

Propositions are the basic building blocks of any theory of logic. Here are some examples of propositions.

Example 1:

1. The main campus of Palawan State University is in Puerto Princesa City.
2. Quezon City is the capital of the Philippines.
3. Benguet is a part of the Cordillera Administrative Region.
4. 1 + 1 = 2.
5. 2 + 2 = 3.

Example 2: Below are some more sample propositions and their corresponding truth values.

p:	4 is an integer.	True
q:	$\sqrt{5}$ is a rational number.	False
r:	Tubattaha Reef is located in Cagayancillo, Palawan.	True
s:	5 is less than 3.	False
t:	7 is an even integer.	False

Note: The small letters p, q, r, s, and t are used to name the propositions. They do not have any significance other than labeling. Just like in example 1, we can write a proposition even without labels.

Example 3: Take a look at this case:

Every even integer greater than 4 is a sum of two odd primes. (Goldbach's conjecture)

So far, no one has been able to prove that the Goldbach's conjecture is true. At the same time, no one has proved that it is false. Nevertheless, it is a proposition because it is either true or false, but not both.

Example 4. Below are sample sentences which are not propositions. The sentences are not declarative and it is meaningless to determine their truth values.

Will you go?
Have a nice day!
Be quiet.

Try this!

Which of the following are propositions? Give the truth value of the propositions, otherwise, write "not a proposition".

1. Check your solution.
2. A square is quadrilateral.
3. August is the 9th month of the year.
4. Have a nice day!
5. $5x + 1 = 2$.

Answers:

1. Not a proposition
2. Proposition- True
3. Proposition- False
4. Not a proposition
5. Not a proposition (We would need a value for x to decide whether this is true or false. A separate lesson for expressions like this will be discussed later in the chapter.)

A **simple proposition** conveys a single idea (like the examples above) while a **compound proposition** conveys more than one idea. Compound propositions are formed from existing propositions using logical operators or connectives. A **logical connective** is a word or symbol that joins two sentences or propositions to produce a new one.

Examples of compound propositions:

s: I am old **and** still strong.
t: **Either** you are sick **or** will come to the party tonight.
n: **If** you study hard **then** you will pass the course.
P: Unicorns are **not** real.

The table below shows the different logical connectives and the corresponding key words and symbol used.

Type	Connective (Key Word)	Symbol
Negation	Not	$\sim$
Conjunction	And	$\wedge$
Disjunction	or	$\vee$
Implication or Conditional	If…then…	$\rightarrow$
Biconditional	…if and only if…	$\leftrightarrow$

Among these, the negation (~) is the only one that is a unary operation because it involves only one operand. The rest are binary operations.

Learning check

Activity # 2

Identify the propositions among the given statements. Write P if the statement is a proposition, and Not P if not. Give the truth value of the propositions, otherwise, write "not a proposition".

1. How far is Unitop from NCCC?
2. There is life on Mars.
3. $x - 2 = 2x$, when $x = -2$.
4. You made it!
5. Iran is a country in the Middle East.
6. There are 31 days in December.
7. How are you feeling?
8. Don't speak when your mouth is full.
9. Mixing blue and yellow produces red.
10. 2x + 1 = 5.

C. The Negation

When an original idea is denied, the resulting proposition is called its **negation**. We use the symbol "~" or "¬" to mean "not".

Example 5: Consider the proposition:

f: 3 is a prime number True

Its negation is ~f, and is written as:

~f**:** 3 is not a prime number False

Statement ~f is obtained by negating proposition f, and the truth values of f and ~f are opposite.

Note that negations, like ~f, do not consist two propositions. Nonetheless, they are considered here as compound proposition. The negation is called a unary operation.

Following is the truth value table for negation:

p	**~p**
T	F
F	T

Note: The **truth table** of a proposition P made up of the individual propositions $p_1, \ldots, p_n$ lists all possible combinations of truth values for $p_1, \ldots, p_n$, T denoting true and F denoting false, and for each of such combination lists the truth value of P.

Example 6: Consider the following statements/ propositions:

Let p: 2 is positive. True

Then q: It is not the case that 2 is positive. False

Example 7. Let p: 5 is even. False

Then $\sim p$: It is not the case that 5 is even. True

(or simply $\sim p$: 5 is not even.)

$\sim p$: 5 is odd. (another way of writing the negation of p.)

Example 8: Let s: You failed the subject.

Then $\sim$s: You passed the subject.

Note: Not all negations imply a negative idea.

Try this!

Write the negation of each of the following propositions.

a. Unicorns are real. Unicorns are not real.
b. $7 + 21 > 35$ $7 + 21 \leq 35$
c. I look good today. I do not look good today.

Learning check

Activity # 3

Determine whether each sentence below is a proposition. If the sentence is a proposition, write its negation. Otherwise, write “not a proposition”.

1. 2 + 5 = 19
2. Waiter, will you serve the nuts- I mean, would you serve the guests the nuts?
3. Peel me a mango.
4. Sarah and Matteo make a good couple.
5. This sentence makes sense.

D. The Conjunction

Definition: If two propositions are joined together by the connective **and**, the resulting compound proposition is called a **conjunction**. The propositions composing conjunction are called **conjuncts**.

The symbol used to mean “and” is $\wedge$.

Note: Variables are used to represent or label propositions for easier symbolism. The most common variables used are p, q, and r.

The truth value of the conjunction of two propositions, say *p* and *q,* is defined in the table below:

p	**q**	**p $\wedge$ q**
T	T	T
T	F	F
F	T	F
F	F	F

Based on the table, we remember that a conjunction is true only when both of its conjuncts are true. Otherwise, the conjunction is false.

Following is a list of words that mean the same as “and” in the study of propositions:

but	however	whereas	furthermore
still	while	nevertheless	although
yet	despite	moreover	

Try this!

I. Consider the following propositions:

a: Prices are high.
b: Salaries are low.
c: Expenditure level is high.

