Title?

Anonymous Author(s)

ABSTRACT

We investigate the 2cm app, a data-exchange platform published for China Mobile Hong Kong 4G Pro Service Plan customers. Telecommunications ISPs' revenue is typically gained by charging users a fixed fee for a maximum amount of data usage in a month, i.e., a monthly data cap [**?**]. 2cm's (2nd exchange market) data exchange platform allows users to submit bids to buy and sell data. This usage model, is, as far as our knowledge, the first data trading platform that allows customers to buy and sell their own data. We describe a distributed auction mechanism for data exchange inspired by the classic PSP throughput problem, and prove that our distributed data exchange mechanism provides incentive compatibility (social choice function), and that we have efficiency using only partial valuation information of each participant in an exchange market.

In applying a distributed PSP implementation to CMHK market, we find that the market is able to achieve an equilibrium as the sellers and buyers have an incentive for a collaborative exchange, and design our mechanism tp provide the functions for effective communication between the connected users. We claim that in this secondary market our formulation holds the desired VCG qualities through the construction of a probable equilibrium [**?**]. We further provide bounds on the auction duration, with respect to the classic throughput problem. and provide simulated results on convergence time to support our (FIND COMPETITVE RATIO!), and a bound on the convergence of our mechanism. We extend the works of cite!cite! i.e. (market influence/EQ,social EQ,payment/allocation models) OR (bandwidth, data bundles, distributed market algorithms) and show the existence of a dynamic global market equilibrium, allowing for a unique set of market dynamics.

1 INTRODUCTION

(NOTE: THE INTRODUCTION IS CURRENTLY A COLLECTION OF NOTES AND PARTIAL SENTENCES) In this work, we propose *a distributed progressive second price (PSP) auction in order to maximize social utility in the CMHK market.* Using the distributed PSP mechanism on CMHKs data exhange platform, we show that for cellular data allocated between multiple users there exists an ϵ-Nash market equilibria. A quality of the PSP auction is that demand information is not known centrally, rather, it is distributed in the buyers' valuations. The mechanism for an auction is defined as *distributed* when the allocations at any element depend only on *local* state: the quantity offered by the seller at that element, and the bids for that element only [2]. In this work, the proposed mechanism allows the distribution of bids, where there are many ISPs each holding thier own local auction; there is no entity that holds a global market knowledge.

In a PSP mechanism, bids consist of (i) an available (required) quantity and (ii) a unit-price (calculated using its own demand functions). Buyers submit bids cyclically until an (ϵ-Nash) equilibrium is reached and a local auction is concluded.

(HOW DO WE MODEL ISP REVENUE? FUTURE WORK?)

The form of the auction mechanism presented here is described as a *locally* pure-strategy progressive game with incomplete, but perfect information. In a distributed setting, local markets have perfect local information, since you know what each move of the opponent is. However, since users are not aware of behavior in other auctions, it is a game of incomplete information. We complete a comprehensive analysis of market dynamics, and will assume complete and perfect information in order to verify properties and validate results of the auction mechanism.

(TO ORGANIZE)

The main contribution of this work is an auction mechanism inspired by the classic PSP throughput problem. In order to apply a distributed PSP implementation to the CMHK market, we analize the behavior of users in a dynamical data exchange market. As both buyers and sellers are able to change their bid strategies, and as each user only has *local* information about the bidding environment, it is clear that an unconstrained market, even with a finite number of users, could suffer from the communication expense from numerous local auctions trading an infinitely divisible resource. We will assume that the cost of participating in the CMHK secondary market is absorbed by the bid fee, which could represent data used in submitting bids, or a fee charged per unit of data, or a flat rate charged at the completion of the purchase. It is worth mentioning that CMHK users are not allowed to resell data purchased from the CMHK market, additionally, the purchased data does not carry to the next service period. Therefore, a market equilibrium, where supply equals demand, requires that users maintain incentive for truthfullness across a distributed setting. To the best of our knowledge, this is the first work to provide a truthful mechanism in a distributed setting for data-exchange based on market behavior.

Remark: It is certainly possible to extend the mechanism, alleviating the restrictions introduced by the CMHK secondary market. We reserve this for future work.

We claim that the CMHK market is able to achieve an equilibrium as the sellers and buyers have an incentive for a collaborative exchange, and as our mechanism provides incentives for optimal, effective communication between the connected users.

Despite the small message space, with many user types, there is no single way to do the transformation from the direct revelation mechanism to the desired one. Our mechanism is designed by guessing right direct-revelation-to-desired-mechanism transformation and building it into the allocation rule to be the primary step in the design. This incentive for a user to truthfully reveal its type is built into the user strategies. As in classic mechanism design, we determine the equilibria as result of incentive compatibility in our mechanism design. We claim that in the CMHK market our formulation not only holds the desired VCG qualities, but minimizes comminication overhead and auction duration, resulting in a convergence time (FIND COMPETITIVE RATIO?) with respect to the classic throughput problem.

The strategies described in this paper have removed the necessity for a user to determine its own valuation function, we intend to show that the market dynamics resulting from the construction of the user strategy space results in truthful bids that are optimal for all users, i.e. bid prices are to the marginal value as determined by market dynamics. This is the (built-in) transformation from the direct-revelation mechanism to the desired message space.

The paper is organized as follows...

1.1 Distributed Progressive Second Price Auctions

Progressive second price auctions (PSPs) were proposed in [2], [9] to provide a dynamic network service pricing scheme to provide consistent services for network bandwith users. [9] conducts a game theoretic analysis, deriving optimal strategies for buyers and brokers, and further shows the existence of networkwide market equilibria based on their game-theoretic model. Constructing necessary and sufficient conditions for the stability of the game allows the sustainability of any set of service level agreement configurations between Internet service providers.

We begin with a brief introduction to the distributed PSP auction for bandwith sharing, first introduced by Lazar and Semret [2]. We define a set of $\mathcal{I} = \{1, \cdots, I\}$ network bandwidth users. Suppose each user $i \in \mathcal{I}$ makes a bid $s_i^j = (p_i^j, d_i^j)$ to the seller of resource j, where p_i^j is the unit-price the user is willing to pay and d_i^j is the quantity the user desires. The *bidding profile* forms a grid, $s \equiv [s_i^j] \in \mathcal{I} \times \mathcal{I}$, and $s_{-i} \equiv [s_1^j, \cdots, s_{i-1}^j, s_{i+1}^j, \cdots, s_I^j]_{j \in \mathcal{I}}$ is the profile of user i's opponents. Using this classic PSP mechanism, [2] shows that given the opponents bids s_{-i}, user i's ϵ-best response to seller j is $s_i^j = (w_i^j, v_i^j)$ and is a Nash move where $\epsilon > 0$ is the bid fee, $B_i = \sum_{j \in \mathcal{I}} b_i^j$ is user i's budget, and every user has an elastic demand function. Based on the profile of bids $s^j = [s_1^j, \cdots, s_I^j]$, the seller applies an allocation rule $a(s^j) = [a_1^j, \cdots, a_I^j]$, where a_i^j is the quantity allocated by j to each user $i \in \mathcal{I}$ and c_i^j is the cost charged to i for allocations awarded in auction j. An allocation is considered feasible if $a_i^j \leq d_i^j$, and $c_i^j \leq p_i^j d_i^j$.

1.1.1 *The PSP Mechanism.*

The PSP auction as given in [2] and [3] is designed for the problem of network bandwidth allocation, and is analyzed as a noncooperative game where $i \in \mathcal{I}$ agents buy the fixed amount of bandwidth d_i^j from sellers $j \in \mathcal{I}$. The market price function (MPF) for a buyer-seller pair is,

$$P_i^j(z, s_{-i}) = \inf \left\{ y \geq 0 : D_i^j(y, s_{-i}) \geq z \right\}, \tag{1}$$

and is the of minimum prices a user bids in order to obtain bandwidth z given opponent profile s_{-i}. The maximum available quantity of data in auction j at unit price y given s_{-i}^j is,

$$D_i^j(y, s_{-i}) = \left[D^j - \sum_{p_k^j > y, k \neq i} d_k^j \right]^+, \tag{2}$$

where D^j is the total amount of bandwith that user j has to offer. For each $i \in \mathcal{I}$, the allocation from auction j is,

$$a_i^j(s) = \min \left(d_i^j, D_i^j(p_i^j, s_{-i}^j) \right). \tag{3}$$

Finally, we have the cost of the allocation,

$$c_i^j(s) = \sum_{k \neq i} p_k^j \left[a_k^j(0; s_{-i}^j) - a_k^j(s_i^j; s_{-i}^j) \right]. \tag{4}$$

It was shown, in [2], that the mechanism may converge to a Nash market equilibria for differentiated services allocated between multiple agents when all players bid their real marginal valuation of the bandwidth resource. In other words, the PSP constraints are sufficient to attain the desirable property of truthfullness through incentive compatibility. The pricing mechanism upholds the *exclusion-compensation principle*, user i pays for its allocation so as to exactly cover the "social opportunity cost" which is given by the declared willingness to pay (bids) of the users who are excluded by i's presence, and thus also compensates the seller for the maximum lost potential revenue [2].

Definition 1.1. (Nash Equilibrium) A Nash equilibrium is defined as a strategy vector, or, in terms of PSP, a bid profile s, from which no player has a unilateral incentive to deviate (Johari, 2004) (EXPAND?)

The PSP rules assume that an agent's valuation is represented by an elastic valuation function.

Definition 1.2. [2] A real valued function, $\theta(\cdot) : [0, \infty) \to [0, \infty)$, is an *(elastic) valuation function* on $[0, D]$ if

- $\theta(0) = 0$,
- θ is differentiable,
- $\theta' \geq 0$, and $\theta_i{}'$ is non-increasing and continuous,
- There exists $\gamma > 0$, such that for all $z \in [0, D]$, $\theta'(z) > 0$ implies that for all $\eta \in [0, z), \theta'(z) \leq \theta'(\eta) - \gamma(z - \eta)$.

The function $\theta'(\cdot)$ on $[0, D]$ is called an (elastic) demand function. In the PSP market, a user is considered truthful if their bid price equals their marginal valuation, i.e. $p_i^j = \theta_i^{j\prime}$.

1.2 Motivation for Privacy/Security

(NEW SECTION)

A property of PSP is that the user valuations are held private.

1. The seller doesn't update with the ISP, and so does not know the market price in the data-exchange market.
2. The auction is distributed, so there is no collusion between sellers and ISP.
3. The mechanism is self-contained, and only requires participation. (THINK!) 4. Better economy, as few users will get data plans with a lot of data, make money, allowing others to have cheaper data plans and not pay a lot for texting, phone calls (Buzzfeed). Free market capitalism!

(CONCERN HERE ABOUT THROTTLING)

2 RELATED WORK

3 THE PROBLEM MODEL

3.1 The CMHK Market

We construct the model for a PSP data auction for mobile users particpating in CMHK's secondary data-sharing market. Define the index set $\mathcal{I} = \{1, \cdots, I\}$ to represent the set of users who purchase or sell data from other users. We again define the bidding profile for any user to be $s \equiv [s_i^j] \in \mathcal{I} \times \mathcal{I}$, and $s_{-i} \equiv [s_1^j, \cdots, s_{i-1}^j, s_{i+1}^j, \cdots, s_I^j]_{j \in \mathcal{I}}$ as the profile of user i's opponents. Our strategy space is the vector space of all buyer i's *possible* bids, $S_i = \Pi_{j\in\mathcal{I}} S_i^j$, and $S_{-i} = \Pi_{j\in\mathcal{I}}\Pi_{k\neq i\in\mathcal{I}} S_i^j$ the set of opponent profiles. We assume that the users are connected. A buyer submits bids directly to sellers. Users prefer to participate in the CMHK secondary market as it allows them to purchase additional data for a cost less than the overage fee set by the ISP. We assume that users are selfish, and therefore rational. Thus we assume that all users submit bids in order to maximize their (private) valuation functions. In general, user preferences are defined by a utility function, u, which represents a users' valuation of an allocation minus the price. Absent the cost or revenue from trading data, CMHK users gain utility from consuming data. We assume that data is a unary resource belonging to the seller, and therefore we identify each local auction with the identity of the seller $j \in \mathcal{I}$. We further show that user valuation satisfies the conditions for an *elastic demand function*, as in [2]. Finally, we assume in the CMHK market that a buyers' budget is always sufficient, as the alternative is to pay the overage fee to the ISP.

