ATOC30004: Assignment 4 (12.5%: Due 1159 pm Friday, May 27)

Wave Dynamics

1. (30 marks total) **Group velocity, carrier waves, envelope waves**

For any given wave supporting dynamical system, the frequency ω is usually a function of the wavenumber. In such circumstances, the frequency of a wave with wavelength $k+\delta k$ can be approximated by $\omega(k)+\frac{\partial\omega}{\partial k}\delta k$ provided δk is small. Now consider the superposition of two similar waves. One wave has the wavenumber and frequency given by $(k+\delta k)$ and $\left[\omega(k)+\frac{\partial\omega}{\partial k}\delta k\right]$, respectively, while the other wave has wavenumber and frequency given by $(k-\delta k)$ and $\left[\omega(k)-\frac{\partial\omega}{\partial k}\delta k\right]$, respectively. Consider the superposition of two such waves given by

$$\psi=\exp\left\{i\left[(k+\delta k)x-\left(\omega(k)+\frac{\partial\omega}{\partial k}\delta k\right)t\right]\right\}+\exp\left\{i\left[(k-\delta k)x-\left(\omega(k)-\frac{\partial\omega}{\partial k}\delta k\right)t\right]\right\}$$
$$=\left(\exp\left\{i\left[(\delta k)x-\frac{\partial\omega}{\partial k}(\delta k)t\right]\right\}+\exp\left\{-i\left[(\delta k)x-\frac{\partial\omega}{\partial k}(\delta k)t\right]\right\}\right)\exp\left[i(kx-\omega t)\right]$$

(a) (6 marks) Prove that this equation can be simplified to the form

$$\psi=2\cos\left[(\delta k)x-(\delta k)\frac{\partial\omega}{\partial k}t\right]\exp\left[i(kx-\omega t)\right] \qquad (1.1)$$

(You must show your working to get full marks for this question)

Answer: Because $\left(\exp\{i\theta\}+\exp\{-i\theta\}\right)=(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)=2\cos\theta$, it follows that

$$\psi=\left(\exp\left\{i\left[(\delta k)x-\frac{\partial\omega}{\partial k}(\delta k)t\right]\right\}+\exp\left\{-i\left[(\delta k)x-\frac{\partial\omega}{\partial k}(\delta k)t\right]\right\}\right)\exp\left[i(kx-\omega t)\right]$$
$$=2\cos\left[(\delta k)x-\frac{\partial\omega}{\partial k}(\delta k)t\right]\exp\left[i(kx-\omega t)\right]$$

as was required.

(1b) (4 marks) What is the formula for the wavelength of the $\cos\left[(\delta k)x-\frac{\partial\omega}{\partial k}(\delta k)t\right]$ wave?

Answer: $\lambda_{envelope}=\frac{2\pi}{\delta k}$

(1c) (4 marks) What is the speed c_x^g of the $\cos\left[(\delta k)x - \frac{\partial\omega}{\partial k}(\delta k)t\right]$ wave?

Answer: $c_x^g = \frac{\frac{\partial\omega}{\partial k}(\delta k)}{\delta k} = \frac{\partial\omega}{\partial k}$