Write the following conjunctions in symbolic form, then determine their truth value assuming that a, b, and c, are true, false, and true, respectively

1. Prices are high, still salaries are low.
2. The expenditure level is high despite salaries being low.
3. Prices are high, moreover, the expenditure level is quite high.

Solutions: After writing the conjunctions in symbols, remember the truth value of each conjunct and go back to the truth value table to determine the truth value of the conjunction.

1. $a \wedge b$ False
   (The truth value table says that if at least one conjunct is false, then the conjunction is false. See second row.)

2. $c \wedge b$ False
   (The truth value table says that if at least one conjunct is false, then the conjunction is false. See third row.)

3. $a \wedge c$ True
   (The truth table says that if both conjuncts are true, then the conjunction is true.)

II. Write the **truth table** for the proposition $(p \wedge q) \wedge \sim p$.

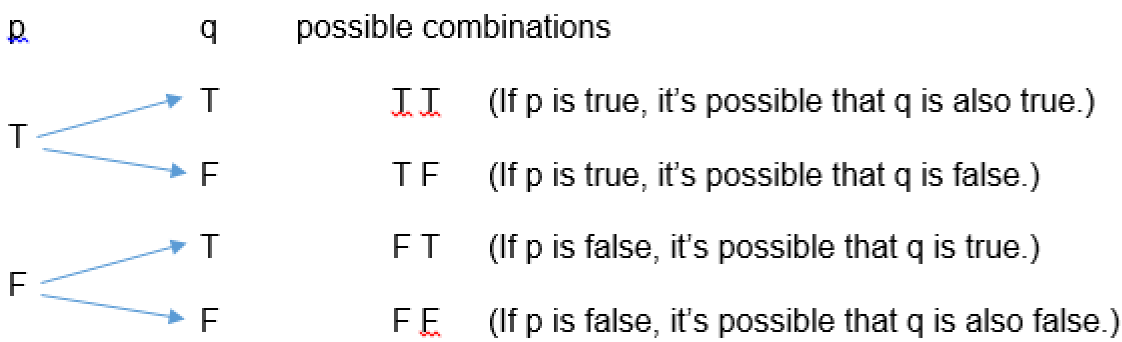
Solution:

Since there are two operands (p and q), there will be $2^2 = 4$ rows in the truth table. If there were three operands (say p, q, and r), the number of rows in the truth table will be $2^3 = 8$. In general, the number of rows of a proposition's truth table is given by 2^n, where n is the number of operands or simple propositions involved.

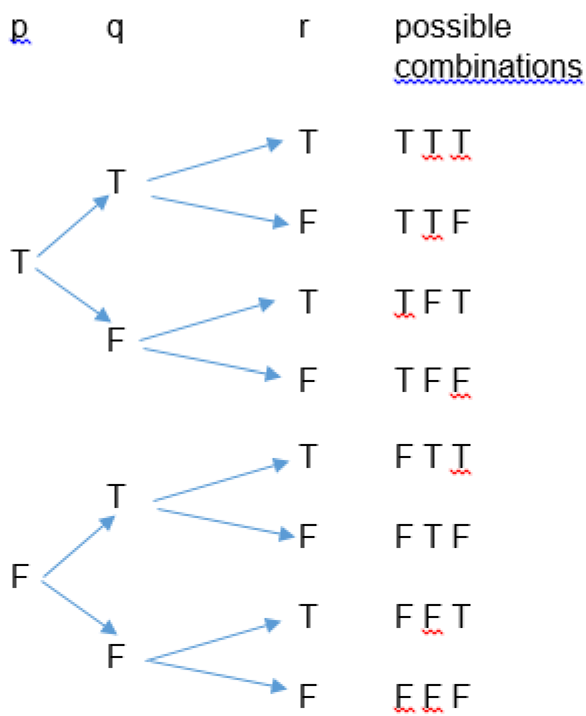
P	Q	$p \wedge q$	$\sim p$	$(p \wedge q) \wedge \sim p$
T	T	T	F	F
T	F	F	F	F
F	T	F	T	F
F	F	F	T	F

Read here to fully understand how the truth table for the compound proposition $(p \wedge q) \wedge \sim p$ is constructed.

- ✓ The entries in the columns of p and q were derived using the tree diagram below:



Had there been three simple propositions involved, the tree diagram will look like this:



- ✓ The entries in the p ∧ q column were derived using the truth table for conjunctions.
- ✓ The entries in the ~p column were derived by taking the opposite truth value of the entries in each row of p.
- ✓ The entries in the (p ∧ q) ∧ ~p column were derived by applying the truth table for conjunctions for the entries in the column of p ∧ q and the column of ~p. So here, we treat (p ∧ q) as a simple proposition, hence the parentheses.

Note: The truth table derived for the problem above resulted to an entry of “all F” in the last column. This happens when the proposition is a fallacy. On the other hand, if the last column led to “all T” entries, then the proposition is a tautology. Moreover, if the last column led to a combination of T and F, then the proposition is a contingency.

Learning check

Activity # 3

Determine the truth values of the following statements. Write True or False. (Hint: You have to know for yourself first the truth value of each conjunct.)

1. Red is a primary color, while orange is a secondary color.
2. 3 and 7 are even, and circles have 4 vertices.
3. $(\sqrt{30} > 5) \wedge (\sqrt{30} < 7)$.
4. Blackpink is a South Korean girl group, but Lalisa Manoban is a Thai.
5. Squares have four equal sides and four right angles.
6. 2 is an even and a composite number.
7. 5 is a rational and a prime number.
8. Negative numbers are integers, and squares are rectangles.
9. $(-3 > -2) \wedge (\sqrt{3} > 2)$.
10. There are five vowels and twenty-one consonants in the English Alphabet.

E. The Disjunction

Definition: A **disjunction** is a compound proposition consisting of two propositions, say, *p* and *q*, that are connected by “or”. Propositions “p” and “q” are called **disjuncts**.

