Dynamical Meteorology and Oceanography ATOC30004

Topic 5: Vorticity

- Absolute and relative vorticity
- The β-plane
- Qualitative description of Rossby waves
- Shallow water potential vorticity and the lee trough
- Baroclinic vorticity generation in convection

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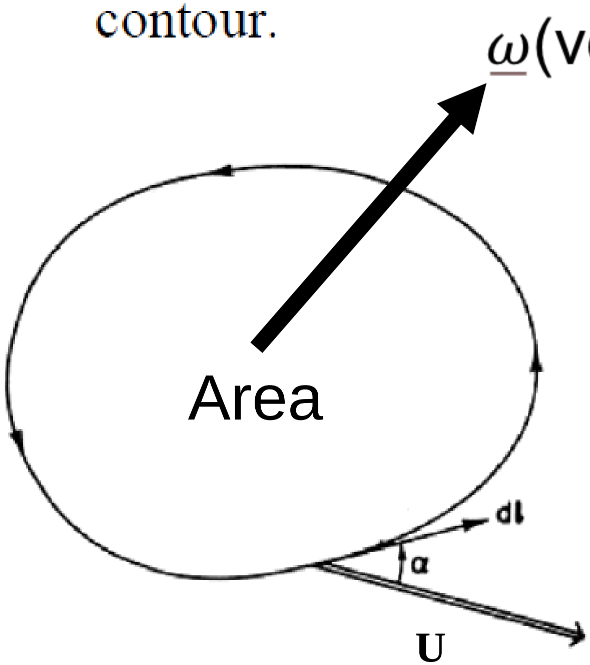
Circulation and Vorticity

- The *circulation*, C, about a closed contour in a fluid is defined as the line integral evaluated along the contour of the component of the velocity vector that is locally tangent to the contour.



$$C \equiv \oint \mathbf{U} \bullet d\mathbf{l} = \oint |\mathbf{U}| \cos\,\alpha\; dl$$

C > 0 ➔ Counterclockwise
C < 0 ➔ Clockwise

In the limit of vanishingly small Area, magnitude-of-Vorticity=circulation/area. Vorticity vector is perpendicular to plane defined by the area and is oriented according to the right-hand rule; i.e. give a thumbs-up and then orientate your Hand so that the fingers indicate the sense of rotation of the flow: your thumb is Then pointing in the same direction as the vorticity vector.

Vorticity

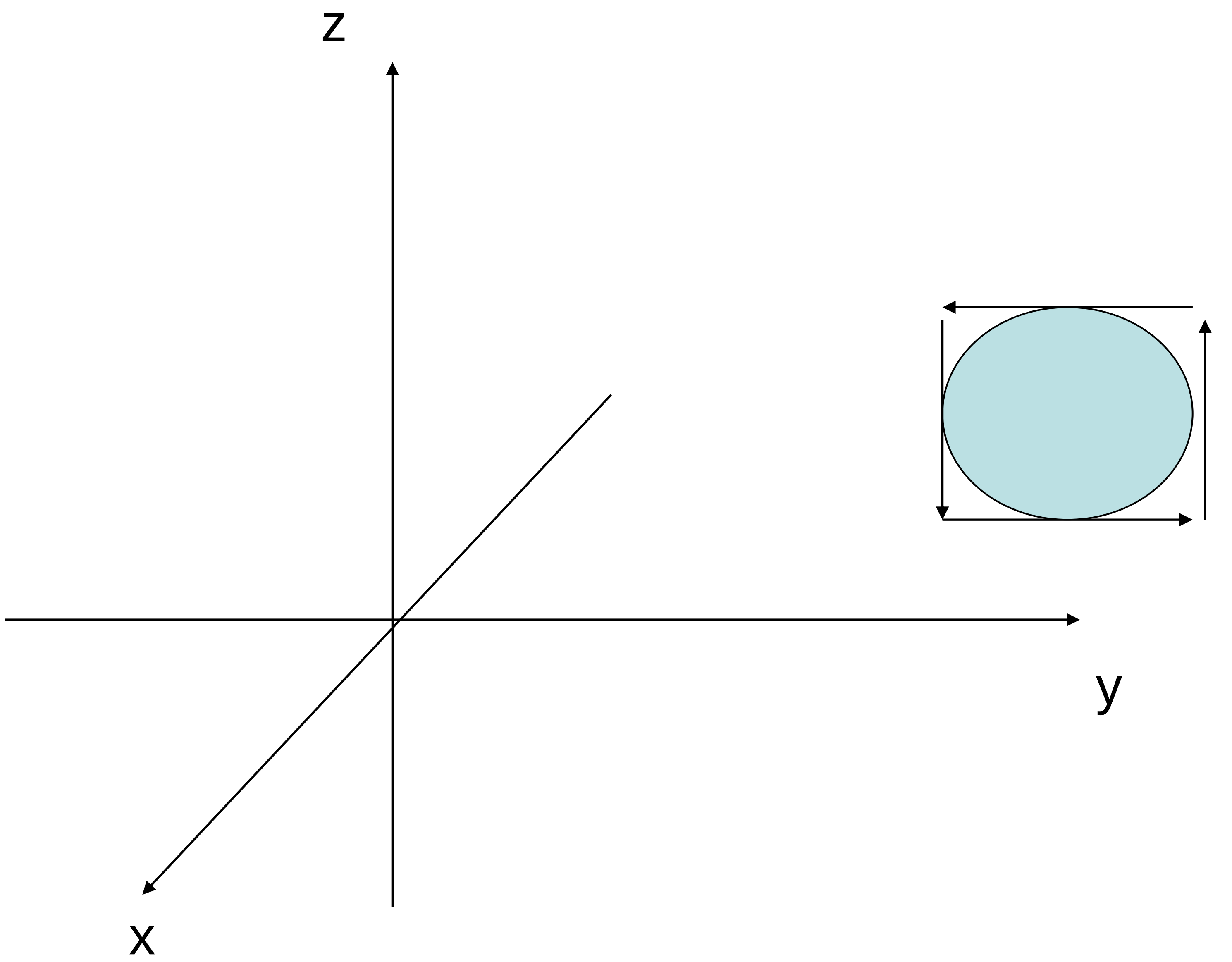
The vorticity is a measure of the circulation / rotation of the fluid.

It is a vector quantity, defined as the curl of the velocity field.

$$\underline{\omega} = \xi \underline{i} + \eta \underline{j} + \zeta \underline{k} = \nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

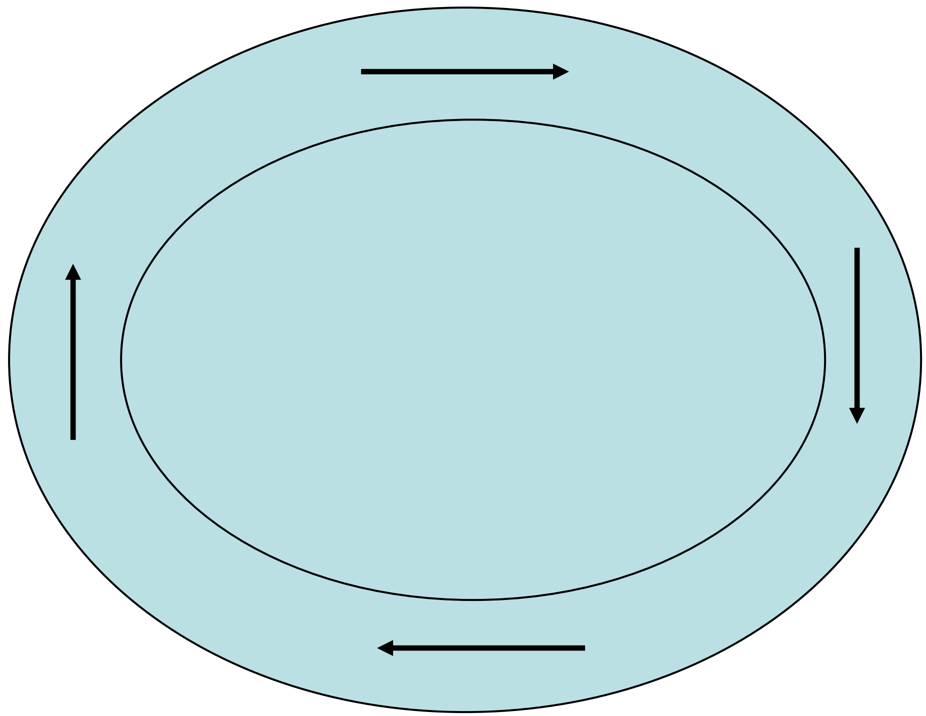
$$\underline{\omega} = \underline{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \underline{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \underline{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Each component of the vorticity refers to rotation in a plane perpendicular to that component.


z
y
x

Examples - A clockwise rotating vortex (in the horizontal plane):

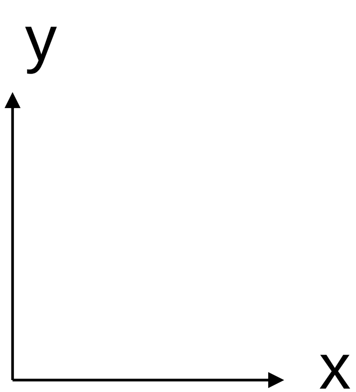
The vertical component of the vorticity is: $\underline{k}\cdot\underline{\omega} = \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$



$$\frac{\partial v}{\partial x} < 0$$

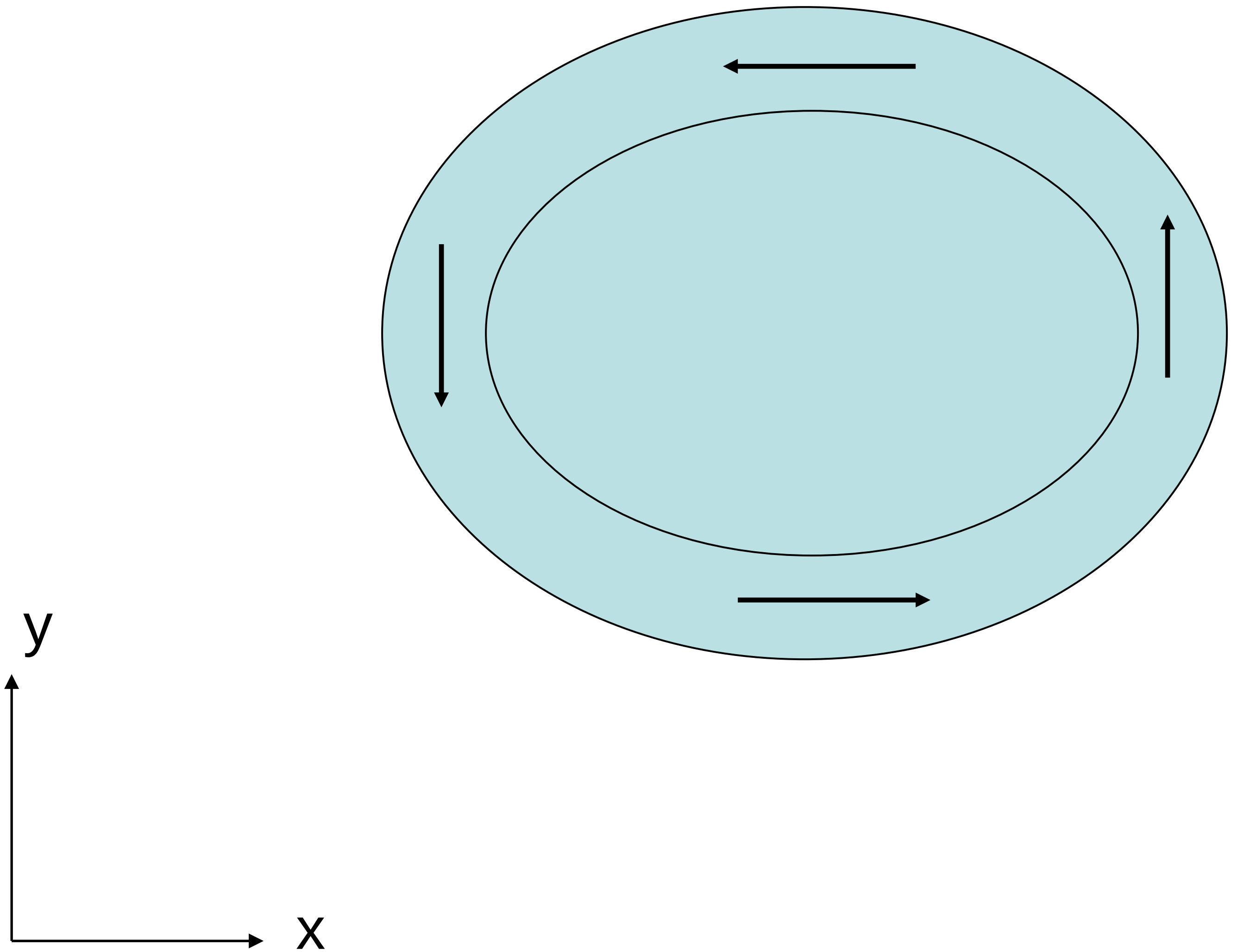
$$\frac{\partial u}{\partial y} > 0$$

$$\Rightarrow \quad \zeta < 0$$



Examples - An anticlockwise rotating vortex

The vertical component of the vorticity is: $\underline{k}\cdot\underline{\omega} = \zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$



$$\frac{\partial v}{\partial x} > 0$$

$$\frac{\partial u}{\partial y} < 0$$

$$\Rightarrow \quad \zeta > 0$$

Thus,

In the northern hemisphere:

