

# 1 Abstract

We investigate the 2cm app, a data-exchange platform published for China Mobile Hong Kong 4G Pro Service Plan customers. Telecommunications ISPs' revenue is typically gained by charging users a fixed fee for a maximum amount of data usage in a month, i.e., a monthly data cap [?]. 2cm's (2nd exchange market) data exchange platform allows users to submit bids to buy and sell data. This usage model, is, as far as our knowledge, the first data trading platform that allows customers to buy and sell their own data. We describe a distributed auction mechanism for data exchange inspired by the classic PSP throughput problem, and prove that our distributed data exchange mechanism provides incentive compatibility (social choice function), and that we have efficiency using only partial valuation information of each participant in an exchange market.

In applying a distributed PSP implementation to CMHK market, we find that the market is able to achieve an equilibrium as the sellers and buyers have an incentive for a collaborative exchange, and design our mechanism to provide the functions for effective communication between the connected users. We claim that in this secondary market our formulation holds the desired VCG qualities through the construction of a probable equilibrium [?]. We further provide bounds on the auction duration, with respect to the classic throughput problem. and provide simulated results on convergence time to support our (FIND COMPETITIVE

RATIO!), and a bound on the convergence of our mechanism. We extend the works of [1] i.e. (market influence/EQ, social EQ, payment/allocation models) OR (bandwidth, data bundles, distributed market algorithms) and show the existence of a dynamic global market equilibrium, allowing for a unique set of market dynamics.

## 2 Introduction

In this work, we propose a *distributed progressive second price (PSP) auction in order to maximize social utility in the CMHK market*. Using the distributed PSP mechanism on CMHKs data exchange platform, we show that for cellular data allocated between multiple users there exists an  $\epsilon$ -Nash market equilibria. A quality of the PSP auction is that demand information is not known centrally, rather, it is distributed in the buyers' valuations. The mechanism for an auction is defined as *distributed* when the allocations at any element depend only on *local* state: the quantity offered by the seller at that element, and the bids for that element only [2]. In this work, the proposed mechanism allows the distribution of bids, where there are many ISPs each holding their own local auction; there is no entity that holds a global market knowledge.

In a PSP mechanism, bids consist of (i) an available (required) quantity and (ii) a unit-price (calculated using its own demand functions). Buyers submit bids cyclically until an ( $\epsilon$ -Nash) equilibrium is reached and a local auction is concluded.

(FEE IS FIXED OR PER-

UNIT?)(HOW DO WE MODEL ISP REVENUE? IMMEDIATE FUTURE)

The form of the auction mechanism presented here is (CAN BE? NEED TO SHOW TO CLAIM 'IS') described as a pure-strategy progressive game with incomplete, but perfect information. (WHAT DOES NASH SAY ABOUT THIS?) (TRY MIXED? CAN ONLY HAVE MIXED WITH A DISTRIBUTED VALUATION.. FUTURE WORK)

The paper is organized as follows...

(MOVED: TO ORGANIZE)

The main contribution of this work is an auction mechanism inspired by the classic PSP throughput problem. In order to apply a distributed PSP implementation to the CMHK market, we analyze the behavior of users in a dynamical data exchange market. As both buyers and sellers are able to change their bid strategies, and as each user only has *local* information about the bidding environment, it is clear that an unconstrained market, even with a finite number of users, could suffer from the communication expense from numerous local auctions trading an infinitely divisible resource. We will assume that the cost of participating in the CMHK secondary market is absorbed by the bid fee, which could represent data used in submitting bids, or a fee charged per unit of data, or a flat rate charged at the completion of the purchase. It is worth mentioning that CMHK users are not allowed to resell data purchased from the CMHK market, additionally, the purchased data does not carry to the next service period. Therefore, a market equilibrium, where supply equals demand, requires that users

maintain incentive for truthfulness across a distributed setting. We complete a comprehensive analysis of market dynamics, (FINISH) To the best of our knowledge, this is the first work to provide a truthful mechanism in a distributed setting for data-exchange based on market behavior.

**Remark:** It is certainly possible to extend the mechanism, alleviating the restrictions introduced by the CMHK secondary market. We reserve this for future work.

We claim that the CMHK market is able to achieve an equilibrium as the sellers and buyers have an incentive for a collaborative exchange, and as our mechanism provides for effective communication between the connected users.

Despite the small message space, with many user types, there is no single way to do the transformation from the direct revelation mechanism to the desired one. Our mechanism is naturally designed by guessing right direct-revelation-to-desired-mechanism transformation and building it into the allocation rule to be the primary step in the design. This incentive for a user to truthfully reveal its type is built into the user strategies. As in classic mechanism design, we determine the equilibria as result of incentive compatibility in our mechanism design. We claim that in the CMHK market our formulation not only holds the desired VCG qualities, but minimizes communication overhead and auction duration, resulting in a convergence time (FIND COMPETITIVE RATIO?) with respect to the classic throughput problem.

This is the (built-in) transformation

from the direct-revelation mechanism to the desired message space.

## 2.1 Distributed Progressive Second Price Auctions

Progressive second price auctions (PSPs) were proposed in [2], [9] to provide a dynamic network service pricing scheme to provide consistent services for network bandwidth users. [9] conducts a game theoretic analysis, deriving optimal strategies for buyers and brokers, and further shows the existence of networkwide market equilibria based on their game-theoretic model. Constructing necessary and sufficient conditions for the stability of the game allows the sustainability of any set of service level agreement configurations between Internet service providers.

We begin with a brief introduction to the distributed PSP auction for bandwidth sharing, first introduced by Lazar and Semret [2]. We define a set of  $\mathcal{I} = \{1, \dots, I\}$  network bandwidth users. Suppose each user  $i \in \mathcal{I}$  makes a bid  $s_i^j = (p_i^j, d_i^j)$  to the seller of resource  $j$ , where  $p_i^j$  is the unit-price the user is willing to pay and  $d_i^j$  is the quantity the user desires. The *bidding profile* forms a grid,  $s \equiv [s_i^j] \in \mathcal{I} \times \mathcal{I}$ , and  $s_{-i} \equiv [s_1^j, \dots, s_{i-1}^j, s_{i+1}^j, \dots, s_I^j]_{j \in \mathcal{I}}$  is the profile of user  $i$ 's opponents. Using this classic PSP mechanism, [2] shows that given the opponents bids  $s_{-i}$ , user  $i$ 's  $\epsilon$ -best response to seller  $j$  is  $s_i^j = (w_i^j, v_i^j)$  and is a Nash move where  $\epsilon > 0$  is the bid fee,  $B_i = \sum_{j \in \mathcal{I}} b_i^j$  is user  $i$ 's budget, and every user has an elastic demand func-

tion. Based on the profile of bids  $s^j = [s_1^j, \dots, s_I^j]$ , the seller applies an allocation rule  $a(s^j) = [a_1^j, \dots, a_I^j]$ , where  $a_i^j$  is the quantity allocated by  $j$  to each user  $i \in \mathcal{I}$  and  $c_i^j$  is the cost charged to  $i$  for allocations awarded in auction  $j$ . An allocation is considered feasible if  $a_i^j \leq d_i^j$ , and  $c_i^j \leq p_i^j d_i^j$ .

### 2.1.1 The PSP Mechanism

The PSP auction as given in [2] and [3] is designed for the problem of network bandwidth allocation, and is analyzed as a noncooperative game where  $i \in \mathcal{I}$  agents buy the fixed amount of bandwidth  $d_i^j$  from sellers  $j \in \mathcal{I}$ . The market price function (MPF) for a buyer-seller pair is,

$$P_i^j(z, s_{-i}) = \inf \left\{ y \geq 0 : D_i^j(y, s_{-i}) \geq z \right\}, \quad (1)$$

and is the of minimum prices a user bids in order to obtain bandwidth  $z$  given opponent profile  $s_{-i}$ . The maximum available quantity of data in auction  $j$  at unit price  $y$  given  $s_{-i}^j$  is,

$$D_i^j(y, s_{-i}) = \left[ D^j - \sum_{p_k^j > y, k \neq i} d_k^j \right]^+, \quad (2)$$

where  $D^j$  is the total amount of bandwidth that user  $j$  has to offer. For each  $i \in \mathcal{I}$ , the allocation from auction  $j$  is,

$$a_i^j(s) = \min \left( d_i^j, D_i^j(p_i^j, s_{-i}^j) \right). \quad (3)$$

Finally, we have the cost of the allocation,

$$c_i^j(s) = \sum_{k \neq i} p_k^j [a_k^j(0; s_{-i}^j) - a_k^j(s_i^j; s_{-i}^j)]. \quad (4)$$

It was shown, in [2], that the mechanism may converge to a Nash market equilibria for differentiated services allocated between multiple agents when all players bid their real marginal valuation of the bandwidth resource. In other words, the PSP constraints are sufficient to attain the desirable property of truthfulness through incentive compatibility. The pricing mechanism upholds the *exclusion-compensation principle*, user  $i$  pays for its allocation so as to exactly cover the “social opportunity cost” which is given by the declared willingness to pay (bids) of the users who are excluded by  $i$ ’s presence, and thus also compensates the seller for the maximum lost potential revenue [2]. The PSP rules assume that an agent’s valuation is represented by an elastic valuation function.

**Definition 2.1.** [2] A real valued function,  $\theta(\cdot) : [0, \infty) \rightarrow [0, \infty)$ , is an (*elastic*) *valuation function* on  $[0, D]$  if

- $\theta(0) = 0$ ,
- $\theta$  is differentiable,
- $\theta' \geq 0$ , and  $\theta'_i$  is non-increasing and continuous,
- There exists  $\gamma > 0$ , such that for all  $z \in [0, D]$ ,  $\theta'(z) > 0$  implies that for all  $\eta \in [0, z]$ ,  $\theta'(z) \leq \theta'(\eta) - \gamma(z - \eta)$ .

The function  $\theta'(\cdot)$  on  $[0, D]$  is called an (elastic) demand function. In the PSP market, a user is considered truthful if their bid price equals their marginal valuation, i.e.  $p_i^j = \theta_i^{j'}$ .

