

# Solution for Lab 4: System Type and Lead-Lag Control of a DC Servo Motor

## 1 Procedure

### 3.1 System Type and Tracking Error

The Lab4\_SystemType.slx model contains five systems:

1. step input, position feedback
2. ramp input, position feedback
3. parabolic input, position feedback
4. step input, speed feedback
5. ramp input, speed feedback

The initial feedback gain for all five systems is 1.

For each system, we observe whether or not the reference is tracked, and if not, how the error evolves. We explore the effect of the feedback gain on the error signal.

1. The step input is tracked with position feedback since the steady state error goes to zero.

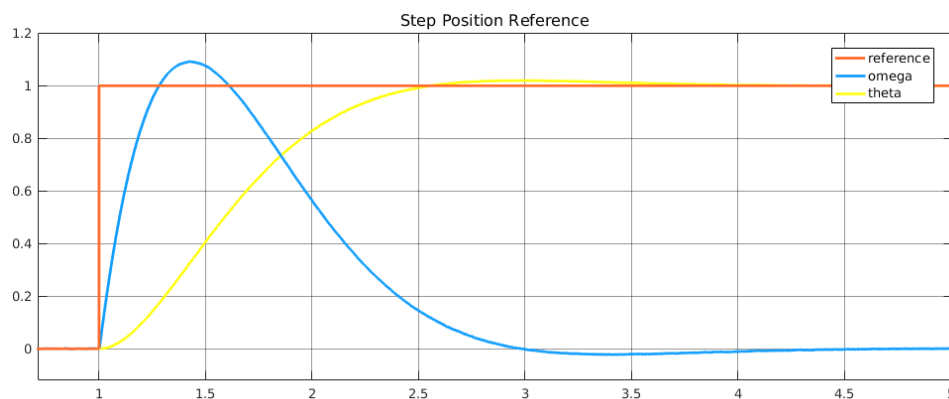


Figure 1: Position (yellow) tracks the reference (red).

2. The ramp input is not tracked with position feedback. The error approaches a constant value, implying that the speed implied by the slope of the reference is attained by the plant.

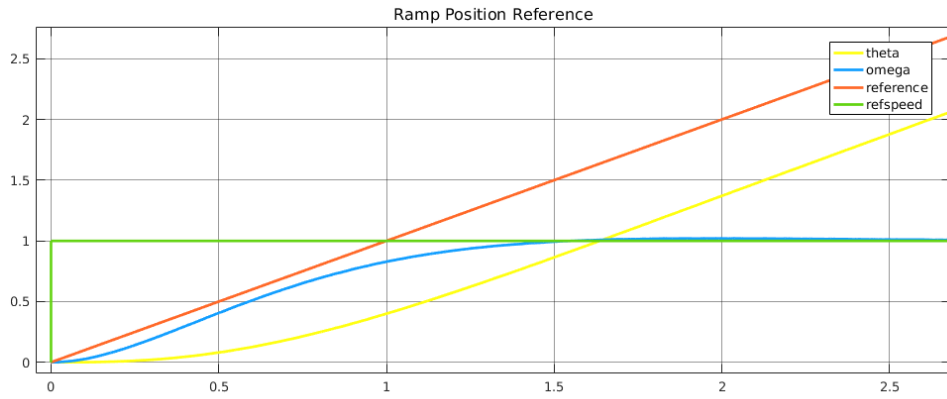
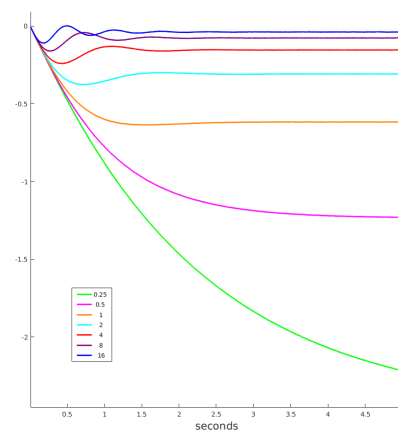
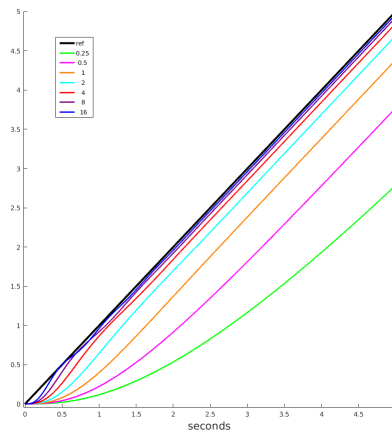


Figure 2: Position(yellow) does not track the reference (red). However, the speed (blue) eventually matches the speed implied by the slope of the ramp (green).



(a) Family of position trajectories for the following values of K: .25, .5, 1, 2, 4, 8. (b) Family of position errors for the same K values.

The effect of increasing the gain is illustrated by the following:

Increasing the feedback gain reduces the limiting value of the position error, but does not completely eliminate it.

