Dynamical Meteorology and Oceanography ATOC30004

Topics

1. The equations of motion for a non-rotating fluid
2. The apparent "Coriolis force" in terms of conservation of angular momentum and the apparent centrifugal force.
3. Vertical wind variations and thermal wind balance
4. Balanced vortices
5. Basic vorticity dynamics
6. Rossby and gravity waves
7. Frontogenesis and Q-vectors

Prof Craig Bishop
Room 449
craig.bishop@unimelb.edu.au

Dynamical meteorology and oceanography: *The study of motions in the atmosphere and ocean, and the processes and fundamental forces that control those motions.*

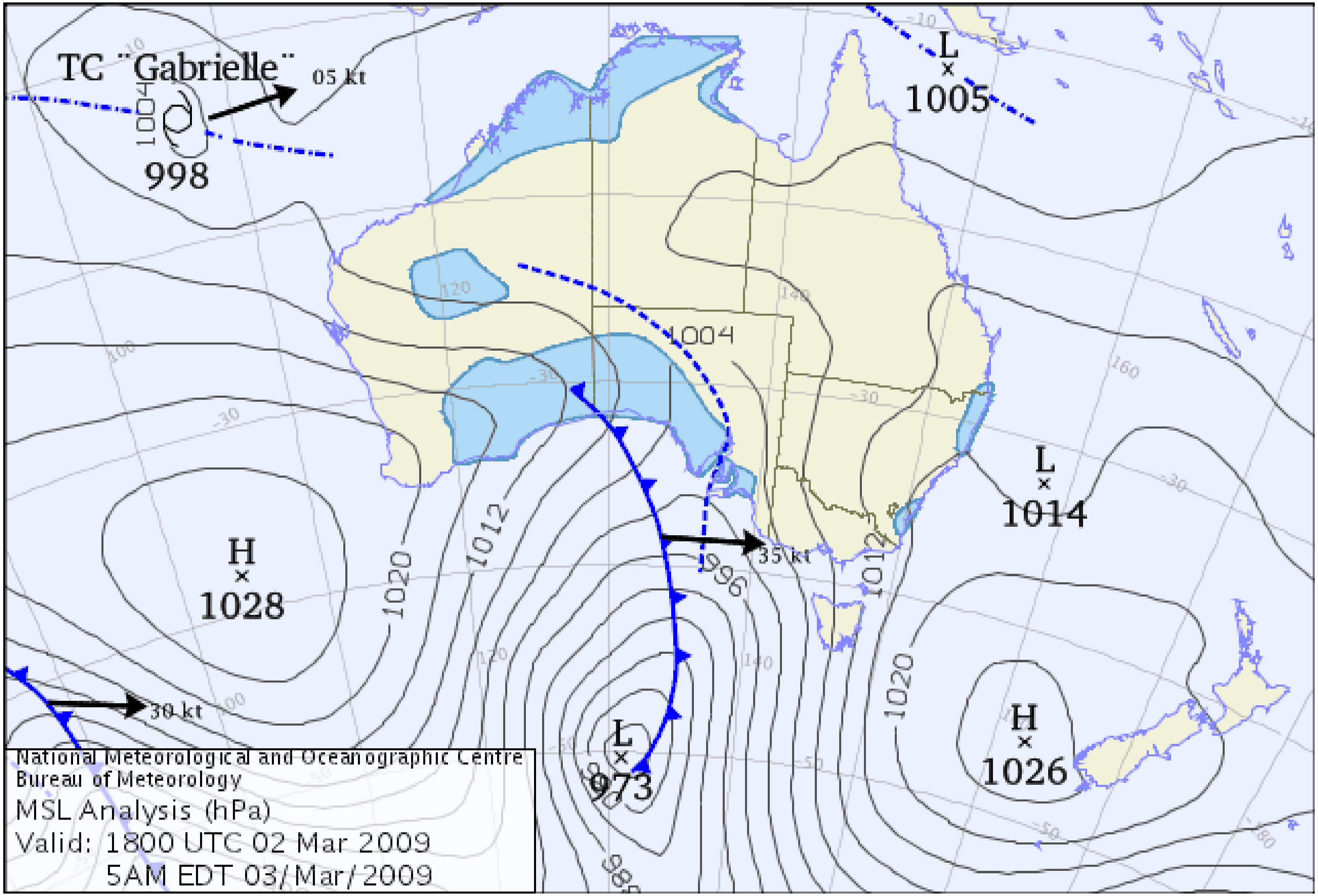
- *Essential to understand how the atmosphere / ocean works.*

- *Many ideas, explanations, and theories are interchangeable (with minor adjustments) between the atmosphere and ocean.*

- *Dynamics is the foundation of modern weather and climate models.*

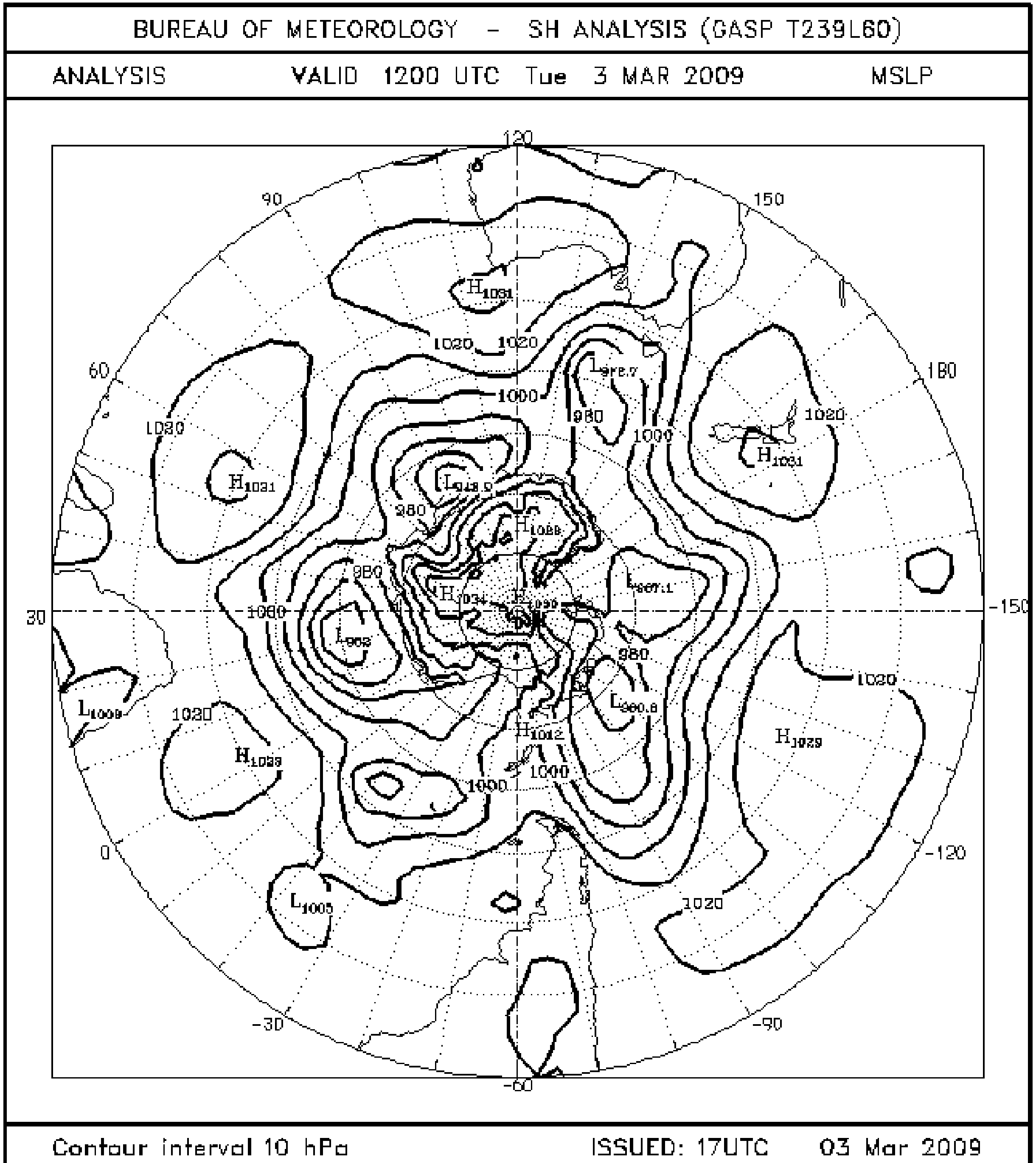
- *Dynamics is perhaps the most loved and most hated field in the atmospheric sciences - but absolutely necessary.*

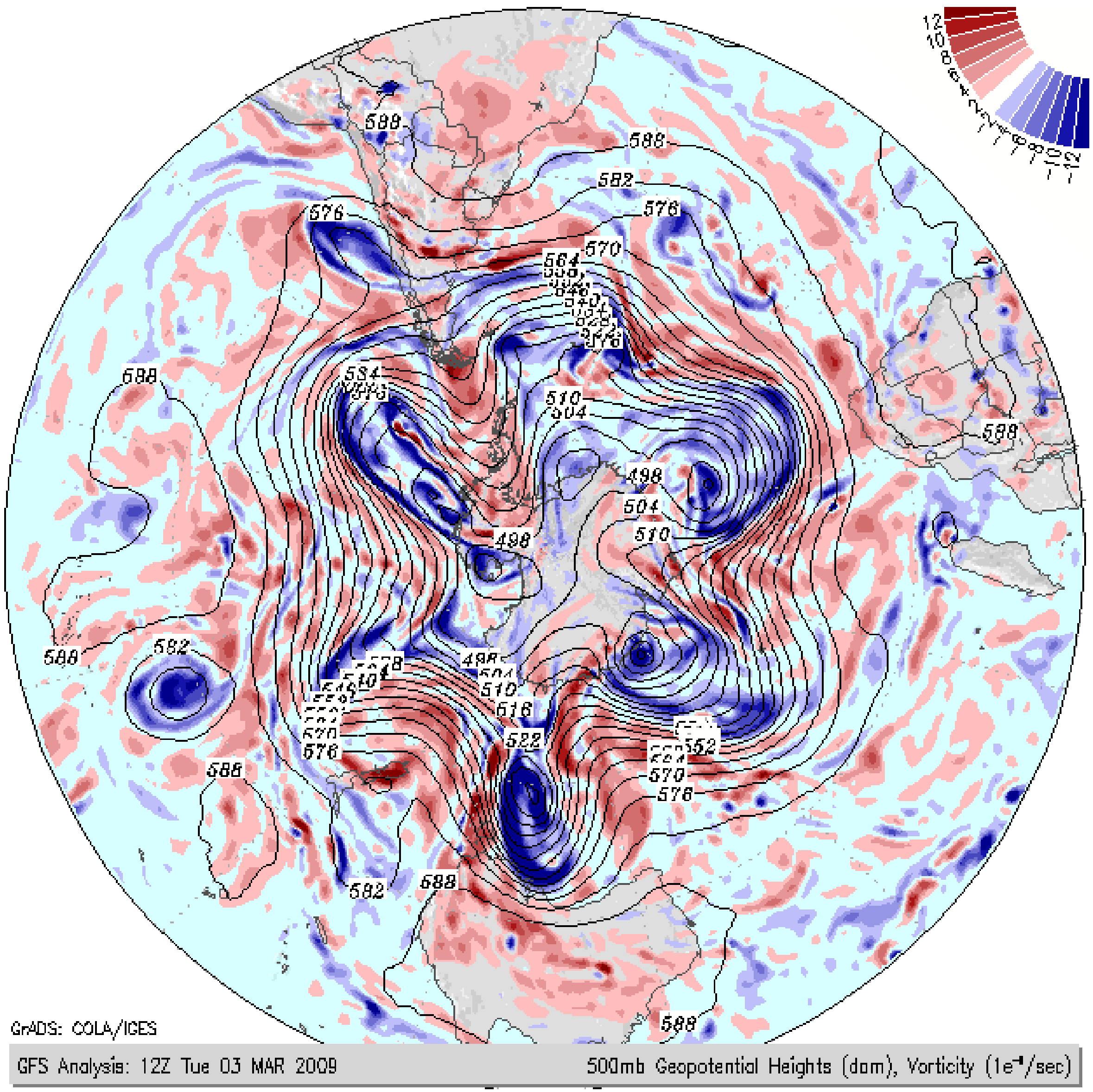
How does wind behave / evolve on the synoptic scale?



