Dynamical Meteorology and Oceanography ATOC30004

Topic 6: Introduction to atmospheric and oceanic wave motion

- Waves in shallow water
- Rossby waves
- Inertial oscillations
- Gravity waves

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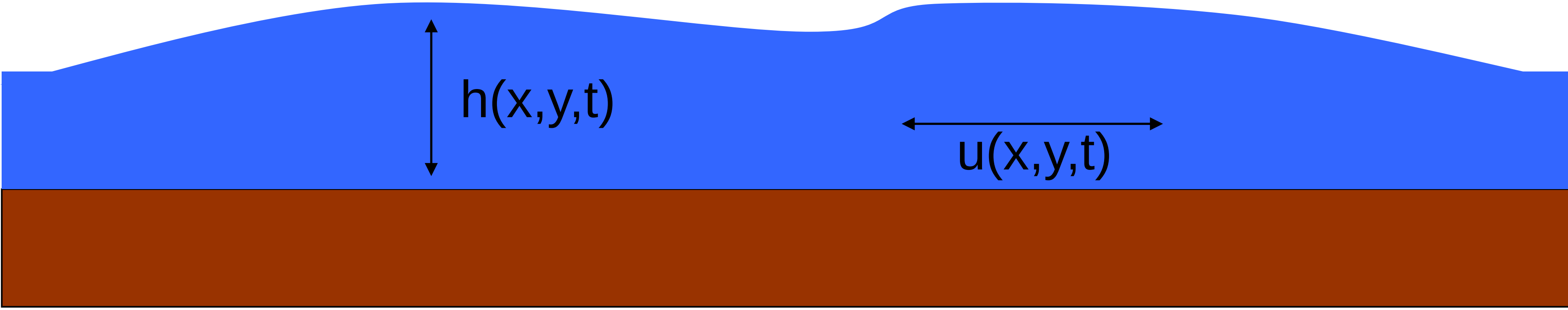
Shallow water flow derivation.

Assumptions:

- Homogeneous fluid of depth, h.
- Constant velocity through depth of fluid (u=u(x,y,t)).
- Hydrostatic (L>>H)



Modifications to full hydrostatic equations of motion:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

Relationship between pressure and fluid depth:

$$\frac{\partial p}{\partial z} = -\rho g$$

Hydrostatic

$$\int_0^h \frac{\partial p}{\partial z}\,.\,dz = -\int_0^h \rho g\,.\,dz$$

Integrate over fluid depth (assume density constant)

$$p(h) - p(0) = -\left[\rho g z\right]_0^h = -\rho g h$$

$$p(0) = \rho g h + p(h)$$

Assume that pressure at top of fluid, p(h), is constant

$$\Rightarrow \frac{\partial p}{\partial x} = \rho g\frac{\partial h}{\partial x}, \qquad \frac{\partial p}{\partial y} = \rho g\frac{\partial h}{\partial y}$$

Pressure gradient (within fluid) is related to horizontal gradient in fluid depth.

Horizontal equations of motion become:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial h}{\partial x}$$

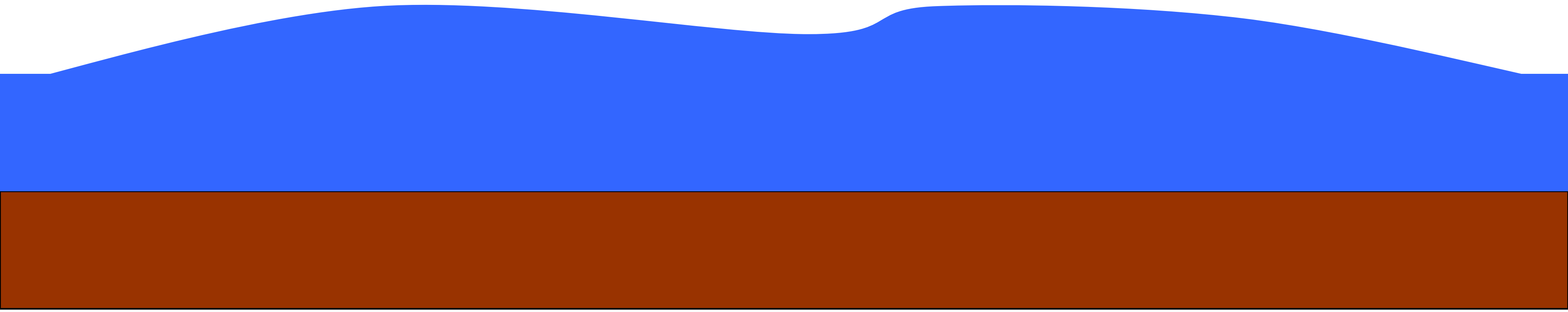
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial h}{\partial y}$$

We made the homogeneous assumption (density = constant) also:

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\nabla_H \cdot \underline{u}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

From topic 5



At the bottom of this fluid, w=0. At the top of the fluid w=$^{Dh}/_{Dt}$

$$\frac{\partial w}{\partial z} = -\nabla_H \cdot \underline{u}_H$$

Integrate over depth of fluid

$$\int_0^h \frac{\partial w}{\partial z} dz = -\int_0^h \nabla_H \cdot \underline{u}_H dz$$

Remember u and v are constant over depth

$$w(h) - w(0) = -h\nabla_H \cdot \underline{u}_H$$

Remember w(0)=0   and w(h)=Dh/Dt

$$\frac{Dh}{Dt} = -h\nabla_H \cdot \underline{u}_H = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\boxed{\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\frac{\partial u}{\partial x} - h\frac{\partial v}{\partial y}}$$

