Dynamical Meteorology and Oceanography ATOC30004

Topic 2: Flow in a rotating frame of reference

- The Coriolis effect and effective gravity
- Scale analysis of the governing equations
- The Rossby number
- Geostrophy

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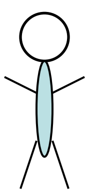
Derivation of momentum equations from Topic 1 was in an "inertial" frame of reference. The Earth, atmosphere, and ocean are all rotating.

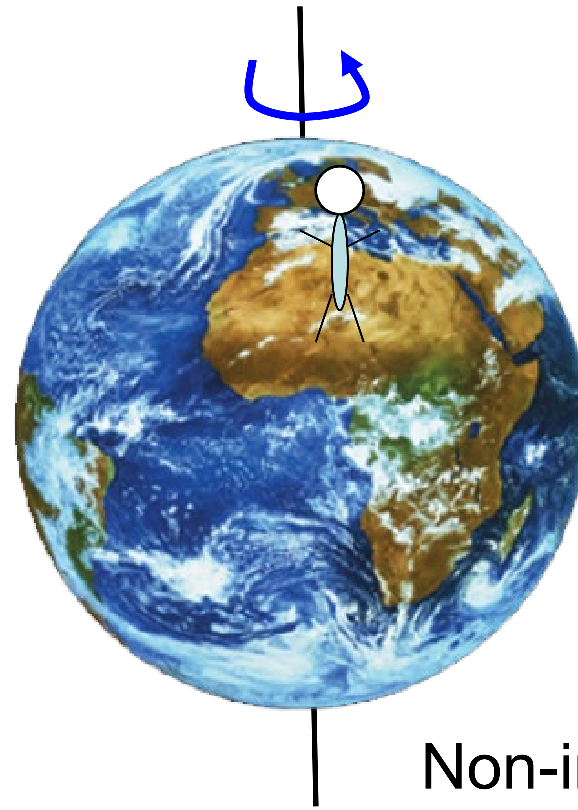
Rotation rate of Earth (in radians)

$$= \frac{2\pi}{23\ hours,\ 56\ minutes\ and\ 4\ seconds} = 7.2921159\ x\ 10^{-5}\ rad\ /\ s = \Omega$$

$$\square\ \frac{2\pi}{day} = \frac{2\pi}{24 \times 3600} = \frac{2\pi}{86,400}$$

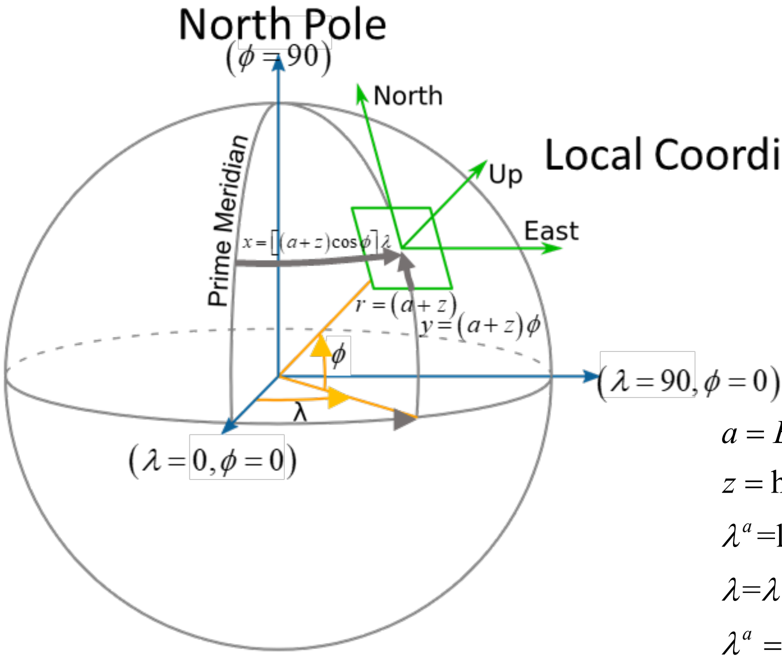


Inertial observer
(not accelerating)

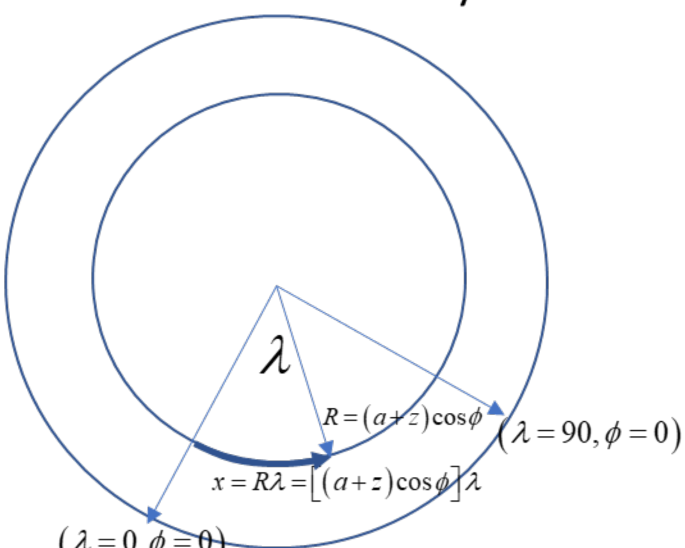


Non-inertial
(constantly accelerating)

Coordinates and Definitions



Overview of coordinate system



View from above North Pole

$a = Earth$ radius to sea level

$z =$ height above sea-level, $(a+z)$ is the distance of a parcel from Earth centre.

λ^a =longitude in a non-rotating reference frame

$\lambda = \lambda^a - \Omega t$, or equivalently

$\lambda^a = \Omega t + \lambda$ (note that λ=0 on the prime Greenwich meridian)

ϕ^a =latitude in a non-rotating reference frame=ϕ = latitude in a rotating reference frame

$$x^a = (a+z)\cos\phi\lambda^a = (a+z)\cos\phi[\Omega t + \lambda]$$

The absolute zonal wind speed is given by

$$u^a = (a+z)\cos\phi\frac{D\lambda^a}{Dt} = (a+z)\cos\phi\left(\Omega + \frac{D\lambda}{Dt}\right) = (a+z)\Omega\cos\phi + u$$

where $u = (a+z)\cos\phi\left(\frac{D\lambda}{Dt}\right)$ is the Earth relative zonal wind speed

$$x = R\lambda = [(a+z)\cos\phi]\lambda$$

$$y = (a+z)\phi$$

$$v = (a+z)\left(\frac{D\phi}{Dt}\right), w = \frac{Dz}{Dt}$$

What is the form of the apparent forces associated with the rotation of the Earth?

In topic 1, we showed that

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \text{apparent\_force\_x}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \text{apparent\_force\_y}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \text{apparent\_force\_z}$$

Here in topic 2, we will derive the quantitative form of the apparent forces by considering the apparent effect of centrifugal force and the conservation of angular momentum.

By way of introduction, this is the form of the apparent forces associated with the rotation of the Earth

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi) \qquad (5.1)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\sin\phi - \frac{vw}{(a+z)} \qquad (5.2)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_{eff} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\cos\phi + \frac{v^2}{(a+z)} \qquad (5.3)$$

Where do these terms come from?

We are going to prove that the terms are all the result of conservation of angular momentum and the centrifugal force: Specifically:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left|\mathbf{F}^{apparent}_{\text{conservation of angular momentum of zonal movement moving northward or upward}}\right|$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \left|\mathbf{F}^{apparent}_{\text{Northward component of centrifugal force of zonal flow}}\right|$$

$$+ \left|\mathbf{F}^{apparent}_{\text{conservation of angular momentum of Northward flow moving upward}}\right|$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \left|\mathbf{F}^{apparent}_{\text{Upward component of centrifugal force of zonal flow}}\right|$$

$$+ \left|\mathbf{F}^{apparent}_{\text{Upward component of centrifugal force of Northward flow}}\right|$$

$$\mathbf{F}_{\lambda}^{ZAM} = \mathbf{F}_{\text{conservation of angular momentum of zonal movement moving northward or upward}}^{apparent}$$

$$\mathbf{F}_{\phi}^{CFZF} = \mathbf{F}_{\text{Northward component of centrifugal force of zonal flow}}^{apparent}$$

$$\mathbf{F}_{\phi}^{NAM} = \mathbf{F}_{\text{conservation of angular momentum of Northward flow moving upward}}^{apparent}$$

$$\mathbf{F}_{r}^{CFZF} = \mathbf{F}_{\text{Upward component of centrifugal force of zonal flow}}^{apparent}$$

$$\mathbf{F}_{r}^{CFNF} = \mathbf{F}_{\text{Upward component of centrifugal force of Northward flow}}^{apparent}$$

$$\mathbf{F}_{\lambda}^{ZAM} = \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\left(v\sin\phi - w\cos\phi\right)\mathbf{e}_{\lambda} \tag{7.1}$$

$$\mathbf{F}_{\phi}^{CFZF} = -\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\left(u\sin\phi\right)\mathbf{e}_{\phi} \tag{7.2}$$

$$\mathbf{F}_{r}^{CFZF} = \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\left(u\cos\phi\right)\mathbf{e}_{r} \tag{7.3}$$

$$\mathbf{F}_{\phi}^{NAM} = -\frac{vw}{(a+z)}\mathbf{e}_{\phi} \cong -\frac{vw}{a}\mathbf{e}_{\phi} \tag{7.4}$$

$$\mathbf{F}_{r}^{CFNF} = \frac{v^2}{a+z}\mathbf{e}_{r} \tag{7.5}$$

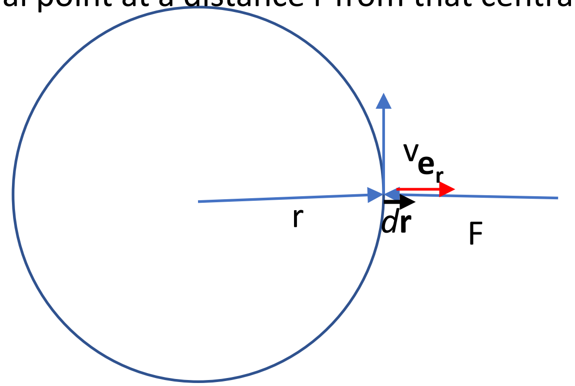
Conservation of angular momentum (review)

$$acceleration, a = \frac{2\pi v}{T},$$ where T is the period and $T = \dfrac{2\pi r}{v}$ so that

$$a = \frac{2\pi v v}{2\pi r} = \frac{v^2}{r}$$

$$\mathbf{F} = ma\mathbf{e}_r = -\frac{mv^2}{r}\mathbf{e}_r$$

Consider a body rotating around a central point at a distance r from that central POINT.



where $\mathbf{e}_r$ is the unit vector pointing radially outward from the centre of the circle.

$$\delta\mathbf{r} = \mathbf{r} + \delta\mathbf{r} - \mathbf{r}$$

Physics tells us that

$$\mathbf{F}.\delta\mathbf{r} = \delta\left(\frac{1}{2}mv^2\right) = KE(r+\delta r) - KE(r)$$

$$-\frac{mv^2}{r}\mathbf{e}_r.\delta\mathbf{r} = -\frac{mv^2}{r}\delta r = \frac{1}{2}m\delta\left(v^2\right)$$

$$=> -\frac{v^2}{r}\delta r = \frac{1}{2}\delta\left(v^2\right)$$

$$=> -\frac{\delta r}{r} = \frac{1}{2}\frac{\delta\left(v^2\right)}{v^2} \qquad (8.1)$$

Noting that

$\dfrac{d\left(\ln(r)\right)}{dr} = \dfrac{1}{r}$ and $\dfrac{d\left(\ln(v^2)\right)}{d\left(v^2\right)} = \dfrac{1}{v^2}$, it follows that for very small δr and very small $\delta\left(v^2\right)$

$$\frac{\delta r}{r} = \frac{d\left(\ln(r)\right)}{dr}\delta r = \delta\ln(r), \qquad (8.2a)$$

$$\frac{\delta\left(v^2\right)}{v^2} = \frac{d\left(\ln(v^2)\right)}{d\left(v^2\right)}\delta\left(v^2\right) = \delta\ln(v^2) = \delta 2\ln(v) = 2\delta\ln(v) \qquad (8.2b)$$

Conservation of angular momentum (review)

Hence, using (8.2) in $-\frac{\delta r}{r}=\frac{1}{2}\frac{\delta\left(v^2\right)}{v^2}$ gives

$-\{\delta[\ln(r)]\}=\delta\ln(v);$ hence,

$$-\{\ln(r+\delta r)-\ln(r)\}=\ln\left[v\left(r+\delta r\right)\right]-\ln\left[v\left(r\right)\right] \qquad (9.1)$$

Note that in this equation (9.1), $v\left(r+\delta r\right)$ indicates the speed of the parcel at the radius $r+\delta r$ and $v(r)$ indicates the speed of the parcel at radius r.

Equation (9.1) implies that

$$\ln\left(\frac{r}{r+\delta r}\right)=\ln\left(\frac{v\left(r+\delta r\right)}{v\left(r\right)}\right) => \frac{r}{r+\delta r}=\frac{v\left(r+\delta r\right)}{v\left(r\right)} \qquad (9.2)$$

$$=> rv(r)=\left(r+\delta r\right)v\left(r+\delta r\right) \qquad (9.3)$$

Because (2.3) is true for any r value and for any small change δr, it follows that the quantity (rv) or (mvr) is conserved;i.e.

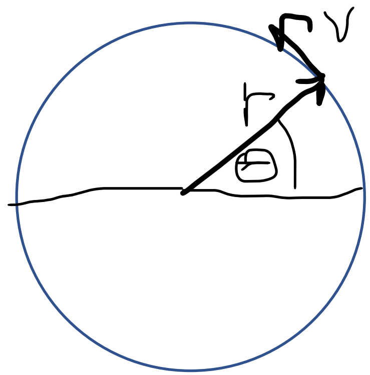
$$r_1 v(r_1)=r_2 v_2\left(r_2\right) \quad \text{(following the parcel)} \qquad (9.4a)$$

$$=>\frac{D\left(rv\right)}{Dt}=0 \qquad (9.4b)$$

The quantity mvr is called the angular momentum.

Following a non-precipitating fluid parcel, both mvr and vr are conserved.