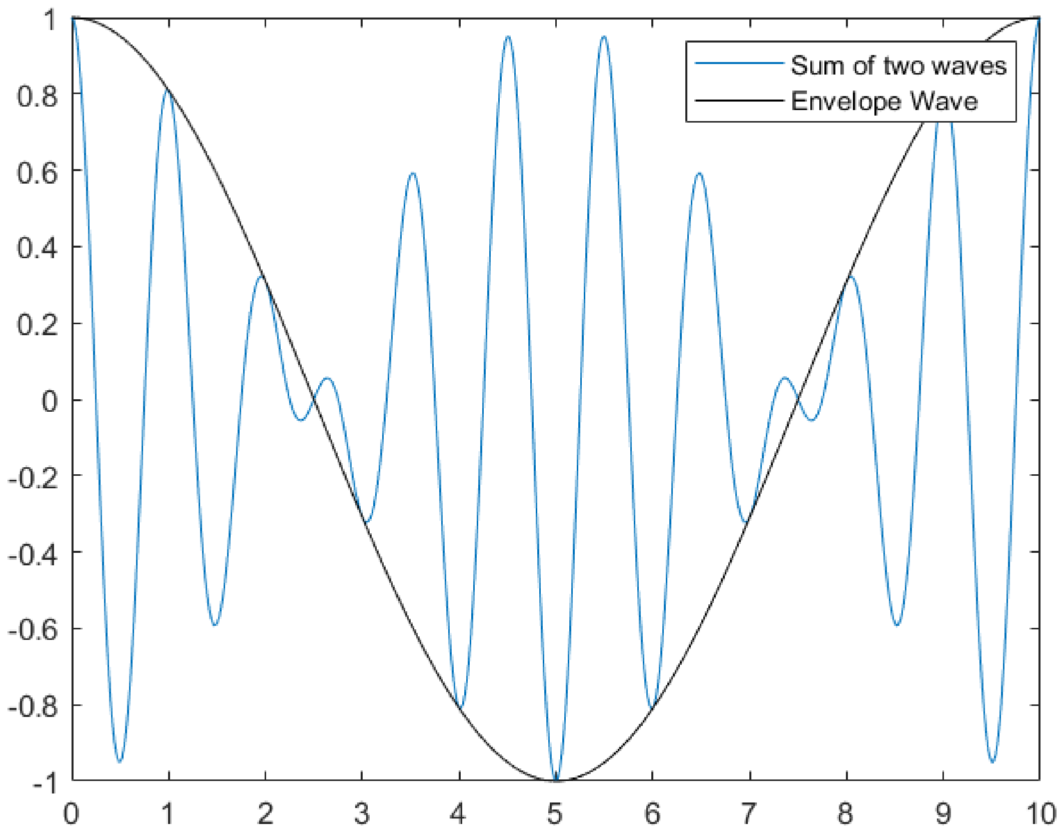
(1d) (4 marks) What is the phase speed c_x of the $\exp\left[i(kx-\omega t)\right]$ wave?

Answer: $c_x = \frac{\omega}{k}$

(1e) (12 marks) Use Matlab to (i) plot the real part of $\psi = 2\cos\left[(\delta k)x - (\delta k)\frac{\partial\omega}{\partial k}t\right]\exp\left[i(kx-\omega t)\right]$ from x=0 to $x = \frac{2\pi}{\delta k}$ at the time t=0 (plot this line in black), and (ii) add to the plot another plot of the envelope wave given by $\psi = 2\cos\left[(\delta k)x - (\delta k)\frac{\partial\omega}{\partial k}t\right]$ (plot this line in black). For simplicity, for plotting purposes, assume that $k = 2\pi$ and that $\delta k = 2\pi/10$.

You can use Matlab/OtherProgram or a hand drawn figure for your answer.

Answer:



I used Matlab but full marks will be given to a simple hand drawn figure that gets across the general idea of the above. You would not be required to draw the envelope wave but it might help you draw the figure by hand if you actually do draw one.

2. (30 marks) Assuming barotopic non-divergent flow, the horizontal flow can be expressed in terms of the streamfunction ψ such that $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$

(2a) (4 marks) If $u = -\frac{\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$ then deduce the mathematical expression for the relative vorticity $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ in terms of ψ .

Answer:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

(2b) (7 marks) Assuming $\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$ about a uniform zonal flow $U = -\frac{\partial \bar{\psi}}{\partial y}$, equation $\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \beta v = 0$ simplifies to

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2}\right) + \beta\frac{\partial \psi'}{\partial x} = 0 \qquad (2.1)$$

where primed quantities represent small amplitude perturbations and where $\beta = \frac{\partial f}{\partial y}$.

Assuming that $\psi' = A\cos(kx + ly - \omega t)$, use equation (2.1) to prove that

$\omega = Uk - \frac{\beta k}{(k^2 + l^2)}$ for such waves. (You must show your working to receive full marks on this question.)