## 3 Related Work

## 4 The Problem Model

### 4.1 The CMHK Market

We construct the model for a PSP data auction for mobile users participating in CMHK’s secondary data-sharing market. Define  $\mathcal{I} = \{1, \dots, I\}$  as the set of users who purchase or sell data from other users. We again define the bidding profile for any user to be  $s \equiv [s_i^j] \in \mathcal{I} \times \mathcal{I}$ , and  $s_{-i} \equiv [s_1^j, \dots, s_{i-1}^j, s_{i+1}^j, \dots, s_I^j]_{j \in \mathcal{I}}$  as the profile of user  $i$ ’s opponents. As a cellular network is fully connected, we assume that a buyer submits bids directly to sellers via the cellular network. We assume that users are selfish, and therefore rational. Users prefer to participate in the CMHK secondary market as it allows them to purchase additional data for a cost less than the overage fee set by the ISP. Thus we assume that all users submit bids in order to maximize their (private) valuation functions. In general, user preferences are defined by a utility function,  $u$ , which represents a users’ valuation of an allocation minus the price. Absent the cost or revenue from trading data, CMHK users gain utility from consuming data. We assume that data is a unary resource belonging to the seller, and therefore we identify each local auction with the identity of the seller  $j \in \mathcal{I}$ , and that the user valuation satisfies the conditions for an *elastic demand function*, as in [2]. Finally, we assume in the CMHK market that a buyers’ budget is always sufficient, as the alternative is to pay the overage fee to the

ISP.

The CMHK market does not allow for brokers [6], we thereby determine that the bid profiles must adhere to some additional restrictions, which we will imply using the PSP bid profile notation. In other words, we assume that buyers and sellers are separated (a seller does not also buy data and vice versa). Thus, we may assume that this is implied in our notation. A user's identity  $i \in \mathcal{I}$  as a **subscript** indicates that the bid belongs to a **buyer**, and a **superscript**,  $j \in \mathcal{I}$ , indicates the bid belongs to a **seller**. Suppose  $i$  is buying from  $j$ . The bid is represented by  $s_i^j = (d_i^j, p_i^j)$ , meaning  $i$  would like to buy from  $j$  a quantity  $d_i^j$  and is willing to pay a unit price  $p_i^j$ . Without loss of generality, we assume that all users bid in all auctions; if a user  $i$  does not submit a bid to  $j$ , or vice versa, we simply set  $s_i^j = (0, 0)$ . Naturally, in a live auction, if a buyer does not submit a bid to a seller, then this implies  $s_i^j = 0$  for both buyer  $i$  and seller  $j$ . Obviously, a buyer that does not submit a bid will not receive opponent profiles from seller  $j$ . We additionally determine that a user who does not submit a bid is holding its previous bid, either zero or nonzero. For the purposes of our analysis, we will assume that a zero bid from a buyer is equivalent to no bid. A seller  $j$  submits a bid  $s_i^j = (d_i^j, p_i^j)$  to the secondary market, with the intent of offering a quantity  $d_i^j \in d^j = [d_i^j]_{i \in \mathcal{I}}$  with reserve unit price  $p_i^j \in p^j = [p_i^j]_{i \in \mathcal{I}}$  to buyer  $i$ . We emphasize that we allow for  $s_i^j$  to stand for a buyer or sellers' bid, the *direction* of the bid (vector) is determined by the user type, whether or not they are

a buyer or a seller. To further clarify our analysis, we will emphasize the separation of buyers and sellers using  $s_i$  and  $s^j$ , indicating if a bid is from a buyer or a seller. In other words, a bid  $s^j = [s_i^j]_{i \in \mathcal{I}}$  is understood as an offer of data by seller  $j$  in the CMHK secondary market. The notational conventions of the bid vectors are essentially slices of the grid,  $s^j = [s_i^j]_{i \in \mathcal{I}}$  denotes a sellers' profile, and  $s_i = [s_i^j]_{j \in \mathcal{I}}$  denotes a buyers' profile. Furthermore, noting that this is a simplification for ease of notation, we let  $D^j = \sum_{i \in \mathcal{I}} d_i^j$  be the total amount of data  $j$  has to sell, and  $D_i = \sum_{j \in \mathcal{I}} d_i^j$  represent the total amount of data requested by buyer  $i$ .

Consider the grid of bid profiles,  $s$ , representing the distributed PSP auction mechanism in the CMHK market, each buyer  $i$  will have information from each seller  $j$ , as well as opponent profiles  $s_{-i}$  from each auction in which it is participating, and therefore in the extreme case, where  $i$  submits bids to all auctions  $j \in \mathcal{I}$ , buyer  $i$  gains access to the full grid  $s$ . However, sellers can only gain information about the market grid by observing buyer behavior in their local auction. In our current formulation, we do not allow a seller to host multiple auctions (FUTURE WORK?). Thus, the buyers are able to directly and globally influence the CMHK market dynamics, with the sellers taking a secondary role (FIND IN ANOTHER PAPER FOR SUPPORT OR OBVIOUS ENOUGH?).

## 4.2 User Valuation

We assume that the CMHK market does not have network bottlenecks, which is purely a bandwidth problem, as in [3], and as bandwidth in wireless networks is not the focus of this paper. We conduct a comprehensive analysis, deriving a globally optimal strategy suited for users with local information in a distributed data-sharing model. Users in the CMHK market do not need to bid on a complete “route”, as in the bandwidth allocation problem [3]. Instead, our mechanism allows a buyer to *opt-out* of auctions by submitting zero bids. This strategy maximizes utility while minimizing the number of positive bids submitted to the overall market. We define an **opt-out function**, associated with a buyer  $i$  as part of its type. Buyer  $i$ , when determining how to acquire a possible allocation  $a$ , will determine its bid quantities by,

$$\sigma_i(a) = [\sigma_i^j(a)]_{j \in \mathcal{I}}. \quad (5)$$

In a general sense, this abstraction applies our *user strategy* to the PSP rules. We define each buyer as a user  $i \in \mathcal{I}$  with quasi-linear utility function  $u_i = [u_i^j]_{j \in \mathcal{I}}$ , a buyers’ utility function is now of the form,

$$u_i = \theta_i \circ \sigma_i(a) - c_i, \quad (6)$$

where the composition of the elastic valuation function  $\theta_i$  with  $e_i$  acts as a distribution function, splitting the buyers’ valuation of allocation  $a$  across local markets (and thus multiple sellers). In this way we extend the PSP rules described in [3] to design equilibria across subsets of local data-exchange markets.

The sellers,  $j \in \mathcal{I}$  are not associated with an opt-out function, we consider their valuation to be a functional extension of the buyers, where  $\theta^j$  is constructed by buyer demand. The sellers strategy can only be to determine the reserve price of their local auction, using only information from buyers who have not opted out. In our analysis, we demonstrate market dynamics, and further show evidence of symmetry in the strategies of buyers and sellers.

### 4.2.1 Valuation under Market Dynamics

The buyer demand largely motivates the market price function, however, the distributed nature of the market prevents any single user from knowing the market demand for a quantity of data. All users have knowledge of market supply, as this is public information, however only buyers are able to determine supply or demand across multiple auctions after time  $t = 0$ , and then only from auctions in which they participate.

**Remark:** It is possible that a seller would be able to derive information about other auctions by examining buyer bids over time, particularly if the seller had knowledge of the buyer strategy. (FUTURE WORK?)

In order to represent the public platform of the secondary market, we model the ISP as a kind of buyer  $\kappa$  who remains at time  $t = 0$ , i.e. does not participate in any auctions. At time  $t = 0$ , a seller  $k$  entering the market will have submitted bid  $s_\kappa^j = (\epsilon, D^j)$  to the CMHK platform. We

assume that the auction begins at time  $t > 0$ , and so  $j$  will increment through a single iteration, initializing bid prices.

A given demand may lead to different prices, which emerge directly from the users valuation of the resources and, possibly, their decision strategies. The seller does not require a priori demand information. It is worth mentioning that the *analysis* of the auction as a game assumes some forms of demand and supply, to derive properties. The mechanism itself does not require any knowledge of user demand or valuation.

As the buyer valuation is elastic, even infinitesimal changes in the market dynamics can be modeled. This, and the homogenous nature of data in the CMHK market, allows for the analysis of additional constraints imposed by the user strategies. Thus our motivation to begin our analysis with buyer valuation  $\theta_i$ . A buyers' valuation of an amount of data represents how much a buyer is willing to pay for that amount. This is equivalent to the bid price that satisfies  $\theta_i$  given a fixed amount of data. We determine the buyers' utility-maximizing bid given quantity  $z \geq 0$  to be a mapping to the lowest possible unit price. We have,

$$f_i(z) \triangleq \inf \{y \geq 0 : \rho_i(y) \geq z, \forall j \in \mathcal{I}\}, \quad (7)$$

where  $\rho_i(y)$  represents the demand function of buyer  $i$  at bid price  $y \geq 0$ , and gives the quantity that buyer  $i$  would buy at a given price. We determine that the market supply function is equivalent to buyer demand, and acts as an "inverse" function of  $f_i$ . We have, for bid price

$y \geq 0$ ,

$$\rho_i(y) = \sum_{j \in \mathcal{I}: p_i^j \leq y} D^j. \quad (8)$$

We note that  $f_i$  is such that  $i$  could still bid in *any* auction  $j \in \mathcal{I}$ . Therefore, in a coordinated bid, the utility-maximizing bid price is the lowest unit cost of the buyer to participate in all auctions, and corresponds to the maximum reserve price amongst the sellers. We will later show, in Proposition 23, that a buyer does not lose utility from coordinating its bids, i.e. using the same bid for all auctions. Using buyer bid quantity  $\sigma_i^j(a)$ , from (5), we have the following Lemma,

**Lemma 4.1.** (*Buyer constraints*) Let  $i \in \mathcal{I}$  be a buyer with bid,  $s_i = (\sigma_i(a), p_i)$ , which must obey the following constraints,

$$\sum_{j \in \mathcal{I}} \sigma_i^j(a) \leq D_i, \quad (9)$$

and, as  $i$  is rational, and therefore utility-maximizing,  $i$  must have sufficient bids  $s_i^j > 0$  so that,

$$\sum_{j \in \mathcal{I}} d_i^j \geq D_i, \quad (10)$$

where  $d_i^j$  denotes the bid quantity of seller  $j$ .