We use the symbol “v” to stand for “or “, hence the disjunction just defined is written as “p v q “.

The truth value of the disjunction of two propositions *p* and *q* is defined in the table below:

p	**Q**	**p v q**
T	T	T
T	F	T
F	T	T
F	F	F

The table indicates that a disjunction is true when at least one of the disjuncts is true and becomes false when both disjuncts are false.

Below is a list of words that express disjunction besides “or”:

else
either … or
unless

Try this!

I. Consider the following propositions:

m: You are good in Mathematics.
n: You are good in English.
o: You like poetry.

Suppose m is true, n is false, and o is true, determine the truth value of each of the following disjunctions.

1. You are good in Math or you like poetry. (m v o)
2. Either you are good in Math or in English. (m v n)

Solutions:

1. True
(The table indicates that a disjunction is true when at least one of the disjuncts is true.)

2. True (The table indicates that a disjunction is true when at least one of the disjuncts is true.)

II. Formulate the symbolic expression $p \vee \sim q$ in words using
p: Lee takes Information Technology.
q: Lee takes mathematics.

Answer: Either Lee takes Information Technology or he does not take mathematics.

III. Given that proposition **p is false**, proposition **q is true**, and proposition **r is false**, determine whether the compound proposition $\sim p \vee \sim (q \wedge r)$ is true or false. (Note: This does not require a truth table.)

Solution: We will replace each proposition by their corresponding truth values and slowly work on performing each operation.

$\sim p \vee \sim (q \wedge r)$ given

$T \vee \sim (F)$	p is false means that $\sim p$ is true; the conjuction of true (q) and false (r) is false
$T \vee T$	the negation of F is T
T	disjunction is true is there is at least one true proposition

Therefore, the truth value of the compound proposition $\sim p \vee \sim (q \wedge r)$ is true.
Note: Some books call compound propositions like this a “statement formula”.

Learning check

Activity # 4

Determine the truth values of the following statements. Write True or False. (Hint: You have to know for yourself first the truth value of each disjunct.)

1. 12 is a prime number or 15 is a prime number.
2. 6 is a multiple of either 2 or 3.
3. $(\sqrt{30} > 5) \vee (\sqrt{30} < 7)$.
4. UP Diliman is in Pasay City or De La Salle University is in Quezon City.
5. It is not true that 1 + 1 = 3 or 2 + 1 = 3.
6. Sugar is sweet or the sun is hot.
7. $\frac{3}{4}$ is an integer or a counting number.
8. Amphibians live either in water or on land.
9. 3 divides 9 or 15.
10. Counting numbers can be prime or even.

F. The Conditional

Definition: If two propositions are connected by the pair “*if … then*” the resulting proposition is called an **implication** or **conditional proposition**. The statement following the word “if” and before “then” is called the **antecedent** or **hypothesis** while the statement after the word “then” is called the **consequent** or **conclusion**.

The symbol used to denote the connective “if … then” is →. Thus, if p is the antecedent and q the conclusion of an implication, the proposition is written a “p → q” and is read, “if p, then q “ or “p implies q”

Following is a list of other expressions that have the same meaning as “if p, then q”:

q if p

q provided p
q given p
q in case p
p only if q
p only when q
p implies q
p is sufficient for q
q whenever p
when p, q
a necessary condition for p is q
a sufficient condition for q is p

Note: Unlike in conjunction and disjunction, where the order of the conjuncts or disjuncts does not affect the truth value of the compound proposition, here in conditional proposition, order matters. It is important that we know which phrase is the antecedent, and which phrase is the consequent because it will affect the truth value of the conditional proposition. It is easiest to determine the antecedent and the consequent when the conditional proposition is expressed in standard form “If p, then q”.

At this point, let us try to translate some conditional propositions into their equivalent “if p, then q” format. Then let’s identify which phrase is the antecedent p and which is the consequent q. Use the list given above and the tips given below to understand how this works.

Tips:

- ✓ The hypothesis is the clause following the word if.
- ✓ The “if p then q” formulation emphasizes the hypothesis, whereas the “p only if q” formulation emphasizes the conclusion; the difference is only stylistic.
- ✓ “When” means the same as “if”.
- ✓ A necessary condition is just that: a condition that is necessary for a particular outcome to be achieved. The conclusion expresses a necessary condition.
- ✓ A sufficient condition is a condition that suffices to guarantee a particular outcome. The hypothesis expresses a sufficient condition.
- ✓ Given the conditional “$p \rightarrow q$” proposition p is referred to as a sufficient condition for q while q is a necessary condition for p.

Example 9:

a. If I get a bonus, then I will buy a car.

Answer: Since the statement is already in the standard form, we just identify p and q.

p: I get a bonus.
q: I will buy a car.

b. Mary will be a good student if she studies hard.

Answer: If Mary studies hard, then she will be a good student.

p: Mary studies hard.

q: Mary will be a good student.

c. John takes calculus only if he has a sophomore, junior, or senior standing.

Answer: If John takes calculus, then he has sophomore, junior, or senior standing.

p. John takes calculus.

q: John has sophomore, junior, or senior standing.

d. When you sing, my ears hurt.

Answer: If you sing, then my ears hurt.

p: You sing.

q: My ears hurt.

e. A necessary condition for PSU Bearcats to win the baseball in STRASUC Meet is that they sign a right- handed relief pitcher.

Answer: If the PSU Bearcats win the baseball in STRASUC Meet, then they signed a right-handed relief pitcher.

p: The PSU Bearcats win the baseball in STRASUC Meet.

q: The PSU Bearcats signed a right- handed relief pitcher.

f. A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

Answer: If Maria goes to the Eiffel Tower, the she visits France.

p:Maria goes to the Eiffel Tower.

q:Maria visits France.