3. The parabolic input is not tracked with position feedback. Position error increases linearly in time.

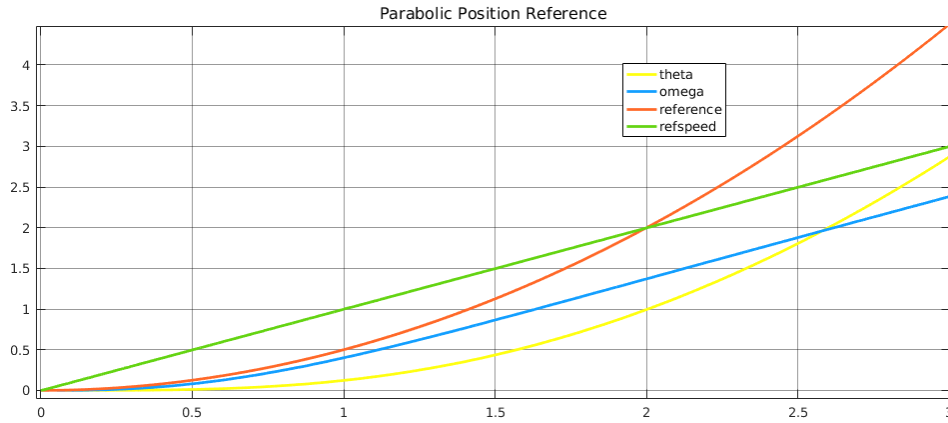
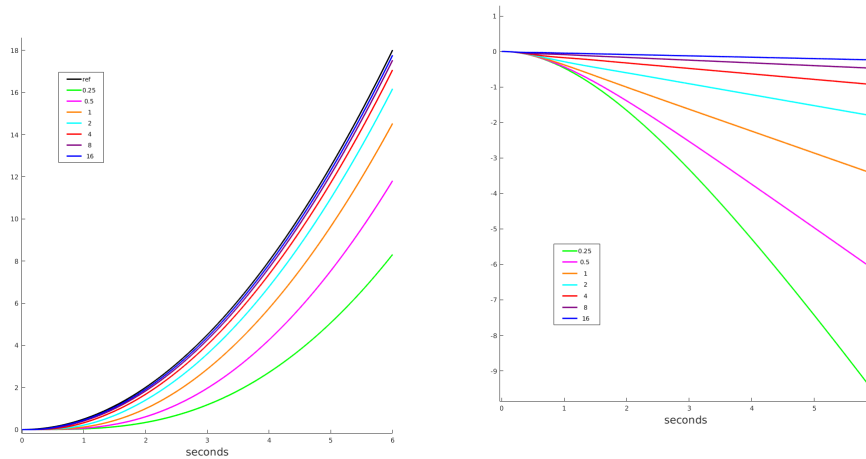


Figure 4: Position(yellow) does not track the reference (red). The speed implied by the derivative of the input eventually differs from the speed (blue) by a constant after

The effect of increasing the gain is illustrated by the following:



- (a) A family of position curves for the same K values. The downturn at the right is due to the maximum speed of the servo being attained.
- (b) A family of position error curves for the same K values. The downturn at the right is due to the maximum speed of the servo being attained.

Increasing the feedback gain can reduce the tracking error and decrease the constant rate at which the error increases, but it is not completely eliminated.

4. The step input is not tracked with speed feedback. The error approaches a constant value.

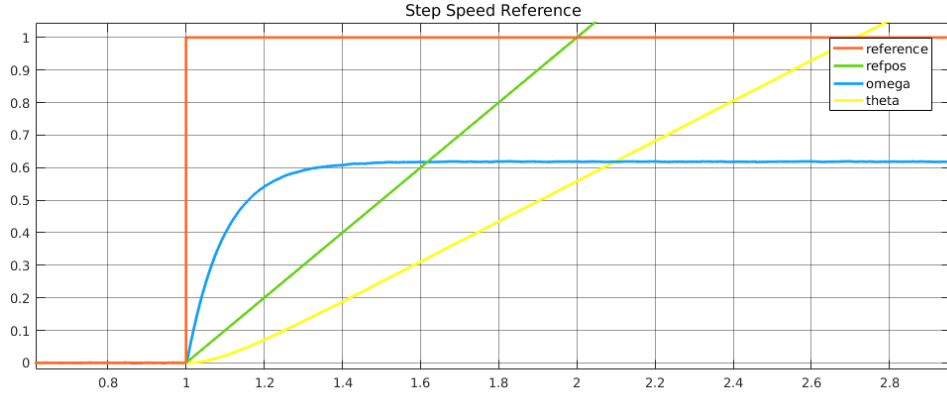
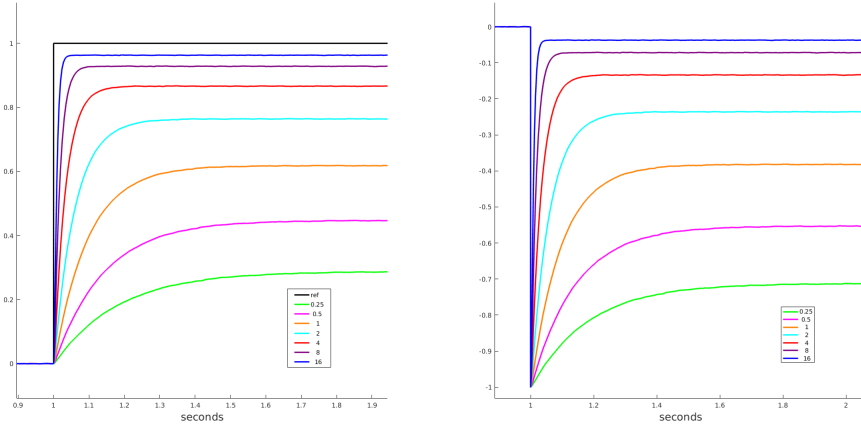


Figure 6: Speed (blue) does not track the reference (red). The difference approaches a constant value.

