Dynamical Meteorology and Oceanography ATOC30004

Topic 4: Balanced vortices

- Acceleration in curved flows
- Gradient wind balance
- Cyclostrophic balance

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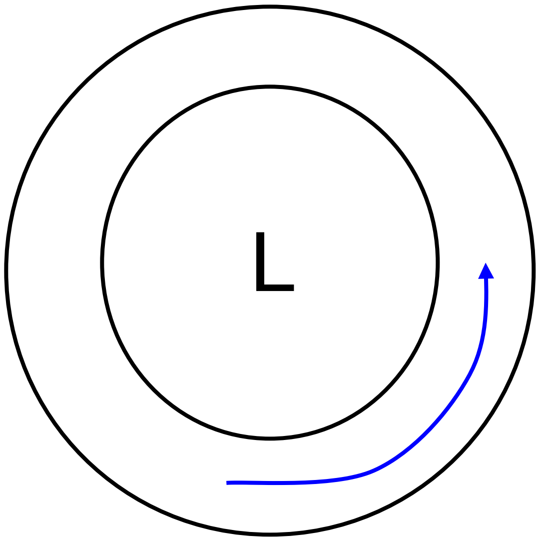
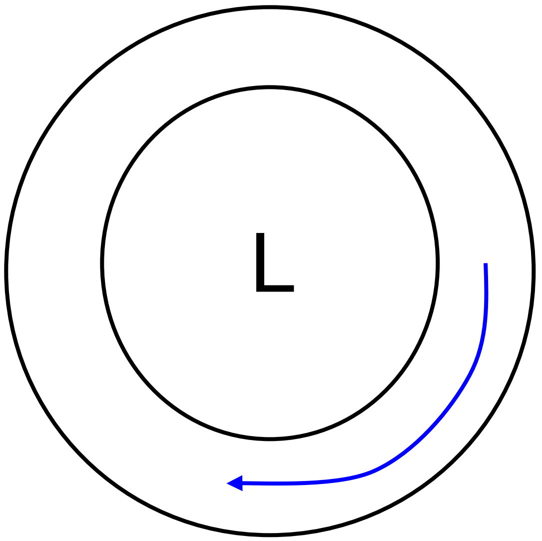
In northern hemisphere:

- Coriolis force is directed to right of winds
- Low pressure is always to left of winds
- Anticlockwise flow is called “cyclonic”
- Clockwise flow is called “anticyclonic”



In southern hemisphere:

- Coriolis force is directed to left of winds
- Low pressure is always to right of winds
- Clockwise flow is called “cyclonic”
- Anticlockwise flow is called “anticyclonic”



Properties of the geostrophic wind.

• There is no net force in the direction of the wind - i.e., no net acceleration of the flow (in rotating reference frame).

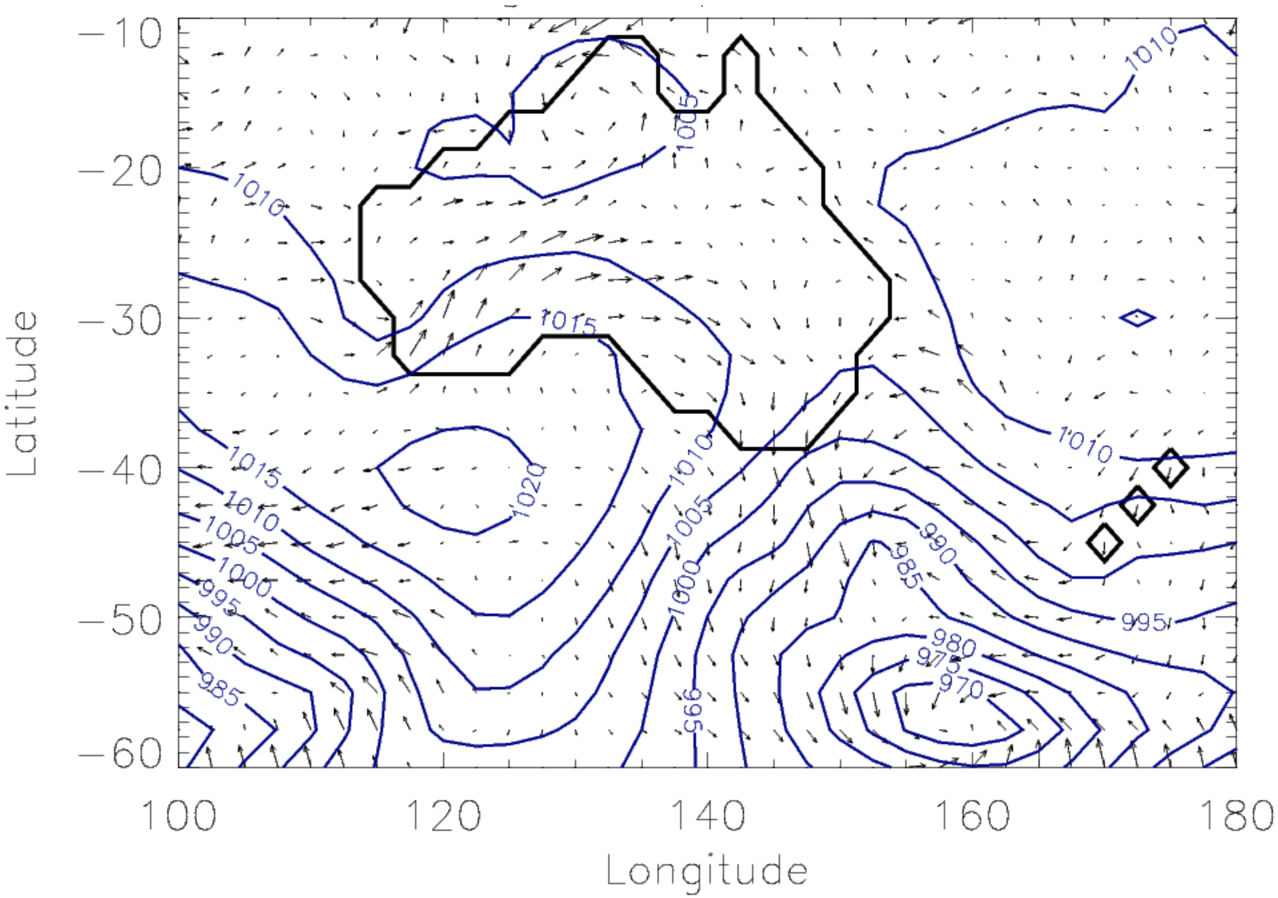
•The geostrophic wind equation is a "diagnostic" relation, it tells us nothing about the evolution of the flow. (The flow evolves due to *ageostrophic flow* [on a flat surface])

• **Is only strictly valid for straight isobars (no flow curvature).**

• The geostrophic wind relation breaks down in the tropics (small / zero coriolis). In that case, the acceleration terms are important.