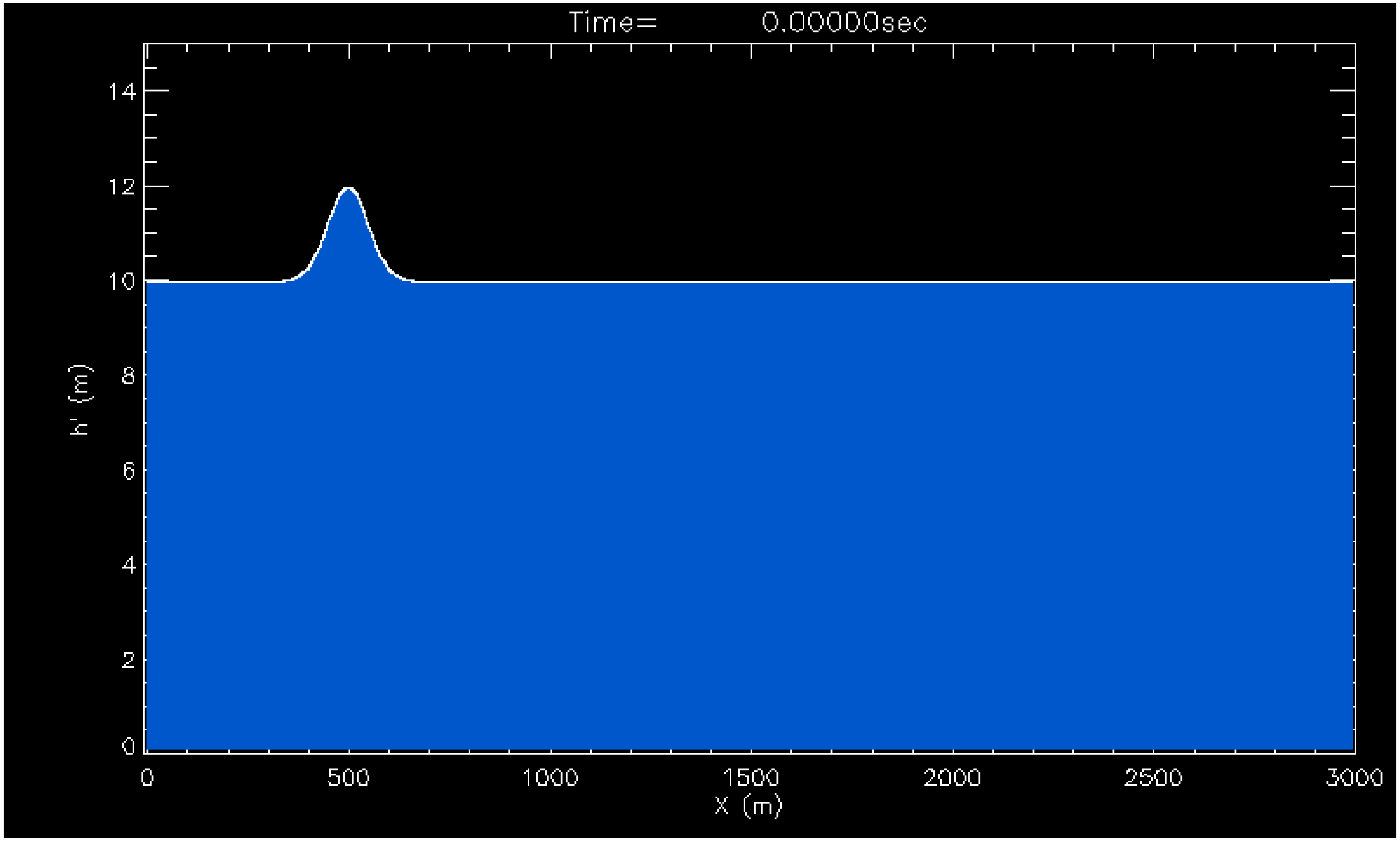
This gives us the shallow water system of equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\frac{\partial u}{\partial x} - h\frac{\partial v}{\partial y}$$

For some atmospheric / oceanic applications the shallow water system provides a realistic representation of dynamic processes. The shallow water system supports wave motion. For waves with periods much shorter than a day, such as ocean waves, the Coriolis terms can be neglected.



What controls characteristics of wave:

- Amplitude and width comes from initial disturbance
- Speed constrained by dynamics.

Consider a sinusoidal wave of constant amplitude, A.

The fluid depth, h=H+h', where H is the mean depth and h' is the wave perturbation.

If we only consider one spatial dimension we can write:

$$\exp(i\theta) = \cos\theta + i\sin\theta, \text{ where } i = \sqrt{-1}$$

$$h'(x,t) = \mathrm{Re}\left\{A\exp\left[i(kx - \omega t + \phi)\right]\right\}$$