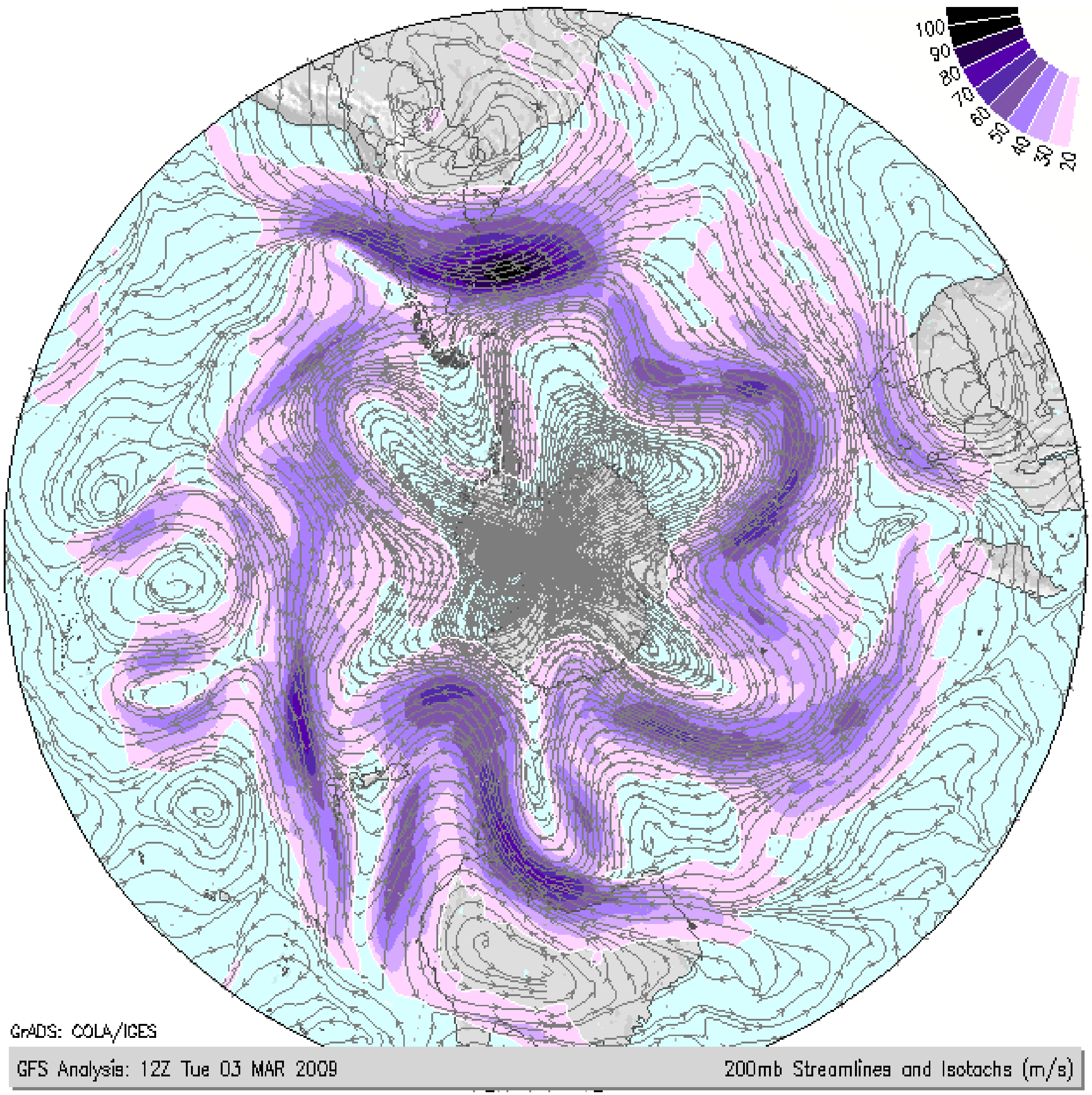
Why are Highs bigger than Lows? Why do they move towards the East (in general)

Formation of “wave trains”

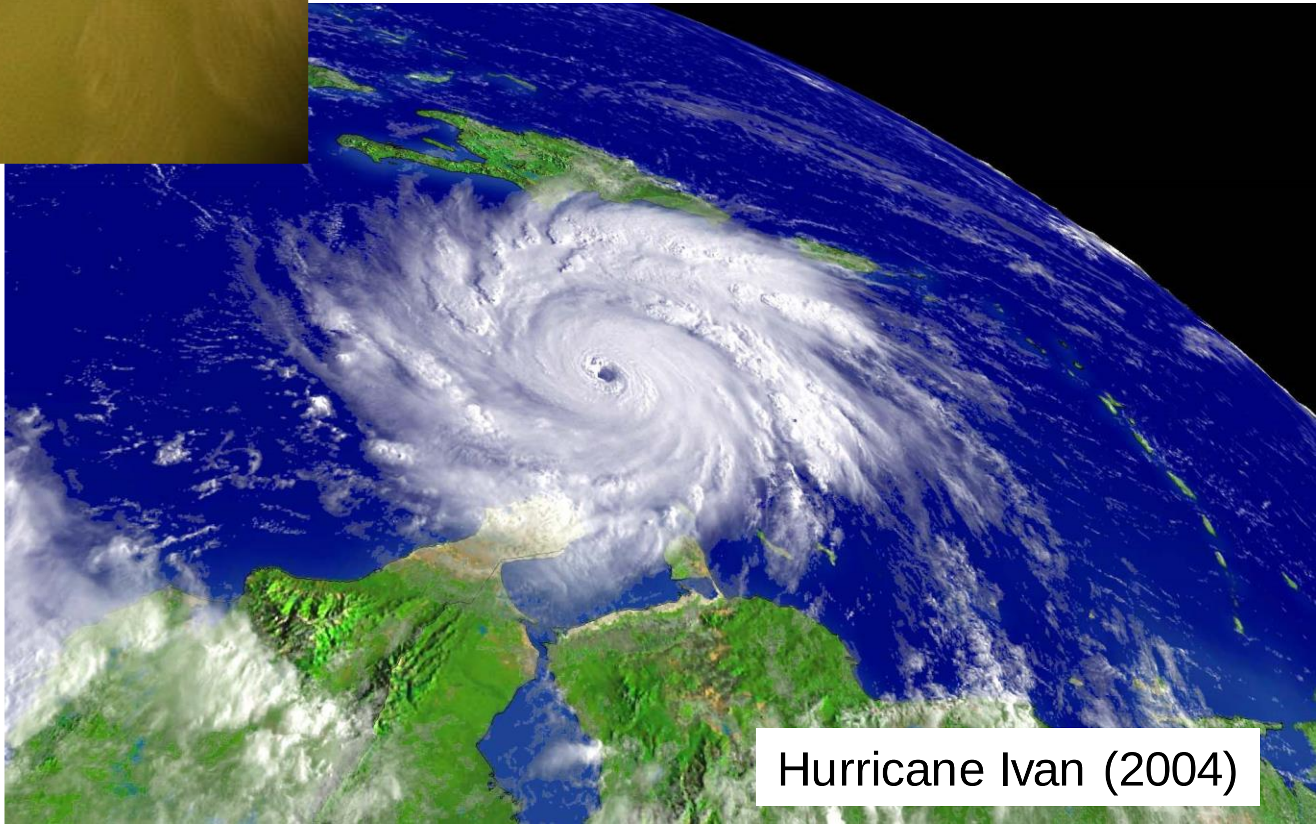



GrADS: COLA/IGES
GFS Analysis: 12Z Tue 03 MAR 2009
500mb Geopotential Heights (dam), Vorticity (1e-4/sec)

