Understanding the Coriolis Force

In the lecture, a qualitative explanation of the Coriolis force was given in terms of centrifugal forces and conservation of angular momentum principles. By rewriting the vectorial momentum equations in terms of unit vectors $\mathbf{e}_\lambda$, $\mathbf{e}_\phi$ and $\mathbf{e}_r$ that point in the Eastward, Northward and vertical directions and which rotate with the planet, the equations governing the rate of change of the wind velocity on a rotating planet in a local coordinate system are found to be as follows: (Note that these are slightly rearranged versions of Holton's equations 2.19-2.21. Nevertheless, they are identical to Holton's equations)

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi)$$
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\sin\phi - \frac{vw}{(a+z)} \qquad \text{(M1)}$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_{eff} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\cos\phi + \frac{v^2}{(a+z)}$$

The above equations are referring to a spherical coordinate system in which

$a = Earth$ radius

$z =$ height above sea-level, $(a+z)$ is the distance of a parcel from Earth centre.

λ^a =longitude in a non-rotating reference frame

$\lambda = \lambda^a - \Omega t$, or equivalently

$\lambda^a = \Omega t + \lambda$ (note that λ=0 on the prime Greenwich meridian)

ϕ^a =latitude in a non-rotating reference frame= ϕ = latitude in a rotating reference frame

$x^a = (a+z)\cos\phi\lambda^a = (a+z)\cos\phi[\Omega t + \lambda]$

The absolute zonal wind speed is given by

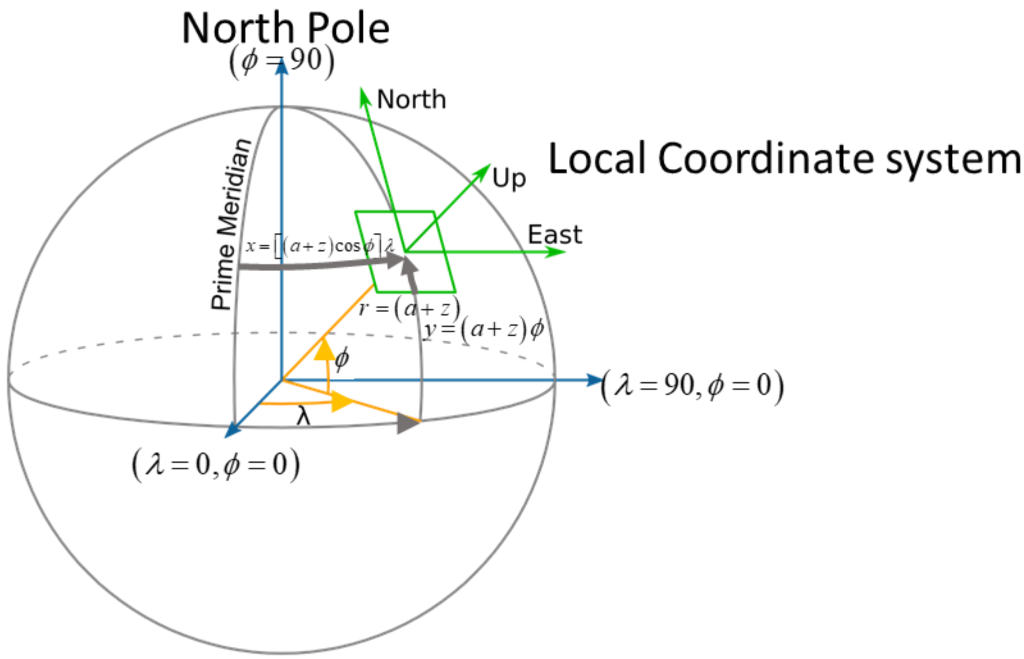
$$u^a = (a+z)\cos\phi\frac{D\lambda^a}{Dt} = (a+z)\cos\phi\left(\Omega + \frac{D\lambda}{Dt}\right) = (a+z)\Omega\cos\phi + u$$

where $u = (a+z)\cos\phi\left(\frac{D\lambda}{Dt}\right)$ is the Earth relative zonal wind speed