**Proof:** A rational buyer does not want to purchase extra data, as this would be equivalent to overpaying, and so (9) holds. We also assume that  $i$  wants to fulfill their data requirement, and so (10) implies that  $i$  submits positive bids to a set of buyers that will be able to fulfill their demand.

The seller is similarly represented, however a seller only has information from

buyers in its own auction, and may only be indirectly influenced by buyers in other auctions. So from the perspective of the seller we have a more direct interpretation of valuation as revenue. We determine the demand function of seller  $j$  at reserve price  $y \geq 0$  to be,

$$\rho^j(y) = \sum_{i \in \mathcal{I}: p_i^j \geq y} \sigma_i^j(a), \quad (11)$$

and define the “inverse” of the buyer demand function for seller  $j$  as potential revenue at unit price  $y$ , we have,

$$f^j(z) \triangleq \sup \{y \geq 0 : \rho^j(y) \geq z, \forall i \in \mathcal{I}\}, \quad (12)$$

and, unsurprisingly,  $f^j$  maps quantity  $z$  to the highest possible unit data price. Now, we determine that, in addition to some natural constraints, the sellers’ highest “possible” price is conditioned by additional restrictions on its pricing function, the motivation for which is that a buyer has the choice to opt-out of any auction, and so seller  $j$  has an incentive to set its reserve price to include enough buyers to sell all its data. Thus we have the following Lemma,

**Lemma 4.2.** (*Seller constraints*) Let  $j \in \mathcal{I}$  be a seller with bid  $s^j$ , then  $j$  must obey the following constraints. A bid  $s_i^j = (d_i^j, p_i^j)$ , to buyer  $i$  must be such that,

$$\sum_{i \in \mathcal{I}} d_i^j \leq D^j, \quad (13)$$

In addition, for a rational seller, the reserve price must satisfy,  $\forall i \in \mathcal{I}$ ,

$$p_i^j \geq \min_{i \in \mathcal{I}} (p_i^j), \quad (14)$$

and,

$$p_i^j \leq \max_{i \in \mathcal{I}} (p_i^j). \quad (15)$$

Finally, as  $j$  is rational,  $j$  must submit enough bids  $s_i^j > 0$  so that,

$$\sum_{i \in \mathcal{I}} \sigma_i^j(a) \geq D^j. \quad (16)$$

**Proof:** The first statement is obvious, a seller cannot sell more data than they have. Then, (14) follows from the assumption that  $j$  is rational, and so utility-maximization acts as revenue maximization, a rational seller will not sell its data at a price less than the lowest offer. Finally, as a rational seller will attempt to maximize profit, and will try and sell all of its data, and so we must have sufficient demand from the buyers. Therefore, (15) and (16) hold. Together, (13) and (16) address the case where  $\sum_{i \in \mathcal{I}} \sigma_i^j(a) > D^j$ . In this case, by (21),  $j$  may submit a partial bid to a buyer  $i$  so that  $\sum_{i \in \mathcal{I}} d_i^j = D^j$ .

We proceed to determine the properties and characteristics of the user valuation functions. Buyer  $i$ ’s valuation is interpreted as a unit valuation  $\theta_i$ , distributed across the CMHK market via bids in local auctions. As such, buyer behavior is influenced by opponent bid profiles from multiple auctions. We therefore determine that the valuation of any user must be modeled as a function of the entire marketplace. Naturally, a buyers’ valuation is aggregated over local markets, and the sellers’ valuation is aggregated over its own auction. We model the valuation of users based on the opt-out user behavior, thereby including the buyers’ influence on the market. We have already introduced the composition  $\theta_i \circ e_i$



as the valuation of the buyers. We further model the valuation of the sellers, based on (11) and (12). We first note that, in general (and so we omit the subscript/superscript notation), the valuation of data quantity  $x \geq 0$  is given by,

$$\theta(x) = \int_0^x f(z) dz,$$

as in [3]. Now, we have the following Lemma,

**Lemma 4.3.** (*User valuation*) *For any buyer  $i \in \mathcal{I}$ , the valuation of a potential allocation  $a$  is,*

$$\theta_i \circ \sigma_i(a) = \sum_{j \in \mathcal{I}} \int_0^{\sigma_i^j(a)} f_i(z) dz. \quad (17)$$

Now, we may define seller  $j$ 's valuation in terms of revenue, so we have,

$$\theta^j = \sum_{i \in \mathcal{I}} \theta^j \circ \sigma_i^j(a) = \sum_{i \in \mathcal{I}} \int_0^{\sigma_i^j(a)} f^j(z) dz. \quad (18)$$

**Proof:** We assume that a buyer wants to minimize the cost of purchasing their data requirement, at the same time ensuring they get the full allocation  $e_i(a)$ , and so must minimally meet  $j$ 's reserve price. A seller will try to maximize profit, and so will try to sell all of its data. By Lemma 4.1 and Lemma 4.2, we have,  $d_i^j = \sigma_i^j(a) \Rightarrow \sum_{i \in \mathcal{I}} d_i^j = \sum_{i \in \mathcal{I}} \sigma_i^j(a) = D^j = D_i$ . It is worth observing that this result does *not* imply that  $s_i = s^j$ . Let  $\xi$  be a unit of data from buyer bid quantity  $\sigma_i^j(a)$ . If  $\xi$  decreases by incremental amount  $x$ , then seller bid  $d_i^j$  must similarly decrease. The lost potential revenue

for seller  $j$  is the price of the unit times the quantity decreased, by definition,  $f^j(\xi)x$ , and so,

$$\theta^j(\xi) - \theta^j(\xi - x) = f^j(\xi)x.$$

Thus (18) holds. As we may use the same argument for (17), we complete the proof with the observation that the function  $f^j$  (resp.  $f_i$ ) is the first derivative of the valuation function with respect to quantity, and therefore, in this context, represent the marginal valuations of the users,  $\theta^{j'}$  (resp.  $\theta_i'$ ). Finally, the sellers' natural utility is the potential profit, or simply  $u^j = \theta^j$ , where we have chosen to omit the original cost of the data paid to the ISP, as it is not a component of our mechanism, and as a discussion of mobile data plans is outside the scope of this paper.

### 4.3 PSP for Data-Exchange

We intend to show that our auction is rational and achieves the desired VCG properties, as does the original formulation. Using [1] as a basis for our model, and [2] as realistic, theoretic, and notational templates, we define optimal strategies for CMHK users, and demonstrate that the development of a set PSP auction mechanisms in a data exchange setting is able achieve a network equilibrium for cellular data.

#### 4.3.1 Data Auction Mechanism

We now proceed to formally define the PSP auction, which determines the actions buyers and sellers in the CMHK market, and which we will denote the

data PSP rules. The rules presented here incorporate of the opt-out function with the mechanism as in [2], which we note greatly simplifies our analysis. The market price function (MPF) for a buyer in the CMHK market can be described as follows:

$$\begin{aligned}\bar{P}_i(z, s_{-i}) &= \sum_{j \in \mathcal{I}} \sigma_i^j \circ P_i^j(z_i^j, s_{-i}^j) \\ &= \sum_{j \in \mathcal{I}} \left( \inf \left\{ y \geq 0 : D_i^j(y, s_{-i}^j) \geq \sigma_i^j(z) \right\} \right),\end{aligned}\quad (19)$$

and is interpreted as the aggregate of minimum prices that buyer  $i$  bids in order to obtain data amount  $z$  given opponent profile  $s_{-i}$ . We note that the total minimum price for the buyer must be an aggregation of the *individual* prices of the buyers as it is possible that the reserve prices of the individual sellers may vary.

**Remark:** We further note that except at points of discontinuity, from Lemma 4.3 we have that  $P_i^j(z) = f_i(z)$ .

The maximum available quantity of data in auction  $j$  at unit price  $y$  given  $s_{-i}^j$  is:

$$D_i^j(y, s_{-i}^j) = \left[ D^j - \sum_{p_k^j > y} d_k^j \right]^+. \quad (20)$$

The resulting data allocation rule is a function of the local market interactions between buyers and sellers over all local auctions, as is composed with  $i$ 's opt-out value, so that for each  $i \in \mathcal{I}$ , the allocation

from auction  $j$  is,

$$\begin{aligned}\bar{a}_i^j(s) &= \sigma_i^j \circ a_i^j(s) \\ &= \min \left\{ \sigma_i^j(a), \frac{d_i^j}{\sum_{p_k^j = p_i^j} d_k^j} D_i^j(p_i^j, s_{-i}^j) \right\},\end{aligned}\quad (21)$$

noting that for the full allocation from all auctions we may simply aggregate over the seller pool. Finally we must have that the cost to the buyer adheres to the second price rule for each local auction, with total cost to buyer  $i$ ,

$$\bar{c}_i(s) = \sum_{j \in \mathcal{I}} p_i^j \left( \bar{a}_i^j(0; s_{-i}^j) - \bar{a}_i^j(s_i^j; s_{-i}^j) \right). \quad (22)$$

## 4.4 User Behavior

### 4.4.1 Buyer Strategy

Although it is possible for a seller to fully satisfy a buyer  $i$ 's demand, it is also reasonable to expect that a seller may come close to using their entire data cap, and only sell the fractional overage. In this case, we determine that some buyers must split their bids among multiple sellers. (DO I WANT ISP UPDATE?, MORE FAIR NOT TO?) We assume that sellers update their bids with the ISP, which is modeled as a "buyer"  $\kappa$ . (CAN I GIVE SELLERS A CHOICE TO UPDATE WITH ISP? WITHOUT KNOWLEDGE -i FAIRNESS?) A new sellers' bid,  $(D^j, \epsilon)$ , is public knowledge, however the buyer does not have price information until joining the auction. (FINISH!) We reason that buyers, in addition to fulfilling

their data requirement, have incentive to join multiple auctions in order to compare prices. Each seller that buyer  $i$  adds to its pool potentially decreases  $i$ 's bid quantity in all other auctions, and so each buyer  $i \in \mathcal{I}$  directly impacts other buyers in auctions where buyers intersect. Indeed, each auction joined has the potential to decrease the price a buyer will pay for its data, in exchange for additional messaging overhead, possible fees, and increased competition from other buyers. We determine that this aspect of a buyers' valuation represents equity, the liabilities from which may be considered work done to find sellers, monetary fees, or effort of communication from participation, and so is naturally absorbed by the bid fee as characterized in [3].