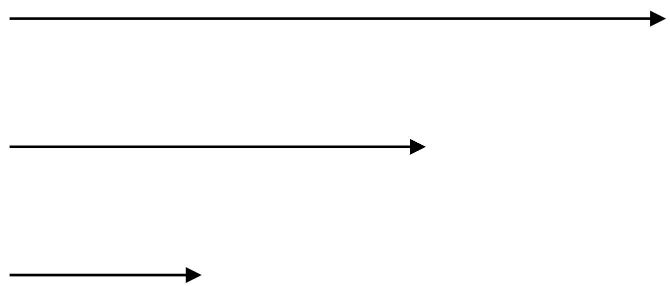
- A cyclonic vortex (LOW) has positive vertical vorticity
- An anticyclonic vortex (High) has negative vertical vorticity

In the southern hemisphere:

- A cyclonic vortex (LOW) has negative vertical vorticity
- An anticyclonic vortex (High) has positive vertical vorticity

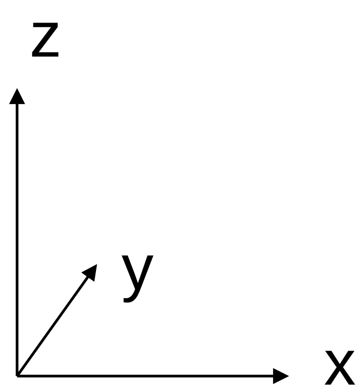
Of course, vorticity is present in other directions and also in unidirectional flows (i.e., not vortices).

e.g., vertical wind shear.



y-component of vorticity:

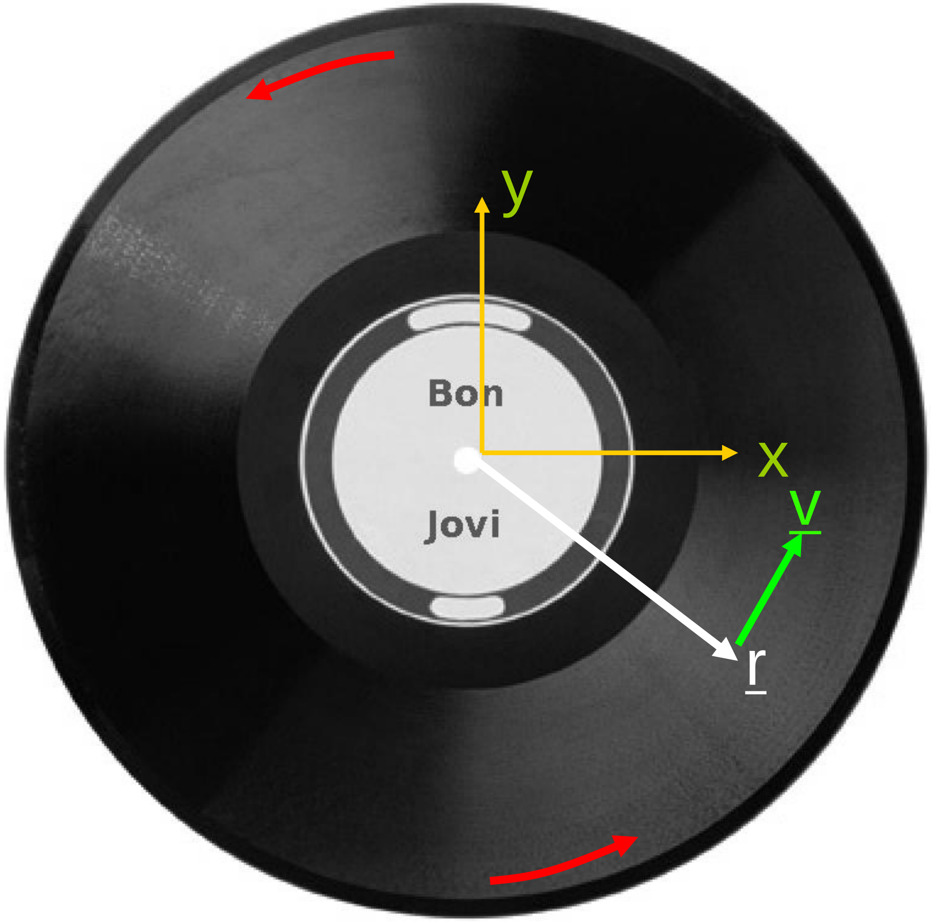
$$\underline{j}\cdot\underline{\omega} = \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$



$$\frac{\partial u}{\partial z} > 0 \qquad \frac{\partial w}{\partial x} = 0$$

$$\Rightarrow \eta > 0$$

Final example - solid body rotation:



Recall $|\underline{v}|=|\underline{\Omega}||\underline{r}|$

$v(x,0)= |\Omega x|$
$u(0,y)= -|\Omega y|$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$= 2\Omega$

For a rotating disk

$v = \Omega r$

Circulation=Circumference*$v = 2\pi r \Omega r = 2\Omega \pi r^2$

Area=πr^2

Vorticity=$\dfrac{\text{Circulation}}{Area}$=$2\Omega$

An object (or fluid flow) in solid body rotation has vorticity equal to twice the angular velocity. (The vorticity vector is aligned with the angular velocity vector, i.e., perpendicular to the plane of rotation.

The concept of vorticity is most illustrative once we derive the equation governing its evolution. Specifically, we aim to derive the vertical vorticity equation, which is useful for synoptic flow.

Start with our equations of motion governing the two horizontal velocity components, assuming the Boussinesq approximation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_*}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_*}\frac{\partial p}{\partial y}$$

Also, allow the coriolis parameter to be a function of *y*. i.e., *f = f(y)*
[Recall f=2Ωsin(φ)]

Take $\frac{\partial}{\partial x}(2)-\frac{\partial}{\partial y}(1)$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho_*}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho_*}\frac{\partial p}{\partial y} \qquad (2)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial f}{\partial y}+$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \qquad (3)$$

Note that $\frac{Df}{Dt}=v\frac{\partial f}{\partial y}$

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)+\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}=0 \qquad (4)$$

Vorticity production due to convergence

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=\boxed{-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)}+\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}$$

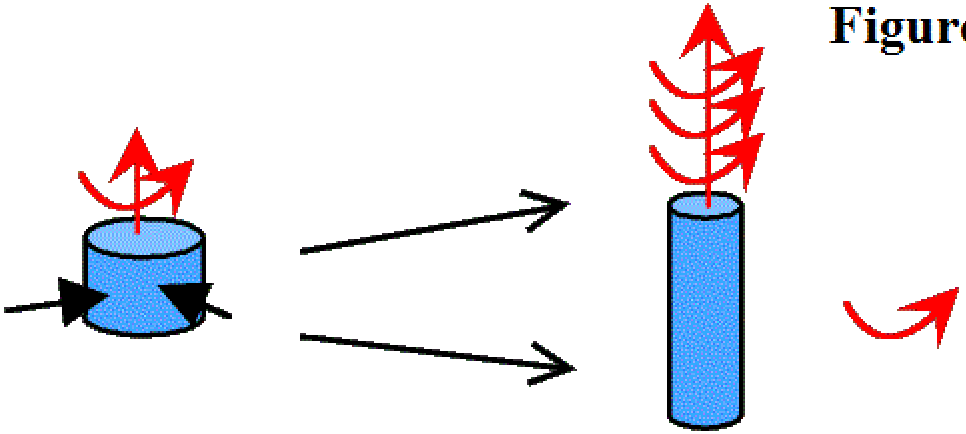


Figure 1a

horizontal convergence --> vertical stretching, faster rotation

Vorticity production due to convergence

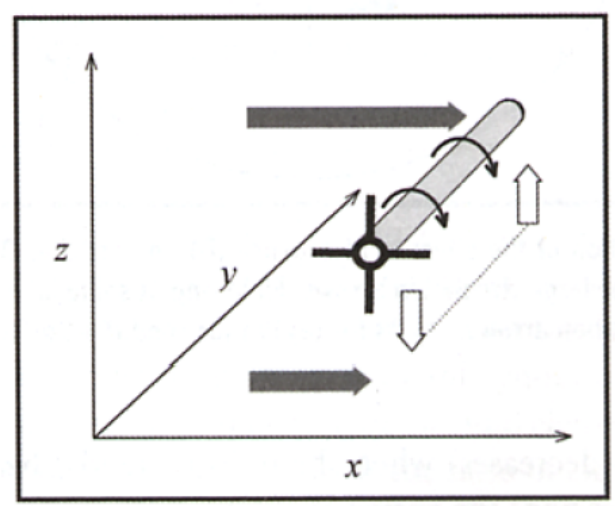
tilting

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)=\boxed{-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f\right)}+\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}$$

What are terms like $\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}$ and $-\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}$ doing?

- Convert vorticity in X and Y directions into the Z-direction by the tilting/twisting effect produced by the vertical velocity (əw/əx and əw/əy).

$$\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}$$



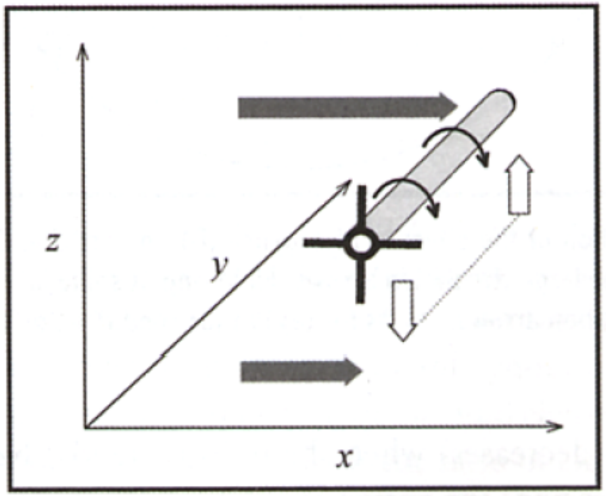
$\frac{\partial u}{\partial z}$ ➔ **vorticity in Y-direction**

$\frac{\partial w}{\partial y}$ ➔ **Tilting by the variation of w in Y-direction**

Interpretation of terms like $\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}$ and $-\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}$

- Convert vorticity in X and Y directions into the Z-direction by the tilting/twisting effect produced by the vertical velocity (∂w/∂x and ∂w/∂y).

$$\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}$$



$\frac{\partial u}{\partial z}$ ➔ **vorticity in Y-direction**

$\frac{\partial w}{\partial y}$ ➔ **Tilting by the variation of w in Y-direction**

Under the assumption of zero vertical shear (i.e., a barotropic flow) or the assumption of a two-dimensional flow (w=0), then this becomes:

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\nabla_H \cdot \underline{u}_H$$

$$\text{where } \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

The (vertical) vorticity equation is:

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\nabla_H \cdot \underline{u}_H$$

$\zeta + f$ is called the ***absolute vorticity***.

In a rotating reference frame we usually refer to ζ as the ***relative vorticity***.

f is the coriolis parameter. Recall this is 2 x the vertical component of the Earth's angular velocity. Thus, f is referred to as the ***planetary vorticity***.

Cyclonic vorticity grows with convergence or decays with divergence.

Vorticity and Action at a distance: Questions for prac:

Vorticity can be thought of as inducing flow at a distance in exactly the same way as electric current and charge produce potentials that "act at a distance". To see this, let us first define a streamfunction ψ such that

$$v = \frac{\partial \psi}{\partial x},\ u = -\frac{\partial \psi}{\partial y}$$

In class problem: Prove that the vertical component of vorticity ζ is then given by

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$$

Those of you who have done physics will recognize that this Poisson type equation turns up when describing the relation between (2D) charge and electric potential

Evolution of vorticity in low Rossby number flows (synoptic scale)

Recall: $R_o \rightarrow 0$ $u \rightarrow u_g$

Under normal assumptions $\nabla \bullet \underline{u}_g = 0$

Thus,

$$\frac{D}{Dt}\left(z + f\right) = 0$$

This is often referred to as the barotropic vorticity equation.

$$\frac{D}{Dt}(z + f) = 0$$

For nondivergent flow (low Rossby number) the absolute vorticity is conserved following the fluid.

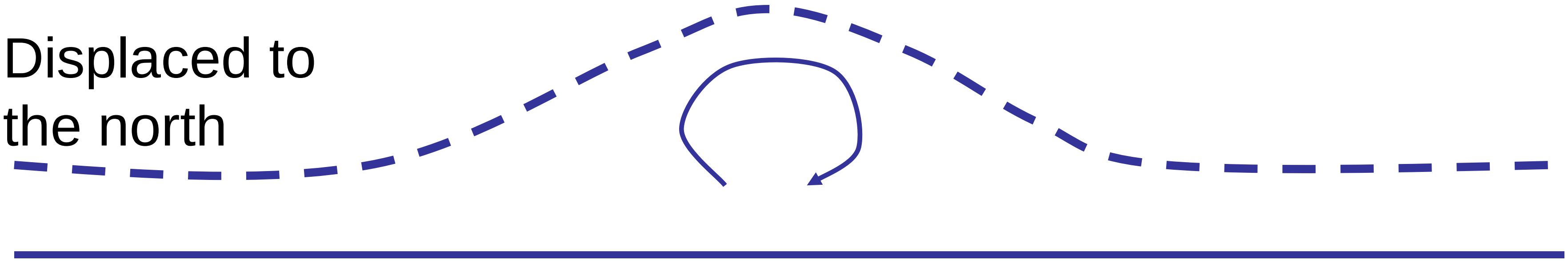
For this to hold the sum of the planetary vorticity (coriolis parameter) and the relative vorticity of a parcel must always be the same.

