1 Abstract

We investigate the 2cm app, a data-exchange platform published for China Mobile Hong Kong 4G Pro Service Plan customers. Telecommunications ISPs' revenue is typically gained by charging users a fixed fee for a maximum amount of data usage in a month, i.e., a monthly data cap [?]. 2cm's (2nd exchange market) data exchange platform allows users to submit bids to buy and sell data. This usage model, is, as far as our knowledge, the first data trading platform that allows customers to buy and sell their own data. We describe a distributed auction mechanism for data exchange inspired by the classic PSP throughput problem, and prove that our distributed data exchange mechanism provides incentive compatibility (social choice function), and that we have efficiency using only partial valuation information of each participant in an exchange market.

In applying a distributed PSP implementation to CMHK's secondary market, we find that the market is able to achieve an equilibrium as the sellers and buyers have an incentive for a collaborative exchange, and design our mechanism tp provide the functions for effective communication between the connected users. We claim that in this secondary market our formulation holds the desired VCG qualities through the construction of a probable equilibrium [?]. We further provide bounds on the auction duration, with respect to the classic throughput problem. and provide simulated results on convergence time to support our (FIND COMPETITVE RATIO!), and a bound on the convergence of our mechanism. We extend the works of cite!cite! i.e. (market influence/EQ,social EQ,payment/allocation models) OR (bandwidth, data bundles, distributed market algorithms) and show the existence of a dynamic global market equilibrium, allowing for a unique set of market dynamics.

2 Introduction

In this work, we propose *a distributed progressive second price (PSP) auction in order to maximize social utility in this secondary market*. Using the distributed PSP mechanism on CMHKs data exhange platform, we show that for cellular data allocated between multiple users there exists an ϵ-Nash market equilibria. A quality of the PSP auction is that demand information is not known centrally, rather, it is distributed in the buyers' valuations. The mechanism for an auction is defined as *distributed* when the allocations at any element depend only on *local* state: the quantity offered by the seller at that element, and the bids for that element only [2]. In this work, the proposed mechanism allows the distribution of bids, where there are many ISPs each holding thier own local auction; there is no entity that holds a global market knowledge.

In a PSP mechanism, bids consist of (i) an available (required) quantity and (ii) a unit-price (calculated using its own demand functions). Buyers submit bids cyclically until an (ϵ-Nash) equilibrium is reached and a local auction is concluded.

(FEE IS FIXED OR PER-UNIT?)(HOW DO WE MODEL ISP REVENUE? IMMEDIATE FUTURE)

The form of the auction mechanism presented here is (CAN BE? NEED TO SHOW TO CLAIM 'IS') described as a pure-strategy progressive game with incomplete, but perfect information. (WHAT DOES NASH SAY ABOUT THIS?) (TRY MIXED? CAN ONLY HAVE MIXED WITH A DISTRIBUTED VALUATION.. FUTURE WORK)

The paper is organized as follows...

2.1 Distributed Progressive Second Price Auctions

2.1.1 Allocation using PSP

We begin with a brief introduction to the generalized distributed PSP auction, first introduced by Lazar and Semret [2]. We define a set of $\mathcal{I} = \{1, \cdots, I\}$ users.

Suppose each user $i \in \mathcal{I}$ makes a bid $s_i^j = (p_i^j, d_i^j)$ to the seller of resource j, where p_i^j is the unit-price the user is willing to pay and d_i^j is the quantity the user desires. The *bidding profile* forms a grid, $s \equiv [s_i^j] \in \mathcal{I} \times \mathcal{I}$, and $s_{-i} \equiv [s_1^j, \cdots, s_{i-1}^j, s_{i+1}^j, \cdots, s_I^j]_{j\in\mathcal{I}}$ is the profile of user i's opponents.

Using this classic PSP mechanism, [2] shows that given the opponents bids s_{-i}, user i's ϵ-best response to seller j is $s_i^j = (w_i^j, v_i^j)$ and is a Nash move where $\epsilon > 0$ is the bid fee, $B_i = \sum_{j\in\mathcal{I}} b_i^j$ is user i's budget, and every user has an elastic demand function.

Based on the profile of bids $s^j = [s_1^j, \cdots, s_I^j]$, the seller applies an allocation rule $a(s^j) = [a_1^j, \cdots, a_I^j]$, where a_i^j is the quantity allocated by j to each user $i \in \mathcal{I}$ and c_i^j is the cost charged to i for allocations awarded in auction j. An allocation is considered feasible if $a_i^j \le d_i^j$, and $c_i^j \le p_i^j d_i^j$.

We intend to show that our auction is rational and achieves the desired VCG properties, as does the original formulation. Using [1] as a basis for our model, and [2] as realistic, theoretic, and notational templates, we define optimal strategies for CMHK users, and demonstrate that the development of a set PSP auction mechanisms in a data exchange setting is able achieve a network equilibrium for cellular data.

2.2 Data Auction Mechanism

We now proceed to formally define the PSP auction, which determines the actions buyers and sellers in the secondary market, which we will denote the *data* PSP rules. The market price function (MPF) for a buyer in the secondary market can be described as follows:

$$\bar{P}_i(z, s_{-i}) = \sum_{j\in\mathcal{I}} P_i^j(z, s_{-i}^j)$$
$$= \sum_{j\in\mathcal{I}} \left(\inf \left\{ y \ge 0 : D_i^j(y, s_{-i}^j) \ge z \right\} \right), \quad (1)$$

and is interpreted as the aggragate of minimum prices that buyer i bids in order to obtain data amount z given opponent profile s_{-i}. We note that the total minimum price for the buyer must be an aggragation of the *individual* prices of the buyers as it is possible that the reserve prices of the individual sellers may vary. The maximum available quantity of data in auction j at unit price y given s_{-i}^j is:

$$D_i^j(y, s_{-i}^j) = \left[D^j - \sum_{p_k^j > y} d_k^j\right]^+, \tag{2}$$

it follows that the inverse price function is aggregated over all local auctions $j \in \mathcal{I}$,

$$\bar{D}_i(y, s_{-i}) = \sum_{j \in \mathcal{I}} \left(\sup \left\{ z \in \left[0, D^j\right] : \bar{P}_i(z, s_{-i}^j) < y \right\} \right). \tag{3}$$

The resulting data allocation rule is a function of the local market interactions between buyers and sellers over all local auctions, as is composed with i's opt-out value, so that for each $i \in \mathcal{I}$, the allocation from auction j is,

$$a_i^j(s) = \min \Bigg\{ D_i, \tag{4}$$

$$\frac{d_i^j}{\sum_{p_k^j = p_i^j} d_k^j} D_i^j(p_i^j, s_{-i}^j) \Bigg\}. \tag{5}$$

Finally, we must have that the cost to the buyer adheres to the second price rule for each local auction, with total cost to buyer i,

$$\bar{c}_i(s) = \sum_{j \in \mathcal{I}} p^j \left(a_i^j(0; s_{-i}^j) - a_i^j(s_i^j; s_{-i}^j) \right). \tag{6}$$

We intend to show that the PSP constraints are sufficient to attain the desirable property of truthfullness through incentive compatibility. We reason, due to our pricing mechanism, that our formulation upholds the *exclusion-compensation principle*, and is a valid progressive second price auction to the extent that buyer i pays for its allocation so as to exactly cover the "social opportunity cost" which is given by the declared willingness to pay (bids) of the users who are excluded by i's presence, and thus also compensates the seller for the maximum lost potential revenue [2].