The buyer strategy bids in auctions with the highest quantity bids first, a natural exploitation of the demand curve. A new seller entering the market with a large quantity bid will be in high demand. This behavior contributes to market price stability, as seller valuation is determined by buyer demand, and as such, the buyer strategy drives sellers towards similar quantity bids, and therefore similar prices. Market equilibrium is achieved when each buyer has equal bids in each auction. Our bidding strategy is inspired by [2], and we also hold buyers to consistent bids, where buyers submit identical bids to a subset of sellers with the highest offers. Thus, we determine when rational (utility-maximizing) buyers opt-out of a local auction. We propose the following strategy,

**Proposition 4.1.** (*Opt-out buyer strat-*

*egy*) Define any auction duration to be  $\tau \in [0, \infty)$ . Let  $i \in \mathcal{I}$  be a buyer and fix all other buyers' bids  $s_{-i}$  at time  $t > 0 \in \tau$ , and let  $a$  be  $i$ 's desired allocation. Define the composition,

$$\sigma_i^j \circ a = \sigma_i^j(a) = \frac{a_i^j}{j},$$

to be the buyer strategy with respect to quantity, and the set,

$$\mathcal{I}_i(n) = \arg \max_{\mathcal{I}' \subset \mathcal{I}, |\mathcal{I}'|=n} \sum_{j \in \mathcal{I}'} D^j,$$

where buyer  $i$  chooses its seller pool by determining  $n$ , where

$$n = \min(j \mid j \in \mathcal{I}_i(n) : jD^j \geq D_i), \quad (23)$$

The buyer strategy produces a minimal subset of sellers  $\in \mathcal{I}$  able to satisfy buyer  $i$ 's demand. For fixed  $n$  we will denote this subset,

$$\mathcal{I}_i \subset \mathcal{I}. \quad (24)$$

Now let  $j^* = n \leq I$  represent the seller with the least amount of data  $\in \mathcal{I}_i$ , i.e.  $D^{j^*} \leq D^j, \forall j \in \mathcal{I}^j$ , and define  $i$ 's bid vector  $e_i$  with respect to its strategy, where

$$\sigma_i^j(a) \triangleq \begin{cases} \sigma_i^{j^*}(a), & j \in \mathcal{I}^j, \\ 0, & j \ni \mathcal{I}^j. \end{cases} \quad (25)$$

Now, (25) holds  $\forall j \in \mathcal{I}$ , and we have that  $\sigma_i$  defines an optimal strategy for buyer  $i$ .

**Proof:** We assume that a buyer will try and fill their data requirement. In the case that there exists a seller who can completely satisfy a buyers' demand,  $j^* = 1, |\mathcal{I}_i| = 1$  and (23) holds. If such a buyer does not exist, as the set  $\mathcal{I}_i$  is ordered by the quantity of the sellers' bids,

$i$  may discover  $j^*$  by computing  $\mathcal{I}_i$ . Suppose that  $D_i > \sum_{j \in \mathcal{I}} D^j$ , then  $j^* > I$  and  $\mathcal{I}_i = \emptyset$ . We model the ISP at time  $t > 0$  as a seller  $\kappa$  with bid  $s^\kappa = (d^\kappa, p^\kappa)$ , where  $d^\kappa > D^j$ ,  $\forall j \in \mathcal{I}_i$ , and  $p^\kappa$  represents the overage fee for data set by the ISP, which we note is also the upper bound of the sellers' pricing function, and so again (23) holds. Now, consider some  $k \neq i \in \mathcal{I}$  where  $p_i^j = p_k^j$ . The allocation rule (21) determines that the data will be split proportionally between all buyers with the same unit price. It is possible that the resulting partial allocation of data to  $i$  and  $k$  would not satisfy some demand. As the two cases  $i$  and  $k$  are the same, we will only consider one. Suppose seller  $j$  updates its bid to reflect the new data quantity, where  $d_i^{j(t+1)} < \sigma_i^{j(t)}(a)$ . First,  $i$  sets its bid to  $s_i^j = 0$ , and from the new subset  $\mathcal{I}_i$ , submits bids (PARALLEL OR SEQUENTIAL?) until  $\sum_{j \in \mathcal{I}_i} d_i^j \geq D_i$ , by (10). Now, we consider the case where a new buyer  $k$  with bid price  $p_k^j > p_i^j$  for some  $j \in \mathcal{I}_i$ , in other words, a new buyer  $k$  may enter the market with a better price, decreasing the value of  $i$ 's bid for  $j \in \mathcal{I}_i$ . In this case, by (23),  $i$  will choose  $\mathcal{I}_i$  so that,

$$\sigma_i^{j(t+1)}(a) = \sigma_i^{j(t)}(a) - \sigma_k^{j(t)}(a),$$

and so  $\mathcal{I}_i$  is large enough to balance the additional demand from  $k$ . Thus, as in each case we have that  $i$  is able to satisfy thier demand, and we determine that the opt-out strategy is optimal.

**Remark:** The bid quantity  $\sigma_i^j(a)$  and the allocation  $\bar{a}_i^j$  are complementary. In fact, the buyer strategy is the first term in the minimum, the second term being owned

by the seller.

Finally, we note that  $\mathcal{I}_i$  is not the only possible minimum subset  $\in \mathcal{I}$  able to satisfy  $i$ 's demand, it is the minimal subset where a coordinated bid is possible. 5.

#### 4.4.2 Seller Strategy

In order to to develop the seller strategy, we examine the incentive of a rational seller with only local information in a dynamic market of many buyers and sellers. A local auction, examined independently, may appear as single market with a single seller and many buyers, but is in fact a subset of the larger data-exchange market, and is subject to the trends and dynamics therewithin. A seller must determine allocations using only bids in its local market, while the buyers' response is based on the allocations and resulting opponent bids from all auctions in its seller pool. In addition, buyers are allowed to bid both dynamically and asynchronously. In order to maximize revenue, the seller must also be able to respond dynamically to address the mutation of competitive bids in its market. In order to do this, we determine that the seller may modify its reserve price in response to the changing market dynamics.

We will show that sellers are able to maximize revenue in restricted subset of buyers in  $\mathcal{I}$ , and as such will attempt to facilitate a local market equilibrium for this subset. A local auction  $j$  converges when all buyer bids remain the same over a time step, that is, if  $\forall i \in \mathcal{I}$ ,  $s_i^{j(t+1)} = s_i^{j(t)}$ , at which point the allocation is stable, the data is sold, and the

auction ends. In the sellers' local environment, we determine that the best course of action is to maximize revenue, and then try to keep its buyer pool stable until convergence occurs. Thus, the seller strategy is complementary to that of the buyers, and is designed to achieve and maintain a local market equilibrium.

We describe a *local* auction strategy for data allocation, where the seller is unaware of the existence of other auctions, and so the seller behavior is the same in the case of a single buyer, a small buyer set, and in the extreme case, where all buyers  $i \in \mathcal{I}$  participate. We again note that the seller must initialize the strategy with a first iteration, and so the auction is defined for time  $t > 0$ . In our model, a local auction may be described as a progressive game of strategy with incomplete, but perfect information, however in our analysis, as before, we will assume complete information. (BUYERS ARRIVE AS A POISSON PROCESS? FUTURE WORK)

**Proposition 4.2.** (*Localized seller strategy (i.e. progressive allocation)*) Define any auction duration to be  $\tau \in [0, \infty)$ . For any seller  $j$ , fix all other bids  $[s_i^k]_{i,k \neq j \in \mathcal{I}}$  at time  $t > 0 \in \tau$ . Define the set,

$$\mathcal{I}^j(n) = \arg \max_{\mathcal{I}' \subset \mathcal{I}, |\mathcal{I}'|=n} \sum_{i \in \mathcal{I}'} p_i^j,$$

where,

$$n = \min(i \mid i \in \mathcal{I}^j(n) : \sum_{i \in \mathcal{I}} d_i^j \geq D^j), \quad (26)$$

where  $n$  produces a minimal subset of buyers that maximizes  $j$ 's revenue at time  $t$ ,

which we will denote, for fixed  $n$ , by,

$$\mathcal{I}^j \subset \mathcal{I}. \quad (27)$$

Define buyer  $i^* = n - 1 \leq I$  as the buyer with the maximum bid price  $\ni \mathcal{I}^j$ . Then, for time  $(t+1)$ , set  $j$ 's reserve price to be,

$$p_i^{j(t+1)} = p_{i^*}^{j(t)} + \epsilon, \quad (28)$$

Let the winner at time  $t$  be determined by,

$$\bar{i} = \max_{i \in \mathcal{I}^j} p_i^{j(t)}, \quad (29)$$

and update  $j$ 's total data to reflect the (tentative) allocation,

$$D^{j(t+1)} = D^{j(t)} - \sigma_{\bar{i}}^{j(t)}(a), \quad (30)$$

Allowing  $t$  to range over  $\tau$ , we have that (26) - (30) produces a local market equilibrium.

**Proof:** We assume that the seller will try to maximize its revenue. In the case where  $|\mathcal{I}^j| = 1$ , then if  $\sigma_i^j(a) = D^j$ , then  $j$ 's market is at equilibrium. Otherwise, we arrive at the case of multiple buyers, which we note includes the case where  $\sigma_i^j(a) < D^j$ , which is reflected trivially here.