Answer:

$$\left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2}\right) = -(k^2 + l^2)A\cos(kx + ly - \omega t) = -(k^2 + l^2)\psi'$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2}\right) = (Uk - \omega)(k^2 + l^2)A\sin(kx + ly - \omega t)$$

$$\beta\frac{\partial \psi'}{\partial x} = -\beta k A\sin(kx + ly - \omega t)$$

Hence,

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}\right)+\beta\frac{\partial\psi'}{\partial x}$$

$$\left[(Uk-\omega)\left(k^2+l^2\right)-\beta k\right]A\sin(kx+ly-\omega t)=0$$

$$=>\left[(Uk-\omega)\left(k^2+l^2\right)-\beta k\right]=0$$

$$=>(Uk-\omega)\left(k^2+l^2\right)=\beta k$$

$$=>\omega-Uk=-\frac{\beta k}{\left(k^2+l^2\right)}$$

$$=>\omega=Uk-\frac{\beta k}{\left(k^2+l^2\right)}$$

As was required (QED)

(2c) (4 marks, 2 marks for part (i), 1 mark for part (ii))

(i) For a arbitrary wave of the form $\psi=A\cos(kx+ly-\omega t)$, prove that the speed of the wave perpendicular to the wave crests is $c=\frac{\omega}{\kappa}$, where $\kappa=\sqrt{k^2+l^2}$ is the total wavenumber.

Wave crests occur on isolines of the linear function
$\theta=kx+ly-\omega t.$ (*)
Specifically, they occur where $\theta=n2\pi$. Isolines of θ are perpendicular to the unit vector $\mathbf{n}=\frac{\nabla_h\theta}{|\nabla_h\theta|}=\frac{k\mathbf{i}+l\mathbf{j}}{\sqrt{k^2+l^2}}$.

The the angle ξ between the +ve x-axis and $\mathbf{n}$ satisfies

$\tan(\xi)=\left(\frac{l}{k}\right)$, $\cos\xi=\frac{k}{\sqrt{k^2+l^2}}$ and $\sin\xi=\frac{l}{\sqrt{k^2+l^2}}$. Hence, if r is the distance along the line going from the origin to the crest of a wave, the coordinates of this point are given by

$x=r\cos\xi=r\frac{k}{\sqrt{k^2+l^2}}$ and $y=r\sin\xi=r\frac{l}{\sqrt{k^2+l^2}}$ (**)

Substituting (**) in (*) gives

$$\theta=kx+ly-\omega t=r\left(\frac{k^2}{\sqrt{k^2+l^2}}+\frac{l^2}{\sqrt{k^2+l^2}}\right)-\omega t=\kappa r-\omega t \quad (***)$$

One crest occues at $\theta=0$. On this crest

$0=\kappa r-\omega t=>r=\frac{\omega}{\kappa}t$ so the speed of the wave perpendicular to the crest is $c=\frac{\omega}{\kappa}$, QED

(ii) Use the Rossby wave dispersion relation $\omega = Uk - \frac{\beta k}{\left(k^2 + l^2\right)}$ and $c = \frac{\omega}{\kappa}$ to give the expression for the Rossby wave speed perpendicular to the crest of the wave. State whether the wave is moving towards the West or the East relative to the basic state flow.

Answer: $c = \frac{\omega}{\sqrt{k^2 + l^2}} = \frac{Uk}{\sqrt{k^2 + l^2}} - \frac{\beta k}{\left(k^2 + l^2\right)^{3/2}}$

(2d) (4 marks) Mountains often excite Rossby waves whose phase speeds are stationary with respect to the Earth. These waves are called *stationary* Rossby waves. For a given zonal wind U and Coriolis derivative β, what is the relationship between $\left(k^2 + l^2\right)$ and U and β that ensures that the phase speed of the Rossby wave relative to the Earth is equal to zero?