Simple expressions for kinematics of objects going around in circles. (Review)



An object travelling around a circle of radius r every T seconds at constant speed, has:

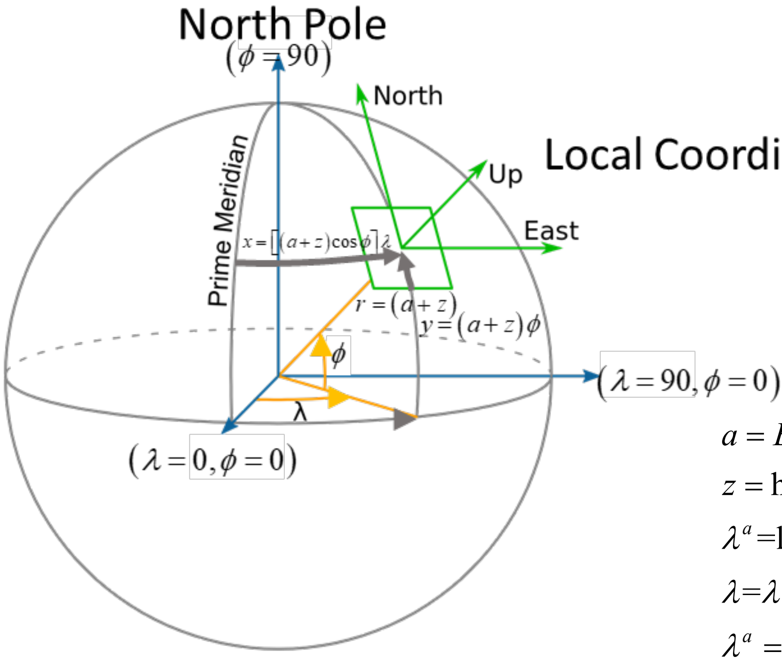
the angular speed, $\dfrac{d\theta}{dt} = \omega = \dfrac{2\pi}{T}$, ($T$ is the time it takes for the object to complete a circle == period)

the speed, $\mathrm{v} = \dfrac{2\pi r}{T}$; $\Rightarrow \omega = \dfrac{\mathrm{v}}{r} \Rightarrow \mathrm{v} = \omega r = \dfrac{d\theta}{dt} r$

and the centripetal acceleration, $\dfrac{2\pi \mathrm{v}}{T} = \dfrac{4\pi^2 r}{T^2} = \dfrac{\mathrm{v}^2}{r} = \omega^2 r.$

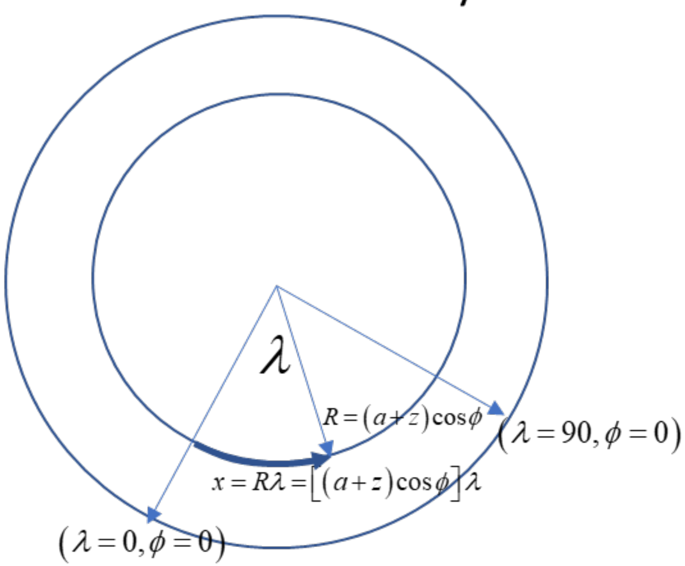
Recall that the centripetal acceleration is directed towards the centre of the circle.

Coordinates and Definitions for apparent force analysis



Local Coordinate system

Overview of coordinate system



View from above North Pole

$a = Earth$ radius to sea level

$z =$ height above sea-level, $(a+z)$ is the distance of a parcel from Earth centre.

λ^a=longitude in a non-rotating reference frame

$\lambda = \lambda^a - \Omega t$, or equivalently

$\lambda^a = \Omega t + \lambda$ (note that λ=0 on the prime Greenwich meridian)

ϕ^a=latitude in a non-rotating reference frame=ϕ = latitude in a rotating reference frame

$x^a = (a+z)\cos\phi\lambda^a = (a+z)\cos\phi[\Omega t + \lambda]$

The absolute zonal wind speed is given by

$$u^a = (a+z)\cos\phi\frac{D\lambda^a}{Dt} = (a+z)\cos\phi\left(\Omega + \frac{D\lambda}{Dt}\right) = (a+z)\Omega\cos\phi + u$$

where $u = (a+z)\cos\phi\left(\frac{D\lambda}{Dt}\right)$ is the Earth relative zonal wind speed

$x = R\lambda = (a+z)\cos\phi\lambda$

$y = (a+z)\phi$

$v = (a+z)\left(\frac{D\phi}{Dt}\right), w = \frac{Dz}{Dt}$

Conservation of angular momentum on planet Earth: the absolute Eastward speed of a parcel

Now consider an air parcel rotating around the North pole at a radius R at a speed u^a. The absolute zonal wind u^a in terms of the distance $(a+z)$ of the parcel from the centre of the Earth and the rate of change of longitude following a fluid parcel.

The distance of the parcel from the axis of rotation is $R=(a+z)\cos(\phi)$. The speed of the parcel relative to an observer in a non-rotating inertial reference frame (sat a stationary reference frame far above the North pole)

is given by

$$u^a = \frac{D\lambda^a}{Dt}R = \frac{D\lambda^a}{Dt}(a+z)\cos(\phi) \qquad (11.1)$$

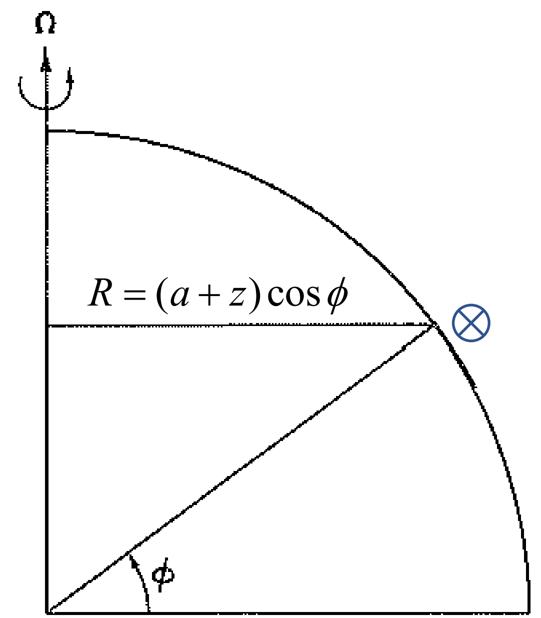
But $\lambda^a = \lambda + \Omega t$ so $\dfrac{D\lambda^a}{Dt} = \Omega + \dfrac{D\lambda}{Dt}$ (11.2)

then using (3.2) in (3.1) gives

$$u^a = \left(\Omega + \frac{D\lambda}{Dt}\right)(a+z)\cos(\phi)$$

$$= \Omega(a+z)\cos(\phi) + \left(\frac{D\lambda}{Dt}\right)(a+z)\cos(\phi) = \underbrace{\Omega(a+z)\cos(\phi)}_{\substack{\textit{Speed}\text{ of Earth} \\ \text{stationary things}}} + u$$

where $u = \left(\dfrac{D\lambda}{Dt}\right)(r+z)\cos(\phi)$ is the Earth relative zonal velocity.



Conservation of angular momentum

Let the zonal component of wind in a non-rotating frame be $u^a = \Omega\left[\left(a+z\right)\cos\phi\right]+u$,

where $u=\left(a+z\right)\cos\phi\dfrac{D\lambda}{Dt}$ is the Earth relative part and $\Omega\left[\left(a+z\right)\cos\phi\right]$ is the part

due to the rotation of the Earth. Angular momentum per unit mass of parcel is

then $u^a\left[\left(a+z\right)\cos\phi\right]$. Conservation of angular momentum per unit mass then gives

$$0=\frac{D\left(u^a\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}=u^a\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}+\left[\left(a+z\right)\cos\phi\right]\frac{D\left(u^a\right)}{Dt}$$

Rearranging the first equation gives

$$\frac{D\left(u^{\mathrm{a}}\right)}{Dt}=-\frac{u^a}{\left(a+z\right)\cos\phi}\frac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}\qquad(12.1)$$

Note that (12.1) is saying that if $\left[\left(a+z\right)\cos\phi\right]$ gets smaller so that $\dfrac{D\left(\left[\left(a+z\right)\cos\phi\right]\right)}{Dt}<0$

then $\dfrac{D\left(u^{\mathrm{a}}\right)}{Dt}>0$ just as we anticipated!

Conservation of angular momentum

Recalling that $u^a = \Omega\left[(a+z)\cos\phi\right] + u$, it follows that

$$\frac{D\left(u^{\mathrm{a}}\right)}{Dt} = \Omega\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt} + \frac{Du}{Dt} \qquad (12.2a)$$

Using (12.2a) in (12.1) gives

$$\frac{Du}{Dt} = -\Omega\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt} - \frac{u^a}{(a+z)\cos\phi}\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt} \qquad (12.2b)$$

$$= -\left(\Omega + \frac{u^a}{(a+z)\cos\phi}\right)\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt} \qquad (12.3)$$

Using $u^a = \Omega\left[(a+z)\cos\phi\right] + u$ in (12.3) gives

$$\frac{Du}{Dt} = -\left(\Omega + \frac{\Omega\left[(a+z)\cos\phi\right] + u}{(a+z)\cos\phi}\right)\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt}$$

$$\Rightarrow \frac{Du}{Dt} = -\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt} \qquad (12.4)$$

Note how similar this equation is to (5.1). We are getting close!

Conservation of angular momentum

Note that

$$\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt}=\frac{D\left(\left[(a+z)\right]\right)}{Dt}\cos\phi+(a+z)\frac{D\left(\left[\cos\phi\right]\right)}{Dt}$$

$$=w\cos\phi+(a+z)\frac{D\phi}{Dt}\frac{d\left(\left[\cos\phi\right]\right)}{d\phi}=w\cos\phi-\left[(a+z)\frac{D\phi}{Dt}\right]\sin\phi$$

Since $(a+z)\dfrac{D\phi}{Dt}=v,$ this simplifies to

$$\frac{D\left(\left[(a+z)\cos\phi\right]\right)}{Dt}=w\cos\phi-v\sin\phi, \qquad (12.5)$$

Using (5) in (4) gives

$$\frac{Du}{Dt}=\left(2\Omega+\frac{u}{(a+z)\cos\phi}\right)(v\sin\phi-w\cos\phi) \qquad (12.6a)$$

The acceleration given by (12.6a) is entirely due to the effect of conservation of angular within our choice of latitude,longitude rotating frame of reference. Hence, we may associate it with the apparent force-per-unit-mass due to conservation of angular momentum;i.e.

$$\mathbf{F}_{\lambda}^{ZAM}=\left(2\Omega+\frac{u}{(a+z)\cos\phi}\right)(v\sin\phi-w\cos\phi)\mathbf{e}_{\lambda} \qquad (12.6b)$$

Thus recovering equation (7.1) and (5.1) as was required.

By way of introduction, this is the form of the apparent forces associated with the rotation of the Earth

Proved!

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \boxed{\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi)} \qquad (5.1)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\sin\phi - \frac{vw}{(a+z)} \qquad (5.2)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_{eff} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\cos\phi + \frac{v^2}{(a+z)} \qquad (5.3)$$

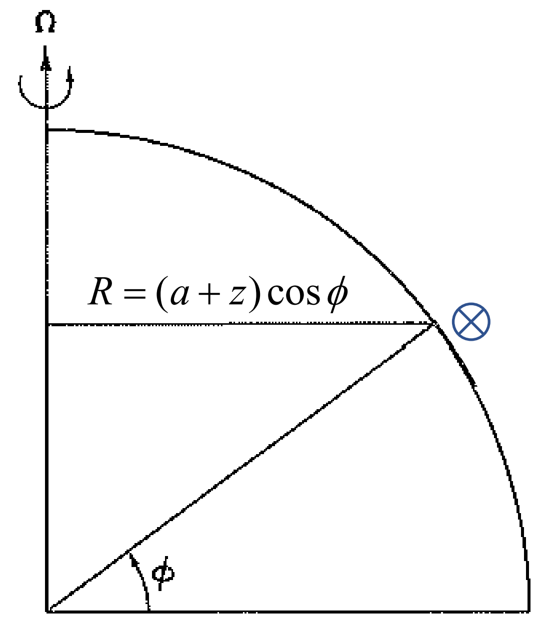
Centrifugal force associated with zonal motion

The centrifugal force is directed perpendicularly outward from the axis of rotation

$$\mathbf{e}_R = \cos\phi\,\mathbf{e}_r - \sin\phi\,\mathbf{e}_\phi \qquad (15.1)$$

The apparent centrifugal force is given by the negative of the centripetal acceleration; ie

$$\mathbf{F}_R^{CFZF} = \frac{\left(u^a\right)^2}{(a+z)\cos\phi}\mathbf{e}_R \qquad (15.2)$$



Recall that $u^a = \left\{\Omega\left[(a+z)\cos\phi\right]\right\} + u$ so

$$\left(u^a\right)^2 = \left\{\Omega\left[(a+z)\cos\phi\right]\right\}^2 + 2u\left\{\Omega\left[(a+z)\cos\phi\right]\right\} + u^2 \qquad (15.3)$$

so that

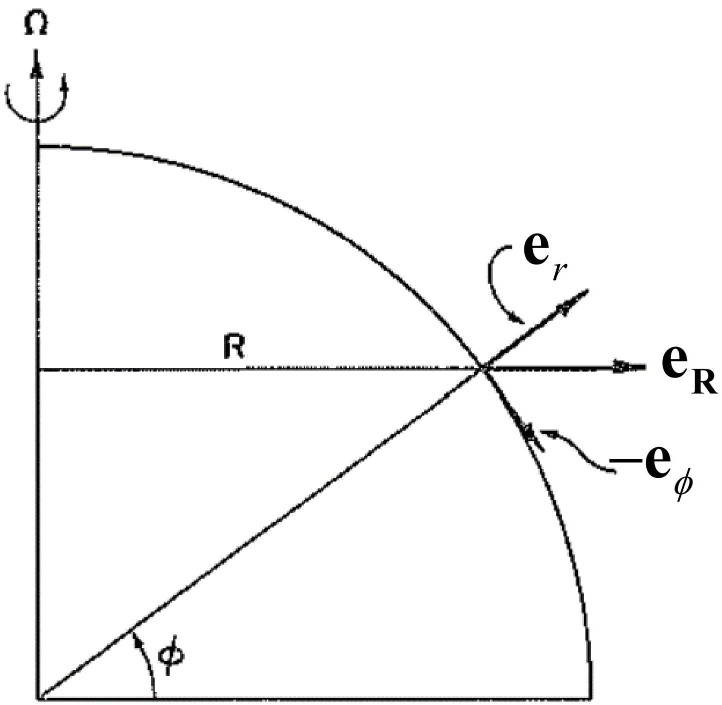
$$\frac{\left(u^a\right)^2}{(a+z)\cos\phi} = \frac{\left\{\Omega\left[(a+z)\cos\phi\right]\right\}^2}{(a+z)\cos\phi} + 2\Omega u + \frac{u^2}{(a+z)\cos\phi} \qquad (15.4)$$

Note that $\dfrac{\left\{\Omega\left[(a+z)\cos\phi\right]\right\}^2}{(a+z)\cos\phi}$ is independent of u. Might this term affect the shape of the Earth?

Effect of centrifugal force on shape of Earth

Suppose all of the Earth was covered by water (no dry land) then assuming there was no zonal current, the centrifugal force on the water would be given by

$$\mathbf{F}_R^{CFZF} = \frac{\left\{\Omega\left[\left(a+z\right)\cos\phi\right]\right\}^2}{\left(a+z\right)\cos\phi}\mathbf{e}_R = \Omega^2\left[\left(a+z\right)\cos\phi\right]\mathbf{e}_R = \Omega^2 R\mathbf{e}_R$$



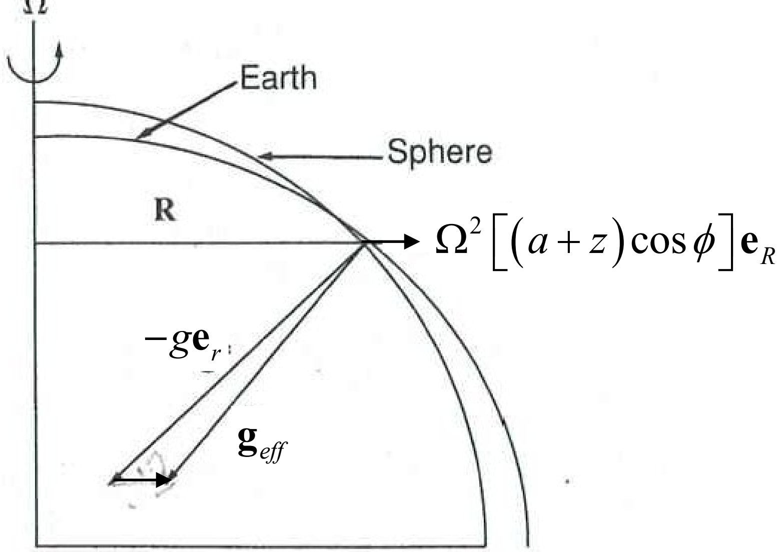
How would this force push the water?

