1 Introduction

China Mobile Hong Kong (CMHK) recently introduced such a secondary market. CMHKs 2cm (2nd exchange market) data exchange platform allows users to submit bids to buy and sell data, with CMHK acting as a middleman both to match buyers and sellers and to ensure that the sellers trading revenue and buyers purchased data are reflected on customers monthly bills. [1]

In this work, we propose a distributed progressive second price (PSP) auction in order to maximize social utility. We show that for cellular data allocated between multiple users there exists an ϵ-Nash market equilibria when all users bid their real marginal valuation of the mobile data offered in the secondary market described in [1]. The PSP auction's (as in all auctions), demand information is not known centrally, rather it is distributed in the buyers' valuations. The mechanism for an auction is defined as *distributed* when the allocations at any element depend only on *local* state: the quantity offered by the seller at that element, and the bids for that element only. In this work, the proposed mechanism allows sellers submit bids to buyers directly; there is no entity that holds a global maret knowledge.

We suppose that each seller (resp. buyer) can submit a bid to the secondary market consisting of (i) an available (required) quantity and (ii) a unit-price (calculated using its own demand functions). Buyers submit bids cyclically until an (ϵ-Nash) equilibrium is reached where ϵ corresponds to a bid fee to be paid to the ISP on completion of the transaction. (FEE IS FIXED OR PER-UNIT? (will change market incentives slightly)) (DO WE MODEL ISP PROFIT? FUTURE WORK)

1.1 Distributed Progressive Second Price Auctions

The distributed PSP auction first introduced in [2] forms a part of the overall market based allocation model. Consider a noncooperative game where a set of $\mathcal{I} = \{1, \cdots, I\}$ users buy a fixed amount of resource D_i from a set of resources $\mathcal{I} = \{1, \cdots, L\}$. Suppose each user $i \in \mathcal{I}$ makes a bid $s_i^l = (p_i^l, d_i^l)$ to the seller of resource l, where p_i^l is the unit-price the user is willing to pay and d_i^l is the quantity the user desires. The *bidding profile* forms a grid, $s \equiv [s_i^l] \in \mathcal{I} \times \mathcal{I}$ (BAD NOTATION? SEEMS OK) and $s_i^l \equiv [s_1^l, \cdots, s_{i1}^l, s_{i+1}^l, \cdots, s_I^l]_{l \in \mathcal{I}}$ is the profile of user is opponents. In addition, the user type now includes a *routing variable*, $r_i \in \{0,1\}^L$ to manage throughput in an arbitrary collection of nodes where $r_i^l > 0$ if and only if $l(i)$ is on i's path, i.e. $l(i) \in \{l \in \mathcal{I} : r_i^l > 0\}$. The market now shares L resources, with allocation:

$$\hat{D}_i^l(y, s_{-i}^l) = \left[D^l - \sum_{p_k^l > y, k \neq i} d_k^l \right]^+ .$$

The market price function (MPF) of user i is defined as:

$$\begin{aligned} \tilde{P}_i(z, s_{-i}) &= \sum_{l \in \mathcal{I}} P_i^l(z_i^l, s_{-i}^l) r_i^l \\ &= \sum_{l \in \mathcal{I}} \left(\inf \left\{ y \geq 0 : \hat{D}_i^l(y, s_{-i}^l) \right\} r_i^l \right), \end{aligned}$$

which is interpreted as the aggragate minimum price a user bids over its route in order to obtain the resource z given the opponents profile s_i. Its inverse function $\tilde{D}_i$ is defined as follows:

$$\tilde{D}_i(y, s_{-i}) = \sup\left\{z \in \left(0, \min_l \hat{D}_i^l / r_i^l\right) : \tilde{P}_i(z, s_{-i}^l) < y\right\},$$

which means the maximum available quantity at a bid price of y given s_i^j. With this notation, the modified PSP allocation rule [4] is defined as:

$$\tilde{a}_i(s) = \min\left\{d_i^{l(i)}, \frac{d_i^{l(i)}}{\sum_{k:p_k^{l(k)}=p_i^{l(i)}} d_k^{l(k)}} \tilde{D}_i(p_i^{l(i)}, s_{-i}^l)\right\},$$

$$\tilde{c}_i(s) = \sum_{j \neq i}\left[\sum_{l \in \mathcal{I}} p_j^{l(j)}\left(\tilde{a}_j(0; s_{-i}^l) - \tilde{a}_j(s_i^l; s_{-i}^l)\right) r_i^l\right],$$

$a_i^{l(i)}$ denotes the quantity user i obtains by a bid price $p_i^{l(i)}$ (when the opponents bid s_i^l) and the charge to user i by the seller is denoted $c_i^{l(i)}$. $\sum_{l \in \mathcal{I}} c_i^{l(i)}$ is interpreted to be the total cost incurred in the system if user i is removed from the auction. Note that the allocation rule is modified from [2] according to [4], so that buyers with identical unit-price p_i are not rejected.

