ATOC30004

Topic 6(contd):

Gravity waves

- Linear theory and gravity waves
- Convection waves
- Mountain waves

As we've seen, in many cases perturbations about the mean state are relatively small amplitude and can be examined using linear theory to understand their properties.

Separate flow fields into mean component and perturbation:

$$u = U + u'$$

$$w = W + w' = w'$$

$$\theta = \bar{\theta} + \theta'$$

$$B = \bar{B} + b = b$$

Start with 2D Boussinesq equations

(neglect horizontal and vertical variations in density term)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + B$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = 0$$

Substitute u = U + u' etc.

$$\frac{\partial (U+u')}{\partial t} + (U+u')\frac{\partial (U+u')}{\partial x} + w'\frac{\partial (U+u')}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + (U+u')\frac{\partial w'}{\partial x} + w'\frac{\partial w'}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial (\bar{\theta}+\theta')}{\partial t} + (U+u')\frac{\partial (\bar{\theta}+\theta')}{\partial x} + w'\frac{\partial (\bar{\theta}+\theta')}{\partial z} = 0$$

If U = const, $\bar{\theta} = \bar{\theta}(z)$ then

$$\frac{\partial u'}{\partial t} + (U+u')\frac{\partial u'}{\partial x} + w'\frac{\partial u'}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + (U+u')\frac{\partial w'}{\partial x} + w'\frac{\partial w'}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \theta'}{\partial t} + (U+u')\frac{\partial \theta'}{\partial x} + w'\frac{\partial \theta'}{\partial z} + w'\frac{\partial \bar{\theta}}{\partial z} = 0$$

Assuming small amplitude disturbances: i.e., products of perturbation quantities are negligibly small.

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \theta'}{\partial t} + U\frac{\partial \theta'}{\partial x} + w'\frac{\partial \bar{\theta}}{\partial z} = 0 \qquad (***)$$

These are the linearised equations of motion.

By multiplying (***) by

Recalling that $b = \frac{g}{\bar{\theta}(z)}\theta'$ then multiplying (***) by $\frac{g}{\bar{\theta}(z)}$ yields

$$\frac{\partial b}{\partial t} + U\frac{\partial b}{\partial x} + w'N^2 = 0, \text{ where } N^2 = \frac{g}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}$$

The linearised 2D Boussinesq equations:

(1) $$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

(2) $$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b$$

(3) $$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

(4) $$\frac{\partial b}{\partial t} + U\frac{\partial b}{\partial x} + w'N^2 = 0$$

Simple example -

U=0, neglect pressure variations. (2) and (4) become:

$$\frac{\partial w'}{\partial t} = b$$

$$\frac{\partial b}{\partial t} + w'N^2 = 0$$

Manipulate these:

$$\frac{\partial w'}{\partial t} = b$$

$$\frac{\partial b}{\partial t} + w'N^2 = 0$$

Take time derivative and manipulate.

$$\frac{\partial^2 w'}{\partial t^2} = \frac{\partial b}{\partial t} = -w'N^2$$

$$\frac{\partial^2 w'}{\partial t^2} + w'N^2 = 0$$

Single equation for w'

$$w' = Ae^{i\omega t} = Ae^{iNt}$$

$$\frac{\partial w'}{\partial t} = i\omega Ae^{i\omega t}, \text{ then } \frac{\partial^2 w'}{\partial t^2} = (i\omega)^2 Ae^{i\omega t} = -\omega^2 Ae^{i\omega t}$$

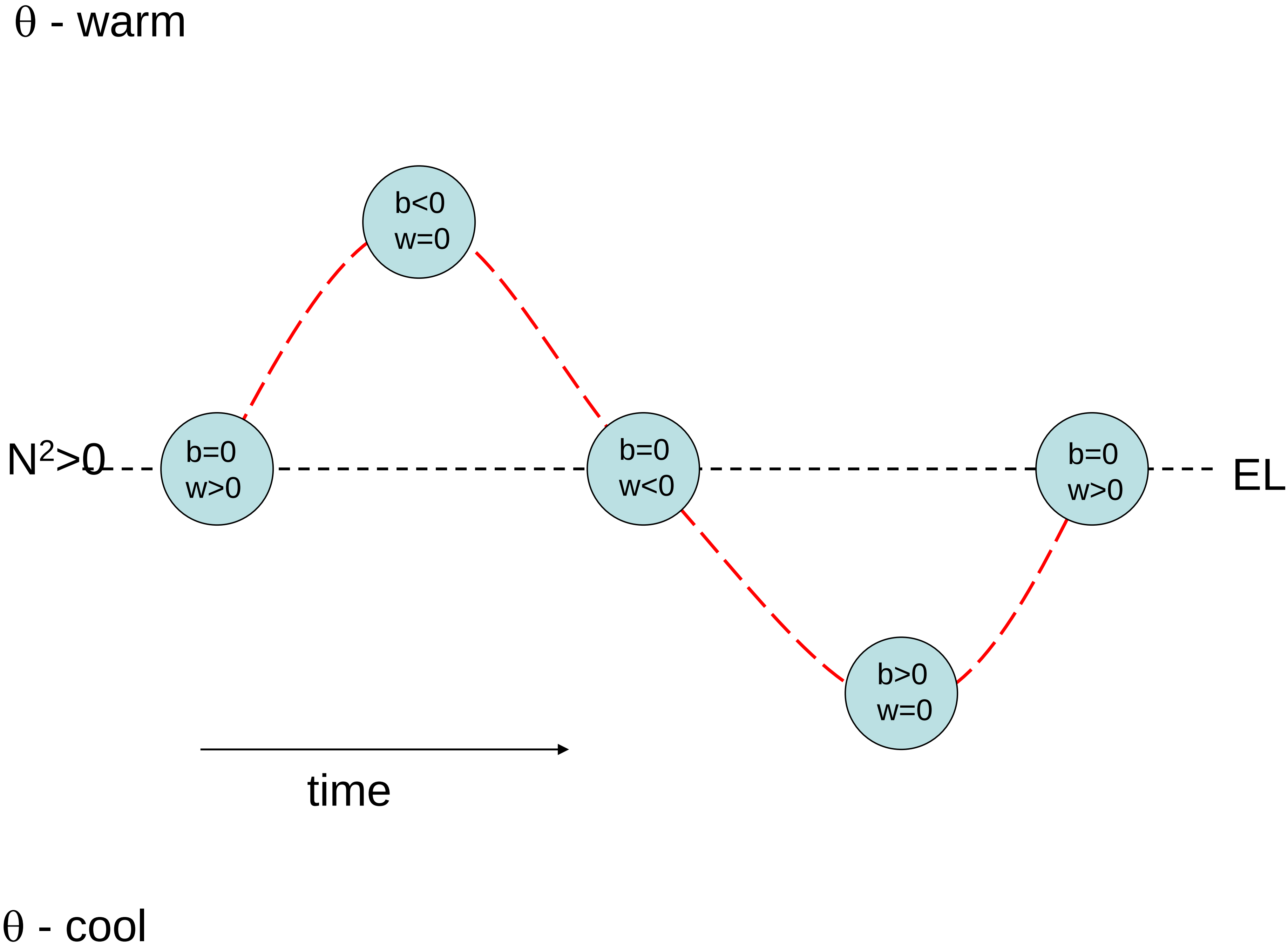
$$-\omega^2 Ae^{i\omega t} + Ae^{iNt} N^2 = 0$$

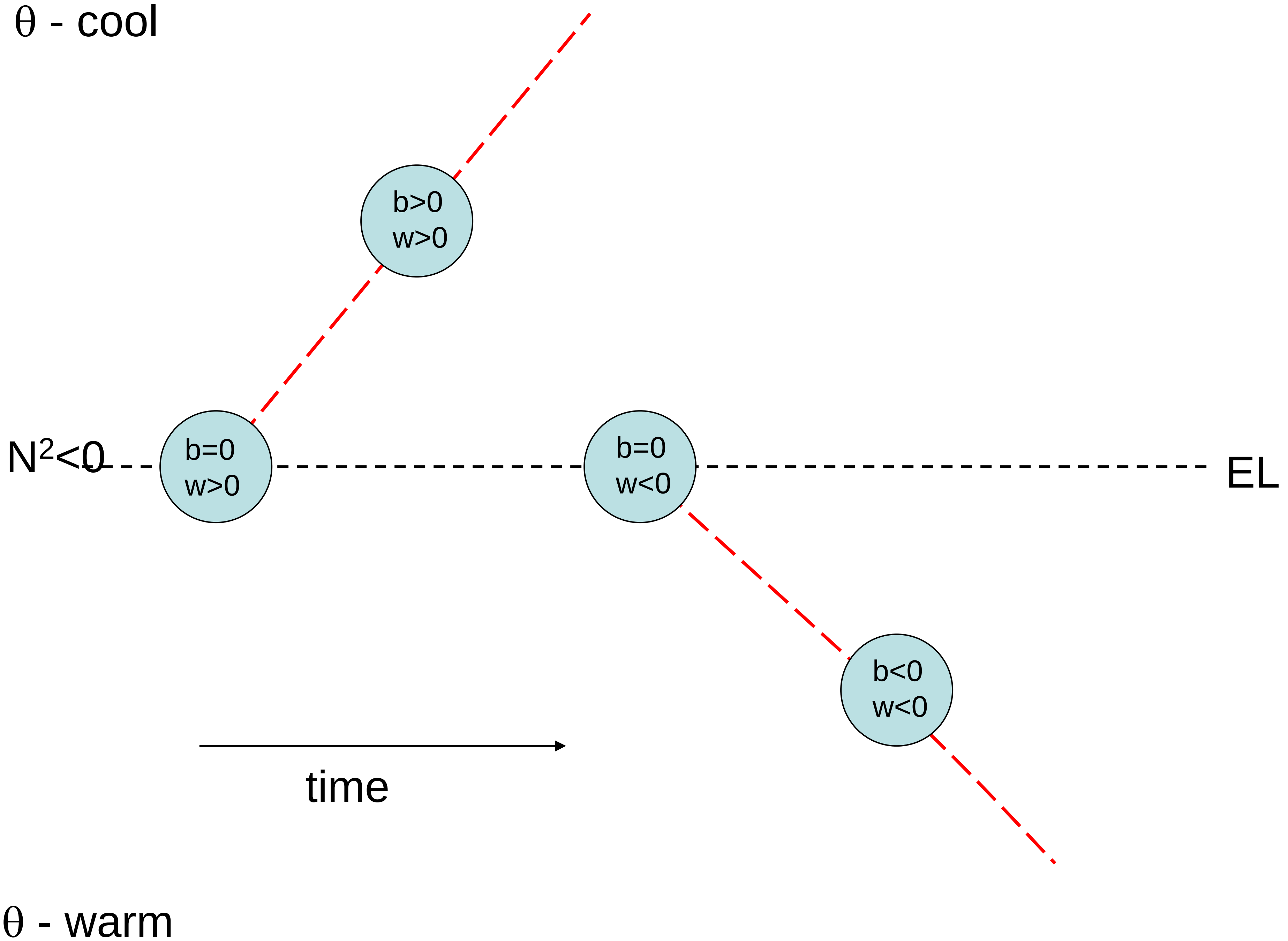
w' has solutions of the form

$$=> \omega^2 = N^2$$

Simple harmonic oscillator with frequency +/-N, provided N is real. Period of oscillation will be $2\pi/N$.

