# 1 Introduction

China Mobile Hong Kong (CMHK) recently introduced such a secondary market. CMHKs 2cm (2nd exchange market) data exchange platform allows users to submit bids to buy and sell data, with CMHK acting as a middleman both to match buyers and sellers and to ensure that the sellers trading revenue and buyers purchased data are reflected on customers monthly bills. [1]

In this work, we propose a distributed progressive second price (PSP) auction in order to maximize social utility. We show that for cellular data allocated between multiple users there exists a Nash market equilibria when all users bid their real marginal valuation of the bandwidth resource. The P2P auction's (as in all auctions), demand information is not known centrally, rather it is distributed in the buyers' valuations. A basic goal is that the mechanism also be *distributed* in that the allocations at any element depend only on *local* state: the quantity offered by the seller at that element, and the bids for that element only.

We suppose that each seller (resp. buyer) can submit a bid to the secondary market consisting of (i) an available (required) quantity and (ii) a unit-price (calculated using its own demand functions). Buyers submit bids cyclically until an ($\epsilon$-Nash) equilibrium is reached where $\epsilon$ corresponds to a bid fee to be paid to the ISP on completion of the transaction.

## 1.1 Progressive Second Price Auctions

The PSP auction first introduced in [2] forms a part of the overall market based allocation model. Consider a noncooperative game where $I$ users buy the fixed amount of resource $D$ from one seller. Suppose each user $i \in \mathcal{I}$ makes a bid $s = (p_i, d_i)$ to the seller, where $p_i$ is the unit-price the user is willing to pay and $d_i$ is the quantity the user desires. $s \equiv [s_i]_{i\in\mathcal{I}}$ is the bidding profile and $s_i \equiv [s_1, \cdots, s_{i1}, s_{i+1}, \cdots, s_N]$ is the profile of user $i$s opponents. The market price function (MPF) of user $i$ is defined as:

$$P_i(z, s_i) = \inf\left\{ y \geq 0 : D - \sum_{p_k > y, k\neq i} d_k \geq z \right\}, \tag{1}$$

which is interpreted as the minimum price a user bids in order to obtain the resource $z$ given the opponents profile $si$. Its inverse function $D_i$ is defined as follows:

$$D_i(y, s_i) = \left[ D - \sum_{p_k > y, k\neq i} d_k \right]^+, \tag{2}$$

which means the maximum available quantity at a bid price of $y$ given $s_i$. With this notation, the PSP allocation rule [4] is defined as

$$a_i(s) = \min\left\{ d_i, \frac{d_i}{\sum_{k:p_k=p_i} d_k} D_i(p_i, s_{-i}) \right\}, \tag{3}$$

$$c_i(s) = \sum_{j\neq i} p_j \left[ a_j(0; s_{-i}) - a_j(s_i; s_{-i} \right], \tag{4}$$

where $a_i$ denotes the quantity user $i$ obtains by a bid price $p_i$ (when the opponents bid $s_i$) and the charge to user $i$ by the seller is denoted $c_i$. $c_i$ is interpreted to be the total cost incurred in the system if user $i$ is removed from the auction. Note that the allocation rule is modified according to [4], so that buyers with identical unit-price $p_i$ are not rejected.

We further modify the allocation rule in [4] in order to suit our model. We argue that as there is no capacity bottleneck in our network as in [2] and [3], we are able to increase the bid quantity and take the maximum of the allocation rule. We will introduce a player type called an *opt-out buyer* in order to perform our analysis. The opt-out buyer maximizes its utility by placing bids with all sellers who are able to fully meet their data requirements, similar to [1]. We propose a *data allocation rule*,

**Proposition 1.1.** *(Data allocation rule)*

$$a_i(s) = \max\left\{d_i, \frac{d_i}{\sum_{k:p_k=p_i} d_k} D_i(p_i, s_{-i})\right\} \tag{5}$$

The modified allocation rule changes the market dynamics by adding a threshold to buyer behavior; the strategy is now to submit bids to a subset of players, rather than incuding an assumption that data infinitely divisibile.