The effect of increasing the gain is illustrated by the following:



(a) Family of speed curves for the same values of K. (b) Family of speed error curves for the same values of K.

Increasing  $K$  reduces the steady state speed error, but does not eliminate it.

5. The ramp input is not tracked with speed feedback. The error increases linearly in time.

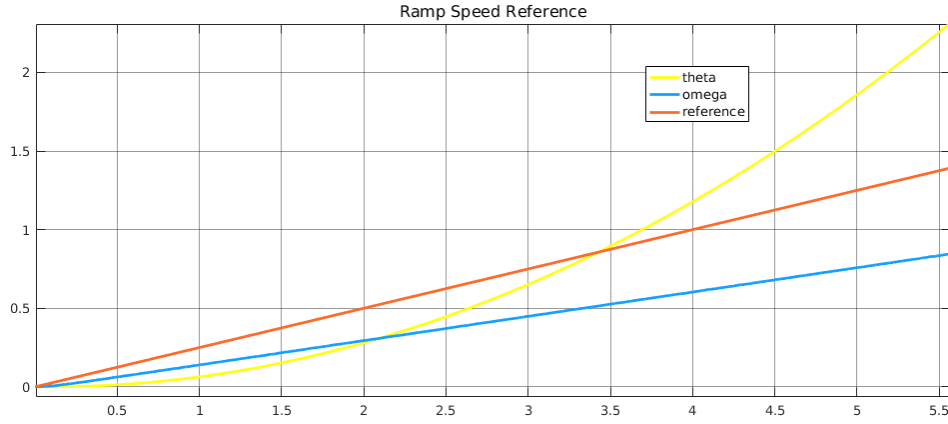
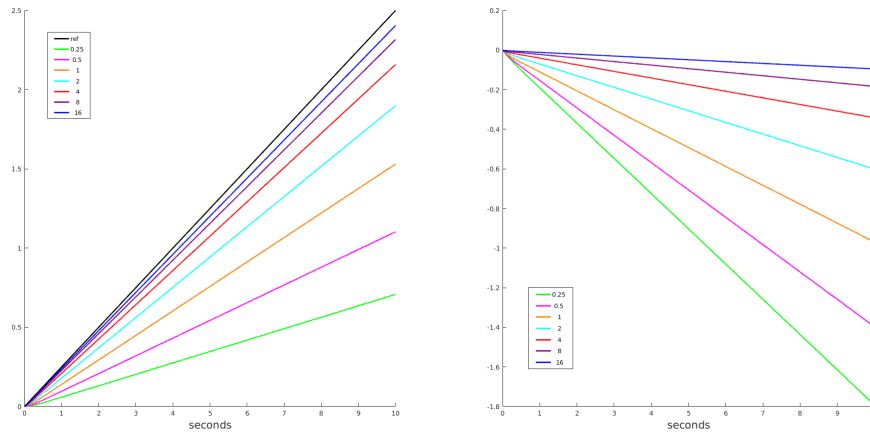


Figure 8: The speed curve (blue) does not track the reference (red). The error increases linearly with time.

The effect of increasing the gain is illustrated by the following:



(a) A family of speed curves for the same values of K. (b) A family of speed error curves for the same values of K.

Increasing the feedback gain reduces the steady state error and the rate at which it increases, but does not completely eliminate it.

### 3.2 Identify $K_s$ and $\tau$

We run the Simulink model Lab4\_Identification.slx which applies a unit step to the open loop system at  $t=1$  and estimate steady state value of the speed to be 1.62, implying a  $K_s$  of 1.62. We then estimate the time at which the output reaches 63.2% of its steady state value, or 1.0238, to be 1.254, implying a  $\tau$  of .254.

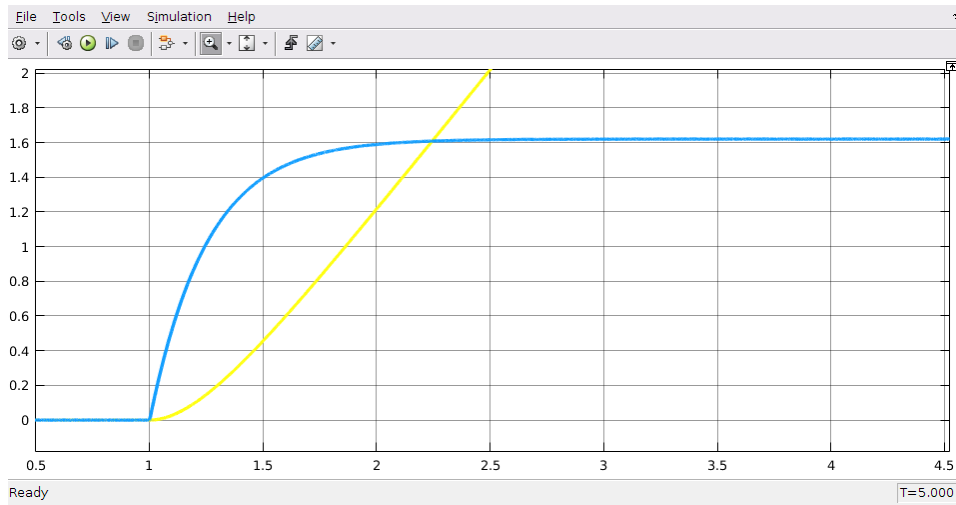


Figure 10: Speed trajectory (blue) for a unit step input.