3 Related Work

4 The Problem Model

4.1 The Secondary Market

We define the set of users, $\mathcal{I} = \{1, \cdots, I\}$, who purchase or sell data from other users. A buyer submits bids directly to sellers, where we assume that all users submit bids in order to maximize their (private) valuation functions. We assume that users are selfish, and therefore rational. Users prefer to participate in the secondary market as it allows them to purchase additional data for a cost less than the overage fee set by the ISP. In general, user preferences are defined by a utility function, which typically represents a users' valuation of an allocation minus the price,

$$\begin{aligned} u : S &\to (-\infty, \infty) \\ s &\to u(s). \end{aligned}$$

Absent the cost or revenue from trading data, CMHK users gain utility from consuming data. We will assume that the user valuation satisfies the conditions for an *elastic demand function*, as in [2]:

Definition 4.1. [2] A real valued function, $\theta(\cdot) : [0, \infty) \to [0, \infty)$, is an *(elastic) valuation function* on $[0, D]$ if

- $\theta(0) = 0$,
- θ is differentiable,
- $\theta' \geq 0$, and $\theta_i{}'$ is non-increasing and continuous,
- There exists $\gamma > 0$, such that for all $z \in [0, D]$, $\theta'(z) > 0$ implies that for all $\eta \in [0, z), \theta'(z) \leq \theta'(\eta) - \gamma(z - \eta)$.

A user's identity $i \in \mathcal{I}$ as a subscript indicates that the user is a buyer, and a superscript indicates the seller. Suppose user i is buying from user j. A bid $s_i^j = (d_i^j, p_i^j)$, from buyer i means i would like to buy from j a quantity d_i^j and is willing to pay a unit price p_i^j. Without loss of generality, we assume that all users bid in all auctions; if a user i does not submit a bid to j, we simply set $s_i^j = (0, 0)$. A seller j places an ask $s_i^j = (d_i^j, p_i^j)$, meaning j is offering a quantity $d_i^j \in d^j = [d_i^j]_{i \in \mathcal{I}}$ with reserve unit price $p^j = [p_i^j]_{i \in \mathcal{I}}$. $D^j = \sum_{i \in \mathcal{I}} d_i^j$ is the total amount of data j has to sell, and $D_i = \sum_{j \in \mathcal{I}} d_i^j$ is the total amount of data required by a buyer i. We emphasize that we allow for s_i^j to stand for a buyer or sellers' bid, determined by whether or not the user is a buyer or seller. In other words, a bid s^j, the bid is understood as an offer in the secondary market; we assume that data is a unary resource belonging to the seller, and therefore can identify the data (for sale) with the identity of the user. To further clarify our analysis, we will emphasize the separation of buyers and sellers by denoting bid vectors as slices of the grid, i.e. $s^j = [s_i^j]_{i \in \mathcal{I}}$ will denote a sellers' profile, and $s_i = [s_i^j]_{j \in \mathcal{I}}$ denotes a buyers' profile. The notational conventions given by the slices s_i and s^j will be used indicate if a bid is from a buyer or a seller. However, we note that this is a simplification for ease of notation, and considering the grid s in a distributed setting, each buyer i will have information from each auction in which it is participating, and therefore in the limit will have access to the full grid s, i.e. the buyer will participate in all auctions. However, sellers can only gain information about the market grid by observing buyer behavior in their local auction. In our current formulation, we do not allow a seller to host multiple auctions(FUTURE WORK).

The main contribution of this work is an auction mechanism inspired by the classic PSP throughput problem. In order to apply a distributed PSP implementation to CMHK's secondary market, we analize the behavior of users in a dynamical data exchange market. As both buyers and sellers are able to change their bid strategies, and as each user only has *local* information about the bidding environment, it is clear that an unconstrained market, even with a finite number of users, could suffer from the communication expense from numerous local auctions trading an infinitely divisible resource. We will assume that the cost of participatingn in the CMHK secondary market is absorbed by the bid fee, which could represent data used in submitting bids, or a fee charged per unit of data, or a flat rate charged at the completion of the purchase. We perform a simple survey of these bid fee models, we provide some idea of the expected revenue of the mobile data ISP (NEED STATS!). It is worth mentioning that CMHK users are not allowed to resell data purchased from the secondary market, additionally, the purchased data expires (does not carry to the next service period). Therefore, a simple definition of market equilibrium, where supply equals demand, is insufficent to complete a comprehensive analysis of the CMHK data-exchange market behavior. We will make an attempt to address why our formulation at least partially considers some of these issues (BAD), such as the impact of the bid fee on user behavior. (MOVE SOME OF THIS UP) Finally, we .... (WHAT? COMPLETE THE THOUGHT)

We claim that the market is able to achieve an equilibrium as the sellers and buyers have an incentive for a collaborative exchange, our mechanism provides the functions for the effective communication between the connected users. It was shown in [2] that a 2-dimensional message space is sufficient for the PSP auction. Using a restricted message space is essential for the distributed nature of our design (EXPLAIN), however, as a given message can come from many possible types, there is no single way to do the transformation from the direct revelation mechanism to the desired one. This is equivalent to guessing the right direct-revelation-to-desired-mechanism transformation and building it into the allocation rule from the start. (FINISH!) We claim that in this secondary market our formulation not only holds the desired VCG qualities, but minimizes comminication overhead (and so possibly fees paid to the ISP) and auction duration, resulting in a convergence time (FIND COMPETITVE RATIO?) with respect to the classic throughput problem.

4.2 User Behavior

We are not concerned with network bottlenecks, which is purely a bandwidth problem, as in [3], however we reason that there remains an optimal user strategy. As a user no longer needs to bid on a complete (but arbitrary) cluster of nodes with minimum bandwidth allocation (defined as a route in [3]), we reason that a buyer may opt-out of auctions, maximizing its utility while minimizing the number of positive bids submitted to the overall market. We define an **opt-out function**, e, as a function that when composed with our user type to describes its market behavior. In a general sense, e applies our user *strategy* to the PSP rules. Our reserve price function is determined by the subset of nodes participating in the auction, where the seller is its own auctioneer. This implies that the influence of the greater market on the individual auctions will be influenced only by the submission of bids from buyers to sellers. As a buyer may have access to multiple auctions, the sellers will be dynamically influenced by the market via the ϵ-best replies from the buyers. We demonstrate that as the valuation function of seller j is dependent on the buyers demand, and further show evidence of symmetry in the strategies of buyers and sellers.

4.2.1 Buyer Strategy

We define each buyer as a user $i \in \mathcal{I}$ with quasi-linear utility function θ_i, the value of the allocation minus the cost,

$$u_i = \theta_i \circ e_i(a) - c_i, \tag{7}$$

where $e_i : [0,\infty)^I \to [0,\infty),\ \forall\ i \in \mathcal{I}$. We extend the P2P rules described in [3] to account for a set of *local* data-exchange markets.

Suppose the total amount of seller j's data on the network at the instance that user i joins the auction is D^j. A sellers' allocation cannot exceed the total amount they have available, i.e. $\sum_{i\in\mathcal{I}} a_i^j \leq D^j$. This will hold simultaneously for each $i \in \mathcal{I}$ if and only if

$$D^j \geq \sum_{i\in\mathcal{I}} D_i. \tag{8}$$

We define the only "seller" to satisfy (8) to be the ISP. We will show that in our algorithm, sellers are restricted to subset of buyers $\in \mathcal{I}$, and provide a buyer strategy defining when a rational (utility-maximizing) buyer will set $s_i^j = 0$. The seller, in our analysis, is a functional extension of the buyer, with valuation θ^j constructed by buyer demand. We assume that buyers and sellers are separated (a seller does not also buy data and vice versa).