Formation of jet streams

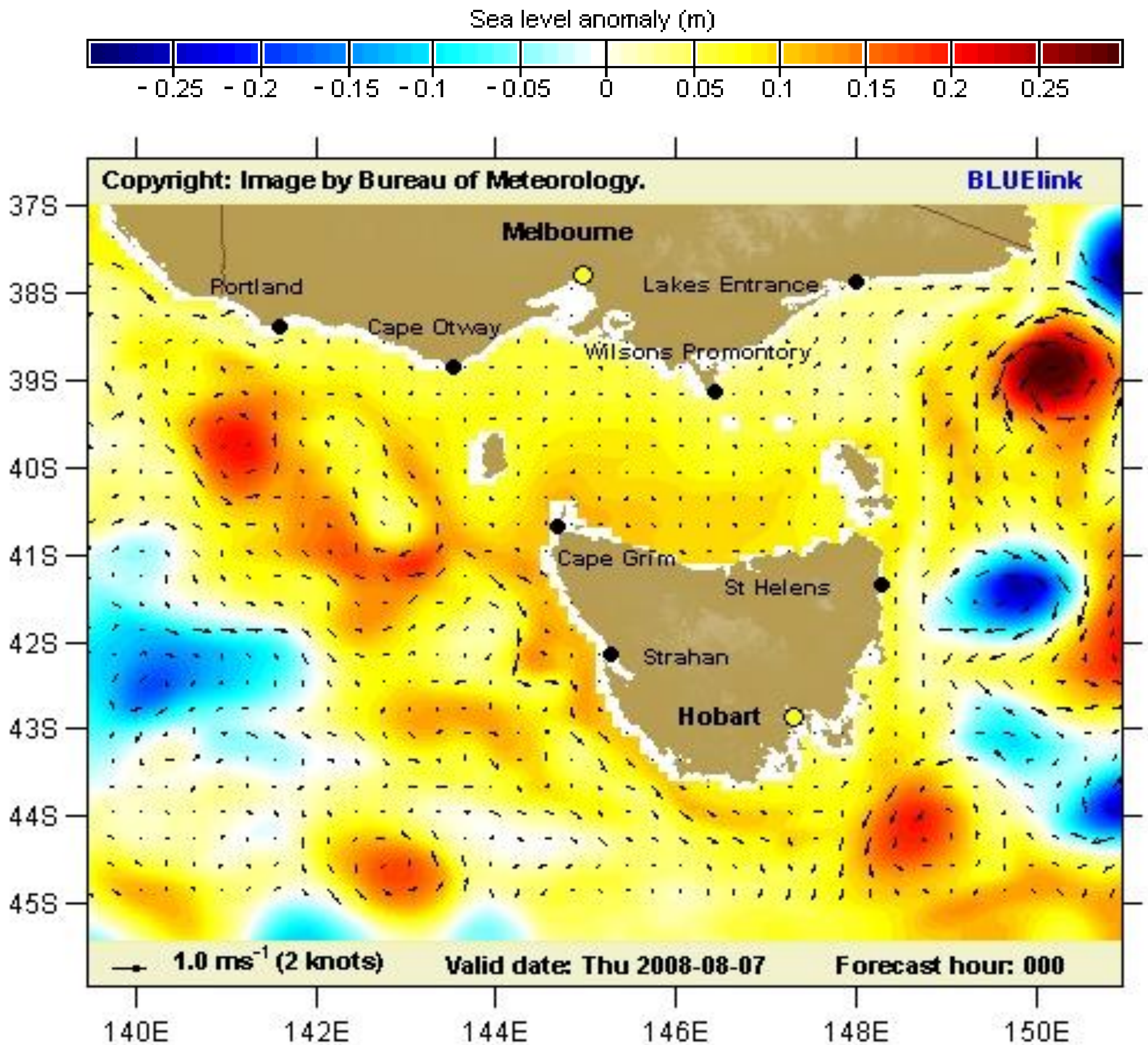


Vortex flows…



Hurricane Ivan (2004)

Behaviour of ocean currents / eddies…



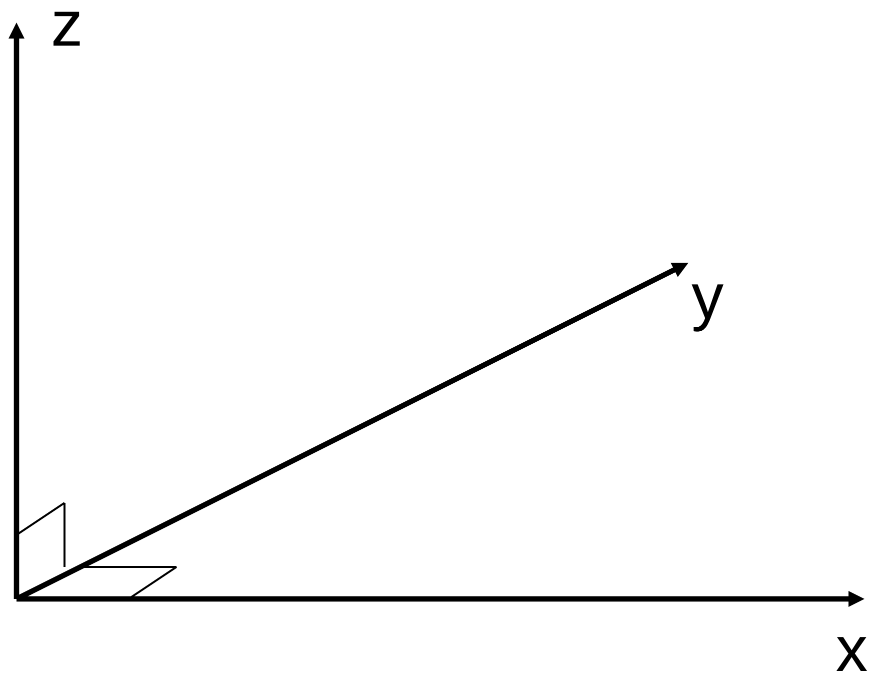
Dynamical Meteorology and Oceanography ATOC30004

Topic 1: The equations of motion of a non-rotating fluid

- Lagrangian and Eulerian representations
- The material derivative
- Mass continuity
- The momentum equations

Prof Craig Bishop
Room 449
craig.bishop@unimelb.edu.au

Usual terminology



x-axis points to East
y-axis points to North
z-axis points upwards

u is *x*-velocity: ZONAL
v is *y*-velocity: MERIDIONAL
w is *z*-velocity: VERTICAL

$\underline{\mathbf{x}}=(x,y,z)=x\underline{\mathbf{i}}+y\underline{\mathbf{j}}+z\underline{\mathbf{k}}$ position vector

$\underline{\mathbf{u}}=(u,v,w)=u\underline{\mathbf{i}}+v\underline{\mathbf{j}}+w\underline{\mathbf{k}}$ velocity vector

$\underline{\mathbf{i}}$, $\underline{\mathbf{j}}$, $\underline{\mathbf{k}}$ are orthogonal unit vectors

Vector definitions / operators

If $\underline{a}=(a_1,a_2,a_3)$ and $\underline{b}=(b_1,b_2,b_3)$

Then:

$\underline{a}+\underline{b}=(a_1+b_1,a_2+b_2,a_3+b_3)$ vector sum, vector result

$\underline{a}\bullet\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$ dot-product, “a-dot-b”, scalar result

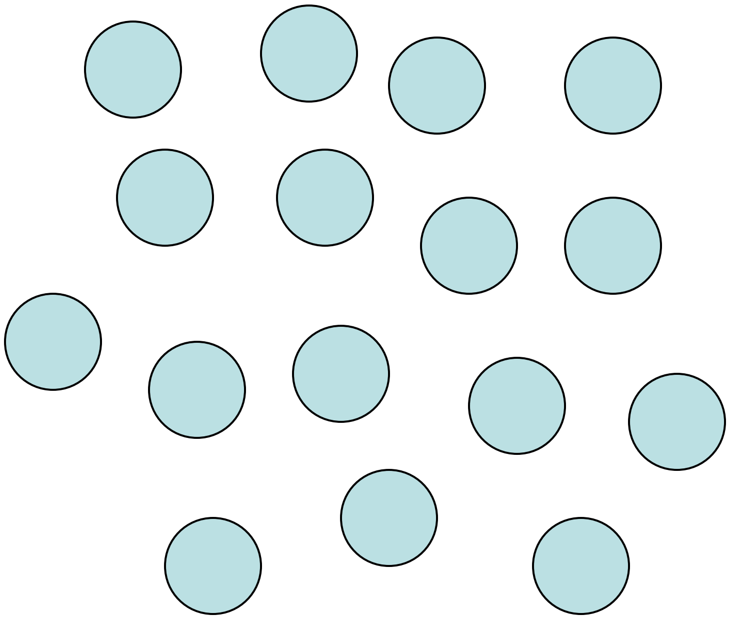
$$\nabla \cdot \underline{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

divergence operator, “div-a”, scalar result

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \frac{\partial \phi}{\partial x}\underline{\mathrm{i}} + \frac{\partial \phi}{\partial y}\underline{\mathrm{j}} + \frac{\partial \phi}{\partial z}\underline{\mathrm{k}}$$

gradient operator, “grad-phi”, vector result (phi a scalar function)

“Lagrangian” perspective



$$\frac{dx}{dt} = u$$

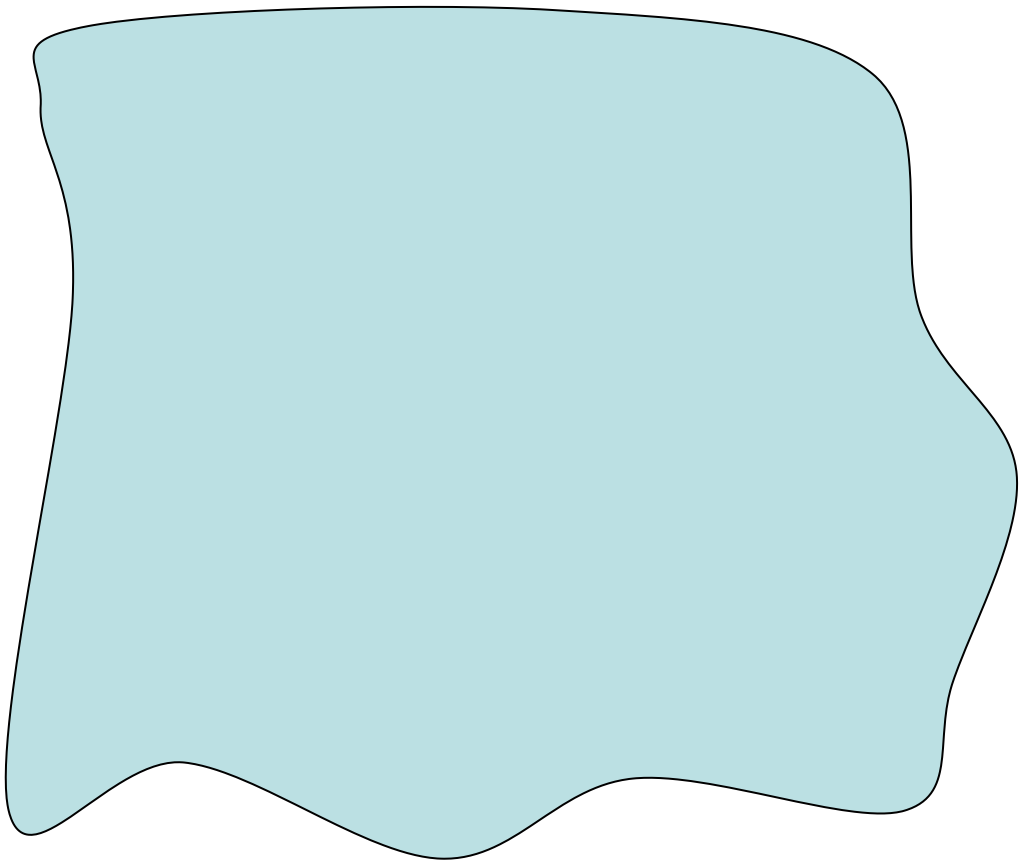
$$\frac{dy}{dt} = v$$

$$\frac{dz}{dt} = w$$

Atmosphere (fluid) made up of numerous “parcels” of air.

- akin to solid body mechanics

“Eulerian” perspective



$$u = u(x, y, z, t)$$

$$v = v(x, y, z, t)$$

$$w = w(x, y, z, t)$$

$$p = p(x, y, z, t)$$

Atmosphere (fluid) is a continuum

- focus on the properties of the fluid as a function of position

Mathematical derivation:

$$\phi = \phi(x, y, z, t)$$

Use chain rule for differentiation:

$$\delta\phi = \frac{\partial\phi}{\partial t}\delta t + \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z$$

Divide both sides by δt

$$\frac{\delta\phi}{\delta t} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial\phi}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial\phi}{\partial z}\frac{\delta z}{\delta t}$$

Take limit $\delta \mathrm{t} \rightarrow 0$

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}u + \frac{\partial\phi}{\partial y}v + \frac{\partial\phi}{\partial z}w$$

D/Dt represents the total rate of change of f following the fluid.

- (strictly d/dt is for variables that are function of t only, so we use D/Dt to avoid confusion.)

The material derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

D/Dt - The time rate of change of a quantity following the fluid.