Caution: A common mistake of students in activities like this is including the words “if” and “then”, or other compounding words in writing the component simple propositions. For instance, in Example 5a, some students would write: p: *If* I get a bonus, and q: *Then* I will buy a car. In Example 5f., some students would write: p: A sufficient condition for Maria to visit France, and q: She goes to the Eiffel Tower. These are incorrect. Observe how compounding words are dropped in writing the component simple propositions in our answers. Also, do not forget to “listen” to your proposition if it sounds right, or if it is grammatically correct.

Example 10: Write the following conditionals in symbolic form.

a. If two numbers are even, then their product is even.
b. x + 1 is odd provided x is even.
c. Two nonzero numbers are equal only if their reciprocals are equal.
d. Joey will pass the discrete mathematics exam if he studies hard.

Solutions:

a. Let m: Two numbers are even.
   n: The product of two numbers are even.

   Then, conditional (a) is written as $m \rightarrow n$

b. Suppose t: $x + 1$ is odd
   s: x is even

   Hence, conditional (b) is written $s \rightarrow t$.

c. Let p: Two nonzero numbers are equal.
   q: The reciprocals of two numbers are equal.

   Thus, conditional (c) can be written as $p \rightarrow q$

d. Suppose u: Joey will pass the discrete mathematics exam.
   v: Joey studies hard.

   Then, conditional number (d) is written: $v \rightarrow u$

Following is the truth value table for implication:

p	**q**	**$p \rightarrow q$**
T	T	T
T	F	F
F	T	T
F	F	T

Note: The table tells us that an implication is false only when the antecedent is true and the consequent false. Otherwise, the implication is always true. A conditional proposition that is true because the hypothesis is false is said to be true by default or vacuously true.

Try this!

I. Suppose propositions m, n, and s are assigned the truth values F, T, and F, respectively. Determine the truth values of the following conditionals.

1. $m \rightarrow n$
2. $n \rightarrow s$
3. $s \rightarrow m$

Solutions:

1. Since the antecedent m is false, it follows that regardless of the truth value of the consequent n, the conditional proposition $m \rightarrow n$ is true. In fact, it is true by default.

2. The antecedent n is true, while the consequent s is false. So the conditional proposition $n \rightarrow s$ is false.

3. Since the antecedent s is false, it follows that regardless of the truth value of the consequent m, the conditional proposition $s \rightarrow m$ is true. In fact, it is true by default.

II. Assuming that **p and r are false** and that **q and s are true,** find the truth value of the statement formula: $(p \rightarrow q) \wedge (q \rightarrow r)$

Solution: We will replace each proposition by their corresponding truth values and slowly work on performing each operation.

$(p \rightarrow q) \wedge (q \rightarrow r)$	given
$(F \rightarrow T) \wedge (T \rightarrow F)$	replace each proposition by corresponding truth value
$(T) \wedge (F)$	apply the rules in the truth table for implication
F	apply the rule in the truth table for conjunction

Therefore, the truth value of the statement formula $(p \rightarrow q) \wedge (q \rightarrow r)$ is false.

III. Formulate the symbolic expression $\sim p \rightarrow (q \vee r)$ in words using

p: Today is Monday.
q: It is raining.
r: It is hot.

Answer: If today is not Monday, then it is raining unless it is hot.
(Answers may vary but the logical connectives must have the same meaning.)

Converse, Inverse, and Contrapositive

Definition: Let p and q be propositions.

i. The proposition $q \rightarrow p$ is called the **converse** of the implication $p \rightarrow q$.
ii. The proposition $\sim p \rightarrow \sim q$ is called the **inverse** of the implication of $p \rightarrow q$.
iii. The statement $\sim q \rightarrow \sim p$ is called the **contrapositive** of the implication $p \rightarrow q$.

Example 11: Let p: Today is Sunday.
q: I will go for a walk.

$p \rightarrow q$: If today is Sunday then I will go for a walk.

Converse: $q \rightarrow p$: If I will go for a walk then today is Sunday.
Inverse: $\sim p \rightarrow \sim q$: If today is not Sunday then I will not go for a walk.
Contrapositive: $\sim q \rightarrow \sim p$: If I will not go for a walk then today is not Sunday.

Learning check

Activity # 5

1. Suppose a conditional proposition is true and its antecedent also true, what is the truth value of its consequent?

2. If an implication is true and its consequent is false, what is the truth value of its antecedent?

3. Let the propositions *r* and *s* be true and false, respectively.
   r: Prices are high
   s: Wages are high
   Determine the truth value of the following conditionals.
   a. Prices are high in case wages are high.
   b. Prices are high only if wages are high.
   c. Wages are high; hence inflation rate is high.

4. Write the converse, inverse, and contrapositive of the following conditional propositions. (Hint: If applicable, write each conditional proposition in standard form first.)
   a. Rose may graduate if she has 120 hours of OJT credits.
   b. A necessary condition for Bill to buy a computer is that he obtains P20,000.
   c. A sufficient condition for Katrina to take the algorithms course is that she passes discrete mathematics.
   d. The program is readable only if it is well- structured.

G. The Biconditional

Definition: When two propositions, say p and q, are connected by the phrase "if and only if", the resulting compound proposition is called a **biconditional** or **double implication**. The symbol used for "if and only if" is "$\leftrightarrow$", hence, the biconditional "p if and only if q" is written as "$p \leftrightarrow q$". This compound proposition is the compact way of writing "$(p \rightarrow q)$ and $(q \rightarrow p)$".

Sometimes a biconditional is read as *"p is a necessary and sufficient condition for q"* or "q is a necessary and sufficient condition for p".

Other words that could replace "if and only if" are:

when and only when
just in case
granted that and only granted that

Truth value table for biconditional:

P	**q**	**p ↔ q**
T	T	T
T	F	F
F	T	F
F	F	T

Note: Observe from the table that a biconditional is true provided the proposition written on both sides of the symbol ↔ are both true or both false.

Bonus knowledge: It is traditional in mathematical definitions to use "if" to mean "if and only if".