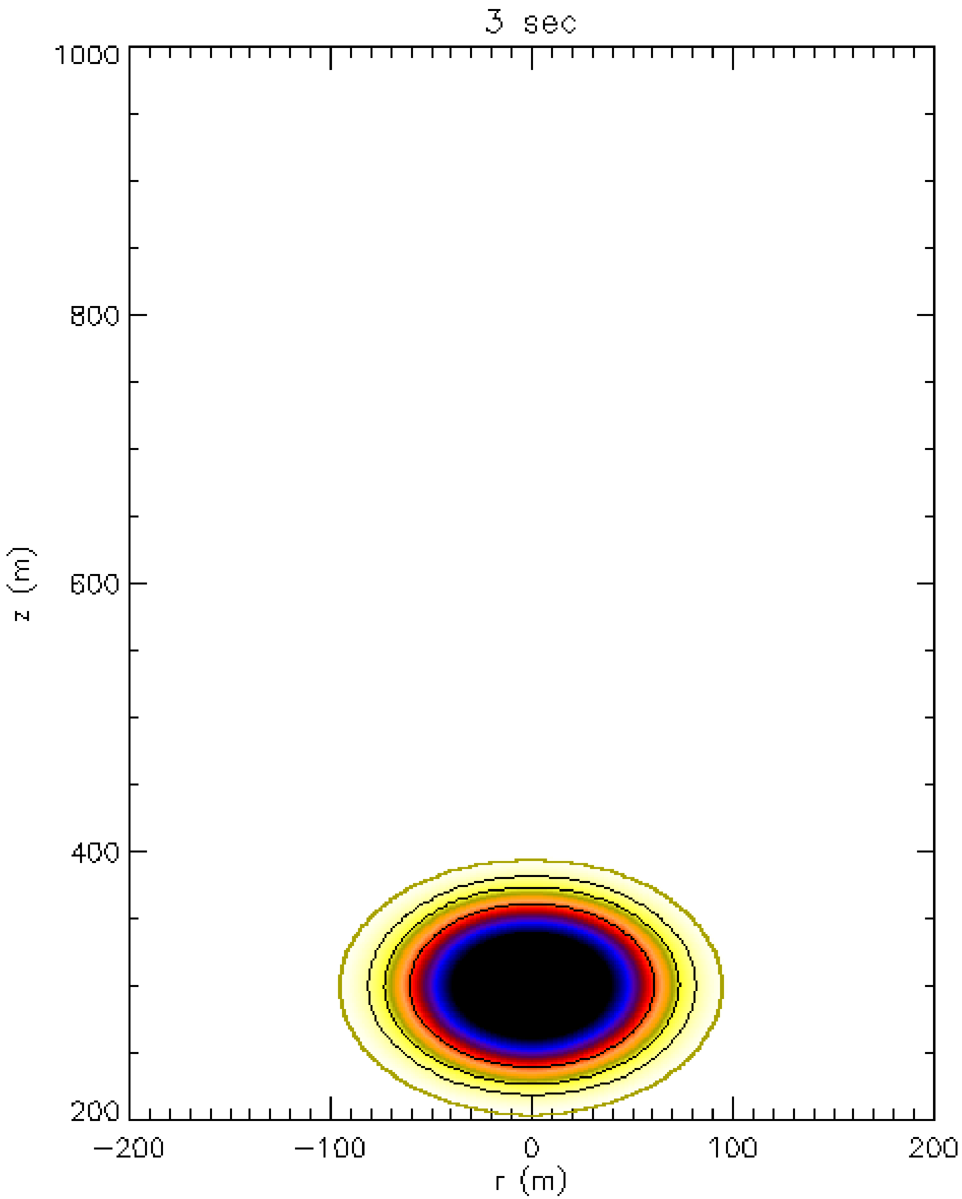
Otherwise, if N is complex then solution grows exponentially with time (i.e.,


θ - warm
b<0
w=0
N²>0
b=0
w>0
b=0
w<0
b=0
w>0
EL
b>0
w=0
time
θ - cool



θ - cool
b>0
w>0
N²<0
b=0
w>0
b=0
w<0
EL
b<0
w<0
time
θ - warm

N=0


3 sec
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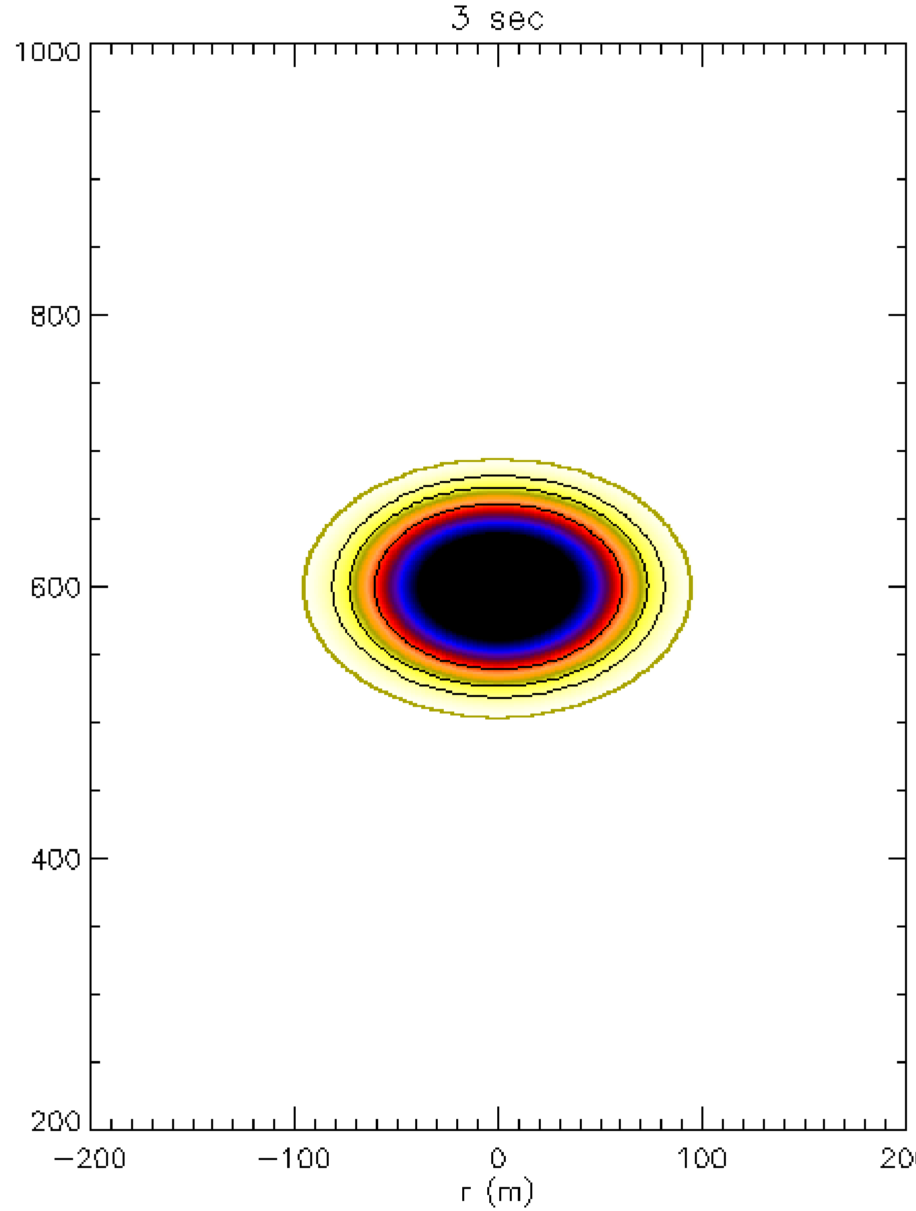
N=0.02 s^{-1}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r (m)

This oscillation is the underlying process behind gravity waves:

Gravity waves:

- oscillations in all dynamical variables
- the restoring force is buoyancy in a stably stratified fluid
- sourced from any ‘unbalanced’ process that produces a vertical parcel displacement
- the waves can ‘propagate’ horizontally and vertically
- ubiquitous in the atmosphere and ocean

Sources:

- orography / bathymetry
- dry, moist, and oceanic convection
- jets and fronts

The linearised 2D Boussinesq equations:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial b}{\partial t} + U\frac{\partial b}{\partial x} + w'N^2 = 0$$

We can use these equations to determine properties of gravity waves.

Want to derive a ‘wave equation’ and ‘dispersion relation’, which relates the frequency, horizontal, and vertical wavelength of gravity waves to properties of the background flow. i.e., what controls the character of these waves.

$$(1)\quad \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$(2)\quad \frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b$$

$$(3)\quad \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$(4)\quad \frac{\partial b}{\partial t} + U\frac{\partial b}{\partial x} + w'N^2 = 0$$

To eliminate pressure, take:

$$\frac{\partial}{\partial z}(1) - \frac{\partial}{\partial x}(2)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right) + U\frac{\partial^2 u'}{\partial x \partial z} - U\frac{\partial^2 w'}{\partial x^2} + \frac{\partial b}{\partial x} = 0 \qquad (5)$$

take $\frac{\partial}{\partial x}(5)$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u'}{\partial z \partial x} - \frac{\partial^2 w'}{\partial x^2}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial^2 u'}{\partial x \partial z} - \frac{\partial^2 w'}{\partial x^2}\right) + \frac{\partial^2 b}{\partial x^2} = 0$$

using (3) this becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 w'}{\partial z^2} + \frac{\partial^2 w'}{\partial x^2}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial^2 w'}{\partial z^2} + \frac{\partial^2 w'}{\partial x^2}\right) - \frac{\partial^2 b}{\partial x^2} = 0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)-\frac{\partial^2 b}{\partial x^2}=0$$

take $\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}$ of this

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)-\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial^2 b}{\partial x^2}=0$$

Take $\frac{\partial^2}{\partial x^2}$ of (4) gives:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial^2 b}{\partial x^2}+N^2\frac{\partial^2 w'}{\partial x^2}=0$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)+N^2\frac{\partial^2 w'}{\partial x^2}=0 \qquad (*)$$

A linear wave equation (involving only w'). We have combined 4 equations into 1!