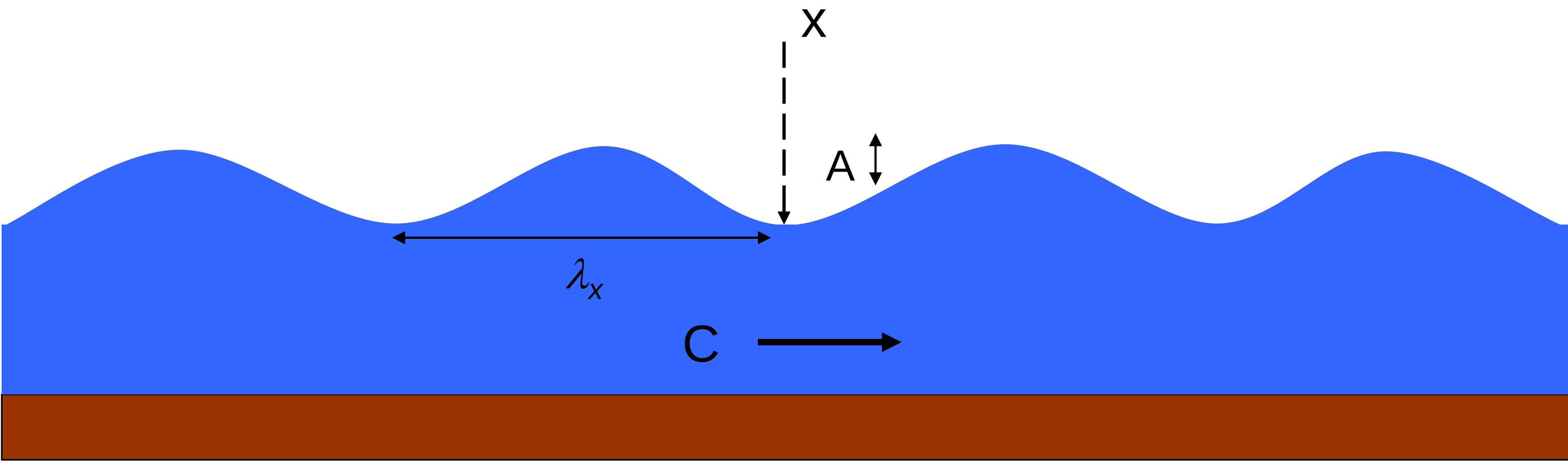
$$= A\cos(kx - \omega t + \phi)$$

$$\text{wavelength} = \lambda_x = 2\pi / k$$

$$\text{period} = T = 2\pi / \omega$$

$$\text{phase speed} = c = \lambda_x / T = \omega / k$$

A is the amplitude
$k = 2\pi / \lambda_x$ is the horizontal wavenumber (λ_x is the horizontal wavelength)
$\omega = 2\pi / T$ is the wave frequency (T is the wave period)
ϕ is a phase difference.



c is wave phase speed, which measures the speed of a point of constant phase, e.g. a peak.

In one wave period, *T*, the wave will travel one wavelength, λ_x, in the horizontal.

An observer at point x will initially measure a trough, then a crest, then a trough in one complete period.

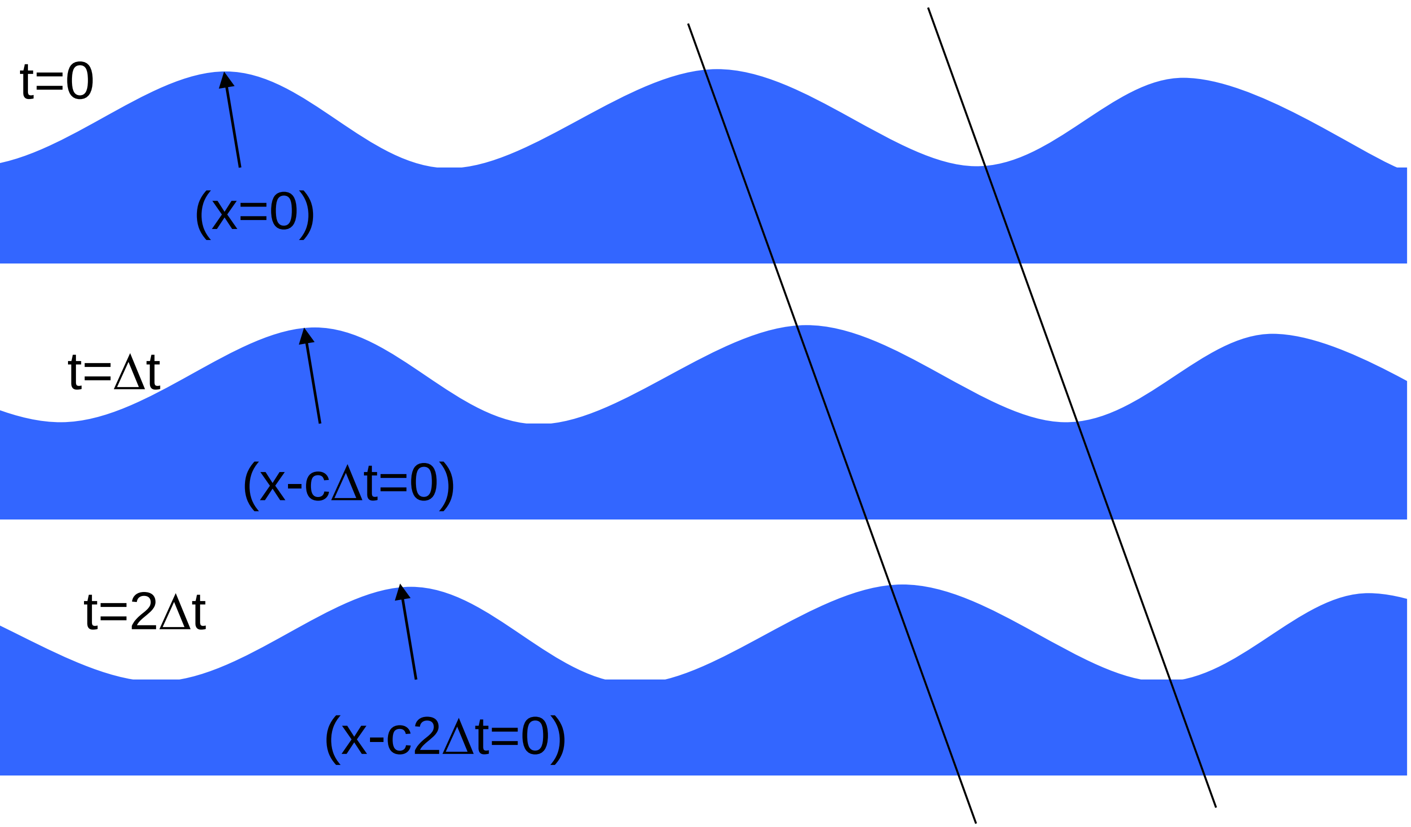
The phase speed is therefore: $$c = \frac{\lambda_x}{T} = \frac{\omega}{k}$$

We can rewrite:

$$\eta'(x,t) = A\cos(kx - \omega t + \phi)$$

as

$$\eta'(x,t) = A\cos(k(x - ct) + \phi)$$



Lines of constant phase

If ϕ=0

Peak when (x-ct) =0

This implies a peak whenever x=ct

Thus, c=dx/dt, is the speed of movement of the point of constant phase.