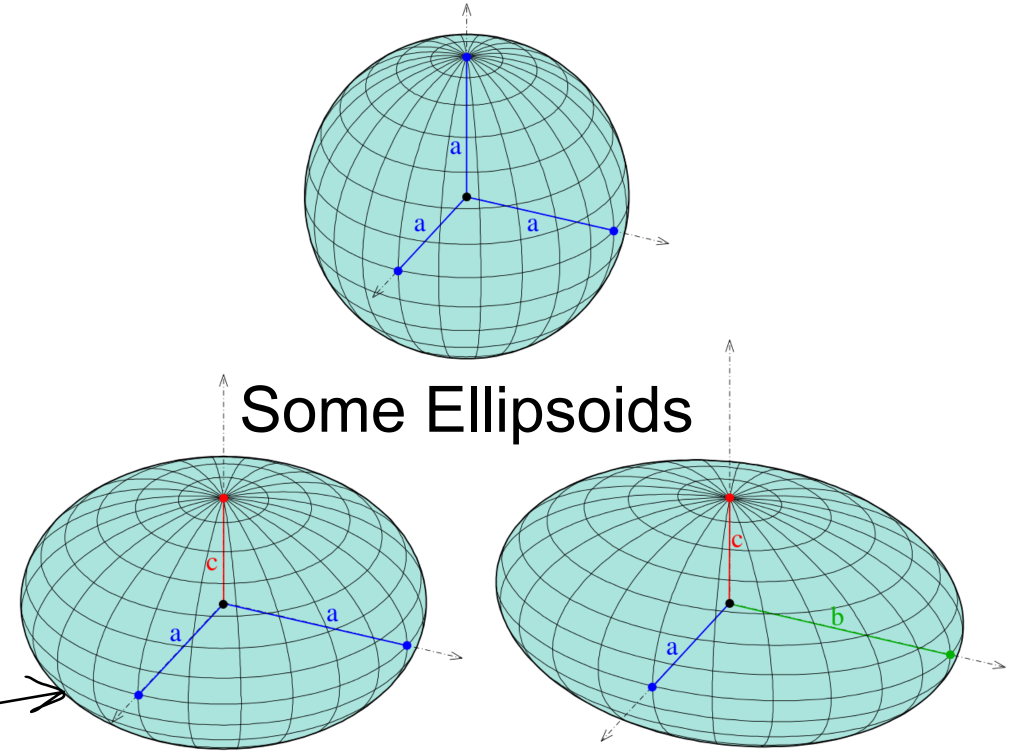
It would push the water towards the equator, right?

This would give the Earth an ellipsoidal shape, right?

One could hypothesize that the Earth would keep getting more ellipsoidal until the sum $\mathbf{g}_{eff}$ of the gravitational force-per-unit-mass $\mathbf{g}$ and $\Omega^2\left[\left(a+z\right)\cos\phi\right]\mathbf{e}_R$ was precisely perpendicular to the surface of the ocean.

Right?





Earth is like this

It turns out that this is exactly what has happened!

The ellipsoidal shape of the Earth ensures the undisturbed ocean lies perpendicular to the vector

$$\mathbf{g}_{eff} = -g\mathbf{e}_r + \Omega^2\left[(a+z)\cos\phi\right]\mathbf{e}_R$$

Because the Earth is almost a sphere, to a high degree of accuracy we can still model the Earth as a sphere provided we assume that the direction of $\mathbf{g}_{eff}$ is perpendicular to the surface of the sphere;

ie, we assume that $\mathbf{g}_{eff} \approx -g_{eff}\mathbf{e}_r$ where $-g_{eff} = -g + \Omega^2\left[(a+z)\cos\phi\right]$.

By absorbing $\Omega^2\left[(a+z)\cos\phi\right]\mathbf{e}_R$ into gravity, we can still model the Earth as a sphere while preventing modelled oceans/atmospheres from flowing towards the equator because of the centrifugal force. It's more accurate to assume the spheroidal geometry at the outset but, for simplicity, we make the spherical approximation here and correct its major flaw by using $\mathbf{g}_{eff}$ instead of $\mathbf{g}$ for gravity.

By way of introduction, this is the form of the apparent forces associated with the rotation of the Earth

Proved!

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \boxed{\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi)} \qquad (5.1)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\sin\phi - \frac{vw}{(a+z)} \qquad (5.2)$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \boxed{- g_{eff}} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\cos\phi + \frac{v^2}{(a+z)} \qquad (5.3)$$

Proved!

Having absorbed this term into an *effective* gravity, the remnant apparent force in the $\mathbf{e}_R$ direction is given by

$$\mathbf{F}_R^{CFZF} = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\right]\mathbf{e}_R \qquad (15.5)$$

Recalling that $\mathbf{e}_R = \cos\phi\mathbf{e}_r - \sin\phi\mathbf{e}_\phi$ so that $\mathbf{e}_R.\mathbf{e}_\phi = -\sin(\phi)$, the component of this force in the Northward $\mathbf{e}_\phi$ direction is given by

$$\mathbf{F}_\phi^{CFZF} = \left(\mathbf{F}_R^{CFZF}.\ \mathbf{e}_\phi\right)\mathbf{e}_\phi$$

$$=> \mathbf{F}_\phi^{CFZF} = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\right]u\left(\mathbf{e}_R.\mathbf{e}_\phi\right)\mathbf{e}_\phi$$

$$\boxed{= -\left\{\left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\right]u\sin\phi\right\}\mathbf{e}_\phi \qquad (15.6)}$$

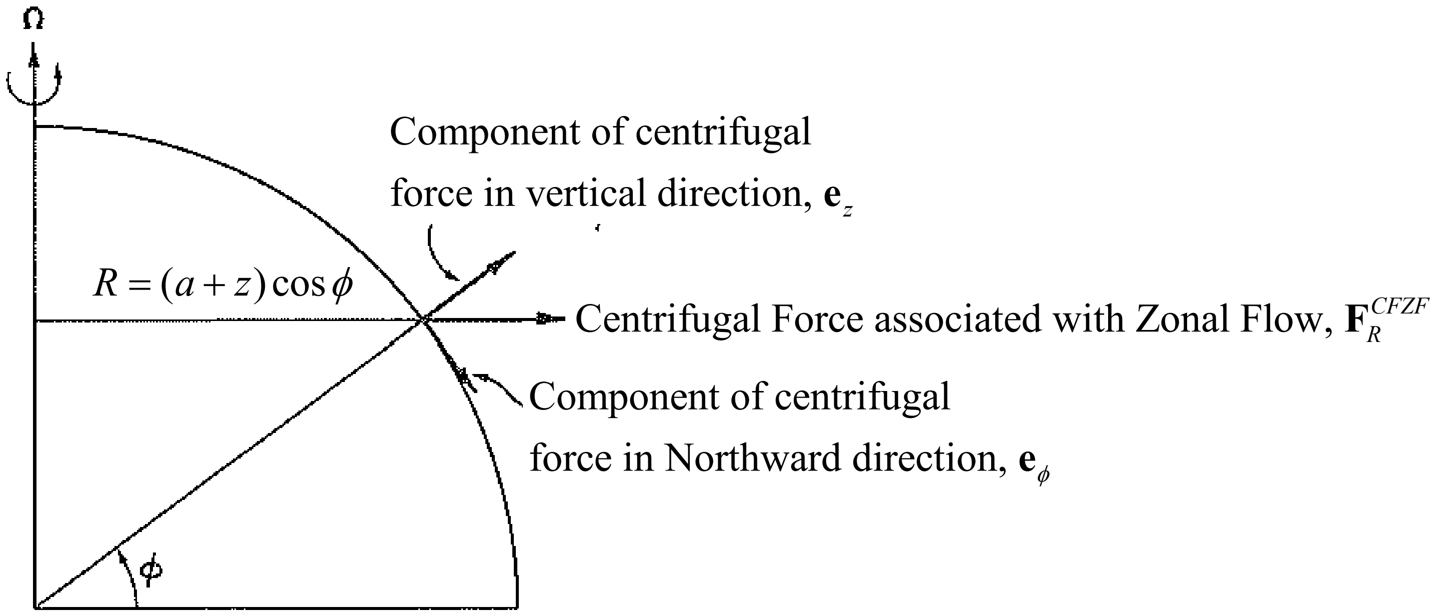
Similarly, since $\left(\mathbf{e}_R.\mathbf{e}_r\right) = -\sin\phi\left(\mathbf{e}_\phi.\mathbf{e}_r\right) + \cos\phi\left(\mathbf{e}_r.\mathbf{e}_r\right) = \cos\phi;$

the component of this force in the $\mathbf{e}_r$ direction is given by

$$\mathbf{F}_r^{CFZF} = \left(\mathbf{F}_R^{CFZF}.\mathbf{e}_r\right)\mathbf{e}_r \qquad (9)$$

$$\boxed{=> \mathbf{F}_r^{CFZF} = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\right]u\left(\mathbf{e}_R.\mathbf{e}_r\right)\mathbf{e}_r = \left[\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)\right]u\cos\phi\mathbf{e}_r} \qquad (15.7)$$

(Apparent) Centrifugal Force associated with zonal motion:Summary



$$-g_{eff}\mathbf{e}_r=\left\{-g+\Omega^2\left[\left(a+z\right)\cos\phi\right]\right\}\mathbf{e}_r \text{ == effective gravitational force}$$

$$\mathbf{F}_\phi^{CFZF}=-\left\{\left[\left(2\Omega+\frac{u}{(a+z)\cos\phi}\right)\right]u\sin\phi\right\}\mathbf{e}_\phi$$

$$\mathbf{F}_r^{CFZF}=\left[\left(2\Omega+\frac{u}{(a+z)\cos\phi}\right)u\right](\mathbf{e}_R.\mathbf{e}_r)\mathbf{e}_r=\left\{\left[\left(2\Omega+\frac{u}{(a+z)\cos\phi}\right)\right]u\cos\phi\right\}\mathbf{e}_r$$

Hence, as anticipated, the centrifugal force associated with rotation distorts the shape of the Earth and creates u associated apparent accelerations in both the Northward direction $\mathbf{e}_\phi$ and the upward direction $\mathbf{e}_r$.

By way of introduction, this is the form of the apparent forces associated with the rotation of the Earth

Proved!

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \boxed{+\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi)} \qquad (5.1)$$

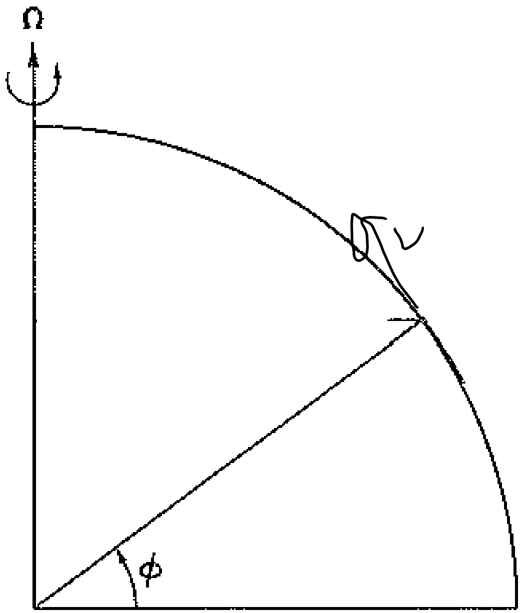
$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \boxed{-\left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\sin\phi} - \frac{vw}{(a+z)} \qquad (5.2)$$

Proved!

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \boxed{-g_{eff} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\cos\phi} + \frac{v^2}{(a+z)} \qquad (5.3)$$

Proved!

Apparent Force Associated with Conservation of Angular Momentum associated with *v*



$$\frac{D\left[v\left(a+z\right)\right]}{Dt}=0=\left(a+z\right)\frac{Dv}{Dt}+v\frac{D\left[\left(a+z\right)\right]}{Dt}=\left(a+z\right)\frac{Dv}{Dt}+vw$$

$$=>\left(a+z\right)\frac{Dv}{Dt}=-vw$$

$$=>\mathbf{F}_{\phi}^{NAM}=\frac{Dv}{Dt}\mathbf{e}_{\phi}=-\frac{vw}{\left(a+z\right)}\mathbf{e}_{\phi}\qquad(11)$$

The centrifugal Force associated with Northward Motion

$$\mathbf{F}_r^{CFNF} = \frac{Dw}{Dt}\mathbf{e}_r = \frac{v^2}{a+z}\mathbf{e}_r$$

We have now derived all the apparent forces from the Principles of

(i) Conservation of angular momentum, and
(ii) The centrifugal force creates apparent accelerations from the perspective of a rotating observer.

By way of introduction, this is the form of the apparent forces associated with the rotation of the Earth

Proved!

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi) \qquad (5.1)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\sin\phi - \frac{vw}{(a+z)} \qquad (5.2)$$

Proved!

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_{eff} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\cos\phi + \frac{v^2}{(a+z)} \qquad (5.3)$$

Proved!

All proved!

Scale analysis of equations

- To gain insight into physical systems, it is often helpful to identify the largest forces/accelerations associated with the system and then simplify the equations by neglecting the less important forces. The procedure for doing this is known as “Scale Analysis”.
- We will see that many of the components of the apparent forces seen in reference frames attached to the surface of rotating planets can be neglected for the spatio-temporal scales associated of “weather-defining” features of the atmospheric or oceanic state.