The CMHK market does not allow for brokers [6], we thereby determine that the bid profiles must adhere to some additional restrictions, which we will imply using the PSP bid profile notation. In other words, we assume that buyers and sellers are separated (a seller does not also buy data and vice versa). Thus, we may assume that this is implied in our notation. A user's identity $i \in \mathcal{I}$ as a **subscript** indicates that the bid belongs to a **buyer**, and a **superscript**, $j \in \mathcal{I}$, indicates the bid belongs to a **seller**. Suppose i is buying from j. The bid is represented by $s_i^j = (d_i^j, p_i^j)$, meaning i would like to buy from j a quantity d_i^j and is willing to pay a unit price p_i^j. Without loss of generality, we assume that all users bid in all auctions; if a user i does not submit a bid to j, or vice versa, we simply set $s_i^j = (0, 0)$. Naturally, in a live auction, if a buyer does not submit a bid to a seller, then this implies $s_i^j = 0$ for both buyer i and seller j. Obviously, a buyer that does not submit a bid will not recieve opponent profiles from seller j. We additionally determine that a user who does not submit a bid is holding its previous bid, either zero or nonzero. For the purposes of our analysis, we will assume that a zero bid from a buyer is equivalent to no bid. A seller j submits a bid $s_i^j = (d_i^j, p_i^j)$ to the secondary market, with the intent of offering a quantity $d_i^j \in d^j = [d_i^j]_{i\in\mathcal{I}}$ with reserve unit price $p_i^j \in p^j = [p_i^j]_{i\in\mathcal{I}}$ to buyer i. We emphasize that we allow for s_i^j to stand for a buyer or sellers' bid, the *direction* of the bid (vector) is determined by the user type, whether or not they are a buyer or a seller. To further clarify our analysis, we will emphasize the separation of buyers and sellers using s_i and s^j, indicating if a bid is from a buyer or a seller. In other words, a bid $s^j = [s_i^j]_{j\in\mathcal{I}}$ is understood as an offer of data by seller j in the CMHK secondary market. The notational conventions of the bid vectors are essentially slices of the grid, $s^j = [s_i^j]_{i\in\mathcal{I}}$ denotes a sellers' profile, and $s_i = [s_i^j]_{j\in\mathcal{I}}$ denotes a buyers' profile. Furthermore, noting that this is a simplification for ease of notation, we let $D^j = \sum_{i\in\mathcal{I}} d_i^j$ be the total amount of data j has to sell, and $D_i = \sum_{j\in\mathcal{I}} d_i^j$ represent the total amount of data requested by buyer i.

Consider the grid of bid proiles, s, representing the distributed PSP auction mechanism in the CMHK market, each buyer i will have information from each seller j, as well as opponent profiles s_{-i} from each auction in which it is participating, and therefore in the extreme case, where i submits bids to all auctions $j \in \mathcal{I}$, buyer i gains access to the full grid s. However, sellers can only gain information about the market grid by observing buyer behavior in their local auction. In our current formulation, we do not allow a seller to host multiple auctions(FUTURE WORK?). Thus, the buyers are able to directly and globally influence the CMHK market dynamics, with the sellers taking a secondary role (FIND IN ANOTHER PAPER FOR SUPPORT OR OBVIOUS ENOUGH?).

3.2 User Valuation

We assume that the CMHK market does not have network bottlenecks, which is purely a bandwidth problem, as in [3], and as bandwidth in wireless networks is not the focus of this paper. We derive a globally optimal strategy suited for users with local information in a distrubuted data-sharing model.

Our mechanism allows a buyer to *opt-out* of auctions by submitting zero bids. This strategy maximizes utility while minimizing the number of positive bids submitted to the overall market. We define an **opt-out function**, associated with a buyer i as part of its type. Buyer i, when determining how to aquire a possible allocation a, will determine its bid quanities by,

$$\sigma_i(a) = [\sigma_i^j(a)]_{j\in\mathcal{I}}. \tag{5}$$

In a general sense, this abstraction applies our *user strategy* to the PSP rules. We define each buyer as a user $i \in \mathcal{I}$ with quasi-linear utility function $u_i = [u_i^j]_{j\in\mathcal{I}}$, a buyers' utility function is of the form,

$$u_i = \theta_i \circ (\sigma_i(a)) - c_i, \tag{6}$$

where the composition of the elastic valuation function θ_i with σ_i distributes a buyers' valuation of allocation a across local markets (and thus multiple sellers). In this way we extend the PSP rules described in [3] to design equilibria across subsets of local data-exchange markets.

The sellers, $j \in \mathcal{I}$ are not associated with an opt-out function, we consider their valuation to be a functional extension of the buyers, where θ^j is constructed by buyer demand. The sellers strategy can only be to determine the reserve price of their local auction, using only information from buyers who have not opted out. In our analysis, we demonstrate market dynamics, and further show evidence of symmetry in the strategies of buyers and sellers.

3.2.1 Valuation under Market Dynamics. The buyer demand largely motivates the market price function, however, the distributed nature of the market prevents any single user from knowing the market demand for a quantity of data. All users have knowledge of market supply, as this is public information, however only buyers are able to determine supply or demand across multiple auctions, and then only from auctions in which they participate.

Remark: It is possible that a seller would be able to derive information about other auctions by examining buyer bids over time, particularly if the seller had knowledge of the buyer strategy. (FUTURE WORK?)

We interpret the collection of local auctions as collection of congestion games, where a buyers' payoff depends on the dynamics of the set of local auctions it "chooses". In a multi-auction market, each auction a buyer joins has the potential to decrease the potential cost of its data. However, increasing the size of the auction implies a certain risk, which we may interpret as a potential and definite liability. Increasing the number of transactions causes additional messaging overhead, fees, and increased competition from other buyers. A transaction also causes potential indirect costs, which may be considered work done to find sellers, or effort of communication from participation. A seller has the potential for greater profit with each new buyer in its auction, taking the same risk. The liability of any user is naturally absorbed into the bid fee ϵ, as described in [3]. Therefore, according to our interpretation, the bid fee is dependent on the association between two users and their market positions, in addition to the underlying network structure. Now, both sellers and buyers must consider the cost of adding additional users to their subsequent pools. (MODEL SEPARATE, OR DYNAMIC, TO OPTIMIZE SIZE OF SUBSETS?)

Elastic valuation functions allow for even infinitesimal changes in the market dynamics to be modeled. This, and the homogenous nature of data in the CMHK market, allows for the analysis of constraints imposed by the user strategies. Buyers may directly impact each other in local market intersections. Thus our motivation to begin our analysis with buyer valuation θ_i. A buyers' valuation of an amount of data represents how much a buyer is willing to pay for that amount. This is equivalent to the bid price, given a fixed amount of data, satisfying θ_i. We determine the buyers' utility-maximizing bid given quantity $z \geq 0$ to be a mapping to the lowest possible unit price. We have,

$$f_i(z) \triangleq \inf \left\{ y \geq 0 : \rho_i(y) \geq z,\ \forall\, j \in \mathcal{I} \right\}, \tag{7}$$

where $\rho_i(y)$ represents the demand function of buyer i at bid price $y \geq 0$, and gives the quantity that buyer i would buy at a given price. We determine that the market supply function corresponds to an extreme of possible buyer demand, and acts as an "inverse" function of f_i. We have, for bid price $y \geq 0$,

$$\rho_i(y) = \sum_{j \in \mathcal{I} : p_i^j \geq y} D^j. \tag{8}$$

We note that f_i is such that i could still bid in *any* auction $j \in \mathcal{I}$. Therefore, in a coordinated bid, the utility-maximizing bid price is the lowest unit cost of the buyer to participate in all auctions, and corresponds to the maximum reserve price amongst the sellers.

A seller only has information from buyers in its own auction, and may only be indirectly influenced by buyers in other auctions. So from the perspective of the seller we have a more direct interpretation of valuation as revenue. We determine the demand function of seller j at reserve price $y \geq 0$ to be,

$$\rho^j(y) = \sum_{i \in \mathcal{I} : p_i^j \geq y} \sigma_i^j(a), \tag{9}$$

and define the "inverse" of the buyer demand function for seller j as potential revenue at unit price y, we have,

$$f^j(z) \triangleq \sup \left\{ y \geq 0 : \rho^j(y) \geq z,\ \forall\, i \in \mathcal{I} \right\}, \tag{10}$$

and, unsurprisingly, f^j maps quantity z to the highest possible unit data price.

The valuation of any user must be modeled as a function of the entire marketplace. Naturally, a buyers' valuation is aggragated over local markets, and the sellers' valuation is aggragated over its own auction. We have already introduced the composition $\theta_i \circ \sigma_i$ as the valuation of the buyers. We further model the valuation of the sellers, based on (9) and (10). We first note that, in general (and so we omit the subscript/superscript notation), the valuation of data quantity $x \geq 0$ is given by,

$$\theta(x) = \int_0^x f(z)\, dz,$$

as in [3]. Now, we have the following Lemma,

Lemma 3.1. *(User valuation) For any buyer $i \in \mathcal{I}$, the valuation of a potential allocation a is,*

$$\theta_i \circ \sigma_i(a) = \sum_{j \in \mathcal{I}} \int_0^{\sigma_i^j(a)} f_i(z)\, dz. \tag{11}$$

Now, we may define seller j's valuation in terms of revenue,

$$\theta^j = \sum_{i \in \mathcal{I}} \theta^j \circ \sigma_i^j(a) = \sum_{i \in \mathcal{I}} \int_0^{\sigma_i^j(a)} f^j(z)\, dz. \tag{12}$$

We have that θ_i and θ^j are elastic valuation functions, with derivatives θ_i and $\theta^{j\prime}$ satisfing the conditions of elastic demand. **Proof:** Let ξ be a unit of data from buyer bid quantity $\sigma_i^j(a)$. If ξ decreases by incremental amount x, then seller bid d_i^j must similarly decrease. The lost potential revenue for seller j is the price of the unit times the quantity decreased, by definition, $f^j(\xi)x$, and so,

$$\theta^j(\xi) - \theta^j(\xi - x) = f^j(\xi)x.$$

Thus (12) holds. As we may use the same argument for (11), as such, we will denote $f_i = f^j = f$ for the remainder of the proof. We observe that the function f is the first derivative of the valuation function with respect to quantity. Letting $\theta_i = \theta^j = \theta$, the existence of the derivative implies θ is continuous, and therefore, in this context, f represents the marginal valuation of the user, θ'. Also, clearly $\theta(0) = \theta(\sigma(0)) = 0$. Now, as we consider data to be an infinitely divisible resource, we have a continuous interval between allocations a and b, where $a \leq b$. Now, as θ is continuous, for some $c \in [a, b]$,

$$\theta'(c) = \lim_{x \to c} \frac{\theta(x) - \theta(c)}{x - c} = f(c),$$

and so $f = \theta'$ is continuous at $c \in [a, b]$, and so as $a \geq 0$, $\theta' \geq 0$. Finally, we have that concavity follows from the demand function. Then, as θ' is non-increasing, we may denote its derivative $\gamma \leq 0$, and taking the derivative of the Taylor approximation, we have, $\theta'(z) \leq \theta'(\eta) + \gamma(z - \eta)$. □

Utility is defined by their valuation, and is the basis for user behavior. The sellers' natural utility is the potential profit, or simply $u^j = \theta^j$, where we have chosen to omit the original cost of the data paid to the ISP, as it is not a component of our mechanism, and as a discussion of mobile data plans is outside the scope of this paper. Now, a rational user will try to maximize its utility, thus, user incentive manifests as a response to market dynamics. A buyer has the choice to opt-out of any auction, and as a seller will try to sell the maximum amount of data, the highest possible reserve price is conditioned by "natural" constraints. Utility-maximization acts as revenue maximization for a rational seller, and as cost minimization for a rational buyer. Thus, for each user $p_i^j \geq \min\left(p_i^j\right)$ and $p_i^j \leq \max\left(p_i^j\right)$, which holds $\forall\ i, j \in \mathcal{I}$ such that $s_i^j > 0$. Now, rational buyer does not want to purchase extra data, as this would be equivalent to overpaying, however i submits positive bids to a set of sellers, and a rational seller will attempt to maximize profit, and so will try and sell all of its data. Therefore,

$$\sum_{i \in \mathcal{I}} \sigma_i^j(a) \geq D^j \quad \text{and} \quad \sum_{j \in \mathcal{I}} d_i^j \geq D_i, \tag{13}$$

which holds $\forall\ i, j \in \mathcal{I}$. We will assume that buyers and sellers do not overbid, and so omit this constraint from our formulation. Thus, at equilibrium all users are satisfied, and $D^j = D_i$, although we observe that this result does *not* imply that $s_i = s^j$.