$u_a = u - u_g \qquad v_a = v - v_g$

ageostrophic wind vectors



Note:

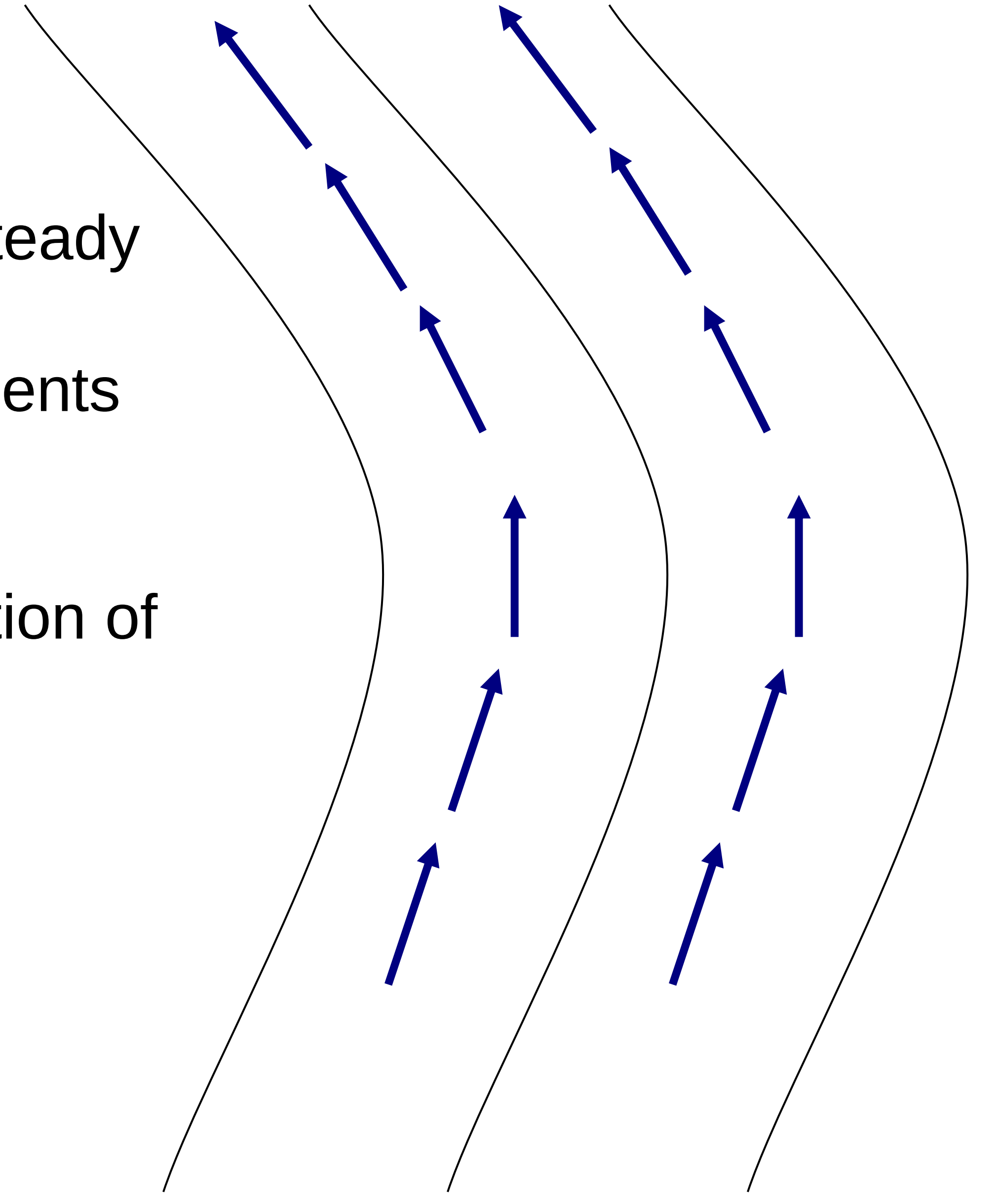
- **ageostrophic wind is strong in regions of curvature**
- **around cyclones ageostrophic flow opposes cyclonic circulation**
- ageostrophic flow is divergent (anticyclones) and convergent (cyclones)

Constant wind speed and steady

Variation in u and v components

Change in direction of wind corresponds to an acceleration of the wind.

To maintain a curved flow, undergoes centripetal acceleration.

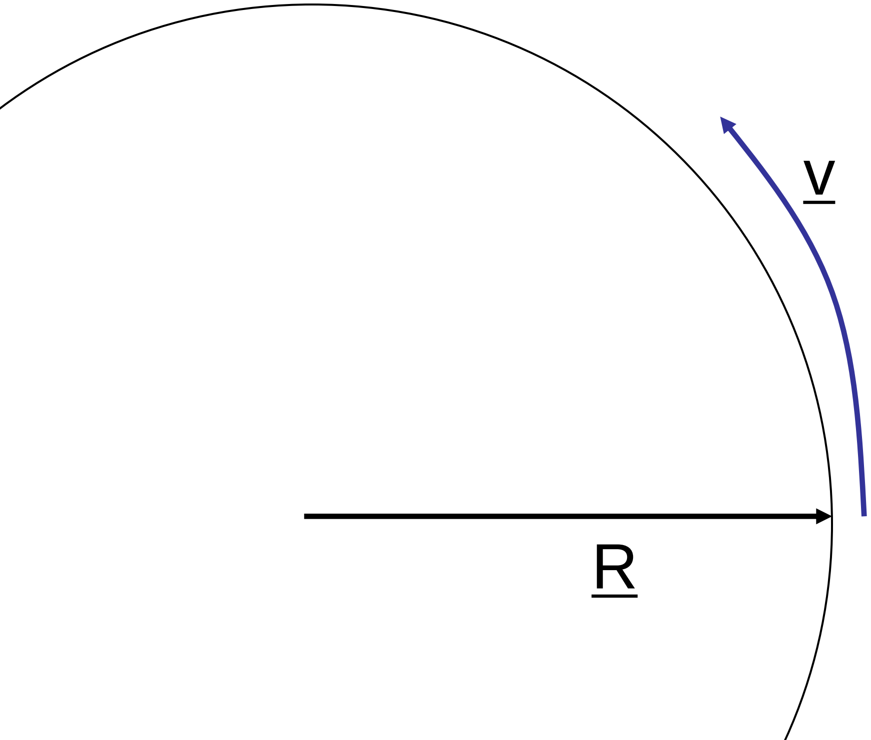


Total acceleration = D(speed)/Dt + centripetal accel.

Derive adjustment to acceleration due to curvature in flow.