- *f* can only change if the parcel moves to a different latitude.
- f increases as you move to the North (in both hemispheres).

$$\frac{D}{Dt}\left(z + f\right) = 0$$

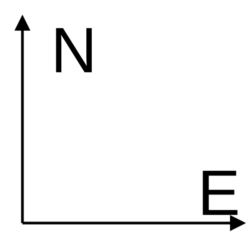
f large

Displaced to
the north



When moved to the north:

f increases
ζ decreases

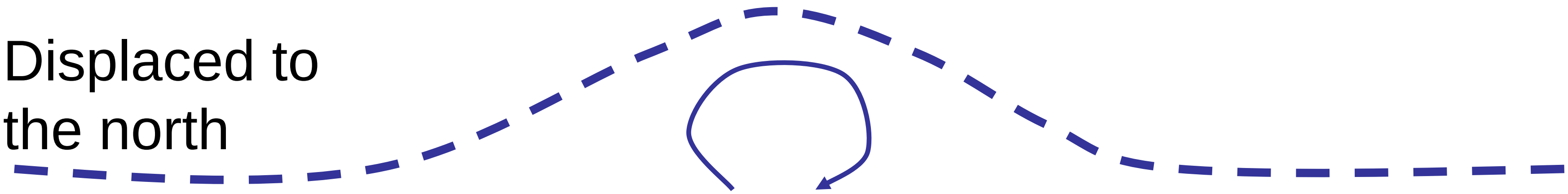


f small

$$\frac{D}{Dt}(z + f) = 0$$

f large

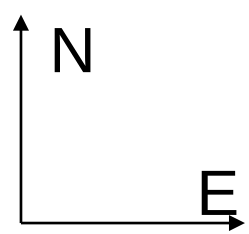
Displaced to
the north



In both hemispheres:

Poleward displacement generates anticyclonic relative vorticity anomalies.

Equatorward displacement generates cyclonic relative vorticity anomalies.



f small

The latitudinal variations in planetary vorticity are commonly simplified using the β-plane approximation:

$f = 2\,\Omega \sin\phi$

using Taylor series:

$f = f(\phi_0) + \delta\phi\, f'(\phi_0) + O(\delta\phi^2)$
$\quad = 2\Omega\sin(\phi_0) + \delta\phi 2\Omega\cos(\phi_0) + O(\delta\phi^2)$

For small $\delta\phi$ $\quad y \approx a\,\delta\phi$, where a is the radius of the Earth.

$f \approx 2\Omega\sin(\phi_0) + (y/a)2\Omega\cos(\phi_0)$
$f \approx f_0 + \beta y$

$$\beta = 2\Omega\cos(\phi_0)/a$$

$$\frac{D}{Dt}(z + f) = 0$$

Can be re-written as:

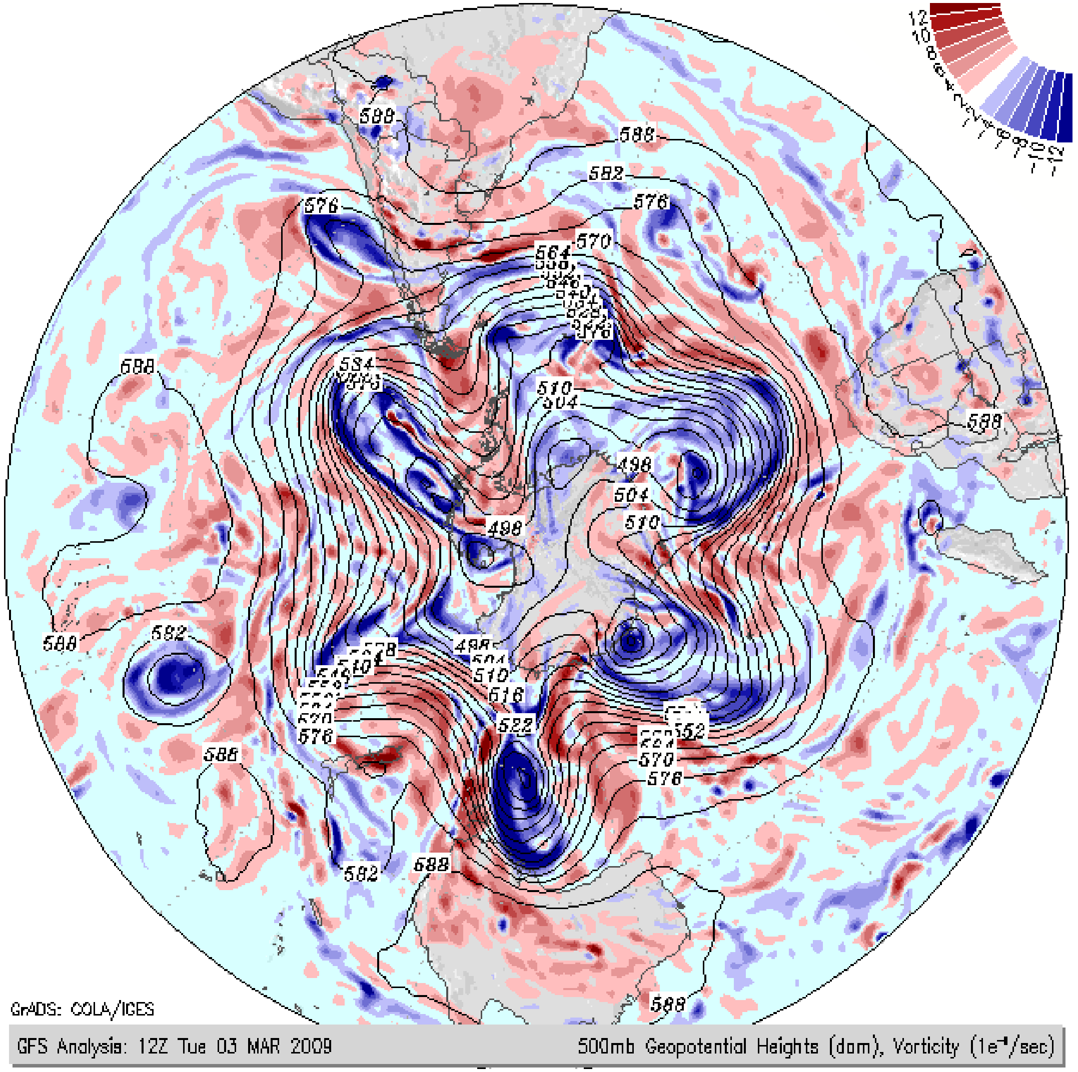
$$\frac{D}{Dt}(\zeta + f_0 + \beta y) = \frac{D\zeta}{Dt} + v\beta = 0$$

$$=> \frac{D\zeta}{Dt} = -v\beta$$

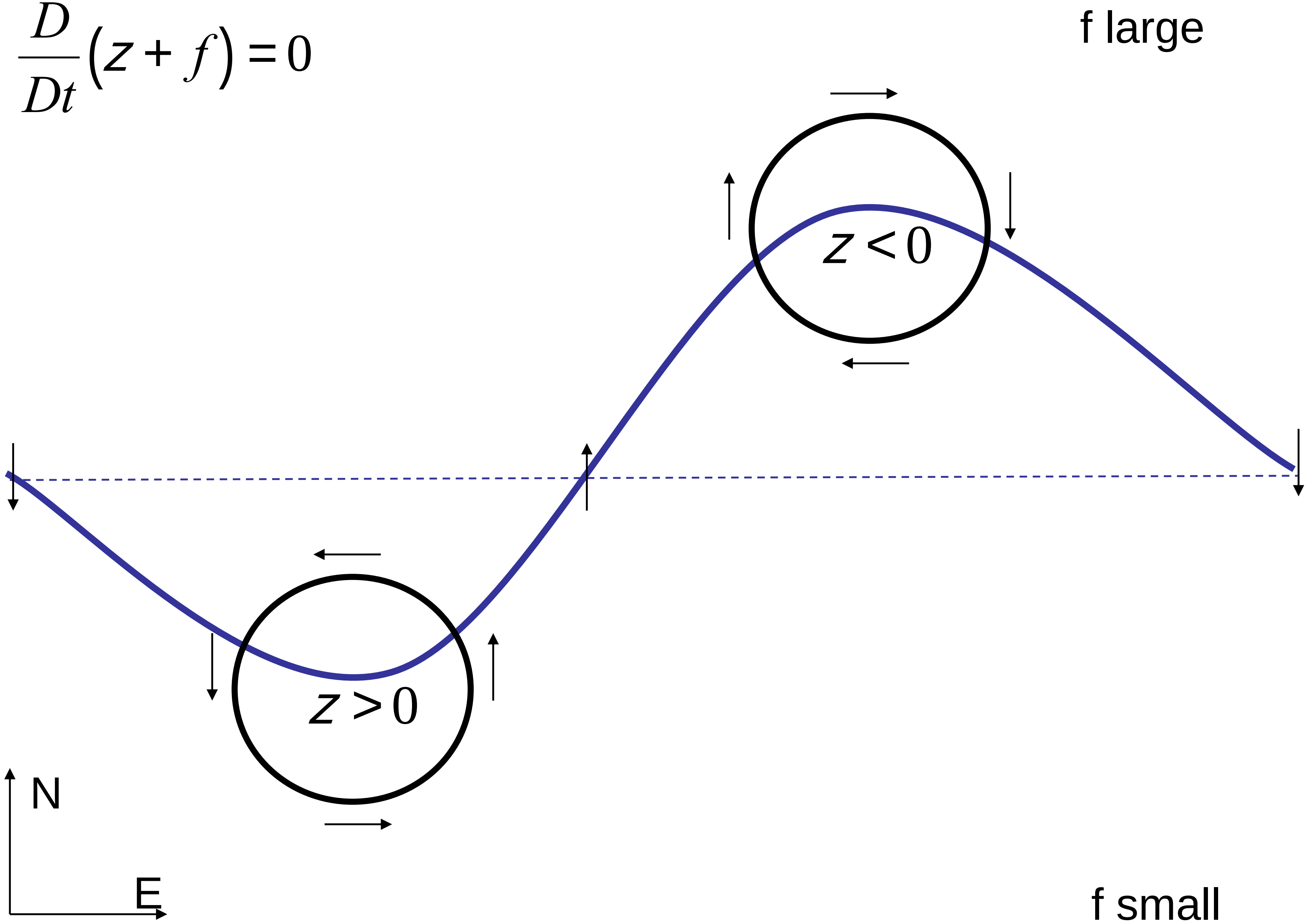
This is a simple representation of the influence of the meridional gradient in planetary vorticity on the generation of relative vorticity, which is a key ingredient for cyclogenesis (and anticyclogenesis).

The conservation of absolute vorticity can be used to explain the propagation of Rossby waves.

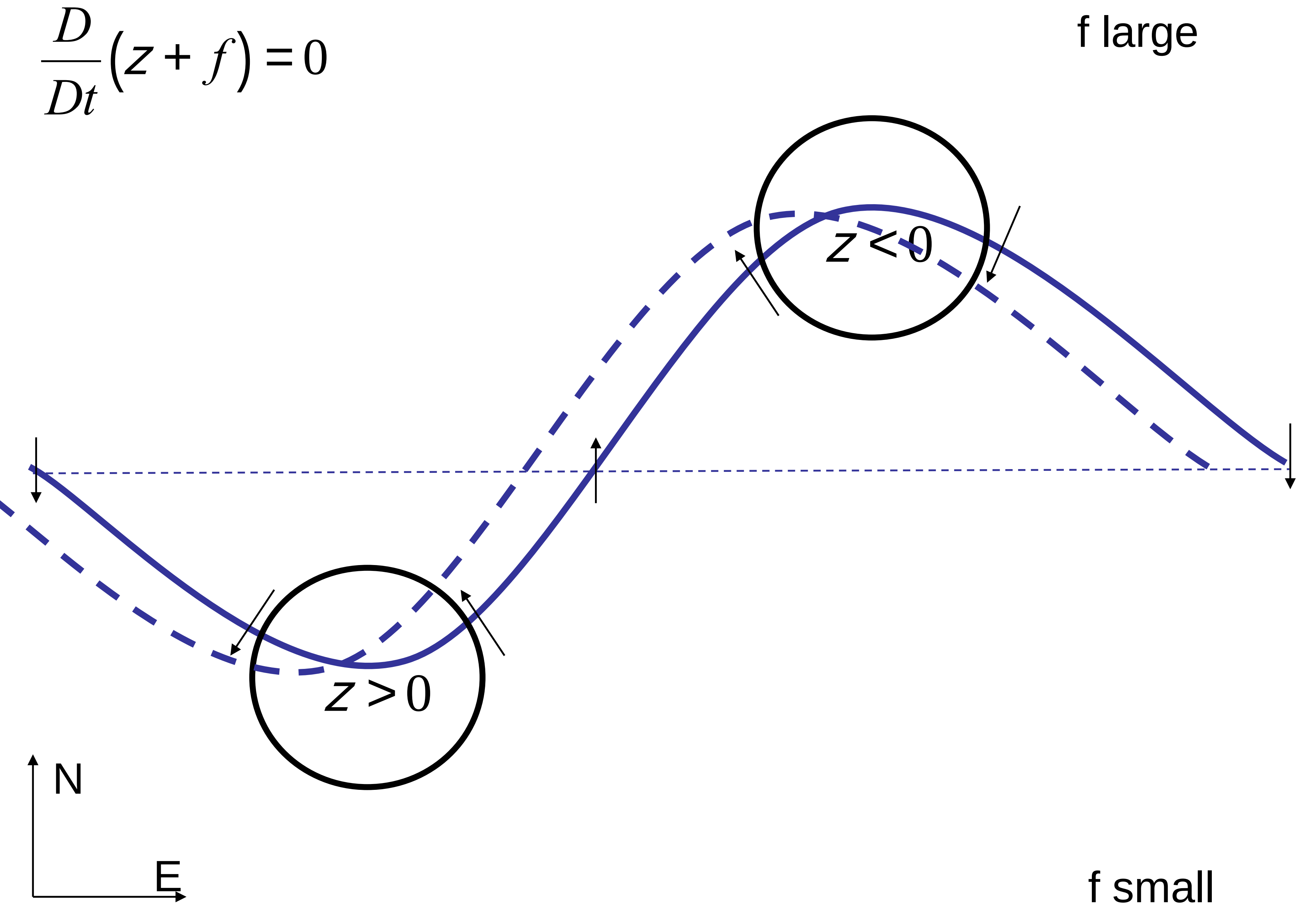
- Qualitative treatment now, quantitative later:



$$\frac{D}{Dt}(z + f) = 0$$



$$\frac{D}{Dt}\left(z + f\right) = 0$$



Rossby wave propagation:

Vorticity anomalies give rise to a velocity field that advects the material surface as shown.