The linear wave equation can be used to determine the possible solutions.
Let’s assume U=0

$$\left(\frac{\partial}{\partial t}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0$$

Substitute travelling wave solutions of the form:

$$w' = \tilde{w}(z)\exp\left[i(kx - \omega t)\right]$$

Where k is horizontal wavenumber. Gives….

$$\frac{\partial^2 \tilde{w}}{\partial z^2} + \left[\left(\frac{N^2}{(\omega)^2} - 1\right)k^2\right]\tilde{w} = 0 \quad \left\{\text{think } m^2 = \left[\left(\frac{N^2}{(\omega)^2} - 1\right)k^2\right]\right\}$$

This is a linear (constant coefficient for N, k, ω fixed) DE.

$$\frac{\partial^2 \tilde{w}}{\partial z^2} + \left(\frac{N^2}{(\omega)^2} - 1 \right) k^2 \tilde{w} = 0$$

Is a simplified form of the ***Taylor-Goldstein equation***. Which determines the vertical structure of wave solutions.

Note that in the above equation we can set $m^2 = \left(\frac{N^2}{(\omega)^2} - 1 \right) k^2$ where m is the vertical wavenumber. If $m^2 > 0$ we obtain solutions that are sinusoidal in the vertical; i.e.

$$\tilde{w}(z) = A \exp\left[i \left(k\sqrt{N^2/(\omega)^2 - 1} \right) z \right] + B \exp\left[-i \left(k\sqrt{N^2/(\omega)^2 - 1} \right) z \right] \qquad \text{for} \quad N^2/(\omega)^2 > 1$$

while if $m^2 < 0$ because $N^2/(\omega)^2 < 1$, m is imaginary and we obtain soultions that are exponential in the vertical.

$$\tilde{w}(z) = C \exp\left[\left(k\sqrt{1 - N^2/(\omega)^2} \right) z \right] + D \exp\left[-\left(k\sqrt{1 - N^2/(\omega)^2} \right) z \right] \qquad \text{for} \quad N^2/(\omega)^2 < 1$$

The first solution (involving A and B) is sinusoidal in the vertical and represents upwards and downwards propagating waves.

The second solution (involving C and D) contains an unphysical solution that grows exponentially with height (C). Part with D decreases exponentially with height.

Thus, there are two possible solutions here. A wave solution [$N^2/(\omega)^2 > 1$] and a vertically decaying solution [$N^2/(\omega)^2 < 1$] (which has C=0, D non-zero).

These two solutions identify a fundamental property of gravity waves.

Waves can only propagate vertically if: $|\omega| < N$

Alternatively, if **$|\omega| > N$** the signal will be ***evanescent***

In situations with U=0 this is a simple property - the frequency of the wave cannot exceed the Brunt-Vaisala frequency. (Think about the restoring force - energy is advected downstream instead of propagating).

If **$\omega = 0$** then we have no wave (for U=0).

So for U=0, the condition is $0 < |\omega| < N$, which states that the frequency of a vertically propagating wave cannot exceed the Brunt-Vaisala frequency.

If we assume purely vertically propagating solutions to the wave equation, they have the form:

$$w' = A\exp\left[i(kx + mz - \omega t)\right]$$

where m is the vertical wavenumber and assume $|U| > 0$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2}\right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0$$

To guess how $|U| > 0$ might affect the term $m^2 = \left(\frac{N^2}{(\omega)^2} - 1\right)k^2 = \frac{N^2 k^2}{(\omega)^2} - k^2$, consider the case of a wave that is not moving relative to the Earth (ω=0) but $U > 0$. In this case, each air parcel would travel from one crest to the next crest in the time (period) $T = \frac{\lambda_x}{U}$. Hence, from the perspective of an air parcel, there is a wave travelling in the negative $x-$direction with frequency $\frac{2\pi}{T} = \frac{2\pi}{\lambda_x}U = kU$.

Now consider the case where there is a wave moving in the positive $x-direction$ (the same direction as the wind) with speed $\frac{\omega}{k}$, then relative to the air-parcel there is a wave moving in the negative x-direction with the speed $U-\frac{\omega}{k}$ and the air parcel goes up and down with the frequency $k\left(U-\frac{\omega}{k}\right)=kU-\omega$. Based on this reasoning, it is unsurprising that a full derivation of the dispersion relation for the $U>0$ case gives

$$m^2=\frac{N^2k^2}{(\omega-Uk)^2}-k^2$$

$$=>\frac{m^2+k^2}{k^2}=\frac{N^2}{(\omega-Uk)^2}$$

This is known as the Dispersion relation, it relates all of the wave properties to one another.

$$w' = A\exp\left[i(kx+mz-\omega t)\right] = A\cos(kx+mz-\omega t) + iA\sin(kx+mz-\omega t)$$

Hence, maxima in w' occur wherever $kx+mz-\omega t = n2\pi.$

Thus, on a crest

$$mz = n2\pi - kx + \omega t$$

$$z = \frac{n2\pi}{m} - \frac{k}{m}x + \frac{\omega}{m}t$$

Hence, the slope of the line at which their is a crest is $-\dfrac{k}{m}$

and the $z-axis$ intercept is $\dfrac{n2\pi}{m} + \dfrac{\omega}{m}t.$

Since the phase of the wave is given by phase=$(kx+mz-\omega t)$, it follows that wave height changes most rapidly in the direction that the phase changes most rapidly. This direction is given by the gradient (or grad) of the phase; i.e. the vector

$$\nabla(kx+mz-\omega t) = k\mathbf{i} + 0\,\mathbf{j} + m\mathbf{k} = k\mathbf{i} + m\mathbf{k}.$$

Hence, the unit vector that is parallel to the wave propagation direction is given by

$$\mathbf{e}_w = \frac{\left(k\mathbf{i}+m\mathbf{k}\right)}{\sqrt{k^2+m^2}}$$

A position vector $\mathbf{r}(s)$ pointing from the origin to one of the wave crests, and which is perpendicular to the wave crests is given by

$\mathbf{r}(s) = s\mathbf{e}_w = \dfrac{(sk\mathbf{i} + sm\mathbf{k})}{\sqrt{k^2+m^2}}$. Hence, following this vector

$x = s\dfrac{k}{\sqrt{k^2+m^2}}$ and $z = s\dfrac{m}{\sqrt{k^2+m^2}}$;

hence, following this position vector

$$\phi = kx + mz - \omega t = \frac{k^2}{\sqrt{k^2+m^2}}s + \frac{m^2}{\sqrt{k^2+m^2}}s - \omega t = \frac{k^2+m^2}{\sqrt{k^2+m^2}}s - \omega t = \left(\sqrt{k^2+m^2}\right)s - \omega t = \kappa s - \omega t$$

where $\kappa = \sqrt{k^2+m^2}$ is called the *total* wavenumber. In space and time, the crests occur where $\phi = \kappa s - \omega t = n2\pi$. Hence3, moving with a crest,

$$s = \frac{1}{\kappa}(n2\pi + \omega t) = \frac{n2\pi}{\kappa} + \frac{\omega}{\kappa}t$$

Hence, the wave velocity perpendicular to the crests is

$$\mathbf{c} = \frac{\omega}{\kappa}\mathbf{e}_w$$

$$k = \frac{2\pi}{\lambda_x}$$

$$m = \frac{2\pi}{\lambda_z}$$

$$\omega = \frac{2\pi}{T}$$

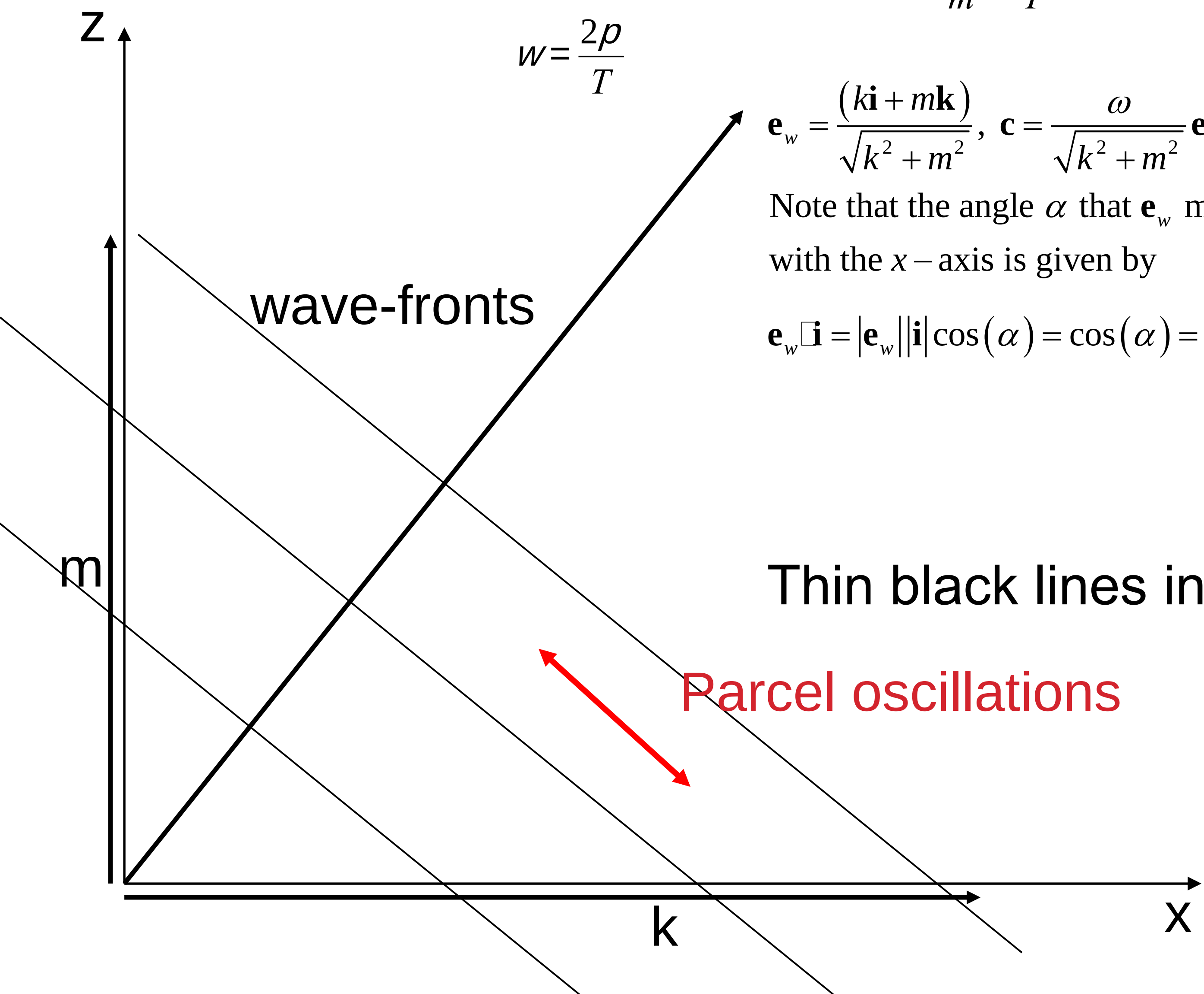
$$c = \frac{\omega}{k} = \frac{\lambda_x}{T}$$ Horizontal phase speed

$$c_z = \frac{\omega}{m} = \frac{\lambda_z}{T}$$ Vertical phase speed

$$\mathbf{e}_w = \frac{(k\mathbf{i} + m\mathbf{k})}{\sqrt{k^2+m^2}}, \ \mathbf{c} = \frac{\omega}{\sqrt{k^2+m^2}}\mathbf{e}_w$$