The full equations: In Holton's book, and in the answers to the questions from the second tutorial we derived the equations governing the three components of the velocity to be

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)(v\sin\phi - w\cos\phi) \quad \text{(Ia)}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - f^* w + \frac{uv\tan\phi}{(a+z)} - \frac{uw}{(a+z)}, \text{ where } f = 2\Omega\sin\phi,\ f^* = 2\Omega\cos\phi$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\sin\phi - \frac{vw}{(a+z)} \quad \text{(Ib)}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu - \frac{u^2\tan\phi}{(a+z)} - \frac{vw}{(a+z)}$$

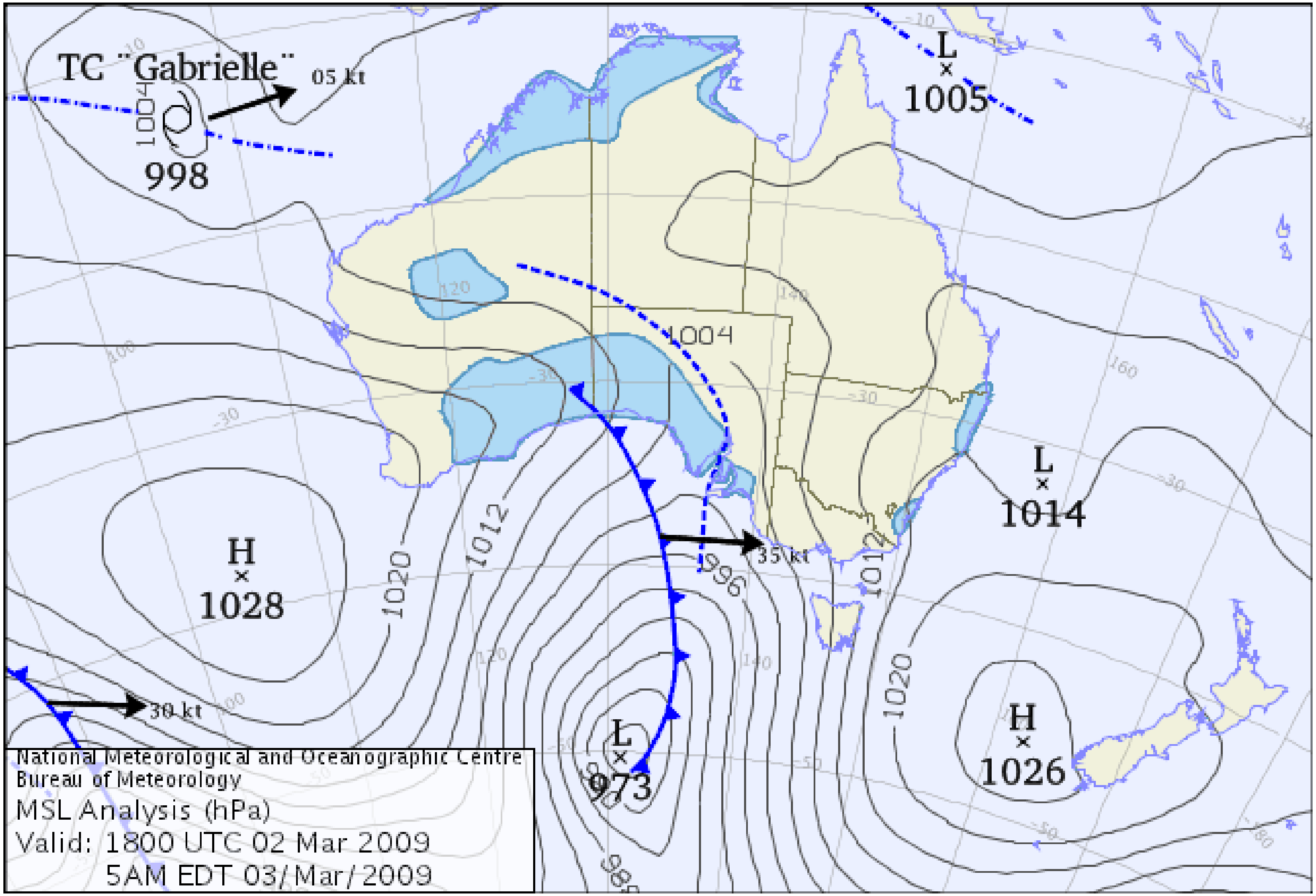
$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_{eff} + \left(2\Omega + \frac{u}{(a+z)\cos\phi}\right)u\cos\phi + \frac{v^2}{(a+z)} \quad \text{(Ic)}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g_{eff} + f^* u + \frac{(u^2+v^2)}{(a+z)}$$

f is called the Coriolis parameter

Where x, y *and* z indicate distance along zonal, meridional and vertical lines. ϕ is latitude, a is the radius of the Earth.

What parts of equations of motion are necessary / dominant / negligible?



Consider synoptic scale flow

Scale analysis of the equations of motion: used to determine which terms / processes are dominant.

Assign typical scales to variables *(use synoptic flow as an example).*

Variable.	Scale.	Value for synoptic flow.
x,y: Horizontal Length	L	10^6 m (1000 km)
z: Depth scale	H	10^4 m (10 km)
u,v: Horizontal Velocity	U	10 m/s
w: Vertical Velocity	W	0.01 m/s
t: Advective Time	T = L / U	10^5 s (~1 Day)
p: Pressure change	δP	1000 Pa (10 hPa)
ρ: Density	R	1 kg/m^3
f,f^*: Coriolis terms	f,f^*	10^{-4} s^{-1}

δP is the horizontal change in pressure over L, not total pressure

Scale analysis of horizontal components of eqns of motion.

Table 2.1 *Scale Analysis of the Horizontal Momentum Equations*

	A	B	C	D	E	F	G
x-Eq.	$\frac{Du}{Dt}$	$-2\Omega v \sin\phi$	$+2\Omega w \cos\phi$	$+\frac{uw}{a}$	$-\frac{uv \tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
y-Eq.	$\frac{Dv}{Dt}$	$+2\Omega u \sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2 \tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
($m\,s^{-2}$)	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

For horizontal equations of motion:

- It can easily be shown that all terms in Du/Dt, Dv/Dt ~ U^2/L

-Leading order terms are coriolis (*f*) and pressure gradient - *these forces approximately balance.*

Keeping terms with magnitude greater than or equal to order 10^{-4} reduces equations to

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Can be written as:

$$\frac{D\underline{u}_H}{Dt} + f\,\underline{k} \times \underline{u}_H = -\frac{1}{\rho}\nabla_H p$$

Scale analysis suggests that $D\underline{u}_H/Dt$ terms are small for synoptic scale flow, i.e.,

$$\frac{D\underline{u}_H}{Dt} << f\,\underline{k} \times \underline{u}_H \quad \Rightarrow \quad \frac{U^2}{L} << fU$$

$$\Rightarrow \frac{U}{fL} << 1$$

U / fL **is called the Rossby number, R_o. It represents the ratio of the acceleration to the coriolis acceleration and provides a measure of the importance of rotation in the flow.**

Scale analysis of vertical component of eqns of motion.

Table 2.2 *Scale Analysis of the Vertical Momentum Equation*

z-Eq.	Dw/Dt	$-2\Omega u \cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\,\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	$f_0 U$	U^2/a	$P_0/(\rho H)$	g	$\nu W H^{-2}$
$\mathrm{m\,s^{-2}}$	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

For vertical equation of motion:

- P_0 is typical change in pressure over depth H ~ 10^5 Pa (1000 hPa)
- It can easily be shown that all terms in Dw/Dt ~ UW/L
- Leading order terms are vertical pressure gradient and gravity - *these forces approximately balance.*

Neglecting small terms:

$$0 \approx -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$=> \qquad \frac{\partial p}{\partial z} = -\rho g$$

This is hydrostatic balance: a balance between the upward pressure gradient force and the downward gravitational force.

But perfect gas law gives $p = \rho RT$

$$=> \qquad \frac{\partial p}{\partial z} = -\frac{p}{RT} g$$

$$=> \qquad RT\frac{\delta p}{p} = RT\frac{d(\ln p)}{dp}\delta p = -g\delta z$$

$$=> \qquad RT\delta(\ln p) = -g\delta z$$

$$=> \qquad RT\left[\ln(p+\delta p) - \ln(p)\right] = -g\left[z(p+\delta p) - z(p)\right] = g\left[z(p) - z(p+\delta p)\right]$$

Hence, the distance between pressure surfaces increases with the temperature. This means, *inter alia*, that the same two pressure Surfaces have a much larger vertical separation distance in the tropics than at the poles.

In summary, approximate equations governing extra-polar flow are

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

(Written in height coordinates.)

Rossby Number $R_o = U / fL$

Carl-Gustaf Rossby

$R_o \ll 1$: Rotation dominant

$R_o \gg 1$: Rotation negligible

Flow system	L	U (m/s)	R_o
Midlatitude cyclone	1000 km	1-10	0.01-0.1
Tropical cyclone*	500 km	50	1
Large convective complex	50 km	10	2
Cumulus cloud	1 km	10	100
Tornado	100 m	100	10000
Bath tub vortex	10 cm	0.1	10000

* Note: $f=10^{-4}\ s^{-1}$ is used here, not strictly appropriate for tropics.

Recall: **Rossby number R_o represents the ratio of the acceleration to the Coriolis acceleration and provides a measure of the importance of rotation in the flow.**

Rossby Number $R_o = U / fL$

Because $U \sim L / T$ the Rossby number can be re-written as:

$$R_o = (1 / f) / T$$

$$= \frac{\text{Time scale of rotation}}{\text{Time scale of motion}}$$

As implied by scale analysis, for R_o <<1 the coriolis acceleration is dominant. Therefore:

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x} \equiv -fv_g$$

$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} \equiv fu_g$$

Can be written as:

$$f\underline{k}\times\underline{u}_g = -\frac{1}{\rho}\nabla_H p$$

$$\underline{u}_g = (u_g, v_g) = \left(-\frac{1}{f\rho}\frac{\partial p}{\partial y}, \frac{1}{f\rho}\frac{\partial p}{\partial x}\right)$$

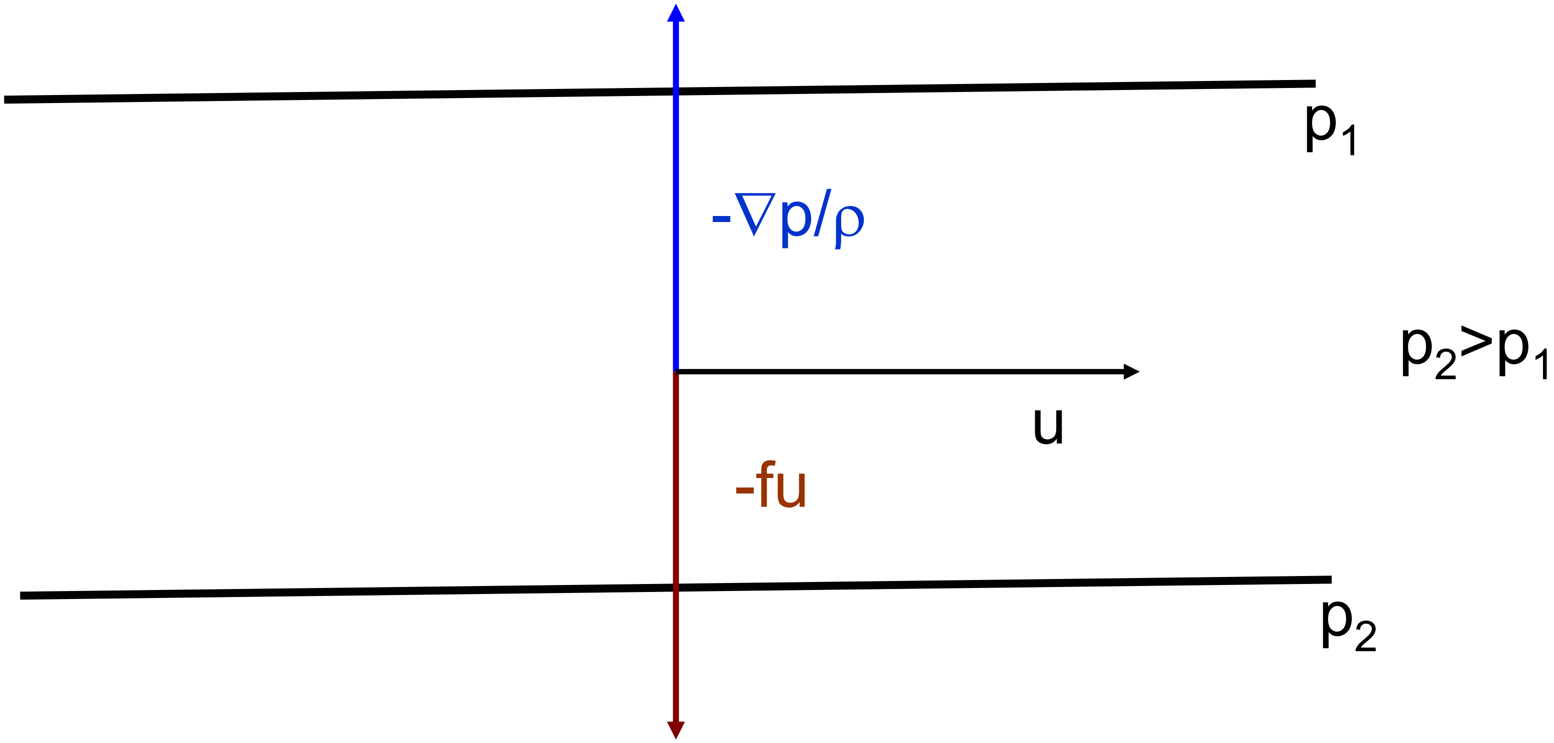
*is the **geostrophic wind***

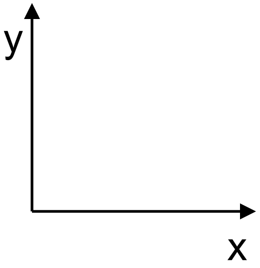
The geostrophic wind represents *a balance between the pressure gradient force and the coriolis force. Alternatively it can be viewed that the pressure gradient forces the coriolis acceleration.*

Geostrophic flow.

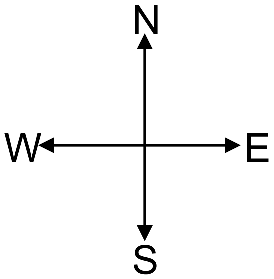
(Northern hemisphere f>0)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f u_g$$





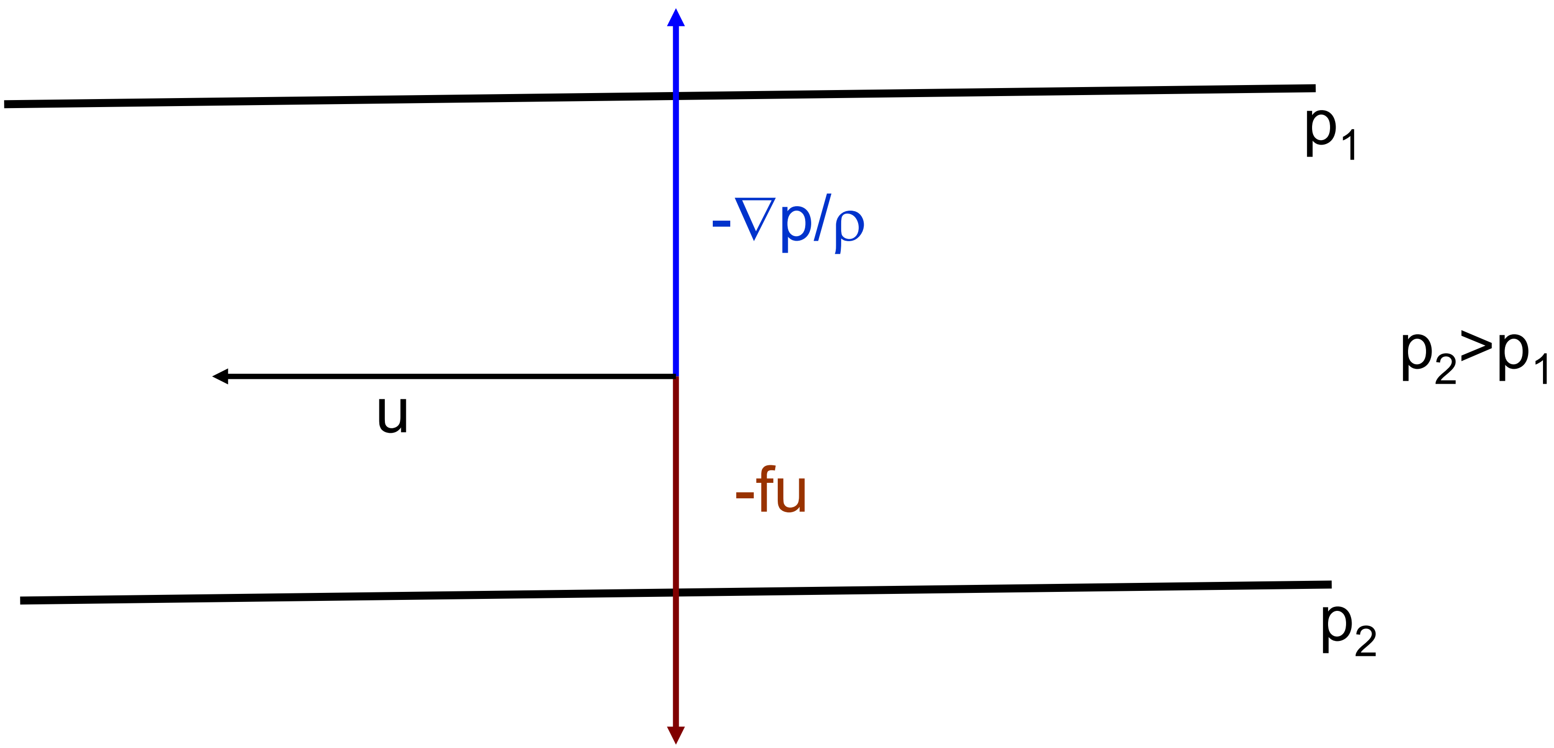
$$-\frac{\partial p}{\partial y} > 0 \Rightarrow -fu < 0 \Rightarrow u > 0$$

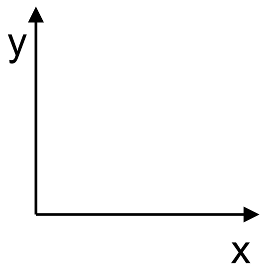


Geostrophic flow.