(NOTE: move this?) We extend the P2P rules as in [3] to include a *local* market price function as determined by the subset of nodes participating in the auction. Therefore the influence of the greater market on the individual auctions will be influenced only by the submission of bids from buyers to sellers. As a buyer may have access to multiple auctions, the sellers will be dynamically influenced by the $\epsilon$-best replies from the buyers. The valuation function of seller $j$ is dependent on the buyers demand, and can be modeled as a function of their potential revenue.

Absent the cost or revenue from trading data, users gain utility from consuming data. We use the $\alpha$-fair utility functions [4] to model the usage utility from consuming $d$ amount of data:

$$\theta(d) = \frac{\sigma d^{1-\alpha}}{1-\alpha} \tag{6}$$

where $\sigma$ is a positive constant representing (the scale) of the usage utility and we take $\alpha \in [0, 1)$. We verify that the user valuation above satisfies the conditions for an *elastic demand function*: (NOTE: this part seems in the wrong place)

**Definition 1.1.** [2] A real valued function $\theta(\cdot)$ is an *(elastic) valuation function* on $[0, D]$ if

- $\theta(0) = 0$;
  *Verification:* (obvious)
- $\theta$ is differentiable;
  *Verification:* The derivative, $\sigma d^{-\alpha}$, is positive assuming non-negative data requirements.
- $\theta' \geq 0$, and $\theta'$ is non-increasing and continuous;
  *Verification:* $U$ is differentiable for all $d$, and therefore continuous. Its derivative is a negative exponential, and so is non-increasing.

- There exists $\gamma > 0$, such that for all $z \in [0, D]$, $\theta'(z) > 0$ implies that for all $\eta \in [0, z), \theta'(z) \leq \theta'(\eta) - \gamma(z-\eta)$.
  *Verification:* Without loss of generality, we may set the scaling constant $\sigma = 1$, and compute the curvature $\gamma(\xi)$, where by definition,
  $$\gamma = \frac{\theta''}{(1+\theta')^{3/2}} = \frac{-\alpha\xi^{-\alpha-1}}{(1+\xi^{-2\alpha})^{3/2}}.$$
  Using a Taylor theorem approximation,
  $$\begin{aligned} z^{-\alpha} &\leq \eta^{-\alpha} + \frac{-\alpha d^{-\alpha-1}}{(1+\xi^{-2\alpha})^{3/2}}(z-\eta) \\ &\leq \eta^{-\alpha} + \frac{\alpha}{2\sqrt{2}\xi}(z-\eta) \\ &\leq \eta^{-\alpha} + \frac{\alpha}{\xi}(z-\eta). \end{aligned}$$
  Now, using Taylor repeatedly, simplifying and taking the limit as $\eta \to z$,
  $$\begin{aligned} z^{-\alpha} - \eta^{-\alpha} &\leq -\alpha\eta^{-\alpha-1}(z-\eta) \\ &\leq \frac{-\alpha}{\xi}(z-\eta). \end{aligned}$$
  And so, since $\xi \leq \eta^{\alpha+1}$, we may set
  $$\gamma \geq \frac{-\alpha\eta^{-(\alpha+1)^2}}{\left(1+(\eta^{-2\alpha(\alpha+1)}\right)^{3/2}},$$
  which holds in the case that $z > 1$, and so assuming that there must be at least one unit of data required for a user to have a valuation, we have that the concavity of $\theta'$ is shown by Squeeze theorem.

We may now define the user's utitlity function as

$$u_i = \theta_i(a_i(s)) - c_i(s). \tag{7}$$

Under the PSP rule, [2] shows that given the opponents bids $s_{-i}$, user $i$'s $\epsilon$-best response $s_i = (w_i, v_i)$ as a Nash move (where $s_i$ is chosen to maximize $i$'s utility with $s_{-i}$ held constant), is given by:

$$v_i = \sup\left\{d \geq 0 : \theta'(d) > P_i(d),\right. \\ \left.\int_0^d P_i(\eta)\ d\eta \leq b_i\right\} - \frac{\epsilon}{\theta_i'(0)} \tag{8}$$

(best quantity reply)

$$w_i = \theta_i'(v_i) \quad \text{(best unit-price reply)}, \tag{9}$$

where $\epsilon > 0$ is the bid fee, $b_i$ is user $i$'s budget, and every user has an elastic demand function.