Although it is possible for a seller to fully satisfy a buyer i's demand, it is also reasonable to expect that a seller may come close to using their entire data cap, and only sell the fractional overage. In this case, we determine that buyers must split their bids among multiple sellers. We propose the following strategy,

Proposition 4.1. *(Opt-out buyer strategy) Define any auction duration to be $\tau \in [0,\infty)$. Let $i \in \mathcal{I}$ be a buyer and fix all other buyers' bids s_{-i} at time $t \in \tau$. Define the composition,*

$$e_i^j \circ a = e_i^j(a) = \frac{a_i^j}{j},$$

to be the buyer strategy with respect to quantity, and the set,

$$\mathcal{I}_i(n) = \underset{\mathcal{I}'\subset\mathcal{I},|\mathcal{I}'|=n}{\arg\max} \sum_{j\in\mathcal{I}'} D^j,$$

where buyer i chooses its seller pool by determining n, where

$$n = \min(j \mid j \in \mathcal{I}_i(n) : jD^j \geq D_i), \tag{9}$$

The buyer strategy produces a minimal subset of sellers $\in \mathcal{I}$, so for any fixed n we will denote this subset,

$$\mathcal{I}_i \subset \mathcal{I}. \tag{10}$$

Now let $j^* = n \leq I$, and define,

$$e_i(a) \triangleq e_i^{j^*}(a) \tag{11}$$

$\forall\ j \in \mathcal{I}_i$. As (11) holds $\forall\ j \in \mathcal{I}_i$, we have that e defines an optimal feasible strategy for buyer i from time t to time $(t+1) \in \tau$.

Proof:

We assume that a buyer wants to fufill their data requirement. In the case that there exists a seller who can completely satisfy a buyers' demand, $j^* = 1$, $|\mathcal{I}_i| = 1$ and (9) holds. If such a buyer does not exist, as the set $\mathcal{I}_i$ is an ordered set, i may discover j^* by computing $\mathcal{I}_i$. If we suppose that $D_i > \sum_{j \in \mathcal{I}} D^j$, then $j^* > I$ and $\mathcal{I}_i = \emptyset$. We model the ISP as a seller κ with bid $s^\kappa = (D^\kappa, P^\kappa)$, where $D^\kappa > D^j$, $\forall\, j \in \mathcal{I}_i$, and P^k represents the overage fee for data set by the ISP, which is also the upper bound of the sellers' pricing function, and so again (9) holds. Now consider some $k \neq i \in \mathcal{I}$ where $p_i^j = p_k^j$. The allocation rule (40) determines that the data will be split proportinally between all buyers with the same unit price. It is possible that the resulting partial allocation of data to i and k would not satisfy some demand. As the two cases i and k are the same, we consider such a seller i. Suppose seller j updates its bid to reflect the new data quantity, so that $d_i^{j(t+1)} < d_i^{j(t)}$ (NOTE: IS THE DIRECTION OF THE BID CONFUSING?, i.e. d_i^j from buyer (rhs) or seller (lhs) unclear? does it matter?) Then, by the definition of ℓ_i, we can only have that $j \ni \mathcal{I}_i$, or that n has been increased, and new seller(s) enter the pool. Additionally, we consider that at time $(t+1)$, we can have a new buyer k, where $p_k^j > p_i^j$, $\forall\, j \in \mathcal{I}_i$, in other words, a new buyer k may enter the market with a better price. In this case, by (9), i will increase the value of n so that

$$e_i^{j(t+1)}(a) = e_i^{j(t)}(a) - e_k^{j(t)}(a),$$

and the subset $\mathcal{I}_i$ is large enough to balance the additional demand from k. Thus, as in each case we have that i is able to satisfy thier demand, and we determine that the opt-out strategy is optimal.

Finally, we note that $\mathcal{I}_i$ is not the only possible minimum subset $\in \mathcal{I}$ able to satisfy i's demand, it is the minimal ordered subset where a coordinated bid is possible, the reasoning for which we will address in further analysis (Section 5).

4.2.2 Buyer Influence

The buyer demand is a key market influence (SILLY SENTENCE), as the buyer valuation is elastic, even infintesimal changes in the market dynamics can be modeled. $\forall\, y \geq 0$, we determine that the market demand for D^j is given by,

$$\rho^j(y) = \sum_{i \in \mathcal{I}: p_i^j \geq y} d_i^j, \tag{12}$$

Buyer i's valuation is interpreted as a unit valuation θ_i, which is distributed across the secondary market via bids in local auctions. Buyer behavior is influenced by opponent bid profiles from multiple auctions. The valuation of any user, however, is a function of the entire marketplace. We first define the "inverse" demand for any buyer i,

$$f_i(z) \triangleq \inf \left\{ y \geq 0 : \rho_i^j(y) \geq z,\ \forall\, j \in \mathcal{I} \right\}. \tag{13}$$

For a given demand ρ^j, f_i maps the z-th unit of data to the lowest price at which i could still bid in *any* auction $j \in \mathcal{I}$. Naturally, this is the lowest unit cost of the buyer to participate in all auctions, and corresponds to the maximum reserve price amongst the sellers.

The seller is represented by a similar function, however a seller only has a single auction, and is indirectly influenced by the buyers in other auctions, and so from the perspective of the seller we have a more direct interpretation of revenue.

We define the "inverse" of the buyer demand function for seller j as revenue at unit price y.

$$f^j(z) \triangleq \sup\left\{y \geq 0 : \rho_i^j(y) \geq z,\ \forall\ i \in \mathcal{I}\right\}, \tag{14}$$

where f^j maps to the highest possible unit data price. Thus we have the following Lemma.

Lemma 4.1. *(User valuation) For any buyer i, the valuation of seller j's data may be modeled as,*

$$\theta_i^j \circ e_i^j(a) = \int_0^{e_i^j(a)} f_i(z)\ dz, \tag{15}$$

it follows that

$$\theta_i \circ e_i(a) = \sum_{j \in \mathcal{I}} \int_0^{e_i^j(a)} f_i(z)\ dz. \tag{16}$$

Similarly, seller j's valuation is,

$$\theta^j \circ e_i(a) = \sum_{i \in \mathcal{I}} \int_0^{e_i^j(a)} f^j(z)\ dz. \tag{17}$$

Proof: We assume that a buyer wants to minimize the cost of purchasing their data requirement, at the same time ensuring they get the full allocation $e_i(a)$, and so must minimally meet j's reserve price. A seller will try to maximize profit for any given allocation a, and will try to sell all of its data, and so $d_i^j = e_i(a) \Rightarrow \sum_{i \in \mathcal{I}} d_i^j = D^j = \sum_{i \in \mathcal{I}} e_i(a)$, The remainder of the proof follows as in [3]. (REALLY THOUGH?) (THERE MUST BE A CLEARER WAY TO DESCRIBE THE RELATIONSHIP)

The sellers' natural utility is the potential profit $u^j = \theta^j \circ e_i(a)$, where θ^j is the potential revenue from the sale of data composed with each buyers' opt-out value, $e_i(a)$. We have chosen to omit the original cost of the data paid to the ISP, as a discussion of mobile data plans is outside the scope of this paper.

As j's behavior is restricted by the bid strategy of the buyers, in addition to natural constraints, and we have the following Lemma.