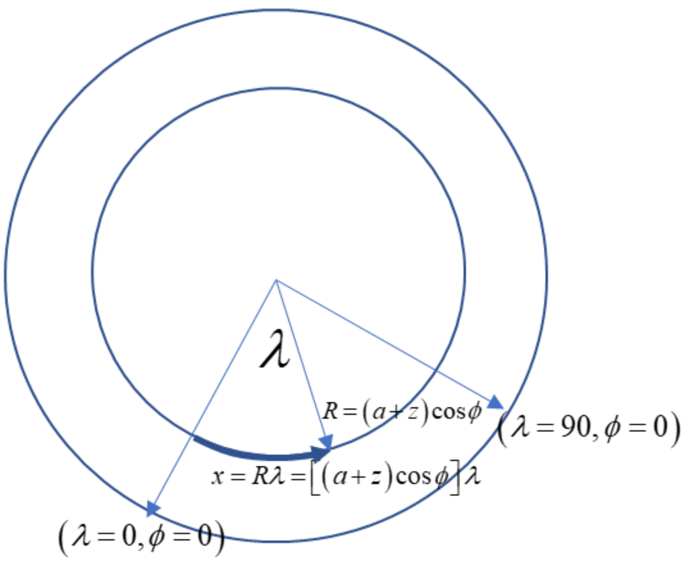
$x = R\lambda = (a+z)\cos\phi\lambda$

$y = (a+z)\phi$

$v = (a+z)\left(\frac{D\phi}{Dt}\right), w = \frac{Dz}{Dt}$



Overview of coordinate system



View from above North Pole

The pressure gradient terms in equation (M1) are easy to understand but what about those other terms? From the perspective of a rotating reference frame, centrifugal forces (the outward force one feels if one swings a heavy object around a circle) and the conservation of angular momentum (that leads to an increase in the rate of rotation of an ice skater when she pulls in her arms) are apparent forces/accelerations. Here, we show that the "apparent" force/acceleration terms in (1) – (3) can be entirely accounted for by conservation of angular momentum and centrifugal force effects. Specifically, we will show that

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left|\mathbf{F}^{apparent}_{\text{conservation of angular momentum of zonal movement moving northward or upward}}\right|$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \left|\mathbf{F}^{apparent}_{\text{Northward component of centrifugal force of zonal flow}}\right|$$

$$+\left|\mathbf{F}^{apparent}_{\text{conservation of angular momentum of Northward flow moving upward}}\right|$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \left|\mathbf{F}^{apparent}_{\text{Upward component of centrifugal force of zonal flow}}\right|$$

$$+\left|\mathbf{F}^{apparent}_{\text{Upward component of centrifugal force of Northward flow}}\right|$$

For a less verbose notation, we will use

$$\mathbf{F}^{ZAM}_{\lambda} = \mathbf{F}^{apparent}_{\text{conservation of angular momentum of zonal movement moving northward or upward}}$$

$$\mathbf{F}^{CFZF}_{\phi} = \mathbf{F}^{apparent}_{\text{Northward component of centrifugal force of zonal flow}}$$

$$\mathbf{F}^{NAM}_{\phi} = \mathbf{F}^{apparent}_{\text{conservation of angular momentum of Northward flow moving upward}}$$