Local change ($\frac{\partial}{\partial t}$) can be associated with a local variation from sources/sinks and/or due to advection / transport from somewhere else.

For a purely conserved quantity (no local sources or sinks) local variation must be related to advection from somewhere else. But following the fluid nothing changes (D/Dt = 0)

A conserved quantity (e.g., potential temperature):

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = 0$$

Moving with the fluid there is no change in θ

$$\frac{\partial \theta}{\partial t} = -u\frac{\partial \theta}{\partial x} - v\frac{\partial \theta}{\partial y} - w\frac{\partial \theta}{\partial z}$$

Local rate of change is due to advection only.

Material derivative can be written as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla$$

e.g.,

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \underline{u} \cdot \nabla\phi$$

Conservation of Mass:

fundamental constraint on atmospheric motion.

In the absence of sources / sinks of mass total mass in a bounded volume cannot change.

e.g., sealed room, balloon, etc.

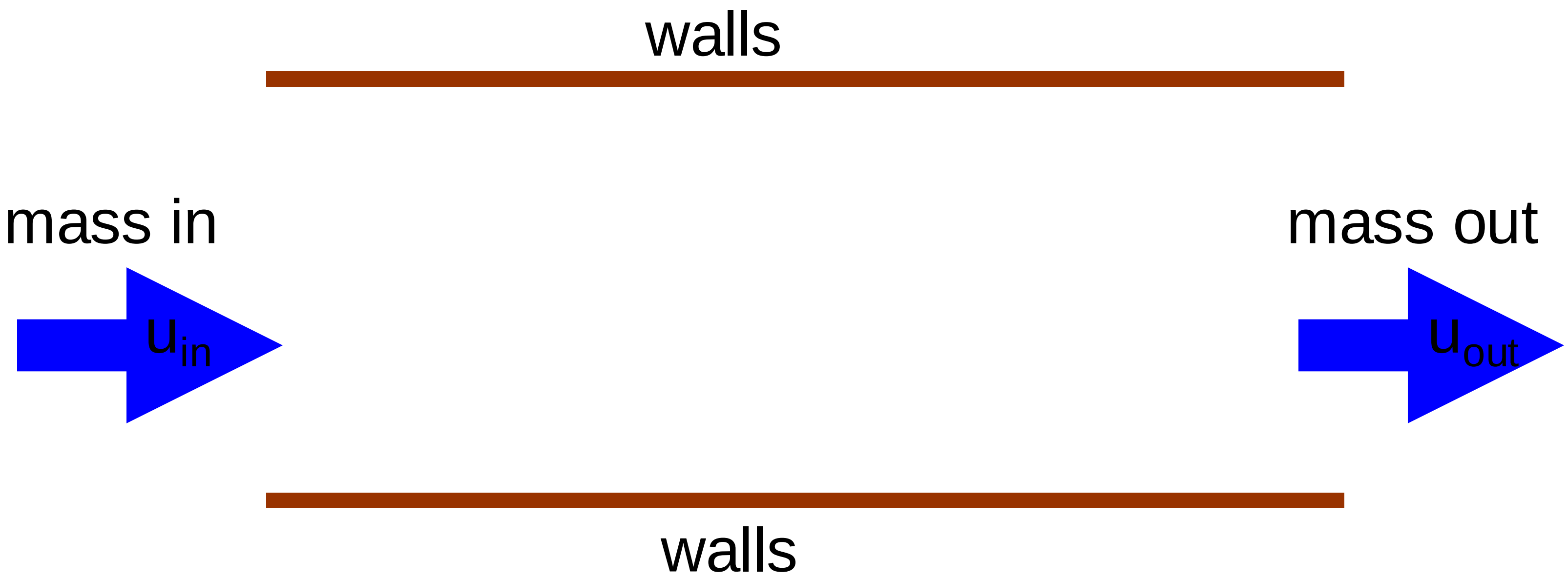
This property also applies similarly to unbounded (arbitrary) volumes. Either:

1. Mass into volume equals mass out of volume.

or

2. The difference between mass-in and mass-out causes a change in density within the volume.

1st - consider a simple channel Δy wide and Δz deep



Mass entering channel in time Δt is:

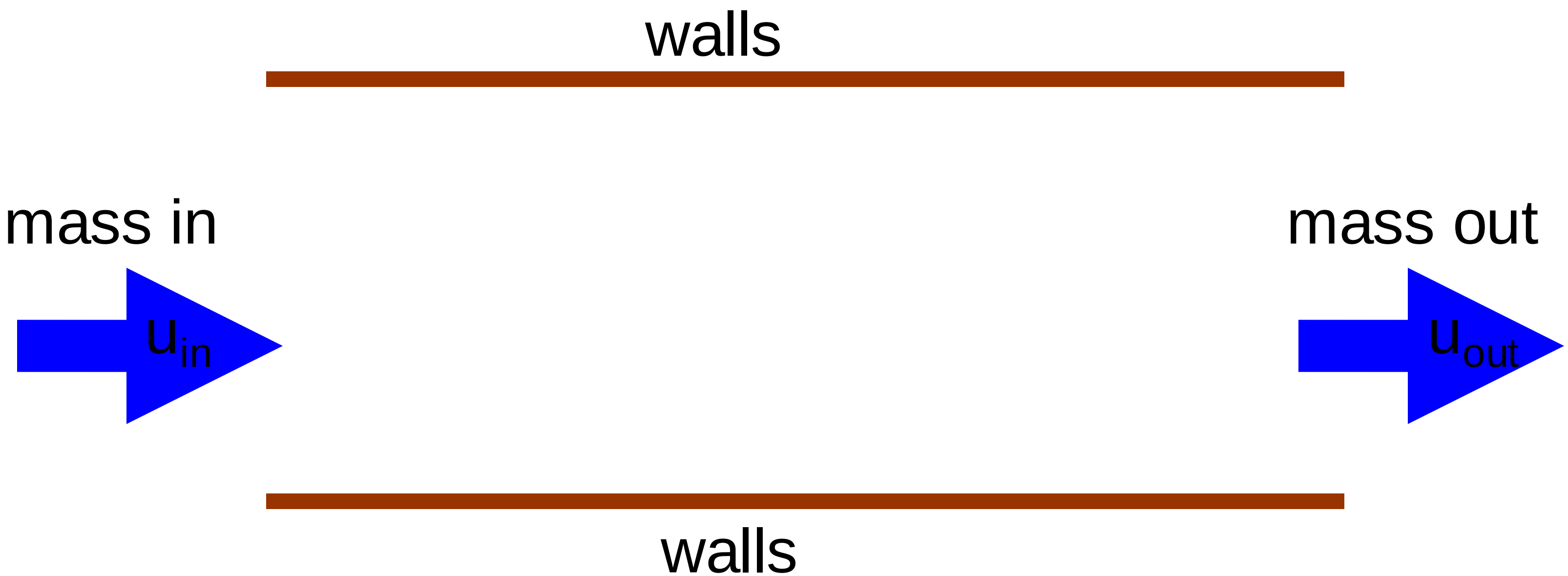
$m_{in} = \rho\ \Delta y \Delta z\ u_{in} \Delta t$

Mass leaving channel is time Δt is:

$m_{out} = \rho\ \Delta y \Delta z\ u_{out} \Delta t$

(Distance travelled by fluid in time Δt is $u\ \Delta t$)

1st - consider a simple channel Δy wide and Δz deep



If flux of mass-in = flux of mass-out

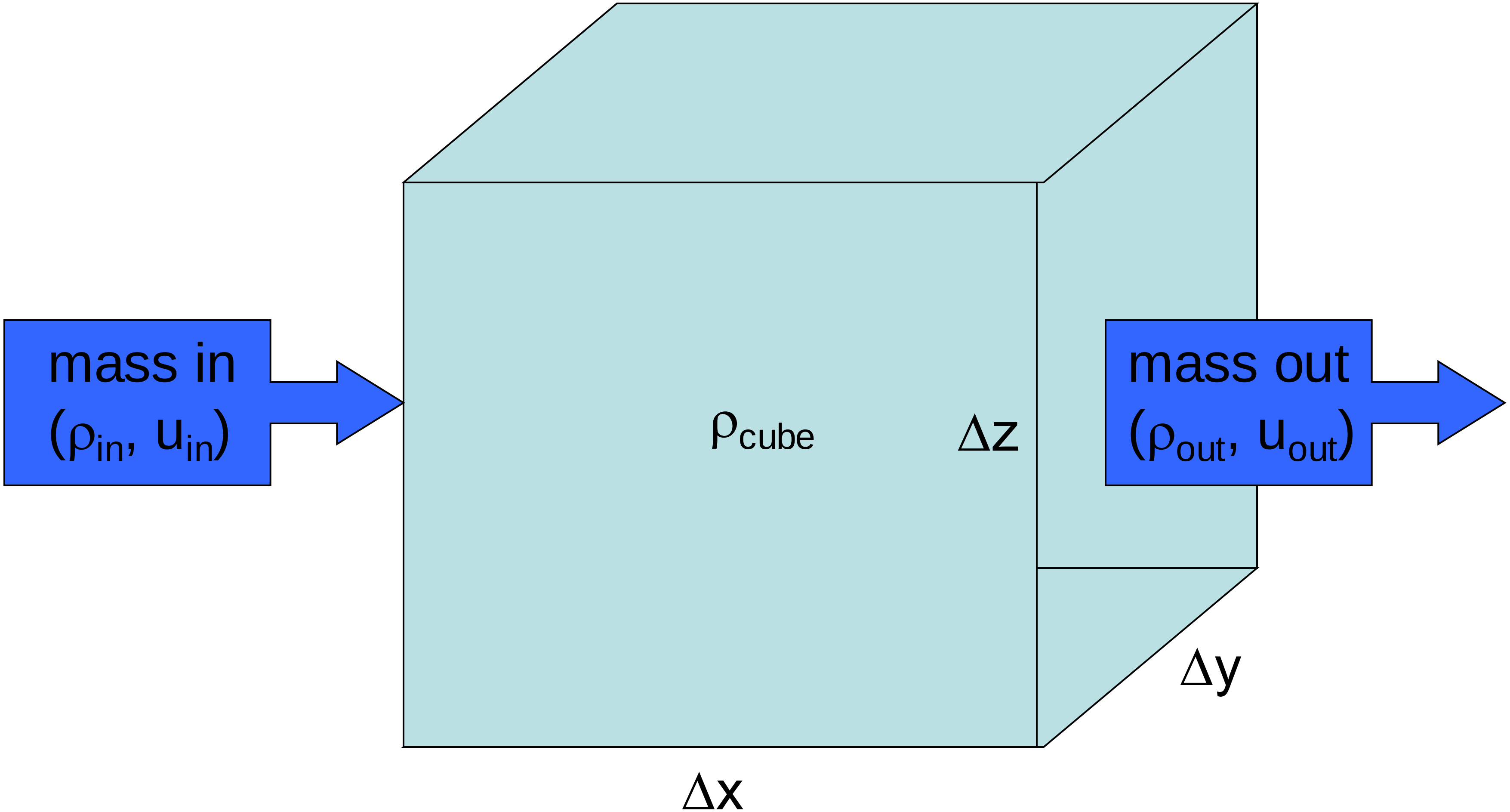
$m_{out} - m_{in} = 0$

$\rho\, \Delta y \Delta z\, u_{out} \Delta t - \rho\, \Delta y \Delta z\, u_{in} \Delta t = 0$

$\rho\, u_{out} - \rho\, u_{in} = 0$ and if density is constant

$u_{out} - u_{in} = 0$

Now consider cube with walls (no flow through walls):



Relax equilibrium assumption: *compressible flow*

mass in $\neq$ *mass out* (at any instant in time)

Compressible flow through cube.