- Centripetal effects due to curvature should not be confused with those associated with rotating reference frame (this has already been incorporated into our equations of motion).

Assume a steady flow, around a curved path. For simplicity assume that the curved path is circular with a constant radius R.



Define new coordinates (radial and azimuthal direction):

$$\underline{\mathrm{R}} = R\mathbf{e}_R$$

$$\underline{\mathrm{v}} = v\mathbf{e}_\phi$$

If horizontal velocity vector in new coordinate is:

$$\underline{u} = u\mathbf{e}_R + v\mathbf{e}_\phi$$

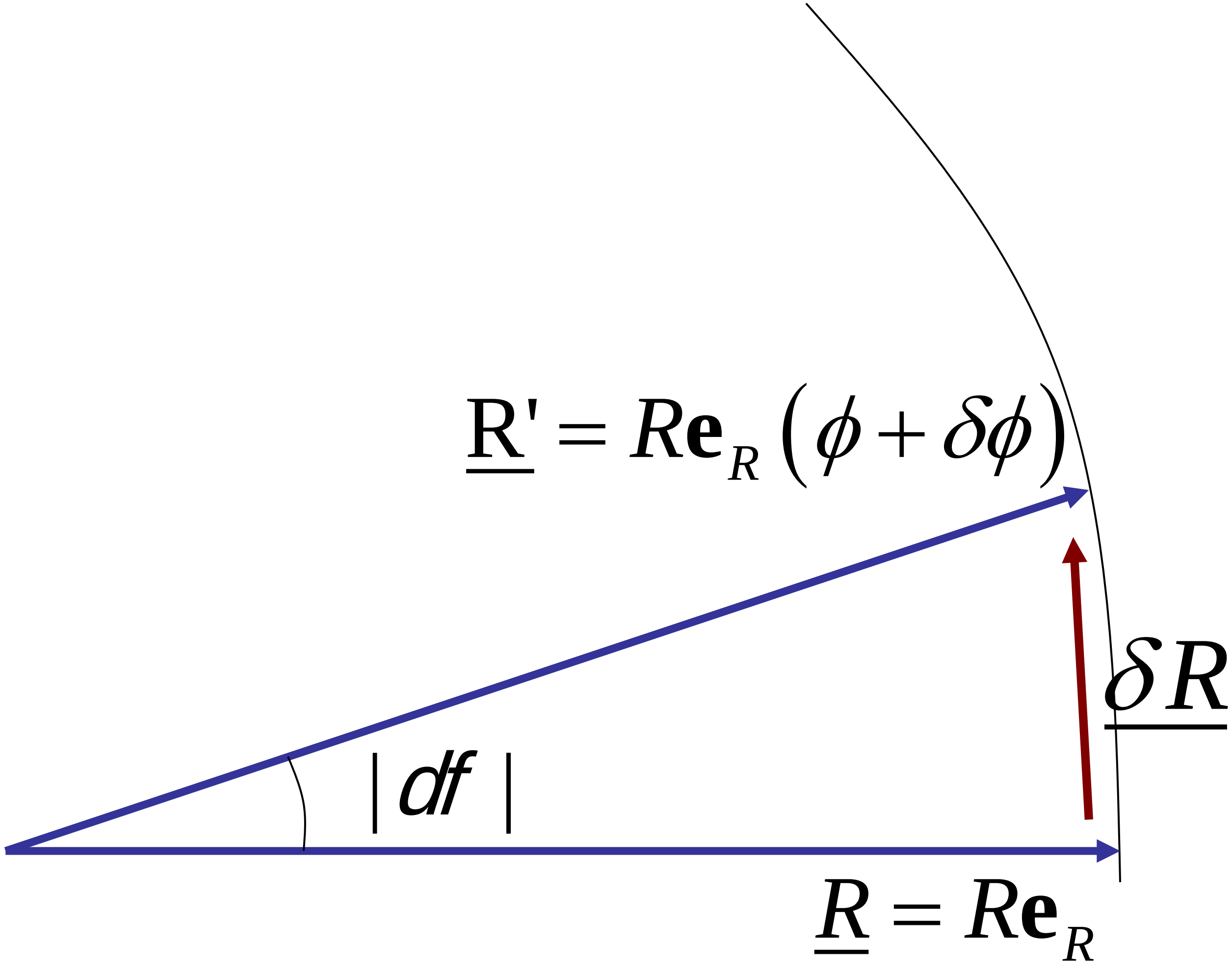
Then

$$\frac{D\underline{u}}{Dt} = \frac{Du\mathbf{e}_R}{Dt} + \frac{Dv\mathbf{e}_\phi}{Dt} = \mathbf{e}_R \frac{Du}{Dt} + u\frac{D\mathbf{e}_R}{Dt} + \mathbf{e}_\phi \frac{Dv}{Dt} + v\frac{D\mathbf{e}_\phi}{Dt}$$

1 2 3 4

(We must make this separation because the unit vectors change direction following the fluid).

Terms 1 & 3 are simply our fluid accelerations. Evaluating 2 and 4 will determine the form of the centripetal accelerations associated with the velocities moving in a curved reference frame.



df and *dr*

Are changes in unit vectors in small time δt

$$|\delta\phi| = \frac{|\underline{\delta R}|}{|\underline{R}|} = \frac{|\underline{\delta r}|}{|\underline{\hat{r}}|} = |\underline{\delta r}|$$