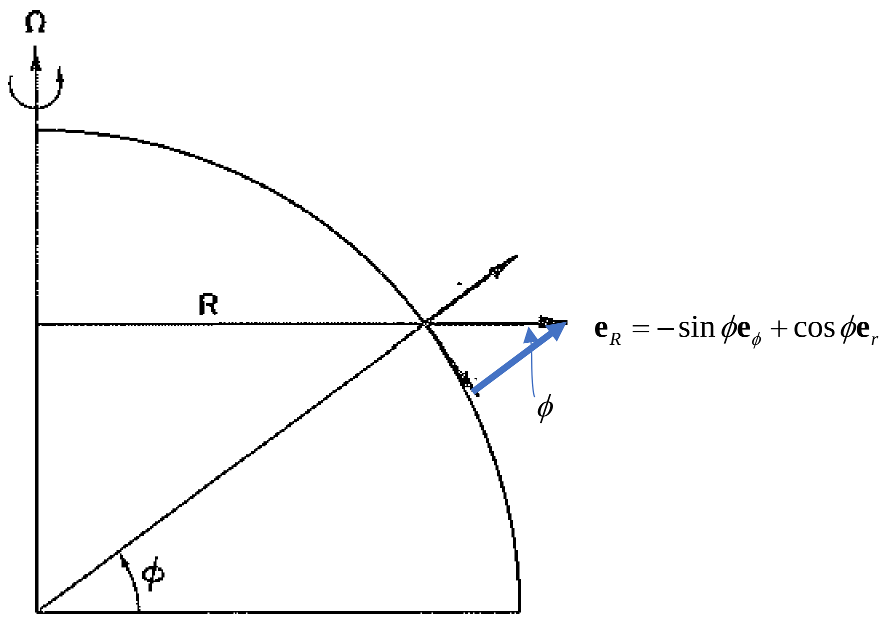
$$\mathbf{F}^{CFZF}_{r} = \mathbf{F}^{apparent}_{\text{Upward component of centrifugal force of zonal flow}}$$

$$\mathbf{F}^{CFNF}_{r} = \mathbf{F}^{apparent}_{\text{Upward component of centrifugal force of Northward flow}}$$

Below we show that

$$\begin{aligned}
\mathbf{F}^{ZAM}_{\lambda} &= \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi)\mathbf{e}_{\lambda} \\
\mathbf{F}^{CFZF}_{\phi} &= -\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(u\sin\phi)\mathbf{e}_{\phi} \\
\mathbf{F}^{CFZF}_{r} &= \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(u\cos\phi)\mathbf{e}_{r} \qquad \text{(M2)} \\
\mathbf{F}^{NAM}_{\phi} &= -\frac{vw}{(a+z)}\mathbf{e}_{\phi} \cong -\frac{vw}{a}\mathbf{e}_{\phi} \\
\mathbf{F}^{CFNF}_{r} &= \frac{v^2}{a+z}\mathbf{e}_{r}
\end{aligned}$$

where $\mathbf{e}_{\lambda}$, $\mathbf{e}_{\phi}$ and $\mathbf{e}_{r}$ are unit vectors that point in the Eastward, Northward and vertical directions, respectively. In the derivation below, we also refer to the unit vector $\mathbf{e}_{R}$ which points perpendicularly outward from the axis of rotation. The relationship between $\mathbf{e}_{R}$, $\mathbf{e}_{\phi}$ and $\mathbf{e}_{r}$ is given in the figure below.



Recall that fluid acceleration in any given direction is identical to the force per unit mass in that same direction. In the following, we will use conservation of angular momentum to infer accelerations and hence apparent forces associated with conservation of angular momentum. To see how this works, consider the following:

Let the zonal component of wind in a non-rotating frame be $u^a = \Omega\left[(a+z)\cos\phi\right] + u$, where $u = (a+z)\cos\phi \dfrac{D\lambda}{Dt}$ is the Earth relative part and $\Omega\left[(a+z)\cos\phi\right]$ is the part due to the rotation of the Earth. Angular momentum per unit mass of parcel is then $u^a\left[(a+z)\cos\phi\right]$. Conservation of angular momentum per unit mass then gives

$$0 = \frac{D\left(u^a\left[(a+z)\cos\phi\right]\right)}{Dt} = u^a \frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt} + \left[(a+z)\cos\phi\right]\frac{D\left(u^a\right)}{Dt}$$

Rearranging the first equation gives

$$\frac{D\left(u^a\right)}{Dt} = -\frac{u^a}{(a+z)\cos\phi}\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt} \qquad (1)$$

Recalling that $u^a = \Omega\left[\left(a+z\right)\cos\phi\right]+u$, it follows that

$$\frac{D\left(u^{a}\right)}{Dt}=\Omega\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}+\frac{Du}{Dt}=-\frac{u^{a}}{\left(a+z\right)\cos\phi}\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt} \qquad (2)$$