We further determine that the set of buyers and sellers participating in a single equilibrium is bounded by the potential indirect costs of participation. We will denote this individual cost to each user as ϱ. The indirect cost is the portion of the bid fee ϵ that is dependent on the underlying network and the individual. Observing that ϱ indirectly effects user utility, and therefore acts to establish a natural budget for each user. We give this constraint as,

$$u \leq \varrho, \tag{14}$$

which may be interpreted as the effort a rational user is willing to expend on its message space, and serves to limit the size of the buyer/seller pools. Additionally, in congestion games, overly-frequent bid updates can result in an auction being overloaded, similarly to a DOS (denial-of-service) attack, thus the "liability" component of ϵ attempts to regulate bid flooding.

Finally, it is worth mention that the *analysis* of the auction as a game assumes some forms of demand and supply, in order to derive properties. The mechanism itself does not require any knowledge of user demand or valuation.

3.3 PSP for Data-Exchange

3.3.1 Data Auction Mechanism. We now proceed to formally define the PSP auction, which determines the actions buyers and sellers in the CMHK market, and which we will denote the *data* PSP rules. The rules presented here incorporate of the opt-out function with the mechanism as in [2], which we note greatly simplifies our analysis. The market price function (MPF) for a buyer in the CMHK market can be described as follows:

$$\begin{aligned} \bar{P}_i(z, s_{-i}) &= \sum_{j \in \mathcal{I}} \sigma_i^j \circ P_i^j(z_i^j, s_{-i}^j) \\ &= \sum_{j \in \mathcal{I}} \left(\inf\left\{ y \geq 0 : D_i^j(y, s_{-i}^j) \geq \sigma_i^j(z) \right\} \right), \end{aligned} \tag{15}$$

and is interpreted as the aggragate of minimum prices that buyer i bids in order to obtain data amount z given opponent profile s_{-i}. We note that the total minimum price for the buyer must be an aggragation of the *individual* prices of the buyers as it is possible that the reserve prices of the individual sellers may vary.

Remark: We further note that except at points of discontinuity, from Lemma 3.1 we have that $P_i^j(z) = f_i(z)$.

(THE ABOVE IS GOOD, BUT DOESN'T FIT MY CONSTRUCTION, CHANGE TO BELOW?) The market price function (MPF) for a buyer in the CMHK market is determined per (7), and is defined as,

$$\begin{aligned} \bar{P}_i(z, s_{-i}) &= \sum_{j \in \mathcal{I}} \sigma_i^j \circ P_i^j(z_i^j, s_{-i}^j) \\ &= \sum_{j \in \mathcal{I}} \left(\inf\left\{ y \geq 0 : D_i^j(y, s_{-i}^j) \geq \sigma_i^j(z),\ \forall\ j \in \mathcal{I} \right\} \right), \end{aligned} \tag{16}$$

and is interpreted as the price that buyer i bids in order to obtain data amount z given opponent profile s_{-i}. The sellers pricing function is according to (10),

$$\begin{aligned} \bar{P}^j(z, s_{-i}) &= \sum_{i \in \mathcal{I}} \sigma_i^j \circ P_i^j(z_i^j, s_{-i}^j) \\ &= \sum_{j \in \mathcal{I}} \left(\sup\left\{ y \geq 0 : D_i^j(y, s_{-i}^j) \geq \sigma_i^j(z),\ \forall\ i \in \mathcal{I} \right\} \right). \end{aligned} \tag{17}$$

We note that the total price cannot be an aggragation of the *individual* bid prices as it is possible that the reserve prices of the individual sellers may vary, which contradicts (7) and (10).

Remark: We further note that except at points of discontinuity, from Lemma 3.1 we have that $P_i^j(z) = f_i(z)$.

The maximum available quantity of data in auction j at unit price y given s_{-i}^j is:

$$\begin{aligned} \bar{D}_i^j(y, s_{-i}^j) &= \sigma_i^j \circ D_i^j(y, s_{-i}^j) \\ &= \left[D^j - \sum_{p_k^j > y} \sigma_k^j(a) \right]^+. \end{aligned} \tag{18}$$

It follows from the upper-semicontinuity of D_i^j that for s_{-i}^j fixed, $\forall\ y, z \geq 0$,

$$\sigma_i^j(z) \leq \sigma_i^j \circ D_i^j(y, s_{-i}^j) \Leftrightarrow y \geq \sigma_i^j \circ P_i^j(z, s_{-i}^j). \tag{19}$$

The resulting data allocation rule is a function of the local market interactions between buyers and sellers over all local auctions, as is composed with i's opt-out value, so that for each $i \in \mathcal{I}$, the allocation from auction j is,

$$\bar{a}_i^j(s) = \sigma_i^j \circ a_i^j(s) = \min\left\{\sigma_i^j(a), \frac{\sigma_i^j(a)}{\sum_{p_k^j = p_i^j} \sigma_k^j(a)} D_i^j(p_i^j, s_{-i}^j)\right\}, \tag{20}$$

noting that for the full allocation from all auctions we may simply aggregate over the seller pool. Finally we must have that the cost to the buyer adheres to the second price rule for each local auction, with total cost to buyer i,

$$\bar{c}_i(s) = \sum_{j\in\mathcal{I}} p_i^j\left(\bar{a}_i^j(0; s_{-i}^j) - \bar{a}_i^j(s_i^j; s_{-i}^j)\right). \tag{21}$$

Remark: The cost to buyer i adds up the willingness of all buyers excluded by player i to pay for quantity $\bar{a}_i^j$. i.e.

$$c_i^j(s) = \int_0^{\bar{a}_i^j} P_i^j(z, s_{-i})\, dz.$$

This is the “social opportunity cost" of the PSP pricing rule.

The formulation is inspired to the thinnest allocation route for bandwidth given in [2]. We note that if a single seller j can satisfy i's demand, then (6) reduces to the original form, defined in [3] as “a simple buyer at a single resource element".

(OWN WORDS!) The cost function will therefore be a stepwise-linear function, which is increasing in slope with each new bidder excluded from the market.

3.4 User Behavior

3.4.1 Buyer Strategy. Although it is possible for a seller to fully satisfy a buyer i's demand, it is also reasonable to expect that a seller may come close to using their entire data cap, and only sell the fractional overage. In this case, a buyer must split its bid among multiple sellers. The buyer strategy bids in auctions with the highest quantities first, a natural exploitation of the demand curve. A new seller entering the market with a large quantity of data will be in high demand. This behavior contributes to market price stability, as seller valuation is determined by buyer demand, the buyer strategy tends towards equal valuation of all local markets, and therefore similar prices. If a buyers' demand is not satisfied, they will need to bid in markets with smaller data quantities, and so will bid on a larger portion of the sellers' bid quantity, increasing their unit price. Market equilibrium is achieved when each buyer has equal bids in each auction. Our bidding strategy is inspired by [2], and we also hold buyers to consistent bids, where buyers sumbit identical bids to a subset of sellers with the highest offers. In the remainder of this section, we will make the assumption of truthful bids from the buyer, although this analysis is left to Section 4. Thus, we determine when rational (utility-maximizing) buyers opt-out of a local auction. We propose the following strategy,

Lemma 3.2. *(Opt-out buyer strategy) Define any auction duration to be $\tau \in [0, \infty)$. Let $i \in \mathcal{I}$ be a buyer and fix all other buyers' bids s_{-i} at time $t > 0 \in \tau$, and let a be i's desired allocation. Define the composition,*

$$\sigma_i^j \circ a = \sigma_i^j(a) = \frac{a_i^j}{j},$$

to be the buyer strategy with respect to quantity. Also define the set,

$$\mathcal{I}_i(n) = \underset{\mathcal{I}'\subset\mathcal{I}, |\mathcal{I}'|=n}{\arg\max} \sum_{j\in\mathcal{I}'} D^j,$$

where buyer i chooses its seller pool by determining n as,

$$\min\left\{n \in \mathcal{I} \mid nD^n \geq D_i\right\}. \tag{22}$$

The buyer strategy produces a minimal subset of sellers $\in \mathcal{I}$ able to satisfy buyer i's demand while ensuring that the size of $\mathcal{I}^j$ does not allow the overhead to outweigh the valuation of the data. For fixed n we will denote this subset,

$$\mathcal{I}_i \subset \mathcal{I}. \tag{23}$$

Now let $j^* = n \leq I$ represent the seller with the least amount of data $\in \mathcal{I}_i$, i.e. $D^{j^*} \leq D^j$, $\forall\, j \in \mathcal{I}^j$, and define i's bid vector σ_i with respect to its strategy, where

$$\sigma_i^j(a) \triangleq \begin{cases} \sigma_i^{j^*}(a), & j \in \mathcal{I}^j, \\ 0, & j \ni \mathcal{I}^j. \end{cases} \tag{24}$$

and define bid price $p_i^j = \theta_i'(\sigma_i^j(a))$. Now, (24) holds $\forall\, j \in \mathcal{I}$, and we have an optimal strategy for buyer i.