In the Secondary Market [1], we intend to show that it is optimal for a buyer to fully satisfy thier demand in a single auction, that is, a buyer will purchase its required data from a single seller. (WHAT?? why do I think this? Answer: defining the routes as the sellers who have the EXACT amount of data that the buyer requires, thus the min route in [2] is the same as the subset of users who have just enough data to meet the buyer's needs)

# 2 Related Work

# 3 The Problem Model

## 3.1 The Secondary Market

We consider the set of $\mathcal{I} = 1, \cdots, I$ users who purchase or sell data from other users. A buyer is matched with sellers who enough leftover data to satisfy thier demand, and will submit bids in order to maximize thier (private) valuation. A user's identity $i \in \mathcal{I}$ as a subscript indicates that the user is a buyer, and a superscript indicates the seller. Suppose user $i$ is buying from user $j$. A bid $s_i^j = (d_i^j, p_i^j)$, meaning $i$ would like to buy from $j$ a quantity $d_i^j$ and is willing to pay a unit price $p_i^j$. Without loss of generality, we assume that all users bid in all auctions; if a user $i$ does not need to buy from j, then this means that the user has the exact amout of data they require and we simply set $s_i^j = (0, 0)$. A seller $j$ places an ask $s_j^j = (d_j^j, p_j^j)$, meaning $j$ is offering a quantity $d_j^j$ , with a reserve unit price of $p_j^j$ . In other words, when the subscript and superscript are the same, the bid is understood as an offer in the secondary market; we assume that data is a unary resource belonging to the seller, and therefore can identify the data (for sale) with the identity of the user.

Based on the profile of bids $s^j = (s_1^j, \cdots s_I^j)$, seller $j$ computes an allocation $(a^j, c^j) = A^j(s^j)$, where $a_i^j$ is the quantity given to user $i$ and $c_i^j$ is the total cost charged to user $i$. $A^j$ is the allocation rule of seller $j$. It is feasible if $a_i^j \leq d_i^j$, and $c_i^j \leq p_i^j d_i^j$.

## 3.2 User Behavior

We define a **opt-out buyer** as a user $i \in \mathcal{I}$ with utility of the form

$$u_i = \theta_i \circ e_i(a) - \sum_j c_i^j, \qquad (10)$$

where $e_i : [0, \infty) \to [0, \infty)$ is the expectation that user $i$ finds a matching seller $j$. An opt-out buyer's valuation depends only on a scalar $e_i(a)$ which is a function of the quantities of all the available data for sale in the secondary market. Buyer $i$'s valuation may now be interpreted as a unit valuation $\theta_i$, scaled by a function of quantity desired from the market. We define, in addition to the valuation and budget of user $i$, a generic **data-provisioning vector** $\varsigma_i$. Define, for any allocation $a$,

$$e_i^j(a) \triangleq a_i^j \varsigma_i^j, \qquad (11)$$

and let

$$e_i(a) \triangleq \max_{j \neq i} e_i^j(a). \qquad (12)$$

Deviating slightly from the bottleneck player defined in [3], in addition to taking the inverse of the route-provisioning vector to suit the data problem, we omit the term $a_j^i$ from the lemma, as it is assumed that buyers and sellers are separated (a seller does not also buy data and vice versa). (NOTE: proof should come easily! make sure!)

Consider a user seeking to prevent data overage by purchasing enough data from a subset of other network users. This user $i$ can be modeled as a opt-out buyer where $\varsigma_i^j$ denotes the fraction of user $j$'s data aquired by user $i$. (TAKE OUT: Buyer $i$ has an incentive to to maximize the amount of requested data from one seller, thereby minimizing the bid fee paid to the ISP and still fufill thier data requirement.) For the sellers that do not meet a $i$'s data requirements, a rational (utility-maximizing) buyer will set $s_i^j = 0$, i.e. they will not place a bid, as in [1]. Also we note that since $i$ is not a seller, $d_i^i = 0$ and $a^i = 0$.

