[letterpaper]article
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Inverse Kinematic Solution for SkoposKt

Friday 29th November, 2024

1 Introduction

This report describes closed form inverse kinematics solutions for SkoposKt. The solution was generated by the IK-BT package from the University of Washington Biorobotics Lab. The IK-BT package is described in `https://arxiv.org/abs/1711.05412`. IK-BT derives your equations using `Python 3.8` and the `sympy 1.9` module for symbolic mathematics.

2 Kinematic Parameters

The kinematic parameters for this robot are

$$[\alpha_{i-1}, \quad a_{i-1}, \quad d_i, \quad \theta_i]$$

$$\begin{bmatrix} \frac{\pi}{2} & a_1 & l_1 & th_1 \\ 0 & a_2 & 0 & th_2 \\ \frac{\pi}{2} & 0 & 0 & th_3 \\ -\frac{\pi}{2} & 0 & l_4 & th_4 \\ 0 & a_5 & 0 & th_5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

3 Forward Kinematic Equations

The forward kinematic equations for this robot are:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & Px \\ r_{21} & r_{22} & r_{23} & Py \\ r_{31} & r_{32} & r_{33} & Pz \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$\text{Column} \quad 1 \begin{bmatrix} c_5 \left(c_3 c_4 \left(c_1 c_2 - s_1 s_2\right) - s_4 \left(c_1 s_2 + c_2 s_1\right)\right) + s_5 \left(-c_3 s_4 \left(c_1 c_2 - s_1 s_2\right) - c_4 \left(c_1 s_2 + c_2 s_1\right)\right) \\ -c_4 c_5 s_3 + s_3 s_4 s_5 \\ c_5 \left(c_3 c_4 \left(c_1 s_2 + c_2 s_1\right) - s_4 \left(-c_1 c_2 + s_1 s_2\right)\right) + s_5 \left(-c_3 s_4 \left(c_1 s_2 + c_2 s_1\right) - c_4 \left(-c_1 c_2 + s_1 s_2\right)\right) \\ 0 \end{bmatrix}$$

$$\text{Column} \quad 2 \begin{bmatrix} c_5 \left(-c_3 s_4 \left(c_1 c_2 - s_1 s_2\right) - c_4 \left(c_1 s_2 + c_2 s_1\right)\right) - s_5 \left(c_3 c_4 \left(c_1 c_2 - s_1 s_2\right) - s_4 \left(c_1 s_2 + c_2 s_1\right)\right) \\ c_4 s_3 s_5 + c_5 s_3 s_4 \\ c_5 \left(-c_3 s_4 \left(c_1 s_2 + c_2 s_1\right) - c_4 \left(-c_1 c_2 + s_1 s_2\right)\right) - s_5 \left(c_3 c_4 \left(c_1 s_2 + c_2 s_1\right) - s_4 \left(-c_1 c_2 + s_1 s_2\right)\right) \\ 0 \end{bmatrix}$$

$$\text{Column} \quad 3 \begin{bmatrix} -s_3 \left(c_1 c_2 - s_1 s_2\right) \\ -c_3 \\ -s_3 \left(c_1 s_2 + c_2 s_1\right) \\ 0 \end{bmatrix}$$

$$\text{Column} \quad 4 \begin{bmatrix} a_1 + a_2 c_1 + a_5 \left(c_3 c_4 \left(c_1 c_2 - s_1 s_2\right) - s_4 \left(c_1 s_2 + c_2 s_1\right)\right) - l_4 s_3 \left(c_1 c_2 - s_1 s_2\right) \\ -a_5 c_4 s_3 - c_3 l_4 - l_1 \\ a_2 s_1 + a_5 \left(c_3 c_4 \left(c_1 s_2 + c_2 s_1\right) - s_4 \left(-c_1 c_2 + s_1 s_2\right)\right) - l_4 s_3 \left(c_1 s_2 + c_2 s_1\right) \\ 1 \end{bmatrix}$$

Note: column numbers use math notation rather than python indeces.

4 Unknown Variables:

The unknown variables for this robot are (in solution order):

1. θ_3
2. θ_4
3. θ_5
4. θ_{12}
5. θ_{45}
6. θ_1
7. θ_2

5 Solutions in Generic Form

The following equations comprise solutions for each unknown.