Proof: We assume that a buyer will try and fill their data requirement. In the case that there exists a seller who can completely satisfy a buyers' demand, $j^* = 1$, $|\mathcal{I}_i| = 1$ and (22) holds. If such a buyer does not exist, as the set $\mathcal{I}_i$ is ordered by the quantity of the sellers' bids, i may discover j^* by computing $\mathcal{I}_i$. Suppose that $D_i > \sum_{j\in\mathcal{I}} D^j$, then $j^* > I$ and $\mathcal{I}_i = \emptyset$. We model the ISP at time $t > 0$ as a seller κ with bid $s^\kappa = (d^\kappa, p^\kappa)$, where $d^\kappa > D^j$, $\forall\, j \in \mathcal{I}_i$, and p^κ represents the overage fee for data set by the ISP, which we note is also the upper bound of the sellers' pricing function. Consider some $k \neq i \in \mathcal{I}$ where $p_i^j = p_k^j$. The allocation rule (20) determines that the data will be split proportionally between all buyers with the same unit price. It is possible that the resulting partial allocation of data to i and k would not satisfy some demand. As the two cases i and k are the same, we will only consider one. Suppose seller j updates its bid to reflect the new data quantity, where $d_i^{j(t+1)} < \sigma_i^{j(t)}(a)$. First, i sets its bid to $s_i^j = 0$, and from the new subset $\mathcal{I}_i$, submits bids until $\sum_{j\in\mathcal{I}_i} \sigma(a)_i^j \geq D_i$, by (13). Now, we consider the case where a new buyer k with bid price $p_k^j > p_i^j$ for some $j \in \mathcal{I}_i$, in other words, a new buyer k may enter the market with a better price, decreasing the value of i's bid for $j \in \mathcal{I}_i$. In this case, by (22), i will choose $\mathcal{I}_i$ so that, $\sigma_i^{j(t+1)}(a) = \sigma_i^{j(t)}(a) - \sigma_k^{j(t)}(a)$, and so $\mathcal{I}_i$ is large enough to balance the additional demand from k. Finally, we consider the case where $|\mathcal{I}^j| = I$, where the demand of buyer i exceeds the supply, and the case where $\sigma_i(\varrho) > \theta_i(\sigma_i(a))$, where the overhead exceeds the current valuation of the data. Then, by (7), the valuation of the data increases until either the demand is satisfied, the debit from the overhead costs are balanced (14), or the upper bound of the sellers' reserve price p^κ is reached. Thus, as in each case we have that i is able to satisfy thier demand, and we determine that the opt-out strategy is optimal. □

Remark: The bid quantity $\sigma_i^j(a)$ and the allocation $\bar{a}_i^j$ are complementary. In fact, the buyer strategy is the first term in the minimum, the second term being owned by the seller.

Finally, we note that $\mathcal{I}_i$ is not the only possible minimum subset $\in \mathcal{I}$ able to satisfy i's demand, in fact, by restricting the size of the set $\mathcal{I}_i$, we would be able to improve the computation time of buyer i, at the cost of increasing the price.

3.4.2 Seller Strategy. In order to to develop the seller strategy, we examine the incentive of a rational seller with only local information in a dynamic market of many buyers and sellers. A local auction, examined independently, may appear as single market with a single seller and many buyers, but is in fact a subset of the larger data-exchange market, and is subject to the trends and dynamics therewithin. A seller must determine allocations using only bids in its local market, while the buyers' response is based on the allocations and resulting opponent bids from all auctions in its seller pool. In addition, buyers are allowed to bid both dynamically and asychronously. In order to maximize revenue, the seller must also be able to respond dynamically to address the mutation of competitive bids in its market. In order to do this, we determine that the seller may modify its reserve price in response to the changing market dynamics.

We will show that sellers are able to maximize revenue in restricted subset of buyers in $\mathcal{I}$, and as such will attempt to facilitate a local market equilibrium for this subset. A local auction j converges when all buyer bids remain the same over a time step, that is, if $\forall\ i \in \mathcal{I}$, $s_i^{j(t+1)} = s_i^{j(t)}$, at which point the allocation is stable, the data is sold, and the auction ends. In the sellers' local environment, we determine that the best course of action is to maximize revenue, and then try to keep its buyer pool stable until convergence occurs. Thus, the seller strategy is complementary to that of the buyers, and is designed to achieve and maintain a local market equilibrium.

We describe a *local* auction strategy for data allocation, where the seller is unaware of the existence of other auctions, and so the seller behavior is the same in the case of a single buyer, a small buyer set, and in the extreme case, where all buyers $i \in \mathcal{I}$ participate. We again note that the seller must initialize the strategy with a first iteration, and so the auction is defined for time $t > 0$. In our model, a local auction may be described as a progressive game of strategy with incomplete, but perfect information, however in our analysis, as before, we will assume complete information. (BUYERS ARRIVE AS A POISSON PROCESS? FUTURE WORK)

LEMMA 3.3. *(Localized seller strategy (i.e. progressive allocation)) Define any auction duration to be $\tau \in [0, \infty)$. For any seller j, fix all other bids $[s_i^k]_{i,k\neq j\in\mathcal{I}}$ at time $t > 0 \in \tau$. Define the set,*

$$\mathcal{I}^j(n) = \underset{\mathcal{I}'\subset\mathcal{I}, |\mathcal{I}'|=n}{\arg\max} \sum_{i\in\mathcal{I}'} p_i^j,$$

where,

$$\min\left\{ n \in \mathcal{I} \;\middle|\; \sum_{i\in\mathcal{I}^j(n)} d_i^j \geq D^j \right\}, \tag{25}$$

so that n produces a minimal subset of buyers that maximizes j's revenue at time t, which we will denote, for fixed n, by,

$$\mathcal{I}^j \subset \mathcal{I}. \tag{26}$$

Define buyer $i^* = n-1 \leq I$ as the buyer with the maximum bid price $\ni \mathcal{I}^j$. Then, for time $(t+1)$, set j's reserve price to be,

$$p_i^{j(t+1)} = p_{i^*}^{j(t)} + \epsilon, \tag{27}$$

Let the winner at time t be determined by,

$$\bar{i} = \max_{i\in I^j} p_i^{j(t)}, \tag{28}$$

and update j's total data to reflect the (tentative) allocation,

$$D^{j(t+1)} = D^{j(t)} - \sigma_{\bar{i}}^{j(t)}(a), \tag{29}$$

Allowing t to range over τ, we have that (25) - (29) produces a local market equilibrium.

Proof: We assume that the seller will try to maximize its revenue. In the case where $|\mathcal{I}^j| = 1$, then if $\sigma_i^j(a) = D^j$, then j's market is at equilibrium. Otherwise, we arrive at the case of multiple buyers, which we note includes the case where $\sigma_i^j(a) < D^j$, which is reflected trivially here.

For auction j with multiple buyers, i^* is the *losing* buyer with the highest unit price offer, determined by (25), Suppose that for some $i \in \mathcal{I}^j$, buyer demand is not met. In this case, by (13) the seller must notify i of a partial allocation by changing the bid vector at index i. With this caveat, and Proposition 3.2, we have that the aggragate demand of subset $\mathcal{I}^j$ is satisfied by seller j, as in Lemma ??. Although the buyers' valuation θ_i is not known to the seller, we will assume that buyers are bidding truthfully, and so the new reserve price $p_{i^*}^j + \epsilon = \theta_{i^*}{}' + \epsilon$. For clarity, let the reserve price be denoted by p_*^j. Now, by the elasticity of (7) and (10), we have that, $\forall\ z \geq 0$, $f_{i^*}(z) < f^j(z) \leq f_i(z)$, which holds $\forall\ i \in \mathcal{I}^j$, and $\forall\ j \in \mathcal{I}_i$. We claim that the choice of reserve price p_*^j does not force any buyers out of the local auction. To show this, we use the assumption of truthful bids, and the fact that since the auction begins at time $t > 0$, buyers will bid at least once. As will be addressed in further analysis, we assume that a new bid price differs from the last bid price by at least ϵ. Suppose the auction starts at equilibrium, so $\sum_{i\in\mathcal{I}^j} \sigma_i^j(a) = D^j$ at time $t = 0$. The reserve price p_*^j set at time $t = 0$ begins the auction with the first bid iteration, and so at $t > 0$, $\forall\ i \in \mathcal{I}^j$, we have that $p_i^j - p_*^j \geq \epsilon$. Now, in the case where at $t = 0$, $\sum_{i\in\mathcal{I}^j} \sigma_i^j(a) > D^j$, by (20), the seller notifies (any) buyer k with the lowest bid price of a partial allocation by changing d_k^j thus by Proposition 3.2, k either decreases its demand or increases its valuation until $\sigma_i^j(a) \leq d_i^j$. Then, as the seller computes the set $\mathcal{I}^j$ at each time step, a new i^* may be chosen and the buyers bid again. Suppose $\exists\ k \in \mathcal{I}^j$ such that $\forall\ l \in \mathcal{I}_k,\ i \ni \mathcal{I}^l\ \forall\ i \neq k \in \mathcal{I}^j$. That is, k is disconnected from all other buyers $i \in \mathcal{I}^j$, and suppose that d_k^j is partial allocation at $t > 0$, and further suppose that there are many $l \in \mathcal{I}_k$ where $|\mathcal{I}^l| > |\mathcal{I}^j|$. The more buyers an auction has, the more likely that cases will occur that cause buyers to rebid, particularly if auctions $l \in \mathcal{I}_k$ have overlapping buyers, then k may opt-out of auction j, i.e. $s_k^{j(t)} \neq s_k^{j(t+1)} = 0$, then the seller may simply return the tentatively allocated data to D^j. Finally, we note that if for some $i \in \mathcal{I}^j\ \exists\ k \in \mathcal{I}^j$ such that $p_i^j = p_k^j$, then the seller again notifies the buyers of a partial allocation by changing d_i^j and d_k^j by (20). Thus we determine the valuation between seller j and buyer i is well-posed, the reserve price (27) is justified, and the local equlibirium created by j is independently stable from time t to $(t+1)$. □

3.4.3 Market Dynamics under Strategy. We conclude this section by examining the relationship between the strategies of buyers and sellers in local auctions. We model the impact of the dynamics of the data-exchange market on a local auction j. As we have shown, the seller is a functional extension of the buyer, with rules determined by the buyers' behavior. This gives an auction j a natural logical extension into the global market through its buyers. We demonstrate that the symmetry between buyer and seller behavior, consequently strategies, stretches into a symmetry across subsets of local auctions. Additionally, we identify a clear bound restricting the influence of local auctions on each other. Defining a single iteration of the auction, where a seller updates bid vector s^j, and the buyers' response s_i, to comprise a single time step, and we have the following Lemma,

Proposition 3.4. *(Valuation across local auctions) For any $i, j \in \mathcal{I}$,*

$$j \in \mathcal{I}_i \Leftrightarrow i \in \mathcal{I}^j. \tag{30}$$

Fix an auction $j \in \mathcal{I}$ with duration τ and define the influence sets of users. The primary influencing set is given as,

$$\Lambda = \bigcup_{i \in \mathcal{I}^j} \mathcal{I}_i, \tag{31}$$

with secondary influencing set,

$$\lambda = \bigcup_{i \in \mathcal{I}^j} \left(\bigcup_{k \in \mathcal{I}_i} \mathcal{I}^k \right) \tag{32}$$

Define $\Delta = \Lambda \cup \lambda$. Fixing all other bids $s_i^j \in \mathcal{I}$, and time $t > 0 \in \tau$, we have that,

$$\sum_{j \in \Lambda} \theta_i^j = \sum_{i \in \lambda} \theta_i^j. \tag{33}$$

Proof: A local auction $j \in \mathcal{I}$, is determined by the collection of buyer bid profiles, where buyer bid $s_i^j > 0 \Rightarrow j \in \mathcal{I}_i$. Using Proposition 3.3 and (30), we have that,

$$i \in \mathcal{I}^j \Leftrightarrow p_i^j > p_{i^*}^j, \tag{34}$$

where (25) defines i^* as the losing buyer with the highest bid price in auction j. By (7) $p_i^j \geq p_{i^*}^j + \epsilon$, thus $p_i^j < p_{i^*}^j$ can only happen during a market shift caused by the underlying dynamics. Consider $k \in \mathcal{I}^j$ at time t where, for example, some buyer(s) enter the auction, and so (34) implies that $\sum_{i \in \mathcal{I}^j} \sigma_i^j(a) > D^j$. Now, $p_i^j < p_{i^*}^j \Rightarrow k \ni \mathcal{I}^j$ and $s_k^j > 0$ will cause k to initiate a shift. By Proposition 3.2, k will set $s_k^j = 0$, and begin to add sellers to its pool. Suppose that at time t, j's market is at equilibrium, i.e. $\sum_{i \in \mathcal{I}^j} \sigma_i^j(a) = D^j$, and fixing all other bids, so no buyer $i \in \mathcal{I}^j$ rebids. Unless k adds a seller with a higher reserve price within $|\mathcal{I}^j|$ time steps, by (29), $D^j = 0$ and the auction ends. Otherwise, at some time $t \in [t+1, \tau]$, we must have that $\sigma_k^j \leq D^j$, and k rejoins auction j or opts-out. Finally, overlooking market shifts and messaging overhead, we have that, $\forall\, i \in \mathcal{I}^j, \nexists\, s_i^j > 0$ where $i \ni \mathcal{I}^j$, and (30) holds.

