Geophysical Fluid Dynamics I P.B. Rhines .. typo corrected vsn 1 Solutions
Problem Set 5
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1. Consider flow of a uniform density fluid ($\rho = \rho_0$) over a mountain range, where the fluid depth, h, is

$$h = H + \tilde{h}(x)$$
$$= H + d\sin(k_0 x)$$

Assume that the fluid flows from west to east (toward positive x), and has uniform potential vorticity (PV). This would be the case, for example if far upstream the depth were uniform (no mountains) and the flow were uniform (constant u-velocity). Use the potential vorticity equation in the form

$$Dq/Dt = 0;$$
$$q = (f+\zeta)/h \approx f/H + \zeta/H - f\tilde{h}\ /H^2$$

for steady flow this becomes

$$\vec{u} \bullet \nabla q = 0\ ;$$

if we linearize the problem for small topography (d/H << 1) then the mean zonal flow is much bigger than the modification of the flow due to the topography: u = U + u'(x), v = v' with u' << U, and the equation becomes

$$U\frac{\partial q}{\partial x} = 0$$

which says that q = constant or at most a function of y. We have shown that the vertical vorticity is the Laplacian of the geostrophic streamfunction (ψ), or of geostrophic pressure (p). Thus the equation becomes

$$\nabla^2\psi = f\tilde{h}/H + const.\ \ or\ \ m(y)$$

Because the upstream flow has uniform PV we don't need the function of y.

- Thus solve for ψ with the above topography. Sketch the streamlines. Note that positive $\tilde{h}$ corresponds to a valley (negative height mountain). Solution of $\psi_{xx} = fdA\sin k_0x/H$ is $\psi = -fdA/(k^2H)\ \sin k_0x - Uy$, which is indep of y except for the mean zonal flow part.

- Find the solution for any topography $\tilde{h}(x)$, assuming the upstream flow is uniform and the PV is uniform. Taking an example as an isolated mountain ridge, show that v, the y-velocity component ('along' the ridge) experiences a change proportional to the *volume* of the ridge, $\int \tilde{h}\ dx$. An example of such a ridge would be $\tilde{h} = d\exp(-x^2/a^2)$. Thus the ridge produces a *vortex sheet* in the fluid, shifting the direction of the zonal flow.

Simply integrate.

2. Thermal wind. In the horizontal – x-direction - vorticity equation, use scale analysis to compare the approximate size of the neglected terms: production of relative horizontal vorticity, say $D\zeta^{(x)}/Dt$, and the term $(\partial p/\partial y)(\partial \rho/\partial z)$ compared to the main thermal wind terms, $(\partial p/\partial z)(\partial \rho/\partial y)$ and $f\ \partial u/\partial z$. Here the x-component of vorticity is $\zeta^{(x)} = \partial w/\partial y - \partial v/\partial z$. [Note that the term $D\zeta^{(x)}/Dt$ the generation of x-vorticity, is important in geostrophic adjustment: in the PS4.2 homework buoyancy twisting makes the fluid slump, with strong horizontal vorticity along the same axis as the twisting, but then Coriolis forces 'arrest' the slumping, negate that component of horizontal vorticity, and settle in with the other component of horizontal vorticity in thermal-wind balance.]

If H is the vertical scale of the motion (*not* necessarily the fluid depth) he horizontal vorticity has scale estimate $\zeta^{(x)} = v_z - w_y \sim \dfrac{U}{H} + \dfrac{W}{L}$. Here if we used the MASS conservation equation $u_x + v_y = w_z$ to estimate W and U, we might think W/U ~ H/L but the first two terms nearly cancel in geostrophic flow, which is nearly 'horizontally non-divergent'. So look elsewhere for a connection between u and w: The vertical vorticity equation

$$\frac{D\zeta^{(z)}}{Dt} = f w_z$$

The vortex stretching effect gives us the connection. Scale analysis of this equation gives

W/U ~ Ro H/L + (1/fT) H/L

thus W/U is much less than the aspect ration, H/L. Ro is the Rossby number U/fL and the 'second Rossby number' based on timescale is 1/fT. Both are small by definition, for nearly geostrophic flows.

Thus the horizontal vorticity is completely dominated by the first term, U/H.

Now let's estimate the ratio of the neglected rate of change of horiz. voriticity to the thermal wind term f∂u/∂z.

$$\frac{D\zeta^{(x)}}{Dt} \sim U/HT + U^2/HL$$

$$f\,\partial u/\partial z \sim fU/H$$

$$ratio \sim 1/fT + U/fL ...small$$

In the stretching term of the x-VORT equation, $\vec{\zeta} \bullet \nabla u$, we keep f∂u/∂z which involves an error of order $\zeta^{(z)}/f$ ~ Ro which will be small.

