Assignment 2.5.1

Jord van Eldik

February 16, 2025

Q1: Well-posedness

In deciding on the wellposedness of both of these problems we first look at our matrix K. As the Toeplitz-matrix is square and invertible by definition we know that solutions exist and are unique. To determine well-posedness we want to look at the relationship between u and f (without noise). In the figure below you can see both plots.

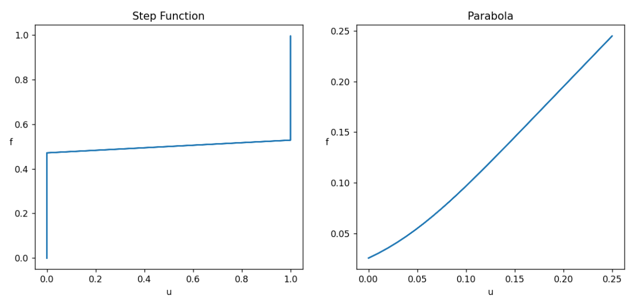


Figure 1: f plotted over u for step function and parabola

REgarding the parabola we see that the relation between u and f is roughly linear, this problem is therefor well-posed. For the step function however, we note very sharp increases around 0 and 1 indicating that this problem is ill posed through ill-conditionedness of this problem.

Q2: Backward Error using Untruncated Pseudo-Inverse

Before moving on to results for my simulations, all code can be found at `https://github.com/JordPBvE/inverse-problems-hw1`. Now, computing the pseudo-inverse of K using the SVD-decomposition we can then compute $||u^\delta - u||_2$, doing this for the given inputs and different noise levels yields the following backward error values.

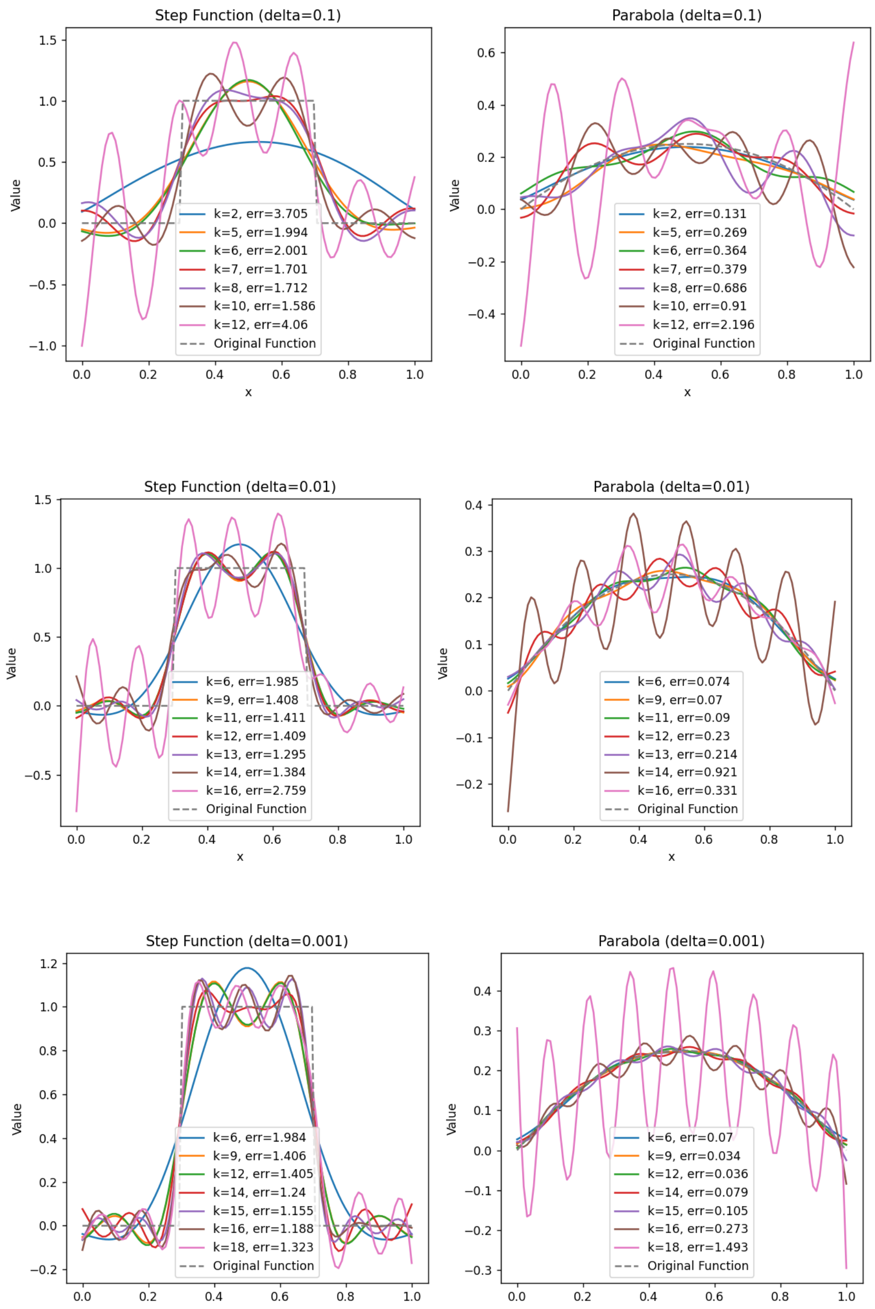
Function	Delta	Backward Error
Step Function	0.1	1.099×10^{16}
Step Function	0.01	1.185×10^{15}
Step Function	0.001	9.988×10^{13}
Parabola	0.1	1.220×10^{16}
Parabola	0.01	1.253×10^{15}
Parabola	0.001	1.035×10^{14}