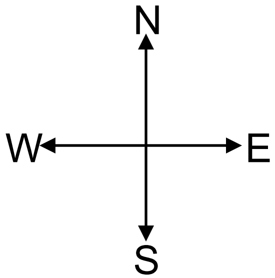
(Southern hemisphere f<0)

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - f u_g$$



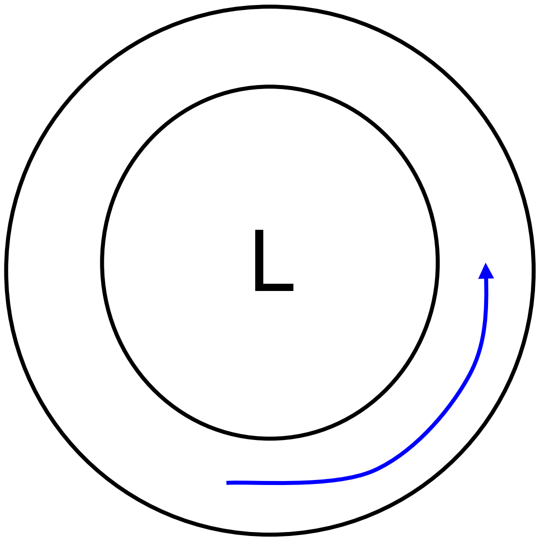


$$-\frac{\partial p}{\partial y} > 0 \Rightarrow -fu < 0 \Rightarrow u < 0$$ *(as f<0)*



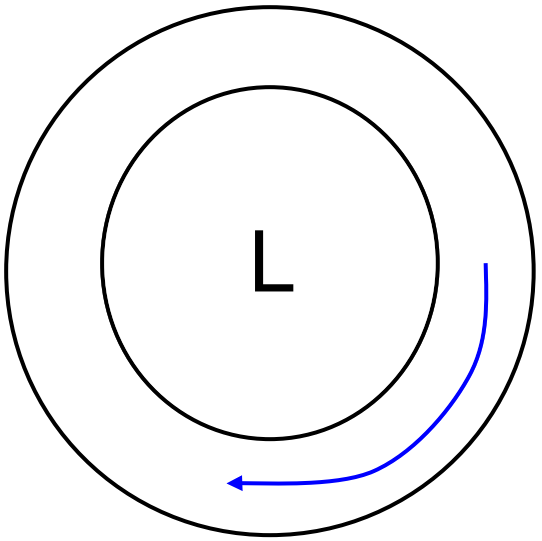
In northern hemisphere:

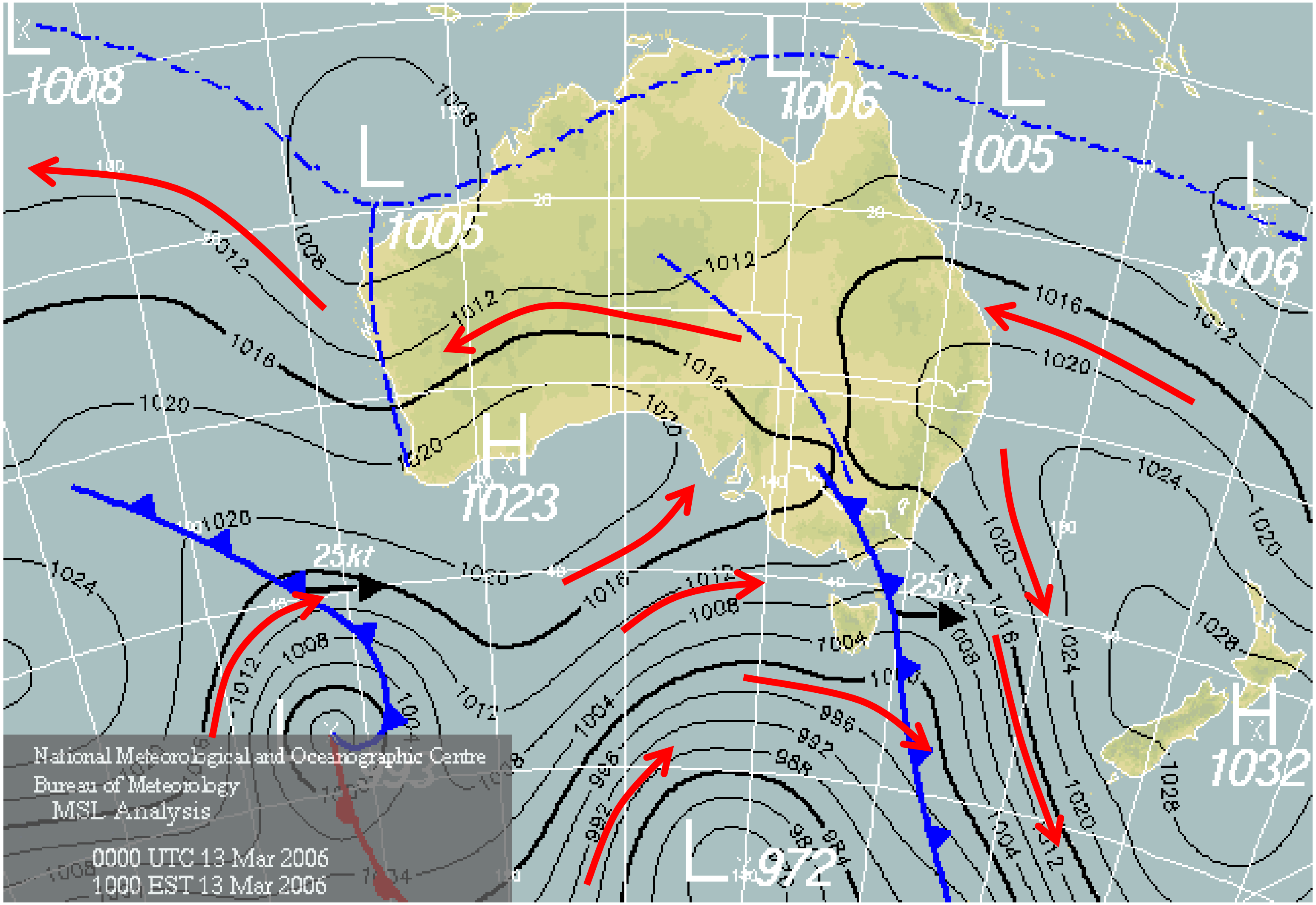
- Coriolis force is directed to right of winds
- Low pressure is always to left of winds
- Anticlockwise flow is called “cyclonic”
- Clockwise flow is called “anticyclonic”



In southern hemisphere:

- Coriolis force is directed to left of winds
- Low pressure is always to right of winds
- Clockwise flow is called “cyclonic”
- Anticlockwise flow is called “anticyclonic”





The speed of the wind is proportional to the strength of the pressure gradient - closer isobars, faster winds. But there is also a latitudinal dependence. What is it?

Properties of the geostrophic wind.

- There is no net force in the direction of the wind - i.e., no net acceleration of the flow (in rotating reference frame).

- The geostrophic wind equation is a “diagnostic” relation, it tells us nothing about the evolution of the flow. (The flow evolves due to *ageostrophic flow* [on a flat surface])

- Is only strictly valid for straight isobars (no flow curvature).

- The geostrophic wind relation breaks down in the tropics (small / zero coriolis). In that case, the acceleration terms are important.

Properties of the geostrophic wind.

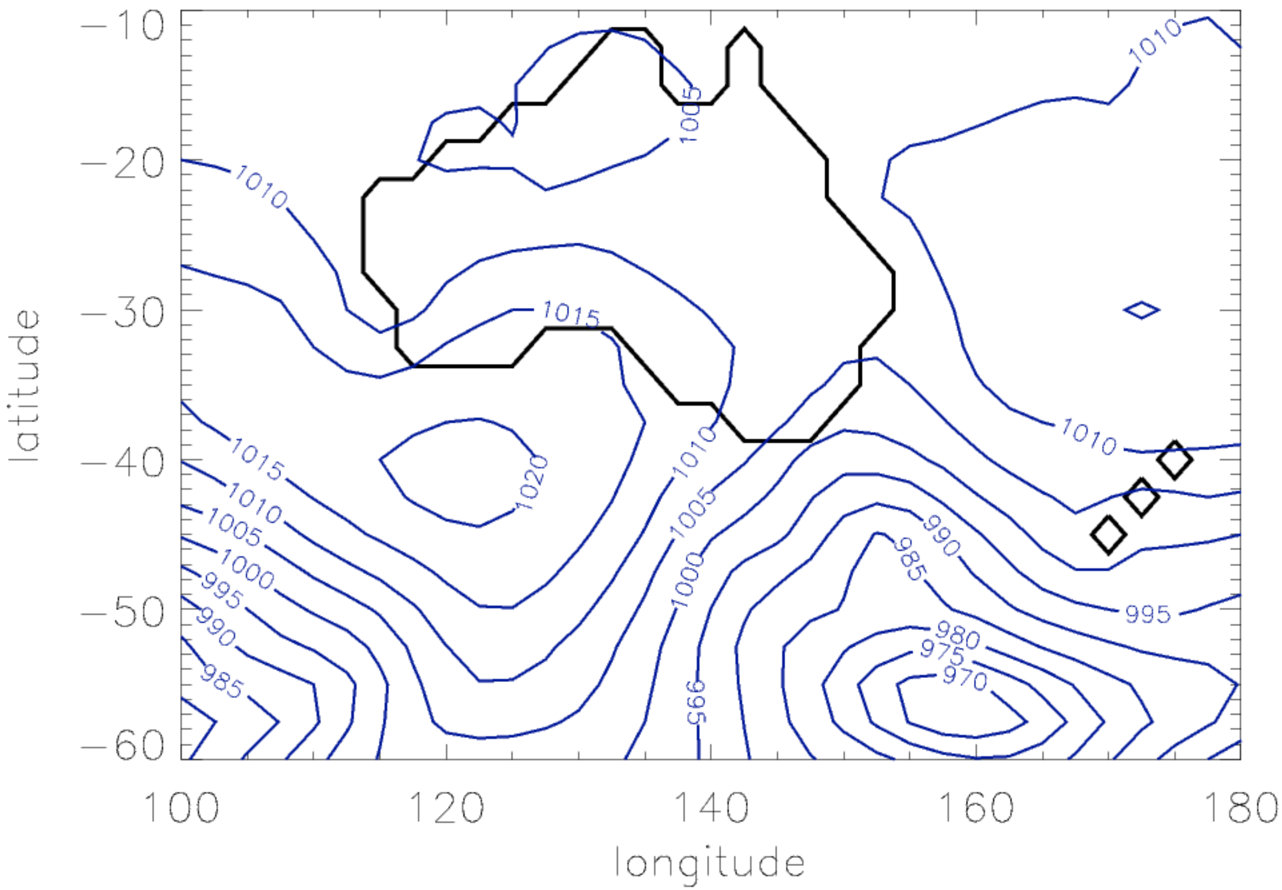
- The geostrophic flow is non-divergent (if the density is horizontally homogeneous and the flow is on an *f*-plane (an f-plane is a plane where it is assumed that the north-south variations in f are negligible):

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{1}{f\rho}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f\rho}\frac{\partial p}{\partial x}\right) = 0$$

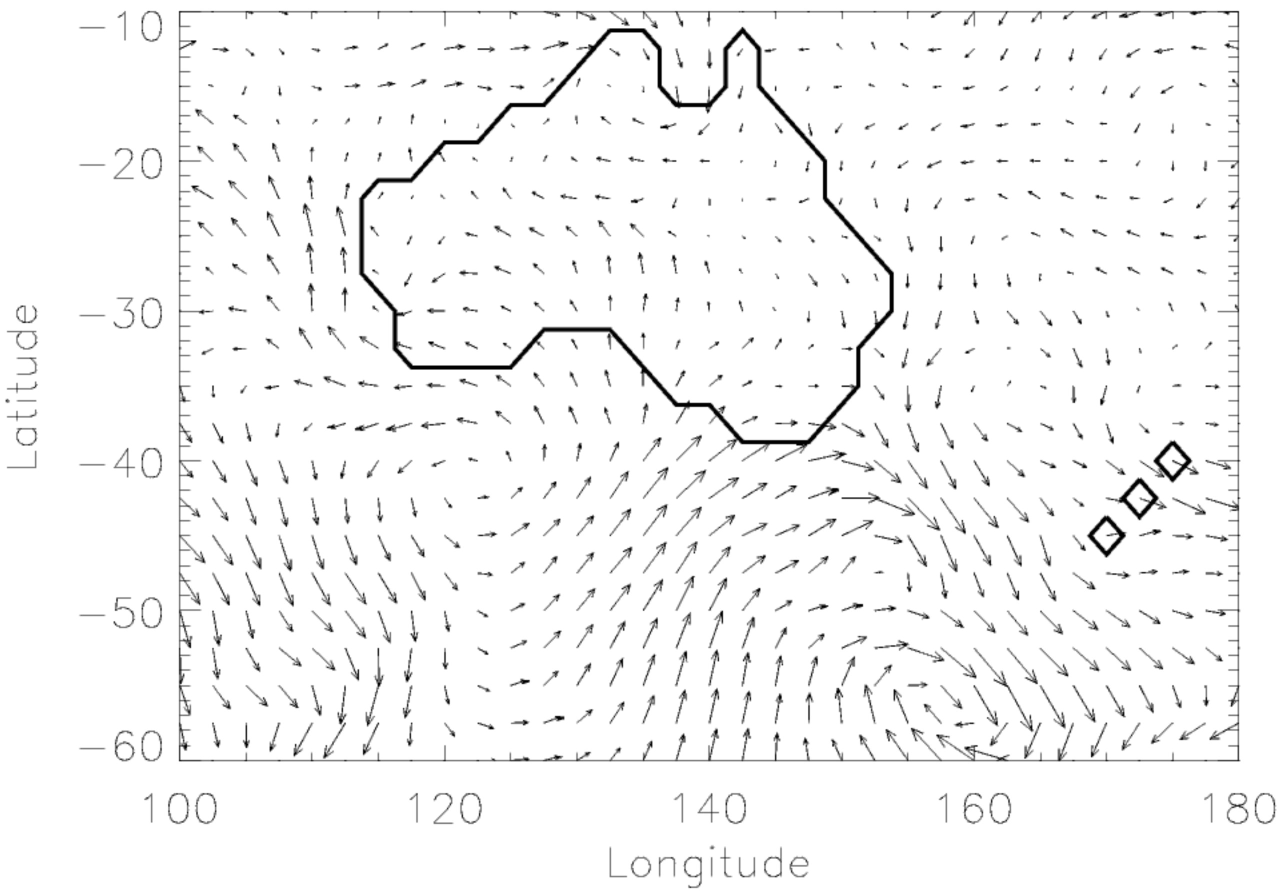
In such flows:

$\frac{\partial w}{\partial z} = 0$ and because *w(0)=0* then *w(z)=0* for all z.