5.1 θ_3

Solution Method: arccos

$$\theta_{3s1} = \operatorname{acos}\left(-r_{23}\right) \tag{3}$$
$$\theta_{3s2} = -\operatorname{acos}\left(-r_{23}\right) \tag{4}$$

5.2 θ_4

Solution Method: arccos

$$\theta_{4s1} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos(\theta_3)}{a_5\sin(\theta_3)}\right) \tag{5}$$

$$\theta_{4s2} = -\operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos(\theta_3)}{a_5\sin(\theta_3)}\right) \tag{6}$$

5.3 θ_5

Solution Method: sinANDcos

$$\theta_{5s1} = \operatorname{atan}_2\left(-r_{22}, r_{21}\right) + \operatorname{atan}_2\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2(\theta_3)\cos^2(\theta_4)}, -\sin(\theta_3)\cos(\theta_4)\right) \tag{7}$$

$$\theta_{5s2} = \operatorname{atan}_2\left(-r_{22}, r_{21}\right) + \operatorname{atan}_2\left(-\sqrt{r_{21}^2 + r_{22}^2 - \sin^2(\theta_3)\cos^2(\theta_4)}, -\sin(\theta_3)\cos(\theta_4)\right) \tag{8}$$

5.4 θ_{12}

Solution Method: atan2(y,x), arccos, best ranked, atan2(y,x)

$$\theta_{12s1} = \operatorname{atan}_2\left(-\frac{r_{33}}{\sin(\theta_3)}, -\frac{r_{13}}{\sin(\theta_3)}\right) \tag{9}$$

5.5 θ_{45}

Solution Method: algebra

$$\theta_{45s1} = \theta_4 + \theta_5 \tag{10}$$

5.6 θ_1

Solution Method: atan2(y,x), arcsin, best ranked, atan2(y,x)

$$\theta_{1s1} \tag{11}$$
$$= \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin(\theta_{12})\cos(\theta_3)\cos(\theta_4) + \sin(\theta_4)\cos(\theta_{12})\right) + l_4\sin(\theta_{12})\sin(\theta_3)}{a_2}, \frac{Px - a_1 + a_5\left(\sin(\theta_{12})\sin(\theta_4) - \cos(\theta_{12}\right.}{a_2}\right.$$

5.7 θ_2

Solution Method: algebra

$$\theta_{2s1} = -\theta_1 + \theta_{12} \tag{12}$$

6 Solutions to Generate all Versions

The following equations are the full set of solutions for each unknown incorporating all combinations of dependencies.

6.1 θ_3

Solution Method: arccos

$$\theta_{3v1} = \operatorname{acos}\left(-r_{23}\right) \tag{13}$$
$$\theta_{3v2} = \operatorname{acos}\left(-r_{23}\right) \tag{14}$$
$$\theta_{3v3} = \operatorname{acos}\left(-r_{23}\right) \tag{15}$$
$$\theta_{3v4} = \operatorname{acos}\left(-r_{23}\right) \tag{16}$$
$$\theta_{3v5} = \operatorname{acos}\left(-r_{23}\right) \tag{17}$$
$$\theta_{3v6} = \operatorname{acos}\left(-r_{23}\right) \tag{18}$$
$$\theta_{3v7} = \operatorname{acos}\left(-r_{23}\right) \tag{19}$$
$$\theta_{3v8} = \operatorname{acos}\left(-r_{23}\right) \tag{20}$$

6.2 θ_4

Solution Method: arccos

$$\theta_{4v1} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s1}\right)}{a_5\sin\left(\theta_{3s1}\right)}\right) \tag{21}$$
$$\theta_{4v2} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s2}\right)}{a_5\sin\left(\theta_{3s2}\right)}\right) \tag{22}$$
$$\theta_{4v3} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s2}\right)}{a_5\sin\left(\theta_{3s2}\right)}\right) \tag{23}$$
$$\theta_{4v4} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s1}\right)}{a_5\sin\left(\theta_{3s1}\right)}\right) \tag{24}$$
$$\theta_{4v5} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s1}\right)}{a_5\sin\left(\theta_{3s1}\right)}\right) \tag{25}$$
$$\theta_{4v6} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s2}\right)}{a_5\sin\left(\theta_{3s2}\right)}\right) \tag{26}$$
$$\theta_{4v7} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s2}\right)}{a_5\sin\left(\theta_{3s2}\right)}\right) \tag{27}$$
$$\theta_{4v8} = \operatorname{acos}\left(-\frac{Py + l_1 + l_4\cos\left(\theta_{3s1}\right)}{a_5\sin\left(\theta_{3s1}\right)}\right) \tag{28}$$