Suppose the total amount of seller $j$'s data on the network at the instance that user $i$ joins the auction is $\chi_j$. For the service to function as desired, the data transfer from each seller cannot exceed the total amount they have available, i.e. $a_i^j \leq \chi_j \varsigma_i^j$. This will hold simultaneously for all $j$ if and only if $\chi_j \geq \max_j \ a_i^j \varsigma_i^j = e_i(a)$. Thus $e_i(a)$ lower-bounds the amount of data that each seller $j$ has in an auction. We determine (explain further?) the valuation of the transaction between seller $j$ and buyer $i$ is well-defined, and the form of (11) is justified. (Does this make sense? and check the MATH!)

In order to form the distributed auction, we set $\varsigma_i^j = 1$ for all sellers who offer enough data to meet the needs required by buyer $i$, and $\varsigma_i^j = 0$ for all other $j$. This restricts the number of auctions in which the user is able to participate. We intend to show that this does not affect thier valuation, and indeed, in this network setting, results in a shared network optima (a global optimum). The formulation is inversely analogous to the thinnest allocation route for bandwidth given in [2]. Reasonably, if only a single seller is available, then (11) reduces to the original form (7), defined in [3] as "a simple buyer at a single resource element".

The seller, in our analysis, is an extension of the buyer, where the valuation $\theta^j$ is dependent on the buyers. The natural utility is the potential profit from buyer $i$, $u^j = \theta^j \circ e_i(a)$, where $\theta^j$ is the potential revenue from the sale of data composed with buyer $i$'s opt-out value $e_i(a)$. We derive the potential revenue as a function of demand $\varrho$ as in [3], $\forall\ y \geq 0$, the demand for $j$ at unit price $y$ is given by

$$\varrho^j(y) \triangleq \sum_{p_j^i \geq y} d_j^i,$$

with inverse

$$f^j(z) \triangleq \sup \left\{ y \geq 0 : \varrho^j(y) \geq z \right\}.$$

Therefore, we have the seller's valuation [3],

$$\theta^j \circ e_i(a) = \int_0^{e_i(a)} f^j(z)\ dz.$$

We note the difference in subscript/superscript notation from [3], we emphasize the separation of buyers and sellers, and our claim that the seller's valuation depends on the opt-out *buyer demand*. The proof remains the same from [3].

# 4 Network Data P2P Analysis

## 4.1 Equilibrium

Consider an opt-out buyer $i \in \mathcal{I}$, participating in many auctions simultaneously. Due to (11), a seller only has an incentive to change its bid value if it increases its expected opt-out value $e_i$. This creates an incentive for $i$ to coordinate its bids to maximize (DOUBLE MAX? DOES THIS MAKE SENSE?) the number of available sellers (with respect to the demand threshold), and therefore maximize its overall utitlity. We show that, without loss of utility, a buyer $i$ can increase its bid quantities $d_i$ to the level where the opt-out value $e_i^j$ is the same for each *qualifying* seller $j$. We show that buyer coordination [3] holds in the secondary data market under our assumptions.

**Lemma 4.1.** *(Opt-out buyer coordination) Let $i \in \mathcal{I}$ be a opt-out buyer. For any profile $s_i = (d_i, p_i)$, let $a \equiv a(s)$ be the resulting data tranfer. For a fixed $s_{-i}$, a better reply for $i$ is $x_i = (z_i, p_i)$ where $\forall\ l \neq i$,*

$$z_i^l = \frac{\big[e_i(a)\big]}{\varsigma_i^l},$$

*and*

$$a_i^l(z_i, p_i) = z_i^l. \tag{13}$$

**Proof:** We will show that

$$u_i(s_i; s_{-i}) \equiv u_i(d_i, p_i) \leq u_i(z_i, p_i) \tag{14}$$

using the modified allocation rule from [4] and our current formulation. $\forall\ l \in \mathcal{I}$,

