Hydrostatic Balance

$$\boxed{\frac{dp}{dz} = -\rho g}$$

- In the absence of atmospheric motions the gravity force must be exactly balanced by the vertical component of the pressure gradient force.
- Because vertical accelerations are very small for large-scale atmospheric motions, this is an excellent approximation for the vertical dependence of pressure in the real atmosphere.

$$\frac{dp}{dz} = -\rho g$$

$$dp = -\rho g\, dz$$

$$p(z) = \int_z^\infty \rho g\, dz$$

Pressure at any point is the weight per square meter of the atmospheric column overlying that point.

For average conditions,

$$p(0) = \int_0^\infty \rho g\, dz = 101.325\ kPa$$

This is the mean sea-level pressure.

We can define a quantity called the geopotential, which is related to gravity. Gravity can be represented as the gradient of the geopotential.

$$\nabla\Phi = -\vec{g}$$

Because $\vec{g} = -g\hat{k},$ then $\Phi = \Phi(z), \quad \frac{d\Phi}{dz} = g$

If the value of the geopotential is set to zero at mean sea level, the geopotential $\Phi(z)$ at height z is the work required to raise a unit mass to height z from mean sea level:

$$\Phi = \int_0^z g\ dz$$

Units of geopotential are J kg^{-1}, which are equivalent to $m^2\ s^{-2}$.

$\Phi = \int_0^z g\ dz$ implies that $d\Phi = g\ dz$

Since $g\ dz = -\frac{1}{\rho}dp = -\alpha\ dp$

then $d\Phi = -\alpha\ dp = -\frac{RT}{p}dp = -RT\ d(\ln p)$

The variation of geopotential with pressure depends on temperature. Integrating in the vertical:

$$\boxed{\Phi(z_2) - \Phi(z_1) = R\int_{p_2}^{p_1} T\ d(\ln p)}$$

This is the hypsometric equation, which relates the difference in geopotential to the layer mean temperature.

Rather than express the hypsometric equation in terms of geopotential, meteorologists often rewrite it in terms of a quantity called geopotential height, which is defined as

$$Z \equiv \Phi(z)/g,$$

Units of geopotential are $m^2 s^{-2}$, so units of geopotential height are m.

where g = 9.8 m s^{-2} is the global average gravity at sea level. The geopotential height is almost identical to the geometric height in the troposphere and lower stratosphere.

Thus the hypsometric equation

$$\Phi(z_2) - \Phi(z_1) = R\int_{p_2}^{p_1} T\ d(\ln p)$$

becomes

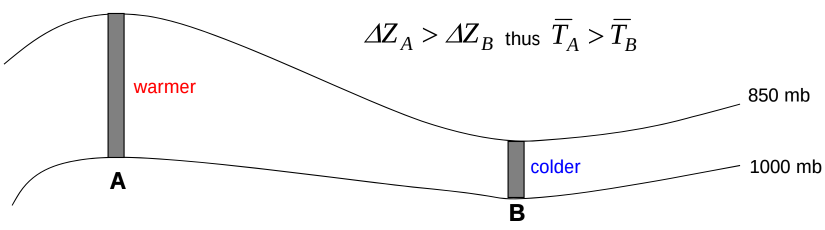
$$\Delta Z \equiv Z_2 - Z_1 = \frac{R}{g}\int_{p_2}^{p_1} T\ d(\ln p)$$

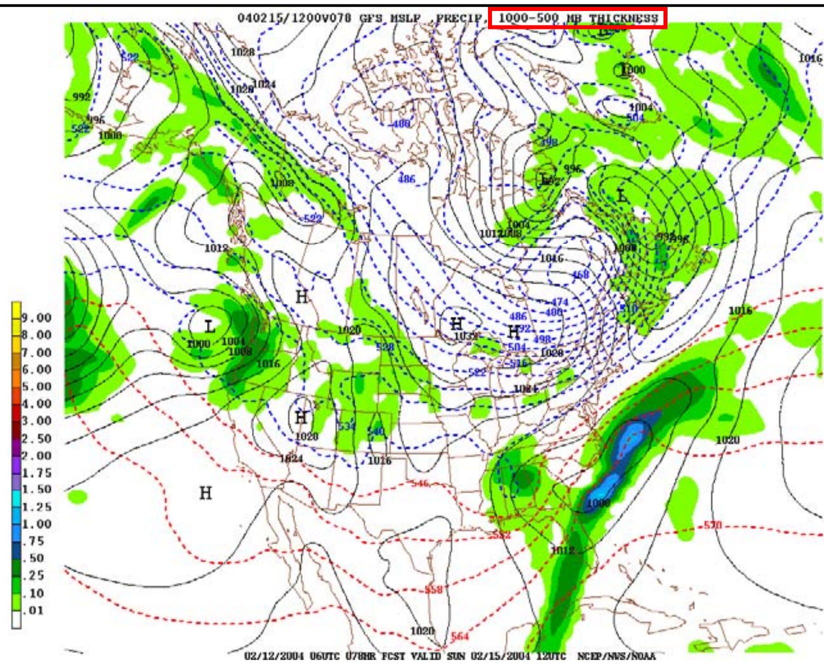
where ΔZ is the thickness of the atmospheric layer between p_1 and p_2.