An allocation rule is feasible [2] if $\forall\ s$,

$$\sum_{i \in \mathcal{I}} a_i(s) \leq Q$$

and $\forall\ i \in \mathcal{I}$,

$$a_i(s) \leq d_i$$
$$c_i(s) \leq p_i d_i.$$

We will introduce a user type called an *opt-out buyer* in order to perform our analysis. The opt-out buyer restricts its pool of sellers by only submitting bids to sellers who have sufficient data to meet their requirement. (NOTE: TAKE MAX ONLY OR NEED TO INTRODUCE A GAP VARIABLE?), similar to [1]. In addition, we define a generic **data-provisioning vector** ς, held the user as part of its type along with its valuation and budget, . We intend to show that the pricing model (23) is sufficient to attain the desirable property of truthfullness through incentive compatibility. (MOVE THIS? NEED OWN WORDS?) We reason that our formulation upholds the *exclusion-compensation principle*, and is a valid progressive second price auction to the extent that buyer i pays for its allocation so as to exactly cover the "social opportunity cost" which is given by the declared willingness to pay (bids) of the users who are excluded by i's presence, and thus also compensates the seller for the maximum lost potential revenue [2].

(NOTE: move this?) We extend the P2P rules as in [3] to include a *local* market price function as determined by the subset of nodes participating in the auction, where the user is the auctioneer. Therefore the influence of the greater market on the individual auctions will be influenced only by the submission of bids from buyers to sellers. As a buyer may have access to multiple auctions, the sellers will be dynamically influenced by the ϵ-best replies from the buyers. The valuation function of seller j is dependent on the buyers demand, and is modeled as a function of their potential revenue [3].

Absent the cost or revenue from trading data, users gain utility from consuming data. We use the α-fair utility functions [1] to model the valuation from consuming d amount of data:

$$\theta(d) = \frac{\sigma d^{1-\alpha}}{1-\alpha} \tag{1}$$

(AUGH! NEED X TO BE SO BIG COMPARED TO ALPHA! can we assume a minimum ask?) where σ is a positive constant representing (the scale) of the usage utility and we take $\alpha \in [0, 1)$. We verify that the user valuation above satisfies the conditions for an *elastic demand function*: (NOTE: this part seems in the wrong place)

Definition 1.1. [2] A real valued function $\theta(\cdot)$ is an *(elastic) valuation function* on $[0, D]$ if

- $\theta(0) = 0$;
  Verification: (obvious)

- θ is differentiable;
  Verification: The derivative, $\sigma d^{-\alpha}$, is positive assuming non-negative data requirements.

- $\theta' \geq 0$, and θ' is non-increasing and continuous;
  Verification: U is differentiable for all d, and therefore continuous. Its derivative is a negative exponential, and so is non-increasing.

- There exists $\gamma > 0$, such that for all $z \in [0, D]$, $\theta'(z) > 0$ implies that for all $\eta \in [0, z), \theta'(z) \leq \theta'(\eta) - \gamma(z - \eta)$.
  Verification: Without loss of generality, we may set the scaling constant $\sigma = 1$, and compute the curvature $\gamma(\xi)$, where by definition,

  $$\gamma = \frac{\theta''}{(1+\theta')^{3/2}} = \frac{-\alpha\xi^{-\alpha-1}}{(1+\xi^{-2\alpha})^{3/2}}.$$

  Using a Taylor theorem approximation,

  $$\begin{aligned} z^{-\alpha} &\leq \eta^{-\alpha} + \frac{-\alpha d^{-\alpha-1}}{(1+\xi^{-2\alpha})^{3/2}}(z-\eta) \\ &\leq \eta^{-\alpha} + \frac{\alpha}{2\sqrt{2}\xi}(z-\eta) \\ &\leq \eta^{-\alpha} + \frac{\alpha}{\xi}(z-\eta). \end{aligned}$$

  Now, using Taylor repeatedly, simplifying and taking the limit as $\eta \to z$,

  $$\begin{aligned} z^{-\alpha} - \eta^{-\alpha} &\leq -\alpha\eta^{-\alpha-1}(z-\eta) \\ &\leq \frac{-\alpha}{\xi}(z-\eta). \end{aligned}$$

  And so, since $\xi \leq \eta^{\alpha+1}$, we may set

  $$\gamma \geq \frac{-\alpha\eta^{-(\alpha+1)^2}}{\left(1+(\eta^{-2\alpha(\alpha+1)}\right)^{3/2}},$$

  which holds in the case that $z > 1$, and so assuming that there must be at least one unit of data required for a user to have a valuation, we have that the concavity of θ' is shown by Squeeze theorem.

In the remainder of this paper we omit the bar in the data allocation rule for simplicity of notation, so $\bar{a}_i = a_i$. We may now define the user's utitlity function as

$$u_i = \theta_i(a_i(s)) - c_i(s). \tag{2}$$

Under the PSP rule, [2] shows that given the opponents bids s_{-i}, user i's ϵ-best response $s_i = (w_i, v_i)$ as a Nash move (where s_i is chosen to maximize i's utility with s_{-i} held constant), is given by:

$$v_i = \sup\left\{d \geq 0 : \theta'(d) > P_i(d), \int_0^d P_i(\eta)\ d\eta \leq b_i\right\} - \frac{\epsilon}{\theta_i'(0)} \quad (3)$$

(best quantity reply)

$$w_i = \theta_i'(v_i) \quad \text{(best unit-price reply)}, \quad (4)$$

where $\epsilon > 0$ is the bid fee, b_i is user i's budget, and every user has an elastic demand function.