Since the $\dfrac{Du}{Dt}$ term in (2) has arisen solely due to the conservation of Eastward angular momentum we can equate it to $\mathbf{F}_{\lambda}^{EAM}$ and hence (2) implies

$$\frac{Du}{Dt}\mathbf{e}_{\lambda}=\mathbf{F}_{\lambda}^{ZAM}=-\left(\Omega+\frac{u^{a}}{\left(a+z\right)\cos\phi}\right)\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}\mathbf{e}_{\lambda} \qquad (3)$$

Using $u^a = \Omega\left[\left(a+z\right)\cos\phi\right]+u$ in (3) gives

$$\frac{Du}{Dt}=-\left(\Omega+\frac{\Omega\left[\left(a+z\right)\cos\phi\right]+u}{\left(a+z\right)\cos\phi}\right)\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}$$

$$\frac{Du}{Dt}=-\left(2\Omega+\frac{u}{\left(a+z\right)\cos\phi}\right)\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt} \qquad (4)$$

Now

$$\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}=\frac{D\left(\left[\left(a+z\right)\right]\right)}{Dt}\cos\phi+\left(a+z\right)\frac{D\left(\left[\cos\phi\right]\right)}{Dt}$$

$$=w\cos\phi+\left(a+z\right)\frac{D\phi}{Dt}\frac{d\left(\left[\cos\phi\right]\right)}{d\phi}=w\cos\phi-\left[\left(a+z\right)\frac{D\phi}{Dt}\right]\sin\phi$$

$$=w\cos\phi-v\sin\phi$$

Since $\left(a+z\right)\dfrac{D\phi}{Dt}=v,$ this simplifies to

$$\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}=w\cos\phi-v\sin\phi, \qquad (5)$$

Using (5) in (4) gives

$$\frac{Du}{Dt}=\left(2\Omega+\frac{u}{\left(a+z\right)\cos\phi}\right)\left(v\sin\phi-w\cos\phi\right) \qquad (6a)$$

Using (6) in (3) gives

$$\mathbf{F}_{\lambda}^{ZAM}=\left(2\Omega+\frac{u}{\left(a+z\right)\cos\phi}\right)\left(v\sin\phi-w\cos\phi\right)\mathbf{e}_{\lambda} \qquad (6b)$$

Now consider the centrifugal force associated with zonal motion.

The centrifugal force is directed in the direction

$$\mathbf{e}_R = \cos\phi\,\mathbf{e}_r - \sin\phi\,\mathbf{e}_\phi \qquad (1)$$

The apparent force is given by

$$\mathbf{F}_R^{CFZF} = \frac{\left(u^a\right)^2}{(a+z)\cos\phi}\mathbf{e}_R \qquad (2)$$

Recall that $u^a = \left\{\Omega\left[(a+z)\cos\phi\right]\right\} + u$ so

$$\left(u^a\right)^2 = \left\{\Omega\left[(a+z)\cos\phi\right]\right\}^2 + 2u\left\{\Omega\left[(a+z)\cos\phi\right]\right\} + u^2 \quad (3)$$

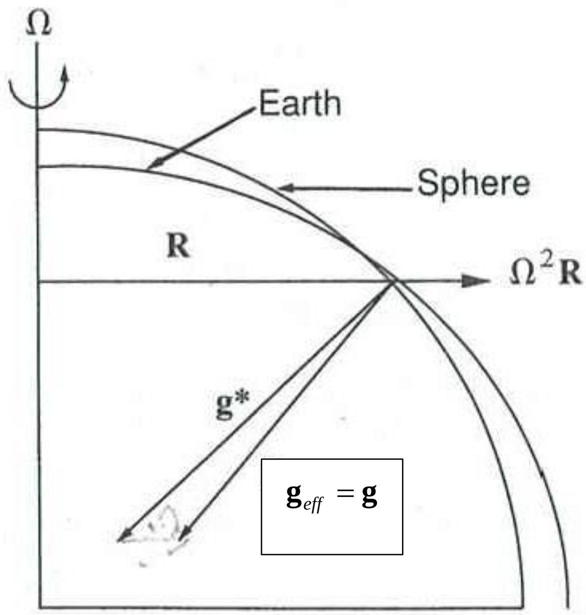
so that

$$\frac{\left(u^a\right)^2}{(a+z)\cos\phi} = \frac{\left\{\Omega\left[(a+z)\cos\phi\right]\right\}^2}{(a+z)\cos\phi} + 2\Omega u + \frac{u^2}{(a+z)\cos\phi} \qquad (4)$$