Lemma 4.2. *(Seller constraints) Let j be a seller with total data amount D^j. First, the seller must satisfy the quantity constraint,*

$$d_i^j \geq e_i^j(a) \tag{18}$$

and

$$\sum_{i \in \mathcal{I}} e_i^j(a) \leq \sum_{i \in \mathcal{I}} d_i^j \leq D^j, \tag{19}$$

which implies, for any $i \in \mathcal{I}$,

$$e_i^j(a) \leq D^j - \sum_{k \in \mathcal{I}, k \neq i} e_k^j(a). \tag{20}$$

In addition, for a rational seller, the reserve price must satisfy, $\forall\ i \in \mathcal{I}$,

$$p_i^j \geq \min_{i \in \mathcal{I}} \left(p_i^j\right). \tag{21}$$

Proof: The first statement is obvious, a seller cannot sell more data than indicated in their bid, (50) and (51) enforce a data constraint for the seller. Finally, (52) follows from the assumption that j is rational, and so utility-maximization acts as revenue maximization; a rational seller will not sell its data at a price less than the lowest offer. (ASSUMPTIONS MADE HERE)

4.2.3 Seller Strategy

In order to to develop the seller strategy, it is necessary we determine that a seller has an incentive to accept fractional (CHANGE TO PARTIAL? PICK ONE) bids (i.e. sell a fraction of their data D^j). Reasonably, there may not exist a buyer such that $D_i = D^j$. [1] reasons that the seller does not know the exact amount of leftover data available, and so they may only sell enough data to ensure that they will not become a buyer while they submit their total data overage to the secondary market. Buyers are allowed to bid both dynamically and asychronously, and as a seller determines allocations using only bids in its local market. Therefore, in order to maximize the revenue gained per unit of data the seller must respond to the varation of competitive bids in its market (MORE? FUTURE WORK? CITE?).

We describe the sellers' *local* auction strategy for allocating its data according to the constraints formed by the buyer strategy. As local auction is progressive, and influenced by the ϵ-best replies of the buyers, we will need the following Lemma. We now define the local auction, which we describe, when coupled with the buyer responses, as a progressive game of strategy with incomplete, but perfect information (SAY MORE?). (BUYERS ARRIVE AS A POISSOIN PROCESS? FUTURE WORK)

Proposition 4.2. *(Localized seller strategy (i.e. fractional allocation)) Define any auction duration to be $\tau \in [0, \infty)$. For any seller j, For any seller j, fix all buyers bids $s_i^j = (d_i^j, p_i^j)$ at time $t \in \tau$. Define*

$$\mathcal{I}^j(n) = \underset{\mathcal{I}' \subset \mathcal{I}, |\mathcal{I}'| = n}{\arg\max} \sum_{i \in \mathcal{I}'} p_i^j,$$

where,

$$n = \min(i \mid i \in \mathcal{I}^j(n) : \sum_{i \in \mathcal{I}} d_i^j \geq D^j), \tag{22}$$

We have, for any fixed n at time t, a minimal subset of buyers that maximizes j's revenue, which we will denote,

$$\mathcal{I}^j \subset \mathcal{I}. \tag{23}$$

Define buyer $i^* = n \leq I$. Then, for time $(t+1)$, set j's reserve price as

$$p_i^j = \theta'_{i^*}(d_{i^*}^j) + \epsilon, \tag{24}$$

Let the winner at time t be determined by,

$$\bar{i} = \max_{i \in I^j} p_i^j, \tag{25}$$

and update j's total data to reflect the (tentative) allocation,

$$D^{j(t+1)} = D^{j(t)} - e_{\bar{i}}^{j(t)}(a), \tag{26}$$

Allowing t to range over τ, we have that (25) - (26) produces a local market equilibrium. (BLAST I RUINED THE EQ)

Proof: We assume that the seller has enough data to satisfy at least one buyer, and that they want to maximize their revenue. In the case of multiple buyers i^* is the *losing* buyer with the highest unit price offer, determined by (22), where i^* loses by: (1) i^* is excluded from I^j or (2) i^*s demand is not met, noting as well that any $i \ni \mathcal{I}^j$ is also a loser. In this case, by (49) the seller must notify the buyer of a fractional allocation.

With this caveat, we have that the aggragate demand of subset $\mathcal{I}^j$ is satisfied by seller j. Although the buyers' valuation θ_i is not known to the seller, we will assume that the buyer is bidding truthfully, and so $\theta_i' + \epsilon = p_i^j + \epsilon$, and as $\mathcal{I}^j \subset \mathcal{I}$, we note that (50) and (52) hold. Now, using (46), we have, $\forall\ z \geq 0$,

$$\int_0^{e_{i^*}^j(a)} f_i(z)\ dz \leq \int_0^{e_i^j(a)} f^j(z)\ dz$$

and so,

$$\theta_i \circ e_{i^*}(a) \leq \theta_i \circ e_i(a),$$

which holds $\forall\ j \in \mathcal{I}_i$. It follows that, using the definition of an ϵ-best reply $s_i^j = (v_i^j, w_i^j)$, for any ϵ-best reply,

$$p_i^j \leq \theta_i'(v_i^j) + \epsilon,$$

$\forall i \in \mathcal{I}^j$. Therefore the choice of p_i^j does not force any buyers out of the local auction. Thus we determine the valuation between seller j and buyer i is well-posed, the reserve price (24) is justified, and the local equlibirium created by j is stable from time t to $(t+1)$. We note the special case where for winner $\bar{i}$, $\exists\ k \in \mathcal{I}^j$ such that $p_{\bar{i}}^j = p_i^k$. In this case the seller again notifies the buyers of a fractional allocation by changing d_i^j and d_k^j to reflect the proportional division. Finally, in the case where winning buyer(s) *opts-out* of the local auction, where $j \in \mathcal{I}_i$ at time t, then at time $(t+1)$ $j \neq \mathcal{I}_i \Rightarrow s_i^{j(t+1)} = (0,0) \Rightarrow i \neq \mathcal{I}^j$, which is mathematically analogous to the definition of opting-out in our scenario. The seller may simply return the tentatively allocated data to D^j, and recompute $\mathcal{I}^j$ and $p_{i^*}^j$,

$$D^{j(t+1)} = D^{j(t)} + \sum_{i\in\Gamma} d_i^{j(t)}$$

where Γ represents the buyers who have opted-out from time t to $(t+1)$, indicated by a zero bid from any buyer $\in \mathcal{I}^j$:

$$\Gamma = \{i \in \mathcal{I}^j : s_i^{j(t+1)} = 0\}.$$

(NOTE - NEED TO SAY MORE HERE... INDUCTION? HOW DOES THE TENTATIVE AWARD WORK IN FRACTIONAL ALLOCATIONS? SAVE FOR NASH EQ PROOF? KEEP WITH t TO t+1?)

We conclude this section by examining the relationship between the strategies of buyers and sellers in local auctions. As we have shown, the seller is a functional extension of the buyer, with rules determined by the buyers' behavior. Consequentially, we have the following Lemma.

Lemma 4.3. *(Valuation across local auctions) For any buyer i, let $s_i^j > 0$. Then,*

$$j \in \mathcal{I}_i \Leftrightarrow i \in \mathcal{I}^j. \tag{27}$$

Define the set of users Λ, where

$$\Lambda = \left(\bigcap_{i\in\mathcal{I}^j} \mathcal{I}_i\right) \cup \mathcal{I}^j, \tag{28}$$

and fix all bids $s_i^j \in \Lambda$. (BETTER RESULT!)