Now, the subset $\mathcal{I}^j \subset \mathcal{I}$ determines j's reserve price $p_{i^*}^j$. We will assume the buyer submits a coordinated, truthful bid. Now, $\mathcal{I}_i \subset \mathcal{I}$ determines the unit price p_i in buyer i's bid. The reserve price (27) of seller j is determined at each shift, and is the lowest price that j will accept to perform any allocation. Let $p_*^j = f^j \circ \sigma_i^j(a)$ denote the reserve price of auction j, noting that $s_i^j = 0$, $\forall\, i \in [s_i^j]_{i \ni \mathcal{I}^j}$, and let $p_i^* = f_i \circ \sigma_i^j(a)$ denote the bid price of buyer i, i.e. $p_i^k = p_i^*$, $\forall\, k \in \mathcal{I}_i$. Using Proposition 3.3, for each $i \in \mathcal{I}^j$, we have from (7), (10), that $p_i^* \geq p_*^k$, $\forall\, k \in \mathcal{I}_i$.

The incentive of each seller $\in \Lambda$ is to sell all of its data at the best possible price. In the simplest case, consider a disjoint local market j, where $\forall\, i \in \mathcal{I}^j$, $s_i^k = 0$, $\forall\, k \neq j \in \mathcal{I}_i \Rightarrow \Lambda = \{j\}$ and $\lambda = \mathcal{I}^j$. Again using (7) and (10), it is clear that $\theta_i = \theta^j$, $\forall\, i \in \mathcal{I}^j$. In all other cases, the sellers $\in \Lambda$ are competing to sell their respective resources to buyers whose valuations are distributed across multiple auctions. The set λ represents all of the buyers influencing auction j, both directly and indirectly. The bid price of buyer $i \in \mathcal{I}^j$ is determined by,

$$p_i^* = \max_{k \in \mathcal{I}_i} \left(f^k \circ \sigma_i(a) \right) = \max_{k \in \mathcal{I}_i} (p_*^k). \tag{35}$$

Λ is the set of sellers directly influencing the bids of buyers in auction j. Now, the reserve price for auction j is such that,

$$p_*^j \leq \min_{i \in \mathcal{I}^j} (p_i^*) - \epsilon, \tag{36}$$

from (27). Now, by Proposition 3.3, in the absence of external influences caused by multi-auction market dynamics, we have that j maintains a local market equilibrium from time t to $(t+1)$. From (31) and (32), Δ is defined by a seller $j \in \mathcal{I}$, where each user $k \in \Delta$ has some direct or indirect influence on j. We may identify Δ by its dominant seller, and we denote $\Delta^j = \Lambda^j \cup \lambda^j$.

Consider the set λ^j. For some buyer $i \in \mathcal{I}^j$, and then for some seller $k \in \mathcal{I}_i$, we have a buyer $l \in \mathcal{I}^k$. By (30), $i, l \in \mathcal{I}^k$, and so the reserve price $p_*^k \leq \min(p_l^*, p_i^*)$, and $k, j \in \mathcal{I}_i \Rightarrow p_i^* \geq \max(p_*^k, p_*^j)$. Suppose that $l \ni \mathcal{I}^j \Leftrightarrow j \ni \mathcal{I}_l$, so that $p_l^* < p_*^j$, and the valuation of buyer l does not impact auction j and vice versa, i.e. $\theta_l^j = 0$. Since $l \in \mathcal{I}^k$, $p_l^* \geq p_*^k \Rightarrow p_*^k < p_*^j$, and $i \in \mathcal{I}^j \Rightarrow p_i^* \geq p_*^j$. Therefore, we have that the ordering implied by (31) and (32) hold, where,

$$p_*^k \leq p_l^* < p_*^j \leq p_i^*, \tag{37}$$

for any buyer $l \in \lambda^j$ such that $l \ni \mathcal{I}^j$. Now, suppose $\exists\, l \in \mathcal{I}^k$ such that $l \in \mathcal{I}^j \Rightarrow p_l^* \geq p_*^j$. In the case where $p_l^* > p_i^*$, we must have that $\exists\, q \in \mathcal{I}_l$ such that $p_*^q > p_*^k$, which implies, again by (35), $q \ni \mathcal{I}_i \Leftrightarrow i \ni \mathcal{I}^q \Rightarrow p_*^q > p_i^*$, therefore $\theta_i^q = 0$, and the reserve price of auction q does not effect the valuation of buyer i, and as $p_*^k < p_*^j \leq p_i^* < p_l^*$, we examine $\mathcal{I}^j$ using (34). Lastly, in the case where $p_i^* > p_l^*$, by the same reasoning, $\theta_l^g = 0$, for some $g \in \mathcal{I}_i$. We have that for any $l \in \mathcal{I}^k$ such that $l \ni \mathcal{I}^j$, $\theta_l^j = 0$, and when $l \in \mathcal{I}^j$, then either $\theta_l^q = 0$, where $q \in \mathcal{I}_l$, or $\theta_l^g = 0$, where $g \in \mathcal{I}_i$, and as $p_*^k < p_*^j \leq p_l^* < p_i^*$, we examine $\mathcal{I}^k$ using (34), a shift in $\mathcal{I}^k$ causes a shift in $\mathcal{I}_i$, so that $\exists\, g \in \mathcal{I}_i$ such that $p_*^g \geq p_*^j$. Thus, we determine a direct influence as $l \in \mathcal{I}^k \cap \Lambda^j$, such that $p_l^* > p_i^*$, and an indirect influence as, for any $l \in \mathcal{I}^k \setminus \Lambda^j$, where $p_l^* > p_i^*$ results in $i^* \in \mathcal{I}^j$ initiating a shift.

Now, consider the subset Λ^j, by Proposition 3.4, a shift occurs in 2 cases. (1) If $i \in \mathcal{I}^j$ decreases its bid quantity so that $\sum_{i \in \mathcal{I}^j} \sigma_i^j(a) < D^j$, and (2) if buyer i^*, defined in Proposition 3.3, increases its valuation so that $p_{i^*}^j < p_*^j$. First, let buyer $i \in \mathcal{I}^j$ be the buyer in auction j with the lowest bid price, the "lowest clearing player", and further suppose $p_i^* > p_*^j + \epsilon$. That is, $\exists\ q \in \mathcal{I}_i$ such that $p_*^q > p_*^j$. Fixing all other bids, a decrease in q's demand will directly impact buyer i. If at the end of the bid iteration, we still have that i is the buyer with the lowest bid price, then (10) holds and j's valuation does not change. Otherwise a new i^* will be chosen upon recomputing $\mathcal{I}^j$, as in Proposition 3.2, and the market will attempt to regain equilibrium. Clearly, if i^* in case (1) or resulting from case (2) increases in valuation, then p_*^j will similarly increase, by (3.1). Consider the seller $k^* \in \mathcal{I}_{i^*}$ at time t, and suppose that $p_*^{k^*} \geq p_*^j$, however, we have that $p_{i^*}^j < p_*^j \Rightarrow i^* \ni \mathcal{I}^j \Rightarrow k^* \ni \Lambda^j \Rightarrow \mathcal{I}^{k^*} \ni \lambda^j$. Now, consider a buyer $l^* \in \mathcal{I}^{k^*}$. We need only consider the case where $\exists\, k \in \Lambda^j$ such that $l^* \in \mathcal{I}^k \subset \lambda^j$ where we determine the influence of Δ^{k^*} on Δ^j by (34).

In each case we have that (7) and (10) hold for some fixed time t, and so, $\forall\ i \in \mathcal{I}^j$,

$$\int_0^{\sigma_i^j(a)} f^j(z)\, dz = \int_0^{\sigma_i^j(a)} f_i(z)\, dz, \tag{38}$$

therefore $\theta_i = \theta^j,\ \forall\ i \in \mathcal{I}^j$. Thus, any bid ouside of our construction has a zero valuation, with respect to buyers $\in \lambda$ and sellers $\in \Lambda$, and therefore cannot cause shifts to occur except through a shared buyer, e.g. some $l \in \mathcal{I}^k$. Thus, in all cases, (7) and (10) hold. Fixing all bids in any auction $q \ni \Lambda^j$, we have, $\forall\ k \in \mathcal{I}_i$,

$$\int_0^{D^k} f^k(z)\, dz = \sum_{i \in \mathcal{I}^k} \int_0^{\sigma_i^k(a)} f^k(z)\, dz, \tag{39}$$

which holds $\forall\ k \in \mathcal{I}_i$, by (30) and Proposition 3.3.. Finally, using (38), (39), $\forall\ i \in \mathcal{I}^j, \forall\ k \in \mathcal{I}_i, \forall\ l \in \mathcal{I}^k$,

$$\int_0^{\sigma_i^k(a)} f_i(z)\, dz = \int_0^{\sigma_i^k(a)} f^k(z)\, dz, \tag{40}$$

and

$$\int_0^{\sigma_l^k(a)} f^k(z)\, dz = \int_0^{\sigma_l^k(a)} f_l(z)\, dz. \tag{41}$$

Thus, with a slight abuse of notation for clarity,

$$\sum_{\lambda} \int_0^{\sigma(a)} f^{\Lambda}(z)\, dz = \sum_{\Lambda} \int_0^{\sigma(a)} f_{\lambda}(z)\, dz, \tag{42}$$

where the result follows by construction, and the continuity of θ'. □

For completeness, in the case where the ISP κ does not adhere to the market dynamics, so $p^\kappa > p^j + \epsilon,\ \forall\ j \in \mathcal{I}$, then we may absorb the overage (difference) as part of the bid fee. (NEED TO DO BETTER WITH TIME?)

3.4.4 Mechanism Realization. In order to represent the public platform of the secondary market, we model the ISP as a kind of buyer κ who remains at time $t = 0$, i.e. does not participate in any auctions. At time $t = 0$, a seller k entering the market will have submitted bid $s_\kappa^j = (\epsilon, D^j)$ to the CMHK platform, and so the seller does not require a priori demand information to enter the market. Also, a new sellers' bid, (D^j, ϵ), is public knowledge. A buyer entering the market at $t = 0$ is assumed to have an initial nonzero bid price, which we may assume (SAY BETTER! ALSO DO I REALLY NEED THE I.I.D?) is initialized as an independently and identically distributed (i.i.d.) random variable $p_i^j = X$ with probability $\mathcal{P}$,

$$\mathcal{P}[\epsilon \leq X \leq \kappa] = \int_\epsilon^{p^\kappa} \mathfrak{f}(x)\, ds,$$

where p^κ is the overage charge of the ISP, and $\mathfrak{f}$ the probability density function of X. Thus the auction begins at time $t > 0$, and at $t = 0$, j will increment through a single iteration, initializing bid prices.

Consider a user seeking to prevent data overage charges by purchasing data from a subset of other network users. The sellers' auction will function as follows: at each bid iteration all buyers submit bids, and the winning bid is the buyer i that has the highest price p_i^j. The seller allocates data to this winner, at which point all other buyers are able to bid again, and the winner leaves the auction (with the exception where multiple bidders bid the same price, where (20) determines they will not fully satisfy their demand, and so we will assume they remain in the auction). The auction progresses as such until all the sellers' data has been allocated.