$$\begin{aligned} z_i^l &= e_i(a)/\varsigma_i^l = \big[\max_{j \neq i} e_i^j(a)\big]/\varsigma_i^l \\ &\geq e_i^j(a) = a_i^j \varsigma_i^j / \varsigma_i^l \\ &\geq \frac{d_i}{\sum_{k:p_k = p_i} d_k} d_i(p_i; s_i) \\ &= \frac{d_i}{\sum_{k:p_k = p_i} d_k} \left[D - \sum_{p_k > y, k \neq i} d_k\right]^+ . \end{aligned} \tag{15}$$

Using the allocation (1.1) rule again, we have

$$\begin{aligned} a_i^l(z_i, p_i) &= \max\Bigg(z_i^l, \\ &\frac{d_i}{\sum_{k:p_k = p_i} d_k} \left[D - \sum_{p_k > y, k \neq i} d_k^l\right]^+\Bigg) \\ &= z_i \\ &= e_i(a)\varsigma_i^l, \end{aligned}$$

and (13) is proven. Then we have that $e_i(a(z_i, p_i)) = a_i^l(z_i, p_i)\varsigma_i^l = e_i(a)\ \forall\ l \neq i$, and so $\theta_i \circ e_i(a(z_i, p_i)) = \theta_i \circ e_i(a)$. Now, by definition of the seller's valuation, $\forall\ l$,

$$e_i(a) \geq e_i^l(a) \Rightarrow z_i^l \varsigma_i^l \geq a_i^l \varsigma_i^l \Rightarrow a_i^l \leq z_i^l \leq a_i^l(z_i, p_i).$$

Additionally, we comment that $x_i$ is feasible, i.e. $\sum_l c_i^l(s) \leq b_i$ as the factor $\varsigma$ separates the auction space, restricting the number of bids a user may place (SAY THIS BETTER!). In effect, we are using the buyer demand to partition the auction space, thereby optimizing the message space for the ISP, and providing an optimal market space to host the buyers and sellers based on their type. (....AND!?) The rest of the proof follows as in [3].

The idea of buyer coordination follows from [2] (FINISH!)

(DOES THIS PROOF EVEN CHANGE?)

**Proposition 4.1.** *(Network incentive compatibility) Let $i \in \mathcal{I}$ be an opt-out buyer, and fix all other users' bids $s_{-i}$, as well as the sellers' bids $s_i^i$ (so $a^i$ is fixed). Let*

$$z_i = \sup\Bigg\{h \geq 0 : \tag{16}$$

$$\theta_i(h) > \sum_{j \neq i} P_i^j\big((h - a_j^i)\varsigma_i^j\big)\varsigma^j\Bigg\}, \tag{17}$$

$$\psi_i = \sup\Bigg\{h \geq 0 : \tag{18}$$

$$\int_0^h \sum_{j \neq i} P_i^j\big((h - a_j^i)\varsigma_i^j\big)\varsigma^j\ dh \leq b_i\Bigg\}, \tag{19}$$

*$e = \max(z_i, \psi_i - \epsilon/\theta_i'(0))^+$, and for each $j \neq i$,*

$$v_i^j = (e - a_j^i)\varsigma_i^j$$

*and*

$$w_i^j = \varsigma_i^j \theta_i'(e).$$

*Then a (coordinated) $\epsilon$-best reply for the opt-out buyer is $t_i = (v_i, w_i)$, i.e., $\forall\ s_i, u_i(t_i; s_{-i}) + \epsilon \geq u_i(s_i; s_{-i})$.*

**Proof:** First suppose $e = z_i$. Since $\theta_i'$ is non-increasing and $\forall j$, $P_i^j$ is non-decreasing, 18 implies $\theta_i'(e) > \sum_{j \neq i} P_i^j(v_i^j)\varsigma_i^j$, and so $\forall j \neq i$,

## 4.2 Efficiency

## 4.3 Convergence

# References

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