Proof: A local auction $j \in \mathcal{I}$, is determined by the collection of buyer bid profiles, where $s_i^j > 0 \Rightarrow j \in \mathcal{I}_i$, and we have that $i \in \mathcal{I}^j$, if and only if $p_i^j \geq p_{i^*}^j$. Suppose $p_i^j < p_{i^*}^j$. By Proposision 4.2, i^* is either included or excluded from $\mathcal{I}^j$. If $i^* \ni \mathcal{I}^j$, then (27) holds; in the latter case, by Proposition 4.1, i^* will add sellers to its pool until its demand is satisfied. Therefore for any $j \in \mathcal{I}_i$, $\nexists\ s_i^j > 0$ where $i \ni \mathcal{I}^j$, and (27) holds. Now, the subset $\mathcal{I}^j \subset \mathcal{I}$ determines j's reserve price $p^j_{i^*}$. Similarly, $\mathcal{I}_i \subset \mathcal{I}$ will determine the (coordinated) unit price p_i in buyer i's (truthful) bid. (22) defines i^* as the losing buyer with the highest bid price, which in turn defines j's reserve price, $p^j_{i^*}$, the lowest price that j will accept to perform any allocation. Using (44) and (45), we have that for each $i \in \mathcal{I}^j$, $p^j_i \geq p^j_{i^*} \geq p^k_i > 0$, which holds $\forall\ k \in \mathcal{I}_i$. Now, fix any buyer $i \in \mathcal{I}^j$, and denote $p^j_{i^*} = p^j$ as the reserve price for auction $j \in \mathcal{I}_i$, (43) implies,

$$\rho^j(p^j) = \sum_{i \in \mathcal{I}^j} e_i(a), \qquad (29)$$

and again using (44) and (45), we have that

$$p^j = f_i \circ e_i(a) = f^j \circ e_i(a). \qquad (30)$$

The sellers $\in \Lambda$ are competing to sell their respective resources. By construction, we have that $j \in \mathcal{I}_i\ \forall\ i \in \mathcal{I}^j$. For any $k \in \bigcap_{i \in \mathcal{I}^j} \mathcal{I}_i$, let $\mathcal{I}^j \cup \mathcal{I}_k$ be a feasible subset, that is, $s^j_i > 0 \in \mathcal{I}^j \cup \mathcal{I}_k$ and $e^j_i(a) \leq e \circ \min(D^j, D_i)$. From (27), $k \in \mathcal{I}_i \Rightarrow i \in \mathcal{I}^k$, so $i \in \mathcal{I}^k \cap \mathcal{I}^j$, and we have from Proposition 4.1 that i will bid so that $e^j_i(a) = e^k_i(a) \Rightarrow f_i(e^k_i(a)) = f_i(e^j_i(a))$, therefore $s^j_i \geq s^k_i > 0$, which holds $\forall\ i \in \mathcal{I}^j$.

We now consider reserve prices among sellers $\in \Lambda$. Fix some $i \in \mathcal{I}^j$, and suppose $\exists\ k \in \mathcal{I}_i$ such that $p^k_i > p^j_i$. As we assume that buyers are bidding their valuation, $p^k = f_i(e^k_i(a)) = f^k(e^k_i(a))$. We have by construction that $j \in \mathcal{I}_i,\ \forall\ i \in \mathcal{I}^j$, and so for any $k \in \mathcal{I}_k \cup \mathcal{I}_i$ we have that $p^k = p^j$. This holds for any $i \in \mathcal{I}^j$ and similarly if $p^k_i < p^j_i$. We have that $f_i = f^k = f^j,\ \forall\ i, j, k \in \Lambda$, for simplicity we denote each as f. Now, from (29), and intersection of subspaces, we determine the aggregate demand from buyer $i \in \mathcal{I}^j$ over the set of intersecting markets Λ to be,

$$\sum_{k \in \mathcal{I}_i \cup \mathcal{I}^j} \rho^k(p^*) = \sum_{k \in \mathcal{I}_i \cup \mathcal{I}^j} e^k_i(a), \qquad (31)$$

where $p^* = \max(p^k),\ \forall\ k \in \Lambda$, and corresponds to the highest reserve price over all the sellers. Now, by the continuity of θ', (AND)

$$\begin{aligned}
&\sum_{i \in \mathcal{I}^j} \Bigg(\sum_{k \in \mathcal{I}_i \cup \mathcal{I}^j} \int_0^{e^k_i(a)} f(z)\ dz \Bigg) \\
&\quad = \sum_{i \in \mathcal{I}^j} \theta_i \circ e_i(a) \\
&\quad = \sum_{k \in \mathcal{I}_i \cup \mathcal{I}^j} \Bigg(\sum_{i \in \mathcal{I}^j} \int_0^{e^k_i(a)} f(z)\ dz \Bigg) \\
&\quad = \sum_{k \in \mathcal{I}_i \cup \mathcal{I}^j} \theta^j \circ e_i(a) \\
&\quad = (\theta_i \circ \theta^j) \circ e_i(a).
\end{aligned}$$

For completeness, in the case where the ISP κ does not adhere to the market dynamics, so $P^\kappa > p^j + \epsilon,\ \forall\ j \in \mathcal{I}$, then we may absorb the overage (difference) as part of the bid fee.

(ADD TIME? INDUCTION GOES HERE? DID NOT SHOW!)

(OWN WORDS) Since the sellers are driven by the buyers' demands, and the buyers are competing in multiple local auctions hosted by the sellers, the two games are inter-dependent, and may be played on the same or on a different scale in valuation, time (AND....).

PSP relies on the *relevation prinicple*; the PSP mechanism is able to demonstrate equlibrium by showing that PSP is incentive-compatible. (MORE?) Arrow's paradox is an impossibility theorem stating that when voters have three or more distinct alternatives (options), no ranked voting electoral system can convert the ranked preferences of individuals into a community-wide (complete and transitive) ranking while also meeting a specified set of criteria: unrestricted domain, non-dictatorship, Pareto efficiency and independence of irrelevant alternatives.

Thus, (WHAT? FINISH, OWN WORDS) The designer of a mechanism generally hopes either to design a mechanism y() that "implements" a social choice function to find the mechanism y() that maximizes some value criterion (e.g. profit) Independence of irrelevant alternatives (IIA) The IIA condition has three purposes (or effects)

Normative Irrelevant alternatives should not matter. Practical Use of minimal information. Strategic Providing the right incentives for the truthful revelation of individual preferences.

4.3 PSP Formulation

Consider a user seeking to prevent data overage by purchasing enough data from a subset of other network users. This user i can be modeled as a opt-out buyer where, as in [3], We intend to show that this network setting results in a shared network optima (a global optimum). The formulation is inspired to the thinnest allocation route for bandwidth given in [2]. We note that if a single seller j can satisfy i's demand, then (7) reduces to the original form, defined in [3] as "a simple buyer at a single resource element".

The sellers' auction will function as follows: at each bid iteration all buyers submit bids, and the winning bid is the buyer i that has the highest price p_i^j. The seller allocates data to this winner, at which point all other buyers are able to bid again, and the winner leaves the auction (with the exception where multiple bidders bid the same price, where (40) determines they will not fully satisfy their demand, and so we will assume they remain in the auction). The auction progresses as such until all the sellers' data has been allocated. We design an algorithm based on the sellers' fractional allocation strategy.

(NOTE - FIX TO UPDATED NOTATION!)