Answer: In order for the wave to be stationary with respect to the surface of the Earth, we must have $\omega = 0$; i.e.

$$0 = Uk - \frac{\beta k}{\left(k^2 + l^2\right)} \Rightarrow U = \frac{\beta}{\left(k^2 + l^2\right)}$$

(2e) (4 marks) Given that $\beta \geq 0$, can stationary Rossby waves exist when U<0 ? (Yes, or No). Why ? (very short answer – you can use an equation)

Answer: No. Because $\frac{\beta}{\left(k^2 + l^2\right)} > 0$

(2f) (3 marks) When waves of many different wavelengths are present simultaneously, it is possible for the "packet" of waves to form a localized disturbance. It is also possible to show that the velocity at which such "packets" or envelopes of "groups" of waves move depends on the wave number. In the *x*-direction, it can be shown this wavenumber dependent x-component of group velocity is given by $\frac{\partial \omega}{\partial k}$. For Rossby waves, in the two-dimensional *y*-independent case with no zonal wind, $\omega = -\frac{\beta}{k}$. Calculate the x-component of the group velocity of a packet of Rossby waves in this case by computing $\frac{\partial \omega}{\partial k}$ for $\omega = -\frac{\beta}{k}$.

Answer: $c_x^g = \frac{\partial \omega}{\partial k} = \frac{\beta}{k^2}$

(2g) (4 marks) Now calculate the phase velocity of the Rossby wave in the case $\omega = -\frac{\beta}{k}$.

Answer: $c_x = \frac{dx}{dt} = \frac{\omega}{k} = -\frac{\beta}{k^2}$

Assuming $\omega = -\frac{\beta}{k}$, correctly complete the following sentences using either the word "Eastward" or the word "Westward".

The phase velocity of the Rossby wave is ___**Westward**

The group velocity of the Rossby wave is ___**Eastward**

3 (40 marks total)

(a) (9 marks) It can be shown that small amplitude perturbations in a stratified flow with Brunt-Vaisala frequency $N = \sqrt{\frac{g}{\overline{\theta}} \frac{\partial \overline{\theta}}{\partial z}}$ is

$$\left(\frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right)^2 \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} \right) + N^2 \frac{\partial^2 w'}{\partial x^2} = 0$$

we then assumed that

$$w' = \widehat{w}(z) \exp\left[i(kx - \omega t) \right] \qquad (3.1)$$

and obtained

$$\frac{\partial^2 \widehat{w}}{\partial z^2} + \left(\frac{N^2}{\left(\omega - Uk \right)^2} - 1 \right) k^2 \widehat{w} = 0 \qquad (3.2)$$

Substitute the function

$$\widehat{w}(z) = A \exp\left[i \left(k \sqrt{\frac{N^2}{\left(\omega - Uk \right)^2} - 1} \right) z \right] \qquad (3.3)$$

into (3.2) to prove that (3.3) is a solution to (3.2). You must show your working to get full credit for this question.

Answer:

$$\frac{\partial^2 \widehat{w}}{\partial z^2} = \left[i \left(k \sqrt{\frac{N^2}{(\omega - Uk)^2} - 1} \right) \right]^2 A \exp\left[i \left(k \sqrt{\frac{N^2}{(\omega - Uk)^2} - 1} \right) z \right]$$

$$= -k^2 \left[\frac{N^2}{(\omega - Uk)^2} - 1 \right] A \exp\left[i \left(k \sqrt{\frac{N^2}{(\omega - Uk)^2} - 1} \right) z \right] = -k^2 \left[\frac{N^2}{(\omega - Uk)^2} - 1 \right] \widehat{w}; \text{ consequently}$$

$$\frac{\partial^2 \widehat{w}}{\partial z^2} + \left(\frac{N^2}{(\omega - Uk)^2} - 1 \right) k^2 \widehat{w} = -k^2 \left[\frac{N^2}{(\omega - Uk)^2} - 1 \right] \widehat{w} + \left(\frac{N^2}{(\omega - Uk)^2} - 1 \right) k^2 \widehat{w} = 0$$

As was required (QED)

(3b) (11 marks)The simplified Buoyancy equation from which this gravity wave solution was derived requires that

$$\frac{\partial b'}{\partial t} + U \frac{\partial b'}{\partial x} = -w' N^2 \qquad (3.4)$$

Assume that

$$w' = B \exp\left[i(kx + mz - \omega t) \right] \text{ and } b' = A \exp\left[i(kx + mz - \omega t) \right] \qquad (3.5)$$

And then use (3.5) in (3.4) to obtain an expression for A in terms of B and ω.