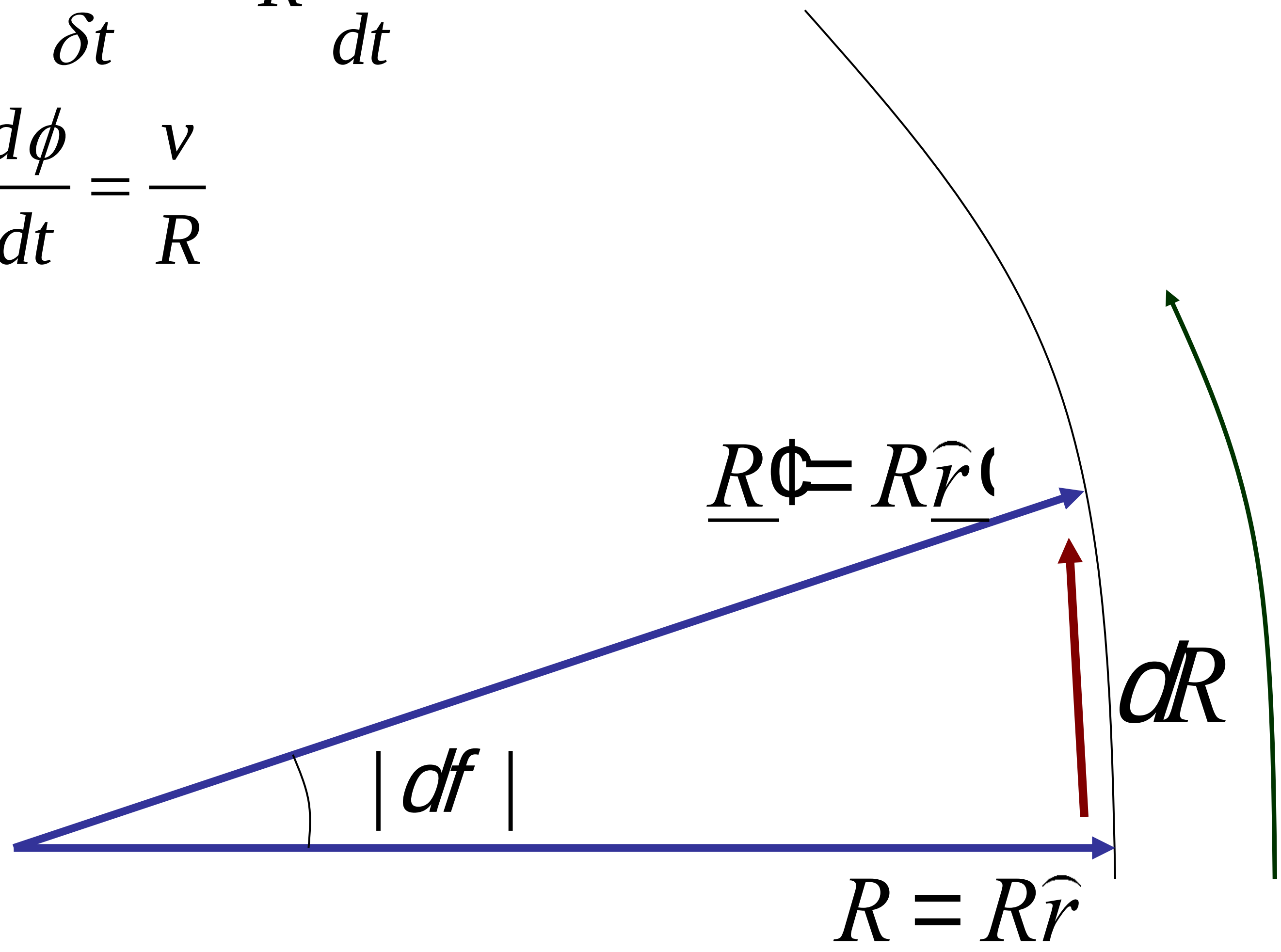
Remember: tanθ≈θ for small θ.
Magnitude of unit vector =1

$$|\delta\phi| = \frac{|\delta\underline{R}|}{|\underline{R}|} = \frac{|\delta\underline{r}|}{|\hat{\underline{r}}|} = |\delta\underline{r}|$$