Note that the angle α that $\mathbf{e}_w$ makes with the $x-$axis is given by

$$\mathbf{e}_w \cdot \mathbf{i} = |\mathbf{e}_w||\mathbf{i}|\cos(\alpha) = \cos(\alpha) = \frac{k}{\sqrt{k^2+m^2}}$$



Thin black lines indicate w’ crests

The dispersion relation can be used to determine the angle of phase lines / wave fronts.

$$\mathbf{e}_w = \frac{(k\mathbf{i} + m\mathbf{k})}{\sqrt{k^2 + m^2}}, \quad \mathbf{c} = \pm \frac{\omega}{\sqrt{k^2 + m^2}} \mathbf{e}_w$$

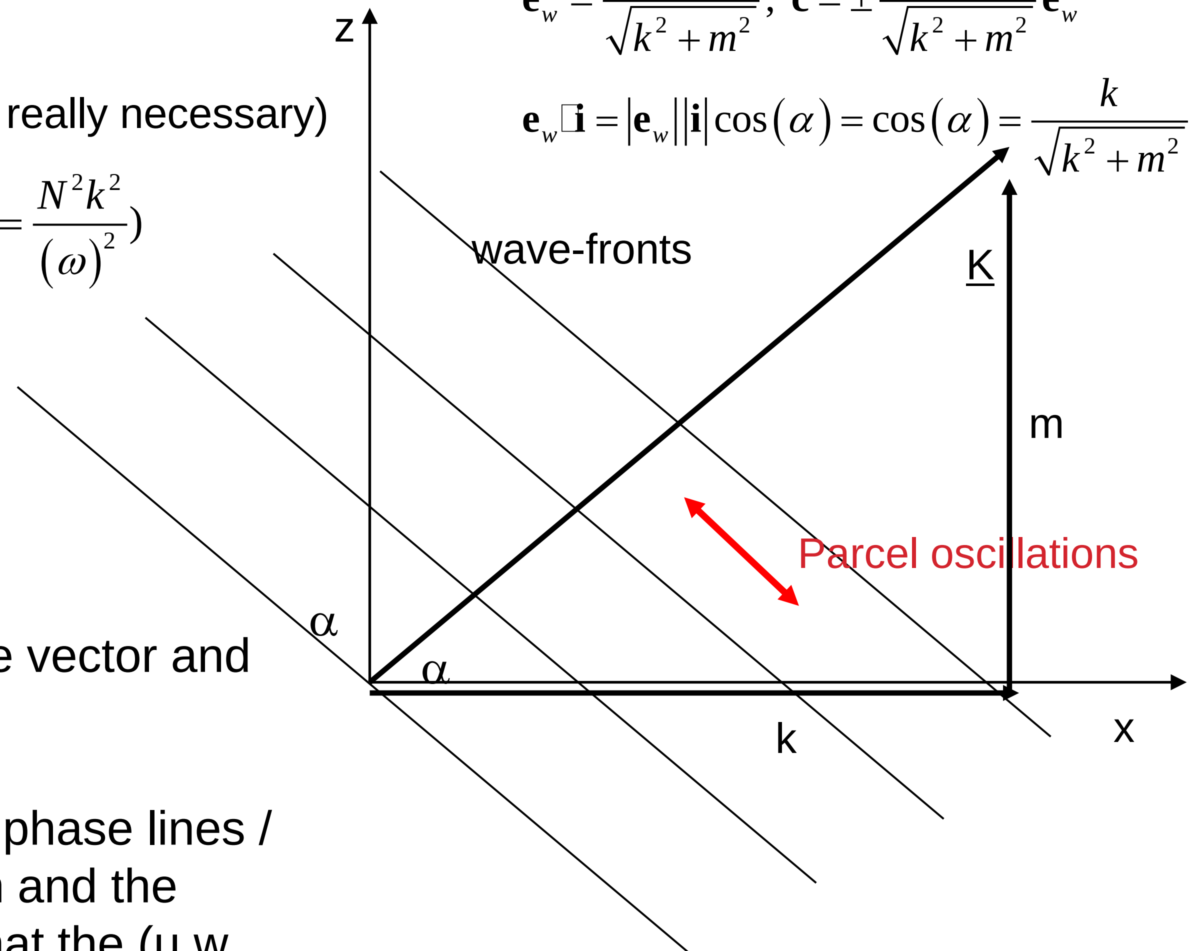
$$\mathbf{e}_w \cdot \mathbf{i} = |\mathbf{e}_w||\mathbf{i}|\cos(\alpha) = \cos(\alpha) = \frac{k}{\sqrt{k^2 + m^2}}$$

Assume U = 0 for simplicity (but not really necessary)

(Recall that $m^2 = \frac{N^2 k^2}{(\omega)^2} - k^2 \Rightarrow m^2 + k^2 = \frac{N^2 k^2}{(\omega)^2}$)

$$\Rightarrow \omega^2 = \frac{N^2 k^2}{m^2 + k^2}$$

$$\Rightarrow \frac{\omega}{N} = \frac{\pm k}{\sqrt{m^2 + k^2}} = \pm\cos(\alpha)$$



α is the angle between the wave vector and the horizontal plane.

α is also the angle between the phase lines / wave fronts / plane of oscillation and the vertical. (How could we prove that the (u,w velocity vector is parallel to the phase lines?)

i.e., as $\omega \rightarrow$ N phase lines / parcel oscillations are oriented vertically.

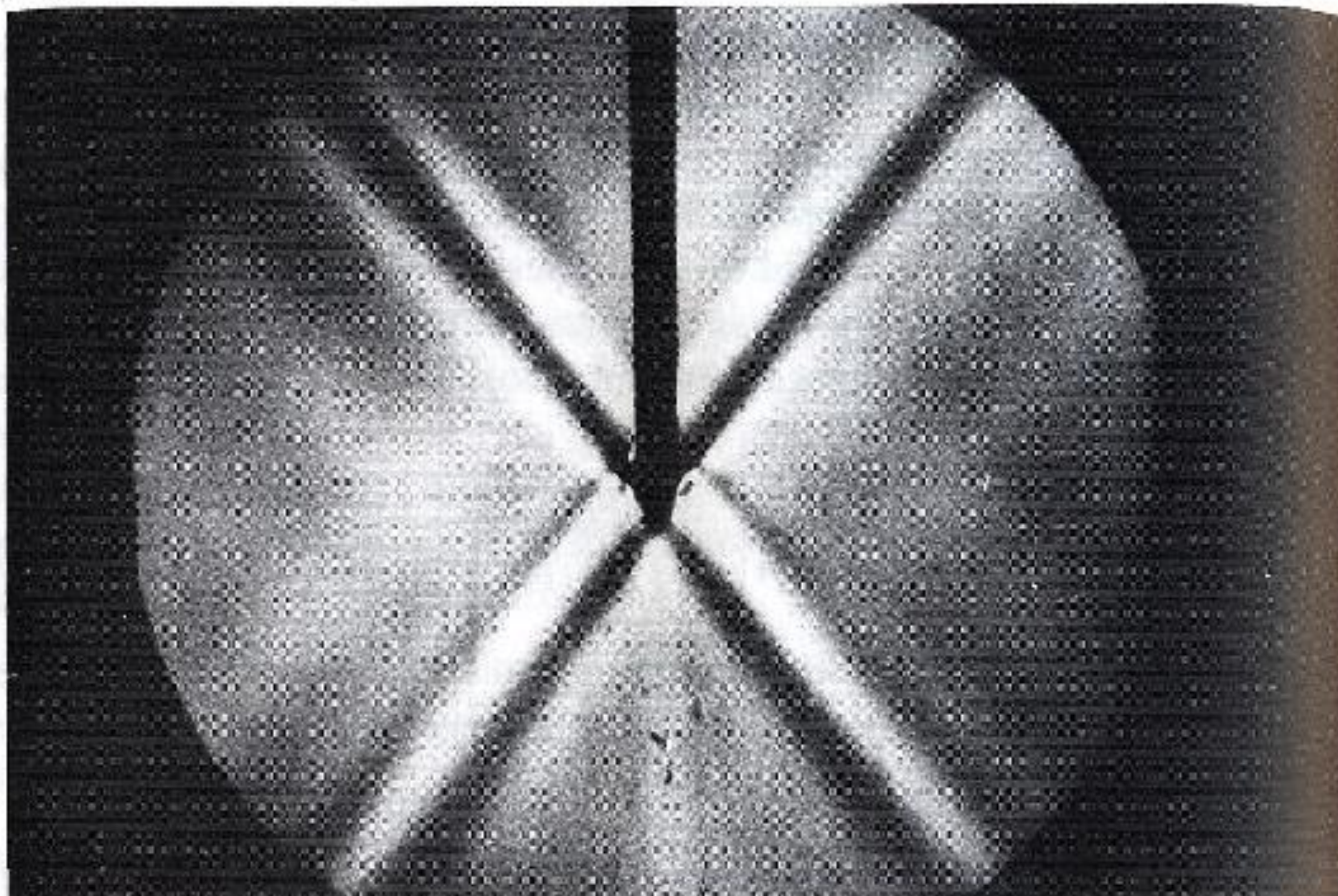


Figure 76. Schlieren picture of waves generated in stratified fluid of uniform Väisälä–Brunt frequency N by oscillation of a horizontal cylinder at frequency $0.70N$. Note that surfaces of constant phase stretch out radially from the source.
[*Photograph by D. H. Mowbray.*]

From Lighthill: Waves in Fluids

Oscillating cylinder has frequency $0.7N$. This will force a gravity wave into the fluid that has a frequency $0.7N$. According to the linear gravity wave theory we have developed here, a wave with this frequency would satisfy

$$\frac{\omega}{N} = \frac{0.7N}{N} = 0.7 = \frac{\pm k}{\sqrt{m^2 + k^2}} = \pm\cos(\alpha)$$

$$=> \alpha = \cos^{-1}\left(0.7\right) = 45.5$$

This angle is close to what is visually seen in the figure on the previous slide.

