ECE 485 Homework 5 - Jordan Carlson V00714886

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1 Question 4.2

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

a) $$p(x) = \frac{1}{h_n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{x - x_i}{h_n}\right)^2\right)} \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right)} dx_i$$

$$= \frac{1}{2\pi h_n \sigma} e^{\left(-\frac{1}{2}\left(\frac{x^2}{h_n^2} + \frac{\mu^2}{\sigma^2}\right)\right)} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2}x_i^2\left(\frac{1}{h_n^2}+\frac{1}{\sigma^2}\right) - 2x_i\left(\frac{x}{h_n^2}+\frac{\mu}{\sigma^2}\right)\right)} dx_i$$

$$= \frac{1}{2\pi h_n \sigma} e^{\left(-\frac{1}{2}\left(\frac{x^2}{h_n^2}+\frac{\mu^2}{\sigma^2}\right) + \frac{\left(\frac{x}{h_n^2}+\frac{\mu}{\sigma^2}\right)^2}{h_n^2\sigma^2/(h_n^2+\sigma^2)}\right)} \int_{-\infty}^{\infty} e^{\frac{\left(-\frac{1}{2}\left(x_i - \left(\frac{h_n^2\sigma^2}{h_n^2+\sigma^2}\right)\left(\frac{x}{h_n^2}+\frac{\mu}{\sigma^2}\right)\right)^2\right)}{h_n^2\sigma^2/(h_n^2+\sigma^2)}} dx_i$$

$$= \frac{\sqrt{2\pi \frac{h_n^2\sigma^2}{h_n^2+\sigma^2}}}{2\pi h_n \sigma} e^{\left(-\frac{1}{2}\left(\frac{x^2}{h_n^2}+\frac{\mu^2}{\sigma^2}\right) + \frac{1}{2}\left(\frac{x}{h_n^2}+\frac{\mu}{\sigma^2}\right)^2 \Big/ \left(h_n^2\sigma^2/(h_n^2+\sigma^2)\right)\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{h_n^2+\sigma^2}} e^{-\frac{1}{2}\left(\frac{x^2 h_n^2}{(h_n^2+\sigma^2)h_n^2} + \frac{\mu^2\sigma^2}{\sigma^2(h_n^2+\sigma^2)} - \frac{2x\mu}{h_n^2+\sigma^2}\right)}$$

$$\boxed{= \frac{1}{\sqrt{2\pi}\sqrt{h_n^2+\sigma^2}} e^{-\frac{1}{2}\left(\frac{(x-\mu)^2}{(h_n^2+\sigma^2)}\right)} = \bar{p}_n(x) \sim N(\mu, \sigma^2 + h_n^2)}$$

b)

$$Var[p_n(x)] = \frac{1}{nh_n^2}\left(E[X^2] - E[X]^2\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\left(-\left(\frac{x-x_i}{h_n}\right)^2\right)} \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right)} dx_i - \frac{h_n^2}{nh_n^2} \frac{1}{\sqrt{2\pi}\sqrt{h_n^2+\sigma^2}} e^{\left(-\frac{1}{2}\frac{(x-\mu)^2}{(h_n^2+\sigma^2)}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{h_n^2+\sigma^2}} e^{\left(-\frac{1}{2}\frac{(x-\mu)^2}{h_n^2/2+\sigma^2}\right)} \frac{1}{nh_n} \frac{1}{2\sqrt{\pi}} - \frac{h_n}{nh_n\sqrt{2\pi}\sigma} e^{\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)}$$

$$\simeq \frac{1}{nh_n 2\sqrt{\pi}\sqrt{2\pi}\sigma} p(x) - 0$$ for small h_n, i.e. $\boxed{Var[p_n(x)] \simeq \frac{p(x)}{2nh_n\sqrt{\pi}}}$

c)

$$p(x) - \bar{p}_n(x) \simeq \frac{1}{2}\left(\frac{h_n}{\sigma}\right)^2\left[1 - \left(\frac{x-\mu}{\sigma}\right)^2\right] p(x)$$

$$p(x) - \bar{p}_n(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)} - \frac{1}{\sqrt{2\pi}\sqrt{h_n^2+\sigma^2}} e^{-\frac{1}{2}\left(\frac{(x-\mu)^2}{h_n^2+\sigma^2}\right)}$$

$$= p(x)\left(1 - \frac{1}{\sqrt{1+(h_n/\sigma^2)}} e^{\frac{h_n^2(x-\mu)^2}{2\sigma^2(h_n^2+\sigma^2)}}\right)$$

$$\simeq p(x)\left(1 - \left(1 - \frac{1}{2}\left(\frac{h_n}{\sigma}\right)^2\right)\left(1 + \frac{h_n^2}{2\sigma^2}\frac{(x-\mu)^2}{(h^2+\sigma^2)}\right)\right)$$

$$\simeq p(x)\frac{1}{2}\left(\frac{h_n}{\sigma}\right)^2\left(1 - \frac{(x-\mu)^2}{h_n^2+\sigma^2}\right) \simeq \boxed{\frac{1}{2}\left(\frac{h_n}{\sigma}\right)^2\left(1 - \left(\frac{x-\mu}{\sigma}\right)^2\right)p(x)}$$

4.6

a) $P_n(e) = \sum \text{error} = \sum p(\omega_1 | \omega_2) + \sum p(\omega_2 | \omega_1)$

for all j in n to $(k-1)/2$ from $j=0$,

with a multiplier of $2^{(-j)} 2^{(j-n)} (\sim B(n, 1/2))$

i.e. $\sum_{j=0}^{(k-1)/n} \binom{n}{j} 2^{-j} 2^{j} 2^{-n} = \boxed{\frac{1}{2^n} \sum^{(k-1)/2} \binom{n}{j}}$

b) $\frac{1}{2^n} \sum_{j=0}^{(k-1)/2} \binom{n}{j}$ is increasing for all $k > 1$