$m_{in} - m_{out} = \Delta m$ (if $m_{in} > m_{out}$ mass in box will increase)

In time Δt:

Δm is change in mass in box. $\Delta m = \Delta\rho \text{Volume} = \Delta\rho\Delta x\Delta y\Delta z$

$m_{in} = \rho_{in}u_{in}\,\Delta t\Delta y\Delta z$
$m_{out} = \rho_{out}u_{out}\,\Delta t\Delta y\Delta z$

$\Rightarrow m_{in} - m_{out} = \rho_{in}u_{in}\,\Delta t\Delta y\Delta z - \rho_{out}u_{out}\,\Delta t\Delta y\Delta z = \Delta\rho\Delta x\Delta y\Delta z$
$\Rightarrow$ ….
$\Rightarrow \Delta\rho / \Delta t = (\rho_{in}u_{in} - \rho_{out}u_{out}) / \Delta x$

In limit as $\Delta t \to 0$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

This result extends easily to three-dimensions:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}$$

and can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

or

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0$$

This is the equation governing *mass continuity.*

If $\mathbf{w}$ is an arbitrary vector field and

V is an enclosed volume with surface S and

$d\mathbf{S}$ is an infinitessimal vector normal to this surface with

a magnitude equal to the infinitessimal surface area it is uniquely associated with, then

$$\oint_S \mathbf{w}.d\mathbf{S} = \int_V (\nabla.\mathbf{w})\, dV$$

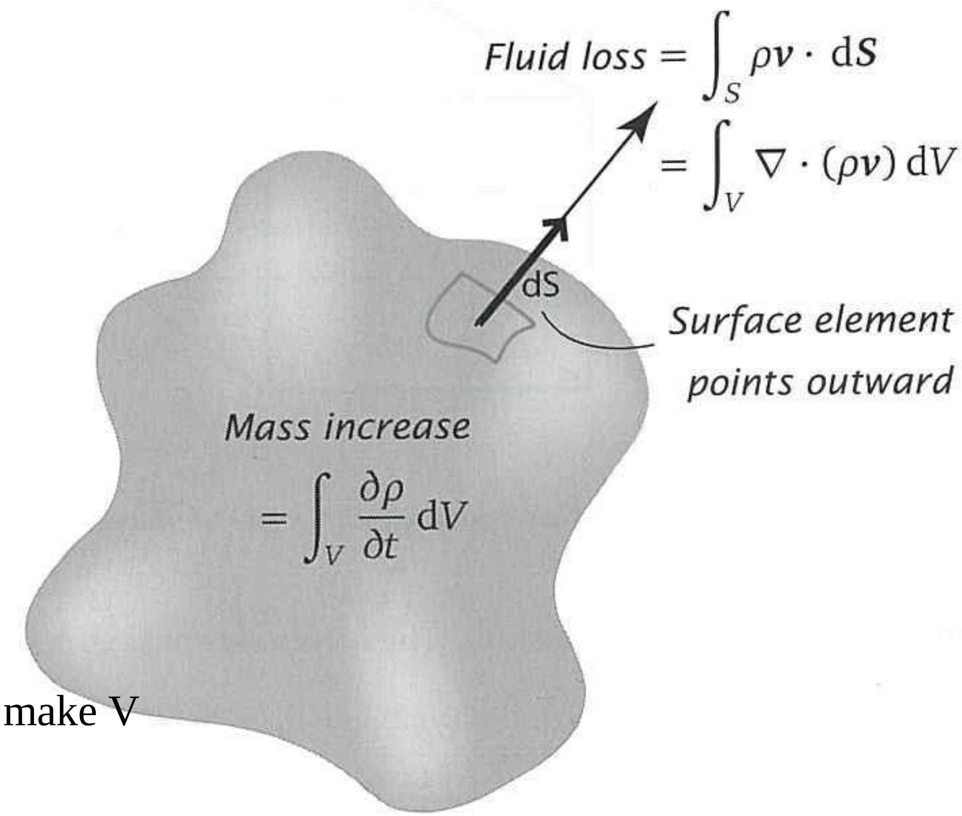


Fig. 1.2 Mass conservation in an arbitrary Eulerian control volume V bounded by a surface S. The mass gain, $\int_V (\partial\rho/\partial t)\,\mathrm{d}V$ is equal to the mass flowing into the volume, $-\int_S (\rho \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = -\int_V \nabla \cdot (\rho \boldsymbol{v})\,\mathrm{d}V$.

Since dV is fixed, we can make V infinitessimal so that

$$\frac{\partial \rho}{\partial t} d\mathrm{V} = -\nabla.(\rho \mathbf{v})\, d\mathrm{V}$$

and then divide by dV to obtain

$$\frac{\partial \rho}{\partial t} = -\nabla.(\rho \mathbf{v})$$

From p10 of text book by GK Vallis

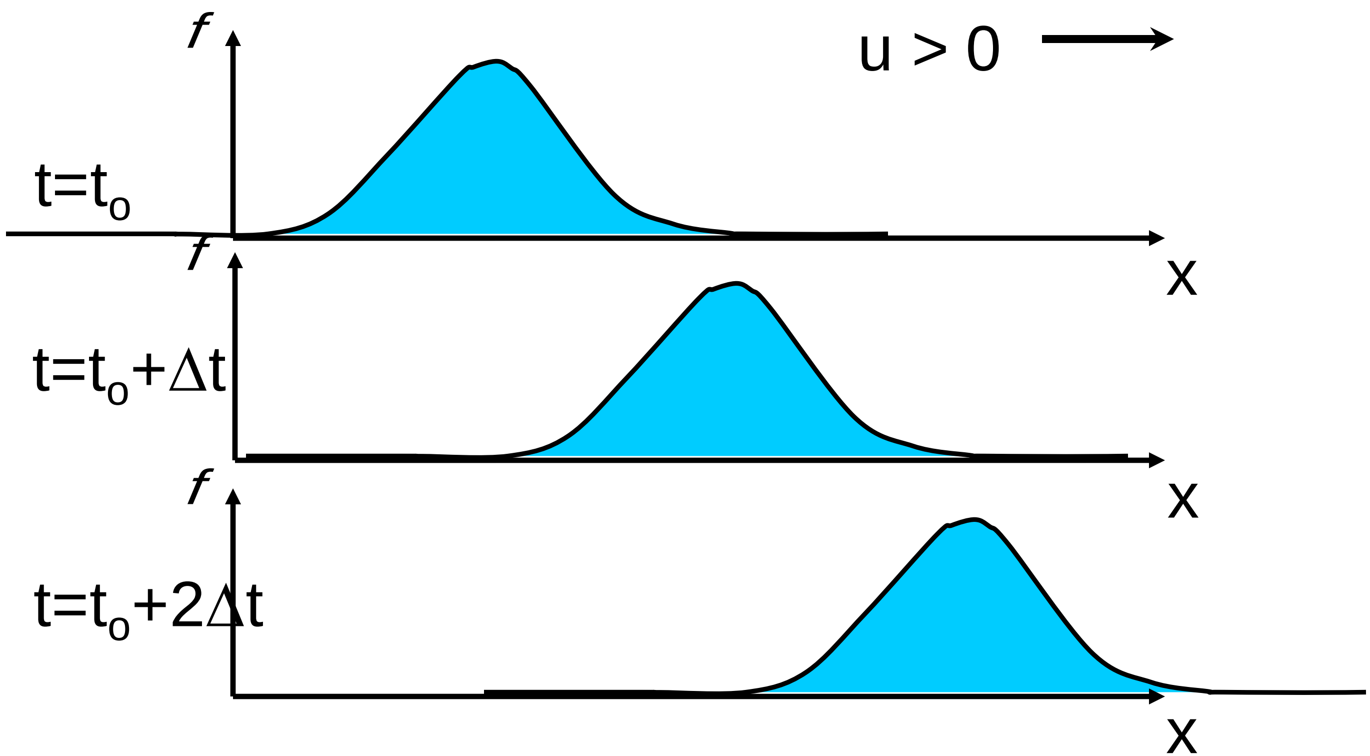
The material derivative - reminder from last lecture.

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

(Rate of change following fluid parcel)

A purely conserved quantity: $$\frac{Df}{Dt} = 0$$

$$\frac{\partial f}{\partial t} = - u\frac{\partial f}{\partial x}$$



The material derivative - reminder from last lecture.

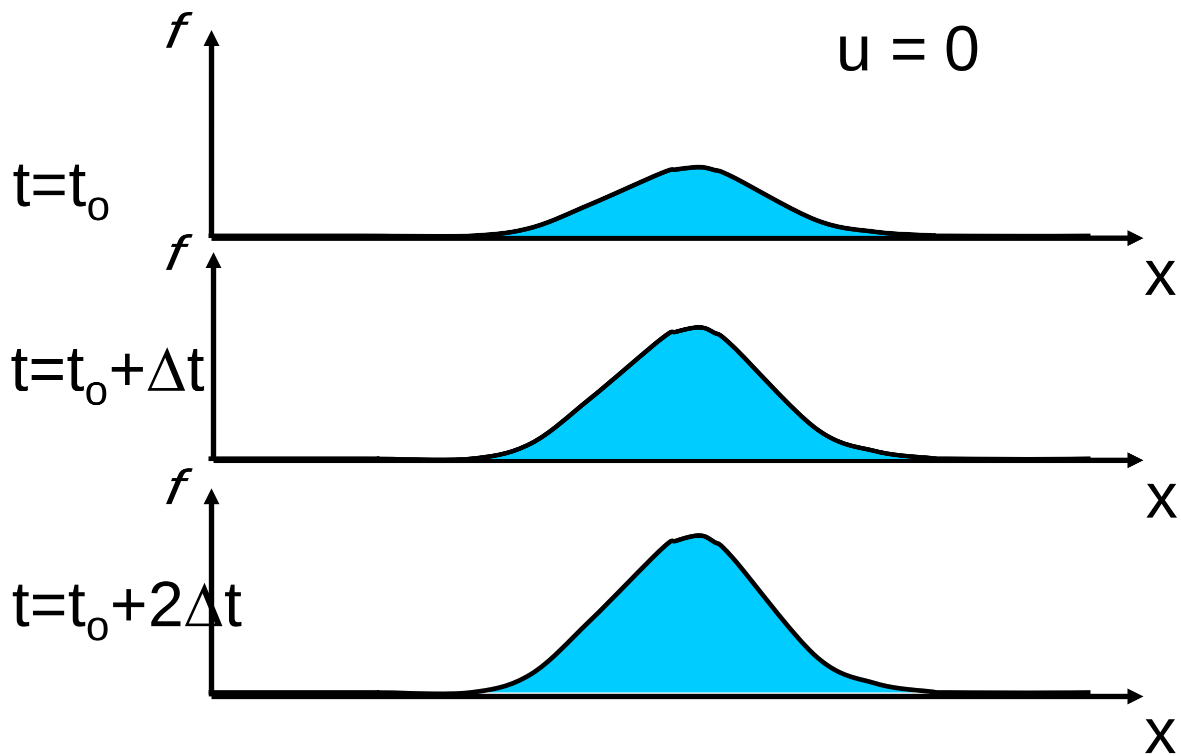
$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

