8.012 Study Notes

An Introduction To Mechanics 2nd. Edition, Kleppner & Kolenkow

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Chapter 1: Vectors and Kinematics

1.2: Vectors

Vectors are represented using *vector notation.* Vectors appear in **boldface**.

$$\mathbf{F} = m\mathbf{a}$$

Here, both $\mathbf{F}$ and $\mathbf{a}$ represent vector quantities. Occasionally, vectors are represented with an arrow, e.g., $\overrightarrow{A}$. Vectors have both a *length* and *direction.* For two vectors to be equal, they must have identical length and direction.
The magnitude or size of a vector $\mathbf{A}$ is written $|\mathbf{A}|$, or simply A. If the length of a vector is one unit, we call it a *unit vector*, and label it with a caret. The vector of unit length parallel to $\mathbf{A}$ is $\hat{\mathbf{A}}$. We have that

$$\hat{\mathbf{A}} = \frac{\mathbf{A}}{A}$$

and

$$\mathbf{A} = A\hat{\mathbf{A}}.$$

1.3: The Algebra of Vectors

1.3.1: Multiplying a Vector by a Scalar

Multiplying a vector $\mathbf{A}$ by a scalar b results in a new vector $\mathbf{C} = b\mathbf{A}$. If $b > 0$, then $\mathbf{C}$ is parallel to $\mathbf{A}$, and its magnitude is b times greater. thus $\hat{\mathbf{C}} = \hat{\mathbf{A}}$ and $C = bA$. If $mathitb < 0$, then $\mathbf{C}$ is opposite in direction to $\mathbf{A}$, and its magnitude is $C = |b|\, A$.

1.3.4: Algebraic Properties of Vectors

Commutative law:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Associative law:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{B} + \mathbf{A}) + \mathbf{C}$$

$$c\,(d\mathbf{A}) = (cd)\,\mathbf{A}$$

Distributive law:

$$c\,(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$

$$(c + d)\,\mathbf{A} = c\mathbf{A} + d\mathbf{A}$$

1.4: Multiplying Vectors

1.4.1: Scalar Product ("Dot Product")

This type of multiplcation is called the *scalar* product because its result is a scalar. It is written, e.g., $\mathbf{A} \cdot \mathbf{B}$. It is defined as

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos\theta$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ when drawn tail to tail. Since $B\cos\theta$ is the projection of $\mathbf{B}$ onto $\mathbf{A}$, it follows that

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A \text{ times the projection of } \mathbf{B} \text{ onto } \mathbf{A} \\ &= B \text{ times the projection of } \mathbf{A} \text{ onto } \mathbf{B} \end{aligned}$$

Note that $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$, and $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. If $\mathbf{A}$ or $\mathbf{B}$ is zero, so is their dot product. Since $\cos\pi/2 = 0$, the dot product of two nonzero vectors can also happen to be zero, if the vectors are perpindicular.

1.4.2: Vector Product ("Cross Product")

This product is called the *vector* product, since two vectors $\mathbf{A}$ and $\mathbf{B}$ are combined to form a third vector $\mathbf{C}$. The symbol for vector product is a cross, lending the name *cross* product:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

The magnitude of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is defined as:

$$C = AB \sin\theta$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ when they are drawn tail to tail. To eliminate any ambiguity, θ is always chosen to be the angle smaller than π. Even if neither vector is zero, their cross product is zero if $\theta = 0$ or π; when the vectors are parallel or antiparallel. It follows that for any vector $\mathbf{A}$,

$$\mathbf{A} \times \mathbf{A} = 0.$$

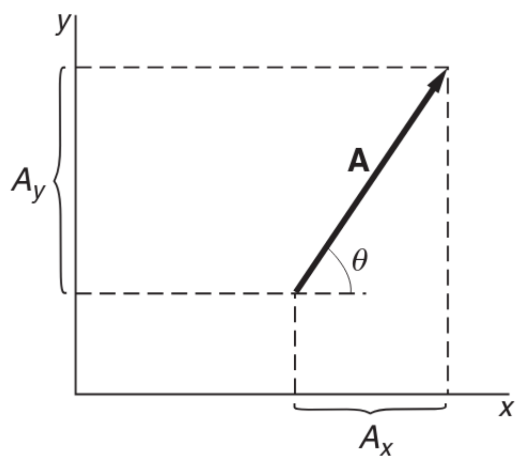
Two vectors **A** and **B** define a unique plane. The direction of **C** is defined to be perpindicular to this plane. **A**, **B**, and **C** form a *right-hand triple.* If **A** lies on the x axis and **B** lies toward the y axis, then **C** lies towards the positive z axis. This definition leads towards the following consequence:

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$$

The cross product is not commutative, but *anticommutative.*

1.5: Components of a Vector

We can uniquely define a vector in terms of its projections along axes of a coordinate system. Consider the vector **A** in the diagram below.



We have that the magnitude of **A** is $\sqrt{A_x^2 + A_y^2}$, and the direction of **A** makes $\theta = \arctan A_y/A_x$. Thus,

$$\mathbf{A} = \left(A_x, A_y\right).$$

Or in three dimensions:

$$\mathbf{A} = \left(A_x, A_y, A_z\right).$$

If two vectors are equal, then their corresponding componenets are equal. Multiplication by a scalar is written

$$c\mathbf{A} = \left(cA_x, cA_y, cA_z\right).$$

Vector addition is written

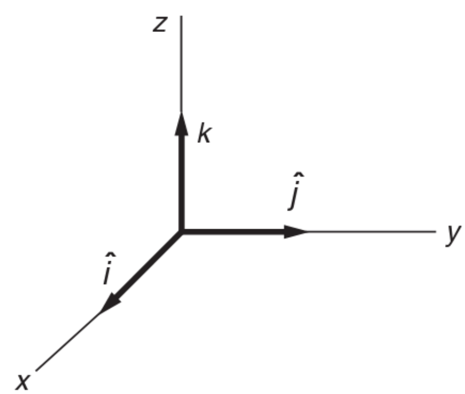
$$\mathbf{A} + \mathbf{B} = \left(A_x + B_x, A_y + B_y, A_z + B_z\right).$$

By writing **A** and **B** as sums of vectors along each of the coordinate axes, it follows that

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z.$$

1.6: Base Vectors

Base vectors are a set of orthogonal unit vectors, one for each dimension. In the Cartesian coordinate system of three dimensions, we have unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, for axes x, y, and z, respectively.



We have the following properties:

$$\begin{aligned}
\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \\
\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0 \\
\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \\
\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}} \\
\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}
\end{aligned}$$

Any vector **A** can be written in terms of its components:

$$\mathbf{A} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$$

To find the component of a vector in any direction, take the dot product with a unit vector in that dimension.
The cross product of two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ is

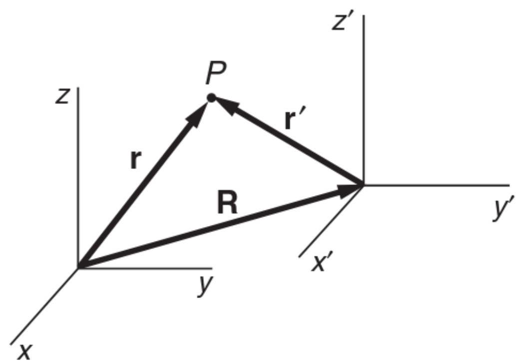
$$\mathbf{A} \times \mathbf{B} = \left(A_y B_z - A_z B_y\right) \hat{\mathbf{i}} - \left(A_x B_z - A_z B_x\right) \hat{\mathbf{j}} + \left(A_x B_y - A_y B_x\right) \hat{\mathbf{k}}$$

1.7: The Position Vector r and Displacement

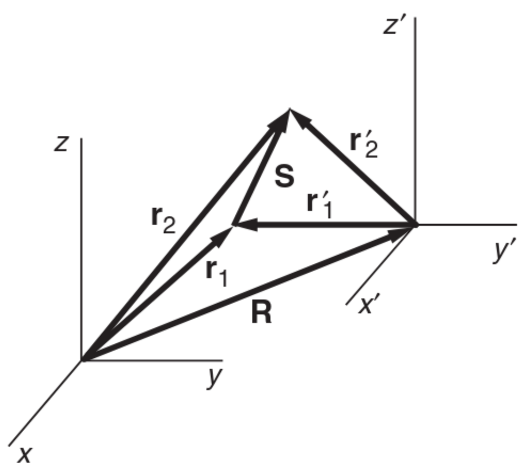
Consider an arbitrary point P at coordinates (x, y, z). Its position is written as