The other neglected term is $p_y \rho_z$ compared to $p_z \rho_y$. The ratio scales like

$$\frac{p_y \rho_z}{p_z \rho_y} \sim \frac{\rho f U\ (\rho N^2/g)}{p_z \rho_y} \sim \frac{\rho f U\ (\rho N^2/g)}{\rho g\ \ \rho f v_z/g}$$

$$\sim \frac{\rho f U\ (\rho N^2/g)}{\rho g \rho f U/gH} \sim N^2 H/g \sim H/H_s$$

where H_s is the scale height of the density field. Thus the thermal wind equation as we have written it has the same accuracy as the Boussinesq approximation. A more accurate thermal wind equation results by replacing the horizontal density gradient ρ_y (for z=const) by the horizontal density gradient taken on an isobaric surface, p=const.

3. Fourier analysis of flow fields. The procedure above basically involves solving for velocity, pressure, streamfunction when we know the vorticity, $\zeta=\nabla^2\psi$. This involves a *Poisson equation*

$$\nabla^2 \psi = F(x,y)$$

which is an inhomogeneous, 'forced' Laplace equation. Let us think about this using some ideas of Fourier analysis.

Suppose we write ψ and F as a Fourier series, a sum of sines and cosines, each with an amplitude factor:

$$\begin{aligned} \psi &= \mathrm{Re}\,al[\sum_n \hat{\psi}_n \exp(ik_n x + il_n y)] \\ F &= \mathrm{Re}\,al[\sum_n \hat{F}_n \exp(ik_n x + il_n y)] \quad k_n = 2\pi n/L,\ l_n = 2\pi n/L \end{aligned} \qquad (1)$$

These sine-waves have wavenumbers which are a multiple of the constant 2π/L, hence ψ an F are periodic in x, over a distance L.

- Show that Poisson's equation can be written in terms of the Fourier coefficients as

$$-(k_n^2 + l_n^2)\hat{\psi}_n = \hat{F}_n \qquad (2)$$

by substitution, and equating coefficients of each exp(i….) to zero.

Thus $\hat{\psi}$ is 'amplified' at small wavenumber and reduced in strength at large wavenumber, compared with $\hat{F}$. We can say ψ *is a low-pass filtered image of F.* This is just what one does in image-analysis to soften or sharpen an image. In engineering, there is a maxim, 'Integration smoothes, differentiation roughens', which is saying the same thing.

The Laplace operator ∇^2 of the height of a surface is proportional to its *curvature* (for small amplitude). The equation for deflection, ψ, of a string (one-dimensional) or elastic membrane (two-dimensional) is the above Poisson equation where F(x,y) represents a distribution of weights or lifting forces. This physical analog helps in visualizing solutions.

- For our fluid case, the kinetic energy of the fluid is ½ $\rho_0 |\nabla\psi|^2$. Show that the contribution to the kinetic energy from a given Fourier component, say KE(k_n, l_n), is $\frac{1}{4}\rho_0(k_n^2+l_n^2)|\hat{\psi}_n|^2$ (where we need to be careful about taking the real part in the definition (1). The idea is that the sum of KE over all wavenumbers equals the integral of the kinetic energy in space,

$$\tfrac{1}{4}\rho_0\sum_n (k_n^2+l_n^2)|\psi_n|^2 = \iint \tfrac{1}{2}\rho_0 |\nabla\psi|^2 dx\ dy$$

This is pretty straightforward except for the miserable ¼ factor. Consider a simple case where there is just one Fourier component, so using the real and imaginary parts of the Fourier coefficient $\hat{\psi}$

$$\begin{aligned}
\psi &= real(\hat{\psi}\exp(ikx)) \\
&= \hat{\psi}_r \cos kx - \hat{\psi}_i \sin kx \\
|\psi|^2 &= (\hat{\psi}_r)^2\cos^2 kx + (\hat{\psi}_i)^2 \sin^2 kx + 2\hat{\psi}_r\hat{\psi}_i \sin kx \cos kx \\
average\ over\ x: \ |\psi|^2 &= \tfrac{1}{2}(\hat{\psi}_r)^2 + \tfrac{1}{2}(\hat{\psi}_i)^2 \\
&= \tfrac{1}{2}|\hat{\psi}|^2 \\
so\ \ \tfrac{1}{2}|\psi|^2 &= \tfrac{1}{4}|\hat{\psi}|^2
\end{aligned}$$

{This is in fact known as Parseval's Theorem. Note, Real(m) = ½ (m+m*) where m* is the complex conjugate of m}).