To a certain extent the wavelength / wavenumber is defined by the initial displacement. What determines the phase speed, c?

- For shallow water flow the phase speed c is constrained by the flow depth.

To prove this we will derive a wave equation.

Neglect the Coriolis force, and assume 1D non-rotating flow (*v*=0, *u(x,t), h(x,t)*) then :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\frac{\partial u}{\partial x}$$

Assume that the wind and the fluid depth can be separated into $h=H+h'$ and $u=U+u'$ where the primes represent the wave perturbations. *H* is the mean fluid depth (a constant) and U is the mean fluid velocity (also a constant). We can re-write the shallow water eqns as:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial h'}{\partial t} + U\frac{\partial h'}{\partial x} + u'\frac{\partial h'}{\partial x} = -H\frac{\partial u'}{\partial x} - h'\frac{\partial u'}{\partial x}$$

At this point we have made no additional approximations:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial h'}{\partial t} + U\frac{\partial h'}{\partial x} + u'\frac{\partial h'}{\partial x} = -H\frac{\partial u'}{\partial x} - h'\frac{\partial u'}{\partial x}$$

Let's now assume that the waves are *small-amplitude. i.e.*:

h' << H   and u' << U

If this is the case, then products of small quantities, i.e., nonlinear terms, will be much smaller than the others (linear terms). Let's neglect the small terms:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \cancel{u'\frac{\partial u'}{\partial x}} = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial h'}{\partial t} + U\frac{\partial h'}{\partial x} + \cancel{u'\frac{\partial h'}{\partial x}} = -H\frac{\partial u'}{\partial x} - \cancel{h'\frac{\partial u'}{\partial x}}$$

The result is:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -g\frac{\partial h'}{\partial x}$$

$$\frac{\partial h'}{\partial t} + U\frac{\partial h'}{\partial x} = -H\frac{\partial u'}{\partial x}$$

This is what's referred to as a *linear approximation.*

We could have done this more formally with a scale analysis. E.g., assign:

U ~ U

X ~ L

T ~ L/U

H ~ H

u' ~ μ

h' ~ η

And assume that η and μ are at least an order of magnitude smaller than H and U

Our linear shallow water equations are:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = -g\frac{\partial h'}{\partial x} \qquad \frac{\partial h'}{\partial t} + U\frac{\partial h'}{\partial x} = -H\frac{\partial u'}{\partial x}$$

<u>For simplicity, assume U=0</u>

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \qquad \frac{\partial h'}{\partial t} = -H\frac{\partial u'}{\partial x}$$

Take derivatives: *wrt x* *wrt t*

$$\frac{\partial^2 u'}{\partial t\,\partial x} = -g\frac{\partial^2 h'}{\partial x^2} \qquad \frac{\partial^2 h'}{\partial t^2} = -H\frac{\partial^2 u'}{\partial x\,\partial t}$$

Eliminate *u'* to get:

$$\frac{\partial^2 h'}{\partial t^2} = gH\frac{\partial^2 h'}{\partial x^2}$$

$$\frac{\partial^2 h'}{\partial t^2} = gH \frac{\partial^2 h'}{\partial x^2}$$

Is the *wave equation* for (non-rotating) shallow water flow.

Use this to determine wave phase speed.

If our wave solution is of form: $h' = A\cos(kx - \omega t + \phi)$

Then:

$$\frac{\partial h'}{\partial t} = \omega A \sin(kx - \omega t + \phi) \qquad \frac{\partial h'}{\partial x} = -kA\sin(kx - \omega t + \phi)$$

$$\frac{\partial^2 h'}{\partial t^2} = -\omega^2 A\cos(kx - \omega t + \phi) \qquad \frac{\partial^2 h'}{\partial x^2} = -k^2 A\cos(kx - \omega t + \phi)$$

From our wave equation:

$$-\omega^2 A\cos(kx - \omega t + \phi) = -gHk^2 A\cos(kx - \omega t + \phi)$$

$$\Rightarrow \omega^2 = gHk^2$$

$$w^2 = gHk^2$$

A relation (like above) which relates the frequency to the wavenumber (and any other property of the flow) is referred to as the "*dispersion relation*".