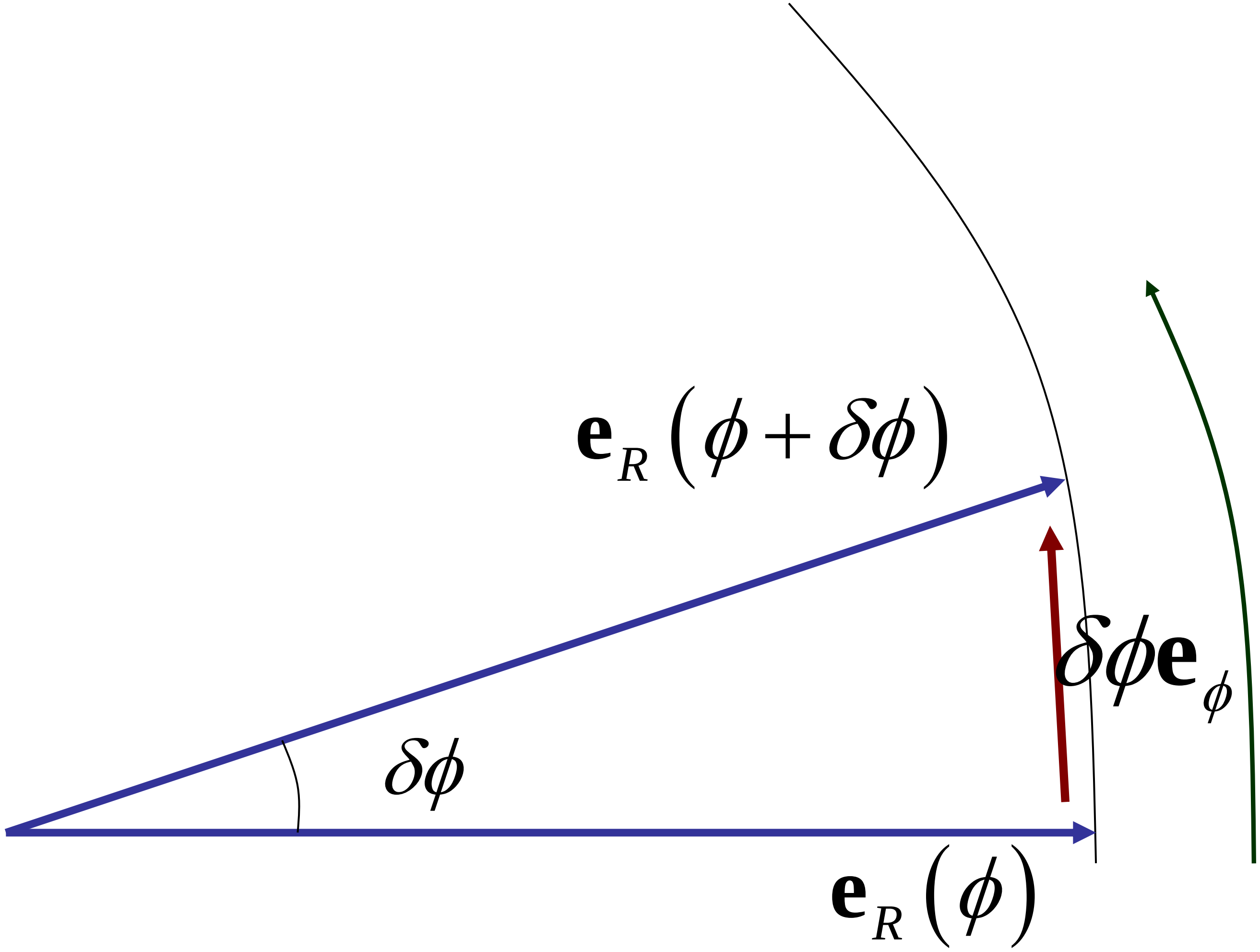
Divide both sides by δt

$$v = \frac{\left(R\delta\phi\right)}{\delta t} = R\frac{d\phi}{dt}$$

$$=> \frac{d\phi}{dt} = \frac{v}{R}$$

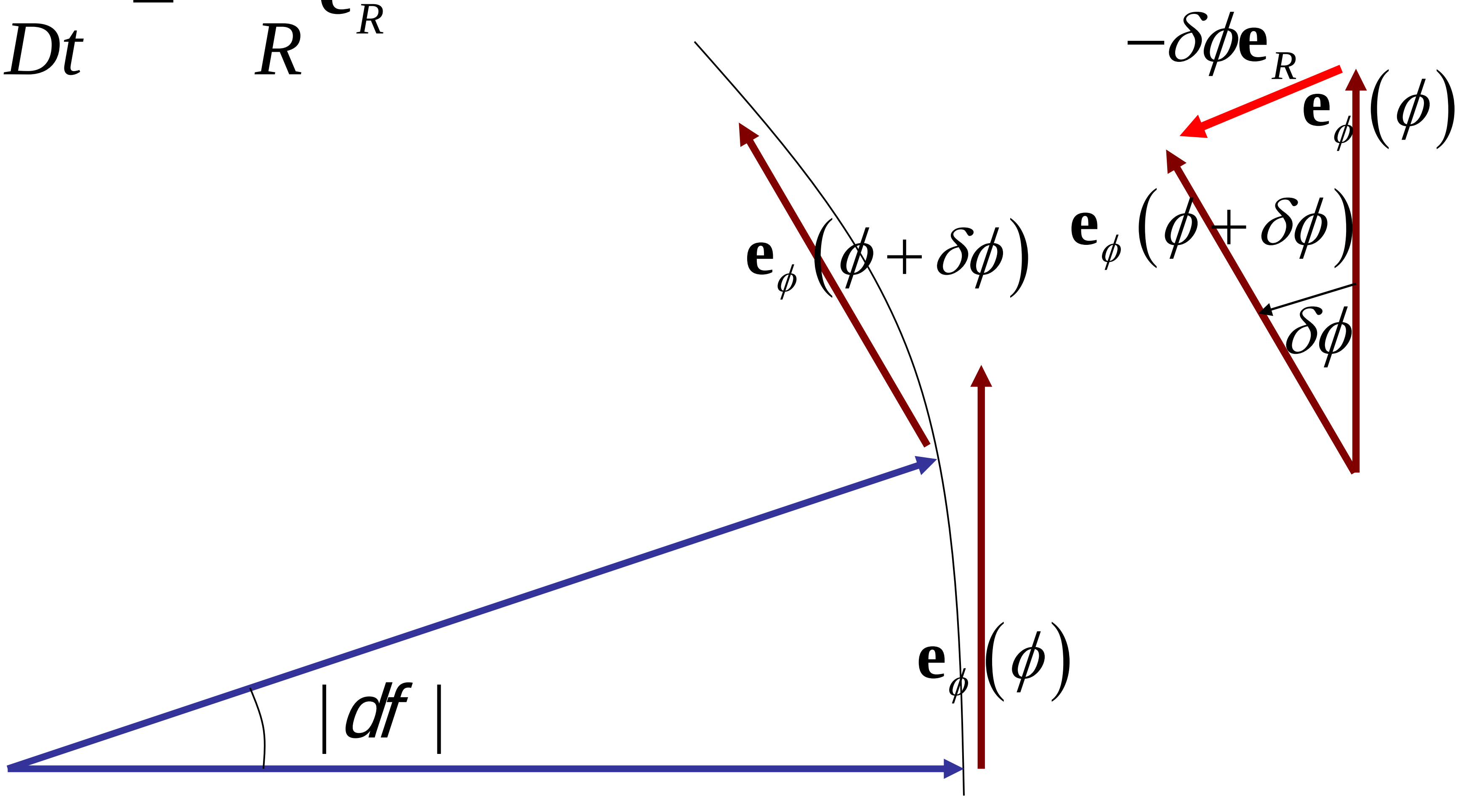


$$\frac{D\mathbf{e}_R}{Dt} = \frac{1\delta\phi}{\delta t}\mathbf{e}_\phi = \frac{D\phi}{Dt} = \frac{v}{R}\mathbf{e}_\phi$$



$$\frac{D\mathbf{e}_\phi}{Dt} \approx -\frac{\delta\phi\mathbf{e}_R}{\delta t} \approx -\frac{d\phi}{dt}\mathbf{e}_R = -\frac{v}{R}\mathbf{e}_R$$

$$\frac{D\mathbf{e}_\phi}{Dt} = -\frac{v}{R}\mathbf{e}_R$$



Thus, the acceleration becomes:

$$\frac{D\underline{u}}{Dt} = \mathbf{e}_R \frac{Du}{Dt} + u\frac{D\mathbf{e}_R}{Dt} + \mathbf{e}_\phi \frac{Dv}{Dt} + v\frac{D\mathbf{e}_\phi}{Dt}$$

$$= \mathbf{e}_R \frac{Du}{Dt} + u\left(\frac{v}{R}\mathbf{e}_\phi\right) + \mathbf{e}_\phi \frac{Dv}{Dt} + v\left(-\frac{v}{R}\mathbf{e}_R\right)$$

$$= \mathbf{e}_R\left(\frac{Du}{Dt} - \frac{v^2}{R}\right) + \mathbf{e}_\phi\left(\frac{Dv}{Dt} + \frac{uv}{R}\right)$$

Let's now consider the radial component of the acceleration ….

It can be shown that the form of the other terms in the radial velocity equation are unchanged:

$$\frac{Du}{Dt} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$

where we now use r for the radial distance instead of R

This equation is written in "cylindrical coordinates", and allows for both azimuthal and radial flow.

Another way to approach this problem is to examine "*natural coordinates*" where the coordinate system is aligned with the wind velocity, v, (and therefore u=0). This system has the same equation as that for cylindrical coordinates

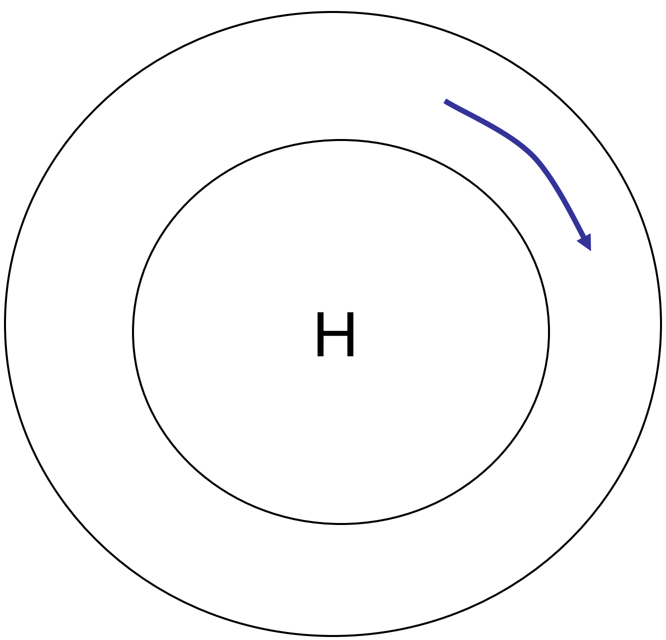
$$\frac{Du}{Dt} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