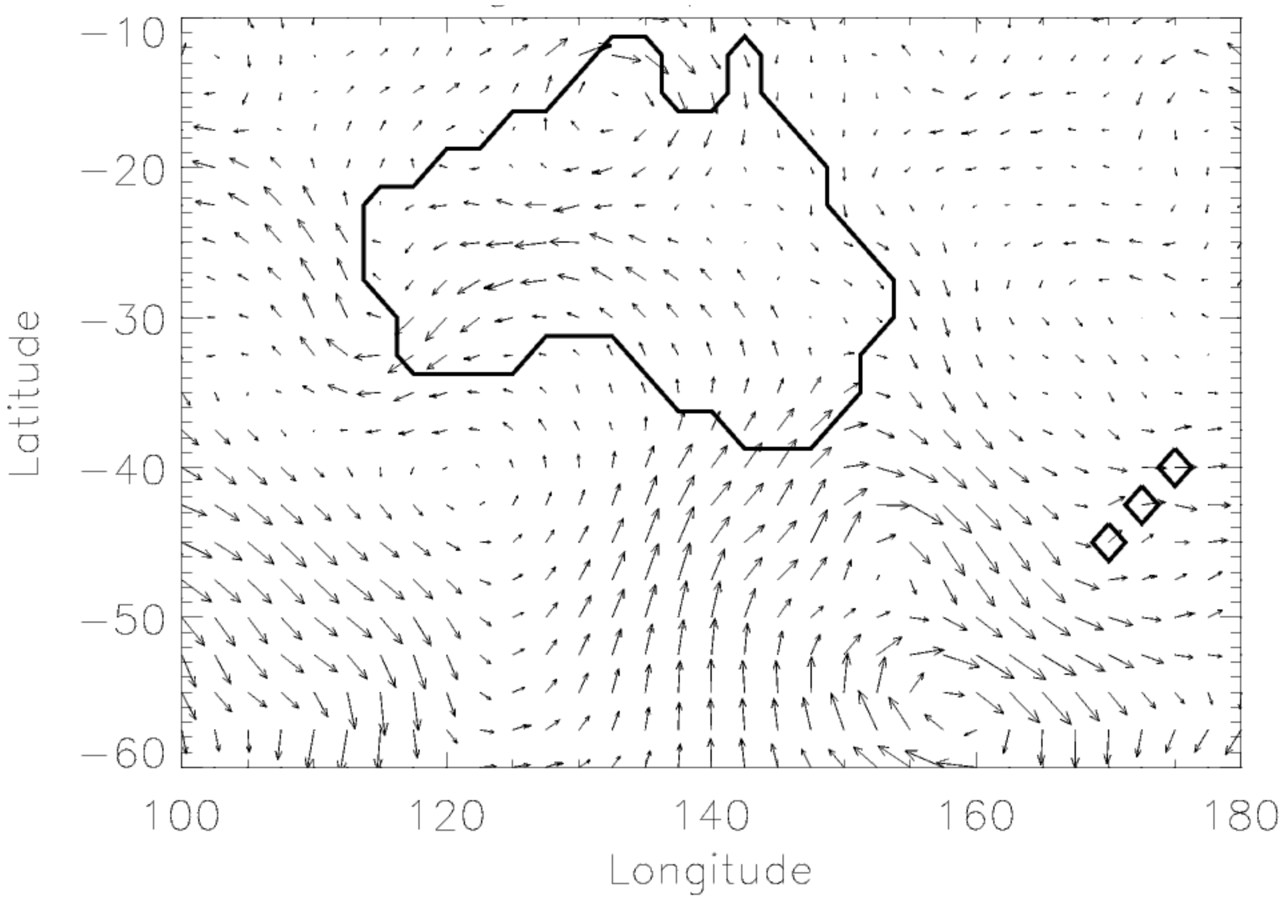
Mean sea-level pressure



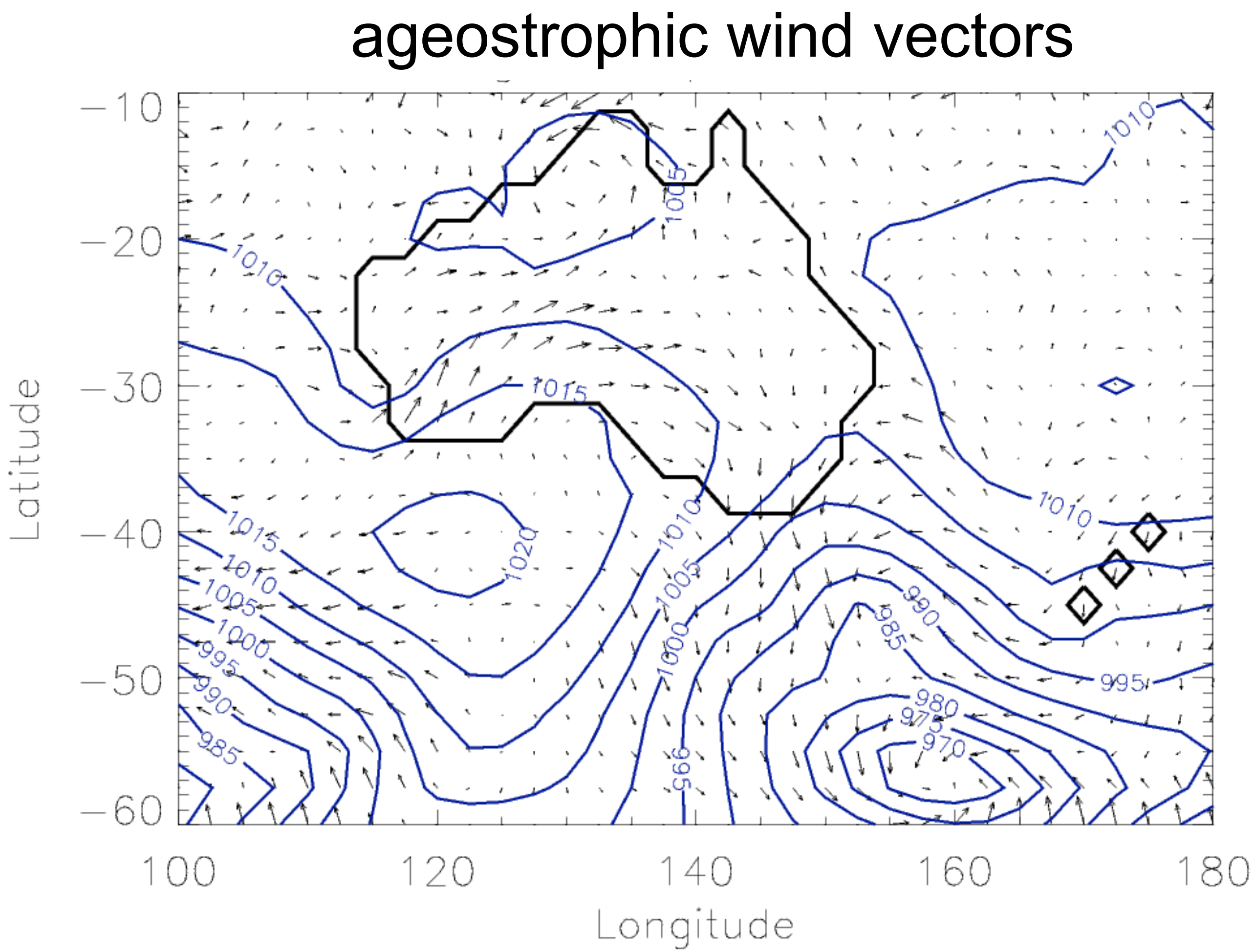
Actual wind vectors



Geostrophic wind vectors



$u_a = u - u_g$ $v_a = v - v_g$

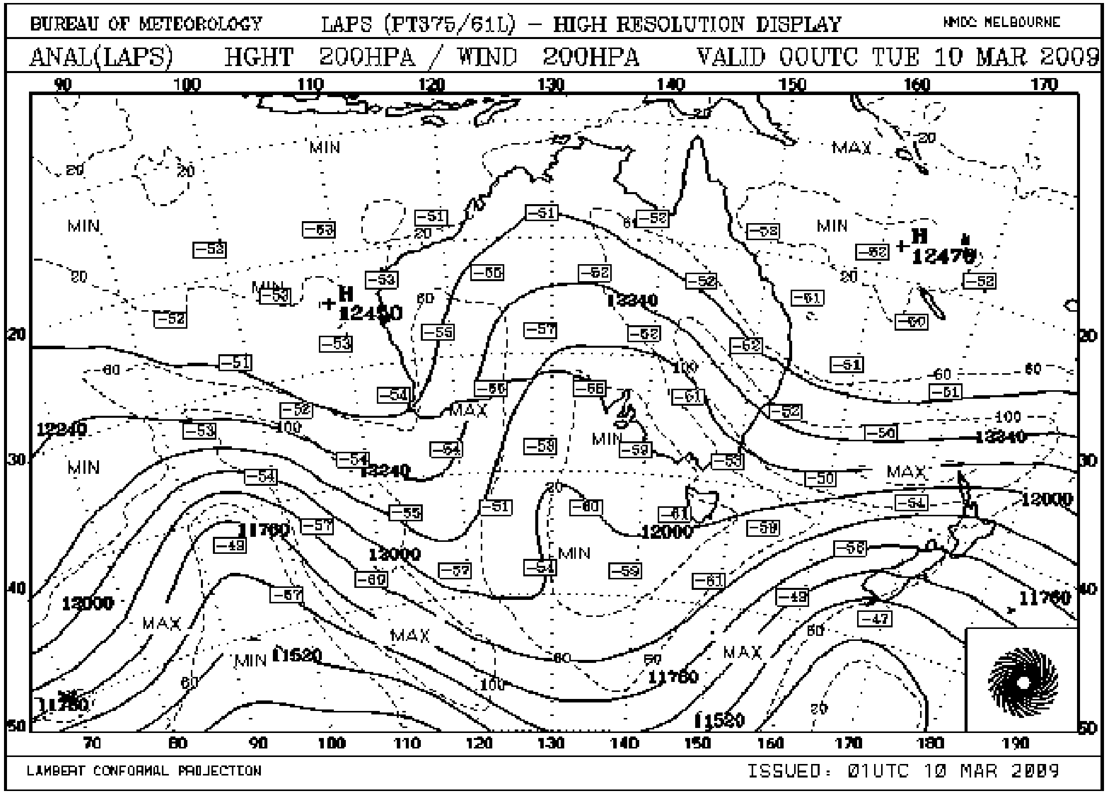


Note:

- ageostrophic wind is strong in regions of curvature
- around cyclones ageostrophic flow opposes cyclonic circulation
- ageostrophic flow is divergent (anticyclones) and convergent (cyclones)

The formulation in height coordinates is convenient and intuitive for most.

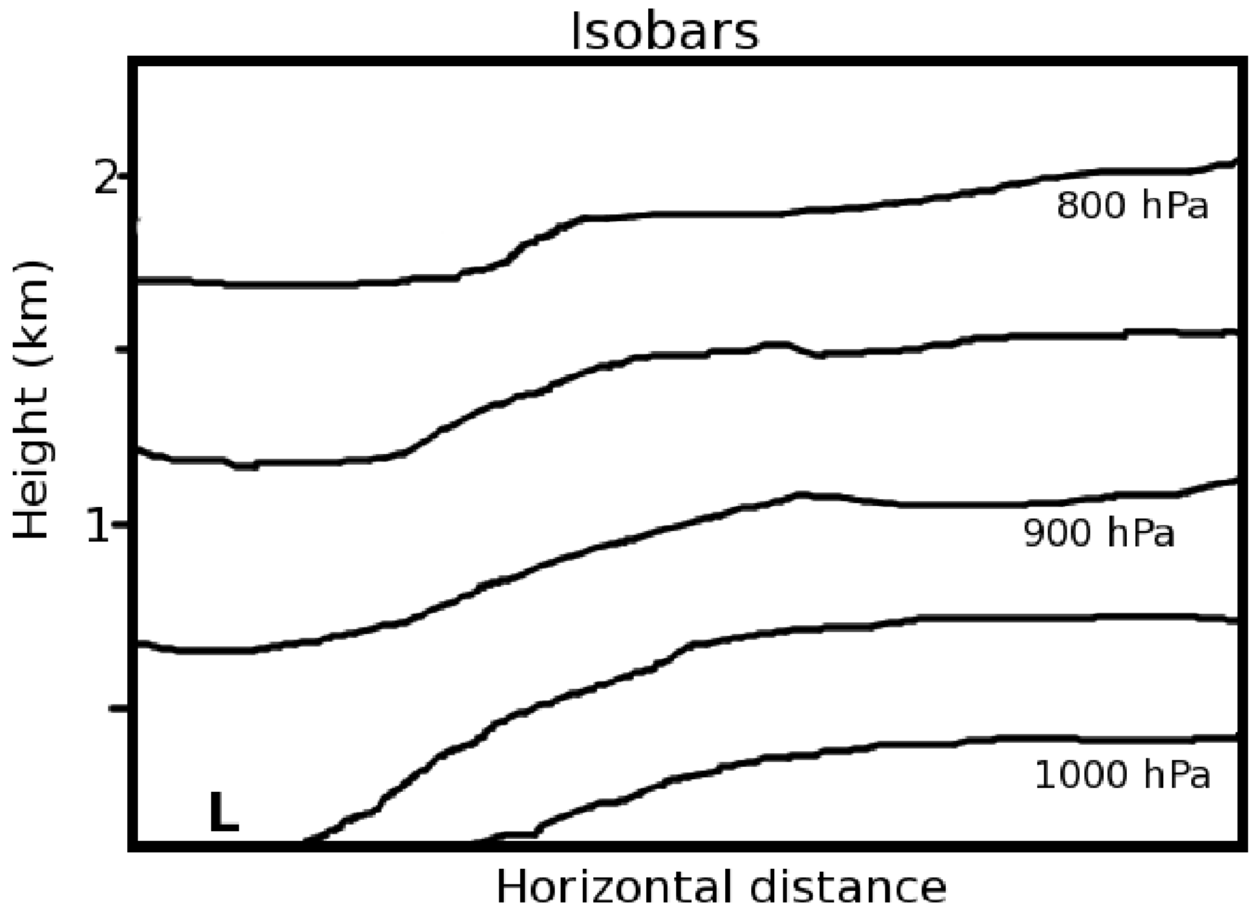
- Easy to apply to surface pressure field.
- Traditionally, upper level meteorological charts are presented on pressure levels (i.e., surfaces of constant pressure). Height coordinates are less useful here.
- Application to the ocean is also difficult.



Solid lines: height of 200 hPa surface.

Cannot determine a horizontal pressure gradient directly because pressure is constant.

Clearly, the height field and pressure field are related as are their gradients.



Low pressure at 1 km - low height of 900 hPa surface. Rightward gradient in pressure - rightward gradient in height. (Rightward==right of page)

In a hydrostatic system, pressure is a monotonic function of height and can be considered a vertical coordinate.

Our x,y,z coordinate system is somewhat arbitrary - its choice is not motivated by any property of the flow. The only constraint is that x,y,z are all orthogonal.

Can define a new coordinate system (X,Y,ζ) where $\zeta = \zeta(x,y,z,t)$ and X,Y,ζ are all orthogonal.

The value of the material derivative D/Dt is the same in each coordinate system (it is a property of the flow, not of the coordinate), but the terms that make up that derivative are different.