6.3 θ_5

Solution Method: sinANDcos

$$\theta_{5v1} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s1}\right)\cos^2\left(\theta_{4v1}\right)}, -\sin\left(\theta_{3s1}\right)\cos\left(\theta_{4v1}\right)\right) \tag{29}$$

$$\theta_{5v2} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s2}\right)\cos^2\left(\theta_{4v2}\right)}, -\sin\left(\theta_{3s2}\right)\cos\left(\theta_{4v2}\right)\right) \tag{30}$$

$$\theta_{5v3} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s2}\right)\cos^2\left(\theta_{4v3}\right)}, -\sin\left(\theta_{3s2}\right)\cos\left(\theta_{4v3}\right)\right) \tag{31}$$

$$\theta_{5v4} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s1}\right)\cos^2\left(\theta_{4v4}\right)}, -\sin\left(\theta_{3s1}\right)\cos\left(\theta_{4v4}\right)\right) \tag{32}$$

$$\theta_{5v5} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s1}\right)\cos^2\left(\theta_{4v4}\right)}, -\sin\left(\theta_{3s1}\right)\cos\left(\theta_{4v4}\right)\right) \tag{33}$$

$$\theta_{5v6} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s2}\right)\cos^2\left(\theta_{4v3}\right)}, -\sin\left(\theta_{3s2}\right)\cos\left(\theta_{4v3}\right)\right) \tag{34}$$

$$\theta_{5v7} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s2}\right)\cos^2\left(\theta_{4v2}\right)}, -\sin\left(\theta_{3s2}\right)\cos\left(\theta_{4v2}\right)\right) \tag{35}$$

$$\theta_{5v8} = \operatorname{atan_2}\left(-r_{22}, r_{21}\right) + \operatorname{atan_2}\left(\sqrt{r_{21}^2 + r_{22}^2 - \sin^2\left(\theta_{3s1}\right)\cos^2\left(\theta_{4v1}\right)}, -\sin\left(\theta_{3s1}\right)\cos\left(\theta_{4v1}\right)\right) \tag{36}$$

6.4 θ_{12}

Solution Method: atan2(y,x), arccos, best ranked, atan2(y,x)

$$\theta_{12v1} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s1}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s1}\right)}\right) \tag{37}$$

$$\theta_{12v2} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s2}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s2}\right)}\right) \tag{38}$$

$$\theta_{12v3} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s2}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s2}\right)}\right) \tag{39}$$

$$\theta_{12v4} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s1}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s1}\right)}\right) \tag{40}$$

$$\theta_{12v5} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s1}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s1}\right)}\right) \tag{41}$$

$$\theta_{12v6} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s2}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s2}\right)}\right) \tag{42}$$

$$\theta_{12v7} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s2}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s2}\right)}\right) \tag{43}$$

$$\theta_{12v8} = \operatorname{atan_2}\left(-\frac{r_{33}}{\sin\left(\theta_{3s1}\right)}, -\frac{r_{13}}{\sin\left(\theta_{3s1}\right)}\right) \tag{44}$$

6.5 θ_{45}

Solution Method: algebra

$$\theta_{45v1} = \theta_{4v1} + \theta_{5v1} \tag{45}$$
$$\theta_{45v2} = \theta_{4v2} + \theta_{5v2} \tag{46}$$
$$\theta_{45v3} = \theta_{4v3} + \theta_{5v3} \tag{47}$$
$$\theta_{45v4} = \theta_{4v4} + \theta_{5v4} \tag{48}$$
$$\theta_{45v5} = \theta_{4v4} + \theta_{5v5} \tag{49}$$
$$\theta_{45v6} = \theta_{4v3} + \theta_{5v6} \tag{50}$$
$$\theta_{45v7} = \theta_{4v2} + \theta_{5v7} \tag{51}$$
$$\theta_{45v8} = \theta_{4v1} + \theta_{5v8} \tag{52}$$