St Andrew's cross

314 *Internal waves* [4·4

z

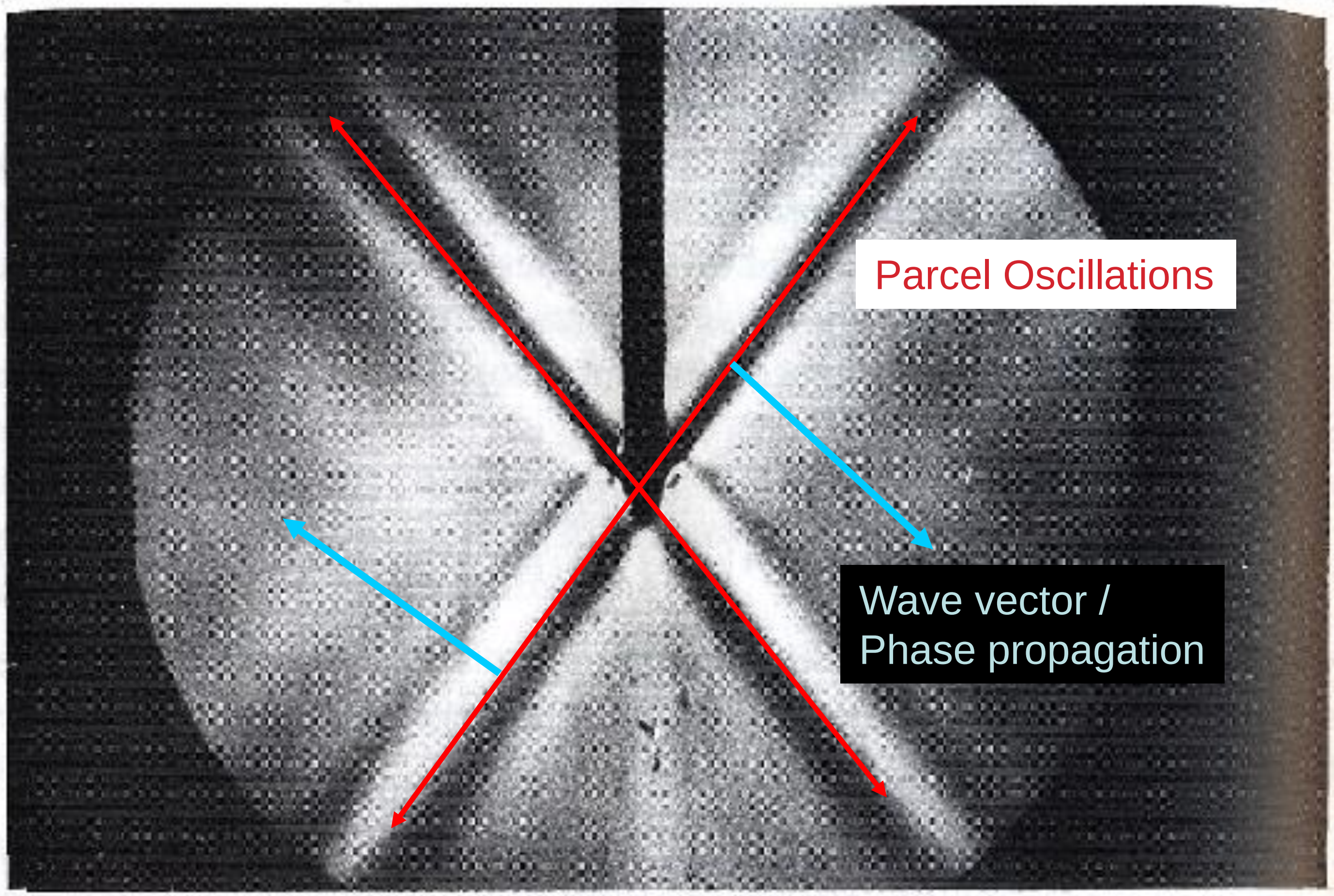
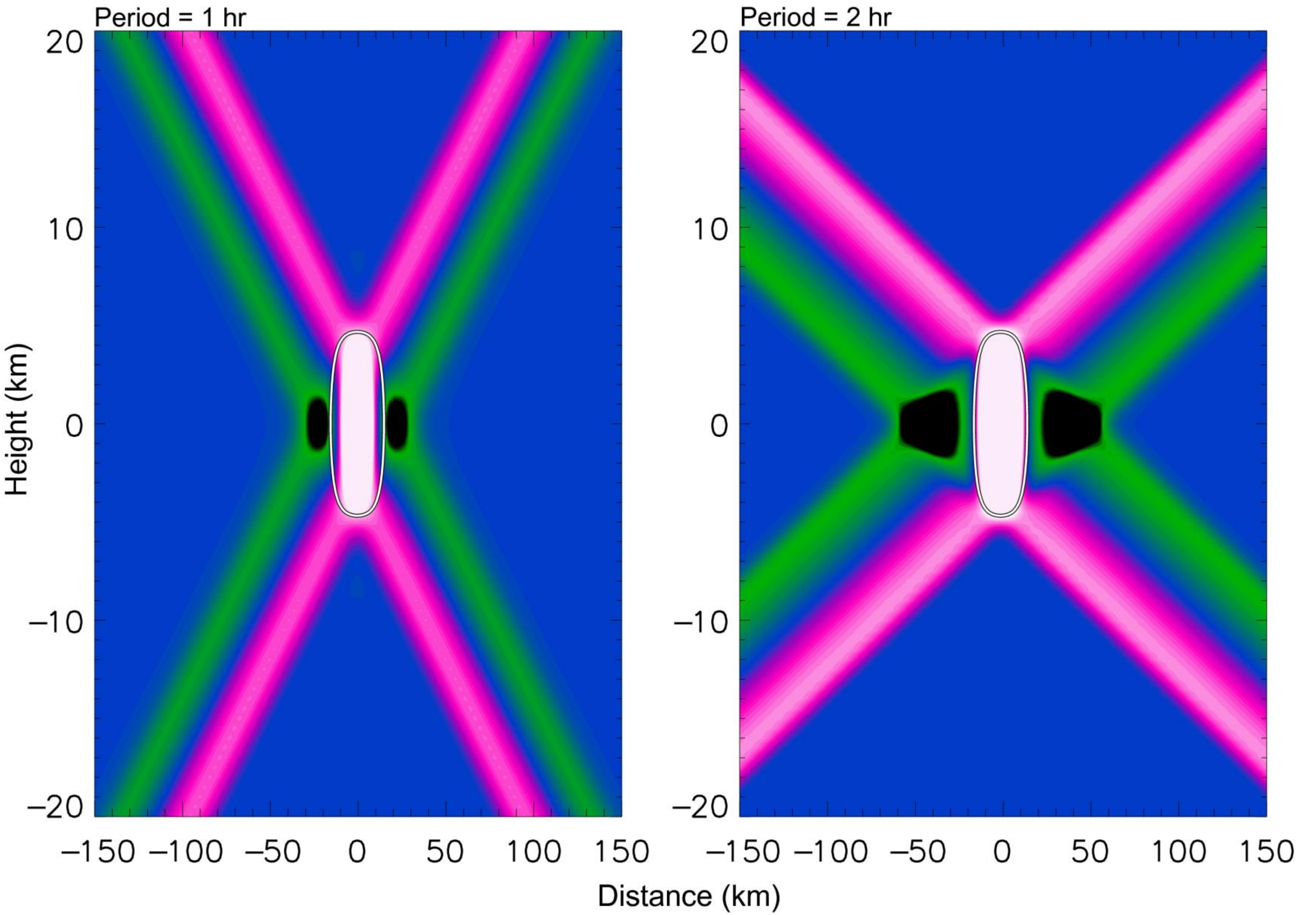


Figure 76. Schlieren picture of waves generated in stratified fluid of uniform Väisälä–Brunt frequency N by oscillation of a horizontal cylinder at frequency $0.70N$. Note that surfaces of constant phase stretch out radially from the source. [*Photograph by D. H. Mowbray.*]