For auction  $j$  with multiple buyers,  $i^*$  is the *losing* buyer with the highest unit price offer, determined by (26). Suppose that for some  $i \in \mathcal{I}^j$ , buyer demand is not met. In this case, by (16) the seller must notify  $i$  of a fractional allocation by changing the bid vector at index  $i$ . With this caveat, and Proposition 4.1, we have that the aggregate demand of subset  $\mathcal{I}^j$  is satisfied by seller  $j$ , as in Lemma 4.2. Although the buyers' valuation  $\theta_i$  is not known to the seller, we will assume that buyers are bidding truthfully, and so the new reserve price  $p_{i^*}^j + \epsilon = \theta_{i^*}' + \epsilon$ . For

clarity, let the reserve price be denoted by  $p_*^j$ . Now, by the point elasticity of (7) and (12) (PROVE POINT ELASTICITY?), we have that,  $\forall z \geq 0$ ,

$$f_{i^*}(z) < f^j(z) \leq f_i(z),$$

which holds  $\forall i \in \mathcal{I}^j$ , and  $\forall j \in \mathcal{I}_i$ . We claim that the choice of reserve price  $p_*^j$  does not force any buyers out of the local auction. To show this, we use the assumption of truthful bids, and the fact that since the auction begins at time  $t > 0$ , buyers will bid at least once. In a truthful reply, the term  $\epsilon/\theta'_i(0)$  ensures that a new bid price differs from the last bid price by at least  $\epsilon$ . Suppose the auction starts at equilibrium, so  $\sum_{i \in \mathcal{I}^j} \sigma_i^j(a) = D^j$  at time  $t = 0$ . The reserve price  $p_*^j$  set at time  $t = 0$  begins the auction with the first bid iteration, and so at  $t > 0$ ,  $\forall i \in \mathcal{I}^j$ , we have that  $p_i^j - p_*^j \geq \epsilon$ . Now, in the case where at  $t = 0$ ,  $\sum_{i \in \mathcal{I}^j} \sigma_i^j(a) > D^j$ , by (21), the seller notifies (any) buyer  $k$  with the lowest bid price of a fractional allocation by changing  $d_k^j$  thus by Proposition 4.1,  $k$  decreases its demand until  $\sigma_k^j(a) \leq d_k^j$ . Then, as the seller computes the set  $\mathcal{I}^j$  at each time step, a new  $i^*$  may be chosen and the buyers bid again. Suppose  $\exists k \in \mathcal{I}^j$  such that  $\forall l \in \mathcal{I}_k$ ,  $i \ni \mathcal{I}^l \forall i \neq k \in \mathcal{I}^j$ . That is,  $k$  is disconnected from all other buyers  $i \in \mathcal{I}^j$ , and suppose that  $d_k^j$  is partial allocation at  $t > 0$ , and further suppose that there are many  $l \in \mathcal{I}_k$  where  $|\mathcal{I}^l| > |\mathcal{I}^j|$ . The more buyers an auction has, the more likely that cases will occur that cause buyers to rebid, particularly if auctions  $l \in \mathcal{I}_k$  have overlapping buyers, then  $k$  may opt-out of auction  $j$ , i.e.  $s_k^{j(t)} \neq s_k^{j(t+1)} = 0$ ,

then the seller may simply return the tentatively allocated data to  $D^j$ . Finally, we note that if for some  $i \in \mathcal{I}^j \ni k \in \mathcal{I}^j$  such that  $p_i^j = p_k^j$ , then the seller again notifies the buyers of a fractional allocation by changing  $d_i^j$  and  $d_k^j$  by (21). Thus we determine the valuation between seller  $j$  and buyer  $i$  is well-posed, the reserve price (28) is justified, and the local equilibrium created by  $j$  is independently stable from time  $t$  to  $(t + 1)$ .

#### 4.4.3 Market Dynamics under Strategy

We conclude this section by examining the relationship between the strategies of buyers and sellers in local auctions. We model the impact of the dynamics of the data-exchange market on a local auction  $j$ . As we have shown, the seller is a functional extension of the buyer, with rules determined by the buyers' behavior. This gives an auction  $j$  a natural logical extension into the global market through its buyers. We demonstrate that the symmetry between buyer and seller behavior, consequently strategies, stretches into a symmetry across subsets of local auctions. Additionally, we identify a clear bound restricting the influence of local auctions on each other. We have the following Lemma,

**Lemma 4.4.** (*Valuation across local auctions*) For any  $i, j \in \mathcal{I}$ ,

$$j \in \mathcal{I}_i \Leftrightarrow i \in \mathcal{I}^j. \quad (31)$$

Fix an auction  $j \in \mathcal{I}$  with duration  $\tau$  and define the influence sets of users. The pri-

mary influencing set is given as,

$$\Lambda = \bigcup_{i \in \mathcal{I}^j} \mathcal{I}_i, \quad (32)$$

with secondary influencing set,

$$\lambda = \bigcup_{i \in \mathcal{I}^j} \left( \bigcup_{k \in \mathcal{I}_i} \mathcal{I}^k \right) \quad (33)$$

Fixing all other bids  $s_i^j \in \mathcal{I}$ , and time  $t > 0 \in \tau$ , we have that,

$$\sum_{j \in \Lambda} \theta_i^j = \sum_{i \in \lambda} \theta_i^j. \quad (34)$$

**Proof:** A local auction  $j \in \mathcal{I}$ , is determined by the collection of buyer bid profiles, where buyer bid  $s_i^j > 0 \Rightarrow j \in \mathcal{I}_i$ , and we have that  $i \in \mathcal{I}^j$ , if and only if  $i$ 's bid is such that  $p_i^j > p_{i^*}^j$ , where (26) defines  $i^*$  as the losing buyer with the highest bid price in auction  $j$ . By (7)  $p_i^j \geq p_{i^*}^j + \epsilon$ , thus  $p_i^j < p_{i^*}^j$  can only happen during a market shift, caused by the underlying dynamics. Consider  $k \in \mathcal{I}^j$  where for example, new buyers enter the auction, or increase their bid prices, so that  $p_i^j < p_{i^*}^j \Rightarrow k \ni \mathcal{I}^j$  and  $s_k^j > 0$  at time  $t$ . By Proposition 4.1,  $k$  will set its bid  $s_i^j = 0$ , and begin to add sellers to its pool. Suppose that at time  $t$ ,  $j$ 's market is at equilibrium, i.e.  $\sum_{i \in \mathcal{I}^j} = D^j$ , and no buyer rebids. Unless  $k$  adds a seller with a higher reserve price within  $|\mathcal{I}^j|$  time steps, by (30),  $D^j = 0$  and the auction will end. Otherwise, at some time  $t \in [t+1, \tau]$ , we must have that  $\sigma_k^j < D^j$ , and  $k$  opts-out. Finally, overlooking market transitions and messaging overhead, we have that,  $\forall i \in \mathcal{I}^j, \nexists s_i^j > 0$  where  $i \ni \mathcal{I}^j$ , and (31) holds.

Now, the subset  $\mathcal{I}^j \subset \mathcal{I}$  determines  $j$ 's reserve price  $p_{i^*}^j$ . We will assume the buyer submits a coordinated, truthful bid. Now,  $\mathcal{I}_i \subset \mathcal{I}$  determines the unit price  $p_i$  in buyer  $i$ 's bid. The reserve price (28) of seller  $j$  is determined at each transition, and is the lowest price that  $j$  will accept to perform any allocation. Let  $p_i^j = f^j \circ \sigma_i^j(a)$  denote the reserve price of auction  $j$ , noting that  $s_i^j = 0, \forall i \in [s_i^j]_{i \ni \mathcal{I}^j}$ , and let  $p_i^* = f_i \circ \sigma_i^j(a)$  denote the bid price of buyer  $i$ , i.e.  $p_i^k = p_i^*, \forall k \in \mathcal{I}_i$ . Using Proposition 4.2, for each  $i \in \mathcal{I}^j$ , we have from (7), (12), that  $p_i^* \geq p_{i^*}^k, \forall k \in \mathcal{I}_i$ .

The incentive of each seller  $\in \Lambda$  is to sell all of its data at the best possible price. In the simplest case, consider a disjoint local market  $j$ , where  $\forall i \in \mathcal{I}^j, s_i^k = 0, \forall k \neq j \in \mathcal{I}_i \Rightarrow \Lambda = \{j\}$  and  $\lambda = \mathcal{I}^j$ . Again using (7) and (12), it is clear that  $\theta_i = \theta^j, \forall i \in \mathcal{I}^j$ . In all other cases, the sellers  $\in \Lambda$  are competing to sell their respective resources to buyers whose valuations are distributed across multiple auctions. The set  $\lambda$  represents all of the buyers influencing auction  $j$ , both directly and indirectly. The bid price of buyer  $i \in \mathcal{I}^j$  is determined by,

$$p_i^* = \max_{k \in \mathcal{I}_i} (f^k \circ \sigma_i^k(a)) = \max_{k \in \mathcal{I}_i} (p_i^k). \quad (35)$$

$\Lambda$  is the set of sellers directly influencing the bids of buyers in auction  $j$ . Now, the reserve price for auction  $j$  is such that,

$$p_{i^*}^j \leq \min_{i \in \mathcal{I}^j} (p_i^*) - \epsilon, \quad (36)$$

by Proposition 4.2. Also, in the absence of transitions, caused by multi-auction market dynamics, we have that  $j$  maintains a local market equilibrium from time  $t$  to  $(t+1)$ . Let buyer  $i \in \mathcal{I}^j$  be the buyer in

auction  $j$  with the lowest bid price, and further suppose  $p_i^* > p_*^j + \epsilon$ . That is,  $\exists q \in \mathcal{I}_i$  such that  $p_*^q > p_*^j$ . Fixing all other bids, a change in  $q$ 's valuation from time  $t$  to  $t+1$  will directly impact buyer  $i$ . If at time  $t+1$  we still have that  $i$  is the buyer with the lowest bid price, then (12) holds and thus  $j$ 's valuation does not change. Otherwise a new buyer with the lowest bid will be chosen upon recomputing  $\mathcal{I}^j$ , as in Proposition 4.1. In each case we have that (7) and (12) hold for some fixed time  $t$ , and so,  $\forall i \in \mathcal{I}^j$ ,

$$\begin{aligned} \int_0^{\sigma_i^j(a)} f^j(z) dz &= \theta_i^j \circ \sigma_i^j(a) \\ &= \int_0^{\sigma_i^j(a)} f_i(z) dz, \end{aligned}$$

therefore  $\theta_i = \theta^j$ ,  $\forall i \in \mathcal{I}^j$ . Furthermore, it follows that,

$$\sum_{i \in \mathcal{I}^j} \int_0^{\sigma_i^j(a)} f^j(z) dz = \sum_{j \in \mathcal{I}_i} \int_0^{\sigma_i^j(a)} f_i(z) dz. \quad (37)$$