Recall:

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x_z} + v\frac{\partial\phi}{\partial y_z} + w\frac{\partial\phi}{\partial z}$$

Where:

$$w = \frac{Dz}{Dt}$$

Then it follows that in new coordinate (x,y,ζ):

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x_\zeta} + v\frac{\partial\phi}{\partial y_\zeta} + \frac{D\zeta}{Dt}\frac{\partial\phi}{\partial\zeta}$$

where 'horizontal' derivatives and velocities are along surfaces of constant ζ

With these properties in mind - let's define a new coordinate system with pressure as the vertical coordinate. *(x,y,p)*

In this system we define: $$\frac{Dp}{Dt} = \omega$$

(in physical space regions of positive ω are regions of negative w).

And therefore:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x_p} + v\frac{\partial}{\partial y_p} + \omega\frac{\partial}{\partial p}$$

So:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_p} + v\frac{\partial u}{\partial y_p} + \omega\frac{\partial u}{\partial p} = \sum FX_P$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_p} + v\frac{\partial v}{\partial y_p} + \omega\frac{\partial v}{\partial p} = \sum FY_P$$

These 'horizontal' velocities are actually along surfaces of constant pressure.

To determine the pressure gradient force…

Since:

$$\delta\phi = \left(\frac{\partial\phi}{\partial x}\right)_z \delta x + \left(\frac{\partial\phi}{\partial z}\right)_x \delta z; \quad \lim \delta x -> 0 \left(\frac{\delta\phi}{\delta x}\right)_p = \lim \delta x -> 0 \left(\frac{\partial\phi}{\partial x}\right)_z \left(\frac{\delta x}{\delta x}\right)_p + \left(\frac{\partial\phi}{\partial z}\right)_x \left(\frac{\delta z}{\delta x}\right)_p$$

$$=> \frac{\partial\phi}{\partial x_p} = \frac{\partial\phi}{\partial x_z} + \frac{\partial z}{\partial x_p}\frac{\partial\phi}{\partial z}, \quad \text{where } \frac{\partial z}{\partial x_p} = \left(\frac{\partial z}{\partial x}\right)_p \quad \text{is the slope of the pressure surface in the x-direction}$$

Substitute $p = \phi$ then

$$\frac{\partial p}{\partial x_p} = \frac{\partial p}{\partial x_z} + \frac{\partial z}{\partial x_p}\frac{\partial p}{\partial z}$$

But $\quad \frac{\partial p}{\partial x_p} = 0 \quad$ and the hydrostatic relation is $\quad \frac{\partial p}{\partial z} = -\rho g$

So

$$0 = \frac{\partial p}{\partial x_z} - \rho g \frac{\partial z}{\partial x_p}$$

$$\Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x_z} = -g\frac{\partial z}{\partial x_p} = -\frac{\partial \Phi}{\partial x_p}$$

where $\Phi = gz$ is the geopotential.

Therefore in pressure coordinates on an f-plane, the equations governing *u* and *v* are:

$$\frac{Du}{Dt} - fv = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_p} + v\frac{\partial u}{\partial y_p} + \omega\frac{\partial u}{\partial p} - fv = -\frac{\partial \Phi}{\partial x_p}$$

$$\frac{Dv}{Dt} + fu = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_p} + v\frac{\partial v}{\partial y_p} + \omega\frac{\partial v}{\partial p} + fu = -\frac{\partial \Phi}{\partial y_p}$$

The hydrostatic equation:

$$\frac{\partial p}{\partial z} = -\rho g, \lim \delta z -> 0 \quad \frac{\delta p}{\delta z} = -\rho g => \lim \delta z -> 0 \quad \frac{1}{\rho} = -\frac{g\delta z}{\delta p} = -\frac{\delta \Phi}{\delta p} = -\frac{\partial \Phi}{\partial p}$$

Hence,

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}$$

Assuming the same scaling used earlier, the geostrophic wind is:

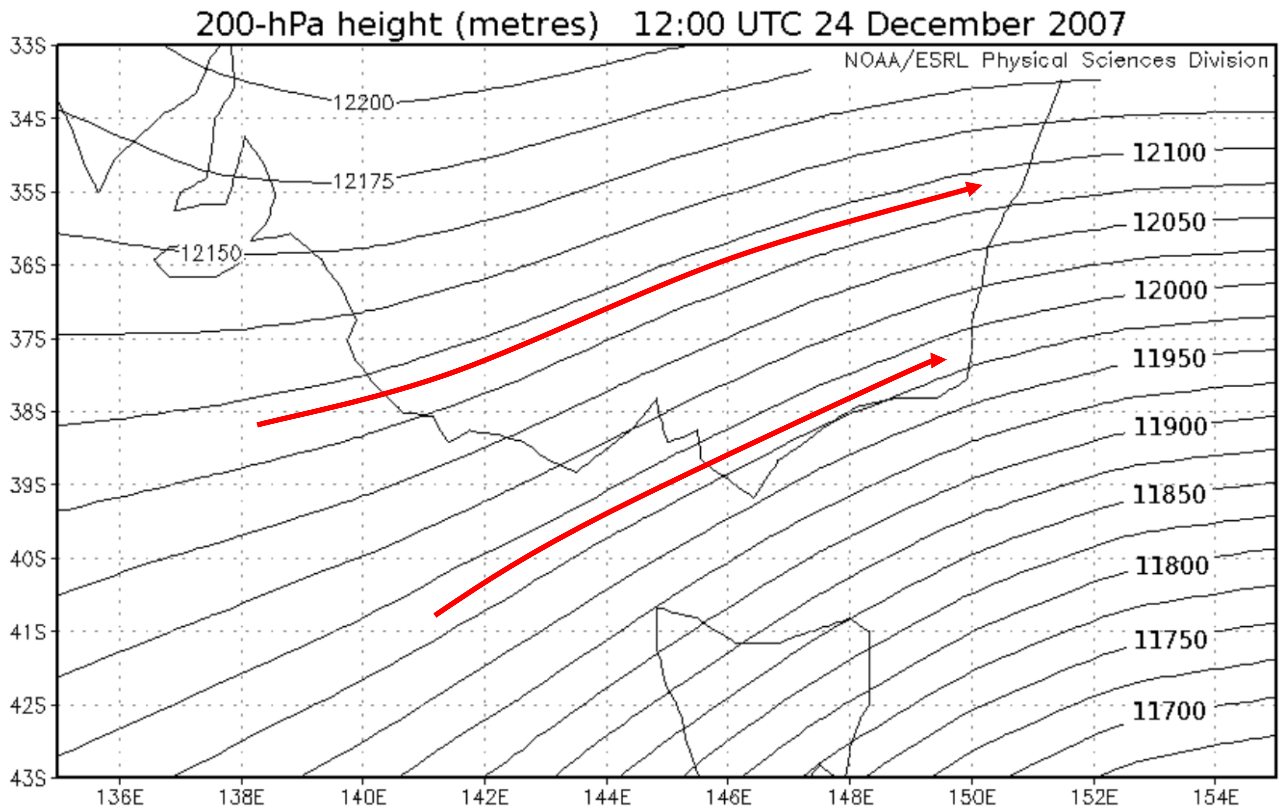
$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y_p}$$

$$v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x_p}$$

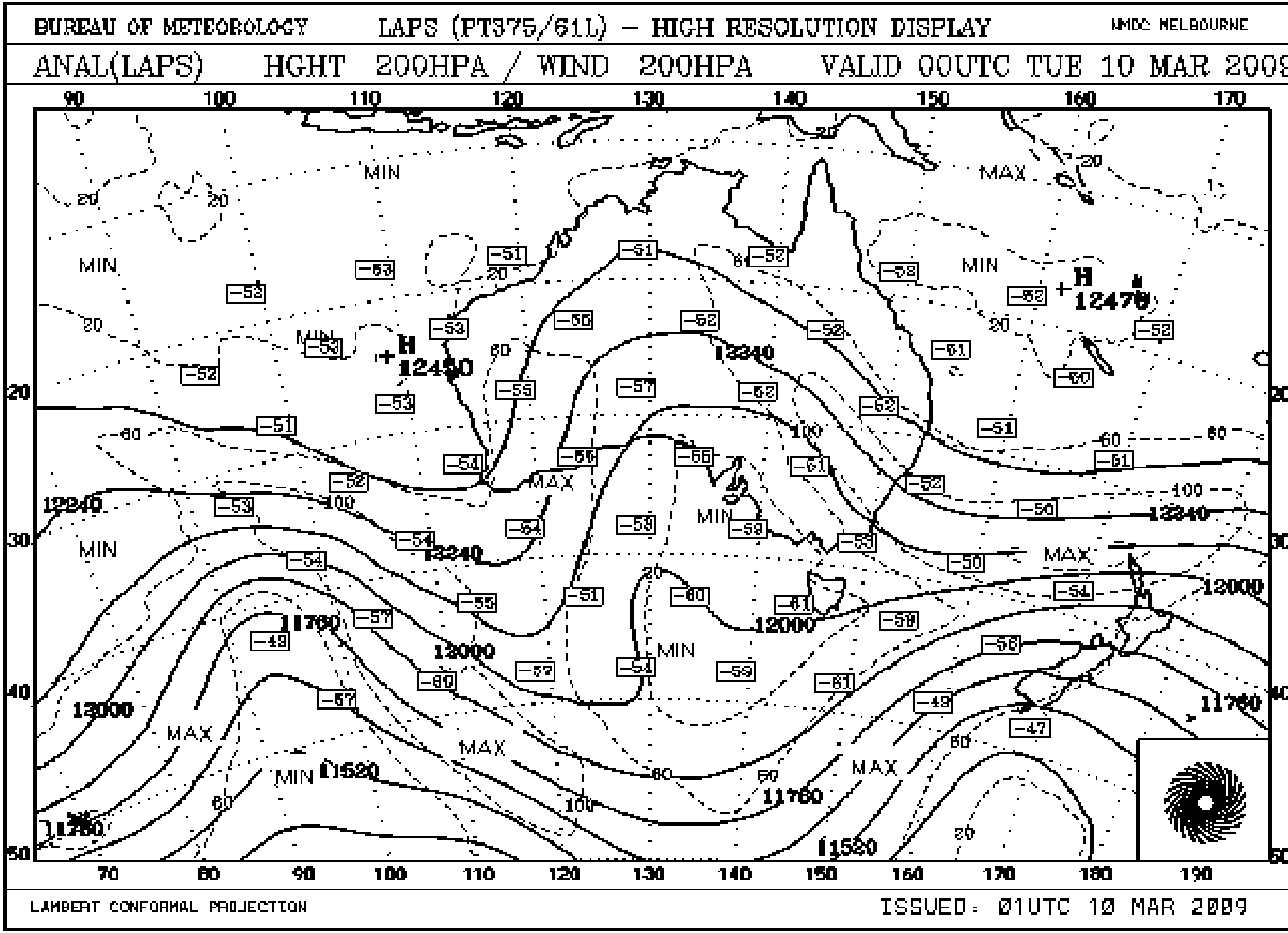
Thus, the horizontal divergence of the geostrophic wind in pressure coordinates is exactly zero on an f-plane (no assumptions about density needed).

This relation is directly analogous to that derived on height coordinates. In the northern hemisphere negative height anomalies will be to the left of the geostrophic wind (and to the right in the southern hemisphere).

Strictly speaking these winds are not 'horizontal' but along pressure surfaces (as are derivatives). However, as shown by scale analysis the angle between pressure surfaces and the ground is very small and can be neglected / ignored in most cases.



200-hPa height (metres) 12:00 UTC 24 December 2007
NOAA/ESRL Physical Sciences Division
12200
12175
12150
12100
12050
12000
11950
11900
11850
11800
11750
11700
33S
34S
35S
36S
37S
38S
39S
40S
41S
42S
43S
136E
138E
140E
142E
144E
146E
148E
150E
152E
154E



At upper levels, less likely to have closed circulations

For completeness, derive mass continuity in pressure coords.

Assume a volume, $\Delta V = \delta x\, \delta y\, \delta z$

Mass = $\rho\, \Delta V$

Conservation of mass dictates that: $\frac{D(\rho \Delta V)}{Dt} = 0$

$$\frac{D(\rho \Delta V)}{Dt} = \frac{D(\rho \delta x \delta y \delta z)}{Dt} = 0$$

$$\delta p = -\rho g \delta z$$

$$\delta z = -\delta p / \rho g$$

$$\Rightarrow \frac{D(-\delta x \delta y \delta p / g)}{Dt} = 0$$

$$\Rightarrow \frac{D(\delta x \delta y \delta p)}{Dt} = 0$$

$$\frac{D(\delta x \delta y \delta p)}{Dt} = 0$$

$$\delta y \delta p \frac{D(\delta x)}{Dt} + \delta x \delta p \frac{D(\delta y)}{Dt} + \delta x \delta y \frac{D(\delta p)}{Dt} = 0$$

divide both sides by $\delta x \delta y \delta p$

$$\frac{1}{\delta x}\frac{D(\delta x)}{Dt} + \frac{1}{\delta y}\frac{D(\delta y)}{Dt} + \frac{1}{\delta p}\frac{D(\delta p)}{Dt} = 0$$

$\frac{Dx}{Dt} = u$ implies $\frac{D\delta x}{Dt} = \delta u$

$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$ implies $\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0}$

Mass continuity in pressure coordinates implies that the total divergence is exactly zero. *(Only assumption to get this is that flow is hydrostatic). This is another advantage of pressure coordinates. (It is a coordinate system based on mass!)*

Equations of motion in pressure coordinates on an f-plane.

$$\frac{Du}{Dt} - fv = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_p} + v\frac{\partial u}{\partial y_p} + \omega\frac{\partial u}{\partial p} - fv = -\frac{\partial \Phi}{\partial x_p}$$

$$\frac{Dv}{Dt} + fu = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_p} + v\frac{\partial v}{\partial y_p} + \omega\frac{\partial v}{\partial p} + fu = -\frac{\partial \Phi}{\partial y_p}$$

$$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p} \quad \text{(recalling that } p = \rho RT \Rightarrow \rho = \frac{p}{RT}\text{)}$$

$$\frac{\partial u}{\partial x_p} + \frac{\partial v}{\partial y_p} + \frac{\partial \omega}{\partial p} = 0$$

Application to the ocean.

Assume the ocean is hydrostatic.

$$\frac{\partial p}{\partial z} = -\rho g$$

and the density is constant (not a bad assumption, variation in density only about 5%).

$$p = -\bar{\rho} g z + p_s$$

Where p_s is the surface (atmospheric) pressure.

If we replace z by h, where h is the depth from the surface, then.