This results in a westward propagation of the Rossby wave (within the background eastward flow).

The westward propagation speed is usually slower than the mean wind. Therefore, synoptic scale Rossby waves usually move eastward but at a speed slower than the mean wind.

The meridional gradient in planetary vorticity acts as the restoring force for the Rossby wave. Analogous to a spring providing the restoring force for a simple harmonic oscillator.

The discussion here is for “neutral” Rossby waves. In the real atmosphere the wave amplification relies on the thermodynamics: so-called baroclinic instability.

A quantitative treatment of Rossby wave propagating will be presented in the next Topic.

The barotropic vorticity equation was at the core of the first efforts at numerical weather prediction.

$$\frac{D}{Dt}\left(\zeta + f\right) = 0 \qquad \text{***}$$

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \quad \text{and} \quad v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x} \qquad \text{**}$$

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{\rho f}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) \qquad \text{*}$$

A closed system of evolving equations.

Determine pressure from vorticity (*), use pressure to define velocity (**), advect vorticity using velocity to get future state (***), and repeat.

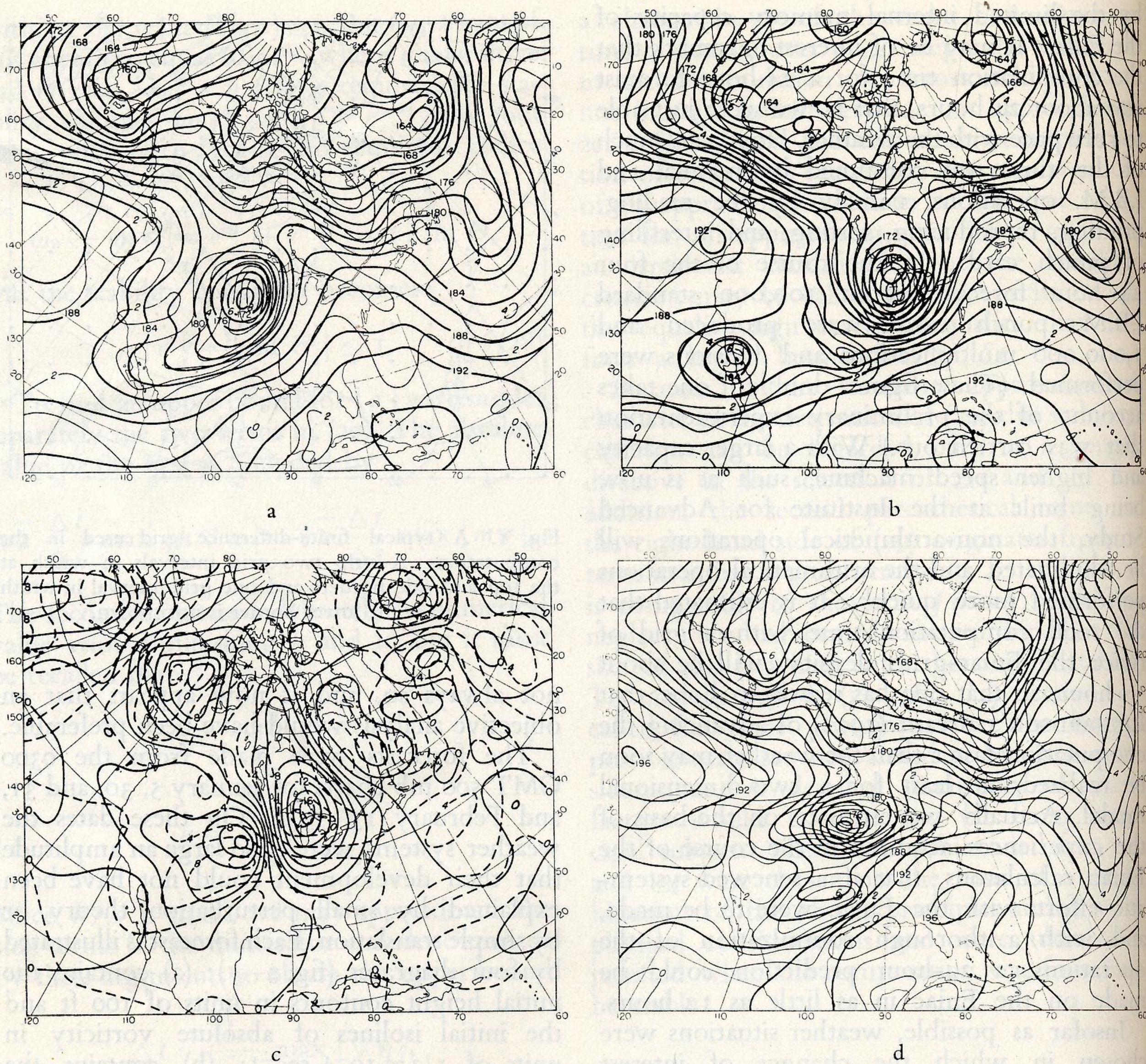


Fig. 2. Forecast of January 5, 1949, 0300 GMT: (a) observed z and η at $t = 0$; (b) observed z and η at $t = 24$ hours; (c) observed (continuous lines) and computed (broken lines) 24-hour height change; (d) computed z and η at $t = 24$ hours. The height unit is 100 ft and the unit of vorticity is $1/3 \times 10^{-4}$ sec^{-1}.

Charney et al., (1950, Tellus)

Return to the (vertical) vorticity equation:

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\nabla_H \cdot \underline{u}_H$$

We had made the assumption that the flow is non-divergent.

Let's relax the non-divergent assumption.

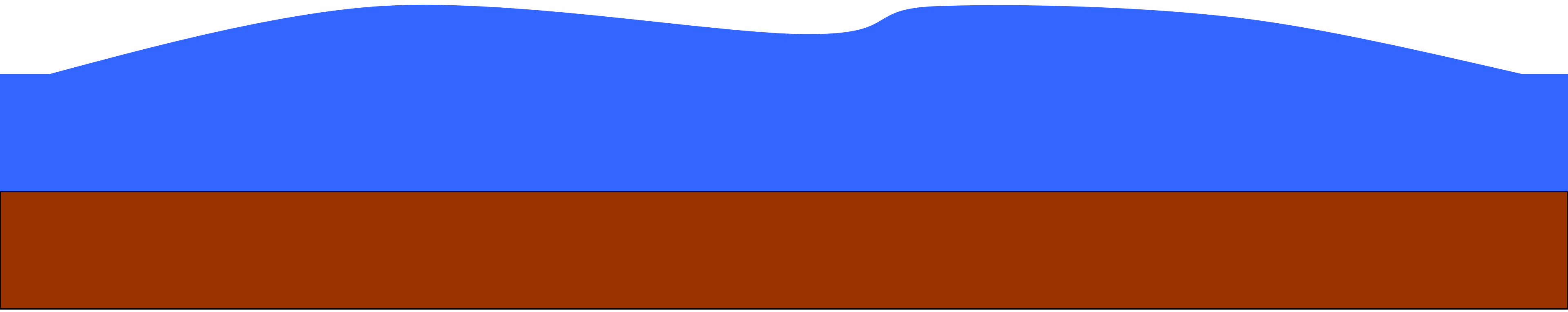
Under the Boussinesq / incompressible approximation:

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\nabla_H \cdot \underline{u}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

Assume a fluid of depth h.

Assume this fluid has uniform horizontal velocity throughout. (We made this assumption earlier to derive vorticity equation)



At the bottom of this fluid, w=0. At the top of the fluid w=$^{Dh}/_{Dt}$

$$\frac{\partial w}{\partial z} = -\nabla_H \cdot \underline{u}_H$$

Integrate over depth of fluid

$$\int_0^h \frac{\partial w}{\partial z} dz = -\int_0^h \nabla_H \cdot \underline{u}_H dz$$

Remember u and v are constant over depth

$$w(h) - w(0) = -h\nabla_H \cdot \underline{u}_H$$

Remember w(0)=0 and w(h)=Dh/Dt

$$\frac{1}{h}\frac{Dh}{Dt} = -\nabla_H \cdot \underline{u}_H$$

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\nabla_H \cdot \underline{u}_H \qquad \frac{1}{h}\frac{Dh}{Dt} = -\nabla_H \cdot \underline{u}_H$$

Gives:

$$\frac{D}{Dt}(\zeta + f) = \frac{(\zeta + f)}{h}\frac{Dh}{Dt}$$

$$\frac{1}{(\zeta + f)}\frac{D}{Dt}(\zeta + f) - \frac{1}{h}\frac{Dh}{Dt} = 0$$

...

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$$

$$\frac{d}{dt}\left(\frac{A}{B}\right) = 0$$

$$\frac{1}{B}\frac{dA}{dt} + A\frac{d}{dt}\left(\frac{1}{B}\right) = 0 \qquad \frac{d}{dt}\left(\frac{1}{B}\right) = -\frac{1}{B^2}\frac{dB}{dt}$$

$$\frac{1}{B}\frac{dA}{dt} - \frac{A}{B^2}\frac{dB}{dt} = 0$$

$$\frac{1}{A}\frac{dA}{dt} - \frac{1}{B}\frac{dB}{dt} = 0$$

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$$

$\left(\frac{\zeta + f}{h}\right)$ is called the (shallow water) potential vorticity (PV).

The conservation of potential vorticity reverts to absolute vorticity conservation for purely two-dimensional flow with constant fluid depth.

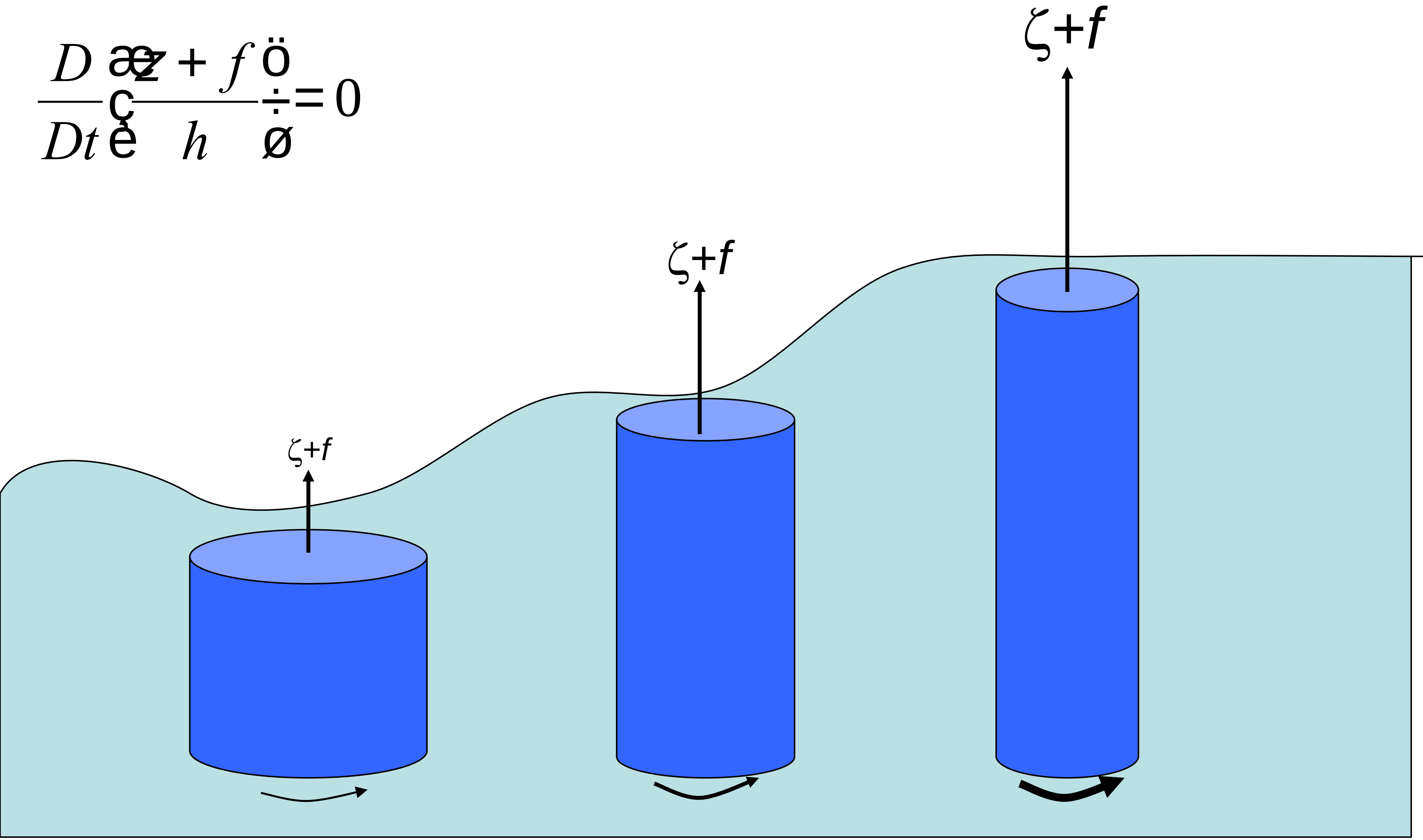
<u>Vortex stretching:</u>

Enhancements in fluid depth (through convergence) will enhance the absolute vorticity magnitude.