Algorithm 1 (Seller progressive allocation)

$p^{j(0)} \leftarrow \epsilon$, $s^{j(0)} \leftarrow (p^j, D^j)$, $\bar{\mathcal{I}} = \emptyset$, compute $\mathcal{I}^{j(0)}$
Update s^j
while $D^j(t) > 0$ **do**
  $\bar{i} \leftarrow \max_{i \in I^j} \sum_{i \in I^j} p_i^j$
  $D^{j(t+1)} \leftarrow D^{j(t)} - \sigma_{\bar{i}}^{j(t)}(a)$
  $p^j \leftarrow p_{i^*}^j + \epsilon$ and $d^j \leftarrow D^{j(t+1)}$
  $s^{j(t+1)} \leftarrow (d^j, p^j)$
  Update s^j
  $\bar{\mathcal{I}} \leftarrow \bar{\mathcal{I}} \cup \bar{i}$
  for $k \in \bar{\mathcal{I}}$ **do**
    if $p_k^j < p_{i*}^j$ **then**
      $D^{j(t+1)} = d_k^j$
      $\bar{\mathcal{I}} \leftarrow \bar{\mathcal{I}} \setminus \{k\}$
  Compute $\mathcal{I}^{j(t)}$
  $\mathcal{I}^{j(t+1)} = \mathcal{I}^{j(t)} \setminus \bar{\mathcal{I}}$
  $t \leftarrow t + 1$

Each time step, s^j is updated it is shared with all participating buyers. At this point buyers have the opportunity to bid again, where a buyer that does not bid again is assumed to hold the same bid, since a buyer dropping out of the auction will set their bid to $s_i^j = (0, 0)$.

Algorithm 2 (Buyer response)

$p_{i(0)} \leftarrow \epsilon$, $s_{i(0)} \leftarrow (p_i, D_i)$, $D_t \leftarrow D_i$, compute $\mathcal{I}_{i(0)}$
Update s_i
while $D_i(t) > 0$ **do**
  $D^j_{i(t+1)} \leftarrow \sum_{j \in \mathcal{I}_i} \sigma_i^{j(t)}(a)$
  if $D^j_{i(t+1)} < D_t$ **then**
    Compute $\mathcal{I}_{i(t)}$
    $p_i \leftarrow \theta_i(\sigma_i(a))$
  $s_{i(t+1)} \leftarrow (\sigma_i(a), p_i)$
  Update s_i
  $D^j_{i(t+1)} \leftarrow D^j_{i(t)}$
  $t \leftarrow t + 1$

Finally, we give a simple example of convergence to a local market equilibrium, where the buyers are assumed to respond with their truthful, ϵ-best replies.

Name	Bid total	Unit price
A	50	1
B	40	1.2
C	26	1.5
D	20	2
E	14	2.2

Let $s^{(1)} = [(65, \epsilon)]_{i \in \mathcal{I}}$ and $s^{(2)} = [(85, \epsilon)]_{i \in \mathcal{I}}$. The buyer bids are as follows:

$$\begin{aligned} s_A &= [(0,0), (50, 1)], \\ s_B &= [(0,0), (40, 1.2)], \\ s_C &= [(0,0), (26, 1.5)], \\ s_D &= [(0,0), (20, 2)], \\ s_E &= [(0,0), (14, 2.2)]. \end{aligned}$$

Then at $t = 1$, we have bid vector $s^{(2)} = [(0, p^{(2)}), (20, p^{(2)}), (26, p^{(2)}), (20, p^{(2)}), (14, p^{(2)})]$, and so $(D^{(2)}, p^{(2)}) = (85, 1 + \epsilon)$, The buyer response is,

$$\begin{aligned} s_A &= [(50, 1), (0,0)], \\ s_B &= [(40, 1.2), (0,0)], \\ s_C &= [(0,0), (26, p^{(2)})], \\ s_D &= [(0,0), (20, p^{(2)})], \\ s_E &= [(0,0), (14, p^{(2)})]. \end{aligned}$$

At $t = 2$, $(D^{(1)}, p^{(1)}) = (65, 1 + \epsilon)$, with bid vector $s^{(1)} = [(25, p^{(1)}), (40, p^{(1)}), (0,0), (0,0), (0,0)]$. $(D^{(2)}, p^{(2)}) = (25, 1 + \epsilon)$. Then,

$$\begin{aligned} s_A &= [(25, p^{(1)}), (25, p^{(2)})], \\ s_B &= [(40, p^{(1)}), (0,0)], \end{aligned}$$

where we have removed bids to indicate winner(s) with a tentative allocation. At $t = 3$, $(D^{(1)}, p^{(1)}) = (50, 1 + \epsilon)$, with bid vector $s^{(1)} = [(25, p^{(1)}), (40, p^{(1)}), (0,0), (0,0), (0,0)]$. $(D^{(2)}, p^{(2)}) = (0, 1+\epsilon)$ and $s^{(2)} = [(25, p^{(1)}), (0,0), (26, p^{(2)}), (20, p^{(2)}), (14, p^{(2)})]$. Then,

$$s_A = [(25, p^{(1)}), (0,0))].$$

At $t = 4$ the auction ends.

Remark: In the case where market resources do not satisfy (13), however as this constraint is not restricted in time, we reason that in the case of insufficient data in the market buyers may wait for additional sellers or purchase from the ISP, κ, as a monopoly sale. Similarly, in the case of insufficient demand, where we may assume that data is held at time $t = 0$ by κ at bid price ϵ.

4 PSP ANALYSIS

4.1 Equilibrium

We intend to show evidence shared network optima (a global optimum). A buyer $i \in \mathcal{I}$ will have incentive to change its bid quantity if it increases its opt-out value σ_i, and therefore its utility (6). We will show that, without loss of utility, buyer i may use a "consistent" bid strategy within its seller pool, i.e. $d_i^j = d_i^k$, $\forall\, j, k \in \mathcal{I}_i$, and as such, Proposition 3.2 supports an optimal strategy with respect to (6). Our result shows that a buyer may select $\mathcal{I}_i$ in order to maximize its utility while maintaining a coordinated bid strategy. Reasonably, if $j^* < I$, a buyer may increase the size of its seller pool $\mathcal{I}_i$, thereby lowering its coordinated bid quantity while obtaining the same (potential) allocation a_i. As buyer i submits identical bids to multiple auctions, the bid price must be as high as the highest reserve price $p_i^j \in \mathcal{I}_i$. Buyer i's bid then has identical bid price $p_i^j \forall\, j \in \mathcal{I}_i$. We further note that i optimal strategy does not require reducing its bid price to a minimum in each auction, where the bid quantity $\sigma_i^j(a)$ is still fufilled. The pricing rule of the PSP auction dictates that a buyer i will pay the cost of excluding other players from the auction, and as i's bid price reflects its valuation of it data requirement D_i across all local markets, we have identical bid prices in each auction where $s_i^j > 0$. Obviously, if $j \ni \mathcal{I}_i$, then $\theta_i^j = 0$.

Lemma 4.1. *(Opt-out buyer coordination) Let $i \in \mathcal{I}$ be a opt-out buyer and fix all sellers' profiles s^j. For any profile $S_i = (D_i, P_i)$, let $a_i \equiv \sum_j a_i^j(s)$ be a tentative data allocation. For any fixed S_{-i}, a better reply for i in any auction is $x_i = \sigma_i \circ (z_i, y_i)$, where $\forall\, j \in \mathcal{I}_i$,*

$$\begin{aligned} z_i^j &= \sigma_i^j(a), \\ y_i^j &= \theta_i'(z_i^j). \end{aligned}$$

Furthermore,

$$a_i^j(z_i, y_i) = z_i^j, \tag{43}$$

and

$$c_i^j(z_i, y_i) = y_i^j, \tag{44}$$

where i's strategy is as in Proposition 3.2.

Proof: As s_{-i} is fixed, we omit it, in addition, we will use $u \equiv u_i \equiv u_i(s_i) \equiv u_i(s_i; s_{-i})$. In full notation, we intend to show

$$u_i((d_i, p_i); s) \le u_i((z_i, y_i); s_{-i}).$$

Now, if there exists a seller who can fully satisfy i's demand, then $|\mathcal{I}_i| = 1$, and the case is trivial as no coordination is necessary for a single bid. Otherwise, buyer i's demand can only be satisfied by purchasing data from multiple sellers. We will show that i may increase $|\mathcal{I}_i|$, and so decreasing d_i^j, $\forall\, j \in \mathcal{I}_i$, without decreasing $\sum_{j\in\mathcal{I}_i} u_i^j$. Buyer i maintains ordered set $\mathcal{I}_i$ where the sellers with the largest bid quantities are considered first; the index of seller j^* defines a minimal subset $\mathcal{I}_i$, satisfying (22). By construction, $d_i^{j^*}$ is the minimum quantity bid offered by any $j \in \mathcal{I}_i$. Thus by (22) and (24), $\forall\, j \in \mathcal{I}_i$, $k \ni \mathcal{I}_i$, $\sigma_i^k(a) \le z_i^j = \sigma_i^j(a)$, and so, using (33),

$$\sigma_i^j(a) \le \left[D^j - \sum_{k\in\mathcal{I}^j : p_k^j > y_i^j} d_k^j\right]^+. \tag{45}$$

Now, the buyer valuation function (11), guarantees that $\forall\, j \in \mathcal{I}_i$, $y_i^j \ge p_{i^*}^j$, where $p_{i^*}^j$ is the reserve price of seller j, defined in Proposision 3.3, and is by definition the minimum price for a buyer bid to be accepted. As $\bar{D}_i^j$ is non-decreasing, $\forall\, j \in \mathcal{I}_i$, $k \ni \mathcal{I}_i$,

$$D_i^j(y_i^j) \ge D_i^j(p_i^j) \ge D_i^j(p_i^k).$$

Thus (45) holds and so, by (20),

$$\begin{aligned} a_i^j(z_i, p_i) &= \min_{i\in\mathcal{I}^j}\left(z_i^j, \left[D^j - \sum_{p_k^j > y_i^j} d_k^j\right]^+\right) \\ &= z_i^j = \sigma_i^j(a) \end{aligned}$$

where the last equality is by definition, and so (43) is proven. From (18), $\bar{D}_i^j(y, s_{-i}) = 0\ \forall\ y < p_{i^*}^j$, and $\bar{D}_i^j(y, s_{-i}) = 0 \le \epsilon \Rightarrow \sigma_i^j(a) = 0 \Rightarrow z_i^k = 0$, $\forall\ k \ni \mathcal{I}_i$, and therefore,

$$\sum_{j\in\mathcal{I}_i} c_i^j(z_i, y_i) = \sum_{j\in\mathcal{I}_i} c_i^j(z_i, p_i),$$

thus (44) simply shows that changing the price p_i^j to y_i^j does not exclude any additional buyers, as the bid p_i^j was already above the reserve price of any seller $j \in \mathcal{I}_i$. We proceed to show that x_i does not result in a loss of utility for buyer i, that is,

$$u_i \le u_i(z_i, y_i).$$

From (43), we have $a_i^j(z_i, y_i) = z_i^j = \sigma_i^j(a(z_i, y_i))$, and so,

$$\theta_i \circ \sigma_i^j(a(z_i, y_i)) = \theta_i \circ \sigma_i^j(a),$$

which holds $\forall\, j \in \mathcal{I}_i$. Therefore, by the definition of utility (6), and the buyers' valuation (11),

$$\begin{aligned} &\theta_i \circ \sigma_i(a(z_i, y_i)) - \theta_i(a) \circ \sigma_i(a) \\ &= u_i(z_i, y_i) - u_i = \sum_{j\in\mathcal{I}_i} c_i^j - c_i^j(z_i, y_i) \\ &= \sum_{j\in\mathcal{I}_i} \int_{a_i^j(z_i,p_i)}^{a_i^j} f_i(d_i^j - x)\, dx. \end{aligned}$$

Then, as $a_i(z_i, p_i) \le z_i^j \le a_i^j$, and noting that $z_i^j > 0 \Rightarrow \theta_i \ge 0 \Rightarrow f_i \ge 0$, we have $u_i(z_i, y_i) - u_i \ge 0, \forall\, j \in \mathcal{I}_i$. □