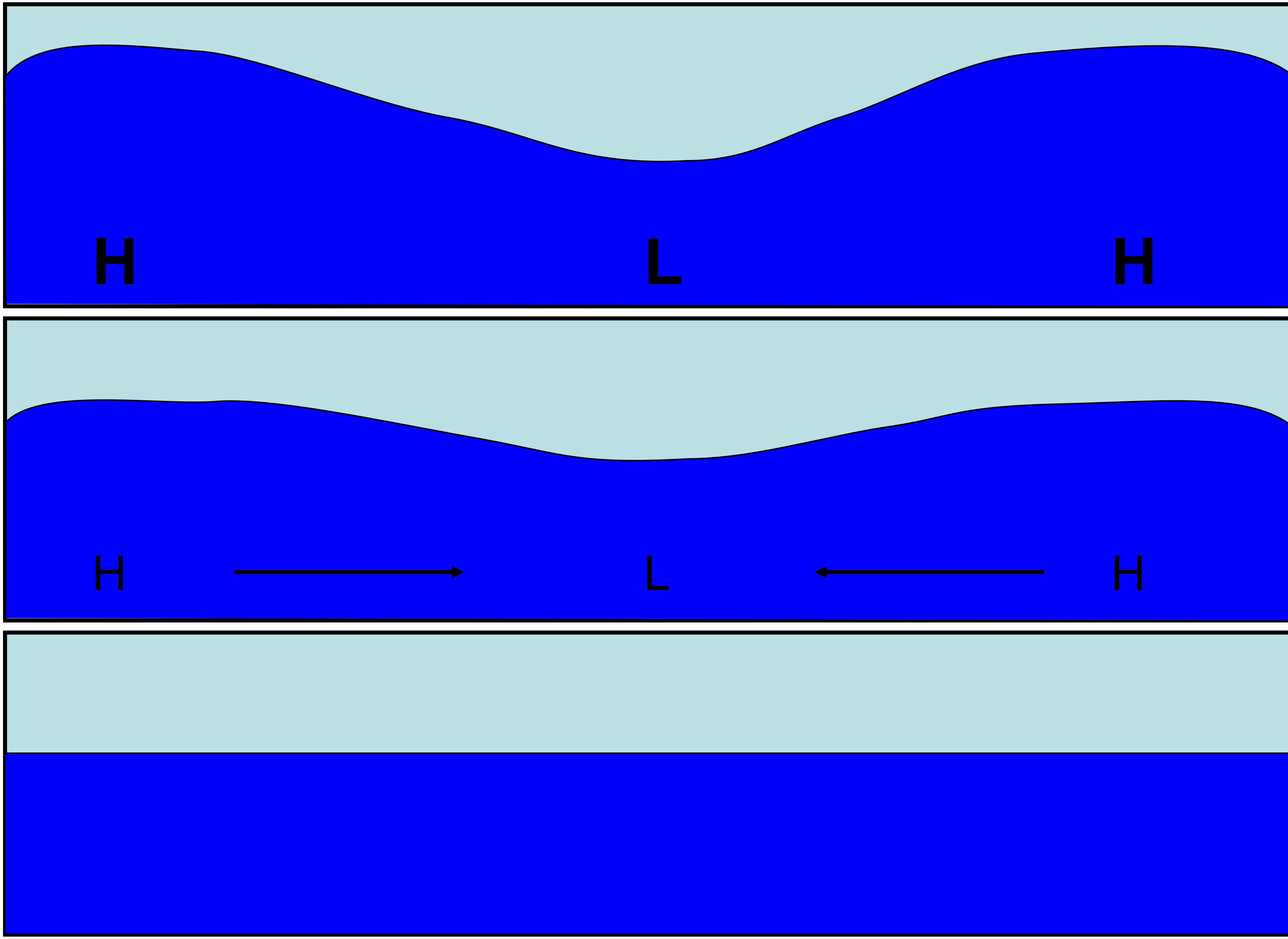
$$p = p_s + \bar{\rho} g h$$

Thus the deeper the fluid above, the higher the pressure.

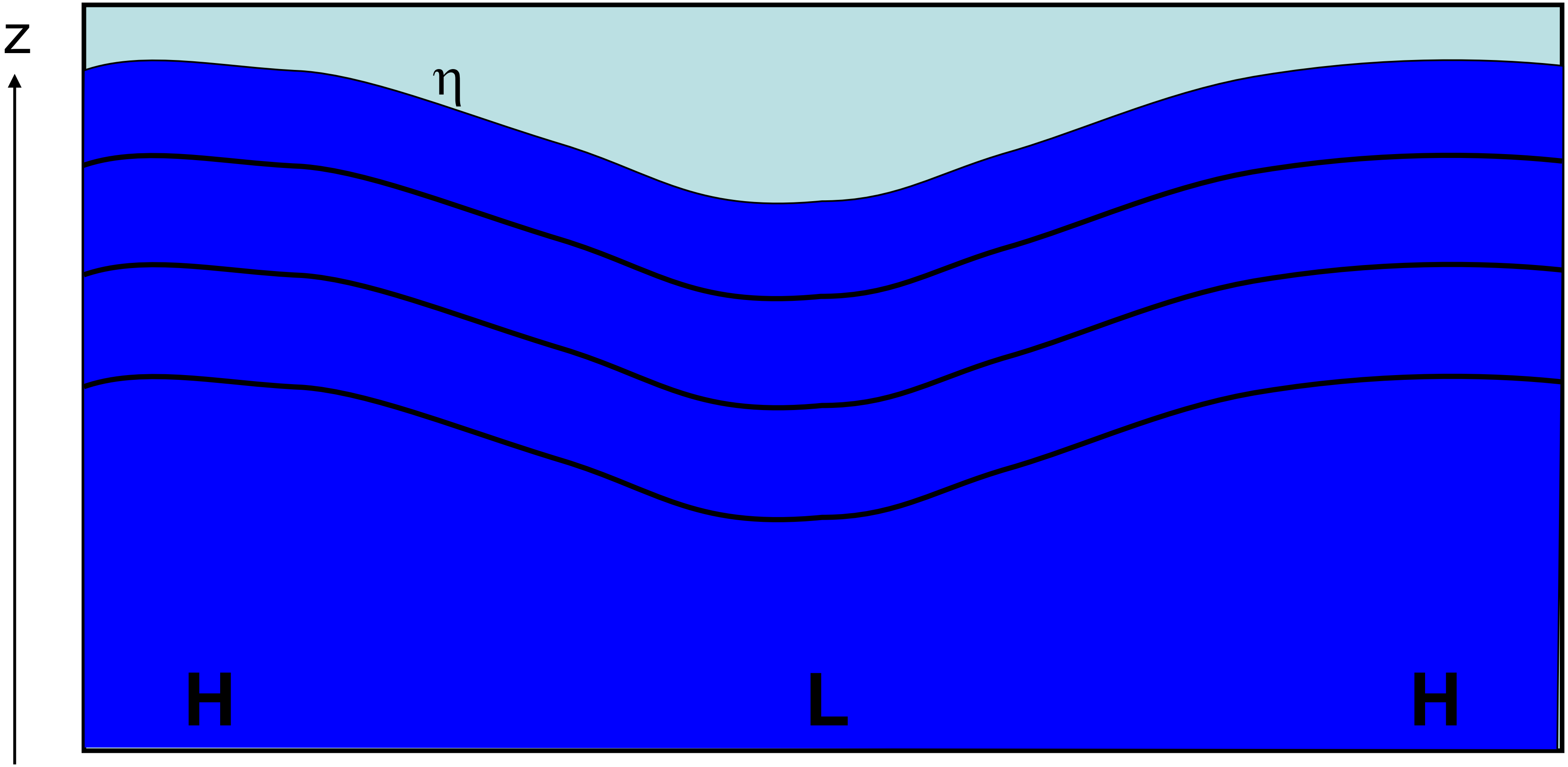
One can use this to show that by descending 10 metres increases the pressure by about 1000 hPa.

Pressure increases almost linearly with depth.

Also, at a given height, small variations in sea-level can cause horizontal gradients in internal pressure.



Fluid moves from high pressure to low pressure in non-rotating frame.



Lines of constant pressure.

Along these pressure surfaces can define a geopotential, gz.
(called dynamic height by oceanographers).

The sea-level, η, represents the height of the pressure surface, where the pressure equals the atmospheric pressure.

Recall:

$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y}$$

$$v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

Is the geostrophic wind in pressure coordinates.

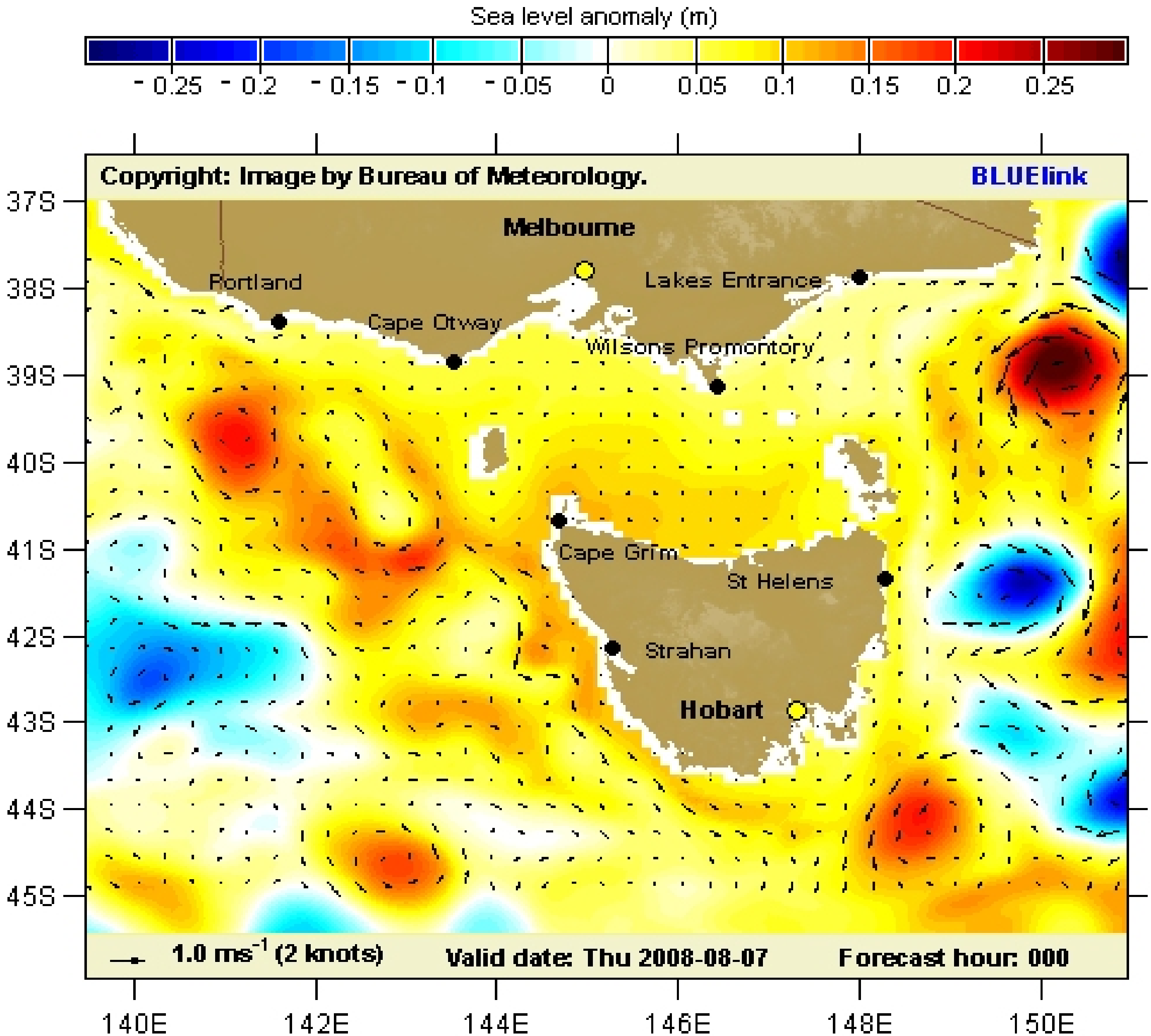
Therefore, we can define surface geostrophic currents to be

$$u_g = -\frac{g}{f}\frac{\partial \eta}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial \eta}{\partial x}$$

Thus, in direct analogy to atmospheric behaviour, geostrophic currents will be cyclonic about a low sea-level anomaly and anticyclonic about a high sea-level anomaly.

Example of geostrophic currents:



Scale analysis of the horizontal equations of motion - ocean.

Assign typical scales to variables *(use previous image as an example).*

Variable.	Scale.	Value for ocean eddies
x,y: Horizontal Length	L	10^5 m (100 km)
u,v: Horizontal Velocity	U	0.1 m/s
t: Advective Time T = L / U	10^6 s	(~10 Days)
η: Sea level variation	D	0.1 m
f : Coriolis terms	f	10^{-4} s^{-1}

Scale analysis of horizontal components of eqns of motion.

$\dfrac{Du}{Dt}$	$-fv$		$= -g\dfrac{\partial \eta}{\partial x}$
$\dfrac{Dv}{Dt}$	$+fu$		$= -g\dfrac{\partial \eta}{\partial y}$
Scales: U^2/L	fU		gD/L
Order: 10^{-7}	10^{-5}		10^{-5}

Rossby number, $R_o = U / fL = 0.01$

Summary of Topic 2.

Modification to the equations of motion in a rotating reference frame. E.g., f-plane approximation:

$$\frac{D\underline{u}}{Dt} + f\,\underline{k} \times \underline{u} = -\frac{1}{\rho}\nabla p - g\underline{k}$$

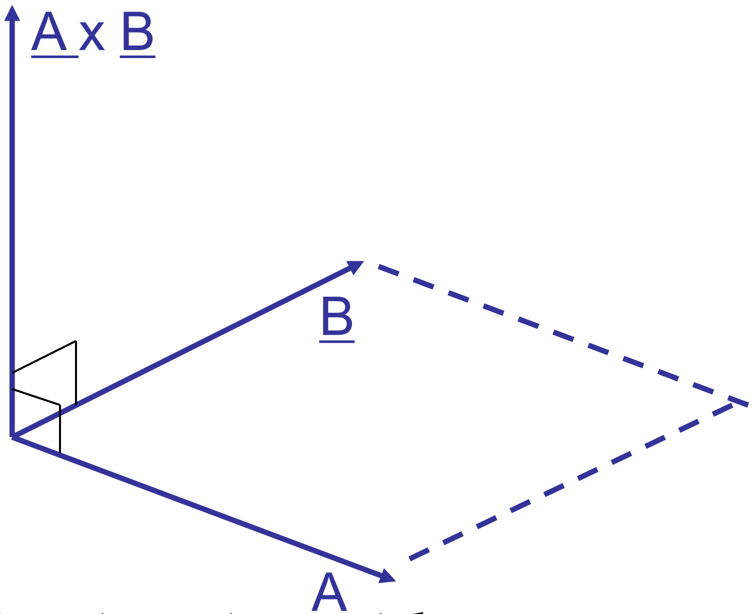
Scale analysis and the geostrophic approximation:

$$f\,\underline{k} \times \underline{u}_g = -\frac{1}{\rho}\nabla_H p$$

The Rossby number: $R_o = U / f L$

Should know / understand representations in both height and pressure coordinates.

Required knowledge: Vector product / Cross product



A x B (*"A-cross-B"*) is the vector perpendicular to the plane created by A and B. Its magnitude is equal to the area of the parallelogram defined by A and B.

$$\underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$$

$$\underline{B} = B_1\underline{i} + B_2\underline{j} + B_3\underline{k}$$

$$\underline{A}\times\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \left(A_2B_3 - A_3B_2\right)\underline{i} - \left(A_1B_3 - A_3B_1\right)\underline{j} + \left(A_1B_2 - A_2B_1\right)\underline{k}$$

$$\left|\underline{A}\times\underline{B}\right| = \left|\underline{A}\right|\left|\underline{B}\right|\sin\left(\theta\right),\ \text{where } \theta \text{ is angle between vectors}$$

'Easy' to remember i,j,k rule:

$$\underline{i}\times\underline{j}=\underline{k} \qquad a\underline{i}\times b\underline{j}=ab\underline{k}$$

$$\underline{j}\times\underline{k}=\underline{i} \qquad c\underline{j}\times d\underline{k}=cd\underline{i}$$

$$\underline{k}\times\underline{i}=\underline{j} \qquad e\underline{k}\times f\underline{i}=ef\underline{j}$$

$$\underline{j}\times\underline{i}=-\underline{k} \qquad g\underline{j}\times h\underline{i}=-gh\underline{k}$$

A x B = - B x A

Right Hand Rule:

If $\underline{A}$ and $\underline{B}$ are two vectors in the same plane, then $\underline{A}\mathrm{x}\underline{B}$ is a vector perpendicular to this plane whose direction is indicated by your thumb provided $\underline{A}$ and $\underline{B}$ are associated with the index and middle fingers of your ***right hand***, respectively.