Reductions in fluid depth (through divergence) will reduce the absolute vorticity magnitude.

Conservation of PV / vortex stretching

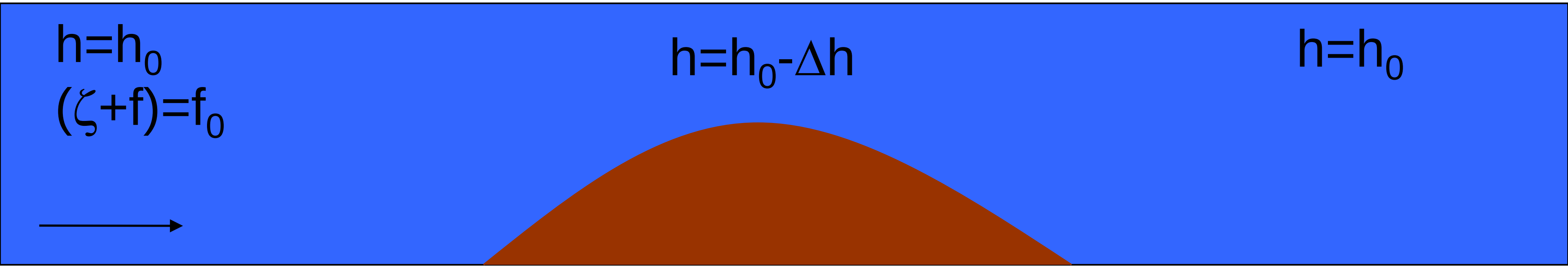
$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$$

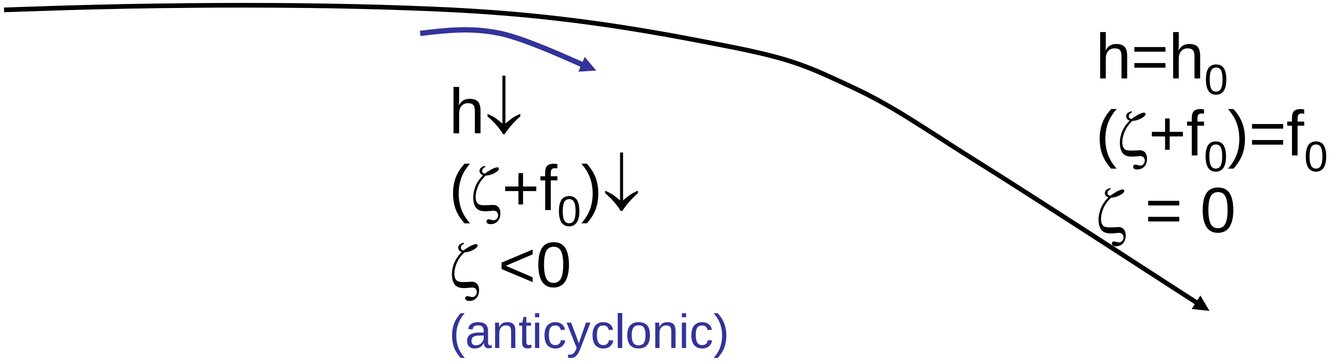


Conservation of PV can also be used to explain flow over large-scale mountain ranges.

Fluid depth gets smaller as air flows over range.

Simplest case, $f = f_0$ = constant everywhere (NH case).

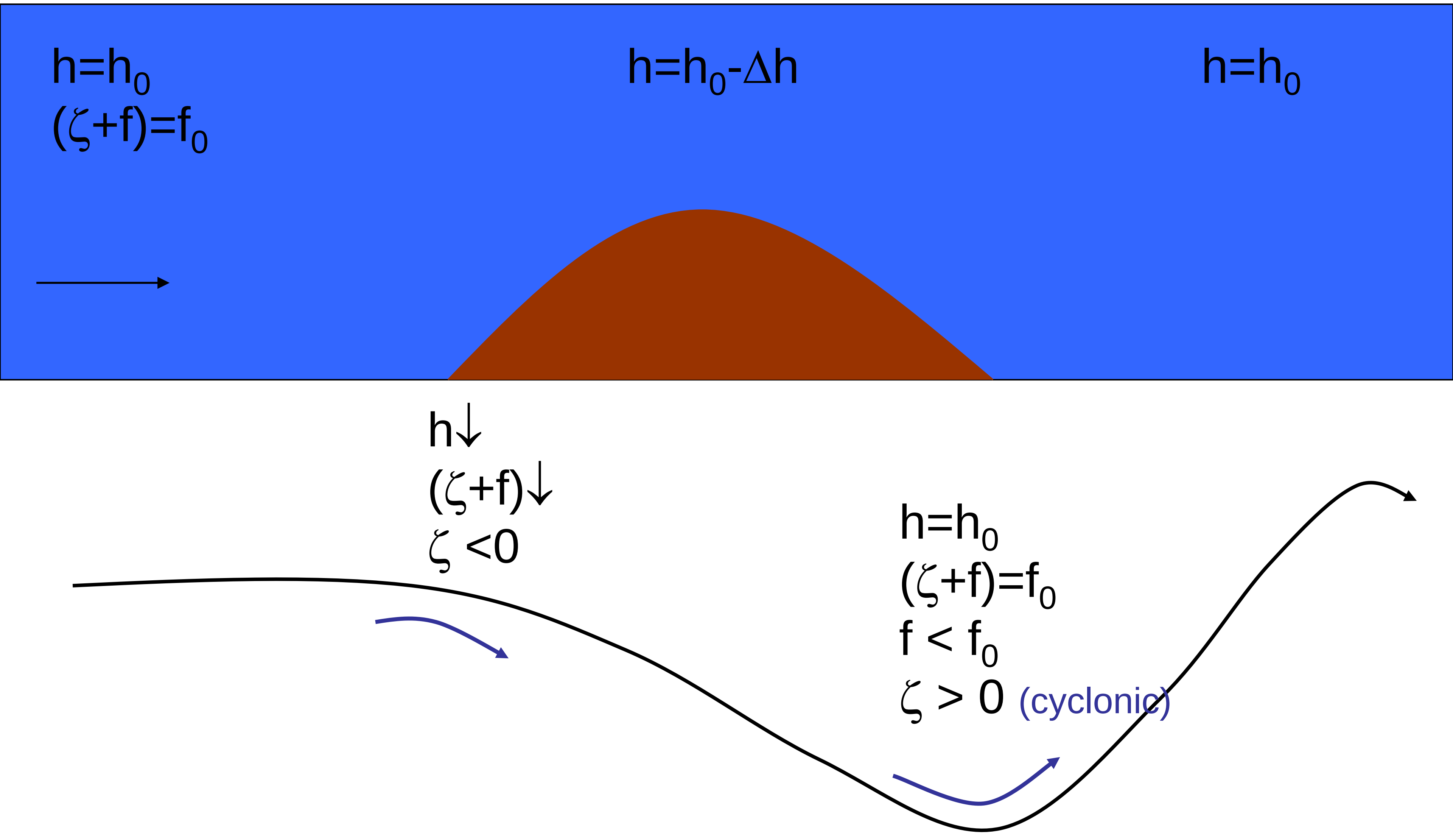




More complicated response if f allowed to vary.

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$$

Northern hemisphere case



Sequence of events (for Northern hemisphere).

1 - As h decreases the absolute vorticity must decrease also. No meridional displacement has occurred (yet) and so the change in absolute vorticity is realized as a decrease in relative vorticity. This causes the trajectory to turn to the right, moving equatorward.

2 - Once the air has passed over the barrier, h returns to its original value and so does the absolute vorticity.However, the equatorward trajectory has left the parcel further to the south than its original value so f is smaller. Thus, to compensate for being further south the relative vorticity must be positive.

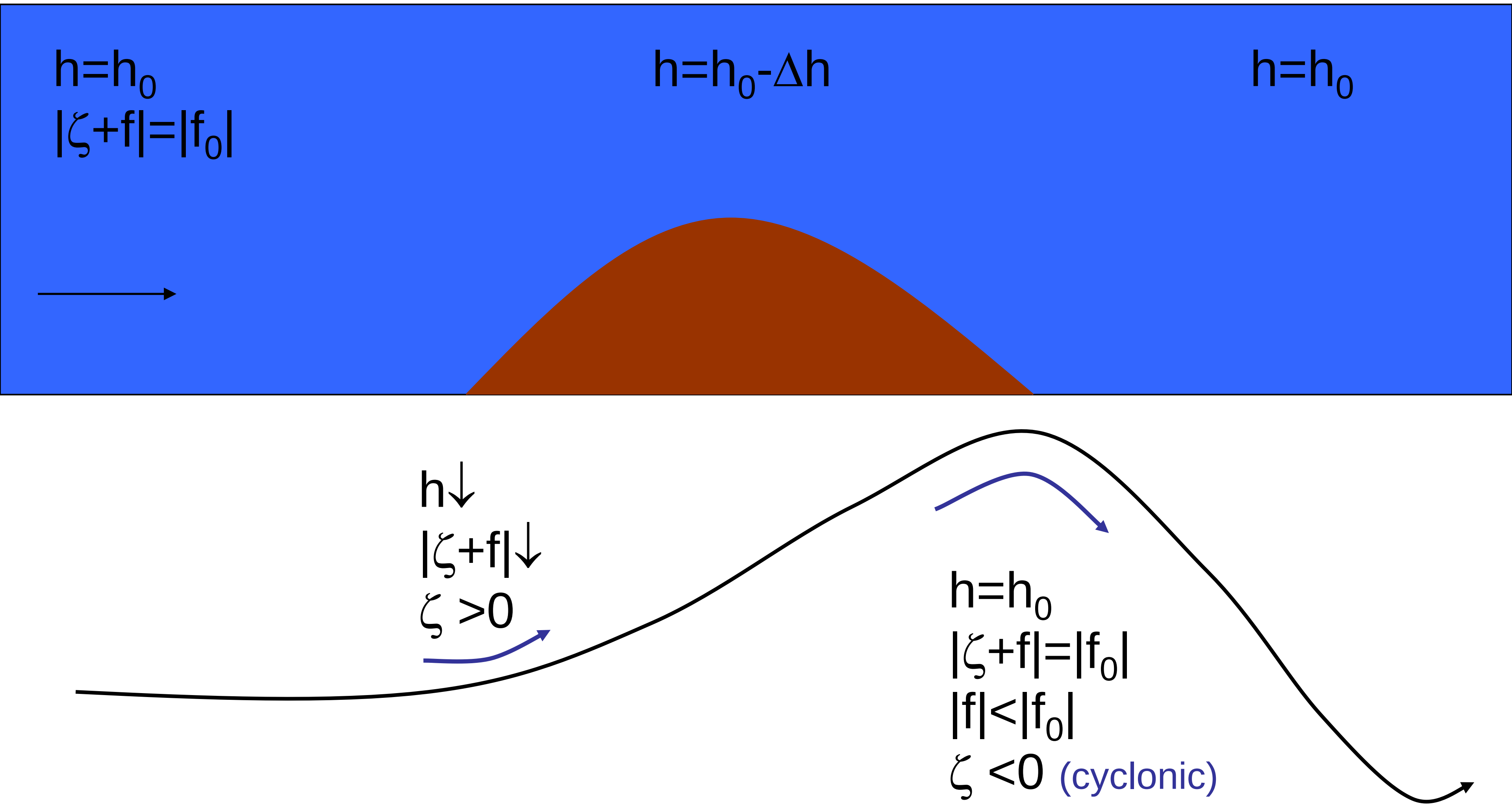
3 - The positive vorticity causes the trajectory to move to the left, i.e., a cyclonic curvature.