x

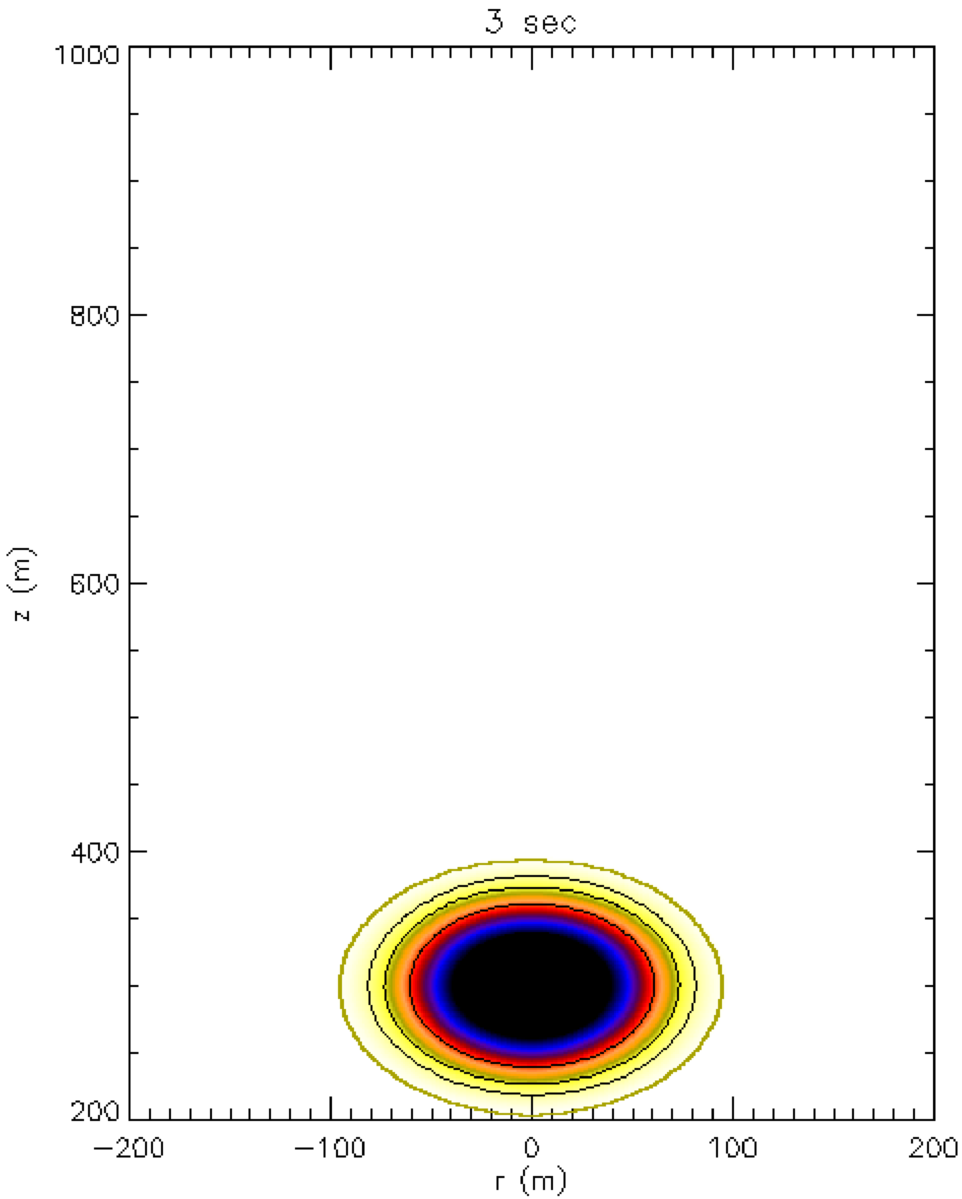
From Lighthill: Waves in Fluids

$N^2=0.01s^{-1}$



Period = 1 hr
Period = 2 hr
Height (km)
20
10
0
−10
−20
−150 −100 −50 0 50 100 150
Distance (km)

N=0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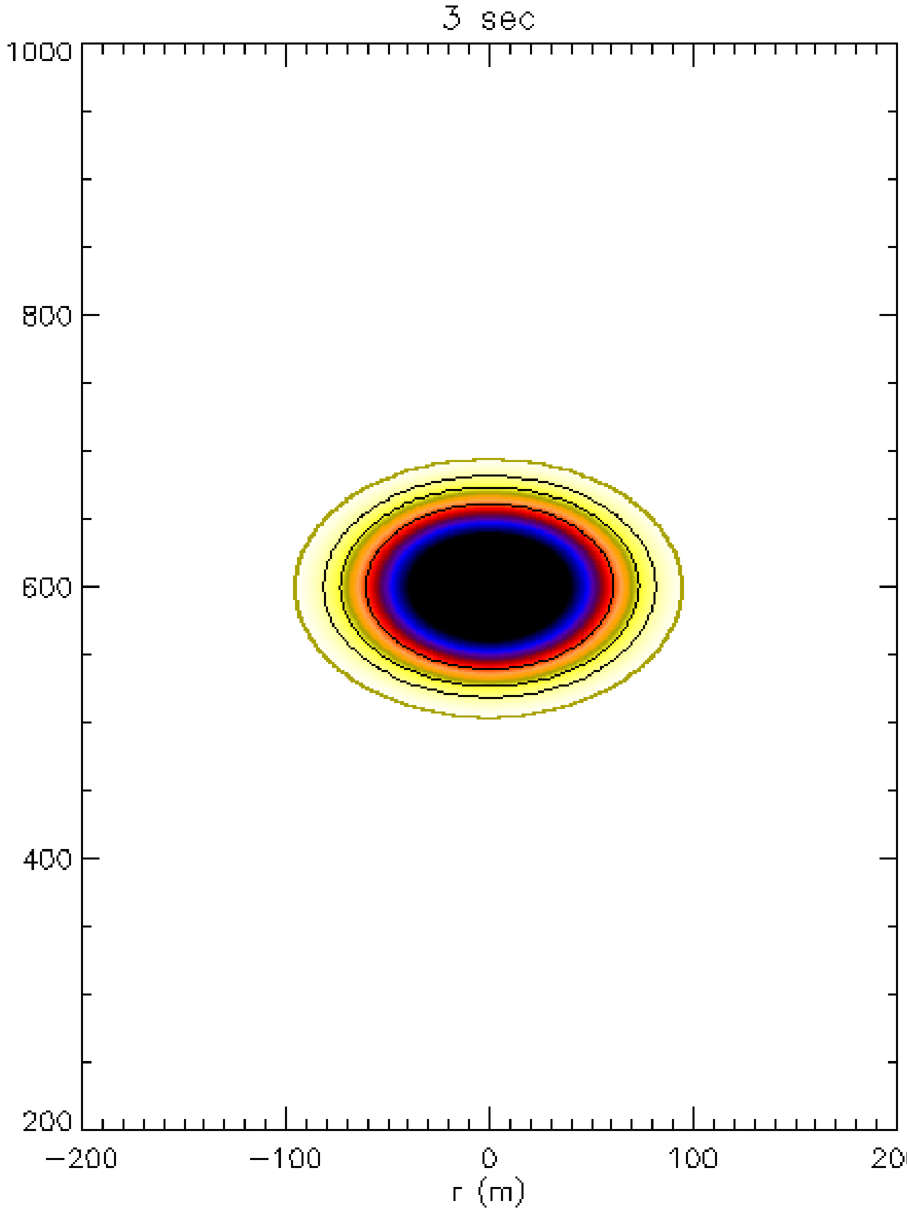
r (m)