In the Secondary Market [1], we intend to show optimal strategies for buyers and sellers which allows the use a PSP auction to achieve a global network equilibrium.

2 Related Work

3 The Problem Model

3.1 The Secondary Market

We consider the set of $\mathcal{I} = \{1, \cdots, I\}$ users who purchase or sell data from other users. A buyer submits bids directly to sellers who enough leftover data to satisfy their demand, and will submit bids in order to maximize their (private) valuation. We assume (SUPPOSE?) that the public information in the secondary market consists of a set of offers that are published by users wishing to sell their data overage. A user's identity $i \in \mathcal{I}$ as a subscript indicates that the user is a buyer, and a superscript indicates the seller. Suppose user i is buying from user j. A bid $s_i^j = (d_i^j, p_i^j)$, meaning i would like to buy from j a quantity d_i^j and is willing to pay a unit price p_i^j. (STILL TRUE?) Without loss of generality, we assume that all users bid in all auctions; if a user i does not submit a bid to j, then this means that the user has the exact amout of data they require, or that seller j does not have enough data to satisfy the buyers' demand, and we simply set $s_i^j = (0, 0)$. A seller j places an ask $s^j = (d^j, p^j)$, meaning j is offering a quantity d^j , with a reserve unit price of p^j . In other words, for a superscript, the bid is understood as an offer in the secondary market; we assume that data is a unary resource belonging to the seller, and therefore can identify the data (for sale) with the identity of the user. To further clarify this restriction, we note that since i is not a seller, the superscript notation will not be used, as $d^i = 0$ and $a^i = 0$. In our current formulation, we do not allow a seller to submit multiple bids to the secondary market (FUTURE WORK).

Based on the profile of bids $s^j = (s_1^j, \cdots s_I^j)$, seller j computes an allocation $(a^j, c^j) = A^j(s^j)$, where a_i^j is the quantity given to user i and c_i^j is the total cost charged to user i. A^j is the allocation rule of seller j. It is feasible if $a_i^j \leq d_i^j$, and $c_i^j \leq p_i^j d_i^j$.

4 Distributed PSP Analysis

4.1 User Behavior

We define a **opt-out buyer** as a user $i \in \mathcal{I}$ with utility function,

$$u_i = \theta_i \circ e_i(a) - c_i, \tag{5}$$

where $e_i : [0,\infty) \to [0,\infty)$ is the expectation (EXPECTATION??? BAD) that user i finds matching seller(s). An opt-out buyers' valuation depends only on a scalar $e_i(a)$ which is a function of the quantities of all the available data for sale in the secondary market.

Suppose the total amount of seller j's data on the network at the instance that user i joins the auction is $\bar{b}^j$. The data transfer from each seller cannot exceed the total amount they have available, i.e. $a_i^j \leq \bar{b}^j$. This will hold simultaneously for each $i \in \mathcal{I}$ if and only if $\bar{b}^j \geq \max_i \ a_i^j$. Therefore a seller j is restricted to subset of buyers $\in \mathcal{I}$. Note that with our formulation, if a seller j does not meet a buyer i's data requirements, a rational (utility-maximizing) buyer will set $s_i^j = 0$, i.e. they will not place a bid, as in [1]. The seller, in our analysis, is an extension of the buyer, where the valuation θ^j is dependent on the buyers, we will assume that buyers and sellers are separated (a seller does not also buy data and vice versa).

Although it is possible for a seller to fully satisfy a buyer i's demand, it is also reasonable to expect that many sellers will come close to using their entire data cap, and only sell the fractional overage. In this case, we determine that a buyer must coordinate their bids among multiple sellers. (SAY MORE?) Buyer i's valuation is interpreted as a unit valuation θ_i, given as the α-fair utility in (1), scaled by a function of quantity desired from the market. We define, in addition to the valuation θ_i and budget b_i, a generic **data-provisioning vector** ς_i, held by buyer i as part of its type. We propose the following strategy,

Proposition 4.1. *(Opt-out buyer strategy) Define, for any allocation a,*

$$e_i^j(a) \triangleq \frac{a_i^j}{\varsigma_i^j}, \tag{6}$$

and define,

$$\ell_i = \underset{\mathcal{I}' \subset \mathcal{I}, |\mathcal{I}'| = n}{\arg\max} \sum_{j \in \mathcal{I}'} d^j.$$

Buyer i chooses its seller pool by determining n, where

$$n = \arg\min_{\ell_i}(j \in \ell_i : \sum_n \frac{d^n}{d_i} = d_i\}, \tag{7}$$

which produces the minimal subset

$$\mathcal{I}_i = \{j \in \mathcal{I} : j < n\} \subset \mathcal{I}. \tag{8}$$

Now let $j^* = n < I$, and define,

$$e_i(a) \triangleq e_i^{j^*}(a). \tag{9}$$

Then, we have that e_i is an optimal feasible strategy for buyer i.