### 3.3 Proportional Feedback

The Simulink model Lab4\_ProportionalFeedback.slx contains a DC servo model with proportional feedback fed with a unit step input at  $t=1$ . We minimize the system's settling time subject to a 5% limit on maximum overshoot.

We use Matlab's `rlocus()` function to plot our DC servo transfer function's root locus. The function `sgrid( $\zeta, \omega_n$ )` with a  $\zeta$  of .6901 plots the boundary of the region in the  $s$  plane that our closed loop poles may occupy. The function `rlocfind()` is then used to find a range of feedback gains corresponding to points on the root locus between the divergence from the real axis and the damping ratio boundary. Feedback gains over this region range roughly from .6 to 1.2. A gain of .9 results in a settling time of about 1.65s with an overshoot of about 1%.

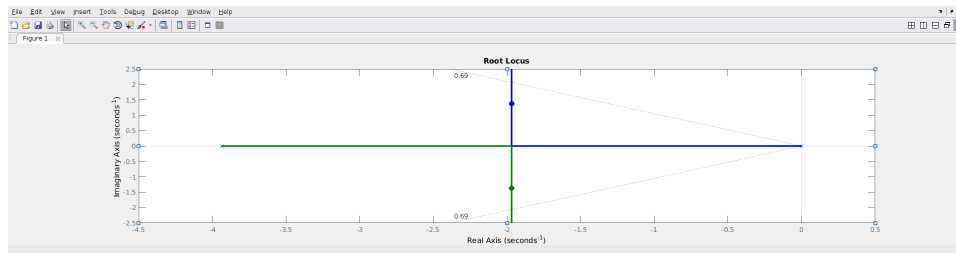
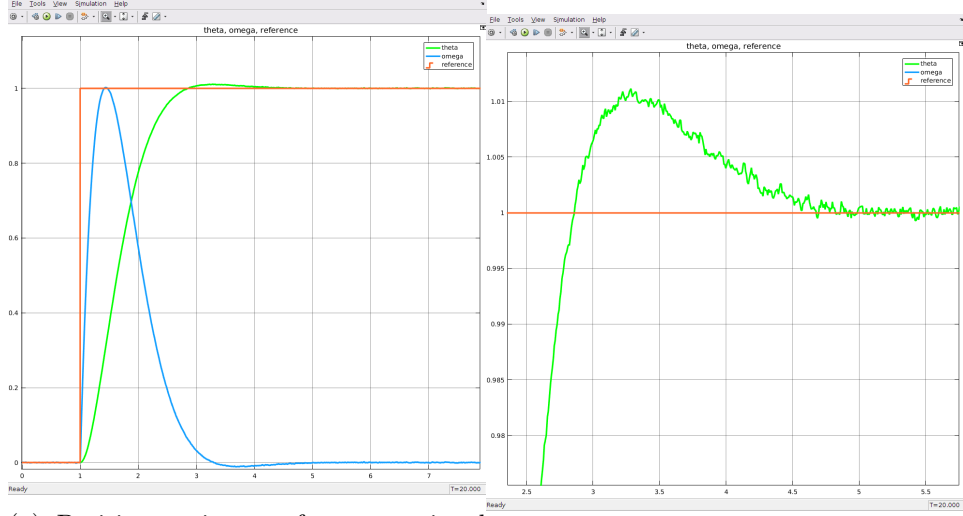


Figure 11: The DC servo system's root locus. The markers indicate the pole locations corresponding to a gain of .9.

### 3.4 Lead Compensation

The specifications for our controller are:



(a) Position trajectory for proportional with  $K = .9$ . (b) Closeup view of peak of the same.

1. Overshoot less than 5%.
2. 2% band settling time less than .6 seconds

We first determine the damping ratio  $\zeta$  and natural frequency  $\omega_n$  corresponding to these constraints.

From the formula for maximum overshoot we have:

$$\zeta = \sqrt{\frac{\ln(M)^2}{\pi^2 + \ln(M)^2}} \quad (1)$$

where  $M$  is the maximum overshoot (.05).

From the formula for the envelope of the oscillations, we have:

$$\omega_n = \frac{1}{\zeta t_s} \log \left( \frac{1}{\epsilon \sqrt{1 - \zeta^2}} \right) \quad (2)$$

alternatively, we could use the less precise formula:

$$\omega_n = \frac{4}{\zeta t_s} \quad (3)$$

where  $t_s$  is the settling time for the 2% band (.6).

We will use the first formula for the natural frequency, yielding  $\omega_n = 10.2288$  and  $\zeta = .6901$ . We now determine the desired location of our dominant closed loop poles according to the formula:

$$s = \sigma + j\omega = \zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2} \quad (4)$$

which yields pole locations of  $-7.0590 \pm j7.4027$ .

Now we will determine the position of the lead controller pole and zero such that the desired closed loop pole locations are on the root locus. The angle condition tells us that

$$\angle G(s)H(s) = \pi \quad (5)$$

which implies

$$\angle z_{lead} - \angle p_{lead} = \pi + \angle p_{ol1} + \angle p_{ol2} \quad (6)$$

indicating that the difference between the lead pole angle and zero angle must be 1.1607 radians. In the limit that the lead compensator pole is at  $-\infty$ , its angle will be zero and the the lead zero angle must be 1.1607, corresponding to -10.2769 on the real axis. This is the farthest left we can place the zero.