(Rate of change following fluid parcel)

A non-conserved quantity (with zero wind):

$$\frac{Df}{Dt} = Source$$

$$\frac{\partial f}{\partial t} = Source$$



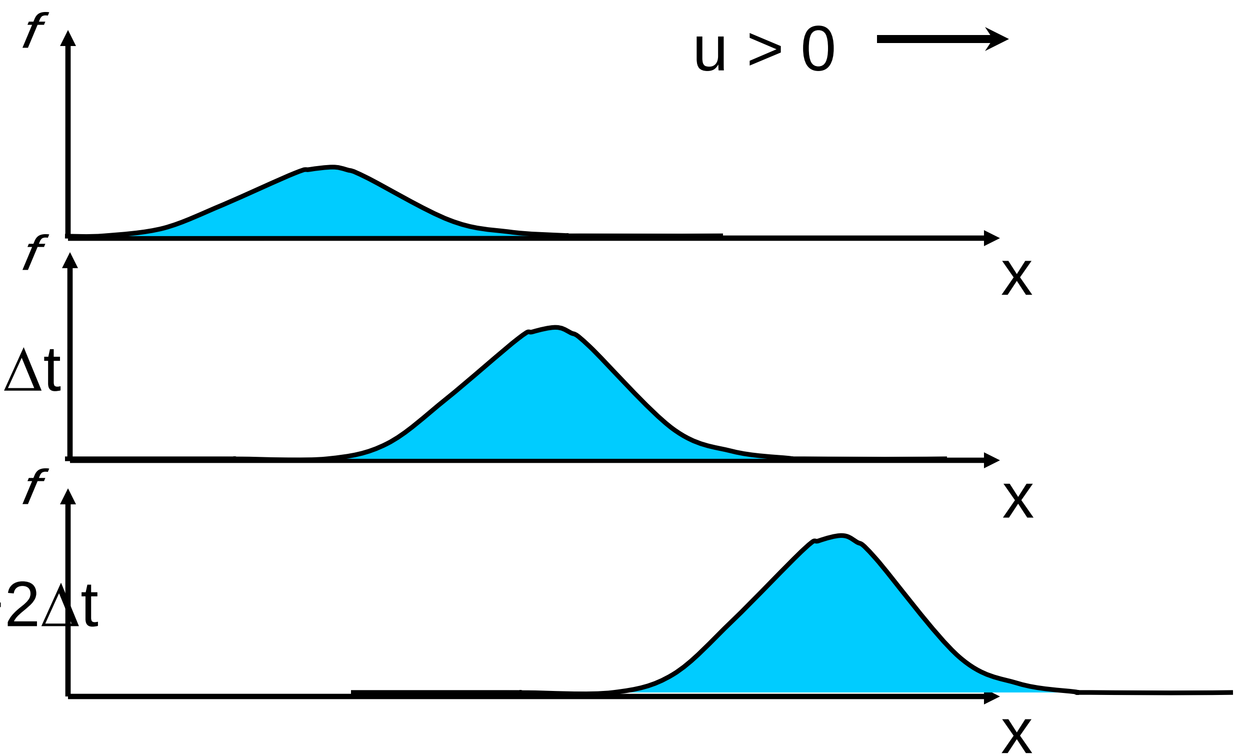
The material derivative - reminder from last lecture.

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

(Rate of change following fluid parcel)

A non-conserved quantity (with wind):

$$\frac{Df}{Dt} = Source$$



$$\frac{\partial f}{\partial t} = -u\frac{\partial f}{\partial x} + Source$$

Mass continuity - Compressible flow:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \underline{u} = 0$$

- Total mass is conserved - but density is not.
- Density can vary in space and time.
- Local variations in density arise from advection and/or flow convergence/divergence.
- The atmosphere is compressible.
- Compressibility allows the existence of sound waves.

Incompressible flow:

$$\frac{Dr}{Dt} = 0$$

- Density is conserved.
- Density can vary in space and time.
- Local variations in density arise from advection only.
- The ocean is nearly incompressible.
- $\frac{Dr}{Dt} = 0$ implies that $\nabla \cdot \underline{u} = 0$
- Is a useful approximation for many atmospheric situations

Other approximations:

Homogeneous flow: Density is always constant everywhere.

Can show that this implies: $$\nabla \cdot \underline{u} = 0$$

Anelastic flow: Density is a function of height, *z*, only.

Can show that this implies: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial(\rho w)}{\partial z} = 0$$

Many (or even most) atmospheric situations are explained well by the anelastic approximation.

Why do we care?

Conservation of mass is a fundamental property of fluid flow.

Examples:

- *Running tap*
- *Gap flows*
- *Clouds / convection*
- *Oceanic gyres*
- *Large-scale atmospheric circulations / cells*

Incompressibility (or anelastic) assumption can be used to determine regions of ascent or descent from horizontal winds.

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\nabla_H \cdot \underline{u}_H$$

The vertical velocity, *w*, is zero at the surface.

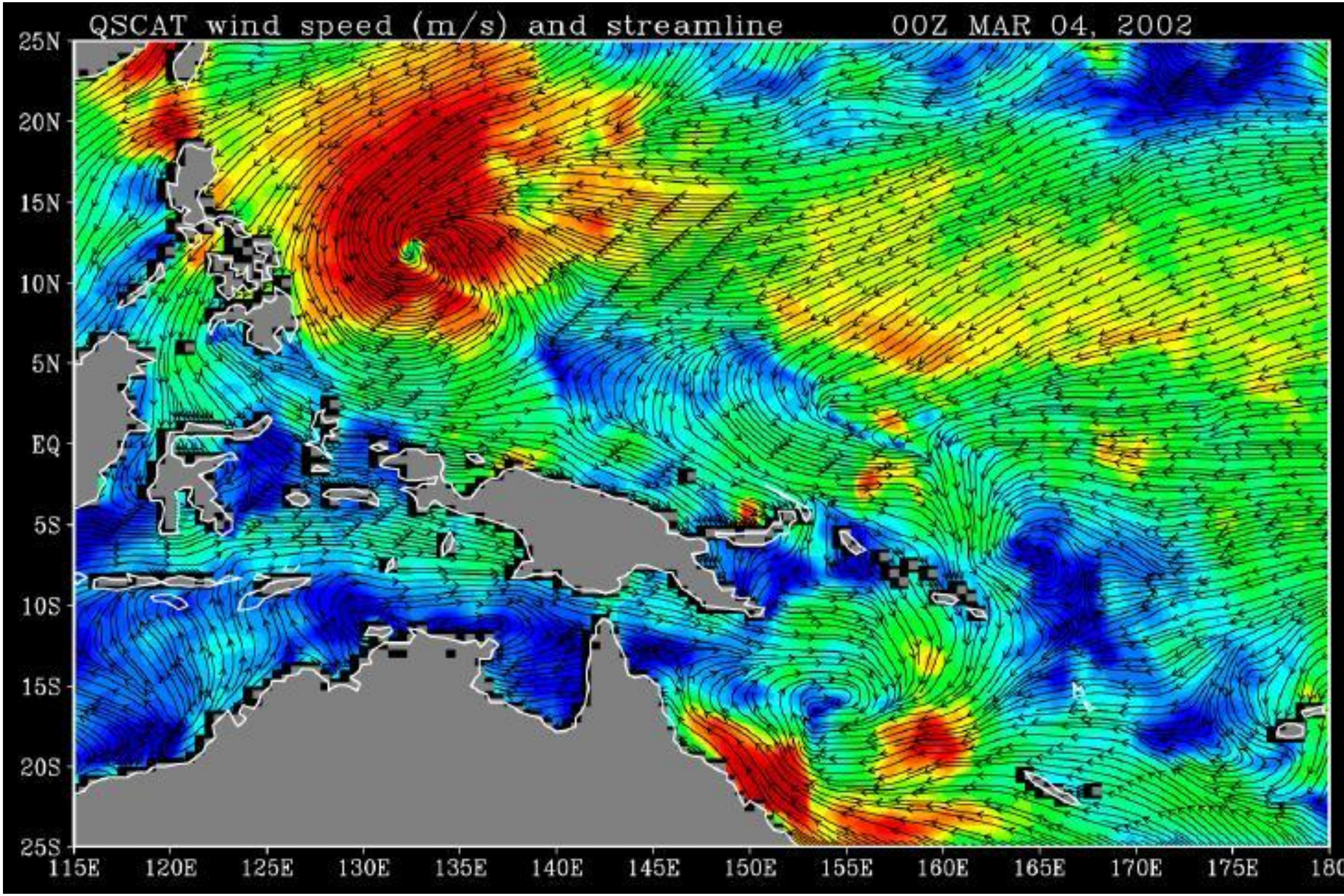
$$\frac{\partial w}{\partial z} > 0$$ near surface implies $w > 0$

Convergent flow (negative horizontal divergence) at (or near) surface implies upward motion.

Analysis of divergence:

- Useful if only surface observations are available
    - AWS or Quickscat winds
- Horizontal wind observations only
    - Radiosondes
- Coarse resolution models
    - poor / noisy representation of vertical velocity

Quikscat - surface winds observed from satellite.