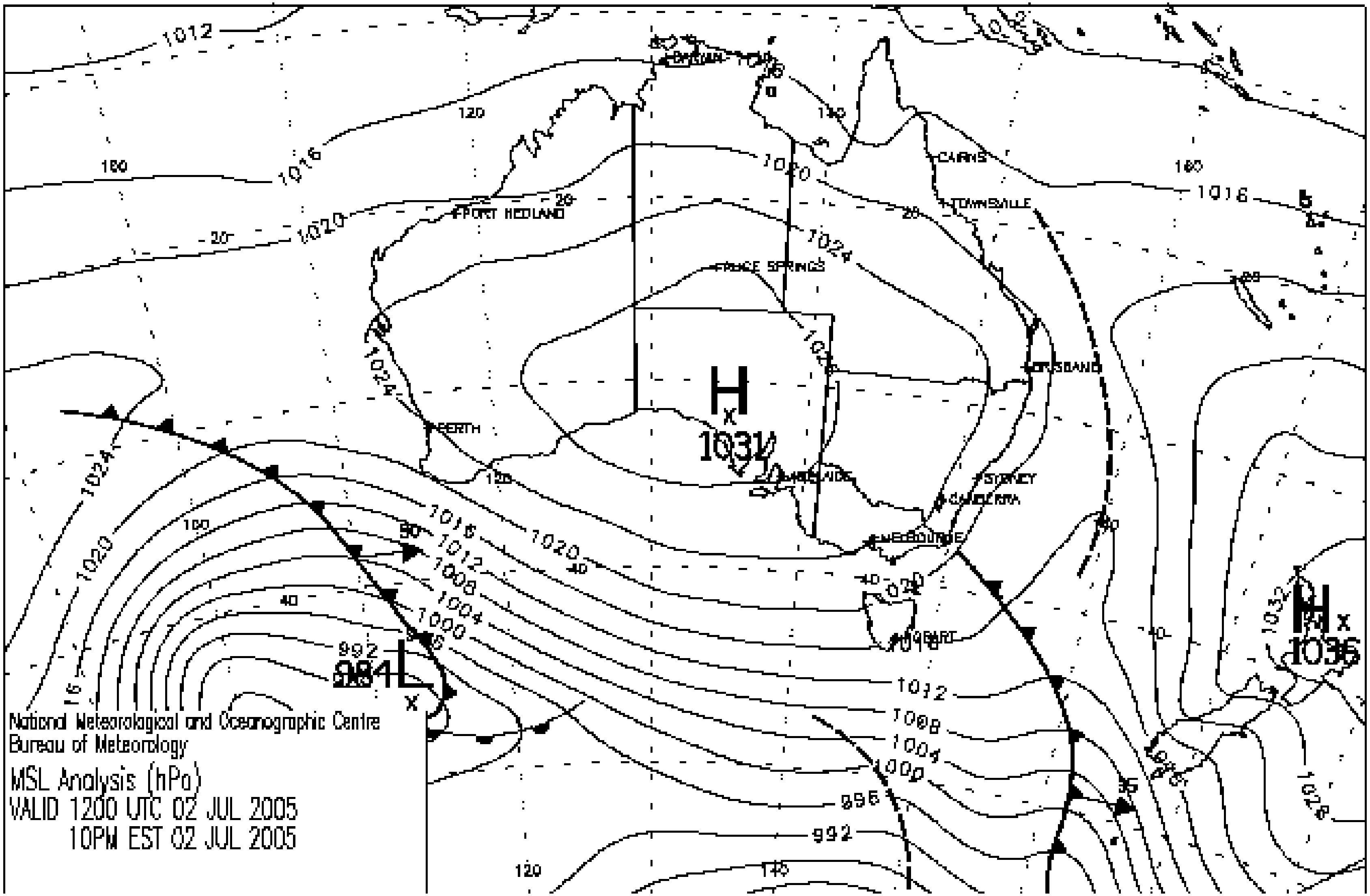
4 - Once the parcel moves back to its original latitude, the northwards trajectory makes it overshoot this latitude and the interplay between the relative and planetary vorticity continue.

5 - The result is that a lee trough usually forms on the downstream side of large mountain ranges. This process can lead to enhanced lee cyclogenesis.

$$\frac{D}{Dt}\left(\frac{\zeta+f}{h}\right)=0$$

Southern hemisphere case ($f<0$)

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MSL Analysis (hPa)
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10PM EST 02 JUL 2005
DARWIN
CAIRNS
TOWNSVILLE
PORT HEDLAND
ALICE SPRINGS
BRISBANE
PERTH
ADELAIDE
SYDNEY
CANBERRA
MELBOURNE
HOBART
H
1031
H
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L
984
1012
1016
1020
1024
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1004
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996
992

Generation of horizontal vorticity by buoyancy anomalies:

Can be directly applied to plume / convective dynamics.

Let's imagine a 2D (x-z) Boussinesq non-hydrostatic system, which is governed by the following equations.

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$b = g\frac{\theta'}{\theta_0} \quad \text{and} \quad \frac{D\theta}{Dt} = 0$$

Let's derive the vorticity equation governing this flow.
Recall, y-vorticity is:

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \qquad *$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b \qquad **$$

To form vorticity equation take: $\frac{\partial}{\partial z}(*) - \frac{\partial}{\partial x}(**)$

$$\frac{\partial^2 u}{\partial z \partial t} - \frac{\partial^2 w}{\partial x \partial t} + u\frac{\partial^2 u}{\partial x \partial z} - u\frac{\partial^2 w}{\partial x^2} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + w\frac{\partial^2 u}{\partial z^2} - w\frac{\partial^2 w}{\partial z \partial x} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} = -\frac{\partial b}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - \frac{\partial w}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = -\frac{\partial b}{\partial x}$$

Recall $\nabla \cdot \underline{u} = 0$ so,

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z} = -\frac{\partial b}{\partial x}$$

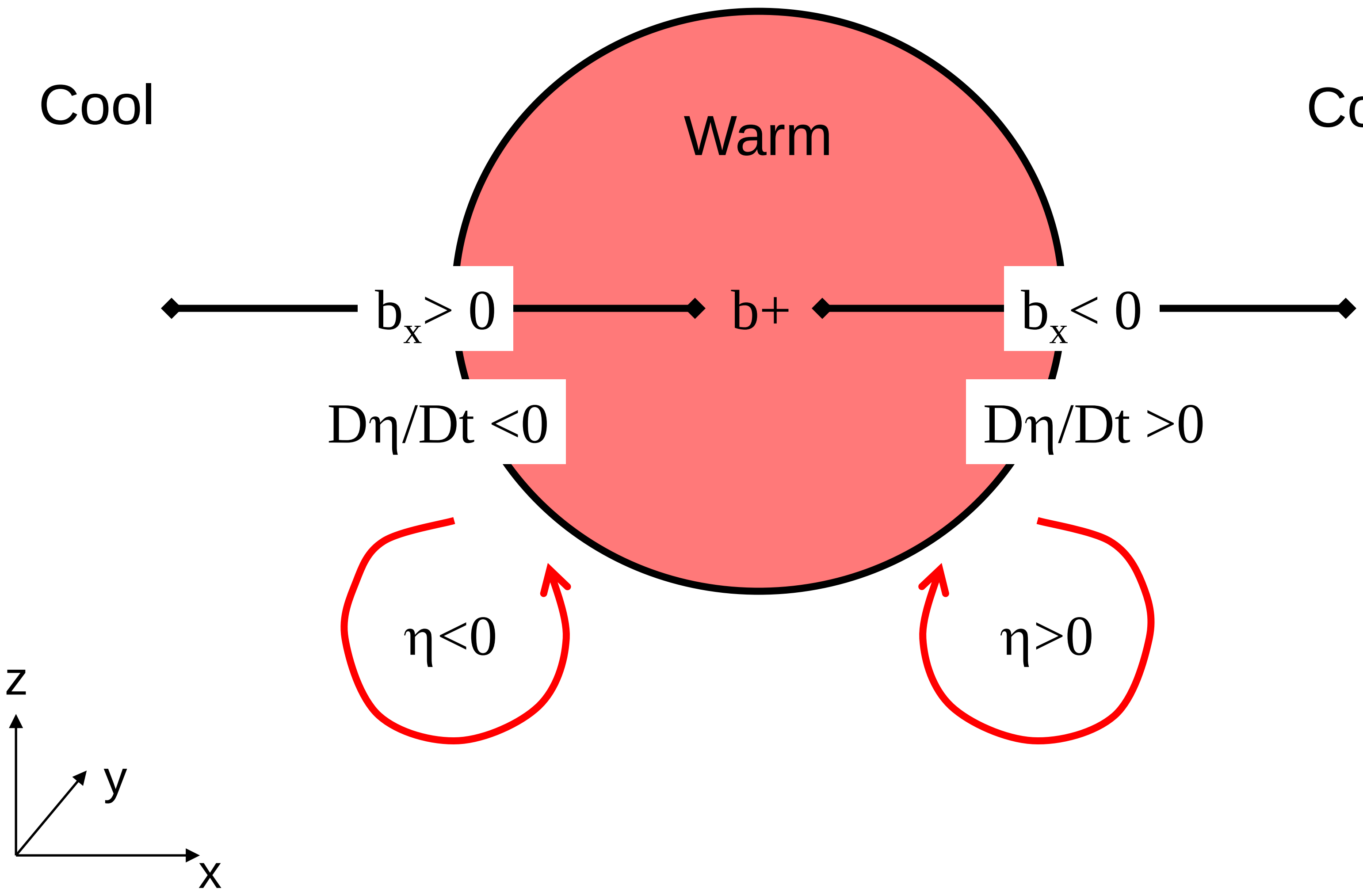
$$\frac{D\eta}{Dt} = -\frac{\partial b}{\partial x}$$

States that horizontal vorticity is conserved if the buoyancy (temperature) is horizontally uniform.

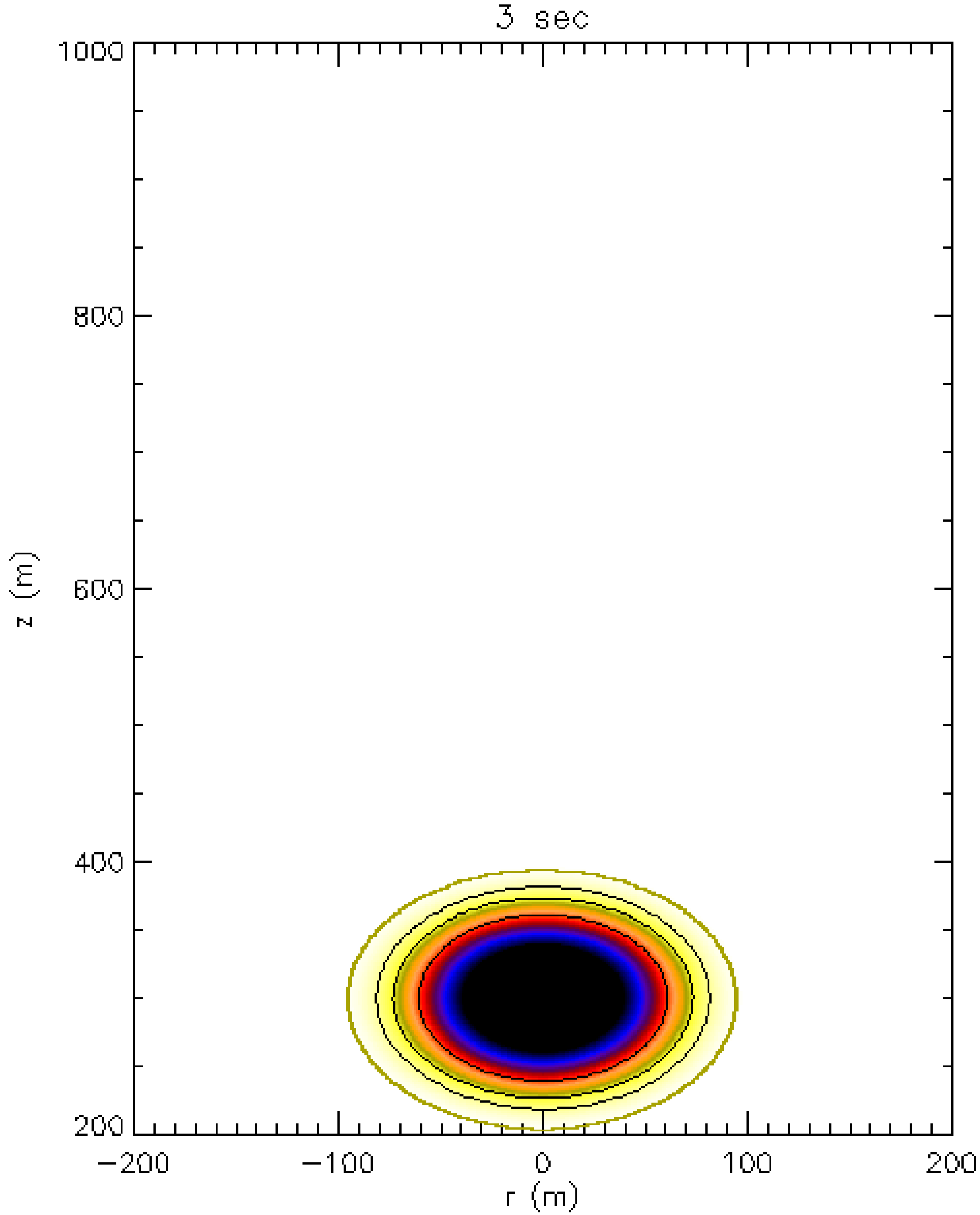
Horizontal vorticity will be generated if there are horizontal gradients in buoyancy (temperature). This is referred to as: *baroclinic vorticity generation*.