Example 12: If two sets A and B are disjoint sets then $A \cap B = \emptyset$.
This definition of disjoint sets means that if A and B are disjoint sets then $A \cap B = \emptyset$ and also if $A \cap B = \emptyset$ then A and B are disjoint sets.

Try this!

I. Write in symbols the biconditional "6x = 2 if and only if x = 1/3", then determine its truth value assuming that the proposition "6x = 2" is true while "x = 1/3" is false.

Solution: Let p: 6x = 2
q: x = 1/3
Then the biconditional in symbols is: p ↔ q. And since p and q have different truth values, the biconditional proposition is false.

II. Construct a truth value table for the statement below:

$$(p \rightarrow q) \leftrightarrow ({\sim}q \rightarrow {\sim}p)$$

Solution: Since the statement formula is composed of two operands p and q, the truth table will have 4 rows.

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

- ✓ The entries in the columns of p, q, $p \to q$, $\sim q$, and $\sim p$ are easily derived using our previous knowledge of tree diagram, rules of implication, and rules of negation.
- ✓ The entries on the $\sim q \to \sim p$ column are derived by applying the rules for implication, but this time, the antecedent is $\sim q$ while the consequent is $\sim p$.
- ✓ The entries on the last column $(p \to q) \leftrightarrow (\sim q \to \sim p)$ are derived by applying the rules for biconditional where the component propositions are $(p \to q)$ and $(\sim q \to \sim p)$. That is, we take $(p \to q)$ and $(\sim q \to \sim p)$ as two separate simple propositions.
- ✓ The last column resulted to "all T" entries so the statement formula $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology.

Learning check

Activity # 6

1. Suppose the biconditional $p \leftrightarrow q$ is given a true value and proposition p is known to be false. What is the truth value of q?
2. Suppose the biconditional $m \leftrightarrow n$ is assumed to be false while proposition n is given a true value. What is the truth value of m?

H. Predicate Logic

The logic that we discussed previously is categorized as the statement or propositional logic. We were interested only in the truth or falsity of the statement. The structure of the statement was not taken into account. But there are many justified arguments whose validity cannot be tested within the framework of propositional logic. For example, consider the following statements:

p: 2 is an even integer
q: 5 is an even integer
r: 0 is an even integer

Observe that the three statements have similar patterns. In English usage, the pattern is noun-verb-adjective. Note that the verb and adjective in all statements are the same. The part which changes is the noun. In mathematics, that component of a statement which changes is called a *variable* and we use small letters, like x, to represent such a component. Thus, we obtain:

s: x is an even integer.

A statement involving a variable, like s, is called a *predicate*.

Note: The truth value of a predicate cannot be evaluated unless the variable, x in the above example, assumes a particular value. Thus, a predicate is not a proposition. But when x is replaced by a number, say 2, or 6, or 0, we obtain a proposition like the statements p and q and r above since we can already determine the truth value of each. In particular, p and r are true propositions, while q is a false proposition. The definition below formalizes this note.

Definition:
Let x be a variable and D be a set; P(x) is a sentence. Then P(x) is called a predicate or propositional function with respect to the set D if for each value of x in D, P(x) is a proposition; that is, P(x) is true or false. Moreover, D is called the domain of discourse and x is called a free variable.

* *The domain of discourse specifies the allowable values for x.*
** *A propositional function P, by itself, is neither true nor false. However, for each x in the domain of discourse, P(x) is a proposition and is, therefore, either true or false.*

Example 1: Here are some predicates in math:

1. Algebra:
   Let D be the set of real numbers.
   $x^2 + 1 = 5$
   $x + 2 > 0$
   $\frac{x}{2} + \frac{x - 1}{3} = 4$

2. Trigonometry:
   Let D be the set of angles in quadrant I.
   $\sin x = \frac{1}{2}$

$\text{Sin}^2 x - \text{Sin } x = 0$
2 Cos x = 1

Example 2: Let P(n) denote the sentence “$n^2 + 2n$ is an odd integer”, where the domain of discourse is the set of positive integers, $\mathbb{Z}^+$. Find the truth value of each of the following:

a. P(1)
b. P(3)
c. P(5)
d. P(2)
e. P(4)
f. P(6)

Solutions: We just replace x in the predicate by each indicated value taken from the domain.

a. P(1): $1^2 + 2(1) = 3$, which is an odd integer. So, P(1) is true.
b. P(3): $3^2 + 2(3) = 15$, which is an odd integer. So, P(3) is true.
c. P(5): $5^2 + 2(5) = 35$, which is an odd integer. So, P(5) is true.
d. P(2): $2^2 + 2(2) = 8$, which is an even integer. So, P(2) is false.
e. P(4): $4^2 + 2(4) = 24$, which is an even integer. So, P(4) is false.
f. P(6): $6^2 + 2(6) = 48$, which is an even integer. So, P(6) is false.

We can also have predicates involving two or more variables

Example 3: Let P(x, y) denote the sentence “x equals y+1”, and let the domain be the set of integers, $\mathbb{Z}$. Find the truth value of each of the following.

a. P(2, 1)
b. P(5, 4)
c. P(6, 4)

Solutions: We just replace x and y accordingly in the predicate by each indicated pair of values taken from the domain. In symbols, our predicate is $x = y + 1$, so we have:

a. P(2, 1): $2 = 1 + 1$
$2 = 2$ So, P(2, 1) is true.

b. P(5, 4): $5 = 4 + 1$
$5 = 5$ So, P(5, 4) is true.

c. P(6, 4): $6 = 4 + 1$
$6 \neq 5$ So, P(6, 4) is false.

Try this!

Let P(x) denote the statement “x ≥ 10” and D be the set of integers. What are the truth values of the following?

1. P(9)
2. P(12)
3. P(-10)
4. P(10)
5. P(0)

Solutions: We just replace x in the predicate by each indicated value taken from the domain.