Algorithm 1 (Seller fractional allocation)

1: $p^{j(0)} \leftarrow \epsilon$
2: $s^{j(0)} \leftarrow (p^j, D^j)$
3: **while** $D^j > 0$ **do**
4: $\quad \mathcal{I}^{j(t+1)} = \mathcal{I}^{j(t)} \setminus \{i \in \mathcal{I}^{j(t)} : d_i^j > D^{j(t)}\}$
5: $\quad \bar{i} \leftarrow \arg\max_{I^j} \sum_{i \in I^j} p_i^j$
6: $\quad D^{j(t+1)} \leftarrow D^{j(t)} - e_{\bar{i}}^{j(t)}(a)$
7: $\quad p^j \leftarrow p_{i^*}^j + \epsilon$ and $d^j \leftarrow D^{j(t+1)}$
8: $\quad s^{j(t+1)} \leftarrow (d^j, p^j)$
9: $\quad$ **if** $\exists\ i : s_i^{j(t+1)} \neq s_i^{j(t)}$ **then**
10: $\qquad D^{j(t+1)} = D^{j(t)}$
11: $\qquad t \leftarrow t+1$
12: $\qquad$ Go to 4.
13: $\quad$ **else**
14: $\qquad \bar{i} \leftarrow e_{\bar{i}}^j(a)$
15: $\qquad t \leftarrow t+1$
16: $\qquad$ Go to 4.

We assume that each time that s^j is updated it is shared with all participating buyers. At this point buyers have the opportunity to bid again, where a buyer that does not bid again is assumed to hold the same bid, since a buyer dropping out of the auction will set their bid to $s_i^j = (0, 0)$. As we will show in our analysis, the buyers are bidding truthfully; the algorithm makes use of the fact that the sellers' valuation is determined by the buyers' market and upholds the PSP mechanism. (CHECK) (ALSO j NEEDS TO UPDATE FOR NEW BUYERS, j'S CONTROL)

(BUYER ALGORITHM? WHY NOT..., CLEANER)

5 PSP Analysis

5.1 Equilibrium

Consider an opt-out buyer $i \in \mathcal{I}$. A PSP auction allows for a buyer to react to opponents bids, and so its incentive is based on the opponent profile.

Due to (7), i only has an incentive to change its bid quantity if it increases its opt-out value e_i, and therefore its utility. We will show that i can coordinate its bids over $\mathcal{I}_i$ so that the opt-out value e_i is the same for each $j \in \mathcal{I}_i$, and therefore, without loss of utility, i may choose a seller pool using a "consistent" strategy, where for any $j \in \mathcal{I}_i$, $d_i^j = d_i^k$, $\forall\ k \in \mathcal{I}_i$, and still have feasible best replies. Our result shows that a buyer may select $\mathcal{I}_i$ in order to maximize its utility while maintaining a coordinated bid strategy. It is intuitive that, if $j^* < I$, a buyer may increase the size of $\mathcal{I}_i$, thereby lowering its bid quantity while obtaining the same (potential) allocation a_i. (OWN WORDS!) One important question to ask is why a bidder should bid with identical unit prices on (j) all auctions and not reduce the unit price to a level where he still wins the amount q i . The reason for this can be found in the pricing rule of the PSP auction.

Lemma 5.1. *(Opt-out buyer coordination) Let $i \in \mathcal{I}$ be a opt-out buyer and fix all sellers' profiles s^j. For any profile $S_i = (D_i, P_i)$, let $a_i \equiv \sum_j a_i^j(s)$ be the resulting data allocation. For any fixed S_{-i}, a better reply for i in any auction* *is* $x_i = (z_i, y_i)$*, where* $\forall\ j \in \mathcal{I}_i$*,*

$$z_i^j = e_i^{j^*}(a),$$
$$y_i^j = \theta_i'(z_i^j).$$

Furthermore,

$$a_i^j(z_i, y_i) = z_i^j, \tag{32}$$

and

$$c_i^j(z_i, y_i) = y_i^j, \tag{33}$$

where i*'s strategy is as in Proposition 4.1.*

Proof: As s_{-i} is fixed, we omit it, in addition, we will use $u \equiv u_i \equiv u_i(s_i) \equiv u_i(s_i; s_{-i})$. In full notation, we intend to show

$$u_i((d_i, p_i); s) \leq u_i((z_i, y_i); s_{-i}).$$

If there exists a seller who can fully satisfy i's demand, then $|\mathcal{I}_i| = 1$, and the case is trivial as no coordination is necessary for a single bid. (SPECIAL CASE OF MONOPOLY? THINK!)

Otherwise, buyer i's demand can only be satisfied by purchasing data from multiple sellers. We will show that i may increase $|\mathcal{I}_i|$, and so decreasing d_i^j, $\forall\ j \in \mathcal{I}_i$, without decreasing u_i. Buyer i maintains ordered set ℓ_i where the sellers with the largest bid quantities are considered first; the index of seller j^* defines a minimal subset $\mathcal{I}_i$, satisfying (9). By construction, $d_i^{j^*}$ is the minimum quantity offered by any $j \in \mathcal{I}_i$, so $d_i^{j^*} \leq d_i^j$, $\forall\ j \in \mathcal{I}_i$; $\mathcal{I}_i$ also defines the maximum quantity bid of any $k \ni \mathcal{I}_i$. Thus by (9) and (11), $\forall\ j \in \mathcal{I}_i$, $k \ni \mathcal{I}_i$,

$$e_i^k(a) \leq z_i^j = e_i^{j^*}(a) \leq e_i^j(a),$$

and so,

$$e_i^{j^*}(a) \leq \left[D^j - \sum_{k \in \mathcal{I}^j : p_k^j > y_i^j} d_k^j\right]^+, \tag{34}$$

The buyer valuation function (47), guarantees that $\forall\ j \in \mathcal{I}_i$, $y_i^j \geq p_{i^*}^j$, where $p_{i^*}^j$ is the reserve price of seller j, defined in Proposision 4.2, and is by definition the minimum price for a buyer bid to be accepted. As D_i is non-decreasing, $\forall\ j \in \mathcal{I}_i$, $k \ni \mathcal{I}_i$,

$$D_i^j(y_i^j) \geq D_i^j(p_i^{j^*}) \geq D_i^j(p_i^k).$$

Furthermore, suppose there exists buyer $k \in \mathcal{I}^j$, such that $e_k^{j^*}(a) \geq e_i^{j^*}(a)$ and so $d^j - d_k^j < z_i^j$, resulting in a partial allocation. Bid $s_k \in S_i$, and so is considered in buyer i's strategy. As buyer i is allowed to choose subset $\mathcal{I}_i$, and $\mathcal{I}_i$ is a minimal set, (9) states that n is such that $e_i^{j*}(a) \geq e_k^{j^*}(a)$ for any k, therefore such a buyer k cannot exist. Thus (34) holds and so, by (40),

$$a_i^j(z_i, p_i) = \min_{i \in \mathcal{I}^j}\left(z_i^j, \left[D^j - \sum_{p_k^j > y_i^j} d_k^j\right]^+\right)$$
$$= z_i^j = e_i^{j^*}(a)$$

where the last equality is by definition, and so (32) is proven. Now, since a buyer is only charged with the cost of excluding other players from the market, the unit price does not influence the final charges. This unit price reflects the valuation of the total resources gained from the multi-auction market, and so a buyer uses this price on all auctions. We have, $\forall\ j\ \in \mathcal{I}_i$, $\forall\ k \ni\ \mathcal{I}_i$,