Hyposmetric Equation and Thickness

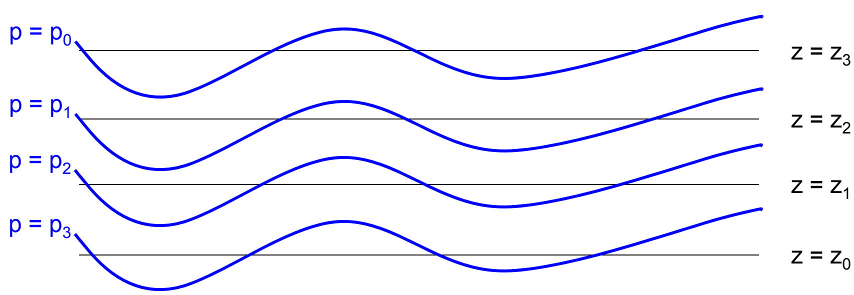
$$\Delta Z \equiv Z_2 - Z_1 = \frac{R}{g}\int_{p_2}^{p_1} T\ d(\ln p\,)$$

The hypsometric equation relates the thickness, or vertical distance between two pressure levels, to the temperature of the intervening layer. The thickness, ΔZ, is proportional to the mean temperature of the layer (using a weighting based on ln p).





Pressure As A Vertical Coordinate



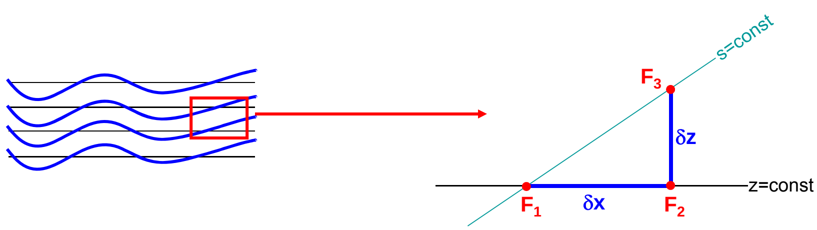
$p_3 > p_2 > p_1 > p_0$

$z_3 > z_2 > z_1 > z_0$

How do we convert our equations from height coordinates (x,y,z) to pressure coordinates (x,y,p)?

Generalized Vertical Coordinates

- The use of pressure as a vertical coordinate is a specific example of the use of generalized vertical coordinates.
- Any quantity *s = s(x,y,z,t)* that changes monotonically with height can be used as a vertical coordinate.
- If we wish to transform equations from *(x,y,z)* coordinates to *(x,y,s)* coordinates, derivatives must be transformed.



Let F = some scalar property, and
s = a generalized vertical coordinate.

We would like to transform derivatives such as $\left(\frac{\partial F}{\partial x}\right)_z$ to $\left(\frac{\partial F}{\partial x}\right)_s$

Derivative in x-direction on a constant z surface

Derivative in x-direction on a constant s surface

$$\frac{F_3 - F_1}{\delta x} = \frac{F_2 - F_1}{\delta x} + \frac{F_3 - F_2}{\delta x} = \frac{F_2 - F_1}{\delta x} + \frac{F_3 - F_2}{\delta z}\frac{\delta z}{\delta x}$$

$$\left(\frac{\delta F}{\delta x}\right)_s = \left(\frac{\delta F}{\delta x}\right)_z + \frac{\delta F}{\delta z}\left(\frac{\delta z}{\delta x}\right)_s$$