1. 9 ≥ 10 makes a false proposition
2. 12 ≥ 10 makes a true proposition
3. -10 ≥ 10 makes a false proposition
4. 10 ≥ 10 makes a true proposition
5. 0 ≥ 10 makes a false proposition

Learning check

Activity # 8

1. Let Q(x) denote the statement “x is an integer”. What are the truth values of the following?

   a. Q(-1)
   b. Q(0)
   c. $Q\left(\frac{8}{2}\right)$
   d. $Q\left(\sqrt{-4}\right)$
   e. $Q\left(\sqrt{4}\right)$

2. Let P(x, y) denote the statement “xy ≤ 15.” Determine the truth values of the following:

   a. P(5, 3)
   b. P(-3, 5)
   c. P(-3, -5)
   d. P(-3, -6)
   e. P(2, 7)

3. Let $P(n): n^2 + 1 = 5$, and D: {-2, 0, 2}. Determine the truth value of P(n) for each value in the domain.

I. Quantifiers

In addition to predicates, we deal with another term- quantifiers. There are two types of quantifiers, universal and existential. As we have learned earlier, a predicate P(x) is not a proposition until x is substituted by a value in the domain of discourse. But in this in this part of the module, we shall learn that although P(x) is not a proposition, it can be turned into a proposition through a process called quantification.

Consider the following statements:
1. Every man is a liar.
2. There exists a real gentleman.

These statements are called quantified statements. The words "every" and "some" are called quantifiers. Moreover, "every" is a universal quantifier and "some" is an existential quantifier.

Definition: Let P be a propositional function with domain of discourse D. The statement **for every x, P(x)** is said to be a **universally quantified statement**.

The symbol for a universal quantifier is ∀.

Thus the statement for every x, P(x) may be written $\forall x\, P(x)$.

Note:
1. Other universal quantifiers are "all", "any", "everything", "nothing".
2. The statement $\forall x\, P(x)$ is **true** if $P(x)$ is true for every x in D.
   The statement $\forall x\, P(x)$ is **false** if $P(x)$ is false for at least one x in D.

A value x in the domain of discourse that makes $P(x)$ false is called a counterexample to the statement $\forall x\, P(x)$.

Example 4:
1. Let $P(x): x^2 \geq x$ and let D: the set of integers.
   What is the truth value of $\forall x: P(x)$?

   The universally quantified proposition $\forall x: P(x)$ means that "For all integers x, $x^2 \geq x$."

Solution: To determine the truth value of this proposition, try to answer this question: "Is it true that the square of an integer is **always** greater than or equal to itself?" If the answer is yes, then the truth value of the proposition is true. But if you can think of at least one value of x that will make the proposition false, then the universally quantified proposition is false.

Some of you might be thinking of substituting a few values to the predicate. That's a good start in finding a counterexample to prove the falsity of this quantified statement. But doing the same thing to prove that this quantified statement is true is useless, actually impossible. Because you will need to substitute ALL values in the domain and get ALL TRUE truth values before you can say that this quantified statement is indeed true. So we advise students to practice logical thinking and reasoning in answering questions like this. For instance, here, since the domain is the set of integers, we know that the square of a positive integer is always greater than the integer, itself. So, the quantified proposition is true for positive integers. What about with negative integers? Well, we know that when you square a negative number, you get a positive number. So the square of a negative integer is also always greater than the negative integer, itself. And what about zero? Here comes the part of the symbol "or equal to" in $\geq$. So, with this, we can say that this quantified proposition "For all integers x, $\mathrm{x}^2 \geq \mathrm{x}$" is true. (Remember, we had to justify its truth to ALL values in the domain!)

Example 5: Let $\mathrm{P(x)}: \mathrm{x} \geq 3$ and let D: the set of real numbers.

What is the truth value of $\forall \mathrm{x}: \mathrm{P(x)}$?

The universally quantified proposition $\forall \mathrm{x}: \mathrm{P(x)}$ means that "For all real numbers x, $\mathrm{x} \geq 3$."

Solution: To determine the truth value of this proposition, try to answer this question: "Is it true that a real number is **always** greater than or equal to 3?" If you can think of at least one value of x (in this case a real number) that will make the proposition false, then the universally quantified proposition is false. We call such value a counterexample. This problem is relatively easy because we can think of a lot of counterexamples. There are many real numbers that are not greater than or equal to 3. But one counterexample is enough to justify the falsity of this quantified statement. We can have counterexample: $x = 0$. Zero (0) is a real number and it is not greater than or equal to three, so the quantified statement "For all real numbers x, $\mathrm{x} \geq 3$" is false.

Note: It is easier to prove the falsity of a universally quantified proposition than its truth, because you would only need one counterexample.

Definition: Let P be a propositional function with domain of discourse D. The statement **there exists x, P(x)** is said to be an **existentially quantified statement**.

The symbol for an existential quantifier is ∃.

Thus the statement there exists x, P(x) may be written $\exists x\, P(x)$.

Note:
1. Other existential quantifiers are "some", "there is".
2. The statement $\exists x\, P(x)$ is **true** if $P(x)$ is true for at least one x in D.
   The statement $\exists x\, P(x)$ is **false** if $P(x)$ is false for every x in D.

Example 6:
1. Let $P(x): x^2 \leq x$ and let D: the set of all real numbers.
   What is the truth value of $\exists x: P(x)$?

   The existentially quantified proposition $\exists x: P(x)$ means that "For some real numbers x, $x^2 \leq x$."

Solution: To determine the truth value of this proposition, try to answer this question: "Is it true that **there is** at least one real number whose square is greater than or equal to itself?" If the answer is yes, then the truth value of the proposition is true. But if you cannot think of at least one value of x that will make the proposition true, then this existentially quantified proposition is false.

