Block diagonalization of 2nx2n matrix representation of symmetric group N

(Procedure)

1.Vector space

* The 2Nx2N matrix representation of the symmetric group S(n) is intended to act on 2N dimensional vectors space, which we call **V** from now on, where each basis vector is associated with a string of zeros and ones of length N.
* Choose a basis vector, (0,0…,1,..0) where the 1 stays at the ith position, to correspond to the string whose binary value is *i-1.*

V is generated by function VectorSpace[n].

2. Subspaces V{i}:

Separate V into subspaces V{i}, for I ϵ [0,n], so that V{i} is the set of all basis vectors corresponding to strings, which have “i” ones.

e.g. For n=2: V{0}=(1,0,0,0)

V{1}=(0,1,0,0),(0,0,1,0)

V{2}=(0,0,0,1)

V{i} are generated by function SubSpaceVi[V,i]

IMPORTANT!!!

* V{i} are themselves invariant (though not irreducible) subspaces since the SWAP operator does not change the number of 1s in the strings of bits.
* For i<N/2, the decomposition of V{i} to invariant irreducible subspaces will be the same as for V{n-i} (almost obvious).
* **KEY Result** (not obvious, can be proved using the lecture notes)**:** there are distinct irreducible modules W{0}, W{1} …, such that **V{i} ≡W{0}+W{1}+..+W{i},** where “≡**”** denotes “isomorphic to”.

3. Procedure.

1. For given N, generate V and separate it into V{i}
2. Find the irreducible invariant subspaces for each V{i}, using the KEY Result
   1. For i=0, V{i} is 1-dimensional thus its only vector corresponds to 1-d invariant subspace (the trivial representation).
   2. For i=1, using the fact that V{1}≡W{0} + W{1}, and that V{0}≡W{0}, we know that V{1} has two irreducible invariant subspaces, one of which is 1-dimensional (the one corresponding to W{0}).
      1. For a given subspace, 1-dimensional subspace can be found by forming a vector from the sum of all the elements of the subspace.
      2. The invariant subspace corresponding to W{1}, can be found by taking the orthogonal complement to the 1-d subspaces. Mathematica has the nice function Nullspace[], which can find the orthogonal complement to some vector subspace.
   3. For i>2, V{i}= W{0} + … +W{i}
      1. Separate the parts of V{i}, V{i,i}≡W{i} and V{i,i-1}≡W{0}+..W{i-1} in the following way:

V{i,i-1} is obtained by function SubModule

* + - 1. As mentioned before V{i} is the set of vectors corresponding to strings with the same number of 1s.
      2. To obtain basis vectors for V{i,i-1} (call these vectors xk) we need to sum the vectors of V{i}, which bisect the vectors of V{i} w.r.t. to the kth 1 in the strings.
      3. Example:

for n=4, i=2;

x1=v1100+ v1010+ v1001,

x2=v1100+ v0110+ v0101

* + 1. The *n-i* vectors xk now form V{i,i-1} and can be decomposed to irreducible spaces by iterating over, with i->i-1, until i=0 is reached
    2. The V{i,i} subspace is just the orthogonal complement of V{i,i-1}, w.r.t. V{i} and it is irreducible itself.

1. In the process you save your new basis vectors and use to form the transformation matrix.