Proof: In the case that there exists a seller who can completely satisfy a buyers' demand, $j^* = 1$, $|\mathcal{I}_i| = 1$ and (7) holds. If such a buyer does not exist, as the set ℓ_i is an ordered set, i may discover j^* by computing ℓ_i. In the case that

$d_i > \sum_{j\in\mathcal{I}} d^j/d_i$, then $j^* > I$ and we consider the buyers' demand infeasible. We also note that $\mathcal{I}_i$ is not the only possible minimum subset $\in \mathcal{I}$ able to satisfy i's demand, it is the minimal subset where a coordinated bid is possible. We will show optimality through the analysis in Section 5.

To determine a strategy for the sellers, we examine seller incentive. The potential revenue for j at unit price y is determined by the demand of the buyers. $\forall\ y \geq 0$, we determine that the demand is given by,

$$\rho^j(y) = \sum_{i\in\mathcal{I}:p_i^j\geq y} d_i^j, \tag{10}$$

with "inverse"

$$f^j(z) \triangleq \sup\left\{y \geq 0 : \rho_i^j(y) \geq z\ \forall\ i \in \mathcal{I}\right\}. \tag{11}$$

For a given demand ρ^j, f^j maps a unit of data to the highest price at which it *could* be sold to any buyer $i \in \mathcal{I}$ (as in [3]). However, j is restricted by the bid strategy of the buyers, as well as some natural constraints, and so we have the following Lemma.

Lemma 4.1. *(Seller constraints) Let j be a seller with budget b^j. Formalizing the quantity threshold, the seller must satisfy the quantity constraint,*

$$d^j \geq e_i(a) \tag{12}$$

and

$$\sum_{i\in\mathcal{I}} e_i^j(a) \leq \sum_{i\in\mathcal{I}} d_i^j \leq b^j. \tag{13}$$

In addition, for a seller who does not sell at a loss, the reserve price must satisfy

$$p^j \geq \min_{i\in\mathcal{I}}\left(\theta_i{}'(d_i^j)\right). \tag{14}$$

Finally, we must have,

$$\sum_{k\in\mathcal{I},k\neq i} e_k^j(a) \leq a_i^j. \tag{15}$$

Proof: The first statement is an assumption, which we may enforce by (6), and as a seller cannot sell more data than their bid, (13) enforces the budget constraint for the seller. (14) follows from the assumption that j does not sell at a loss, and finally, (15) follows from Proposition 4.1.

We may now define the seller j's valuation (potential).

Lemma 4.2. *(Seller valuation) For any $i \in \mathcal{I}$,*

$$\theta_i^j = \int_0^{e_i^j(a)} f^j(z)\ dz. \tag{16}$$

it follows that

$$\theta^j \circ e = \sum_{i\in\mathcal{I}} \int_0^{e_i^j(a)} f^j(z)\ dz. \tag{17}$$

Proof: We assume that a seller will try to maximize profit, and so there is an incentive to sell all of its data, and so from Lemma 4.2, $\sum_{i\in\mathcal{I}} d_i^j = \sum_{i\in\mathcal{I}} e_i$, The remainder of the proof follows as in [3].

The sellers' natural utility is the potential profit $u^j = \theta^j \circ e$, where θ^j is the potential revenue from the sale of data composed with each buyers' opt-out value, $e_i(a)$. This is intuitive as a seller who does not meet the demand of *any* buyer will not participate in any auction. We have chosen to omit the original cost of the data paid to the ISP, as a discussion of mobile data plans is outside the scope of this paper.

In order to to implement a PSP auction, it is necessary that we determine that a seller has an incentive to accept fractional bids (i.e. sell fraction of their data d^j). Reasonably, there may not exist a buyer such that $d_i = d^j$, in addition, a seller may accept fractional bids in order to maximize the revenue gained per unit of data by increasing market competition [**?**]. Finally, in the case that the seller does not know the exact amount of leftover data available, then they may only sell enough data to ensure that they will not become a buyer while they submit their total data overage to the secondary market. (FUTURE WORK?)

Proposition 4.2. *(Localized seller strategy) Define any auction duration to be $\tau \in [0, \infty)$. For any seller j, fix bid $s^j = (d^j, p^j)$ at time $t \in \tau$. Let the winner be determined by,*

$$\bar{i} = \arg\max_{I^j} \sum_{i \in I^j} p_i^j. \tag{18}$$

Also define

$$\ell^j = \arg\max_{\mathcal{I}' \subset \mathcal{I}, |\mathcal{I}'| = n} \sum_{i \in \mathcal{I}'} p_i^j,$$

where,

$$n = \arg\min_{\ell^j} (i \in \ell^j : \sum_{i \in \mathcal{I}^j} d_i^j > b^j), \tag{19}$$

which gives j the minimal subset

$$\mathcal{I}^j = \{i \in \mathcal{I} : i < n\} \subset \mathcal{I} \tag{20}$$

of buyers that maximizes j's revenue. Define buyer $i^* = n \leq I$ in the ordered set ℓ^j. Then, for time $(t+1)$, set j's reserve price as

$$p^j = \theta'_{i^*}(d_{i^*}^j) + \epsilon/2, \tag{21}$$

and update j's budget,

$$\bar{b}^{j(t+1)} = b^{j(t)} - d_{i^*}^{j(t)}. \tag{22}$$

Allowing t to range over τ, we have that (18) - (22) produces a local market equilibrium.