N=0.02 s^{-1}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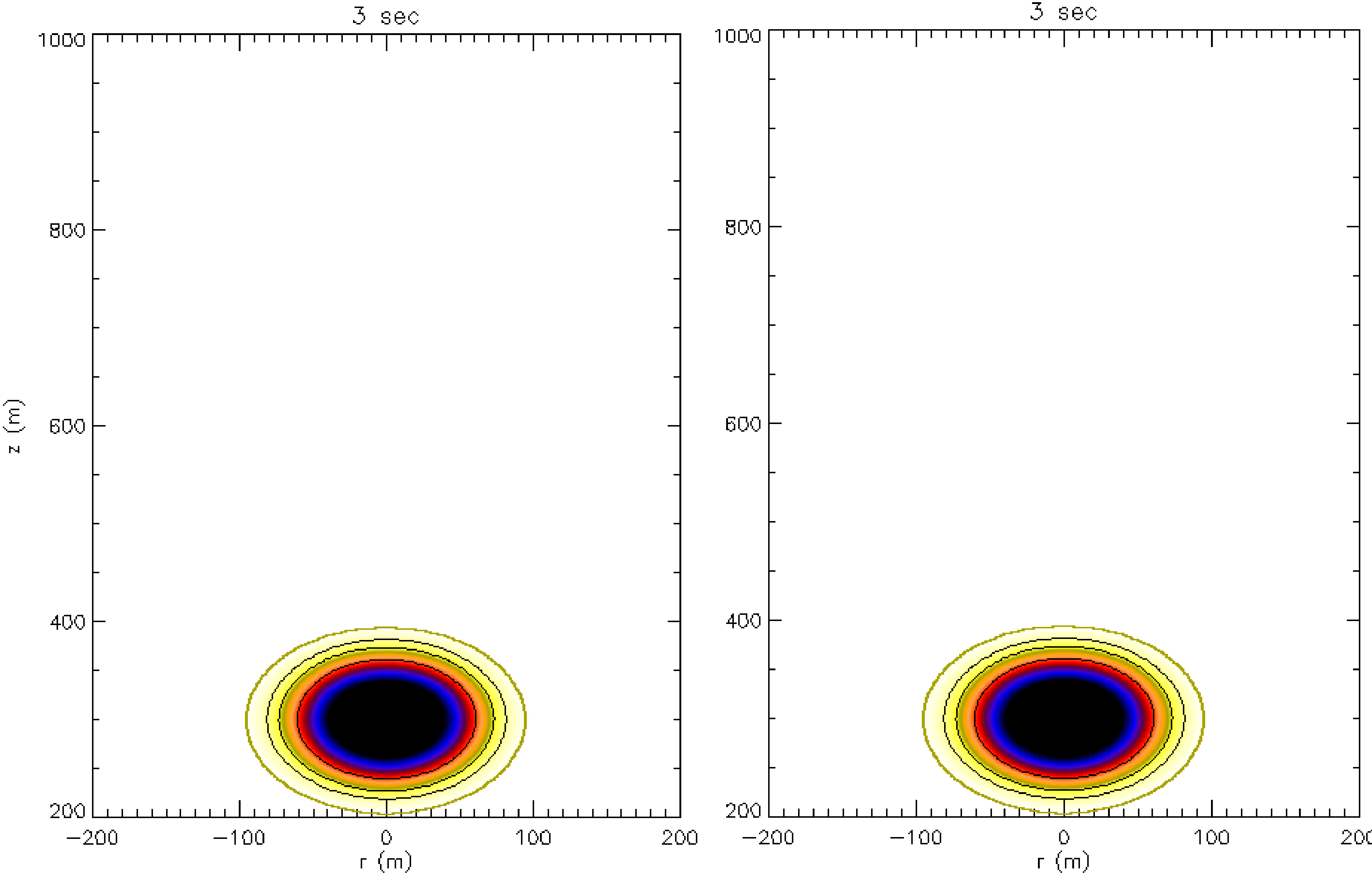
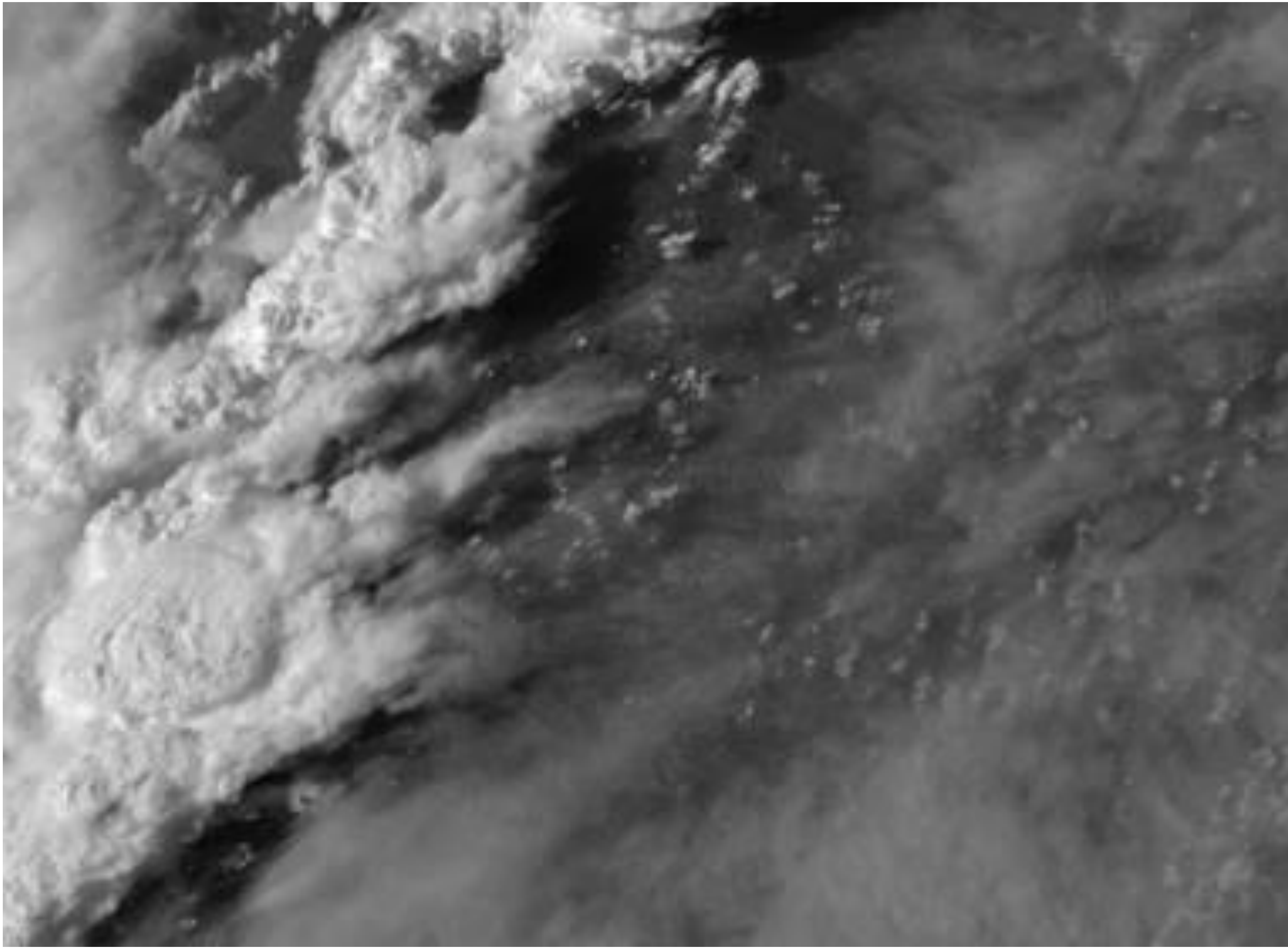
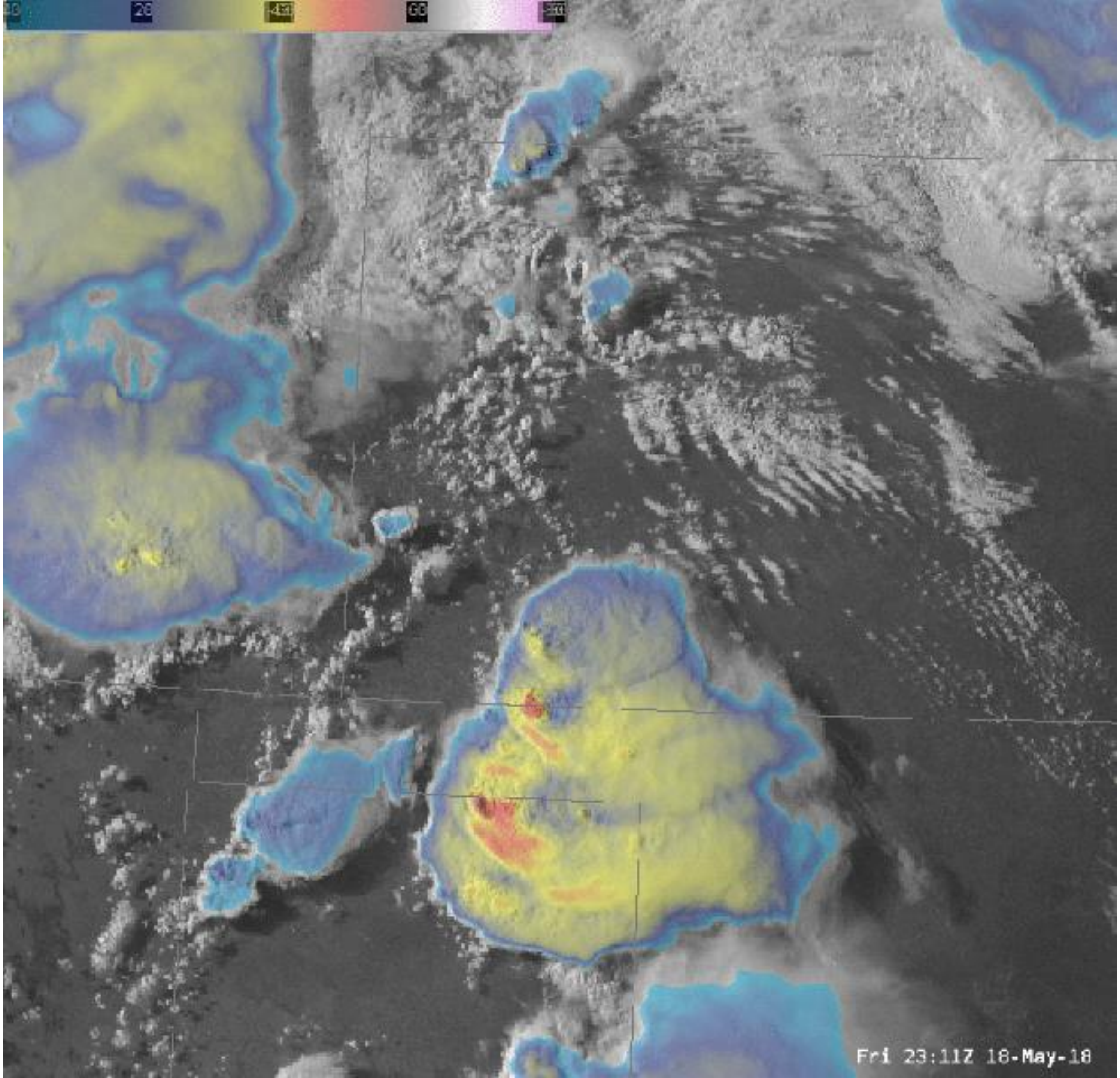
r (m)

N=0 N=0.02 s^{-1} z> 500 m



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Mountain waves.

Steady flow over a corrugated surface.

Let’s return to a simplified form of the **Taylor-Goldstein equation** with U≠0:

$$\frac{\partial^2 \tilde{w}}{\partial z^2} + \left(\frac{N^2 k^2}{(\omega - Uk)^2} - k^2 \right) \tilde{w} = 0$$

where it follows from before for sinusoidal solutions that:

$$m^2 = \frac{N^2 k^2}{(\omega - Uk)^2} - k^2 \;\; => m^2 + k^2 = \frac{\frac{1}{k^2} N^2 k^2}{\frac{1}{k^2} (\omega - Uk)^2} = \frac{N^2}{\left(\frac{\omega - Uk}{k} \right)^2} = \frac{N^2}{(c - U)^2} = \frac{N^2}{(U - c)^2}$$

Let's make a few approximations:

1 - Hydrostatic: $\lambda_x >> \lambda_z$ or equivalently k << m.

$$m^2 + k^2 = \frac{N^2 k^2}{(w - Uk)^2} \quad \text{becomes} \quad m^2 = \frac{N^2}{(U - c)^2}$$

2 - Steady solution. Waves remain stationary with respect to ground, i.e., directly over terrain. (c=0) or equivalently (ω=0).

