ATOC30004 Dynamical Meteorology and Oceanography.

Assignment 3, 2023

Turnitin

(20 marks for each question)

1. By integrating the hydrostatic equation for pressure coordinates $\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$, prove that

   a. (8 points) $\Phi(p_{20000}) - \Phi(p_{100000}) = R\overline{T}\ln(5)$ where $\overline{T} = \frac{\int_{20000}^{100000} T d\ln p}{\int_{20000}^{100000} d\ln p}$

   where, for example, $\Phi(p_{25000})$ refers to the geopotential height of the surface at which the pressure p=250 hPa = 25000Pa

   Answer:

$$\Phi(p_{25000}) - \Phi(p_{100000}) = -\int_{20000}^{100000} d\Phi = -\int_{20000}^{100000} \frac{\partial \Phi}{\partial p} dp = \int_{20000}^{100000} \frac{RT}{p} dp = \int_{20000}^{100000} RT \frac{d\ln p}{dp} dp =$$

$$= \int_{200000}^{100000} RT \frac{d\ln p}{dp} dp = R\int_{200000}^{100000} Td\ln p = R\frac{\int_{200000}^{100000} Td\ln p}{\int_{20000}^{100000} d\ln p} \int_{20000}^{100000} d\ln p = R\overline{T}\int_{20000}^{100000} d\ln p$$

$$= R\overline{T}\left(\ln(100000) - \ln(20000)\right) = R\overline{T}\ln(5)$$

where $\overline{T} = \frac{\int_{20000}^{100000} Td\ln p}{\int_{20000}^{1000} d\ln p}$

   b. (12 points) Use $\Phi(p_{20000}) - \Phi(p_{100000}) = R\overline{T}\ln(5)$ to compute the vertical wind changes $u_g(p_{200hPa}) - u_g(p_{1000hPa})$ and $v_g(p_{200hPa}) - v_g(p_{1000hPa})$ in the case that $\frac{\partial \overline{T}}{\partial x} = \frac{2K}{10^6 m}$ and $\frac{\partial \overline{T}}{\partial y} = \frac{5K}{10^6 m}$. Assume that f takes the typical Southern Hemisphere mid-latitude value of $f = -10^{-4} s^{-1}$.

Answer:

$$\Phi(p_{20000}) - \Phi(p_{100000}) = R\bar{T}\ln(5), \text{ so}$$

$$u_g(p_{200hPa}) - u_g(p_{1000hPa}) = -\frac{1}{f}\frac{\partial\Phi}{\partial y}(p_{20000}) + \frac{1}{f}\frac{\partial\Phi}{\partial y}(p_{100000}) = -\frac{R}{f}\frac{\partial\bar{T}}{\partial y}\ln(5)$$

$$= \frac{287}{10^{-4}} \times \frac{5}{10^{6}}\ln(5) = \frac{2.87\times10^{2}\times5}{10^{2}}\ln(5) = 23.07ms^{-1}$$

$$v_g(p_{200hPa}) - v_g(p_{1000hPa}) = \frac{1}{f}\frac{\partial\Phi}{\partial x}(p_{20000}) - \frac{1}{f}\frac{\partial\Phi}{\partial x}(p_{100000}) = \frac{R}{f}\frac{\partial\bar{T}}{\partial x}\ln(5)$$

$$= -\frac{287}{10^{-4}} \times \frac{2}{10^{6}}\ln(5) = -\frac{2.87\times10^{2}}{10^{2}}\ln(5) = -9.24ms^{-1}$$

2. A parcel of air at 850 hPa is 18 degrees C and the surrounding environment is 15 degrees C.
   (a) Determine the buoyancy force per unit mass acting on this parcel and its direction (10 marks).

Answer :

Buoyancy force $B = \frac{g}{\bar{T}}\left(T - \bar{T}\right) = 3\frac{9.8}{293.15} = 0.101ms^{-2}$

This force is in the posative z-direction; ie the force is upward.