Example: a warm temperature anomaly (thermal).

Baroclinic vorticity generation by positive buoyancy anomaly.

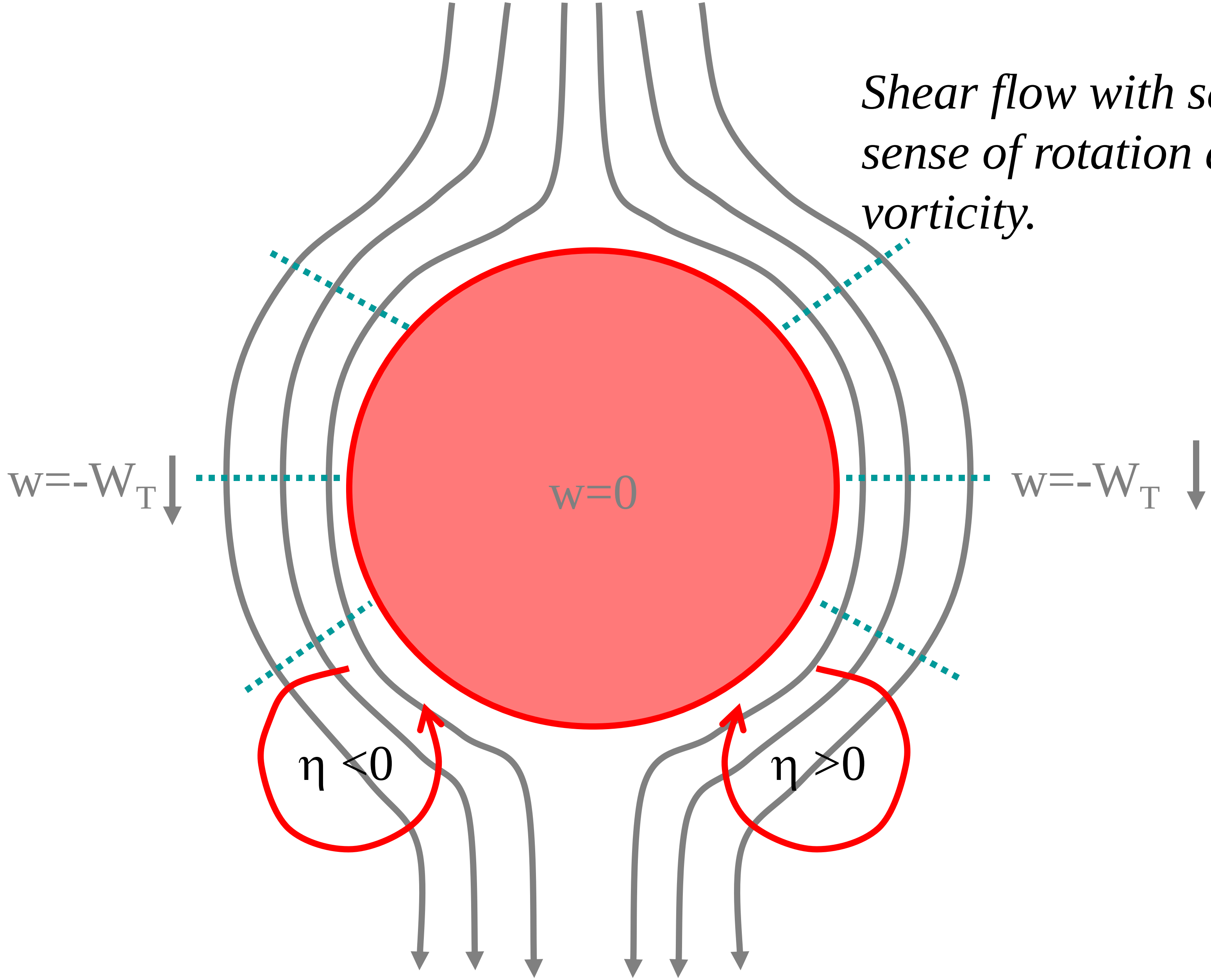


Warm bubble:



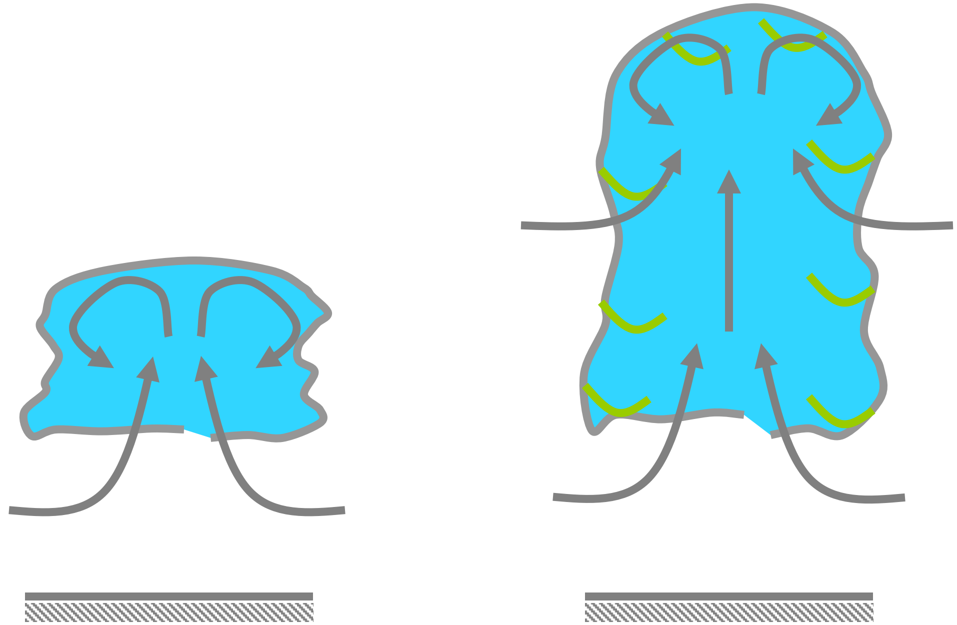
In reference frame moving with thermal (speed W_T).

Shear flow with same sense of rotation as vorticity.

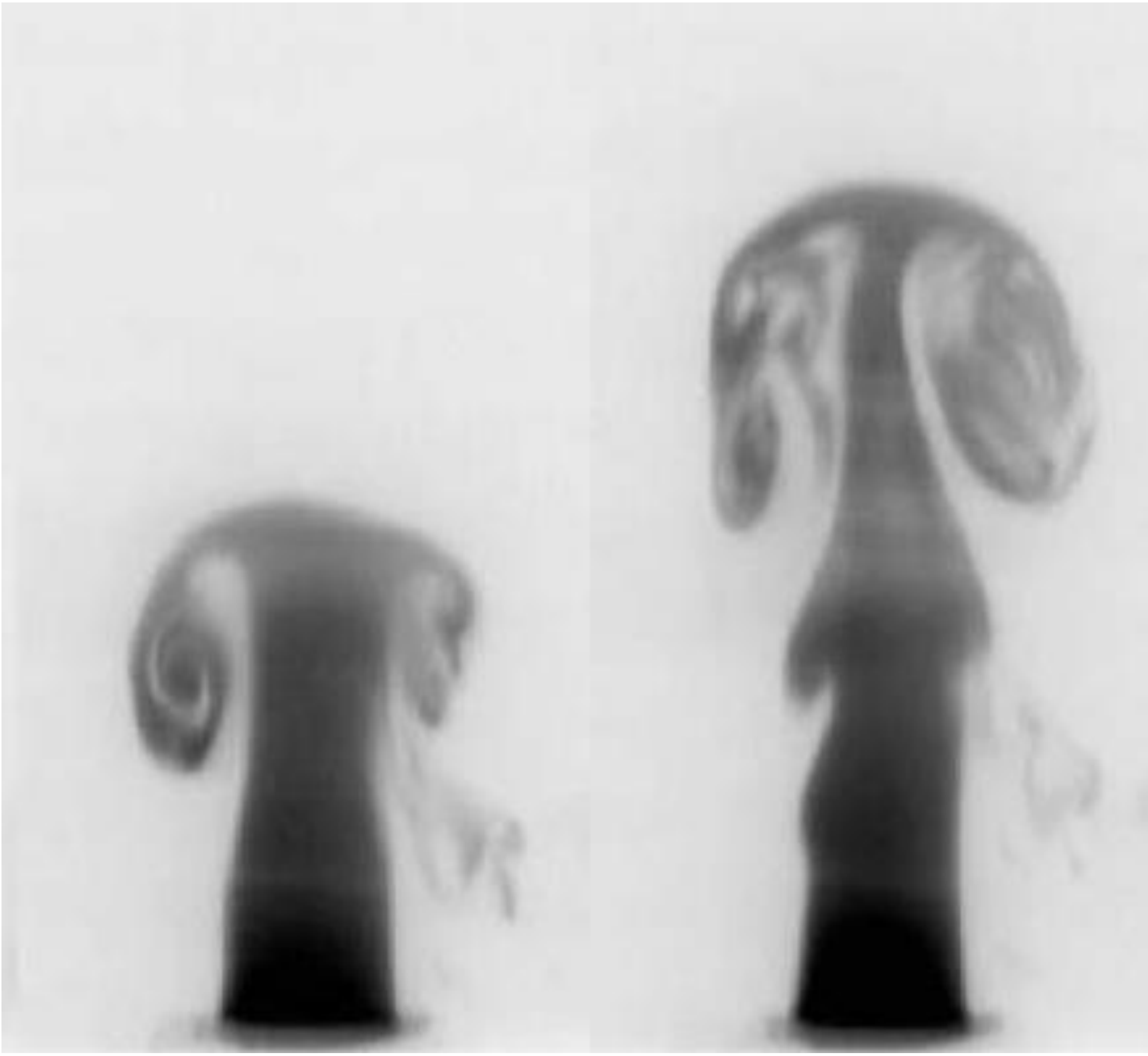


Deep convection.

- Thermal initiates ascent, which leads to condensation
- Latent heat release provides source of buoyancy
- Buoyancy is generated through depth of cloud.
- Individual updrafts develop vorticity field in same way as thermals.
- Conservation of mass and vorticity-induced circulation draws extra fluid into cloud.
- Multiple updrafts often form - complicating flow

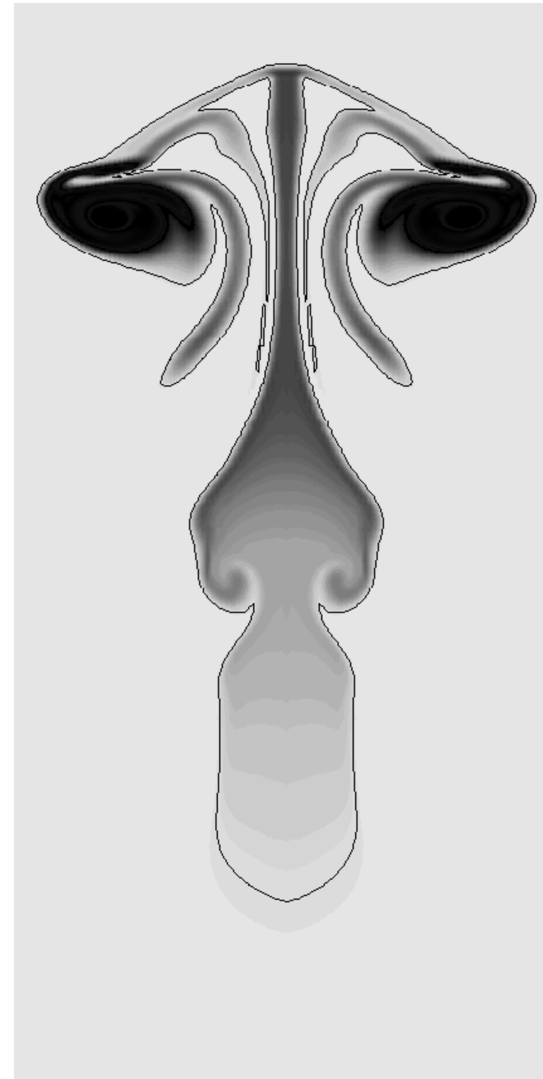


Laboratory Results



Bond & Johari (2005)