Consider the set  $\lambda$ . For some buyer  $i \in \mathcal{I}^j$ , and then for some seller  $k \in \mathcal{I}_i$ , we have a buyer  $l \in \mathcal{I}^k$ . By (31),  $i, l \in \mathcal{I}^k$ , and so the reserve price  $p_*^k \leq \min(p_i^*, p_l^*)$ , and  $k, j \in \mathcal{I}_i \Rightarrow p_i^* \geq \max(p_*^k, p_*^j)$ . Suppose that  $l \ni \mathcal{I}^j \Leftrightarrow j \ni \mathcal{I}_l$ , so that  $p_l^* < p_*^j$ , and the valuation of buyer  $l$  does not impact auction  $j$  and vice versa, i.e.  $\theta_l^j = 0$ . Since  $l \in \mathcal{I}^k$ ,  $p_l^* \geq p_*^k \Rightarrow p_*^k < p_*^j$ , and  $i \in \mathcal{I}^j \Rightarrow p_i^* \geq p_*^j$ , so,

$$p_*^k \leq p_l^* < p_*^j \leq p_i^*,$$

for any buyer  $l \in \lambda$  such that  $l \ni \mathcal{I}^j$ . Now, suppose  $\exists l \in \mathcal{I}^k$  such that  $l \in$

$\mathcal{I}^j \Rightarrow p_l^* \geq p_*^j$ . In the case where  $p_l^* > p_i^*$ , we must have that  $\exists q \in \mathcal{I}_l$  such that  $p_*^q > p_*^k$ , which implies, again by (35),  $q \ni \mathcal{I}_i \Leftrightarrow i \ni \mathcal{I}^q \Rightarrow p_*^q > p_i^*$ , therefore  $\theta_i^q = 0$ , and the reserve price of auction  $q$  does not effect the valuation of buyer  $i$ . Lastly, in the case where  $p_i^* > p_l^*$ , by the same reasoning,  $\theta_l^q = 0$ . We have that for any  $l \in \mathcal{I}^k$  such that  $l \ni \mathcal{I}^j$ ,  $\theta_l^j = 0$ , and when  $l \in \mathcal{I}^j$ , then either  $\theta_l^q = 0$  for  $q \in \mathcal{I}_l$  or  $\theta_l^q = 0$  for some  $q \in \mathcal{I}_i$ . Thus, any bid outside of our construction has a zero valuation, with respect to buyers  $\in \lambda$  and sellers  $\in \Lambda$ , and therefore cannot cause transitions to occur except through a shared buyer, e.g. some  $l \in \mathcal{I}^k$ . Thus, in all cases, (7) and (12) hold. Fixing all bids in any auction  $\mathcal{I}^q$ , where  $q \in \mathcal{I}_l$  and  $q \ni \Lambda$ , we have, for any  $i \in \mathcal{I}^j$ ,

$$\int_0^{D^k} f^k(z) dz = \sum_{i \in \mathcal{I}^k} \int_0^{\sigma_i^k(a)} f^k(z) dz, \quad (38)$$

which holds  $\forall k \in \mathcal{I}_i$ , by (31) and Proposition 4.2.. Finally, using (37), (38),

$$\int_0^{\sigma_i^k(a)} f_i(z) dz = \int_0^{\sigma_i^k(a)} f^k(z) dz,$$

and

$$\int_0^{\sigma_l^k(a)} f^k(z) dz = \int_0^{\sigma_l^k(a)} f_l(z) dz,$$

and the result follows by construction, and the continuity of  $\theta'$ :

$$\sum_{i \in \lambda} \int_0^{\sigma_i^j(a)} f^j(z) dz = \sum_{j \in \Lambda} \int_0^{\sigma_i^j(a)} f_i(z) dz. \quad (39)$$

For completeness, in the case where the ISP  $\kappa$  does not adhere to the market dynamics, so  $p^\kappa > p^j + \epsilon$ ,  $\forall j \in \mathcal{I}$ ,



then we may absorb the overage (difference) as part of the bid fee. (NEED TO DO BETTER WITH TIME?)

As a result of user behavior, and subsequent strategies, we determine that the data-exchange market is semi-partitioned, consisting of unions as in Lemma 4.4. The unions are inter-dependent in a predictable way. However, as each auction may be played on the same or on a different scale in valuation, time and quantity, the rate at which market fluctuations occur is impossible to predict (NEED HELP!). Arrow's paradox is an impossibility theorem stating that when buyers have three or more distinct alternatives (auctions), no deterministic ranking system can convert the ranked preferences of users into a market-wide (complete and transitive) ranking while also meeting a specified set of criteria: unrestricted domain, non-dictatorship, Pareto efficiency and independence of irrelevant alternatives. It follows that the case where  $\theta_i = \theta^j$  as in (7) and (12) does not exist, and is purely a potential, theoretical ideal. (DOES IT????)

Nonetheless, we claim that our mechanism is normative, that irrelevant alternatives should not matter, it is practical, uses minimal information, strategy, and provides the right incentives for the truthful revelation of individual preferences. (EXPAND)

## 4.5 Mechanism Realization

Consider a user seeking to prevent data overage by purchasing enough data from a subset of other network users. This user  $i$

can be modeled as a opt-out buyer where, as in [3], We intend to show that this network setting results in a shared network optima (a global optimum). The formulation is inspired to the thinnest allocation route for bandwidth given in [2]. We note that if a single seller  $j$  can satisfy  $i$ 's demand, then (6) reduces to the original form, defined in [3] as "a simple buyer at a single resource element".

The sellers' auction will function as follows: at each bid iteration all buyers submit bids, and the winning bid is the buyer  $i$  that has the highest price  $p_i^j$ . The seller allocates data to this winner, at which point all other buyers are able to bid again, and the winner leaves the auction (with the exception where multiple bidders bid the same price, where (21) determines they will not fully satisfy their demand, and so we will assume they remain in the auction). The auction progresses as such until all the sellers' data has been allocated. We design an algorithm based on the sellers' fractional allocation strategy.

(NOTE - FIX TO UPDATED NOTATION!)

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**Algorithm 1** (Seller fractional allocation)