Proof: We assume that the seller has enough data to satisfy at least one buyer. As the set of buyers is computed at each iteration, we are guaranteed a a subset $\mathcal{I}^j \subset \mathcal{I}$ that is at equilibrium as it is designed so that the data offered by seller j equals the aggregate demand of the buyers (CITE! KNOWN?). Although the buyers' valuation θ_i is not known to the seller, we assume that the buyer is bidding truthfully, and so $p^j = \theta_i{}' + \epsilon/2 = p_i^j + \epsilon/2$, therefore p^j is known, and as $\mathcal{I}^j \subset \mathcal{I}$, we note that (13) and (14) hold. Now, using (16), we have, $\forall\ z \geq 0$,

$$\int_0^{e_{i^*}^j(a)} f^j(z)\ dz \leq \int_0^{e_i^j(a)} f^j(z)\ dz$$

and so,

$$\theta^j \circ e_{i^*}(a) \leq \theta^j \circ e_i(a),$$

which holds $\forall\ i \in \mathcal{I}^j$. From (19), $i^* \ni \mathcal{I}^j$, and by definition of an ϵ-best reply and (21), Therefore the choice of p^j does not force any buyers out of the local equilibrium. Thus we determine the valuation of the transaction between seller j and buyer i is well-posed, and the reserve price (21) is justified, and that the seller does not force a change to the local market equilibrium from time t to $(t+1)$. (NEED TO SAY MORE HERE... INDUCTION?)

4.2 Data Auction Mechanism

We may now proceed to formally define the PSP auction as a composition of the buyers and sellers in the secondary market. The market price function (MPF) for a buyer in the secondary market can now be described as follows:

$$\begin{aligned}\bar{P}_i(z, s_{-i}) &= \sum_{j \in \mathcal{I}_i} P_i^j(z_i^j, s_{-i}^j) \\ &= \sum_{j \in \mathcal{I}_i} \left(\inf \left\{ y \geq 0 : \bar{D}_i^j(y, s_{-i}^j) \right\} \right),\end{aligned} \tag{23}$$

where

$$\begin{aligned}\bar{D}_i^j(y, s_{-i}^j) &= D_i^j(y, s_{-i}^j) \ \circ \ e_i \\ &= \left[d^j - \sum_{p_k^j > y} d_k^j \right]^+ / \varsigma_i^j,\end{aligned} \tag{24}$$

and is interpreted as the aggragate of minimum bid prices of buyer i for bids over $\mathcal{I}_i$ given an opponent profile s_{-i}. In this expression, we note that the price is an aggragation of the *individual* prices of the buyers as it is possible that $\mathcal{I}_i \neq \mathcal{I}^j$, and so the reserve prices of the individual sellers may vary. The inverse function $\bar{D}_i$ is defined as

$$\begin{aligned}\bar{D}_i(y, s_{-i}) = \sum_{j \in \mathcal{I}_i} \Bigg(\sup \Bigg\{ z \in \left(0, \bar{D}_i^j\right) : \\ \bar{P}_i(z, s_{-i}^j) < y \Bigg\} \Bigg).\end{aligned} \tag{25}$$

We may interpret $\bar{D}_i(y, s_i)$ as the maximum quantity provided by the subset $\mathcal{I}_i$ at a bid price of y given s_{-i}. The data allocation rule is given as,

$$\begin{aligned}\bar{a}_i(s) = a_i(s) \circ e_i = \sum_{j \in \mathcal{I}_i} \Bigg(\min \Bigg\{ d_i^j / \varsigma_i^j, \\ \frac{d_i^j}{\sum_{k:p_k^j = p_i^j} d_k^j} \bar{D}_i^j(p_i^j, s_{-i}^j) \Bigg\} \Bigg)\end{aligned} \tag{26}$$

with cost to the buyer,

$$\bar{c}_i(s) = \sum_{j \in \mathcal{I}_i} p^j \left(\bar{a}_j(0; s_{-i}^j) - \bar{a}_j(s_i^j; s_{-i}^j) \right). \tag{27}$$

The main contribution of this work is comprised by the modified PSP rules along with the user strategies, which we use to describe an auction mechanism inspired by the classic PSP throughput problem. In order to apply PSP to the data-sharing secondary market described in [1] the idea of a "route" is repurposed, and by the use of the opt-out buyer, we are abel to achieve a market equilibrium. We claim that in the secondary market our formulation not only holds the desired VCG qualities, but minimizes the message space and auction duration, resulting in a convergence time (FIND COMPETITVE RATIO?) with respect to the classic throughput problem.