Here, our domain is the set of real numbers, not just the set of integers. If it were just the set of integers, we can take from our example 4 that this proposition is false. (Can you tell why?). However, here, we can substitute fractions so by just trying out, say $x = \frac{1}{2}$, we will have:

$$\left(\frac{1}{2}\right)^2 \leq \frac{1}{2}$$

$$\frac{1}{4} \leq \frac{1}{2} \qquad \text{(true)}$$

With this, we say that the existentially quantified statement "For some real numbers x, $x^2 \leq x$" is true.

Note: It is easier to show that an existentially quantified proposition is true, rather than false, because you would only need one example that shows it is true.

Example 7: Let $P(x): x^2 < x$ and let D: the set of all integers.

What is the truth value of $\exists x: P(x)?$

The existentially quantified proposition $\exists x: P(x)$ means that "For some integers x, $x^2 < x$."

Solution: To determine the truth value of this proposition, try to answer this question: "Is it true that **there is** at least one integer whose square is less than itself?" If the answer is yes, then the truth value of the proposition is true. But if you cannot think of at least one value of x that will make the proposition true, then the existentially quantified proposition is false.

From example 4, we learned that in fact, the square of all integers are always greater than or equal to the integer, itself. It means that the square of any integer will NEVER be less than the integer itself. So this existentially quantified proposition "For some integers x, $x^2 < x$" is false.

✓ *Things to Remember!*

1. To prove that the universally quantified statement $\forall x\, P(x)$ is **true**, show that for every x in the domain of discourse, the proposition $P(x)$ is true. Showing that $P(x)$ is true for **only** a particular value of x **does not** prove that $\forall x\, P(x)$ is true.
2. To prove that the existentially quantified statement $\exists x\, P(x)$ is **true**, find one value of x in the domain of discourse for which the proposition $P(x)$ is true. **One value suffices** (is enough).
3. To prove that the universally quantified statement $\forall x\, P(x)$ is **false**, find one value of x (a counterexample) in the domain of discourse for which the proposition $P(x)$ is false. **One counterexample is enough.**
4. To prove that the existentially quantified statement $\exists x\, P(x)$ is **false**, show that for every x in the domain of discourse, the proposition $P(x)$ is false. Showing that $P(x)$ is false for **only** a particular value of x **does not prove** that $\exists x\, P(x)$ is false.

This time, let us try to translate quantified statements to symbols.mConsider the following statements:

1. All numbers are integers.
2. Some numbers are integers.

The word "all" in (1) can be symbolized by "∀" and if the noun "numbers" is replaced by variable s, we can rewrite (1) as

$$\forall x : x \text{ is an integer} \qquad (2)$$

Remember that the phrase "x is an integer" is the predicate and if this predicate is symbolized as I(x), then (2) becomes

$$\forall x : I(x) \qquad (3)$$

The symbols in last line is read as "For all x, x is an integer" and this is still synonymous to statement (1).

For the 2nd statement the word "some" can be represented by the symbol "∃". Following the pattern in statement (1), statement (2) can finally be written as,

$$\exists x : I(x) \qquad (4)$$

This is read as "For some x, x is an integer."

Try this!

1. Evaluate the truth value of each quantified proposition, assuming the set {0, 2, - 2} as the domain.

   a. $\forall x : x(x^2 - 4) = 0$
   b. $\exists x : x + 1 > 0$
   c. $\exists x : x(x^2 - 4) = 0$

Solutions: Since the domain is composed by only three elements, it is still tolerable to plug in each of these values, if necessary, to the quantified propositions and decide its truth value.

a. $P(0): 0(0^2 - 4) = 0$ true
   $P(2): 2(2^2 - 4) = 0$ true
   $P(-2): -2[(-2)^2 - 4] = 0$ true

Since all values in the domain yielded a true truth value, we can say that this universally quantified proposition is true.

b. $P(0): 0 + 1 > 0$ true

Since this is an existential quantifier, just one proof of truth of the statement is enough to say that this existentially quantified proposition is true.

c. $P(0): 0(0^2 - 4) = 0$ true

Since this is an existential quantifier, just one proof of truth of the statement is enough to say that this existentially quantified proposition is true.

2. Let $P(n)$ be the propositional function "n divides 77". Write each proposition below in words and tell whether it is true or false. The domain of discourse is $\mathbb{Z}^+$.
   a. $P(11)$
   b. $\forall n\, P(n)$

c. $\exists n\, P(n)$

Solutions:

a. We replace n by 11 and write:
"11 divide 77" , which is a true statement.

b. This is a universally quantified proposition.
"For all positive integers n, n divides 77" or simpy, "All positive integers divide 77".

Counterexample: 8 (this may vary), since 8 is a positive integer which does not divide 77.So the universally quantified proposition "All positive integers divide 77" is false.

c. This is an existentially quantified proposition.
"For some positive integers n, there exists n that divides 77" or simpy, "There exists at least one positive integer that divides 77".

Proof: 11 (this may vary), since 11 is a positive integer which divides 77. So the existentially quantified proposition "There exists at least one positive integer that divides 77" is true.

Learning check

Activity # 9

1. Translate the following in symbolic form and state their domain.
   a. All fishes are swimmers.
   b. There are fruits that are sweet.
   c. Every number is a real number.
   d. Any mammal is not two-legged.

2. Evaluate the truth value of the following propositions, using the set {1, 3, 5, 7} as the domain.

   a. $\forall x : x^2$ is odd
   b. $\exists x : (x + 1)^2 > x^2 + 1$
   c. $\exists x : 2(x - 1)$ is even
   d. $\forall x : \frac{2x+1}{3}$ belongs to the domain

EVALUATION

Let us now determine how much you understand from this module!