Answer:

$$w' = B \exp\left[i(kx + mz - \omega t) \right] \text{ and } b' = A \exp\left[i(kx + mz - \omega t) \right]$$

$$\frac{\partial b'}{\partial t} + U \frac{\partial b'}{\partial x} = -w' N^2$$

$$\frac{\partial b'}{\partial t} = -i\omega A \exp\left[i(kx + mz - \omega t) \right]$$

$$U \frac{\partial b'}{\partial x} = ikUA \exp\left[i(kx + mz - \omega t) \right]$$

$$\frac{\partial b'}{\partial t} + U \frac{\partial b'}{\partial x} = (-i\omega + ikU) A \exp\left[i(kx + mz - \omega t) \right] = -w' N^2 = -N^2 B \exp\left[i(kx + mz - \omega t) \right]$$

$$\Rightarrow (-i\omega + ikU) A = -N^2 B$$

$$\Rightarrow (\omega - kU) A = -iN^2 B$$

$$\Rightarrow A = \frac{-iN^2 B}{(\omega - kU)} = \frac{N^2 B}{i(\omega - kU)} = \frac{iN^2 B}{(kU - \omega)} \text{ (all of these forms ok)}$$

(3c) (9 marks total, 2 marks, 7 marks for the second part) If the wind blows over sinusoidal topography with wavenumber k from West to East at the speed U, then it is likely that a standing gravity wave will be produced. Standing waves are not moving with respect to the Earth and hence have $\omega = 0$ even though they can be moving with respect to the flow within which they are embedded. Using your answer to the previous question, state the relationship between *A* and *B* when $\omega = 0$ and *U>0*.

Answer:

$$A = \frac{iN^2 B}{kU}$$

Now find expressions for the real part of *w'* and *b'* in this stationary wave case and where the amplitude *B* of *w'* is real

Answer: Assuming that the amplitude *B* of *w'* is real

$$\mathrm{Re}(w') = \mathrm{Re}\left[B\exp\left[i(kx+mz-\omega t)\right]\right] = B\cos\left[(kx+mz-\omega t)\right]$$

$$\mathrm{Re}(b') = \mathrm{Re}\left\{A\exp\left[i(kx+mz-\omega t)\right]\right\} = \mathrm{Re}\left\{\frac{iN^2 B}{kU}\exp\left[i(kx+mz-\omega t)\right]\right\}$$

$$= \mathrm{Re}\left\{\frac{iN^2 B}{kU}\left[\cos(kx+mz-\omega t) + i\sin(kx+mz-\omega t)\right]\right\}$$

$$= -\frac{N^2 B}{kU}\sin(kx+mz-\omega t)$$

(3d) (11 marks total, 9 marks for figure, 2 marks for buoyancy question) For this special case of a stationary gravity wave $(\omega = 0)$ forced by Eastward flow over a sinusoidal mountain with wavenumber *k* with *U* precisely equal to the value that ensures that $(-Uk)^2 = \frac{N^2}{2}$ so that $m = k\sqrt{\frac{N^2}{(-Uk)^2} - 1} = k$ (also assume that *k>0),* then for the case of positive *B,* make a sketch on the *x*-*z* plane of the crests and troughs of the buoyancy field *b* using your answer to (3c). Label the crests as b_{crest} and the troughs as b_{trough}. Now plot the corresponding crests and troughs of *w'* on the same plot. Label the crests as w'_{crest} and the troughs as w'_{trough} The crests of *w'* will be displaced a quarter wavelength away from the troughs of *b*. Does the positive vertical motion at a w'_{crest} create positive or negative buoyancy? (Give your answer in the form "Positive vertical motion creates ______ buoyancy.")