6.6 θ_1

Solution Method: atan2(y,x), arcsin, best ranked, atan2(y,x)

$$\theta_{1v1} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v1}\right)\cos\left(\theta_{3s1}\right)\cos\left(\theta_{4v1}\right) + \sin\left(\theta_{4v1}\right)\cos\left(\theta_{12v1}\right)\right) + l_4\sin\left(\theta_{12v1}\right)\sin\left(\theta_{3s1}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{53}$$

$$\theta_{1v2} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v2}\right)\cos\left(\theta_{3s2}\right)\cos\left(\theta_{4v2}\right) + \sin\left(\theta_{4v2}\right)\cos\left(\theta_{12v2}\right)\right) + l_4\sin\left(\theta_{12v2}\right)\sin\left(\theta_{3s2}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{54}$$

$$\theta_{1v3} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v3}\right)\cos\left(\theta_{3s2}\right)\cos\left(\theta_{4v3}\right) + \sin\left(\theta_{4v3}\right)\cos\left(\theta_{12v3}\right)\right) + l_4\sin\left(\theta_{12v3}\right)\sin\left(\theta_{3s2}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{55}$$

$$\theta_{1v4} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v4}\right)\cos\left(\theta_{3s1}\right)\cos\left(\theta_{4v4}\right) + \sin\left(\theta_{4v4}\right)\cos\left(\theta_{12v4}\right)\right) + l_4\sin\left(\theta_{12v4}\right)\sin\left(\theta_{3s1}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{56}$$

$$\theta_{1v5} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v5}\right)\cos\left(\theta_{3s1}\right)\cos\left(\theta_{4v4}\right) + \sin\left(\theta_{4v4}\right)\cos\left(\theta_{12v5}\right)\right) + l_4\sin\left(\theta_{12v5}\right)\sin\left(\theta_{3s1}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{57}$$

$$\theta_{1v6} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v6}\right)\cos\left(\theta_{3s2}\right)\cos\left(\theta_{4v3}\right) + \sin\left(\theta_{4v3}\right)\cos\left(\theta_{12v6}\right)\right) + l_4\sin\left(\theta_{12v6}\right)\sin\left(\theta_{3s2}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{58}$$

$$\theta_{1v7} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v7}\right)\cos\left(\theta_{3s2}\right)\cos\left(\theta_{4v2}\right) + \sin\left(\theta_{4v2}\right)\cos\left(\theta_{12v7}\right)\right) + l_4\sin\left(\theta_{12v7}\right)\sin\left(\theta_{3s2}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{59}$$

$$\theta_{1v8} = \operatorname{atan}_2\left(\frac{Pz - a_5\left(\sin\left(\theta_{12v8}\right)\cos\left(\theta_{3s1}\right)\cos\left(\theta_{4v1}\right) + \sin\left(\theta_{4v1}\right)\cos\left(\theta_{12v8}\right)\right) + l_4\sin\left(\theta_{12v8}\right)\sin\left(\theta_{3s1}\right)}{a_2}, \frac{Px - a_1 + a_5 (\sin (\theta_{12v} \cdots}{\cdots}\right. \tag{60}$$

6.7 θ_2

Solution Method: algebra

$$\theta_{2v1} = \theta_{12v1} - \theta_{1v1} \tag{61}$$
$$\theta_{2v2} = \theta_{12v2} - \theta_{1v2} \tag{62}$$
$$\theta_{2v3} = \theta_{12v3} - \theta_{1v3} \tag{63}$$
$$\theta_{2v4} = \theta_{12v4} - \theta_{1v4} \tag{64}$$
$$\theta_{2v5} = \theta_{12v5} - \theta_{1v5} \tag{65}$$
$$\theta_{2v6} = \theta_{12v6} - \theta_{1v6} \tag{66}$$
$$\theta_{2v7} = \theta_{12v7} - \theta_{1v7} \tag{67}$$
$$\theta_{2v8} = \theta_{12v8} - \theta_{1v8} \tag{68}$$