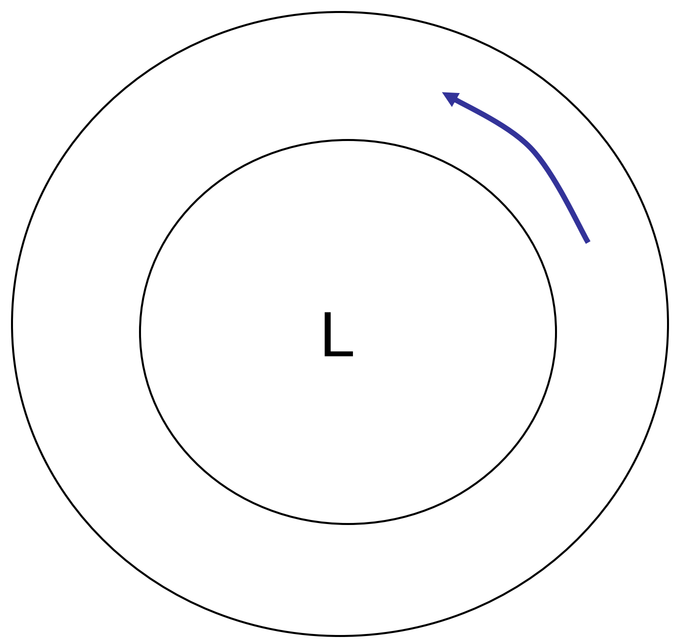
Here r is the "local" flow curvature, and r is aligned perpendicular to the flow.

Balanced vortices

Assume steady, low Rossby number flow in the vicinity of a circular pressure anomaly (High or Low).

$$\frac{v^2}{r} + fv = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

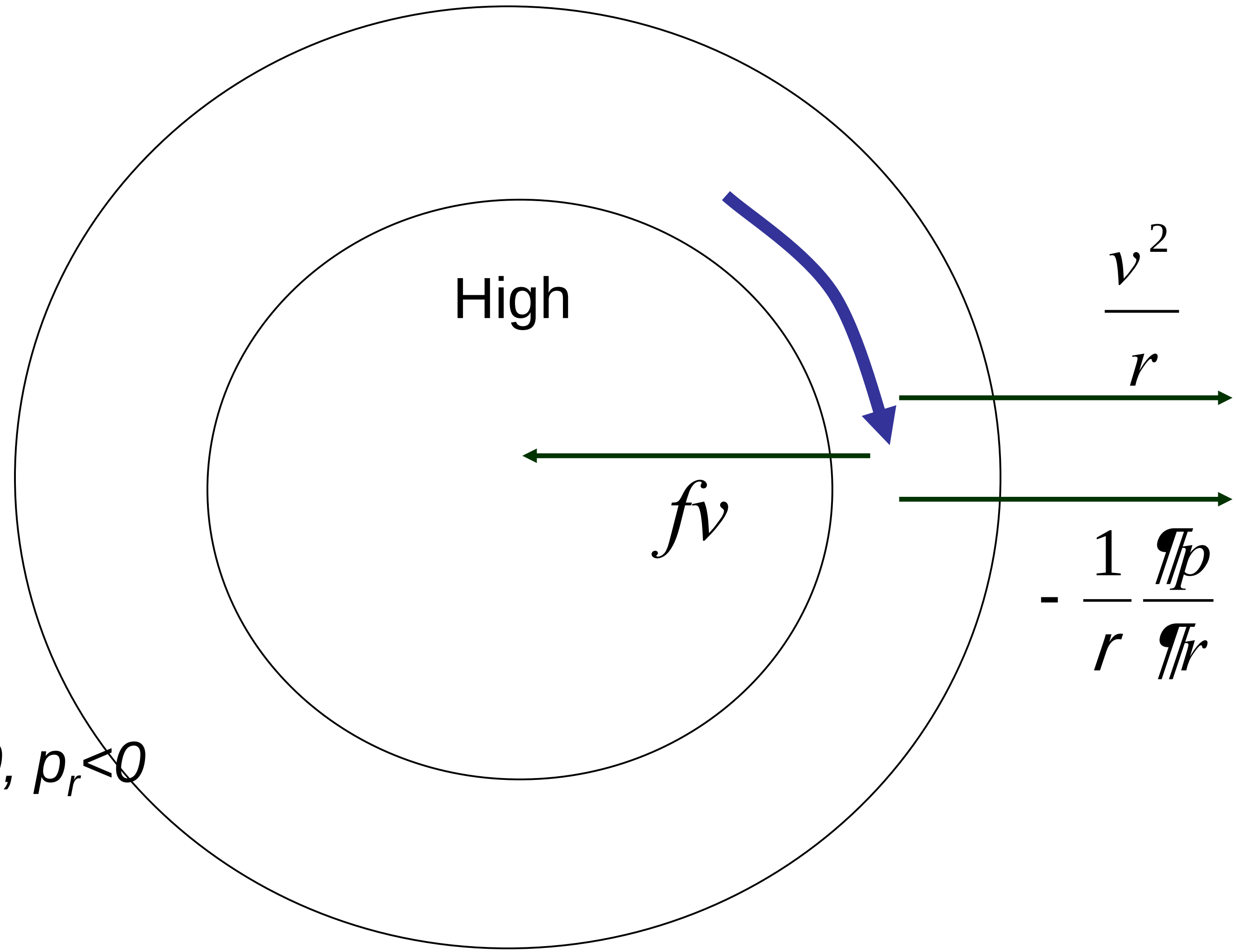




In NH: $f > 0$, $v < 0$, $p_r < 0$ $f > 0$, $v > 0$, $p_r > 0$

$$\frac{v^2}{r} + fv - \frac{1}{r}\frac{\partial p}{\partial r} = 0$$

Balance of forces: coriolis, centrifugal, pressure gradient.