Table 1: Backward Error for Different Delta Values

We find that due to the existence of very small singular values in the Σ-matrix we obtain very large values in Σ^{-1}, leading to noise causing these enormous errors.

Q3: Backward Error using Truncated Pseudo-Inverse

A solution to the blowing up of these inverse singular values we can choose to truncate the SVD decomposition, thus removing the singular values we deem too small, and therefor insignificant. What results is that after inverting, this blowing up of the error term will be reduced. In the following graphs we find the results we achieve for a number of values of k, which denotes the rank of the truncated SVD matrices.



First of all we found that the preferred k-value for each noise level did not depend on which input data we chose, but only depended on the chosen delta parameter. The optimal values are as follows. We see that the smaller the noise parameter is, the higher the optimal truncation parameter.

Noise Level	Optimal k-value
$\delta = 0.1$	$k = 7$
$\delta = 0.01$	$k = 13$
$\delta = 0.001$	$k = 15$

Q4: Explanation for Observations

Looking at the inverse singular values for the decompostition of our matrix K we can plot these values, giving the following graph. Noting from our simulations that the optimal truncation parameters dont exceed 20 we plot a zoomed in version.

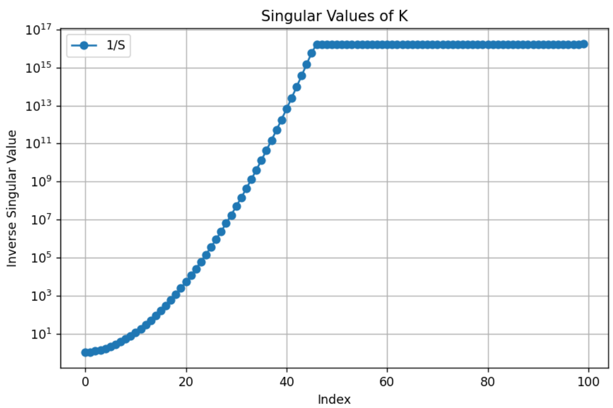


Figure 2: Enter Caption

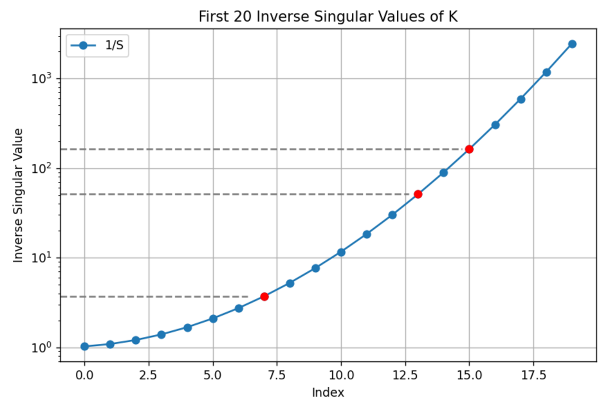


Figure 3: Enter Caption

What we can do is highlight the cutoff values we foudn in Q3 (note that truncating sets all inverse singular values to the right of the cutoff to zero). We know that the resiprocal of the order of magnitude of the noise are 10^1, 10^2 and 10^3 respectively. What we want from our truncation is that we remove the component of f^δ that corresponds with (the order of magnitude component of) this noise. The found values of k do just that.