4.1.1 Incentive Compatibility. The property of truthfullness is an essential component of equilibrium in second-price markets. The strategies described in this paper have removed the necessity for a user to determine its own valuation function, we intend to show that the market dynamics resulting from the construction of the user strategy space results in truthful bids that are optimal for all users, i.e. bid prices are to the marginal value as determined by market dynamics. To achieve incentive compatibility, we find that the opt-out buyer must choose this subset so that its overall marginal value is greater than its market price. We have so far only made the *assumption* of truthful bids throughout our analysis. As was shown in Lemma 3.4, a buyer only has incentive to change its bid as a result of a market shift or partial allocation. We now show the necessary condition of a truthful reply, new bid prices must differ from the last by at least ϵ, and prove that the resulting bid is an ϵ-best response, supporting our strategic analysis. In a truthful reply, the term $\epsilon/\theta_i'(0)$ ensures that a new bid price differs from the last bid price by at least ϵ, thereby ensuring that a buyer does not change its bid without correcting the effects of unstable shifts. We argue that if truthfulness holds *locally* for both buyers and sellers, i.e. $p_i = \theta_i{}'\ \forall\ j \in \mathcal{I}_i$ and $p^j = \theta^j{}'\ \forall\ i \in \mathcal{I}^j$, then there exists a market equilibrium extending over a subset of connected local markets. For a buyer i, define the set of possible ϵ-best replies,

$$S^\epsilon(s) = \{s_i \in S_i(s_{-i}) : u(s_i; s_{-i}) \ge u_i(s_i'; s_{-i}) - \epsilon,\ \forall\ s_i' \in S_i(s_{-i})\}, \tag{46}$$

and the set of *truthful* bids,

$$T_i = \{s_i \in S_i(s_i) : z = \sum_{j\in\mathcal{I}_i} \sigma_i^j(a) \wedge p_i = \theta_i'(z)\}, \tag{47}$$

where $\wedge$ denotes the logical "and" operator. We note that the "strategic" set T_i is restricted by Proposition 3.2. We have the following Proposition,

Proposition 4.2. *(Incentive compatibility across local auctions) Let Λ, λ be defined as in Lemma (3.4), and fix time $t > 0 \in \tau$, and fix s^j, $\forall\, j \in \Lambda$, and for some buyer $i \in \mathcal{I}^j$, let s_l also be fixed $\forall\ l \ni i \in \lambda$. Define,*

$$\chi_i = \left\{x \in [0, D_i] : \theta_i{}'(x) > \max_{j\in\Lambda} P_i^j(x)\right\}, \tag{48}$$

and $z = \sup(\chi_i - \epsilon/\theta_i'(0))^+$, and for each $j \in \Lambda$,

$$v_i^j = \sigma_i^j(z),$$

and

$$w_i^j = \theta_i'(z).$$

Then a (coordinated) ϵ-best reply for the opt-out buyer is $t_i = (v_i, w_i) \in T_i \cap S_i^\epsilon(s_{-i})$, i.e., $\forall\ s_i, u_i(t_i; s_{-i}) + \epsilon \ge u_i(s_i; s_{-i})$. With reserve prices $p^j > 0$, there exists a "truthful" strategy game embeddeded $\in \Delta$. Therefore, a fixed point $\in \Delta$ is a fixed point in the multi-auction game.

Proof: We claim that t_i is an ϵ-best reply for buyer i. That is,

$$u_i(t_i; s_{-i}) + \epsilon \ge u_i(s_i; s_{-i}).$$

As a result of auction initialization, a seller j's valuation defines its reserve price to be determined by a buyer $i \ni \lambda$, even if this price is zero, we have that $p^j = \epsilon \ge 0\ \forall\ j \in \Lambda$. Let $z = \sup(\chi_i^j)$, and again let $p_*^j = f^j \circ \sigma_i^j(a)$ denote the reserve price of auction j, and $p_i^* = f_i \circ \sigma_i^j(a)$ denote the (coordinated) bid price of buyer i. We have that $i \in \mathcal{I}^j$, and (7) defines $\theta_i'(z)$ as being max of the reserve prices p_*^j, $\forall\ j \in \mathcal{I}_i$, therefore (48) is such that,

$$\theta_i'(z) > \max_{j\in\Lambda} P_i^j(v_i^j),$$

which implies, as θ_i' is non-increasing and $P_i^j \geq 0$, we have $\forall\, j \in \mathcal{I}_i$,

$$w_i^j > P_i^j(v_i^j)$$
$$\Rightarrow v_i^j \leq D_i^j(w_i^j) = D^j - \rho^j(w_i^j).$$

And so, by (20),

$$a_i^j(t_i; s_{-i}) = v_i^j$$
$$\Rightarrow \sum_{j \in \Lambda} a_i^j(t_i; s_{-i}) = z.$$

Therefore, $\forall\, j \in \Lambda$ and $\forall\, i \in \lambda$ such that (40) and (41) hold,

$$\int_0^{v_i^j} \bar{P}_i(x)\, dx = \sum_{j \in \Lambda} \int_0^{\sigma_i^j(z)} P_i^j(x)\, dx.$$

It follows that,

$$u_i(t_i; s_{-i}) = \int_0^z \theta_i'(x)\, dx - \sigma_i \circ \int_0^z \bar{P}_i(x)\, dx.$$

Suppose $\exists\, s_i = (d_i, p_i)$ such that $u_i^j(s_i; s_{-i}) > u_i^j(t_i; s_{-i}) + \epsilon$. Propositions 4.1 and 3.2, define the coordinated bid, $\nu_i = (\zeta_i, p_i)$, using (40) and (41), for each $j \in \Lambda$, $\sigma_i^j(a_i^j(\nu_i; s_{-i})) = \zeta_i^j$, then clearly $u_i(\nu_i, s_{-i}) \geq u_i(s_i, s_{-i}) \Rightarrow u_i(t_i; s_{-i}) - u_i(s_i; s_{-i}) > \epsilon$. Denoting ζ_i^j (fixed) as ζ,

$$\int_z^{\zeta} \theta_i{}'(x)\, dx - \int_z^{\zeta} \bar{P}_i(x)\, dx > \epsilon.$$

For concave valuation functions, the first-order derivative of θ at point 0 gives the maximum slope of the valuation function, and so the factor $\epsilon/\theta'(0)$ guarantees that new bids will differ by at least ϵ, and as such, buyer i will remain in any local auction with reserve price determined by (27). We therefore verify that,

$$\int_z^{z+\epsilon/\theta_i'(0)} \theta_i'(x)\, dx \leq \epsilon,$$

and as $P_i^j \geq 0$, we have that, from the construction of ζ,

$$\int_{z+\epsilon/\theta_i'(0)}^{\zeta} \theta_i'(x)\, dx - \int_{z+\epsilon/\theta_i'(0)}^{\zeta} \bar{P}_i(x)\, dx > 0.$$

If $\zeta > z + \epsilon/\theta_i'(0)$, then for some $\delta > 0$, $\theta_i(z + \epsilon/\theta_i'(0) + \delta) > P_i^j(z + \epsilon/\theta_i'(0) + \delta)$, contradicting (48). Now, if $\zeta \leq z$, then $\theta_i'(z + \epsilon/\theta_i'(0)) < P_i^j(z + \epsilon/\theta_i'(0))$, also a contradiction of (48), and so buyer s_i cannot exist. Finally, as we may consider $\Delta \subset \mathcal{I}$ to be a multi-auction game, our user strategies form a "truthful" local game with strategy space restricted to ϵ-best replies from buyers $\in \lambda$. Therefore we have that a fixed point in the "truthful" game is a fixed point for the auction. ◻

The strategy space of is comprised of a collection of bid, or "strategy", vectors that together, describe the collection of congestion games. A change in buyer i's utility, resulting from a change in strategy, equals the change in the local market objective of each seller $j \in \mathcal{I}_i$. These local objectives are known as potential functions, and are formulated by mapping the incentives of all users in a local auction to a single function. The goal of our analysis is to therefore construct a global potential function that encompasses all local markets. Then, we may determine a Nash equilibrium by finding a local optima of the potential function. Additionally, as the potential function also iterates, it may be used in an analysis of convergence. The convergence of a Nash equilibrium results from the progression of ϵ-best replies, where each subsequent bid is a unilateral improvement, provided that t_i is continuous in opponent profiles. From the original proof by [2], we observe that the collection of unconstrained truthful bids may be a subset of the collection of ϵ-best replies, i.e. $T_i \subset S_i^{\epsilon}$. For this work, it suffices to show the continuity of the set of truthful ϵ-best replies in the set of opponent bid profiles. In order to address continuity in a global sense, we must demonstrate continuity in the construction of our model. Thus, we extend our analyis to be all-inclusive, and determine the existence and "uniqueness" of a global market objective by rigor of mathematical construction. Thus, we begin with the definition of correspondence,

Definition 4.3. (Correspondence) A correspondence is mathematically defined as an ordered triple (X, Y, R), where R is a relation from X to Y, i.e. any subset of the Cartesian product $X \times Y$.

In an economic model, a correspondence (S_i, S_{-i}, R) defines a map from S_i to the power set S_{-i}, where R is a binary relation, i.e. $R \subset S_i \times S_{-i}$. The classic example of a correspondence in our model is the buyers' best response B_i^{ϵ}, where, for the multi-auction, S_i and S_{-i} are built by repeatedly using the cartesian product over bid profiles. The power set $S_{-i} = \Pi_j \left(\Pi_{k \neq i} S_i^j\right)$ arises naturally from the product of ordered sets. The best response is a reaction correspondence defined by the mixed-strategy game. Denoting $B_i^{\epsilon} = T_i \cap S_i^{\epsilon}$, is the set of truthful ϵ-best replies in opponent bid profiles S_{-i}.

Remark: The ease by which the game is constructed is a consequence of the the cartesian product on a 2-dimensional message space.

A natural induced topology of this space is the product topology, e.g. the canonical map $S_i \to \Pi_{j \in \mathcal{I}} S_i$.

Motivated by the symmetric nature of supply and demand, we determine the game-theoritical argument is complemented by an abstract-theoretical analysis. In fact, we may even be philosophically motivated, as the truth value of a bid is determined only by how it relates to markets, and whether it provides an accurate correspondence.

Thus, we address the sellers bids, and include the following corollary to complement our result from Proposition 4.2.

Corollary 4.4. *Data-bid correspondence (seller cooperation) For a fixed time $t \in (0, \tau]$, seller bid s^j is consistent with a truthful ϵ-best reply.*

Proof: We claim there exists a binary equality relation $i \sim j$ that naturally evolves in the strategy space. For a seller j, let $y = \theta_i'(\sigma_i^j(a))$ for a buyer i. We use the the axiom of set equality, based on first-order logic with equality, which states that, $\forall\, i \in \mathcal{I},\ \forall\, j \in \mathcal{I}, (i \in \mathcal{I}^j \Leftrightarrow j \in \mathcal{I}_i) \Rightarrow i \sim j$, and is a logical consequence of (30). Then, for any allocation a, we may define the relation, $i \sim j$,

$$\left(\bar{D}_i^j(y), \theta^j{}'(\sigma_i^j(a))\right) = \left(\sigma_i^j(a), y\right). \tag{49}$$

Formally, the axiom states that a set is *uniquely* determined by its members. It follows that $\sim$ defines equality of bids using a static analysis with respect to equilibrium, where all users who are not changing thier bids are considered equal.