We choose to place the zero at -8.668, at an angle of 1.3568 radians, leaving .1960 radians for the lead controller pole angle. This corresponds to a pole location of -44.3358.

We now use the gain condition to find the appropriate gain for our lead controller.

$$K \frac{|s + z_1||s + z_2| \cdots |s + z_m|}{|s + p_1||s + p_2| \cdots |s + p_n|} = 1 \quad (7)$$

In our case,

$$K \frac{|\frac{1.62}{.254}||s + 8.668|}{|s + 44.3358||s + 0||s + 3.9370|} = 1 \quad (8)$$

substituting our desired pole location for s and solving for K, we get 64.6407.

We use the Simulink model Lab4\_Lead.slx to demonstrate the controller's performance. Two systems receive a unit step input as a position reference. Factoring out the zero and pole from lead compensator transfer function so that it will have unity steady state gain gives us a compensator transfer function

$$\frac{.1154s + 1}{.02256s + 1} \quad (9)$$

and a gain of 12.6377. For the system with the compensator in the feedback path, we have a peak overshoot of 5.0% and a settling time of .59 seconds, which meets our requirements. For the system with the compensator in the forward path, the maximum overshoot is 15.7% and the settling time is .51 seconds.

The difference in performance can be understood by considering the difference in the closed loop zeros. Let the open loop servo transfer function G(s) be  $\frac{N_{servo}(s)}{D_{servo}(s)}$  and the lead compensator transfer function H(s) be  $\frac{N_{lead}(s)}{D_{lead}(s)}$  Where N is numerator and D is the denominator of the transfer functions.



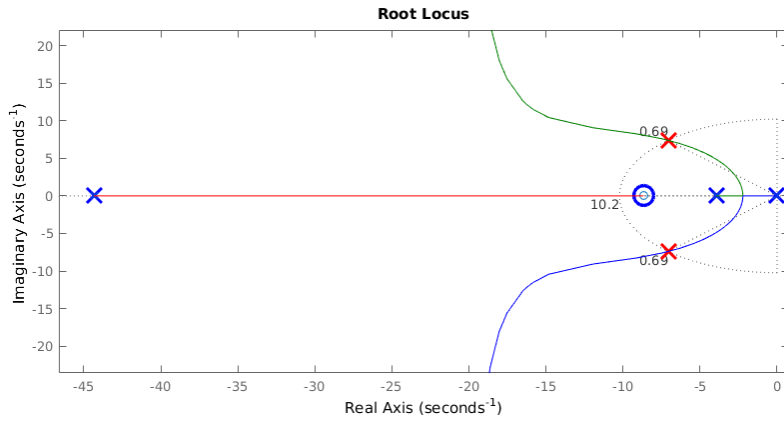
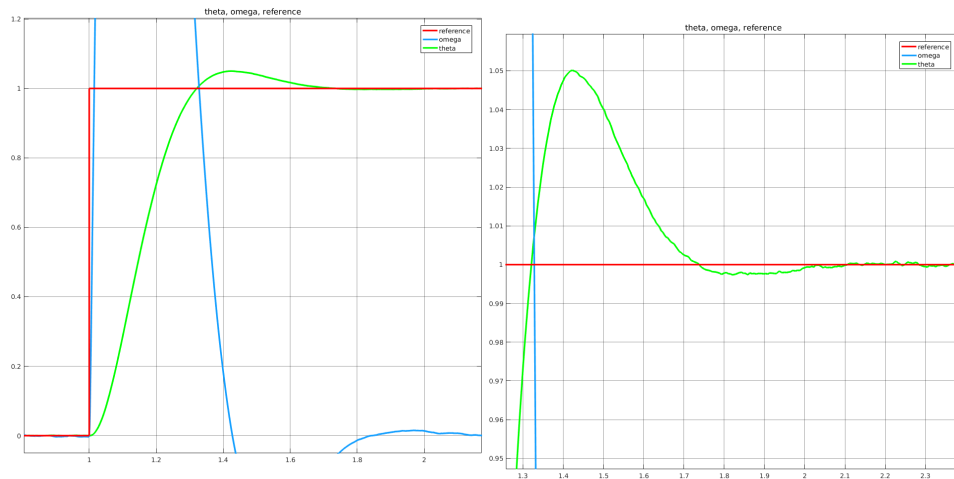
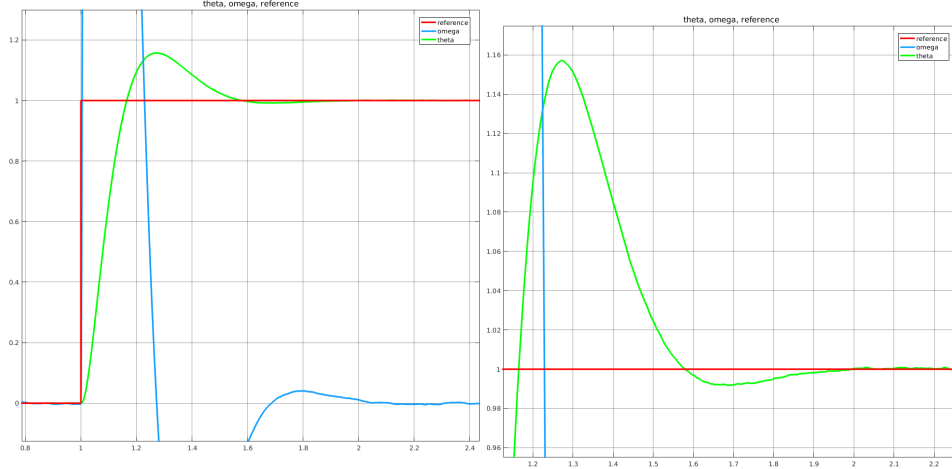


Figure 13: The root locus with open loop poles and zeros in blue and the dominant closed loop poles in red on the  $s$  plane with damping ratio and natural frequency constraints.