4.3 VCG Formulation

Consider a user seeking to prevent data overage by purchasing enough data from a subset of other network users. This user i can be modeled as a opt-out buyer where, as in [3], ς_i^j denotes the fraction of user

j's data aquired by user i. In order to form the distributed auction, we set 1 if seller j has enough data to satisfy i's demand, and $\varsigma_i^j = d_i$ otherwise. We intend to show that this does not affect their valuation, and indeed, in this network setting, results in a shared network optima (a global optimum). The formulation is inspired to the thinnest allocation route for bandwidth given in [2]. We note that if a single seller j can satisfy i's demand, then (5) reduces to the original form, defined in [3] as "a simple buyer at a single resource element".

The sellers' auction will function as follows: at each bid iteration all buyers submit bids, and the winning bid is the buyer i that has the highest price p_i^j. The seller allocates data to this winner, at which point all other buyers are able to bid again, and the winner leaves the auction (with the exception where multiple bidders bid the same price, where (26) determines they will not fully satisfy their demand, and so we will assume they remain in the auction). The auction progresses as such until all the sellers' data has been allocated. We design an algorithm based on the sellers' fractional allocation strategy.

Algorithm 1 (Seller fractional allocation)

1: $p^{j(0)} \leftarrow \min_{i \in \mathcal{I}^j} p_i^j$
2: $s^{j(0)} \leftarrow (p^j, d^j)$
3: **while** $d^j > 0$ **do**
4: $\quad \mathcal{I}^{j(t+1)} = \mathcal{I}^{j(t)} \setminus \{i \in \mathcal{I}^{j(t)} : d_i^j > \bar{b}^{j(t)}\}$
5: $\quad i^* \leftarrow \arg\max_{I^j} \sum_{i \in I^j} p_i^j$
6: $\quad \bar{b}^{j(t+1)} \leftarrow \bar{b}^j - d_{i^*}^j$
7: $\quad p^j \leftarrow p_{i^*}^j + \epsilon/2$ and $d^j \leftarrow \bar{b}^j$
8: $\quad s^{j(t+1)} \leftarrow (d^j, p^j)$
9: $\quad$ **if** $\exists\, i : s_i^{j(t+1)} \neq s_i^{j(t)}$ **then**
10: $\quad\quad \bar{b}^{j(t+1)} = \bar{b}^{j(t)}$
11: $\quad\quad$ Go to 4.
12: $\quad$ **else**
13: $\quad\quad i^* \leftarrow \bar{a}_i(s)$
14: $\quad\quad$ Go to 4.

We assume that each time that s^j is updated it is shared with all participating buyers. At this point buyers have the opportunity to bid again, where a buyer that does not bid again is assumed to hold the same bid, since a buyer dropping out of the auction will set their bid to $s_i^j = (0, 0)$. As we will show in our analysis, the buyers are bidding truthfully; the algorithm makes use of the fact that the sellers' valuation is determined by the market and upholds the PSP mechanism. (CHECK)

5 VCG Analysis

5.1 Equilibrium

Consider an opt-out buyer $i \in \mathcal{I}$. (I THINK WE NEED PRICE TO COME IN SOON... MODIFIED ALLOCATION

RULE!) Due to (5), i only has an incentive to change its bid quantity if it increases its opt-out value e_i. We show that, without loss of utility, i can coordinate its bid quantities d_i to the level where the opt-out value e_i is the same for each qualifying seller $j \in \mathcal{I}_i$. We argue that without loss of utility, i may choose a seller pool where a minimal subset of sellers guaranteed (CHECK) to satisfy demand where i is able to play a "consistent" strategy and still have feasible best replies, and buyer coordination holds in the secondary data market under our assumptions.

Lemma 5.1. *(Opt-out buyer coordination) Let $i \in \mathcal{I}$ be a opt-out buyer. For any profile $s_i = (d_i, p_i)$, let $a_i \equiv \sum_j a_i^j(s)$ be the resulting data tranfer. For a fixed s_{-i}, a better reply for i is $x_i = (z_i, p_i)$, where $\mathcal{I}_i$ is computed as in the buyer strategy,*

$$z_i^j = e_i(a)$$

and

$$a_i^j(z_i, p_i) = z_i^{j^*}, \tag{28}$$

Proof: As s_{-i} is fixed, we omit it, in addition, we will use $u \equiv u_i \equiv u_i(s_i) \equiv u_i(s_i; s_{-i})$. We have, by (6) and (24), $\forall\ j \in \mathcal{I}_i$,

$$\begin{aligned} z_i^{j^*} = z_i^j = e_i(a) = e_i^{j^*}(a) \\ \leq \left[d^{j^*} - \sum_{p_k^j > y} d_k^{j^*} \right]^+ / \varsigma_i^{j^*}, \end{aligned}$$

and so

$$a_i^j(z_i, p_i) = a_i^{j^*}(z_i, p_i) = z_i^{j^*} = e_i(a).$$

In order to determine that i has no loss of utility, we will show that

$$u_i(d_i, p_i) \leq u_i(z_i, p_i).$$

We address two cases:

1. *There exists a seller who can fully satisfy i's demand.*

In this case, $|\mathcal{I}_i| = 1$, and the case is trivial as no coordination is necessary for a single bid.