   (b) If the vertical acceleration was exactly twice the buoyancy force per unit mass, and the density is equal to 1 $kg/\left(m^3\right)$ then what would be the vertical derivative of the perturbation pressure (10 marks)?

Answer

From topic 3, the equation governing vetrtical motion is

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + B$$

Hence, if the vertical acceleration $\frac{Dw}{Dt} = 2B$ it follows that

$$2B = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + B$$

$$\Rightarrow B = -\frac{1}{\rho}\frac{\partial p'}{\partial z} = -\frac{\partial p'}{\partial z} \Rightarrow \frac{\partial p'}{\partial z} = -B = -0.101\left(Pa\right)m^{-1}$$

3. The 3-dimensional vorticity $\vec{\zeta} = \nabla \times \mathbf{u}$ is a vector that is equal to the curl of the vector field defining the magnitude and direction of the wind **u** at every point in space. In a Cartesian coordinate system with **u=(**u,v,w) corresponding to the components of the wind in the x,y (horizontal) and z (vertical) coordinates, respectively it is defined by the equation

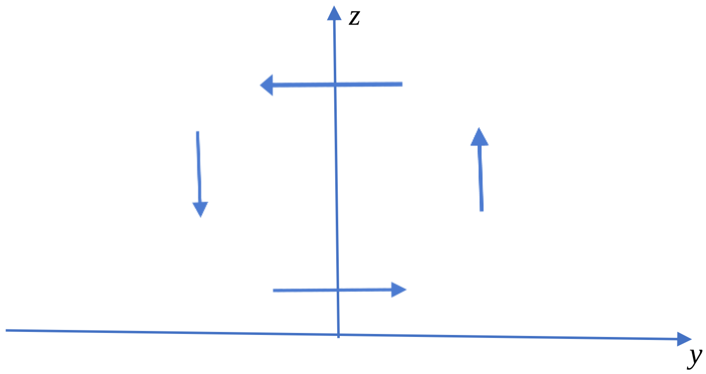
$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \qquad (1.1)$$

(3a) (1 mark) Use (1.1) to define the component of vorticity in the direction of the unit vector **j** which points in the direction of increasing y.

Answer: We get the **j-**component of vorticity from

$$\mathbf{j}.\nabla \times \mathbf{u} = \mathbf{j}.\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

(3b) (2 marks) By sketching wind vectors associated with (v,w) on an y-z cross-section of a plane with constant x values, (as shown below) give an example of a vector wind field that has a *positive* **i**-component. Assume that w=0 at z=0 and that z=0 where the y-axis crosses the z-axis.



(3c) (1 mark) Which of the following quantities is referred to as the vertical component of relative vorticity by meteorologists.

(i) $\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$, or (ii) $\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)$, (iii) f, or (iv) $\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$

(Circle the correct answer)

(3d) (3 marks) Starting with the simplified Barotropic, Boussinesq and inviscid forms of the horizontal momentum equation and assuming that $0=\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}$, it can be shown that

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right) \qquad (1.2)$$

Furthermore, under these quasi-barotropic, Boussinesq conditions the rate of change of the depth h of a column of the fluid is given by

$$\frac{1}{h}\frac{Dh}{Dt}=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad (1.3)$$

Using (1.3) in (1.2) yields

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=\frac{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)}{h}\frac{Dh}{Dt} \qquad (1.4)$$

Use the Calculus product rule, to express the derivative

$$\frac{D}{Dt}\left(\frac{\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f}{h}\right)$$ in terms of the two terms given by the product rule.