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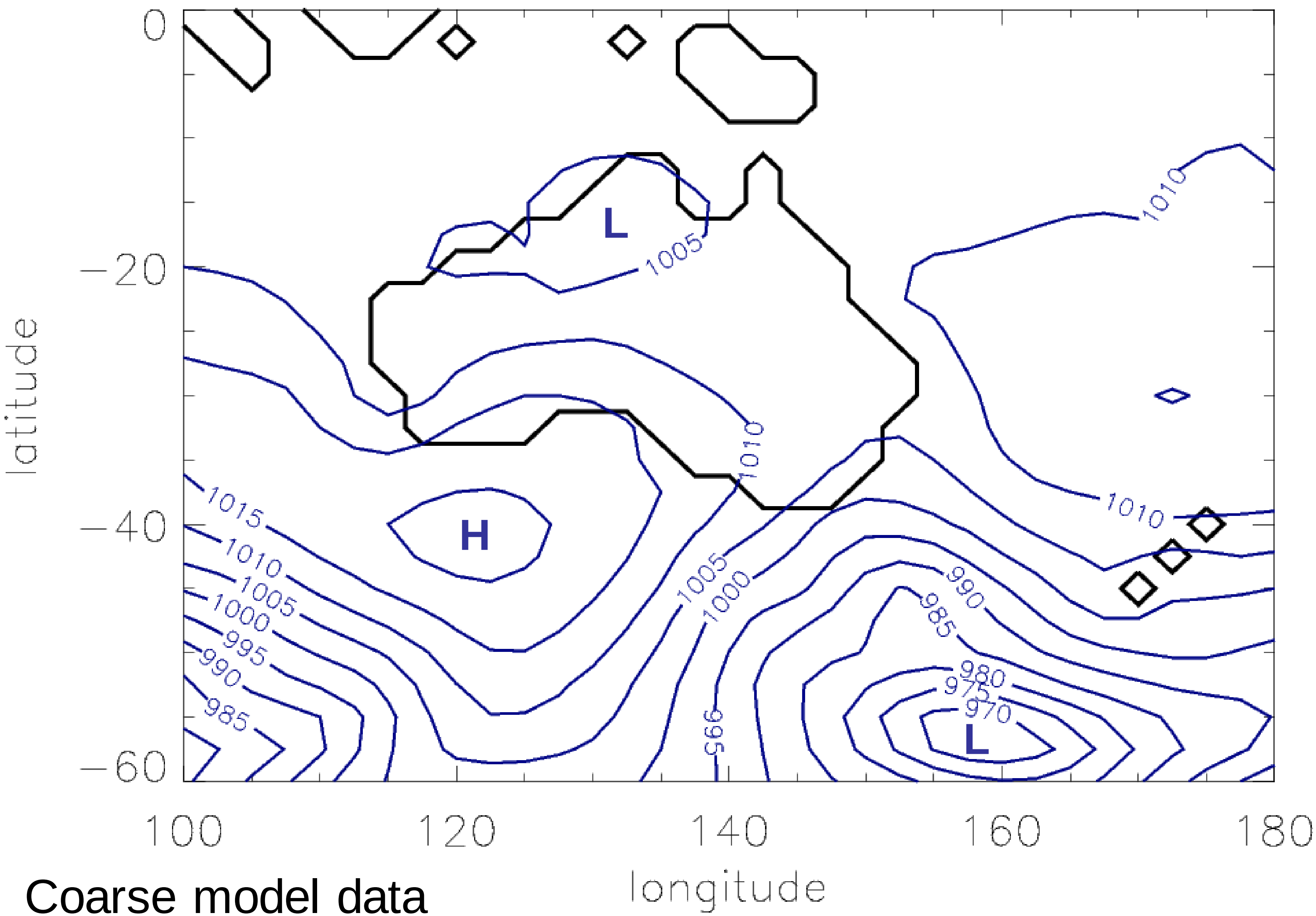
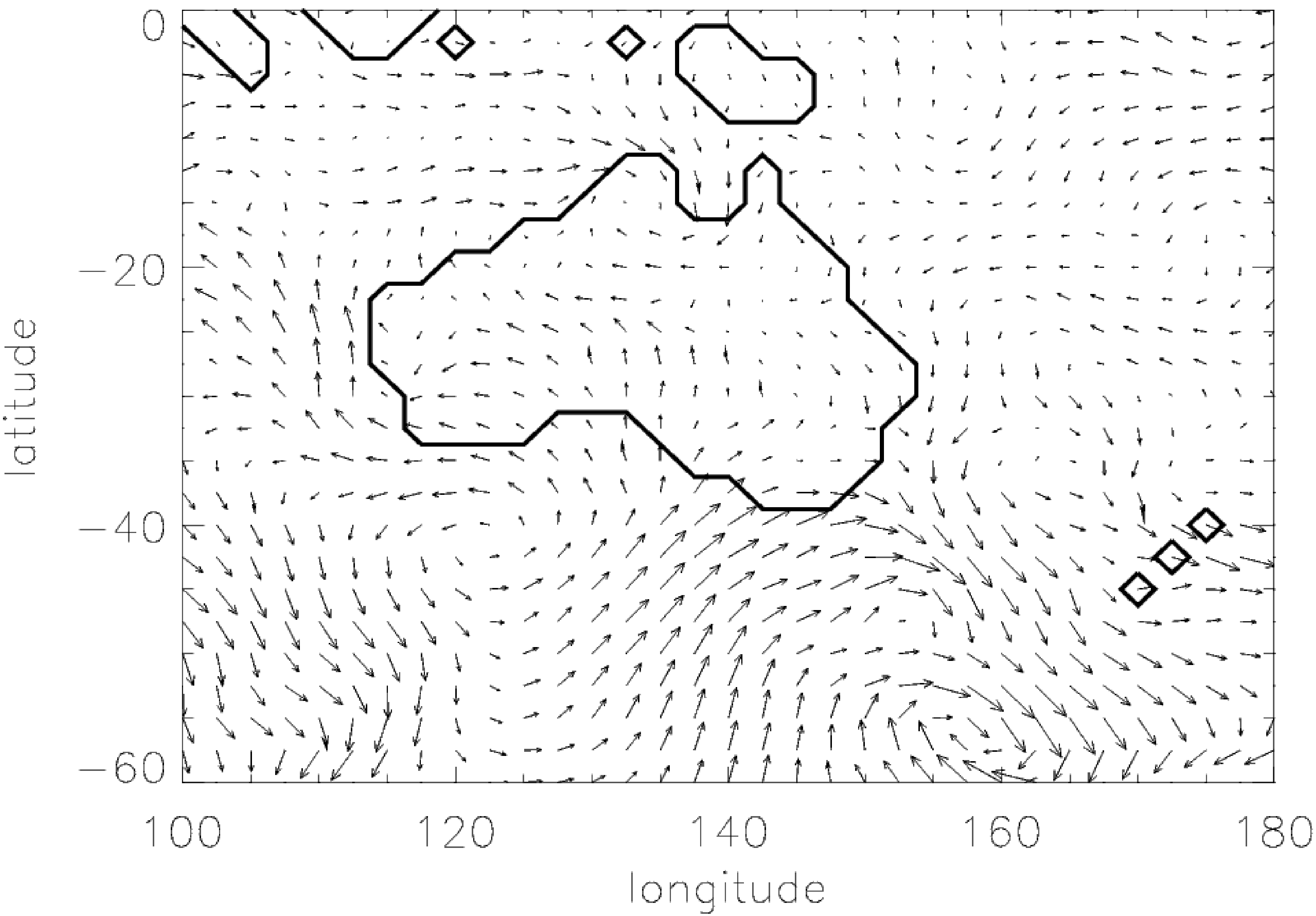
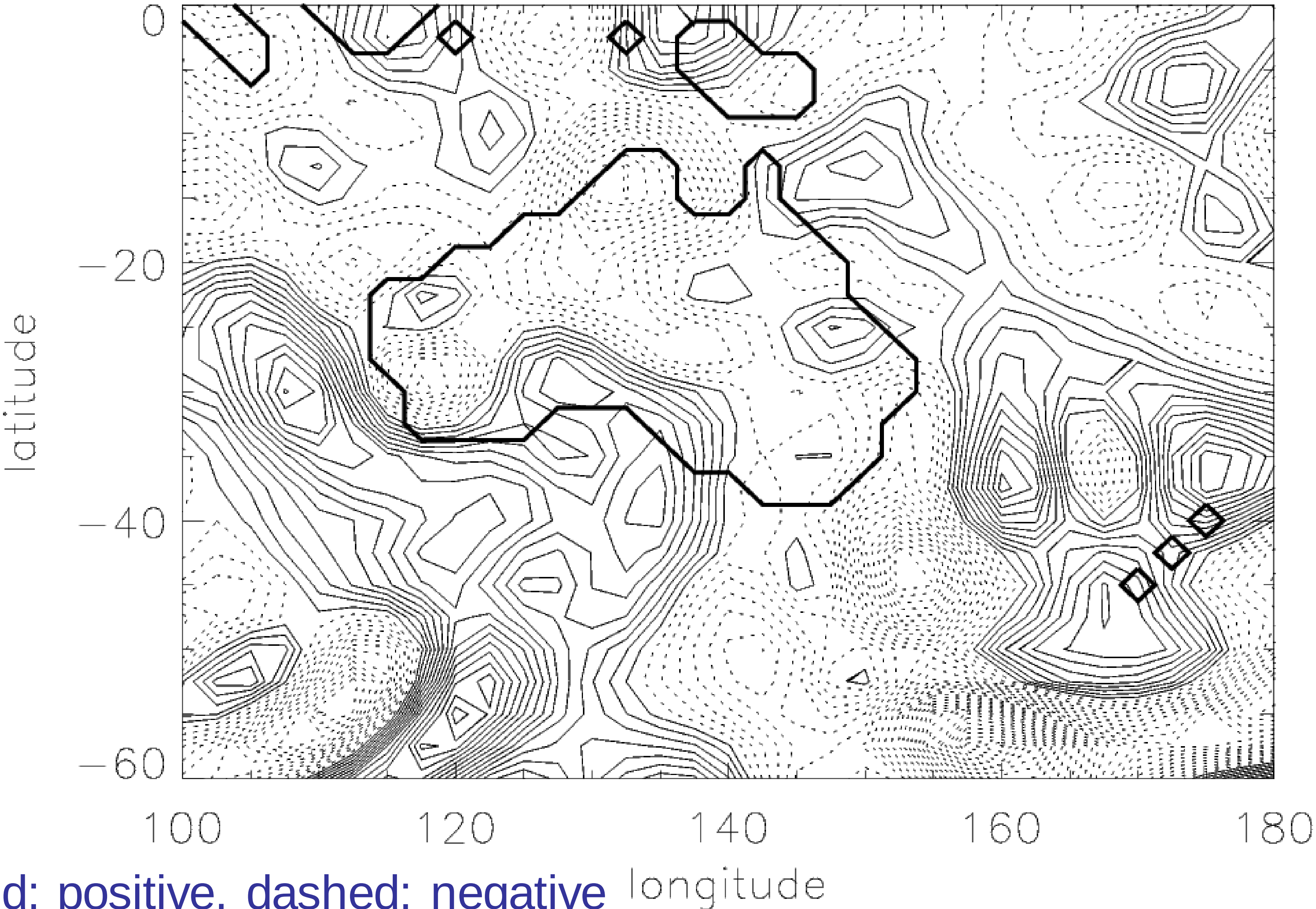