2. *Buyer i's demand can only be satisfied by a minimal subset of sellers.*

Buyer i maintains ordered set ℓ_i where the sellers with the largest bid are considered first, the seller j^* defines the minimal subset $\mathcal{I}_i$ where a coordinated bid is possible. From (26) and (28), we have that, $\forall\ j \in \mathcal{I}_i$,

$$e_i^j(z_i, p_i) = \left[d^j - \sum_{p_k^j > y} d_k^j \right]^+ / \varsigma_i^j$$

We have $e_i^j(a(z_i, p_i)) = a_i^{j^*}(z_i, p_i)/\varsigma_i^{j^*} = e_i(a)$, which implies that

$$\theta_i \circ e_i(a(z_i, p_i)) = \theta_i \circ e_i(a).$$

Therefore, by the definition of utility (2),

$$\begin{aligned} \theta_i \circ e_i(a(z_i, p_i)) - \theta_i(a)) \\ = c_i^j(s) - c_i^{j^*}(z_i, p_i). \end{aligned}$$

Now using the definition of the buyers' valuation (1), we have $\forall\ i \in \mathcal{I}$,

$$\begin{aligned} u_i(z_i, p_i) - u_i \\ = \sum_{\mathcal{I}_j} \left(\theta_i(a_i^j) - \theta_i(a_i^{j^*}(z_i, p_i)) \right) \end{aligned}$$

Now, from (26), $\forall\ j \in \mathcal{I}_i$, $a_i(z_i, p_i) \leq z_i^j \leq a_i^j$ and $\theta_i \geq 0 \Rightarrow u_i(z_i, p_i) - u_i \geq 0$, $\forall\ i \in \mathcal{I}$, as is shown by the definition of the buyers' utility, (CAN USE THIS! VERY STRONG)

$$\sum_{\mathcal{I}_j} \left(\frac{\sigma(a_i^j)^{1-\alpha}}{1-\alpha} - \frac{\sigma(a_i^j(z_i, p_i))^{1-\alpha}}{1-\alpha} \right) \geq 0.$$

Additionally, since (6) does not increase the demand of seller i, $\sum_j c_i^j(s) \leq b_i$, i.e. x_i is feasible. In effect, we are using the buyer demand to partition the auction space, thereby optimizing the message space (EXPLAIN) for the ISP, and providing an optimal market space to host the buyers and sellers based on their type.

We proceed to claim that the optimality of truth-telling holds in our formulation, where the market functions as a hybrid of [3] and [1]. The seller finds the optimal opt-out buyer i , which is the buyer with the largest demand such that the marginal value is greater than the market price. The role of the market price is played by the sum of the market prices at the different auctions, weighted by the data provisioning vector values. The actual bids are obtained by transforming the opt-out buyers' strategy back into the corresponding quantities to bid. As with a single resource, truth-telling is optimal for the buyer, i.e. in each auction, the buyer sets the bid price to the marginal value.

Proposition 5.1. *(Buyer incentive compatibility) Let $i \in \mathcal{I}$ be an opt-out buyer, and fix all other buyers' bids s_{-i}, as well as the sellers' bids s^j (so a_i is fixed). Let*

$$z_i = \sup \left\{ h \geq 0 : {\theta_i}'(h) > \bar{P}_i^j(h) \right\}, \tag{29}$$

$$\chi_i = \sup \left\{ h \geq 0 : \int_0^h \bar{P}_i(h) \ dh \leq b_i \right\}, \tag{30}$$

$e = \min(z_i, \chi_i - \epsilon/\theta_i'(0))^+$, and for each $j \in \mathcal{I}_i$,

$$v_i^j = e = v_i^{j^*}$$

and

$$w_i^j = w_i^{j^*} = \theta_i'(e).$$

Then a (coordinated) ϵ-best reply for the opt-out buyer is $t_i = (v_i, w_i)$, i.e., $\forall\ s_i, u_i(t_i; s_{-i}) + \epsilon \geq u_i(s_i; s_{-i})$.

Proof: For ease of notation, we may assume that bid $s_i^j = s_i^{j^*}$ and that each $j \in \mathcal{I}_i$. First suppose $e = z_i$. Since θ_i' is non-increasing and, P_i^j is non-decreasing, (23) implies $\theta_i'(e) > \sum_j P_i^j(v_i^j)$, due to e being supremum, i.e. greater than the marginal value, same as in [3]. This, along with i's strategy, gives $\forall\ y, z \geq 0$,

$$\begin{aligned} \sum_j y &> \bar{P}_i(z, s_{-i}) \\ &\Rightarrow y > P_i^j(z, s_{-i}) \\ &\Rightarrow z \leq D_i^j(y, s_{-i})/\varsigma_i^j, \end{aligned}$$

and so,

$$\begin{aligned} w_i^j &> P_i^j(v_i^j) \\ &\Rightarrow D_i^j = \frac{[d^j - d^j(w_i^j)]}{\varsigma_i^j}. \end{aligned} \tag{31}$$