KE, is known as the *kinetic energy spectrum* or the *velocity spectrum*, and it tells us the distribution of energy among wavenumbers. In the atmosphere, the kinetic energy spectrum peaks near zonal-wavenumbers 3 to 5. (that is, 3 to 5 sine-waves around a latitude circle). Thus we see that *the wavenumber spectrum of pressure or of ψ (one is proportional to the other) , is a 'low-pass filtered' version of the velocity spectrum (or kinetic energy spectrum), KE.* That is: if *we define* $P = \dfrac{1}{4}(\rho_0 f)^2 |\hat{\psi}_n|^2$ then $KE = (1/\rho_0 f^2)(k_n^2+l_n^2)P$ and similar to the result above, the spatial field of geostrophic pressure is a smoothed field compared with spatial maps of the u and v velocity components.

- Find the relation between the wavenumber spectrum of vertical vorticity, $\nabla^2\psi$, and the spectra of kinetic energy, KE, and pressure, P.

to within some constants, the wavenumber spectrum of vorticity is $|k|^4$ times the spectrum of ψ, KE spectrum is $|k|^2$ times the spectrum of ψ and pressure spectrum is $|k|^{-2}$ times the spectrum of ψ.

- Suppose that the fluid flow is now stratified and geostrophic, with constant buoyancy frequency, N. Include a vertical dependence, and vertical wavenumber (periodic in the depth H of the fluid region),

$$\psi = \mathrm{Re}\,al[\sum_n \hat{\psi}_n \exp(ik_n x + il_n y + im_n z)] \qquad m_n = n2\pi/H$$

We want to know the horizontal wavenumber content of the perturbation density, ρ'(x,y,z). Show that it will be similar to pressure if the vertical variation has just a single wavenumber, yet different from the pressure if there is a rich vertical distribution of wavenumbers.

The hydrostatic equation relates perturbation pressure p' and perturbation density, ρ'. We see that if eddies with different horizontal scale have the same vertical scale, H, their horizontal-wavenumber spectra will be similar. These are eddies of many different widths but the same height. If instead wider eddies are taller, then this 'self-similar' eddy field will be more symmetrical in (m,k) wavenumber space.

If we use $p_z = -g\rho'$ then plots of p(z) will be 'noisier' or 'finer scale' than plots of ρ', unless there is just a single wavenumber component, say $\sin(\pi z/H)$: then p and ρ' have exactly the same appearance (but for a phase shift).

Construct the spectrum APE (of available potential energy), from its definition given in Gill p. 140 (sec 6.7), which is $\frac{1}{2}\frac{g^2\rho'^2}{\rho_0 N^2}$ per unit volume, writing it in terms of the pressure spectrum, and show how the spectral ratio APE/KE depends on your assumption about the relation between vertical and horizontal wavenumbers. This is a more exact way, compared to scale analysis, of looking at energetics. You could think about how u and v velocities appear in horizontal maps (u(x,y) at a fixed z…) compared with density (ρ'(x,y) at fixed z). The above remarks show that ρ' is like pressure if there is just a single vertical mode (single vertical Fourier component), yet ρ' may be more like u and v if there is a rich set of vertical modes. This has to be analyzed by plotting the spectra on the (m, k) plane. So the APE/KE ratio will be 'red' if there is a single vertical mode, with more APE than KE at small horizontal wavenumbers and less at large horizontal wavenumbers. If ρ' is more like u and v in its spectrum than it is like p, the APE/KE ratio will tend to be flat….this is like saying NH/FL ~ 1 for all different height scales H and horizontal length scales L.

• Illustrate some of these results by constructing a 2-dimensional flow field from an assumed 'random' distribution of Fourier components and making plots of ψ(x,y), ζ(x,y), u(x,y). If you have access to Matlab, or a similar calculating/plotting engine, use many Fourier components (many wavenumbers) and plot contour maps of ψ, p, u, v, and ζ. But if not just try two, say $k = k_0$ and $k = 4k_0$. How would you expect the spectrum of *particle displacement* $\int \vec{u}\ dt$ to behave? The particle displacement has more low frequency content than the velocity: it wanders farther from the zero-axis. In fact, if we introduce the idea of u being a random variable with zero mean and standard deviation U^2, the displacement, the time integral of U(t), wanders very from its origin. This is the 'drunkards' random walk problem. The variance if X grows like the square root of time, $(UTt)^{1/2}$ and visits the origin more and more rarely. Here T is the time scale of U(t), say 1/T is where the spectrum of U is maximum.