$$\mathbf{r} = (x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

If we move the point from x_1, y_1, z_1 to x_2, y_2, z_2, then the *displacement* defines a vector $\mathbf{S}$ with $S_x = x_2 - x_1$, $S_y = y_2 - y_1$, and $S_z = z_2 - z_1$. $\mathbf{S}$ points from the original position to the final position, and only encodes the *relative* position of each. Thus $\mathbf{S}$ is a *true* vector: the values of the coordinates of its initial and final points depend on the coordinate system, but $\mathbf{S}$ itself does not.



Consider the above figure. Vectors $\mathbf{r}$ and $\mathbf{r}'$ denote the position of the same point in space, but drawn in different coordinate systems. $\mathbf{R}$ is the vector from the origin of the unprimed system to the origin of the primed coordinate system. We thus have $\mathbf{r} = \mathbf{R} + \mathbf{r}'$, or equally, $\mathbf{r}' = \mathbf{r} - \mathbf{R}$.



$\mathbf{S}$ above is a displacement vector. We show that it's independent of coordinate system chosen to represent it.

$$
\begin{aligned}
\mathbf{S} &= \mathbf{r}_2 - \mathbf{r}_1 \\
&= \left(\mathbf{R} + \mathbf{r}'_2\right) - \left(\mathbf{R} + \mathbf{r}'_1\right) \\
&= \mathbf{r}'_2 - \mathbf{r}'_1
\end{aligned}
$$

1.8: Velocity and Acceleration

1.8.1: Motion in One Dimension

Let x be the value of the coordinate of a particle moving on a line. We assume a continuous record of position versus time. The *average velocity* $\bar{v}$ of the point between time t_1 and t_2 is defined by

$$\bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}.$$

The *instantaneous velocity* v is the limit of the average velocity as the time interval approached zero.

$$v = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

In the notation of Gottfried Leibniz, we write:

$$v = \frac{dx}{dt}$$

Newton would have written

$$v = \dot{x}$$

where the dot stands for d/dt. Newton's notation will only be used for derivatives with respect to time. The derivative of a function $f(x)$ can be written $f'(x) \equiv df(x)/dx$. Simililarly, the *instantaneous acceleration* a is

$$
\begin{aligned}
a &= \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \\
&= \frac{dv}{dt} = \dot{v}.
\end{aligned}
$$

Using $v = dx/dt$,

$$a = \frac{d^2 x}{dt^2} = \ddot{x}.$$

The concept of speed may be useful. Speed s is the magnitude of velocity: $s = |\mathbf{v}|$.

1.8.2: Motion in Several Dimensions

Consider a particle moving in the $x - y$ plane. We know the particle's coordinate at every value of time. The instantaneous position of the particle at some time t_1 is

$$\mathbf{r}(t_1) = \big(x(t_1), y(t_1)\big)$$

or

$$\mathbf{r}(t_1) = (x_1, y_1)$$

The displacement of the particle between t_1 and t_2 is

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = (x_2 - x_1, y_2 - y_1)$$

The displacement of the particle during the interval Δt is

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t).$$

The above equation is equivalent to the two scalar equations

$$\begin{aligned} \Delta x &= x(t + \Delta t) - x(t) \\ \Delta y &= y(t + \Delta t) - y(t) \end{aligned}$$

The *velocity* $\mathbf{v}$ of the particle as it moves is

$$\begin{aligned} \mathbf{v} &= \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} \\ &= \frac{d\mathbf{r}}{dt} \end{aligned}$$

which is equiavalent to the two scalar equiations

$$\begin{aligned} v_x &= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \\ v_y &= \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt} \end{aligned}$$

Adding a third dimension, we tack on

$$v_z = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$$

Another approach to calculating velocity is to start with $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and differentiate.

$$\frac{d\mathbf{r}}{dt} = \frac{d(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{dt}$$

Since the base vectors are constant in time, we treat them as contants when we differentiate.