Now, let $y_i = {\theta_i}'(z_i)$ and suppose $z_i = 0$, then $v_i = 0 \Rightarrow a_i(t_i; s_{-i}) = 0$ and

i's coordinated bids (Lemma 5.1) gives $\bar{D}_i(0, s_{-i}) = 0$. By choice of i's valuation function θ,

$$w_i = \theta_i{}'(v_i) = \sigma v_i^{-\alpha} > \sigma z_i^{-\alpha} = \theta_i{}'(z_i) = y_i,$$

and since D_i^j is non-decreasing, $D_i^j(y_i, s_{-i}) \geq z_i \geq v_i$. Thus, by (26) and (31),

$$a_i^j(t_i; s_{-i}) = v_i^j$$
$$e_i^j \circ a(t_i; s_{-i}) = e.$$

Therefore,

$$\begin{aligned} &u_i(t_i; s_{-i}) \\ &= \int_0^{\epsilon} \theta_i{}'(\eta)\ d\eta - \sum_j \int_0^{v_i^j} P_i^j(z)\ dz \\ &= \int_0^{\epsilon} \theta_i{}'(\eta)\ d\eta - \sum_j \int_0^{\epsilon} P_i^j(\eta/\varsigma_i^j)\ d\eta. \end{aligned}$$

Now suppose $\exists\ s_i = (d_i, p_i)$ such that $u_i(s_i; s_{-i}) > u_i(t_i; s_{-i}) + \epsilon$. Let $\xi = \min_k e_i^k \circ a_i^k(s)$, and $\forall\ j, \zeta_i^j = \xi/\varsigma_i^j$ and $s_{i^*} = (\zeta_i, p_i)$. Then from (28), $a_i^j(s_{i^*}; s_{-i}) = \zeta_i^j$, therefore

$$\begin{aligned} &u_i(s_{i^*}; s_{-i}) \\ &= \int_0^{\xi} \theta_i{}'(\eta)\ d\eta - \sum_j \int_0^{\xi} P_i^j(\eta/\varsigma_i^j)\ d\eta. \end{aligned}$$

By Lemma 5.1, $u_i(s_{i^*}, s_{-i}) > u_i(t_i; s_{-i}) + \epsilon$, which is equivalent to

$$\int_{\epsilon}^{\xi} \theta_i{}'(\eta)\ d\eta - \sum_j \int_0^{\xi} P_i^j(\eta/\varsigma_i^j)\ d\eta > \epsilon.$$

The rest of the calculation follows as in [3] with the modified framework, i.e. both $\bar{\epsilon} = \epsilon + \epsilon/\theta_i{}'(0)$ and $\xi > \bar{\epsilon}$ show a contradiction. (DO I NEED TO SHOW THE CONTRADICTION?)

We address the case where $e = \chi_i$, that is, the seller j^*'s data price is equal to the budget of buyer i under the strategy given in (6), similarly to [3], in this case any bid s_{i^*} where $u_i(s_{i^*}) > u_i(t_i)$ is not feasible, as the buyer would go over its budget.

Finally, suppose that $j^* = I$, that is, $\mathcal{I}_i = \mathcal{I}$. In this case, if $u_i(s_{i^*}) > u_i(t_i)$, then the demand cannot be satisfied by the sellers and the bid is not feasible.

We proceed to examine the strategy of the seller. We argue that if truthfullness holds for both buyers and sellers, i.e. $p_i = \theta_i{}'\ \forall j$ and $p^j = \theta^{j\prime}\ \forall i$, then there exists a market equilibrium.

Theorem 5.1. *(Data Nash Equilibrium) Using the rules of the data auction mechanism, the secondary market described in [1] converges to a ϵ-Nash equilibrium. In the network auctin game with the PSP rule applied independently by each buyer, with fixed s^j and reserve prices $p^j > 0$, for all sellers $j \in \mathcal{I}$, and all buyers applying the opt-out strategy. Then the secondary market wil converge to an ϵ-Nash equilibrium.*

Proof: We will prove the theorem by asserting that our formulation upholds the VGC mechanism that is given in the properties of modern PSP. Namely,

- Incentive compatibility
- Efficiency
- Convergence

(WANT TO SHOW ALGORITHM DOES NOT RUIN VCG :
1. selling off data piecewise over time,
2. using the min price of sellers in the auction i.e. $\theta_i{}'(d^j_{i*}) = p^j$ is OK,
3. that bids are still feasible AND optimal
4. the algorithm achieves global economic equilibrium)
TRY:
Sellers only act when the resources obtained by the buyers influence their respective reserve prices, which agrees with the seller stragety of attempting to sell their data in the first iteration. Therefore we claim there exists a market stability and therefore, the existence of a Nash equilibrium. As the valuation of the sellers is derived by the demand of the buyers, who are bidding equivalent bids over a minimum subset of buyers, then the seller strategy, along with the seller constraint (**??**) results in a global market equilibrium.

5.2 Efficiency

5.3 Convergence

References

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