$$\boxed{\begin{aligned}\left(\frac{\partial F}{\partial x}\right)_s &= \left(\frac{\partial F}{\partial x}\right)_z + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s \\ \left(\frac{\partial F}{\partial y}\right)_s &= \left(\frac{\partial F}{\partial y}\right)_z + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial y}\right)_s\end{aligned}}$$

$$\left(\frac{\partial F}{\partial x}\right)_s = \left(\frac{\partial F}{\partial x}\right)_z + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s$$

$$\left(\frac{\partial F}{\partial y}\right)_s = \left(\frac{\partial F}{\partial y}\right)_z + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial y}\right)_s$$

can be written in vector form as

$$\nabla_s F = \nabla_z F + \frac{\partial F}{\partial z}\nabla_s z \quad \text{where}$$

$$\nabla_s F = \left(\frac{\partial F}{\partial x}\right)_s \hat{i} + \left(\frac{\partial F}{\partial y}\right)_s \hat{j}$$

$$\nabla_z F = \left(\frac{\partial F}{\partial x}\right)_z \hat{i} + \left(\frac{\partial F}{\partial y}\right)_z \hat{j}$$

$$\boxed{\nabla_s F = \nabla_z F + \frac{\partial F}{\partial z}\nabla_s z}$$

We will use this equation to transform the horizontal derivatives in the momentum equation from z-coordinates to p-coordinates.

Horizontal momentum equation scaled for midlatitude large-scale motions.

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p$$

Rate of change of velocity following the fluid motion.

Coriolis acceleration

Pressure gradient force (per unit mass)

To transform to pressure coordinates, we need to transform the pressure gradient term:

$$\cancel{\nabla_p p} = \nabla_z p + \frac{\partial p}{\partial z}\nabla_p z$$

$$\nabla_z p = -\frac{\partial p}{\partial z}\nabla_p z$$

$$\nabla_z p = \rho g \nabla_p z$$

$$-\frac{1}{\rho}\nabla_z p = -g\nabla_p z = -\nabla_p \Phi$$

$$\boxed{\nabla_p p = 0}$$

$$\boxed{\frac{\partial p}{\partial z} = -\rho g}$$

Geopotential gradient

$$\boxed{\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \nabla_p \Phi}$$

or

$$\boxed{\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - g\nabla_p Z}$$

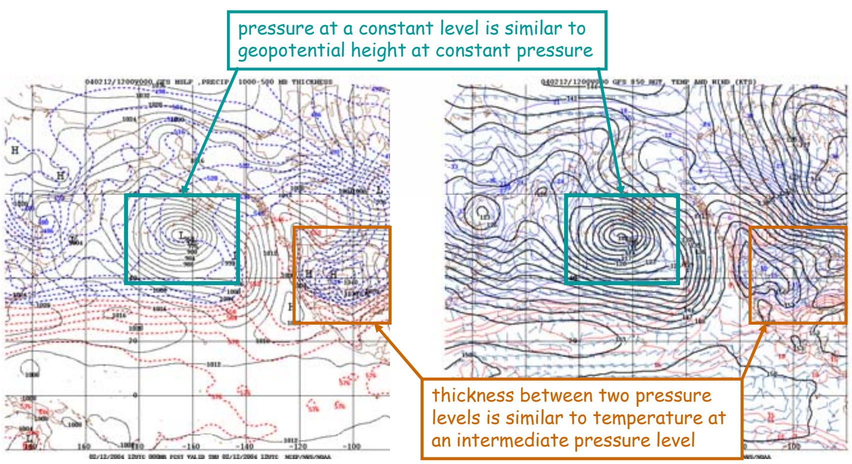
Geopotential height gradient

Characteristics of pressure (isobaric) coordinates:

1) Vertical velocity is expressed as ω = dp/dt. Rising air moves from higher to lower pressure, so upward motion occurs when ω < 0.
2) The geopotential height gradient takes the place of the pressure gradient.
3) Low geopotential height on an isobaric surface are analogous to low pressure on a surface chart.
4) Expansion of the total derivative takes the following form:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial f}{\partial y} + \frac{\partial p}{\partial t}\frac{\partial f}{\partial p}$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \omega\frac{\partial f}{\partial p}}$$



Surface Map
solid: pressure at sea level
dashed: thickness between 1000 hPa and 500 hPa

850 hPa Map
black: geopotential height at 850 hPa
color: temperature at 850 hPa