$$y_i^j \geq p_i^j \geq p_{i^*}^j \geq p_i^k,$$

and we observe that from (38), $\bar{D}_i^j(y, s_{-i}) = 0 \ \forall \ y < p_{i*}^j$, and so $y = 0 \leq \epsilon \Rightarrow e_i^j(a) = 0$, and so clearly $z_i^k = 0, \ \forall \ k \ni \mathcal{I}_i$, and therefore,

$$\sum_{j \in \mathcal{I}_i} c_i^j(z_i, y_i) = \sum_{j \in \mathcal{I}_i} c_i^j(z_i, p_i),$$

thus (33) simply shows that changing the price p_i^j to y_i^j does not exclude any additional buyers, as the bid p_i^j was already above the reserve price of any seller $j \in \mathcal{I}_i$. We proceed to demonstrate that x_i does not result in a loss of utility for buyer i; we will show that

$$u_i \leq u_i(z_i, y_i).$$

From (32), we have $a_i^j(z_i, y_i) = z_i^j = e_i^j(a(z_i, y_i))$, which implies that

$$\theta_i \circ e_i^j(a(z_i, y_i)) = \theta_i \circ e_i^j(a).$$

Therefore, by the definition of utility (7), and the buyers' valuation (47),

$$\begin{aligned}
&\theta_i \circ e_i(a(z_i, y_i)) - \theta_i(a)) \\
&= u_i(z_i, y_i) - u_i = c_i^j - c_i^j(z_i, y_i) \\
&= \int_{a_i^j(z_i,p_i)}^{a_i^j} f_i(D_i - z) \ dz.
\end{aligned}$$

Then, as $a_i(z_i, p_i) \leq z_i^j \leq a_i^j$, and noting that $\theta_i \geq 0 \Rightarrow f_i \geq 0$, we have $u_i(z_i, y_i) - u_i \geq 0, \forall \ j \in \mathcal{I}_i$. Finally, as we do not increase the aggragate demand of the buyer, $\sum_{j \in \mathcal{I}_i} c_i^j(s) \leq \sum_{j \in \mathcal{I}_i} b_i^j$, and so x_i is feasible.

In effect, we are using the buyer demand to partition the auction space based on their type (EXPLAIN).

5.1.1 Incentive Compatibility

We proceed to claim that the optimality of truth-telling holds in our formulation, where the market functions as a hybrid of [3] and [1]. The opt-out buyers' market is comprised of the minimal subset of sellers with the largest amounts of available data, described in the buyer strategy as $\mathcal{I}_i$. To achieve incentive compatibility, we find that the opt-out buyer must choose this subset so that its overall marginal value is greater than its market price. The buyers' market price is calculated as the maximum of the reserve prices of the sellers in the opt-out buyers' pool. the market prices at the The actual bids are obtained from the opt-out buyers' strategy. The quantity to bid is given by the auction mechanism, i.e. (37) and (38), as the maximim possible quantity of data that a buyer i can bid over its seller pool while maintaining its aggragate marginal valuation greater than the aggragate of minimum prices maintained by the sellers in i's pool. As with a single resource, [3] and [2], we show that truth-telling is optimal for the buyer, i.e. in each auction, the buyer sets the bid price to the marginal value.

Seller j's reserve price is determined by a buyer $i \ni \mathcal{I}^j$, and therefore, even if this price is zero, then $p^j = \epsilon \geq 0$.

We argue that if truthfulness holds *locally* for both buyers and sellers, i.e. $p_i = {\theta_i}' \ \forall \ j \in \mathcal{I}_i$ and $p^j = {\theta^j}' \ \forall \ i \in \mathcal{I}^j$, then there exists a local market equilibrium (NOTE: NOT GLOBAL! YET). We have the following Proposition. For completeness, we use the full notation. (BAD SENTENCE)

Proposition 5.1. *(Incentive compatibility in local auctions) For any seller j, let time $t \in \tau$ be fixed and for any buyer $i \in \mathcal{I}^j$, let s_{-i} also be fixed.)VALIDATE: FOR ALL j) Define,*

$$\eta_i = \sup\left\{x \geq 0 : {\theta_i}'(x) > \bar{P}_i^j(x)\right\}, \quad (35)$$

$$\chi_i = \sup\left\{x \geq 0 : \int_0^x \bar{P}_i(x)\ dx \leq \sum_{j\in\mathcal{I}_i} b_i^j\right\}, \quad (36)$$

$z = \min(\eta_i, \chi_i - \epsilon/\theta_i'(0))^+$, and for each $j \in \mathcal{I}_i$,

$$v_i^j = e_i^{j^*}(z)$$

and

$$w_i^j = \theta_i'(z).$$

Then a (coordinated) ϵ-best reply for the opt-out buyer is $t_i = (v_i, w_i)$, i.e., $\forall\ s_i, u_i(t_i; s_{-i}) + \epsilon \geq u_i(s_i; s_{-i})$. With reserve prices $p^j > 0$, there exists a "truthful" local game embeddeded in each local auction, and thus an equilibrium point for the local auction.

Proof: For the buyer, we show that t_i is an ϵ-best reply. That is,

$$u_i(t_i; s_{-i}) + \epsilon \geq u_i(s_i; s_{-i}).$$

Let $z = \eta_i^j$. We have that $i \in \mathcal{I}^j$, and (44) defines $\theta_i'(z)$ as being max of the reserve prices p_i^j, $\forall\ j \in \mathcal{I}_i$, therefore (35) is such that,

$$\theta_i'(z) > \bar{P}_i^j(z),$$

which implies, as (MORE) θ_i' is non-increasing and $P_i^j \geq 0$, we have $\forall\ j \in \mathcal{I}_i$,

$$w_i^j > P_i^j(v_i^j)$$
$$\Rightarrow v_i^j \leq D_i^j(w_i^j) = D^j - \rho^j(w_i^j).$$

And so, by (40),

$$a_i^j(t_i) = v_i^j$$
$$\Rightarrow \sum_{j\in\mathcal{I}_i} a_i^j(t_i) = z.$$

Therefore,

$$\sum_{j\in\mathcal{I}_i}\int_0^{v_i^j} P_i^j(x)\ dx = \int_0^z \bar{P}_i^j(x)\ dx$$
$$= \sum_{j\in\mathcal{I}_i}\int_0^{e_i^j(z)} P_i^j(x)\ dx.$$

Suppose $\exists\ s_i = (d_i, p_i)$ such that $u_i^j(s_i; s_{-i}) > u_i^j(t_i; s_{-i}) + \epsilon$. The buyer coordinated strategy, from Propositions 5.1 and 4.1, gives $s_{i^*} = (e_i \circ a_i(s),\ \theta_i'(z)) = (z_i, c_i)$, where for each $j \in \mathcal{I}$, $a_i^j(s_{i^*}; s_{-i}) = z_i^j$, then clearly $u_i(s_{i*}, s_{-i}) \geq u_i(s_i, s_{-i}) \Rightarrow u_i(t_i; s_{-i}) - u_i(s_i; s_{-i}) > \epsilon$. Denoting z_i^j (fixed) as ζ,

$$\int_z^\zeta {\theta_i}'(x)\ dx - \int_z^\zeta \bar{P}_i(x)\ dx > \epsilon.$$

For concave valuation functions, the first-order derivative of θ at point 0 gives the maximum slope of the valuation function, and so the factor $\epsilon/\theta'(0)$ guarantees that new bids will differ by at least ϵ, and as such, buyer i will remain in any local auction with reserve price determined by (**??**). We therefore verify that,

$$\int_z^{z+\epsilon/\theta_i'(0)} \theta_i'(x)\ dx \leq \epsilon,$$

and as $P_i^j \geq 0$, we have that, from the construction of ζ,

$$\int_{z+\epsilon/\theta_i'(0)}^\zeta \theta_i'(x)\ dx - \int_{z+\epsilon/\theta_i'(0)}^\zeta \bar{P}_i(x)\ dx > 0.$$

If $\zeta > z + \epsilon/\theta_i'(0)$, then for some $\delta > 0$, $\theta_i(z + \epsilon/\theta_i'(0) + \delta) > P_i^j(z + \epsilon/\theta_i'(0) + \delta)$, contradicting (35). Now, if $\zeta \leq z$, then $\theta_i'(z + \epsilon/\theta_i'(0)) < P_i^j(z + \epsilon/\theta_i'(0))$, also a contradiction of (35), and so buyer s_i cannot exist.