7 Solution Graph (Edges)

The following is the abstract representation of solution graph for this manipulator (nodes with parent -1 are roots). Future: graphic representation. :

```
Edge:th_12 depends on: th_3  (th_12-->th_3)
Edge:th_2 depends on: th_12  (th_2-->th_12)
Edge:th_45 depends on: th_4  (th_45-->th_4)
Edge:th_5 depends on: th_3  (th_5-->th_3)
Edge:th_2 depends on: th_1  (th_2-->th_1)
Edge:th_4 depends on: th_3  (th_4-->th_3)
Edge:th_1 depends on: th_3  (th_1-->th_3)
Edge:th_1 depends on: th_12  (th_1-->th_12)
Edge:th_1 depends on: th_4  (th_1-->th_4)
Edge:th_45 depends on: th_5  (th_45-->th_5)
Edge:th_5 depends on: th_4  (th_5-->th_4)
```

8 Solution Set

The following are the sets of joint solutions (poses) for this manipulator:

```
('th_3s2', 'th_4v2', 'th_5v7', 'th_12v7', 'th_45v7', 'th_1v7', 'th_2v7')
('th_3s2', 'th_4v3', 'th_5v3', 'th_12v3', 'th_45v3', 'th_1v3', 'th_2v3')
('th_3s2', 'th_4v2', 'th_5v2', 'th_12v2', 'th_45v2', 'th_1v2', 'th_2v2')
('th_3s2', 'th_4v3', 'th_5v6', 'th_12v6', 'th_45v6', 'th_1v6', 'th_2v6')
('th_3s1', 'th_4v4', 'th_5v5', 'th_12v5', 'th_45v5', 'th_1v5', 'th_2v5')
('th_3s1', 'th_4v1', 'th_5v8', 'th_12v8', 'th_45v8', 'th_1v8', 'th_2v8')
('th_3s1', 'th_4v1', 'th_5v1', 'th_12v1', 'th_45v1', 'th_1v1', 'th_2v1')
('th_3s1', 'th_4v4', 'th_5v4', 'th_12v4', 'th_45v4', 'th_1v4', 'th_2v4')
```

9 Solution Set v3

th_3s2	th_4v2	th_5v7	th_12v7	th_45v7	th_1v7	th_2v7
th_3s2	th_4v3	th_5v3	th_12v3	th_45v3	th_1v3	th_2v3
th_3s2	th_4v2	th_5v2	th_12v2	th_45v2	th_1v2	th_2v2
th_3s2	th_4v3	th_5v6	th_12v6	th_45v6	th_1v6	th_2v6
th_3s1	th_4v4	th_5v5	th_12v5	th_45v5	th_1v5	th_2v5
th_3s1	th_4v1	th_5v8	th_12v8	th_45v8	th_1v8	th_2v8
th_3s1	th_4v1	th_5v1	th_12v1	th_45v1	th_1v1	th_2v1
th_3s1	th_4v4	th_5v4	th_12v4	th_45v4	th_1v4	th_2v4

10 Jacobian Matrix

$$
{}^{6}J_6 =
\begin{array}{l}
\text{Column} \quad 1 \\
\begin{bmatrix} a_2 c_2 s_{45} + a_2 c_3 c_{45} s_2 + a_5 c_3 s_5 - l_4 s_3 s_{45} \\ a_2 c_2 c_{45} - a_2 c_3 s_2 s_{45} + a_5 c_3 c_5 - c_{45} l_4 s_3 \\ -a_2 s_2 s_3 + a_5 s_3 s_4 \\ c_{45} s_3 \\ -s_3 s_{45} \\ c_3 \end{bmatrix} \\
\text{Column} \quad 2 \\
\begin{bmatrix} a_5 c_3 s_5 - l_4 s_3 s_{45} \\ a_5 c_3 c_5 - c_{45} l_4 s_3 \\ a_5 s_3 s_4 \\ c_{45} s_3 \\ -s_3 s_{45} \\ c_3 \end{bmatrix} \\
\text{Column} \quad 3 \\
\begin{bmatrix} -c_{45} l_4 \\ l_4 s_{45} \\ a_5 c_4 \\ -s_{45} \\ -c_{45} \\ 0 \end{bmatrix} \\
\text{Column} \quad 4 \\
\begin{bmatrix} a_5 s_5 \\ a_5 c_5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\text{Column} \quad 5 \\
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
\text{Column} \quad 6 \\
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{array}
\tag{69}
$$