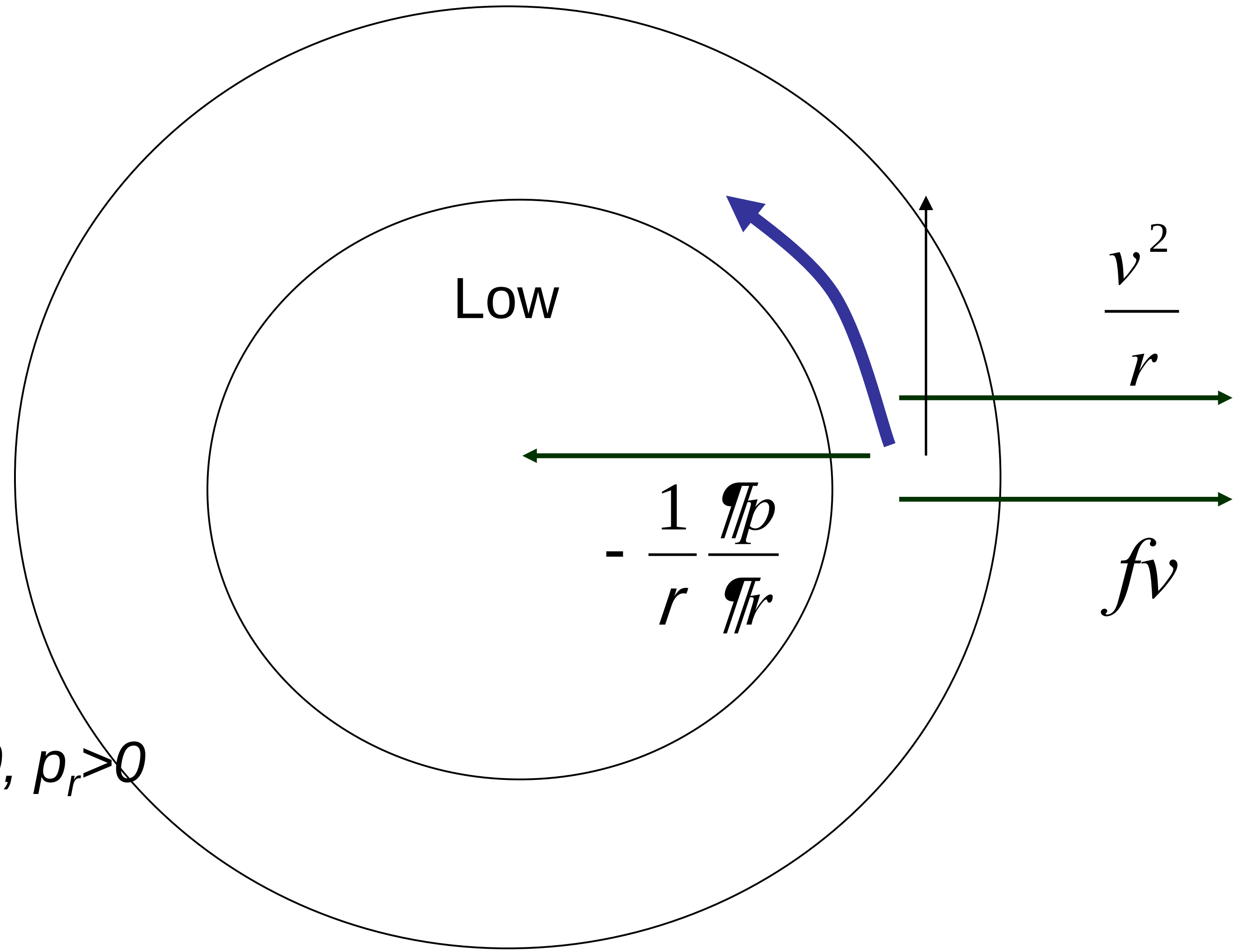


In NH: $f > 0$, $v < 0$, $p_r < 0$

$fv < 0$

$$\frac{v^2}{r} + fv - \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

Balance of forces: coriolis, centrifugal, pressure gradient.



In NH: $f > 0$, $v > 0$, $p_r > 0$

$fv > 0$

The balance between the pressure gradient force, the centrifugal force, and the coriolis force is known as:

Gradient wind balance

Gradient wind balance explains flow around curved trajectories in a rotating reference frame.

$$\frac{v^2}{r} + fv - \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

In limit as $r \rightarrow \infty$ this balance reverts to geostrophic balance.

Let’s re-write relation by assuming: $fv_g = \frac{1}{r}\frac{\partial p}{\partial r}$

$$\frac{v^2}{r} + fv - \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$\frac{v^2}{r} + fv = fv_g$$

$$\Rightarrow frv_g = frv + v^2$$

$$\Rightarrow \frac{v_g}{v} = 1 + \frac{v}{fr}$$

$v_g = geostrophic\ wind$

$v = \text{gradient wind}$

Recall that $v > 0$ corresponds to anti-clockwise flow (in either hemisphere).

This re-arrangement tells us a couple of things:

$$\frac{v_g}{v} = 1 + \frac{v}{fr} \approx 1 + sign\left(\frac{v}{fr}\right) Ro, \text{ where } Ro = \frac{U}{fL} \text{ is the Rossby numbe}$$

For cyclonic flow $v > 0$ (NH), $|v_g| > |v|$. *i.e., the gradient wind is slower than the geostrophic wind would be. This is referred to as "subgeostrophic".* (We saw this earlier, with the ageostrophic wind opposing the cyclonic flow).

For anti-cyclonic flow $v < 0$ (NH), $|v_g| < |v|$. *i.e., the gradient wind is faster than the geostrophic wind would be. This is referred to as "supergeostrophic".*

(The above two properties hold for NH and SH).

For small Rossby number, e.g., midlatitude synoptic flow (R_o~0.1), the difference between the gradient wind and geostrophic wind is only about 10%.

For larger Rossby number, e.g., tropical cyclones (R_o~1), the differences are much larger. (stronger curvature).

Now, let's find some simple solutions to the gradient wind equations

$$\frac{v^2}{r} + fv - \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$v^2 + frv - \frac{r}{\rho}\frac{\partial p}{\partial r} = 0$$

is of form: $Av^2 + Bv + C = 0$

which can be solved using: $v = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$$v = -\frac{fr}{2} \pm \frac{1}{2}\sqrt{f^2 r^2 + 4\frac{r}{\rho}\frac{\partial p}{\partial r}}$$

$$v = -\frac{fr}{2} \pm \frac{fr}{2}\sqrt{1 + \frac{4}{f^2 r \rho}\frac{\partial p}{\partial r}} = -\frac{fr}{2} \pm \frac{fr}{2}\left(1 + \frac{4}{f^2 r \rho}\frac{\partial p}{\partial r}\right)^{1/2}$$

gives the most intense possible anticyclone. Note that the vorticity of this most intense anticyclonic vortex is given by

$$\text{vorticity=lim area->0,} \frac{\text{circulation}}{area} = \frac{-\frac{fr}{2}(2\pi r)}{\pi r^2} = -\frac{f\pi r^2}{\pi r^2} = -f$$

It can be shown (see next page) that the positive root is the only solution that satisfies $v \rightarrow v_g$ when $r \rightarrow \infty$

$$v = -\frac{fr}{2} + \frac{fr}{2}\sqrt{1 + \frac{4}{f^2 r \rho}\frac{\partial p}{\partial r}}$$

Solution to the gradient wind equation.