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```

1:  $p^{j(0)} \leftarrow \epsilon$ 
2:  $s^{j(0)} \leftarrow (p^j, D^j)$ 
3: while  $D^j > 0$  do
4:    $\mathcal{I}^{j(t+1)} = \mathcal{I}^{j(t)} \setminus \{i \in \mathcal{I}^{j(t)} : d_i^j > D^{j(t)}\}$ 
5:    $\bar{i} \leftarrow \arg \max_{I^j} \sum_{i \in I^j} p_i^j$ 
6:    $D^{j(t+1)} \leftarrow D^{j(t)} - e_{\bar{i}}^{j(t)}(a)$ 
7:    $p^j \leftarrow p_{i^*}^j + \epsilon$  and  $d^j \leftarrow D^{j(t+1)}$ 
8:    $s^{j(t+1)} \leftarrow (d^j, p^j)$ 
9:   if  $\exists i : s_i^{j(t+1)} \neq s_i^{j(t)}$  then
10:     $D^{j(t+1)} = D^{j(t)}$ 
11:     $t \leftarrow t + 1$ 
12:    Go to 4.
13:   else
14:     $\bar{i} \leftarrow e_{\bar{i}}^j(a)$ 
15:     $t \leftarrow t + 1$ 
16:    Go to 4.

```

---

We assume that each time step that  $s^j$  is updated it is shared with all participating buyers. At this point buyers have the opportunity to bid again, where a buyer that does not bid again is assumed to hold the same bid, since a buyer dropping out of the auction will set their bid to  $s_i^j = (0, 0)$ . As we will show in our analysis, the buyers are bidding truthfully; the algorithm makes use of the fact that the sellers' valuation is determined by the buyers' market and upholds the PSP mechanism. (CHECK) (ALSO  $j$  NEEDS TO UPDATE FOR NEW BUYERS,  $j$ 'S CONTROL)

(BUYER ALGORITHM? WHY NOT..., CLEANER)

## 5 PSP Analysis

### 5.1 Equilibrium

Consider an opt-out buyer  $i \in \mathcal{I}$ . A PSP auction allows for a buyer to react to opponents bids, and so its incentive is based on the opponent profile.

Due to (6),  $i$  only has an incentive to change its bid quantity if it increases its opt-out value  $e_i$ , and therefore its utility. We will show that  $i$  can coordinate its bids over  $\mathcal{I}_i$  so that the opt-out value  $e_i$  is the same for each  $j \in \mathcal{I}_i$ , and therefore, without loss of utility,  $i$  may choose a seller pool using a “consistent” strategy, where for any  $j \in \mathcal{I}_i$ ,  $d_i^j = d_i^k$ ,  $\forall k \in \mathcal{I}_i$ , and still have feasible best replies. Our result shows that a buyer may select  $\mathcal{I}_i$  in order to maximize its utility while maintaining a coordinated bid strategy. It is intuitive that, if  $j^* < I$ , a buyer may increase the size of  $\mathcal{I}_i$ , thereby lowering its bid quantity while obtaining the same (potential) allocation  $a_i$ . (OWN WORDS!) One important question to ask is why a bidder should bid with identical unit prices on (j) all auctions and not reduce the unit price to a level where he still wins the amount  $q_i$ . The reason for this can be found in the pricing rule of the PSP auction.

**Lemma 5.1.** (*Opt-out buyer coordination*) Let  $i \in \mathcal{I}$  be a opt-out buyer and fix all sellers' profiles  $s^j$ . For any profile  $S_i = (D_i, P_i)$ , let  $a_i \equiv \sum_j a_i^j(s)$  be the resulting data allocation. For any fixed  $S_{-i}$ , a better reply for  $i$  in any auction

is  $x_i = (z_i, y_i)$ , where  $\forall j \in \mathcal{I}_i$ ,

$$\begin{aligned} z_i^j &= e_i^{j*}(a), \\ y_i^j &= \theta'_i(z_i^j). \end{aligned}$$

Furthermore,

$$a_i^j(z_i, y_i) = z_i^j, \quad (40)$$

and

$$c_i^j(z_i, y_i) = y_i^j, \quad (41)$$

where  $i$ 's strategy is as in Proposition 4.1.

**Proof:** As  $s_{-i}$  is fixed, we omit it, in addition, we will use  $u \equiv u_i \equiv u_i(s_i) \equiv u_i(s_i; s_{-i})$ . In full notation, we intend to show

$$u_i((d_i, p_i); s) \leq u_i((z_i, y_i); s_{-i}).$$

If there exists a seller who can fully satisfy  $i$ 's demand, then  $|\mathcal{I}_i| = 1$ , and the case is trivial as no coordination is necessary for a single bid. (SPECIAL CASE OF MONOPOLY? THINK!)

Otherwise, buyer  $i$ 's demand can only be satisfied by purchasing data from multiple sellers. We will show that  $i$  may increase  $|\mathcal{I}_i|$ , and so decreasing  $d_i^j$ ,  $\forall j \in \mathcal{I}_i$ , without decreasing  $u_i$ . Buyer  $i$  maintains ordered set  $\ell_i$  where the sellers with the largest bid quantities are considered first; the index of seller  $j^*$  defines a minimal subset  $\mathcal{I}_i$ , satisfying (23). By construction,  $d_i^{j*}$  is the minimum quantity offered by any  $j \in \mathcal{I}_i$ , so  $d_i^{j*} \leq d_i^j$ ,  $\forall j \in \mathcal{I}_i$ ;  $\mathcal{I}_i$  also defines the maximum quantity bid of any  $k \ni \mathcal{I}_i$ . Thus by (23) and (??),  $\forall j \in \mathcal{I}_i$ ,  $k \ni \mathcal{I}_i$ ,

$$e_i^k(a) \leq z_i^j = e_i^{j*}(a) \leq e_i^j(a),$$

and so,

$$e_i^{j*}(a) \leq \left[ D^j - \sum_{k \in \mathcal{I}^j: p_k^j > y_i^j} d_k^j \right]^+, \quad (42)$$

The buyer valuation function (17), guarantees that  $\forall j \in \mathcal{I}_i$ ,  $y_i^j \geq p_{i*}^j$ , where  $p_{i*}^j$  is the reserve price of seller  $j$ , defined in Proposition 4.2, and is by definition the minimum price for a buyer bid to be accepted. As  $D_i$  is non-decreasing,  $\forall j \in \mathcal{I}_i$ ,  $k \ni \mathcal{I}_i$ ,

$$D_i^j(y_i^j) \geq D_i^j(p_{i*}^j) \geq D_i^j(p_i^k).$$

Furthermore, suppose there exists buyer  $k \in \mathcal{I}^j$ , such that  $e_k^{j*}(a) \geq e_i^{j*}(a)$  and so  $d^j - d_k^j < z_i^j$ , resulting in a partial allocation. Bid  $s_k \in S_i$ , and so is considered in buyer  $i$ 's strategy. As buyer  $i$  is allowed to choose subset  $\mathcal{I}_i$ , and  $\mathcal{I}_i$  is a minimal set, (23) states that  $n$  is such that  $e_i^{j*}(a) \geq e_k^{j*}(a)$  for any  $k$ , therefore such a buyer  $k$  cannot exist. Thus (42) holds and so, by (21),

$$\begin{aligned} a_i^j(z_i, p_i) &= \min_{i \in \mathcal{I}^j} \left( z_i^j, \left[ D^j - \sum_{p_k^j > y_i^j} d_k^j \right]^+ \right) \\ &= z_i^j = e_i^{j*}(a) \end{aligned}$$

where the last equality is by definition, and so (40) is proven. Now, since a buyer is only charged with the cost of excluding other players from the market, the unit price does not influence the final charges. This unit price reflects the valuation of the total resources gained from the multi-auction market, and so a buyer uses this price on all auctions. We have,  $\forall j \in \mathcal{I}_i$ ,  $\forall k \ni \mathcal{I}_i$ ,

$$y_i^j \geq p_i^j \geq p_{i*}^j \geq p_i^k,$$

and we observe that from (20),  $\bar{D}_i^j(y, s_{-i}) = 0 \forall y < p_{i*}^j$ , and so  $y = 0 \leq \epsilon \Rightarrow e_i^j(a) = 0$ , and so clearly  $z_i^k = 0, \forall k \ni \mathcal{I}_i$ , and therefore,

$$\sum_{j \in \mathcal{I}_i} c_i^j(z_i, y_i) = \sum_{j \in \mathcal{I}_i} c_i^j(z_i, p_i),$$

thus (41) simply shows that changing the price  $p_i^j$  to  $y_i^j$  does not exclude any additional buyers, as the bid  $p_i^j$  was already above the reserve price of any seller  $j \in \mathcal{I}_i$ . We proceed to demonstrate that  $x_i$  does not result in a loss of utility for buyer  $i$ ; we will show that

$$u_i \leq u_i(z_i, y_i).$$

From (40), we have  $a_i^j(z_i, y_i) = z_i^j = e_i^j(a(z_i, y_i))$ , and so,

$$\theta_i \circ e_i^j(a(z_i, y_i)) = \theta_i \circ e_i^j(a),$$

which holds  $\forall j \in \mathcal{I}_i$ . Therefore, by the definition of utility (6), and the buyers' valuation (17),

$$\begin{aligned} & \theta_i \circ e_i(a(z_i, y_i)) - \theta_i(a) \circ e_i(a) \\ &= u_i(z_i, y_i) - u_i = \sum_{j \in \mathcal{I}_i} c_i^j - c_i^j(z_i, y_i) \\ &= \sum_{j \in \mathcal{I}_i} \int_{a_i^j(z_i, p_i)}^{a_i^j} f_i(d_i^j - z) dz. \end{aligned}$$

Then, as  $a_i(z_i, p_i) \leq z_i^j \leq a_i^j$ , and noting that  $\theta_i \geq 0 \Rightarrow f_i \geq 0$ , we have  $u_i(z_i, y_i) - u_i \geq 0, \forall j \in \mathcal{I}_i$ . Finally, as we do not increase the aggregate demand of the buyer,  $\sum_{j \in \mathcal{I}_i} c_i^j(s) \leq \sum_{j \in \mathcal{I}_i} b_i^j$ , and so  $x_i$  is feasible.

In effect, we are using the buyer demand to partition the auction space based on their type (EXPLAIN).

### 5.1.1 Incentive Compatibility

We proceed to claim that the optimality of truth-telling holds in our formulation, where the market functions as a hybrid of [3] and [1]. The opt-out buyers' market is comprised of the minimal subset of sellers with the largest amounts of available data, described in the buyer strategy as  $\mathcal{I}_i$ . To achieve incentive compatibility, we find that the opt-out buyer must choose this subset so that its overall marginal value is greater than its market price. The buyers' market price is calculated as the maximum of the reserve prices of the sellers in the opt-out buyers' pool. The actual prices at the The actual bids are obtained from the opt-out buyers' strategy. The quantity to bid is given by the auction mechanism, i.e. (19) and (20), as the maximum possible quantity of data that a buyer  $i$  can bid over its seller pool while maintaining its aggregate marginal valuation greater than the aggregate of minimum prices maintained by the sellers in  $i$ 's pool. As with a single resource, [3] and [2], we show that truth-telling is optimal for the buyer, i.e. in each auction, the buyer sets the bid price to the marginal value.

Seller  $j$ 's reserve price is determined by a buyer  $i \ni \mathcal{I}^j$ , and therefore, even if this price is zero, then  $p^j = \epsilon \geq 0$ .