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

Acceleration $\mathbf{a}$ then, is

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}} \\ &= \frac{d^2\mathbf{r}}{dt} \end{aligned}$$

Let a particle undergo displacement $\Delta\mathbf{r}$ in time Δt. As $\Delta t \to 0$, $\Delta\mathbf{r}$ becomes tangent to trajectory.

$$\begin{aligned} \Delta\mathbf{r} &\approx \frac{d\mathbf{r}}{dt}\Delta t \\ &= \mathbf{v}\Delta t \end{aligned}$$

becomes exact in the limit $\Delta t \to 0$. Thus the instantaneous velocity is at every point tangent to the trajectory.

1.9 Formal Solution of Kinematical Equations

If acceleration is a known function of time, velocity can be found as

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{a}(t)$$

by integration with respect to time. Component-wise, this is

$$\frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$$

We can split this into individual components, e.g.,

$$\frac{dv_x}{dt} = a_x$$

If we know the velocity at some time t_0, we can integrate with respect to time to find the velocity at a later time t_1.

$$\int_{v_0}^{v_1} dv_x = \int_{t0}^{t_1} a_x dt$$
$$v_x(t_1) - v_x(t_0) = \int_{t_0}^{t_1} a_x(t) dt$$

then

$$v_x(t_1) = v_x(t_0) + \int_{t_0}^{t_1} a_x(t) dt$$

Taking all components into account, we have

$$\mathbf{v}(t_1) = \mathbf{v}(t_0) + \int_{t_0}^{t_1} \mathbf{a}(t) dt.$$

The velocity at an arbitrary time t is then

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_{t_0}^{t} \mathbf{a}(t') dt'.$$

Position is the same. If

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t)$$

then

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(t') dt'.$$

One important case is *uniform acceleration*. If $\mathbf{a} =$ constant and $t_0 = 0$, we have

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t$$
$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{0}^{t} (\mathbf{v}_0 + \mathbf{a}t') dt'.$$

or...

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2.$$

1.10: More about the Time Derivative of a Vector

Consider a vector $\mathbf{A}(t)$ that changes with time. The change in $\mathbf{A}(t)$ over interval t to $t+\Delta t$ is

$$\Delta\mathbf{A} = \mathbf{A}(t+\Delta t) - \mathbf{A}(t)$$

We define the time derivative of $\mathbf{A}$ by

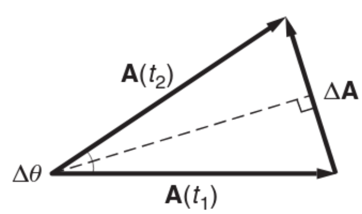
$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t\to 0} \frac{\mathbf{A}(t+\Delta t) - \mathbf{A}(t)}{\Delta t}$$

1.11: Rotating Vectors

If $d\mathbf{A}/dt$ is always perpindicular to $\mathbf{A}$, $\mathbf{A}$ must *rotate*. This cannot change its magnitude, since no component of the derivative is parallel to $\mathbf{A}$.
Suppose a vector $\mathbf{A}(t)$ has constant magnitude A, and the only way it can change in time is by rotating. The direction of $d\mathbf{A}/dt$ is always perpindicular to $\mathbf{A}$. The magnitude of $d\mathbf{A}/dt$ can be found by the following argument.
The change in $\mathbf{A}$ in the interbal t to $t+\Delta t$ is

$$\Delta\mathbf{A} = \mathbf{A}(t+\Delta t) - \mathbf{A}(t)$$

Consider



Then, using the angle $\Delta\theta$ from the image,

$$|\Delta\mathbf{A}| = 2A\sin\frac{\Delta\theta}{2}$$

For $\Delta\theta \ll 1$, $\sin\Delta\theta/2 \approx \Delta\theta/2$. We have then

$$|\Delta\mathbf{A}| = A\Delta\theta$$

and thus

$$\left|\frac{\Delta\mathbf{A}}{\Delta t}\right| \approx A\frac{\Delta\theta}{\Delta t}$$

Taking the limit as $\Delta t \to 0$,

$$\left|\frac{d\mathbf{A}}{dt}\right| = A\frac{d\theta}{dt}$$

$d\theta/dt$ is called the *angular speed* of $\mathbf{A}$.