Remark: Equality is both an equivalence relation and a partial order, and therefore is reflexive, transitive, symmetric and antisymmetric.
Now, we may define the mapping $s \mapsto [s]$,

$$1_\vartheta \equiv \theta_i'(z) - \theta^{j\prime}(z) > \epsilon, \tag{50}$$

o noting that equality in the bid quantity is implicity satisfied and $z = \bar{D}_i^j(y) \geq 0$. We have that ϑ is a price relation for a buyer-seller pair. Without loss of generality, let $S = \Pi_{j \in \mathcal{I}} \left(\Pi_{i \in \mathcal{I}} S_i^j \right.$. The indicator function is the canonical mapping, $1_\vartheta : S \to \{0, 1\}$. Then, as the product topology is preserved, the set of all indicator functions on S naturally forms the power set $\mathcal{P}(S) = S_i \times S_{-i}$. Additionally, the set of all equivalence classes defines the quotient space, $S/\sim \equiv \left\{[k] : k \in \mathcal{I}\right\}$, forming a partition $P = \left\{[s] : s \in S\right\}$ of S.

The result follows by the symmetry of supply and demand, i.e. (7) and (10), and as the buyers' bids are truthful, and by (27), Lemma 3.4 and Lemma 22. □

Proposition 4.5. *(Continuity of ϵ-best reply on Δ) Let Δ be defined as in Lemma (3.4). For any buyer $i \in \lambda^j$, the collection of bids B_i is continuous in S_{-i}*

Proof: Define $\sigma_i \circ \overline{P} = \max_{i \in \mathcal{I}^j} \theta_i'(0)$, and $\bar{P}_i(z, s_i) = \underline{P} = \epsilon - \varrho$, where ϵ is the bid fee, and ϱ is i's liability estimate for auction $j \in \mathcal{I}$. We observe that $\sigma_i \circ B_i^\epsilon$ is simply B_i^ϵ restricted to seller pool $\mathcal{I}_i$, i.e. $\sigma_i \circ B_i^\epsilon \equiv B_i^\epsilon|_{\mathcal{I}_i}$. Thus, we have $\sigma_i \circ T_i = ([0, D^k]_{k \in \mathcal{I}^j} \times [0, \sigma_i \circ \overline{P}]^{|\mathcal{I}^j|})$ is a product of closed subsets of compact sets. Now, we have that a closed subset of a compact set is compact and the resulting product topology gives Tychonoff's theorem. every product of a compact space is compact, we have $\sigma_i \circ B_i^\epsilon$ is compact subset of B_i^ϵ. Now, letting $\overline{P} = \max_{i \in \lambda^j} \theta_i'(0)$, and we have by definition of Δ and the product,

$$\begin{aligned}
&\sigma_i \circ S_i(s_{-i}) \equiv g_i|_{\Lambda^j} : S_i \mapsto S_i \\
&\quad \Rightarrow \left(\bigcup_{i \in \mathcal{I}^j} [0, D^k]_{k \in \mathcal{I}_i}, [0, \overline{P}] \right) = \bigcup_{i \in \mathcal{I}^j} \left([0, D^k]_{k \in \mathcal{I}_i} \times [0, \overline{P}] \right) \\
&\quad = ([0, D^k]_{k \in \Lambda^j}, [0, \overline{P}]) \in \Lambda^j \times \lambda^j \subset T.
\end{aligned}$$

The result follows from the fact that t_i is continuous in s_i, as was proven in [3], and as a finite union of compact sets is a compact set. □ on any subset $\{s_{-i} \in S_i : \forall\, z > 0, \overline{P} \geq P_i(z, s_{-i}) \geq \underline{P}\}$, where $0 < \underline{P} \leq \overline{P} < \infty$ (HERE) The dimension of a linear space is defined as the maximal number of linearly independent vectors or, equivalently, as the minimal number of vectors that span the space

We have that all bids represent ϵ-best replies, and, as was proven in [2], the sellers' positive reserve price implies that bids are truthful. Finally, by properties determined by the construction of a mixed strategy symmetric game with a 2-dimensional message space, we may now restrict our analysis to the set of continuous, truthful, ϵ-best replies, B^ϵ.

Corollary 4.6. *Hemicontinuity of* Δ

The data-sharing market consists of inter-dependent sets of these multi-auction games around possible fixed points. Clearly, the union of all possible sets $\bigcup_{j \in \mathcal{I}} \Delta^j$ covers $\mathcal{I}$. We claim that the shared buyers between the different subsets Δ form a sufficiently connected set that the heirarchy described in Proposition 3.4 holds. Then, there can only be a single primary fixed point, where the sellers' reserve price is an equilibrium price in the global market. We first address the analytical approach, and demonstrate properties of Δ as a finite-dimensional linear topological space. We have that the reserve price of the sellers, and the bid price of the buyers is constant within an interval of length 2ϵ. We have that (HERE) We have the following Corollary, (MIGHT NEED TO REDEFINE.. NOT CLEAR IS A SEQUENCE, SHOULD BE SEQUENCE OF PRICES INSTEAD OF USERS?)

Corollary 4.7. *(Primary fixed point) Let the set of shared buyers be denoted as, $\underline{\lambda} = \bigcap_{j \in \mathcal{I}} \lambda^j$, and the set of all sellers as, $\overline{\Lambda} = \bigcup_{j \in \mathcal{I}} \Lambda^j$. If $\overline{\Lambda}$ is not a partition, i.e. $\nexists\, j, k \in \overline{\Lambda}$ such that $\Lambda^j \cap \Lambda^k = \emptyset$, then, for a fixed time t, $\exists\, j \in \overline{\Lambda}$ such that $p_*^j \geq p_*^k$, $\forall\, k \neq j \in \overline{\Lambda}$.*

(CAN PROBABLY USE ALL BUYERS HERE... BECAUSE OF INF)

Proof: We assume a finite number of users, with continuous valuation functions bounded both above and below. From the assumption that $\overline{\Lambda}$ is not a partition, we have that the limits exist with respect to bid price p_i^j,

$$\limsup_{j \to I} \overline{\Lambda} = \bigcap_{j \geq 1} \bigcup_{k \geq j} \Lambda^k,$$

is the primary seller j, and we have the market price p_*^j from,

$$\liminf_{i \to I} \overline{\lambda} = \bigcup_{i \geq 1} \bigcap k \geq i \lambda^k,$$

and the result follows from Lemma 3.1, Proposition 3.4 and Proposition 4.2. (NEED TO SHOW THEY ARE EQUAL...) (USE BOREL-CANTELLI WITH I.I.D? FUTURE WORK?)

We show that our bidding strategy is part of a Nash equilibrium. We first show the existence of a *static* Nash equilibrium, where the sellers reserve prices are fixed.

Lemma 4.8. *(Static Nash Equilibrium) Let Δ be defined as in Lemma (3.4), and let the duration of auction j be $\tau \in (0, \infty)$, and fix the sellers reserve prices at $t \in (0, \tau)$, $\forall\, j \in \mathcal{I}$. Using the rules of the data auction mechanism applied independently by each user, where users are acting according to their respective strategies, the multi-auction game converges to an ϵ-Nash equilibrium.*

Proof: (CAN'T DO THIS, NOT THE SAME TYPE OF GAME?) As θ_i' is continuous, as was shown in Lemma 3.1, and $t = [t_i^j] \in \lambda^j \times \Lambda^j$ is continuous in s on $T_k = \Pi_{k \in \Lambda^j} T_k^j$. Now, t represents a continuous mapping of $[0, \sum_{k \in \Lambda^j} D^k]_{i \in \lambda^j}$ onto itself, and we may use Brouwer's fixed point theorem, as in [3],

As a result of user behavior, and subsequent strategies, we determine that the data-exchange market behaves in a predictable way. However, each auction may be played on the same or on a different scale in valuation, time and quantity, and so the rate at which market fluctuations occur is impossible to predict (NEED HELP!). Arrow's paradox is an impossibility theorem stating that when buyers have three or more distinct alternatives (auctions), no deterministic ranking system can convert the ranked preferences of users into a market-wide (complete and transitive) ranking while also meeting a specified set of criteria: unrestricted domain, non-dictatorship, Pareto efficiency and independence of irrelevant alternatives. It follows that the case where $\theta_i = \theta^j$ as in (7) and (10) will only occur if each set $\Lambda \cup \lambda$ is disjoint.

Nonetheless, we claim that our mechanism is normative, that irrelevant alternatives should not matter, it is practical, uses minimal information, strategy, and provides the right incentives for the truthful revelation of individual preferences.

The rules of the PSP multi-auction drive market mutations that evolve and are regulated by the user strategies. (HERE)

(DEFINITION.. USE?) In the General Symmetric Game, p is an evolutionarily stable mixed strategy if there is a (small) positive number y such that when any other mixed strategy q invades p at any level x < y, the fitness of an organism playing p is strictly greater than the fitness of an organism playing q. (EXPAND)

Theorem 4.9. *(Dynamic Nash Equilibrium) Using the rules of the data auction mechanism, the CMHK [1] converges to a ϵ-Nash equilibrium. In the network auction game with the data-PSP rules applied independently by each user according to their respective strategies, the secondary market converges to an ϵ-Nash equilibrium.*

Proof:

4.2 Efficiency

Formally, the mechanism is efficient, if, at equilibrium, the allocation maximizes $\sum_i \theta_i(a_i)$. (NEED OWN WORDS) The objective in designing the auction is that, at equlibrium, resources al- ways go to those who value them most. Indeed, the PSP mechanism does have that property. This can be loosely argued as follows: for each player, the marginal valuation is never greater than the bid price of any opponent who is getting a non-zero allocation. Thus, whenever there is a player j whose marginal valuation is less than player i 's and j is getting a non-zero allocation, i can take some away from j , paying a price less than i 's marginal valuation, i.e. increasing u i , but also increasing the total value, since i 's marginal value is greater. Thus at equilibrium, i.e. when no one can unilat- erally increase P their utility, the total value is maximized.

4.3 Convergence

REFERENCES

[1] L. Zheng, C. Joe-Wong, C. W. Tan, S. Ha and M. Chiangs, *Secondary markets for mobile data: Feasibility and benefits of traded data plans*, 2015 IEEE Conference on Computer Communications (INFOCOM), Kowloon, 2015, pp. 1580-1588.

[2] A. A. Lazar and N. Semret, *Design and Analysis of the Progressive Second Price Auction for Network Bandwidth Sharing,* Telecommunication Systems, Special Issue on Network Economics, 2000.

[3] N. Semret, *Market Mechanisms for Network Resource Sharing, Ph.D. thesis.* Columbia University, 1999.

[4] Bruno Tuffin, *Revisited Progressive Second Price Auction for Charging Telecommunication Networks.* [Research Report] RR-4176, INRIA, 2001.

[5] Clare W. Qu, Peng Jia, and Peter E. Caines, *Analysis of a Class of Decentralized Decision Processes: Quantized Progressive Second Price Auctions*, 46th IEEE Conference on Decision and Control, New Orleans, LA, 2007, pp. 779-784.

[6] *China Mobile Hong Kong Company Limited.* https://www.hk.chinamobile.com/en/corporate_information/Data_N_VAS/DataTopup/index.html. 2018

[7] Andreu Mas-Colell, Michael Dennis Whinston, and Jerry R. Green, *Microeconomic Theory.* Oxford University Press. 1995. p. 794.

[8] Matthias Roggendorf and Fernando Beltrán, *Simulation Model for the Dynamic Allocation of Network Resources in a Competitive Wireless Scenario.* Mobility Aware Technologies and Applications, Springer Berlin Heidelberg. 2005. pp. 54 - 64.

[9] N. Semret, R. R. F. Liao, A. T. Campbell and A. A. Lazar, *Pricing, provisioning and peering: dynamic markets for differentiated Internet services and implications for network interconnections*, in IEEE Journal on Selected Areas in Communications, vol. 18, no. 12, pp. 2499-2513, Dec. 2000.

[10] Patrick Maillé. *Market Clearing Price and Equilibria of the Progressive Second Price Mechanism*, RAIRO - Operations Research, EDP Sciences, 2007, 41 (4), pp.465-478.