$$m^2 = \frac{N^2}{(U - c)^2} \quad \text{becomes} \quad m^2 = \frac{N^2}{U^2}$$

Thus (for N^2>0), $$|m| = \left|\frac{N}{U}\right|$$

The solution for U>0 is therefore: $$|m| = \left|\frac{N}{U}\right|$$

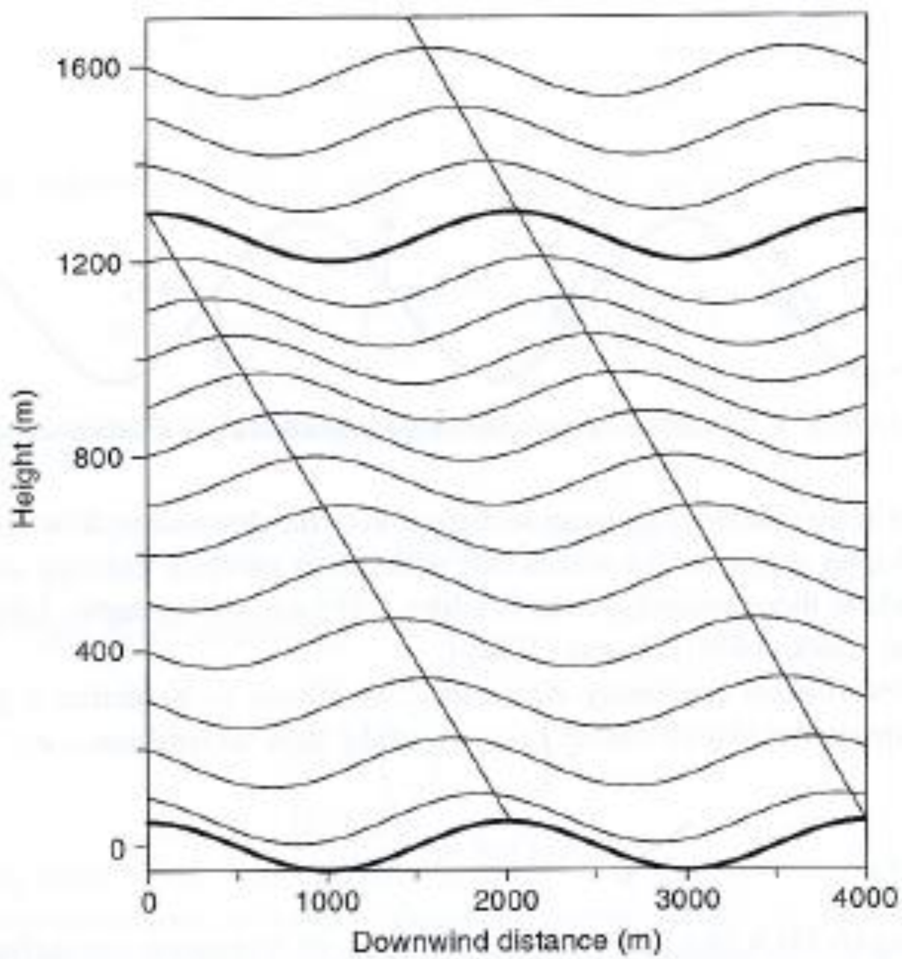
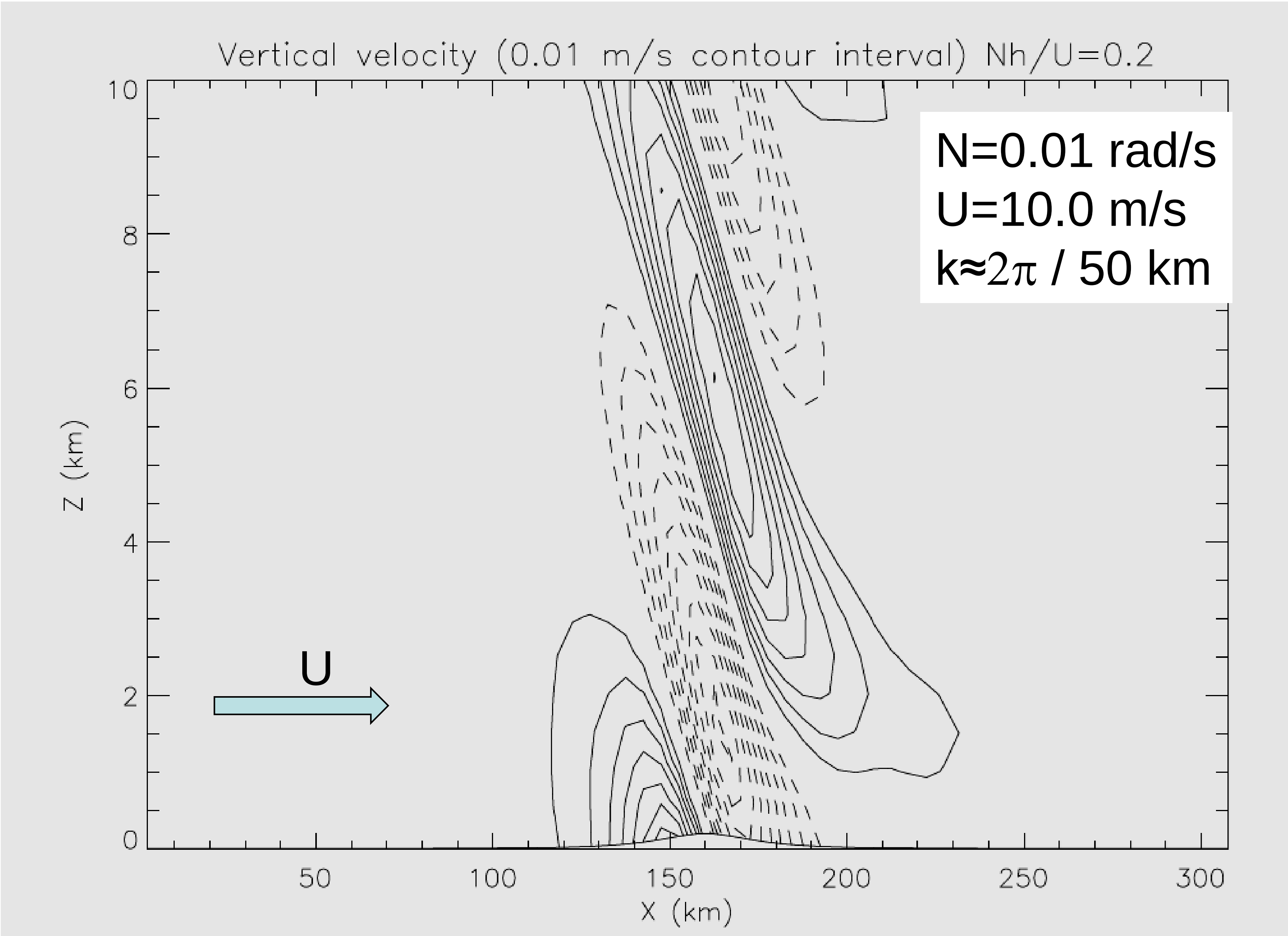


FIGURE 3.8 Streamline displacements over a surface corrugation with $H = 50$ m and wavelength $\lambda_s = 2000$ m. $N = 0.023\ s^{-1}$ and $u_0 = 4\ m\ s^{-1}$.

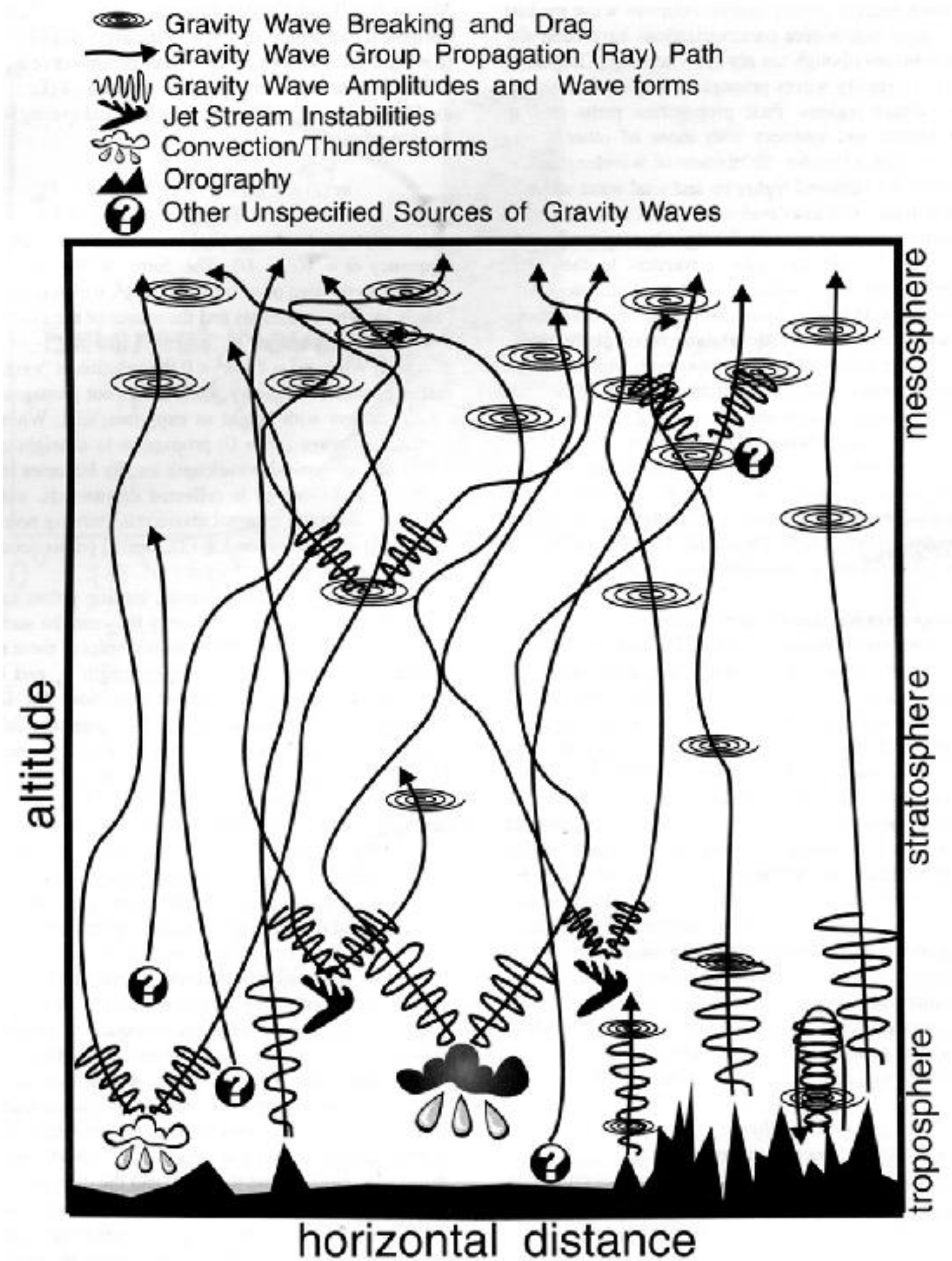
The horizontal wavelength is defined by the scale of the terrain. The vertical wavelength is defined by the atmospheric conditions.

Nappo

This result generalizes to localised mountains also.



Hydrostatic dispersion relation ($|m|=N/U$) gives: $\lambda_z \approx 6.3$ km



Kim et al. 2003

Summary – Gravity waves:

•Linear theory determines the dispersion relation, which constrains the properties of the waves.

•For U=0, the wave frequency must be between 0 and N for vertically propagating solutions.

•The ratio of wave frequency to N determines spatial character of waves

•For mountain waves (hydrostatic) N and U determine vertical structure of wave solution.

Summary wave dynamics.

Shallow water and Rossby waves

- Linear approximation
- Formation of wave equation
- Dispersion relation and phase speed

Inertial oscillations

Gravity waves

Prac questions: Friday, May 20.

1. Consider air blowing across a sinusoidal mountain of wavenumber k at the speed $U > 0$.
If the gravity wave is stationary (ω=0, $c = 0$), what is the slope of the wave?
2. Derive a wave equation for the buoyancy field associated with the w' field.
3. Referencing the notes from Holton posted in the modules section prove that

$$\left[\exp\left[i\left(\delta kx - \delta\omega t\right)\right] + \exp\left[-i\left(\delta kx - \delta\omega t\right)\right]\right]\exp\left[i\left(kx - \omega t\right)\right]$$

$$= 2\cos\left(\delta kx - \delta\omega t\right)\exp\left[i\left(\delta kx - \delta\omega t\right)\right]$$

The above analysis shows that the envelopes containing wave energy move at
the group velocity is given by

$$\mathbf{c}_g = \frac{\partial\omega}{\partial k}\mathbf{i} + \frac{\partial\omega}{\partial l}\mathbf{j} + \frac{\partial\omega}{\partial m}\mathbf{k}$$

4. Compute the vertical component of the group velocity for the
general gravity wave dispersion relation

$$m^2 = \frac{N^2k^2}{\left(\omega - Uk\right)^2} - k^2$$

(Hint:Rearrange the above equation to make ω the subject)
5. Draw the w' and b' phase lines for a gravity whose vertical component of group velocity
is positive in the limit of ω=0.