Simulation



Pyrocumulus

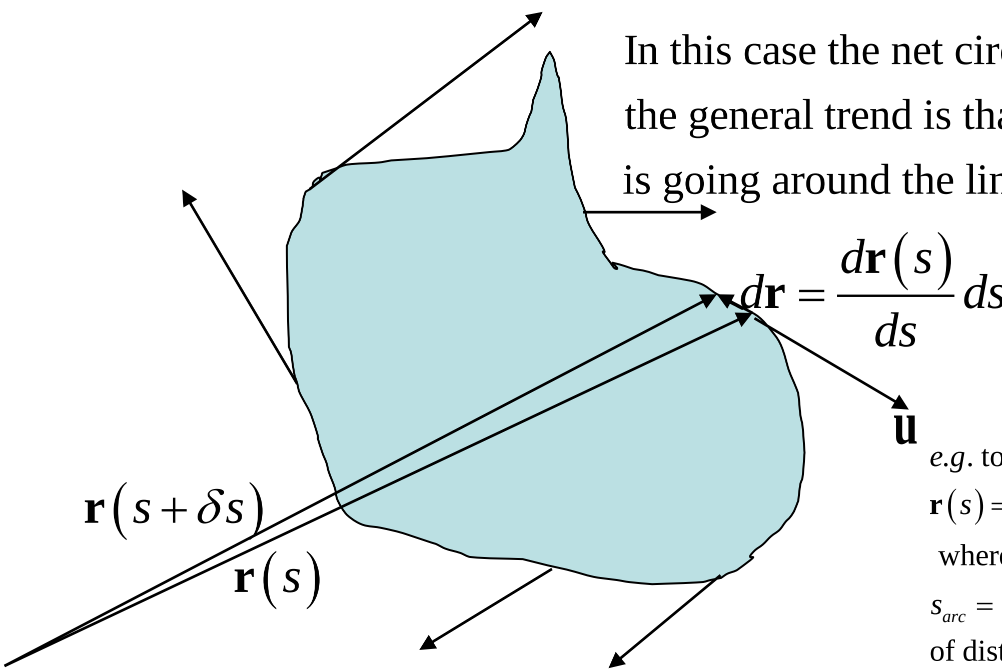
Plume

Source: Univ. of Washington

If the position vector $\mathbf{r}(s)$ traces out a closed loop L as s changes from s_{begin} to s_{end} in an anticlockwise direction then we can define infinitessimal vectors that tangential to the curve using $d\mathbf{r} = \frac{d\mathbf{r}(s)}{ds}ds$. If in addition, there is vector field $\mathbf{u}$ (the velocity field) then we can define, Circulation=$\oint \mathbf{u}.d\mathbf{r}$

In the situation shown here $\mathbf{u}.d\mathbf{r} < 0$ locally

The sum of all the $\mathbf{u}.d\mathbf{r}$ values around the circle is the circulation. In this case the net circulation looks like it will be negative because the general trend is that the part of the flow tangential to the line is going around the line in a clockwise direction.



e.g. to define a circle we would let

$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$

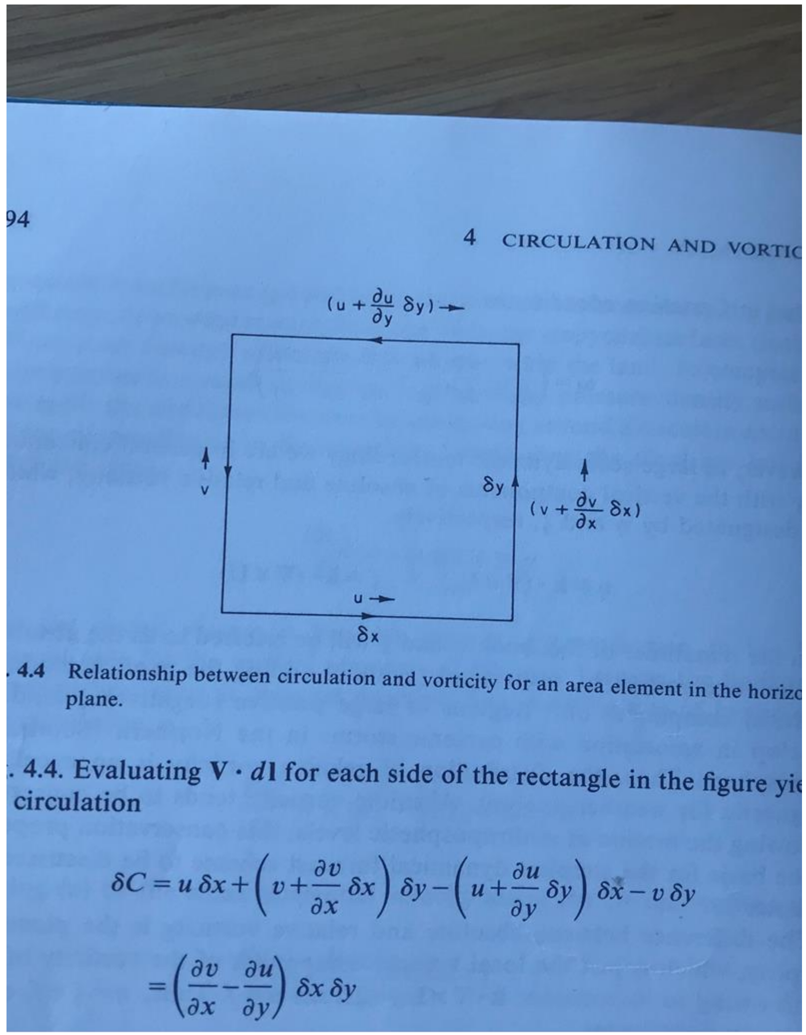
where $x(s) = \cos(s)$ and $y(s) = \sin(s)$ where $s = \theta$

$s_{arc} = r\theta = rs$ hence an equivalent description of the circle in ter,s of distance around the circle s_{arc} is given by

$\mathbf{r}(s_{arc}) = x(s_{arc})\mathbf{i} + y(s_{arc})\mathbf{j}$

where $x(s_{arc}) = \cos(\frac{s_{arc}}{r})$ and $y(s_{arc}) = \sin(\frac{s_{arc}}{r})$

Simple illustration of relationship between circulation and vorticity from Holton.



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$(u + \frac{\partial u}{\partial y}\,\delta y) \rightarrow$

v

δy

$(v + \frac{\partial v}{\partial x}\,\delta x)$

$u \rightarrow$

δx

. 4.4 Relationship between circulation and vorticity for an area element in the horizo plane.

. 4.4. Evaluating $\mathbf{V} \cdot d\mathbf{l}$ for each side of the rectangle in the figure yie circulation

$$\delta C = u\,\delta x + \left(v + \frac{\partial v}{\partial x}\,\delta x\right)\delta y - \left(u + \frac{\partial u}{\partial y}\,\delta y\right)\delta x - v\,\delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x\,\delta y$$

A more in-depth look:
Vorticity and Action at a Distance

If the position vector $\mathbf{r}(\xi,\eta)$ traces out the surface enclosed by the closed loop L then the infinitessimal vector $d\mathbf{A}$ given by

$$d\mathbf{A} = \frac{\partial \mathbf{r}(\xi,\eta)}{\partial \xi} \times \frac{\partial \mathbf{r}(\xi,\eta)}{\partial \eta}\, d\xi d\eta$$

is perpendicular to the surface and has a length equal to the area of the infinitessimal rectangle defined by $\dfrac{\partial \mathbf{r}(\xi,\eta)}{\partial \xi}$ and $\dfrac{\partial \mathbf{r}(\xi,\eta)}{\partial \eta}$.

Provided the coordinates ξ,η are chosen so that $d\mathbf{A}$ points outward from the plane defined by an anticlockwise rotating $\mathbf{r}(s)$ that defines L, then it can be shown that the surface integral

$$\iint_A \nabla \times \mathbf{u}.d\mathbf{A} = \oint_L \mathbf{u}.d\mathbf{r}$$

Hence, the vorticity in the direction normal to the surface at a point (denoted by $(\nabla \times \mathbf{u})_n$) can be defined in terms of the circulation around an infinitessimal area surrounding that point; i.e.

$$(\nabla \times \mathbf{u})_n = \frac{\oint_L \mathbf{u}.d\mathbf{r}}{A} \text{ in the limit as } A -> 0$$

If the position vector $\mathbf{r}(s)$ traces out a closed loop L as s changes from s_{begin} to s_{end} in an anticlockwise direction then we can define infinitessimal vectors that t angential to the curve using $d\mathbf{r} = \dfrac{d\mathbf{r}(s)}{ds}ds$. If in addition, there is vector field $\mathbf{u}$ (the velocity field) then we can define, Circulation=$\oint \mathbf{u}.d\mathbf{r}$

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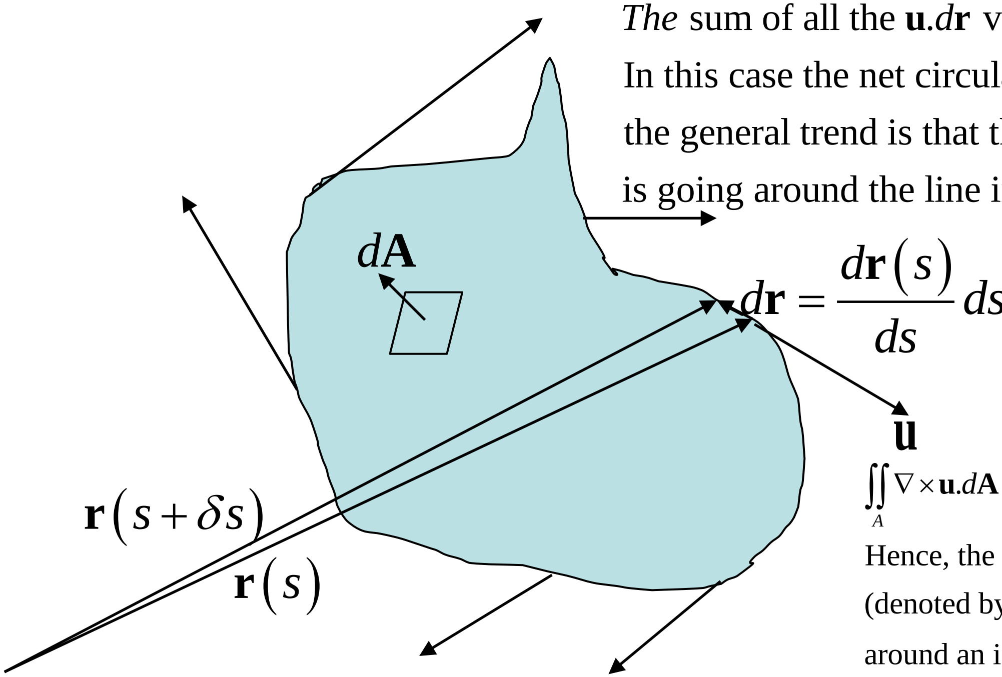
In this case the net circulation looks like it will be negative because the general trend is that the part of the flow tangential to the line is going around the line in a clockwise direction.



$$\iint_A \nabla\times\mathbf{u}.d\mathbf{A} = \oint_L \mathbf{u}.d\mathbf{r}$$

Hence, the vorticity in the direction normal to the surface at a point (denoted by $(\nabla\times\mathbf{u})_n$) can be defined in terms of the circulation around an infinitessimal area surrounding that point; i.e.

$$(\nabla\times\mathbf{u})_n = \frac{\oint_L \mathbf{u}.d\mathbf{r}}{A}$$

Vorticity and Action at a distance: Questions for prac:

Vorticity can be thought of as inducing flow at a distance in exactly the same way as current may be thought of inducing a magnetic field at a distance. To see this, let us first define a streamfunction ψ such that

$$v = \frac{\partial \psi}{\partial x}, \; u = -\frac{\partial \psi}{\partial y}$$

Prac prob 1a: Prove that the vertical component of vorticity ζ is then given by

$$\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$$

It can be shown that in cylindrical coordinates $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$$

Prac prob 1b: If $v = \frac{\partial \psi}{\partial x}, \; u = -\frac{\partial \psi}{\partial y}$, prove that the horizontal divergence of such a wind field has zero divergence. (For that reason, it is called the non-divergent part of the wind field).

Prac prob 2: Prove that if $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and $\psi = C\ln(r)$ then $\nabla^2 \psi = 0$.

Prac prob 3: Compute the tangential wind speed $V = \frac{\partial \psi}{\partial r}$ associated with this stramfunction field.

Prac prob 4: Sketch the wind field

Prac prob 5: Compute the circulation around a circle with radius r that is centered at $r = 0$ using

$$\text{Circulation} = \oint V dr$$

6. Does the Circulation depend on radius?

Comment: Stoke's theorem states that

$$\text{Circulation} = \oint \mathbf{u}.d\mathbf{r} = \oint V dr = \int_A \nabla \times \mathbf{u}.d\mathbf{A} = \int_A \zeta \, dA = \bar{\zeta}\left(\pi r^2\right)$$

Hence, the average vorticity in the circle is given by $\bar{\zeta} = \dfrac{\text{Circulation}}{Area}$

7. Prove that when $\psi = C \ln\left(r\right)$ then $\zeta = \nabla^2 \psi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial \psi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \psi}{\partial \theta^2}$ is equal to zero everywhere except at $r = 0$.

8. Hence, the vorticity must be infinite at $r = 0$ but because the area over which it is infinite is infinitessimal the circulation around it is finite. Discuss the similarity between this situation and that associated with current and magnetic field. Do we indeed have a basis to view vorticity as "acting at a distance" and creating far field flows around itself?

Topic 5 - Major concepts:

- Vorticity
- Conservation of absolute vorticity
- The β - plane approximation
- Mechanism of Rossby wave propagation
- Conservation of potential vorticity
- Baroclinic vorticity generation

Next up: - wave motion - applying these concepts to determine characteristics of atmospheric / oceanic waves.