(a) Position vs. time with lead controller in feedback path.

(b) Close up of position trajectory.



(a) Position vs. time with lead controller in forward path.

(b) Close up of position trajectory.

The closed loop transfer function for the system with the compensator in the feedback is

$$\frac{KG(s)}{1 + KG(s)H(s)} \quad (10)$$

or, in terms of the  $N(s)$  and  $D(s)$ :

$$\frac{KN_{servo}(s)D_{lead}(s)}{D_{servo}(s)D_{lead}(s) + KN_{servo}N_{lead}} \quad (11)$$

and the transfer function for the system with the compensator in the forward path is

$$\frac{KG(s)H(s)}{1 + KG(s)H(s)} \quad (12)$$

or, in terms of the  $N(s)$  and  $D(s)$ :

$$\frac{KN_{servo}(s)N_{lead}(s)}{D_{servo}(s)D_{lead}(s) + KN_{servo}N_{lead}} \quad (13)$$

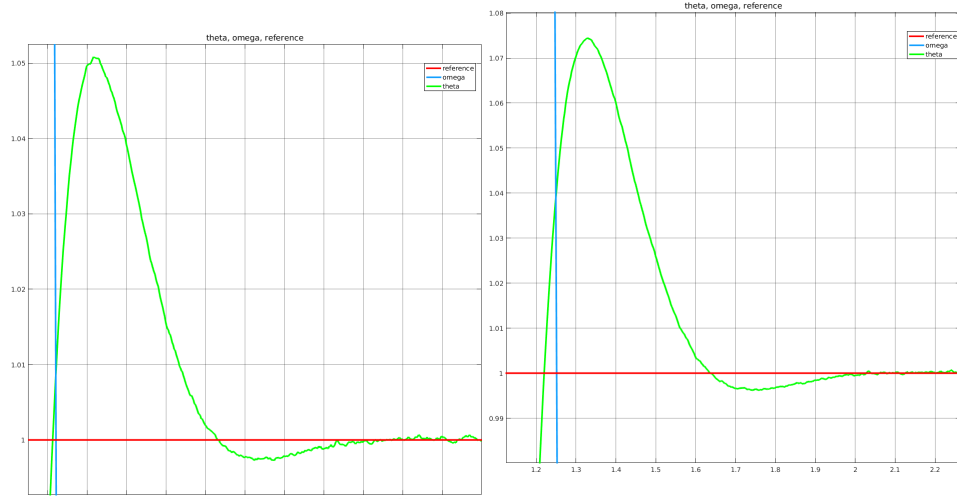
Though we have positioned the poles that we hope will be dominant through our lead compensator design, the remaining pole's contribution to the response may not be negligible. Though the poles are the same for both configurations, the magnitude of each component of the response will depend on the zeros. This can be seen from a partial fraction decomposition of the closed loop transfer function.

We can improve the performance of our forward path lead compensator by designing a system with a higher gain. This time, we choose to put the lead zero at -9.955 and the pole at -206.3. With the lead transfer function written

$$\frac{.1005s + 1}{.004847s + 1} \quad (14)$$

the error gain is 15.595.

The system with the compensator in the feedback path has a peak overshoot of about 5.1% and a settling time of .58 seconds. With the compensator in the forward path, the peak overshoot is only 7.4% and the settling time is .52s.



(a) Overshoot for higher gain lead compensator in forward path. Overshoot has been reduced with the higher gain compensator.

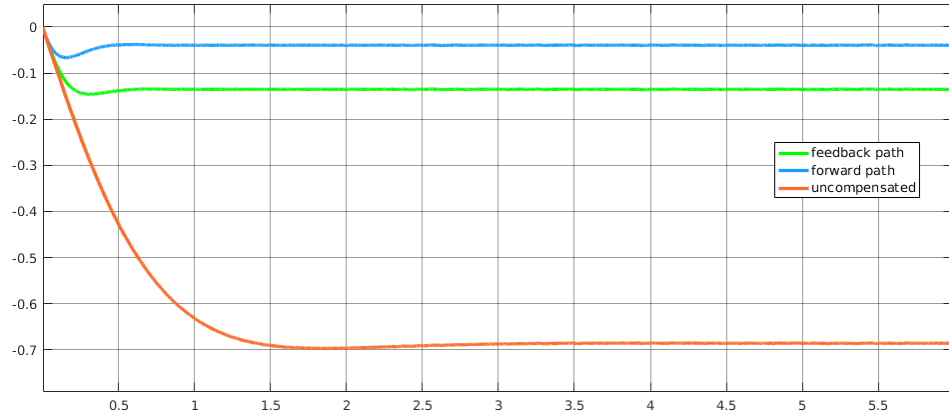


Figure 17: Velocity error for a unit ramp position reference.