Note that to sketch the relative phases of the waves you do not need to know the specific value of *A* or *U.* Assume that *t*=0 for your plot.

Answer: (9 marks for figure)

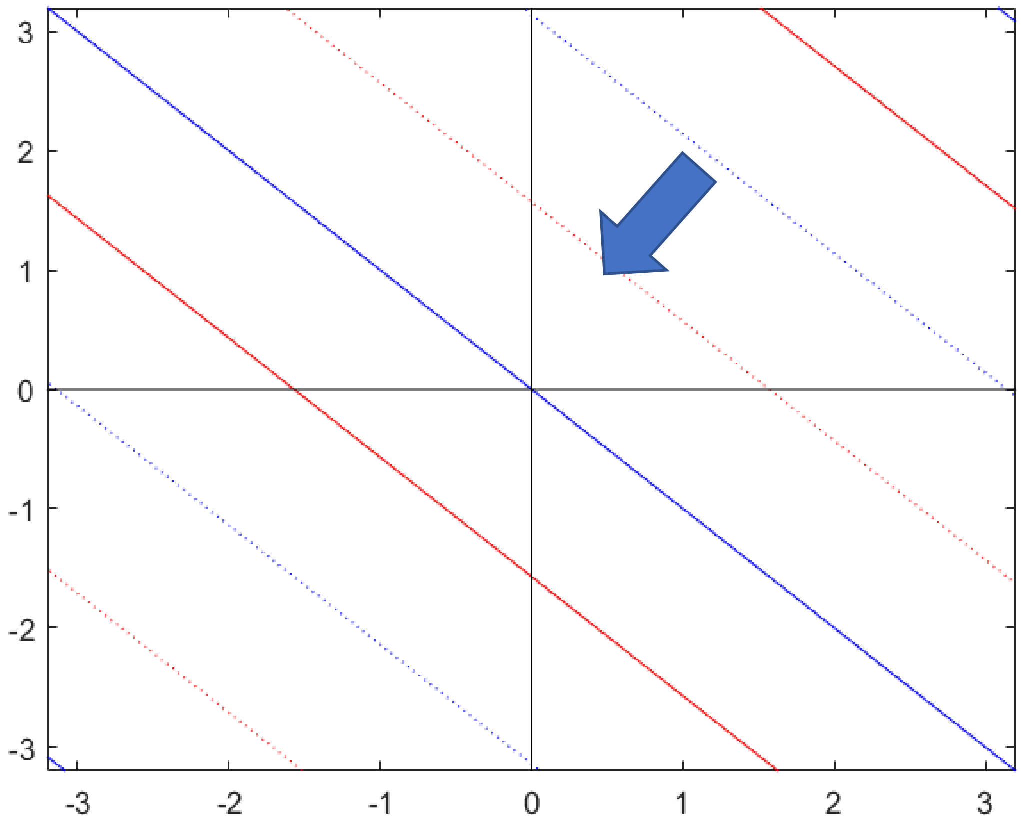


Figure caption: Solid (dashed) red lines indicate buoyancy *b'* crests (troughs). Solid (dashed) blue lines indicate *w'* crests (troughs). I drew these lines with Matlab, a quicker and completely acceptable method would be to hand draw lines like this and label each of them directly by hand.

Answer: (2 marks for this) "Positive vertical motion creates *negative* buoyancy."

Based on the relative positions of the Buoyancy troughs and the *w'* crests, draw on your diagram the direction that you think the wave is propagating relative to the ambient flow. Relative to an observer moving with the flow, is the propagation

(i) Downward and Westward? Answer: (0 marks for this)
(ii) Upward and Eastward?
(iii) Downward and Eastward?
(iv) Upward and Westward?

(Circle the answer that you think is correct).