We argue that if truthfulness holds *locally* for both buyers and sellers, i.e.  $p_i = \theta_i' \forall j \in \mathcal{I}_i$  and  $p^j = \theta^{j'} \forall i \in \mathcal{I}^j$ , then there exists a local market equilibrium (NOTE: NOT GLOBAL! YET). We have the following Proposition. For completeness, we use the full notation. (BAD SENTENCE)

**Proposition 5.1.** (*Incentive compatibility in local auctions*) For any seller  $j$ , let time  $t > 0 \in \tau$  be fixed and for any buyer  $i \in \mathcal{I}^j$ , let  $s_{-i}$  also be fixed. )VALIDATE: FOR ALL  $j$ ) Define,

$$\eta_i = \sup \left\{ x \geq 0 : \theta_i'(x) > \bar{P}_i^j(x) \right\}, \quad (43)$$

$$\chi_i = \sup \left\{ x \geq 0 : \int_0^x \bar{P}_i(x) dx \leq \sum_{j \in \mathcal{I}_i} b_i^j \right\}, \quad (44)$$

$z = \min(\eta_i, \chi_i - \epsilon/\theta_i'(0))^+$ , and for each  $j \in \mathcal{I}_i$ ,

$$v_i^j = e_i^{j*}(z)$$

and

$$w_i^j = \theta_i^j(z).$$

Then a (coordinated)  $\epsilon$ -best reply for the opt-out buyer is  $t_i = (v_i, w_i)$ , i.e.,  $\forall s_i, u_i(t_i; s_{-i}) + \epsilon \geq u_i(s_i; s_{-i})$ . With reserve prices  $p^j > 0$ , there exists a “truthful” local game embedded in each local auction, and thus an equilibrium point for the local auction.

**Proof:** For the buyer, we show that  $t_i$  is an  $\epsilon$ -best reply. That is,

$$u_i(t_i; s_{-i}) + \epsilon \geq u_i(s_i; s_{-i}).$$

Let  $z = \eta_i^j$ . We have that  $i \in \mathcal{I}^j$ , and (7) defines  $\theta_i^j(z)$  as being max of the reserve prices  $p_i^j$ ,  $\forall j \in \mathcal{I}_i$ , therefore (43) is such that,

$$\theta_i^j(z) > \bar{P}_i^j(z),$$

which implies, as (MORE)  $\theta_i^j$  is non-increasing and  $P_i^j \geq 0$ , we have  $\forall j \in \mathcal{I}_i$ ,

$$\begin{aligned} w_i^j &> P_i^j(v_i^j) \\ \Rightarrow v_i^j &\leq D_i^j(w_i^j) = D^j - \rho^j(w_i^j). \end{aligned}$$

And so, by (21),

$$\begin{aligned} a_i^j(t_i) &= v_i^j \\ \Rightarrow \sum_{j \in \mathcal{I}_i} a_i^j(t_i) &= z. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j \in \mathcal{I}_i} \int_0^{v_i^j} P_i^j(x) dx &= \int_0^z \bar{P}_i^j(x) dx \\ &= \sum_{j \in \mathcal{I}_i} \int_0^{e_i^j(z)} P_i^j(x) dx. \end{aligned}$$

Suppose  $\exists s_i = (d_i, p_i)$  such that  $u_i^j(s_i; s_{-i}) > u_i^j(t_i; s_{-i}) + \epsilon$ . The buyer coordinated strategy, from Propositions 5.1 and 4.1, gives  $s_{i*} = (e_i \circ a_i(s), \theta_i^j(z)) = (z_i, c_i)$ , where for each  $j \in \mathcal{I}$ ,  $a_i^j(s_{i*}; s_{-i}) = z_i^j$ , then clearly  $u_i(s_{i*}, s_{-i}) \geq u_i(s_i, s_{-i}) \Rightarrow u_i(t_i; s_{-i}) - u_i(s_i; s_{-i}) > \epsilon$ . Denoting  $z_i^j$  (fixed) as  $\zeta$ ,

$$\int_z^\zeta \theta_i^j(x) dx - \int_z^\zeta \bar{P}_i(x) dx > \epsilon.$$

For concave valuation functions, the first-order derivative of  $\theta$  at point 0 gives the maximum slope of the valuation function, and so the factor  $\epsilon/\theta'(0)$  guarantees that new bids will differ by at least  $\epsilon$ , and as such, buyer  $i$  will remain in any local auction with reserve price determined by (??). We therefore verify that,

$$\int_z^{z+\epsilon/\theta_i'(0)} \theta_i^j(x) dx \leq \epsilon,$$

and as  $P_i^j \geq 0$ , we have that, from the construction of  $\zeta$ ,

$$\int_{z+\epsilon/\theta_i'(0)}^\zeta \theta_i^j(x) dx - \int_{z+\epsilon/\theta_i'(0)}^\zeta \bar{P}_i(x) dx > 0.$$

If  $\zeta > z + \epsilon/\theta'_i(0)$ , then for some  $\delta > 0$ ,  $\theta_i(z + \epsilon/\theta'_i(0) + \delta) > P_i^j(z + \epsilon/\theta'_i(0) + \delta)$ , contradicting (43). Now, if  $\zeta \leq z$ , then  $\theta'_i(z + \epsilon/\theta'_i(0)) < P_i^j(z + \epsilon/\theta'_i(0))$ , also a contradiction of (43), and so buyer  $s_i$  cannot exist.

We consider the other term in the minimization; let  $z = \chi_i$ . We have that For any coordinated bid  $s_i$  such that  $u_i(z_i, c_i) > u_i(t_i) + \epsilon$ , then  $\zeta > z \Rightarrow c_i(s_i) > b_i$ , and so is infeasible. Finally, by Lemma 4.4, for a single buyer-seller interaction, we have that  $f_i = f^j \Rightarrow \theta_i \circ e_i^j = \theta^j \circ e_i^j$ . Therefore  $\theta'_i = \theta^{j'}$ . We have by Proposition 4.2 and (30), that  $\theta^j = \max_{i \in \mathcal{I}^j} \theta_i$ , and so for any  $i \in \mathcal{I}^j$ ,  $\theta^j(z) \geq \theta_i(z)$ , and so  $j$  cannot set a higher reserve price, and also cannot sell more data than the buyers' bids, so we have that  $j$ 's bid is both truthful and optimal.

—REF—

$$\begin{aligned} \bar{P}_i(z, s_{-i}) &= \sum_{j \in \mathcal{I}} P_i^j(z_i^j, s_{-i}^j) \\ &= \sum_{j \in \mathcal{I}} \left( \inf \left\{ y \geq 0 : D_i^j(y, s_{-i}^j) \geq e_i^j(z) \right\} \right), \end{aligned} \quad (45)$$

$$D_i^j(y, s_{-i}^j) = \left[ D^j - \sum_{p_k^j > y} d_k^j \right]^+, \quad (46)$$

$$\begin{aligned} \bar{D}_i(y, s_{-i}) &= \sum_{j \in \mathcal{I}} \left( \sup \left\{ z \in [0, D^j] : \right. \right. \\ &\quad \left. \left. \bar{P}_i(z, s_{-i}^j) < y \right\} \right). \end{aligned} \quad (47)$$

$$\begin{aligned} a_i^j(s) &= \min \left\{ e_i^j(D_i), \frac{d_i^j}{\sum_{p_k^j = p_i^j} d_k^j} D_i^j(p_i^j, s_{-i}^j) \right\}. \end{aligned} \quad (48)$$

$$(49)$$

$$\bar{c}_i(s) = \sum_{j \in \mathcal{I}} p^j \left( a_i^j(0; s_{-i}^j) - a_i^j(s_i^j; s_{-i}^j) \right). \quad (50)$$

$$\rho^j(y) = \sum_{i \in \mathcal{I}: p_i^j \geq y} d_i^j, \quad (51)$$

$$f_i(z) \triangleq \inf \{ y \geq 0 : \rho_i^j(y) \geq z, \forall j \in \mathcal{I} \}. \quad (52)$$

$$f^j(z) \triangleq \sup \{ y \geq 0 : \rho_i^j(y) \geq z, \forall i \in \mathcal{I} \}, \quad (53)$$

$$\theta_i \circ e_i^j = \int_0^{e_i^j(a)} f_i(z) dz, \quad (54)$$

$$\theta_i \circ e = \sum_{j \in \mathcal{I}} \int_0^{e_i^j(a)} f_i(z) dz. \quad (55)$$

$$\theta^j \circ e = \sum_{i \in \mathcal{I}} \int_0^{e_i^j(a)} f^j(z) dz. \quad (56)$$

$$d_i^j \geq e_i^j(a) \quad (57)$$

$$\sum_{i \in \mathcal{I}} e_i^j(a) \leq \sum_{i \in \mathcal{I}} d_i^j \leq D^j, \quad (58)$$

$$e_i^j(a) \leq D^j - \sum_{k \in \mathcal{I}, k \neq i} e_k^j(a). \quad (59)$$

$$p_i^j \geq \min_{i \in \mathcal{I}} (p_i^j). \quad (60)$$

(SELLER FINISH)

This forms a “truthful” local game embedded within  $j$ 's auction with strategy space restricted to  $\epsilon$ -best replies from

buyers  $\in \mathcal{I}^j$ . Therefore we have that a fixed point in the “truthful” local game is a fixed point for the auction. We further argue that as the set  $\mathcal{I}^j$  is computed at each bid iteration, that our result holds for time  $(t + 1) \in \tau$ .

**Lemma 5.2.** (*Static Data Nash Equilibrium*)

**Theorem 5.1.** (*Data Nash Equilibrium*)  
Using the rules of the data auction mechanism, the CMHK [1] converges to a  $\epsilon$ -Nash equilibrium. In the network auction game with the data-PSP rules applied independently by each user according to their respective strategies, the secondary market converges to an  $\epsilon$ -Nash equilibrium.

**Proof:** 2. using the min price of sellers in the auction i.e.  $\theta_i'(d_{i*}^j) = p^j$  is OK,  
3. that bids are still feasible AND optimal  
4. the algorithm achieves global economic equilibrium)

**NEED TO COVER:**

1. Change in buyer valuation
2. New buyers
3. Not enough buyers
4. Not enough data

**TRY:**

Sellers only act when the resources obtained by the buyers influence their respective reserve prices, which agrees with the seller strategy of attempting to sell their data in the first iteration. Therefore we claim there exists a market stability and therefore, the existence of a Nash equilibrium. As the valuation of the sellers is derived by the demand of the buyers, who are bidding equivalent bids over a

minimum subset of buyers, we claim that the seller strategy, along with the seller constraint (??) results in a global market equilibrium. We have shown that the local equilibrium created by  $j$  is stable from time  $t$  to  $(t + 1)$ . Now, suppose that buyer  $i^*$  computes its best response  $s_i^j = (v_i^j, w_i^j)$ . Finally, suppose that a buyer  $k$  enters the market such that for some buyer  $l \in \mathcal{I}^j$ ,

$$\sum_{i \in \mathcal{I}^j} p_i^j e_i^j(a) + p_k^j e_k^j(a) \geq \sum_{i \in \mathcal{I}^j} p_i^j e_i^j(a) - p_l^j e_l^j(a),$$

that is,

NOTES: (today)

2. finish seller incentive compat
3. work on progression
4. check reserve price = monopoly price

## 5.2 Efficiency

Formally, the mechanism is equilibrium-efficient if at equilibrium, the allocation maximizes  $\sum_i \theta_i(a_i)$ . (NEED OWN WORDS)  
The objective in designing the auction is that, at equilibrium, resources always go to those who value them most. Indeed, the PSP mechanism does have that property. This can be loosely argued as follows: for each player, the marginal valuation is never greater than the bid price of any opponent who is getting a non-zero allocation. Thus, whenever there is a player  $j$  whose marginal valuation is less than player  $i$ 's and  $j$  is getting a non-zero allocation,  $i$  can take some away from  $j$ , paying a price less than  $i$ 's marginal valuation, i.e. increasing  $u_i$ , but also increasing the total value, since  $i$ 's marginal value is greater. Thus at equilibrium, i.e. when no one can unilaterally increase  $P$  their utility, the total value is maximized.

## 5.3 Convergence

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