We consider the other term in the minimization; let $z = \chi_i$. We have that For any coordinated bid s_i such that $u_i(z_i, c_i) > u_i(t_i) + \epsilon$, then $\zeta > z \Rightarrow c_i(s_i) > b_i$, and so is infeasible. Finally, by Lemma 4.3, for a single buyer-seller interaction, we have that $f_i = f^j \Rightarrow \theta_i \circ e_i^j = \theta^j \circ e_i^j$. Therefore $\theta_i' = \theta^{j\prime}$. We have by Proposision 4.2 and (**??**eqn 30), that $\theta^j = \max_{i \in \mathcal{I}^j} \theta_i$, and so for any $i \in \mathcal{I}^j, \theta^j(z) \geq \theta_i(z)$, and so j cannot set a higher reserve price, and also cannot sell more data than the buyers' bids, so we have that j's bid is both truthful and optimal.

——————————-REF——————————-

$$\bar{P}_i(z, s_{-i}) = \sum_{j \in \mathcal{I}} P_i^j(z_i^j, s_{-i}^j)$$
$$= \sum_{j \in \mathcal{I}} \left(\inf \left\{ y \geq 0 : D_i^j(y, s_{-i}^j) \geq e_i^j(z) \right\} \right), \tag{37}$$

$$D_i^j(y, s_{-i}^j) = \left[D^j - \sum_{p_k^j > y} d_k^j \right]^+, \tag{38}$$

$$\bar{D}_i(y, s_{-i}) = \sum_{j \in \mathcal{I}} \left(\sup \left\{ z \in \left[0, D^j\right] : \bar{P}_i(z, s_{-i}^j) < y \right\} \right). \tag{39}$$

$$a_i^j(s) \tag{40}$$
$$= \min \left\{ e_i^j(D_i), \frac{d_i^j}{\sum_{p_k^j = p_i^j} d_k^j} D_i^j(p_i^j, s_{-i}^j) \right\}. \tag{41}$$

$$\bar{c}_i(s) = \sum_{j \in \mathcal{I}} p^j \left(ai^j(0; s_{-i}^j) - a_i^j(s_i^j; s_{-i}^j) \right). \tag{42}$$

$$\rho^j(y) = \sum_{i \in \mathcal{I} : p_i^j \geq y} d_i^j, \tag{43}$$

$$f_i(z) \triangleq \inf \left\{ y \geq 0 : \rho_i^j(y) \geq z, \ \forall\, j \in \mathcal{I} \right\}. \tag{44}$$

$$f^j(z) \triangleq \sup \left\{ y \geq 0 : \rho_i^j(y) \geq z, \ \forall\, i \in \mathcal{I} \right\}, \tag{45}$$

$$\theta_i \circ e_i^j = \int_0^{e_i^j(a)} f_i(z) \, dz, \tag{46}$$

$$\theta_i \circ e = \sum_{j \in \mathcal{I}} \int_0^{e_i^j(a)} f_i(z) \, dz. \tag{47}$$

$$\theta^j \circ e = \sum_{i \in \mathcal{I}} \int_0^{e_i^j(a)} f^j(z) \, dz. \tag{48}$$

$$d_i^j \geq e_i^j(a) \tag{49}$$

$$\sum_{i \in \mathcal{I}} e_i^j(a) \leq \sum_{i \in \mathcal{I}} d_i^j \leq D^j, \tag{50}$$

$$e_i^j(a) \leq D^j - \sum_{k \in \mathcal{I}, k \neq i} e_k^j(a). \tag{51}$$

$$p_i^j \geq \min_{i \in \mathcal{I}} \left(p_i^j \right). \tag{52}$$

———————————————————————-

(SELLER FINISH)
This forms a "truthful" local game embeddeded within j's auction with strategy space restricted to ϵ-best replies from

buyers $\in \mathcal{I}^j$. Therefore we have that a fixed point in the "truthful" local game is a fixed point for the auction. We further argue that as the set $\mathcal{I}^j$ is computed at each bid iteration, that our result holds for time $(t+1) \in \tau$.

Lemma 5.2. *(Static Data Nash Equilibrium)*

Theorem 5.1. *(Data Nash Equilibrium) Using the rules of the data auction mechanism, the secondary market described in [1] converges to a ϵ-Nash equilibrium. In the network auction game with the data-PSP rules applied independently by each user according to their respective strategies, the secondary market converges to an ϵ-Nash equilibrium.*

Proof: 2. using the min price of sellers in the auction i.e. $\theta_i{}'(d^j_{i^*}) = p^j$ is OK,

3. that bids are still feasible AND optimal

4. the algorithm achieves global economic equilibrium)

NEED TO COVER:

1. Change in buyer valuation
2. New buyers
3. Not enough buyers
4. Not enough data

TRY:

Sellers only act when the resources obtained by the buyers influence their respective reserve prices, which agrees with the seller stragety of attempting to sell their data in the first iteration. Therefore we claim there exists a market stability and therefore, the existence of a Nash equilibrium. As the valuation of the sellers is derived by the demand of the buyers, who are bidding equivalent bids over a minimum subset of buyers, we claim that the seller strategy, along with the seller constraint (**??**) results in a global market equilibrium. We have shown that the local equlibirium created by j is stable from time t to $(t+1)$. Now, suppose that buyer i^* computes its best response $s^j_i = (v^j_i, w^j_i)$ Finally, suppose that a buyer k enters the market such that for some buyer $l \in \mathcal{I}^j$,

$$\sum_{i \in \mathcal{I}^j} p^j_i e^j_i(a) + p^j_k e^j_k(a) \geq \sum_{i \in \mathcal{I}^j} p^j_i e^j_i(a) - p^j_l e^j_l(a),$$

that is,

NOTES: (today)

2. finish seller incentive compat
3. work on progression
4. check reserve price = monopoly price

5.2 Efficiency

(NEED OWN WORDS) The objective in designing the auction is that, at equilbrium, resources al- ways go to those who value them most. Indeed, the PSP mechanism does have that property. This can be loosely argued as follows: for each player, the marginal valuation is never greater than the bid price of any opponent who is getting a non-zero allocation. Thus, whenever there is a player j whose marginal valuation is less than player i 's and j is getting a non-zero allocation, i can take some away from j , paying a price less than i 's marginal valuation, i.e. increasing u i , but also increasing the total value, since i 's marginal value is greater. Thus at equilibrium, i.e. when no one can unilat- erally increase P their utility, the total value is maximized.

5.3 Convergence

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