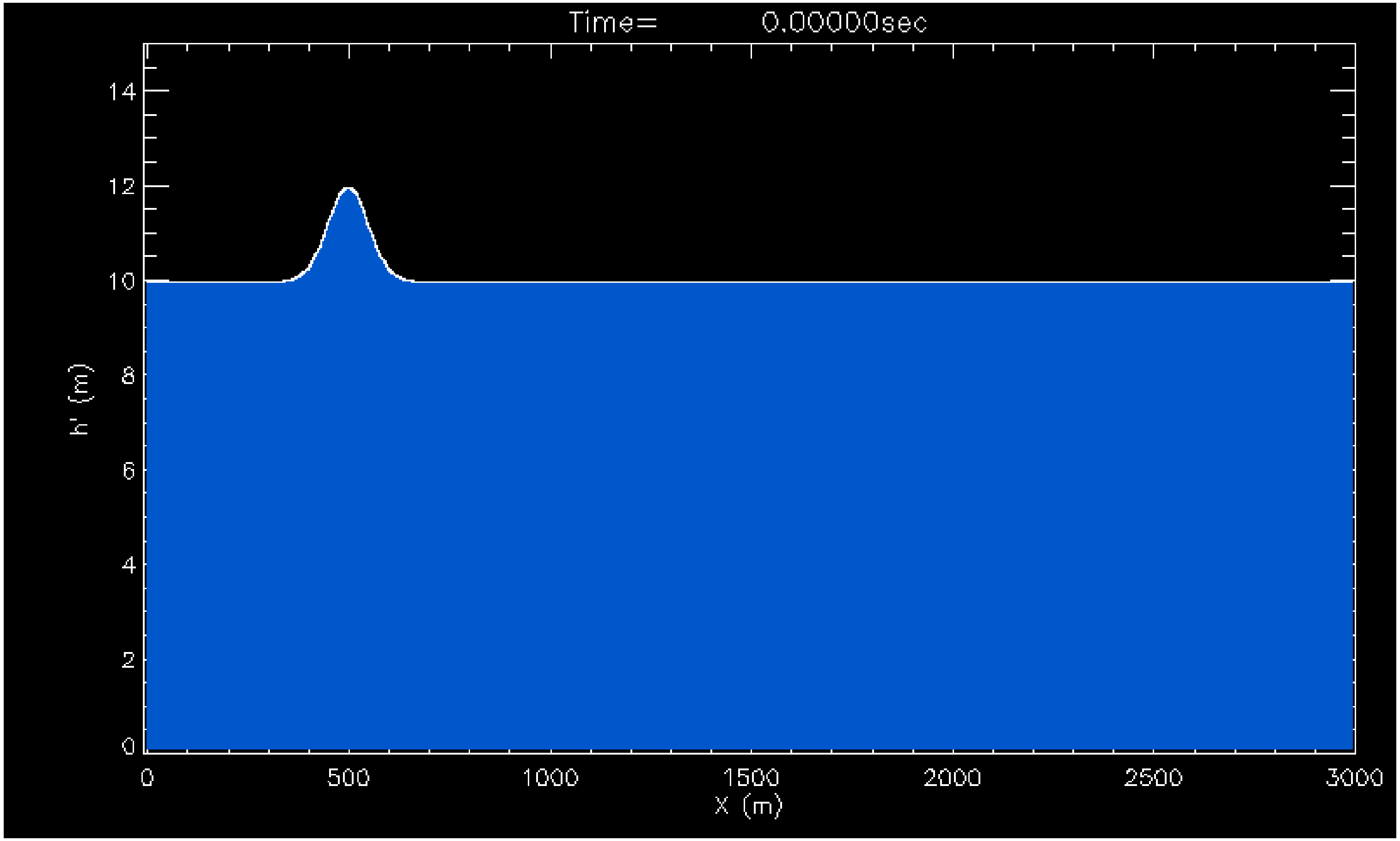
The dispersion relation tells us that the frequency and wavenumber can not be independent, they are constrained by aspects of the fluid.

$$w^2 = gHk^2$$

$$\frac{w^2}{k^2} = gH$$

$$c = \frac{w}{k} = \pm\sqrt{gH}$$

The phase speed is fixed and will only vary with the fluid depth (and gravity).



Time= 0.00000sec
h' (m)
X (m)
0
2
4
6
8
10
12
14
0
500
1000
1500
2000
2500
3000

Shallow water waves:

$$c = \frac{\omega}{k} = \pm\sqrt{gH}$$

In a fluid with constant depth, H, all waves will travel at the same speed. Therefore, shallow water waves are non-dispersive.

The phase speed is NOT related to wave amplitude. (When we examine waves using linearized equations the wave amplitude is irrelevant.)

Phase speed is only related to the properties of the background state. (This is equivalent to assuming the wave amplitude is infinitely small).

Other types of water waves are dispersive. For example, it can be shown for deep water waves the dispersion relation is:

$$\omega^2 = gk$$

$$c = \frac{\omega}{k} = \frac{g}{\omega} = \sqrt{\frac{g}{k}}$$

Different wavelength waves travel at different speeds.

Finally, in addition to determining the phase speed. We can also use the linearized equations to determine the velocity perturbations associated with the passage of the wave:

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \quad \text{if} \quad h' = A\cos(kx - \omega t + \phi)$$

$$\frac{\partial u'}{\partial t} = Akg\sin(kx - \omega t + \phi)$$

$$\Rightarrow \; u' = A\frac{k}{\omega}g\cos(kx - \omega t + \phi)$$

$$\Rightarrow \; u' = A\frac{g}{c}\cos(kx - \omega t + \phi) \qquad **$$

$$\Rightarrow \; u' = \pm A\sqrt{\frac{g}{H}}\cos(kx - \omega t + \phi)$$

As shown by **, the velocity perturbations are in-phase with depth perturbations when the wave propagates in the positive x- direction (c>0), and 180 degree out of phase when propagating in negative x-direction (c<0).

The amplitude of the velocity perturbations is proportional to depth perturbations, and inversely proportional to the square root of the mean fluid depth.

Rossby wave propagation.

Return to the conservation of absolute vorticity (low Rossby number / geostrophic flow).

$$\frac{D}{Dt}(z + f) = 0$$

Recall the β-plane approximation, which approximates latitudinal variations in the coriolis parameter.

$$\frac{D}{Dt}(z + f_0 + by) = 0$$

or

$$\frac{Dz}{Dt} + bv = 0$$

$\beta = (2\Omega/a)\cos(\phi_0)$, where a is the radius of the Earth, and ϕ_0 is a constant latitude.
At 45 degrees, $\beta \sim 1.6 \times 10^{-11}\ \mathrm{m^{-1}s^{-1}}$

$$\frac{D\zeta}{Dt} + \beta v = 0$$

Recall:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and}$$

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \quad \text{and} \quad u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}$$

$$\nabla_H \cdot \left(-\frac{1}{\rho_0 f_0}\frac{\partial p}{\partial y}\mathbf{i} + \frac{1}{\rho_0 f_0}\frac{\partial p}{\partial x}\mathbf{j} \right) = -\frac{1}{\rho_0 f_0}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho_0 f_0}\frac{\partial^2 p}{\partial y \partial x} = 0$$