**3.5 Lead-Lag Compensation** To our second lead compensator we add a lag compensator. We are required to bring the steady state velocity error below .02 in a reasonable time without significantly altering the transient response.

Our target  $K_v$  is 50, given by  $\frac{1}{e_{ss}}$ .

$$50 = \lim_{s \rightarrow 0} sKG_{lead}(s)G_{lag}(s)G_{sys}(s) \quad (15)$$

We will use this relation to determine the parameter  $\beta$  of our lag compensator

$$\frac{s + \frac{1}{T}}{s + \frac{1}{T\beta}}$$

$$50 = K \times 1 \times \beta \times K_s \quad (16)$$

from this, we have  $\beta$  equals 1.979, which we round up to 2. In order to choose  $T$ , we must balance the preservation of lead compensator transient performance against the speed of our lag compensator action. Choices of  $T$  that place the lag compensator pole and zero very near the imaginary axis will reduce the impact on transient performance. Placing the zero and pole close to the the origin enables the difference in the distance between these two points and the target pole location to be kept small, and likewise for the difference in angle. In this way, the impact of the lag compensator on the root locus is minimized. However, placing the lag pole and zero in this way will result in a large time constant for the lag component of the response, resulting in a long delay before the error is reduced to its desired value.

We start by choosing a  $T$  such that

$$\frac{|s_1 - \frac{1}{T}|}{|s_1 - \frac{1}{2T}|} \quad (17)$$

is within 1% of 1. Choose  $T = 10$ , which meets this criterion. The transient performance has not been significantly affected - the peak overshoot is about 8% with the lag compensator and the settling time is .535 seconds.

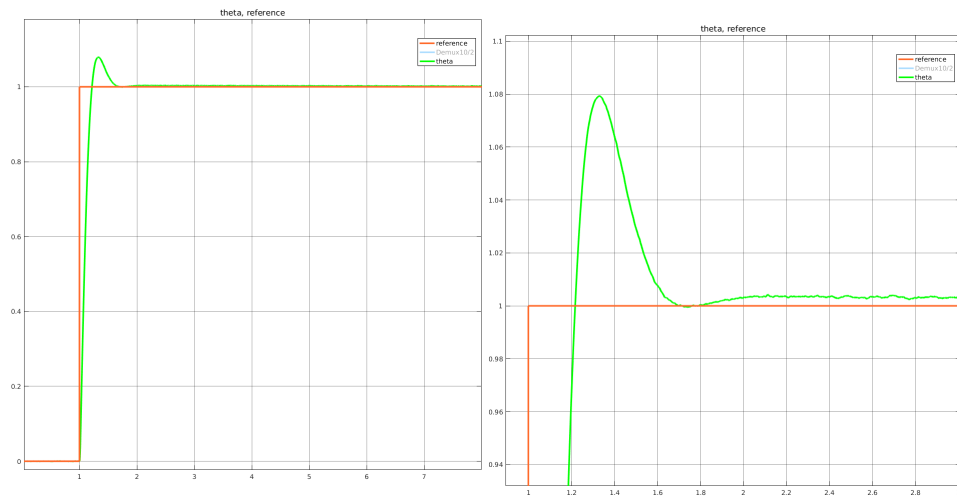
The unit ramp error has been reduced to .02, though it takes nearly 30 seconds approach this limit closely.

The specifications were not clear regarding the relative importance of minimizing overshoot vs attaining the target velocity error quickly. We have tended to prioritize transient performance over reduction of velocity error.

## 2 Exercises

1. **System Type and Tracking Error** From the final value theorem we know:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) \quad (18)$$



(a) Step response of lead-lag compensated system.

(b) Close up of the same.

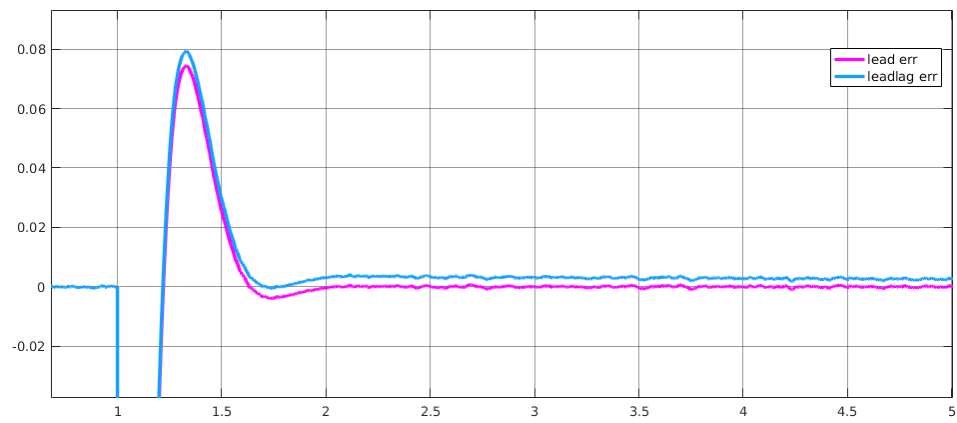


Figure 19: Comparison of error for lead and lead-lag compensation.