The first term in this equation is usually added to the gravitational force to obtain an effective gravity

$$\mathbf{g}_{eff} = -g\mathbf{e}_r + \frac{\left\{\Omega\left[(a+z)\cos\phi\right]\right\}^2}{(a+z)\cos\phi}\mathbf{e}_R. \qquad (5)$$

Oceans and even the Earth itself have adjusted their shape so that the undisturbed ocean or flat Earth lies perpendicular to $\mathbf{g}_{eff}$. *Hence,* this is the "downward" acceleration we measure on the Earth and is, in fact, the $\mathbf{g}$ that we will refer to in the remainder of this course.



Having absorbed this
term into an *effective* gravity, the remnant apparent acceleration in the $\mathbf{e}_R$ direction is given by

$$\mathbf{F}_R^{CFZF} = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\right]\mathbf{e}_R \qquad (6)$$

Recalling that $\mathbf{e}_R.\mathbf{e}_\phi = -\sin(\phi)$, the projection of this acceleration in the $\mathbf{e}_\phi$ direction

$$\mathbf{F}_\phi^{CFZF} = \left(\mathbf{F}_R^{CFZF}.\mathbf{e}_\phi\right)\mathbf{e}_\phi \qquad (7)$$

$$\Rightarrow \mathbf{F}_\phi^{CFZF} = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\right]\left(\mathbf{e}_R.\mathbf{e}_\phi\right)\mathbf{e}_\phi$$

Since $\mathbf{e}_R = -\sin\phi\mathbf{e}_\phi + \cos\phi\mathbf{e}_r$, it follows that $\left(\mathbf{e}_R.\mathbf{e}_\phi\right) = -\sin\phi\left(\mathbf{e}_\phi.\mathbf{e}_\phi\right) + \cos\phi\left(\mathbf{e}_r.\mathbf{e}_\phi\right) = -\sin\phi;$
Hence,

$$\mathbf{F}_\phi^{CFZF} = -\left\{\left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\right]\sin\phi\right\}\mathbf{e}_\phi \qquad (8)$$

With $\mathbf{e}_R = -\sin\phi\mathbf{e}_\phi + \cos\phi\mathbf{e}_r$, it follows that $\left(\mathbf{e}_R.\mathbf{e}_r\right) = -\sin\phi\left(\mathbf{e}_\phi.\mathbf{e}_r\right) + \cos\phi\left(\mathbf{e}_r.\mathbf{e}_r\right) = \cos\phi$; hence, the projection of this acceleration in the $\mathbf{e}_r$ direction is given by

$$\mathbf{F}_r^{CFZF} = \left(\mathbf{F}_R^{CFZF}.\mathbf{e}_r\right)\mathbf{e}_r \qquad (9)$$

$$=> \mathbf{F}_r^{CFZF} = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\right]\left(\mathbf{e}_R.\mathbf{e}_r\right)\mathbf{e}_r = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\right]\cos\phi\mathbf{e}_r \qquad (10)$$

Conservation of angular momentum associated with Northward flow implies that

$$\frac{D\left[v(a+z)\right]}{Dt} = 0 = (a+z)\frac{Dv}{Dt} + v\frac{D\left[(a+z)\right]}{Dt} = (a+z)\frac{Dv}{Dt} + vw$$

$$=> (a+z)\frac{Dv}{Dt} = -vw$$

$$=> \mathbf{F}_\phi^{NAM} = \frac{Dv}{Dt}\mathbf{e}_\phi = -\frac{vw}{(a+z)}\mathbf{e}_\phi \qquad (11)$$

The centrifugal force associated with Northward motion is given by

$$\mathbf{F}_r^{CFNF} = \frac{Dw}{Dt}\mathbf{e}_r = \frac{v^2}{a+z}\mathbf{e}_r$$