Then why not define a geostrophic streamfunction

$$\psi = \frac{p}{\rho_0 f_0} \text{ then } \frac{1}{\rho_0 f_0}\frac{\partial p}{\partial x} = \frac{\partial \psi}{\partial x} = v_g \text{ and } -\frac{1}{\rho_0 f_0}\frac{\partial p}{\partial y} = -\frac{\partial \psi}{\partial y} = u_g$$

$$\nabla_H \cdot \left(-\frac{\partial \psi}{\partial y}\mathbf{i} + \frac{\partial \psi}{\partial x}\mathbf{j} \right) = 0$$

If we consider large scale flow then (a) it is approximately geostrophic and hence, (b) it is approximately non-divergent, hence

$$v = \frac{\partial \psi}{\partial x} \quad \text{and} \quad u = -\frac{\partial \psi}{\partial y}$$

where ψ is called a streamfunction (lines of constant streamfunction are *streamlines*). The streamlines are isobars for geostrophic flow. Then

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$\frac{D\zeta}{Dt} + \beta v = 0$$

Can therefore be written as:

$$\frac{D}{Dt}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \beta \frac{\partial \psi}{\partial x} = 0$$

This is still a nonlinear equation. If we separate $u=U+u'$, $v=V+v'$, $\psi=\Psi+\psi'$, with U V Ψ all constants. We can linearize the above equation (in the same way as we did for the shallow water equations) to become:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2}\right) + \beta \frac{\partial \psi'}{\partial x} = 0$$

This is the wave equation governing non-divergent flow on a β-plane. The perturbation stream function can be interpreted as a pressure perturbation.

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}\right)+\beta\frac{\partial\psi'}{\partial x}=0 \qquad (*)$$

This linearized equation can be used to determine the dispersion relation for Rossby waves.

Assume simplified perturbations of the form: $\psi' = A\cos(kx-\omega t)$

Then (*) reduces to:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi'}{\partial x^2}\right)+\beta\frac{\partial\psi'}{\partial x}=0$$

$$\Rightarrow \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(-k^2A\cos(kx-\omega t)\right)-\beta kA\sin(kx-\omega t)=0$$

$$\Rightarrow -k^2\omega A\sin(kx-\omega t)+Uk^3A\sin(kx-\omega t)-\beta kA\sin(kx-\omega t)=0$$

$$\Rightarrow -k^2\omega+Uk^3-\beta k=0$$

$$\Rightarrow \omega=Uk-\beta/k$$

The dispersion relation for Rossby waves in a constant mean flow, U.

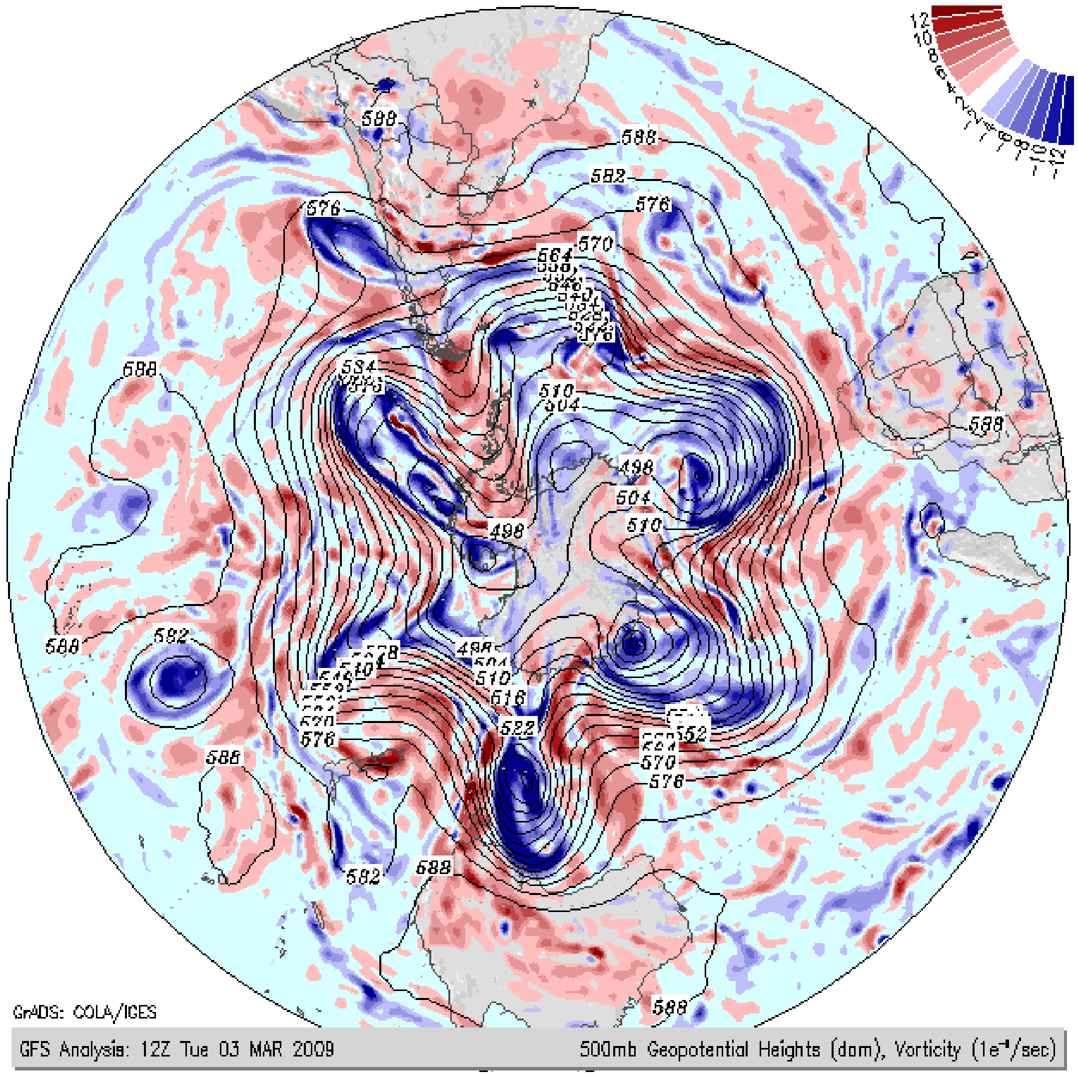
$$\omega = Uk - \beta / k$$

The phase speed:

$$c = \frac{\omega}{k} = U - \beta / k^2$$

- Rossby waves propagate to the west relative to the mean wind (recall this from Topic 5).
- Rossby waves rely on the β effect.
    - If β=0 then the disturbance does not *propagate* it is simply advected by the mean flow.
- Rossby waves are dispersive: different wavelengths travel at different speeds.
    - Long waves travel fastest, short waves slowest.

Integer multiple of Rossby wave wavelength is equal to circumference of Earth at given latitude.



E.g., n=6

Circumference = a2πcos(ϕ).

$$l_x = 2p/k = \frac{2pa\cos(f)}{n}$$

Rossby wave dispersion relation.

Assume no background flow - what is period / phase speed of large-scale Rossby waves?

$$w = -b \big/ k$$

$\beta = \frac{df}{dy} = \frac{1}{a}\frac{df}{d\phi} = \frac{2\Omega\cos(\phi)}{a}$ and $\frac{1}{k}$ may be deduced by noting that since

$\lambda_x = \frac{Circumference \text{ at latitude } \phi}{n} = \frac{2\pi\left[a\cos(\phi)\right]}{n} = 2\pi \big/ k$, it follows that

$\frac{1}{k} = \frac{a\cos(\phi)}{n}$

Then: $$|\omega| = \frac{2\Omega\cos^2(\phi)}{n} \quad \Rightarrow \quad T = \frac{\pi n}{\Omega\cos^2(\phi)}$$

At 45° $\cos^2(\phi)=0.5$, and therefore $T = \frac{2\pi n}{\Omega}$

$2\pi/\Omega=1$ Day. **Thus, T= n days.**

Rossby wave properties for different horizontal wavelengths.

n	1	2	3	4	5	6
λ (1000 km)	28.4	14.2	9.5	7.1	5.7	4.7
T (days)	1	2	3	4	5	6
C (m/s)	330	82	37	21	14	9

(@ 45 degrees latitude, U=0)

Long planetary waves (n= 1, 2) are unrealistically fast and period is too short. Assumptions (e.g., non-divergence) are violated at these scales.

Waves with scales $<\approx$ 10000 km, are reasonably consistent with observations in the real atmosphere of flow-relative phase speed and period.

Steady Rossby waves

Because

$$c = U - \beta / k^2$$

In Westerly flow it is possible for steady Rossby waves, c=0, to occur.

- In this case the wave propagates into the mean flow at an *intrinsic speed* that is equal to the mean wind.
- Such conditions would allow long-lived pressure disturbances over a particular location (e.g., a High).

Steady waves therefore have: $U = \beta / k^2$ or $\lambda_x = 2\pi \sqrt{\frac{U}{\beta}}$

U (m/s)	20	40	60	80
λ (km)	7000	9000	12000	14000

Using $\beta \approx 1.6 \times 10^{-11}\ m^{-1}s^{-1}$

This implies that for typical synoptic systems, stationary waves will only form in a relatively slow background flow.

INERTIAL OSCILLATIONS.

We have discussed (at length) how the coriolis force will turn an object to the left in the SH and to the right in the NH.

- This turning should continue as long as the object is moving.
- This sets up the "inertial oscillation".

For the purposes of this exercise start with the horizontal equations of motion:

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

The "turning" will occur for any moving body (solid or fluid). Let's assume solid body mechanics and remove the influence of pressure gradient forces:

$$\frac{du}{dt} - fv = 0$$

$$\frac{dv}{dt} + fu = 0$$

$$\frac{du}{dt} - fv = 0 \qquad (*)$$

$$\frac{dv}{dt} + fu = 0 \qquad (**)$$

Take the derivative of (*) wrt time gives:

$$\frac{d^2u}{dt^2} - f\frac{dv}{dt} = 0$$

Remove terms involving *v* using (**):

$$\frac{d^2u}{dt^2} + f^2u = 0 \qquad (***)$$

At initial time, $t=t_0$, assume $u=u_0$. Therefore, (***) has a solution:

$$u = u_0 \cos(ft)$$

If v=0 at t=0 then: $\frac{du}{dt} - fv = 0$ and $u = u_0 \cos(ft)$ give:

$$v = -u_0 \sin(ft)$$

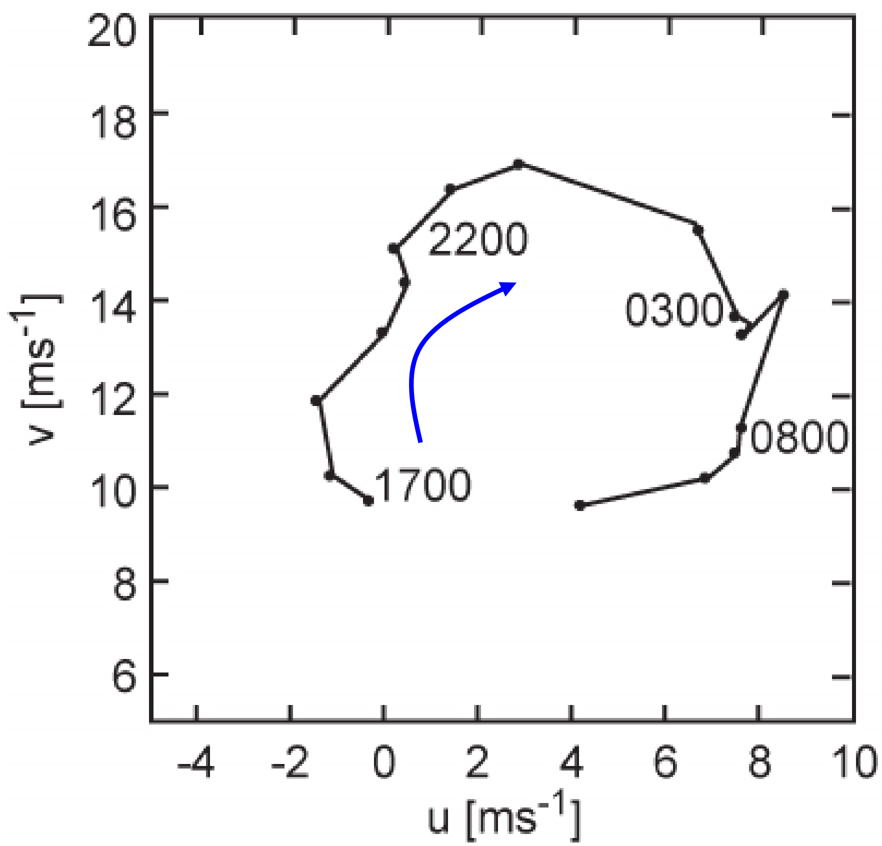
Finally, integrating *u* and *v* with respect to time (assuming object initially located at origin) gives:

$$x = \frac{u_0}{f} \sin(ft)$$

$$y = \frac{u_0}{f} \left[\cos(ft) - 1 \right]$$

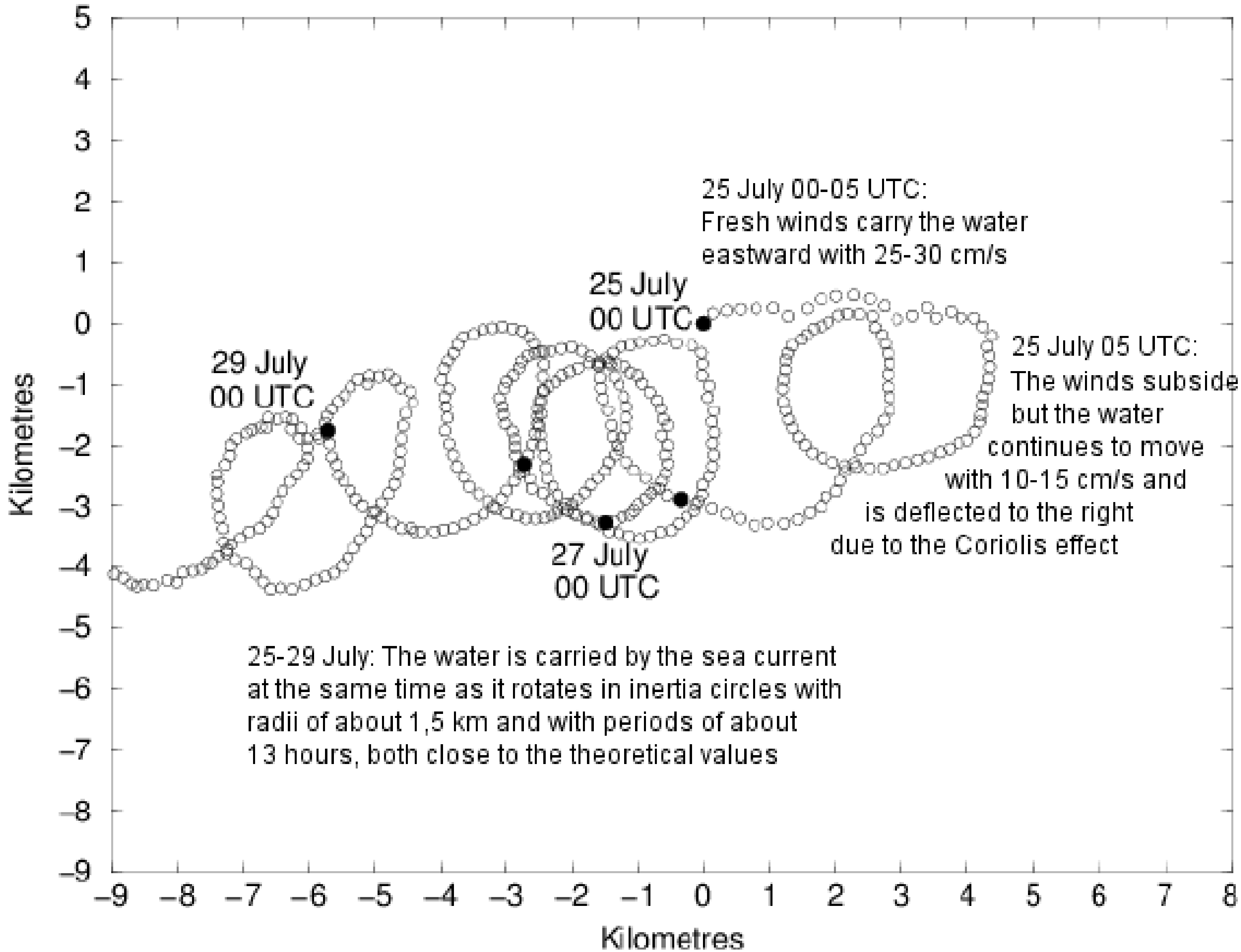
This is the "inertia circle" trajectory.

Size of circle varies with latitude (smallest at poles), period of oscillation is inertial period ($2\pi/f$). (1/2 day at pole, infinite at equator). Its rotation direction is anticyclonic.



From Lynch (F6.3), atmospheric wind hodograph (versus time) at 38^{o}N

Rotation is anticyclonic.



A drifting buoy set in motion by strong westerly winds in the Baltic Sea in July 1969.