Coarse model data

Wind vectors

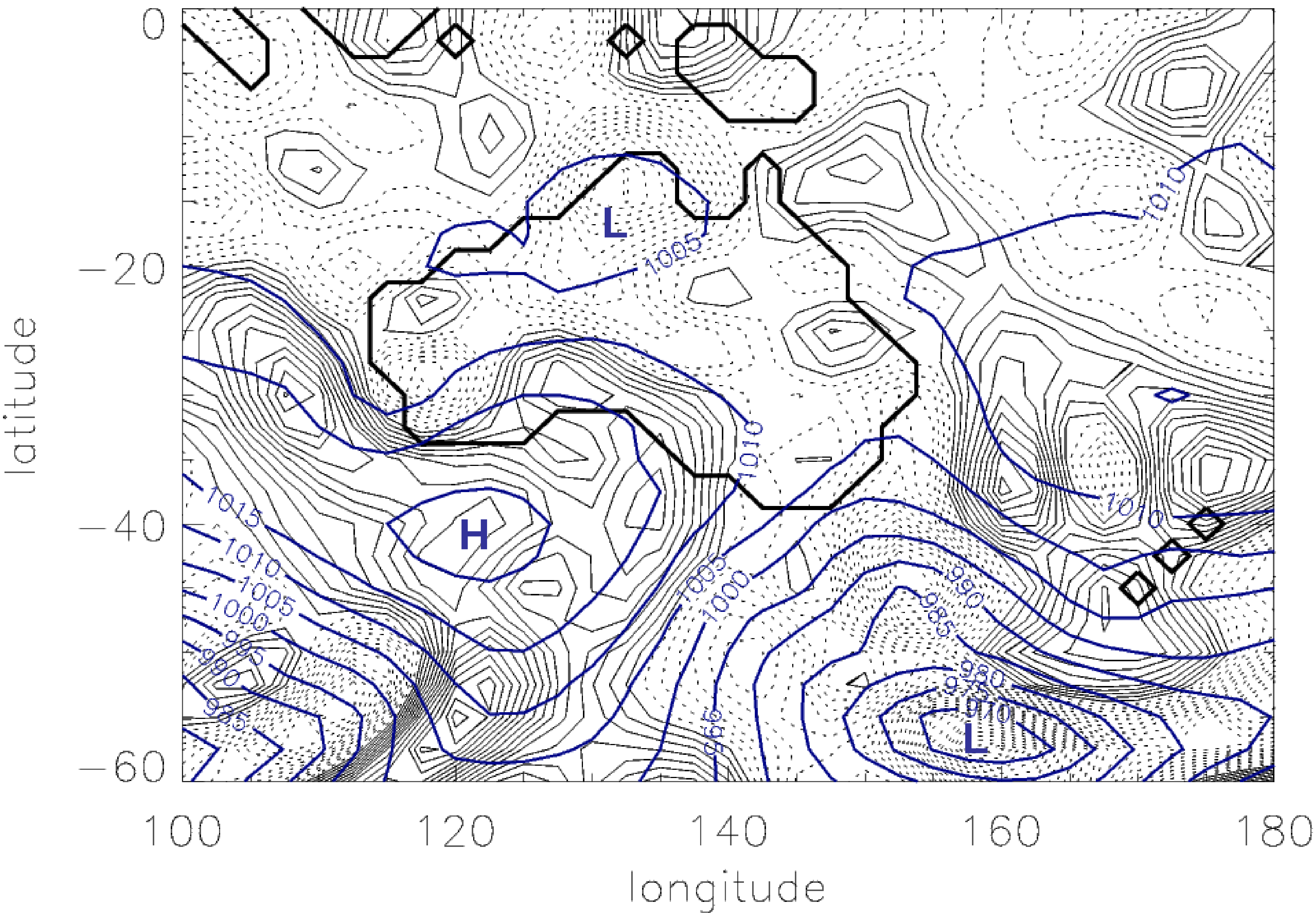


Horizontal divergence at surface



solid: positive, dashed: negative

Horizontal divergence at surface & MSLP



Momentum equations (in non-rotating reference frame).

What causes winds / currents?

For winds / currents to form and/or evolve, there must be an acceleration of the flow.

Acceleration: rate of change of velocity.

Because we are talking about fluid flow this rate of change MUST be the material derivative; i.e., the TOTAL rate of change.

Define a three-dimensional fluid acceleration vector: $\underline{a}$

$$\underline{a} = \frac{D\underline{u}}{Dt} = \frac{Du}{Dt}\underline{\mathrm{i}} + \frac{Dv}{Dt}\underline{\mathrm{j}} + \frac{Dw}{Dt}\underline{\mathrm{k}}$$

What controls acceleration?

<u>Newton's 2nd Law of motion:</u>

The acceleration of an object is directly proportio to the net force acting on it and is inversely propo to its mass. The direction of the acceleration is in direction of the applied net force.

Important points: <u>*net force*</u>

<u>*acceleration in direction of net force*</u>

$$\sum \underline{F} = \frac{D\left(m\mathbf{u}\right)}{Dt} = m\underline{a} \text{ (provided } \frac{Dm}{Dt} = 0)$$

Newton's 2nd law applied to fluid flow:

$$\frac{D\left(m_{parcel}\right)\underline{u}}{Dt} = m_{parcel}\frac{D\underline{u}}{Dt} = \left(dV\rho\right)\frac{D\underline{u}}{Dt} = \sum_{i=1}^{n}\underline{F}_i dV = dV\sum_{i=1}^{n}\underline{F}_i$$

$$=> dV\rho\frac{D\underline{u}}{Dt} = dV\sum_{i=1}^{n}\underline{F}_i$$

where $\underline{F}_i dV$ is the ith force on the volume and

$\underline{F}_i$ is the corresponding force per unit volume.

Dividing both sides of the equation by dV gives

$$\rho\frac{D\underline{u}}{Dt} = \sum_{i=1}^{n}\underline{F}_i$$

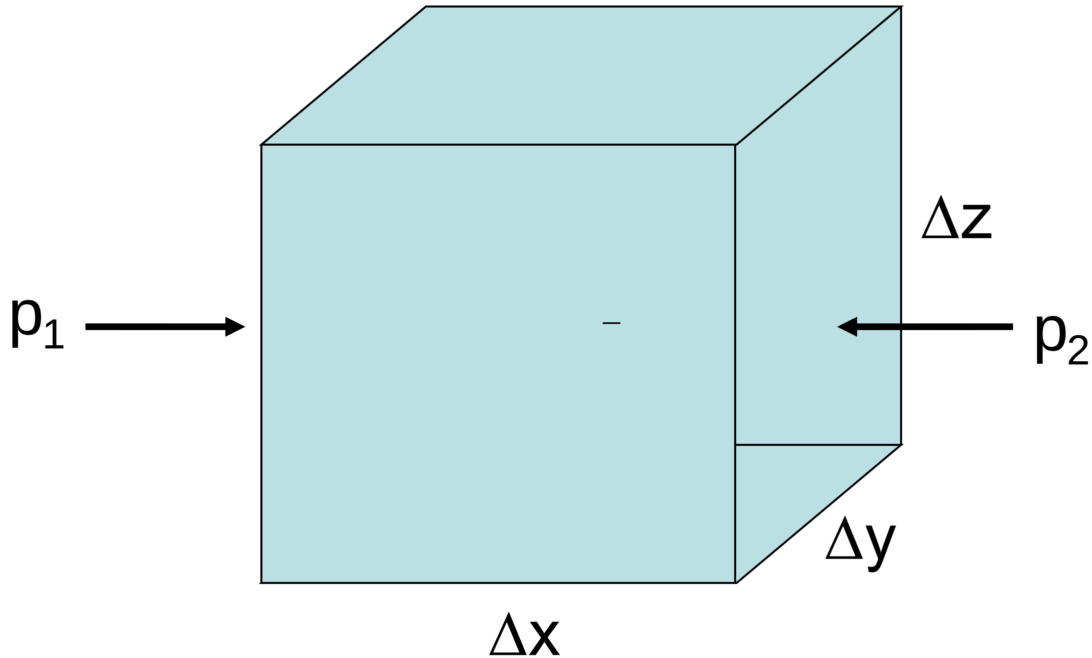
Forces: *pressure gradient, gravity, friction.*

Pressure gradient force per unit volume

(typically just abbreviated to **pressure gradient force**)

Recall: pressure is force per unit area.

Force on a face of cube = pressure times area



In *x*-direction, net force = $p_1\Delta y\Delta z - p_2\Delta y\Delta z$

Force per unit volume = $(p_1\Delta y\Delta z - p_2\Delta y\Delta z)/\Delta x\Delta y\Delta z$

$$= -\,{}^{\partial p}\!/\!{}_{\partial x}$$

To prove this step, note that presure, $p,$ is a function of all spatiotemporal coordinates $p(x, y, z, t)$.

$p_1 = p(x_1, y, z, t)$ and $p_2 = p(x_1 + \Delta x, y, z, t)$

so

$$\lim \Delta x -> 0 \left[\frac{p_1 - p_2}{\Delta x} \right] = \lim \Delta x -> 0 \frac{p(x_1, y, z, t) - p(x_1 + \Delta x, y, z, t)}{\Delta x}$$

$$= \lim \Delta x -> 0 - \frac{p(x_1 + \Delta x, y, z, t) - p(x_1, y, z, t)}{\Delta x}$$

$$= -\frac{\partial p}{\partial x}$$

Pressure force (per unit volume) is

$$-\nabla p = -\frac{\partial p}{\partial x}\underline{\mathrm{i}} + -\frac{\partial p}{\partial y}\underline{\mathrm{j}} + -\frac{\partial p}{\partial z}\underline{\mathrm{k}}$$

Force is directed from high to low pressure.

In the absence of any other forces:

$$\rho\frac{D\underline{u}}{Dt} = -\nabla p \qquad \text{or} \qquad \frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p$$

One additional important contributor: gravity

In the absence of pressure (or other forces) gravity will induce a downward acceleration ($-Dw/Dt$) equal to g = 9.8 m/s^2.

The momentum equations (in the absence of friction) are therefore:

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\underline{\mathrm{k}}$$

*** *Remember this* ***

The momentum equations

(inviscid, sometimes called the Euler equations)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

This is a set of coupled, nonlinear, partial differential equations.

- *while complicated (and a little scary) they tell us that the fluid velocity can change at a point in space due to an imposed pressure gradient and / or gravitational force, or due to simple advection.*

One more force / effect on fluid flow: **Viscosity**

Viscosity: internal friction of fluid.

It can be shown that the viscous force per unit volume is:

$$\nu\nabla^2\underline{u} = \nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2},\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2},\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

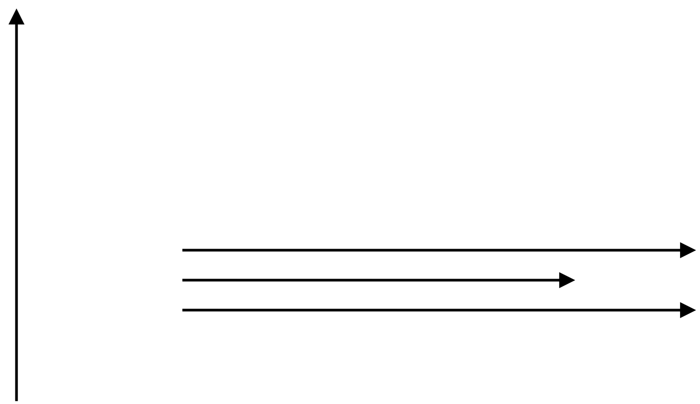
where ν is the kinematic viscosity.

With viscosity included, the momentum equations become:

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\underline{k} + \nu\nabla^2\underline{u}$$

(Navier-Stokes equations)

The kinematic viscosity of air is small ($1\times10^{-5}\ m^2s^{-1}$). Many (or most) atmospheric/oceanic flows can be explained by inviscid processes.The effect of viscosity / turbulence are diffusive.



Important points from Topic 1.

<u>The material derivative of any fluid property:</u>

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$$

The continuity of mass: $$\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0$$

The incompressible approximation: $$\nabla\cdot\underline{u} = 0$$

The momentum equations: $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\underline{\mathrm{k}}$$