All Low's have $\frac{\partial p}{\partial r} > 0$ and therefore the solution is real for all system intensities.

High's have $\frac{\partial p}{\partial r} < 0$ and therefore the solution is only real if:

$$\left|\frac{\partial p}{\partial r}\right| \leq \frac{f^2 r \rho}{4}$$

This condition places an upper limit on the intensity of a High with the maximum wind speed equal to $|v| = fr/2$

Proof that positive root ensures geostrophic balance as $r -> \infty$

$$v = -\frac{fr}{2} \pm \frac{fr}{2}\sqrt{1 + \frac{4}{f^2 r\rho}\frac{\partial p}{\partial r}} = -\frac{fr}{2} \pm \frac{fr}{2}\left(1 + \frac{4}{f^2 r\rho}\frac{\partial p}{\partial r}\right)^{1/2}$$

$$= -\frac{fr}{2} \pm \frac{fr}{2}[1 + x]^{1/2} = -\frac{fr}{2} \pm \frac{fr}{2}\left[1 + \frac{1}{2}x\right],$$ using Taylor expansion for very small x, $x = \frac{4}{f^2 r\rho}\frac{\partial p}{\partial r}$

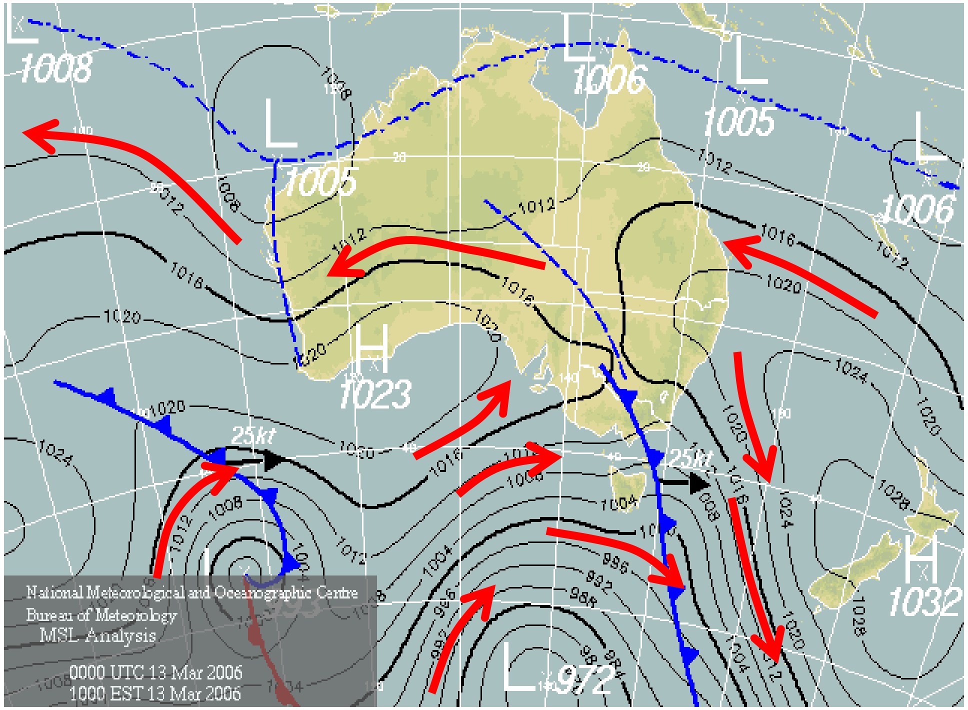
$$= -\frac{fr}{2} + \frac{fr}{2}\left[1 + \frac{1}{2}x\right] = \frac{fr}{4}x = \frac{fr}{4}\frac{4}{f^2 r\rho}\frac{\partial p}{\partial r} = \frac{1}{f\rho}\frac{\partial p}{\partial r}$$

so $\frac{1}{\rho}\frac{\partial p}{\partial r} \geq -\frac{f^2 r}{4}$ if $\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{f^2 r}{4}$ then $v = -\frac{fr}{2}$

Highs are limited to have weaker pressure gradients than Lows, implying weaker winds, and larger sizes.

Constraint also placed by rotation; f $\rightarrow$ 0 as approach equator.

Additional *agradient* flow causes size difference between High's and Low's also.



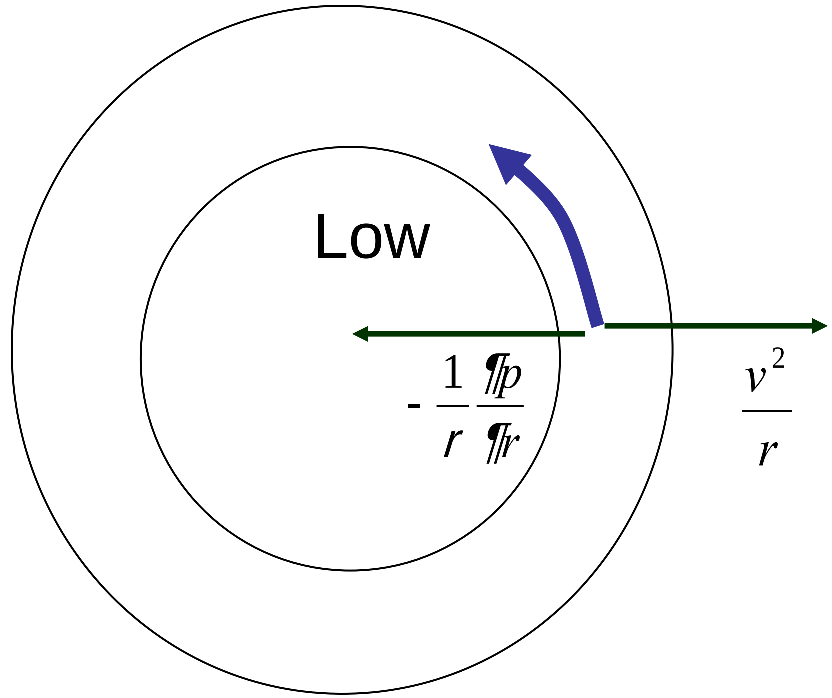
Additional class of balanced vortex: **Cyclostrophic balance.**

Large Rossby number, or equivalently negligible rotation (equator).

$$\frac{v^2}{r} + fv - \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$\frac{v^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$v = \pm\sqrt{\frac{r}{\rho}\frac{\partial p}{\partial r}}$$



Balance between centrifugal force and pressure gradient force.

Real solution only for Low pressure in vortex core.

Azimuthal velocity can go in either direction.

Bath tub vortex.

Clockwise vortex in Northern Hemisphere.