Answer: $$\frac{D}{Dt}\left(\frac{\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f}{h}\right)=\frac{1}{h}\frac{D}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)-\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)\frac{1}{h^2}\frac{Dh}{Dt}$$

(3e) (3 marks) By using equation (1.4) in your answer to (1d) show that

$$\frac{D}{Dt}\left(\frac{\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f}{h}\right)=0 \qquad (1.5)$$

(You must show your working to get credit)

Answer:

$$\frac{D}{Dt}\left(\frac{\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f}{h}\right)=\frac{1}{h}\frac{D}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)-\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)\frac{1}{h^2}\frac{Dh}{Dt}$$

$$=\frac{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)}{h^2}\frac{Dh}{Dt}-\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)\frac{1}{h^2}\frac{Dh}{Dt}=0$$

As was required. (QED)

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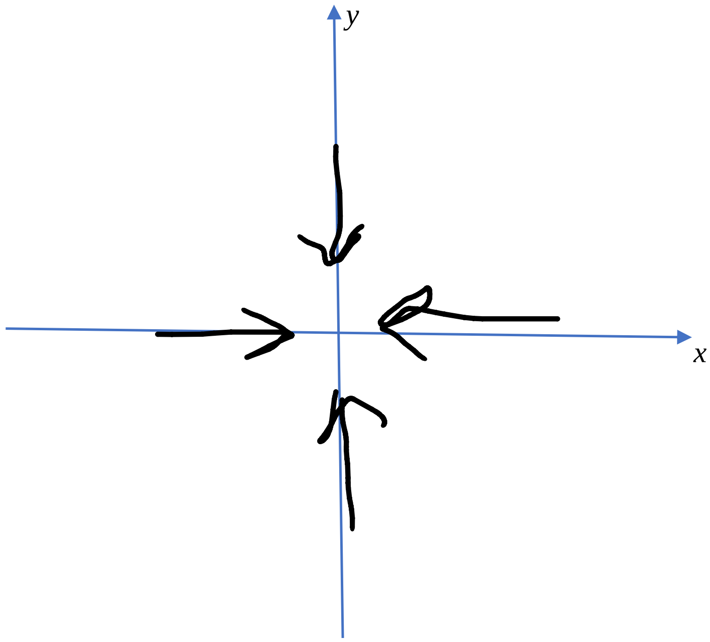
(3f) (1 mark) Is the quantity $\left(\frac{\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f}{h}\right)$ called

(i) Relative vorticity
(ii) Absolute vorticity
(iii) Potential vorticity
(iv) Planetary vorticity
(v) Parceval's vorticity

(Circle the option that you think is correct).

(3g) (4 marks) Using equation (1.4) as a guide give an example of a (u,v) vector field that would *increase* cyclonic vorticity assuming that $\left|\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right| \ll |f|$. (You should only have to plot 4 – 8 vectors to indicate such a field)



(3h) (6 marks) Using equation (1.7) as a guide: Suppose there is an air column with a height *h* of 8 km that has zero relative vorticity while it is travelling from West to East at a latitude where $f = -10^{-4} s^{-1}$ along a flat 2 km high altitude plateau. Later, the air column moves off the Plateau and over the sea and its height *h* increases to 10 km. If equation (1.7) is precisely correct, what is the relative vorticity of the air column after it has moved over the ocean?

$(i)\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.25 f = -0.25x10^{-4} s^{-1}$

$(ii)\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.5 f = -0.5x10^{-4} s^{-1}$

$(iii)\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{f}{2} = 0.5x10^{-4} s^{-1}$

$(iv)\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{5}{4} f = -1.25x10^{-4} s^{-1}$

(Circle the answer that you think is correct)

$$\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_{Initial} = 0, \quad \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_{Final} = ?$$

Conservation of potential vorticity states that

$$\left(\frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f}{h}\right)_{Initial} = \left(\frac{f}{8}\right) = \left(\frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f}{h}\right)_{Final} = \left(\frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f}{10}\right)_{Final}$$

Hence,

$$\left(\frac{f}{8}\right) = \frac{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_{Final} + f}{10}$$

$$=> \frac{10}{8} f = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_{Final} + f$$

$$=> \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_{Final} = \frac{10}{8} f - \frac{8}{8} f = \frac{2}{8} f = 0.25 f = -0.25x10^{-4}$$