1. Let the universal set U = {x | x ∈ natural numbers, $0 \leq x \leq 9$}, A = {2, 4, 7, 9}, B = {1, 3, 5, 7, 9}, C = {2, 3, 4, 5}, D = {1, 6, 7}. Find the ff:

   a. $A' \cup C$
   b. $(B \cap C') \cup A$
   c. $(U \cap B')'$
   d. $A \cap C \cap D'$

2. Among 500 graduating students, 210 of them smoke, 258 drink alcoholic beverages, 216 eat between meals, 122 smoke and drink alcoholic beverages, 83 eat between meals and drink alcoholic beverages, 97 smoke and eat between meals, and 52 engage in these bad health practices. How many

   a. smoke but do not drink alcoholic beverages?
   b. eat between meals and drink alcoholic beverages but do not smoke?
   c. neither smoke nor eats between meals?

3. Let A = {w, x, y, z} and B = {a, b}. Use the roster notation to write each of the following sets, and indicate the number of elements that are in each set:

   a. A x B
   b. B x A
   c. A x A
   d. B x B

4. In a Music club with 15 members, 7 people played piano, 6 people played guitar, and 4 people did not play either of these instruments.

   a. How many people played both piano and guitar?
   b. How many people played piano only?
   c. How many people played guitar only?
   d. How many people did play piano or guitar?

5. Write each symbolic statement as an English sentence. Use p, q, r, s, and t as defined below.

   p: Taylor Swift is a singer.

q: Taylor Swift is not a songwriter.
r: Taylor Swift is an actress.
s: Taylor Swift plays the piano.
t: Taylor Swift does not play the guitar.

a. $(p \vee r) \wedge q$
b. $p \rightarrow (q \wedge {\sim}r)$
c. $t \leftrightarrow ({\sim}r \wedge {\sim}p)$

6. Write the following compound statements in symbolic form.
   a. Today is Friday, still it is raining.
   b. It is not raining, yet I am going to a movie.
   c. I am going to the basketball game or I am going to a movie.
   d. If it is raining, then I am going to the basketball game.

I. Fill in the blanks with appropriate word, phrase, or symbol(s).
1. A sentence that can be judged either true or false is called ___________.
2. A statement that conveys only one idea is called _________.
3. A statement that consists of two or more simple statements is called a ___________.
4. Words such as all, none (or no) and some are examples of _____________.
5. The negation is symbolized by ~ and is read as ___________.
6. The conjunction is symbolized by ∧ and is read as ___________.
7. the disjunction is symbolized by ∨ and is read as __________.
8. The conditional is symbolized by → and is read as __________.
9. The biconditional is symbolized by ↔ and is read as _________.
10. In logic, words such as and, or, and if…then… are called ____________.

II. Write the negation of each statement.
1. Some telephones can take photographs.

   ____
2. All houses have two stories.

   ____
3. Some cars are hybrids.

   ____
4. All golf courses are green.

   ____
5. Some drivers are not safe.

   ____

III. Indicate whether the statement is a simple or a compound statement. If it is a compound statement, indicate whether it is negation, conjunction, disjunction, conditional, or biconditional by using both the word and its appropriate symbol.

Examples: John likes scuba diving. (simple statement)
Greg will go to the circus or to the zoo. (compound, disjunction, $\vee$)

1. If you have cold, then you should eat some chicken soup.
2. Time will go backwards if and only if you travel faster than the speed of light.
3. Louis Armstrong did not play the drums.
4. Bobby joined the Army, and he got married.
5. It is false that Mr. Cruz is a high school teacher and a grade school teacher.
6. The typhoon did million worth of damage to the province.
7. A judge announces to a convicted offender, “I hereby sentence you to five months of community services or a fine of Php10,000”.

ANSWER KEY

A. Set Operations

Try this! ★
1. A ∪ B = {nausea, night sweats, nervousness, dry mouth, swollen feet, weight gain, blurry vision, fever, trouble sleeping}
2. A ∪ C = {nausea, night sweats, nervousness, dry mouth, swollen feet, blurry vision, weight loss, eczema}
3. A ∪ B ∪ C = {nausea, night sweats, nervousness, dry mouth, swollen feet, weight gain, blurry vision, fever, trouble sleeping, weight loss, eczema}

Try this! ★★
1. A ∩ B = {nausea, nervousness}
2. B ∩ C = {nausea, blurry vision}
3. A ∩ B ∩ C = {nausea}

Try this! ★★★
1. A' ∩ C' = {weight gain, trouble sleeping, motor mouth, darting eyes, uncontrollable falling down}
2. (A ∩ B)' ∩ C = {dry mouth, blurry vision, fever, weight loss, eczema}
3. B' ∪ (A ∩ C') = {night sweats, dry mouth, swollen feet, weight loss, eczema, motor mouth, darting eyes, uncontrollable falling down, nervousness}

Try this! ★★★★
1. A – B = {night sweats, dry mouth, swollen feet}
2. B – C = {weight gain, nervousness, fever, trouble sleeping}
3. (A – B) – C = {night sweats, swollen feet}

Try this! ★★★★★
1. Y x Z = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}
2. Z x Y = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}
3. Y x Y = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)}
4. Y x Z has 6 elements, Z x Y has 6 elements, and Y x Y has 9 elements.

B. Propositions

Try this!
1. not a proposition
2. proposition, true

3. proposition, false
4. not a proposition
5. not a proposition

C. The Conjunction

Try this!
1. a ∧ b, false
2. c ∧ b, false
3. a ∧ c, true

D. The Disjunction

Try this!
1. true
2. true

E. The Conditional

Try this!
1. true
2. true

F. The Biconditional

Try this!
Let p: 6x = 2 and q: x = 1/3.
p ↔ q is false.

G. The Negation

p	q	~p	~q	p → q	~q →~p	(p → q) ↔ (~q →~p)
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

H. Predicate Logic

Try this!
1. false
2. true
3. false
4. true
5. false

I. Quantifiers

Try this!
1. true
2. true
3. true

Try this!

Evaluate the truth value of each quantified proposition, assuming the set {0, 2, - 2} as the domain.

a. $\forall x : x(x^2 - 4) = 0$
b. $\exists x : x + 1 > 0$
c. $\exists x : x(x^2 - 4) = 0$

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