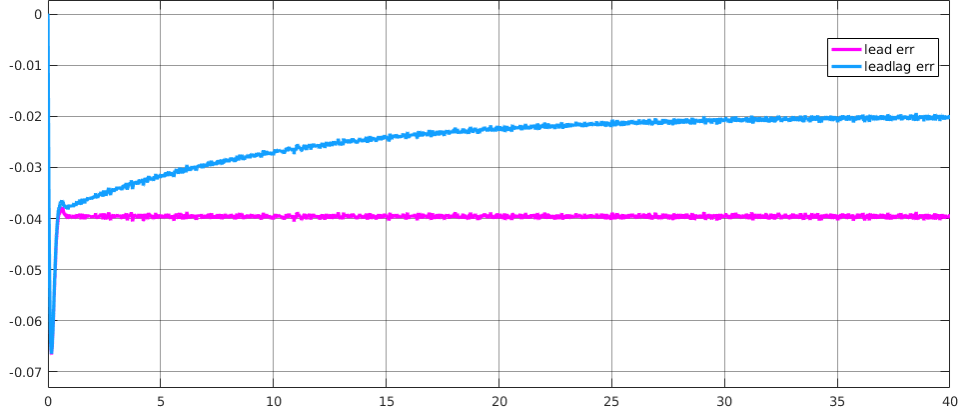


Figure 20: Comparison of error for lead and lead-lag compensation.

or more specifically, in our case:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sU(s)G'(s) \quad (19)$$

where  $G'(s)$  is the closed loop transfer function for  $G(s)$ , the input to position transfer function for the servo system.

For a position feedback system with a step input, we have:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \frac{R}{s} \frac{K_s K_1}{\tau s^2 + s + K_1 K_s} = R \quad (20)$$

therefore, the steady-state value is equal to the reference value.

For a position feedback system with a ramp input, we have:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \frac{1}{s^2} \frac{K_s K_1}{\tau s^2 + s + K_1 K_s} = \infty \quad (21)$$

As expected, the position trajectory has no steady state value.

Using the Final Value Theorem we can find the steady state error:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \frac{1}{s^2} \left( 1 - \frac{K_s K_1}{\tau s^2 + s + K_1 K_s} \right) \quad (22)$$

$$= \lim_{s \rightarrow 0} \frac{1}{s} \frac{\tau s^2 + s}{\tau s^2 + s + K_1 K_s} \quad (23)$$

$$= \lim_{s \rightarrow 0} \frac{\tau s^2 + s}{\tau s^3 + s^2 + K_1 K_s s} \quad (24)$$

$$= \frac{1}{K_1 K_s} \quad (25)$$

which confirms our suspicion that there is no value of  $K_1$  which will completely eliminate the steady-state error.

For a position feedback system with a parabolic input, we can see from the foregoing that the limit of the error will be infinity:

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \frac{1}{s^3} \left( 1 - \frac{K_s K_1}{\tau s^2 + s + K_1 K_s} \right) \quad (26)$$

with the extra  $s$  in the denominator, we end up with:

$$\lim_{s \rightarrow 0} \frac{1}{2K_1 K_s + O(s^2)} = \infty \quad (27)$$

However, if we look instead at the final value of the derivative of the error, we cancel this extra  $s$  and show that the derivative of the error has a constant value, or that the error increases linearly in time.

For speed feedback systems, we must use the input to speed transfer function instead.

For a speed feedback system with a step reference, we have the final value:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \frac{R}{s} \frac{K_s K_1}{\tau s + s + K_1 K_s + 1} = R \frac{K_s K_1}{K_s K_1 + 1} \quad (28)$$

and a steady state error of

$$\frac{R}{K_s K_1 + 1} \quad (29)$$

For a speed feedback system with a ramp reference, we have an additional  $s$  in the denominator, making the error infinite

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \frac{1}{s^2} \left( 1 - \frac{K_s K_1}{\tau s + K_1 K_s + 1} \right) \quad (30)$$

$$= \lim_{s \rightarrow 0} \frac{1}{s} \frac{\tau s + 1}{\tau s + K_1 K_s + 1} = \infty \quad (31)$$

However, we can again cancel this extra  $s$  by considering the derivative of the error, showing that the error also increases linearly in this case.

Writing an open loop transfer function  $G(s)$  as  $\frac{N(s)}{D(s)}$ , we can see that for the closed loop transfer function  $G'(s) = \frac{K_1 G(s)}{1 + K_1 G(s)}$  we will have

$$\frac{K_1 N(s)}{D(s) + K_1 N(s)} \quad (32)$$

In calculating the steady state error using the Final Value Theorem, it is necessary to calculate  $1-G'(s)$ , which is

$$\frac{D(s)}{D(s) + K_1 N(s)} \quad (33)$$

The steady state error is then

$$\lim_{s \rightarrow 0} sU(s) \frac{D(s)}{D(s) + K_1 N(s)} \quad (34)$$

This expression will be finite if the lowest order (in  $s$ ) term of  $D(s)$  is high enough order that the by the time the derivatives required by repeated application of L'Hospital's Rule generate a constant term in  $D(s)$ , there is at least one term without an  $s$  in the denominator of (33). Therefore, systems with more powers of  $s$  to be factored out of the denominator of their open loop transfer functions will have finite errors for  $U(s)$  with a correspondingly greater number of integrators and will track references with fewer integrators than that number for which they have finite error. This explains why position feedback is able to track an input with one more integrator than speed feedback for the servo system.

Differentiating the expression for steady state error offsets an integral in the reference signal, shows that adding an integrator in the input has the effect of adding an integrator to the error signal for systems that do not track the reference.

The type of the system is determined by the powers of  $s$  which can be factored out of the open loop transfer function denominator.