$P_n(e;0) = P_n(e;1) = 0$, $\boxed{P_n(e; >1) = \frac{1}{2^n} \sum_{j=0}^{((k>1)-1/2)/2} \binom{n}{j} > 0}$

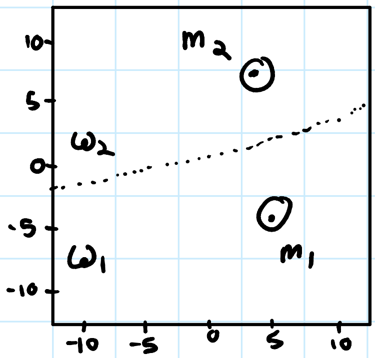
c) $P_n(e) = P_n(B(n, 1/2)) \rightarrow P_n(e) = (x_i)^k (1 - x_i)^{n-k}$

as $x_i < 1$, $x_k < (k-1)/2$, $P_n(e) = (x_i)^{\frac{(a/\sqrt{n}-1)}{2}} (1-x_i)^{1-(a/\sqrt{n}-1)/2}$

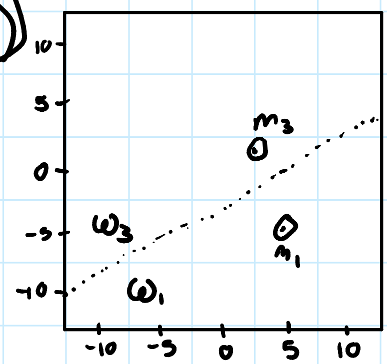
so $\boxed{\text{as } n \to \infty,\ P_n(e) \text{ decreases to } 0}$

4.9

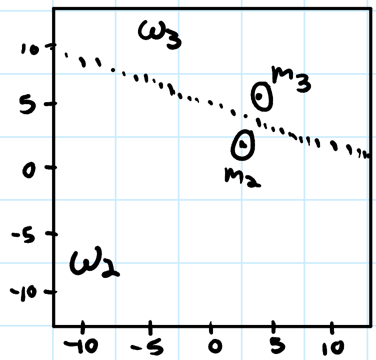
a)



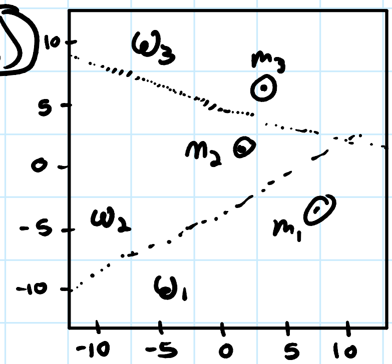
b)



c)



d)



4.11

a) Sample size for the same density in $\boxed{R^d = n_1^d}$, so for $n_1 = 100$,

$n_{20} = 100^{20} \rightarrow \boxed{n_2 = 10^{40}}$

b) Given uniform distribution of the points, the inter-point euclidean distances <u>must also be roughly uniform</u>. The few points still have a large radii as <u>euclidian distance = 5^d</u>, which will result in large distances given the 'd' exponent.

c) $l_d(P) = P^{1/d}$

$l_5(0.01) = .01^{1/5} = .391$

$l_5(0.1) = .1^{1/5} = .725$

$l_{20}(.01) = .01^{1/20} = .861$

$l_{20}(.1) = .1^{1/20} = .861$

(d) $L_\infty(x,y) = \lim_{k\to\infty} \sqrt[1/k]{\sum_{i=1}^{d} |x_i - y_i|^k}$

$= \max(x_i - y_i)$ for all d.

e.g. for $[0,0,\dots,0,1]^d$, $L_\infty = 1$, but the axes will be nearly aligned for the most parallel axes as $k \to \infty$. As each d brings an outlier point, the L_2 from the new point may be effective to determine the relationship between points in d, the L_∞ metric likely gives a more useful figure to use for data analysis and implementation.

4.12

(a) $E(f(x) - \hat{f}(x))^2 = E\left(\sum_{j=1}^{d} a_j x_j - \sum_{j=1}^{d} \hat{a}_j x_j\right)^2$

$= E\left(\sum_{j=1}^{d} a_j x_j - \sum_{j=1}^{d}\left(\underset{a_j}{\operatorname{argmin}} \sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{d} a_j x_{ij}\right)^2\right) x_j\right)^2$

projecting the gaussian's n dimensional parameters to d dimensions as

$y = f(x) + N(0,\sigma^2)^n$,

$E(f(x) - \hat{f}(x))^d) \simeq Var(N(0,\sigma^2)^n)$

$E(f(x) - \hat{f}(x)^2)^d = n\,E(f(x) - \hat{f}(x)^2) = d\sigma^2$

$\boxed{E(f(x) - \hat{f}(x))^2 = d\sigma^2 / M}$

(b) Similarly but with function $\hat{f}(x) = \sum_{m=1}^{M} \hat{a}_m B_m(x)$

$$E\left(f(x) - \hat{f}(x)\right)^2 = E\left(\sum_{m=1}^{M} a_m B_m(x) - \sum_{m=1}^{M} \hat{a}_m B_m(x)\right)^2$$

$$= E\left(\sum_{m=1}^{M} a_m B_m(x) - \sum_{m=1}^{M} \left(\operatorname{argmin}_{a_i} \sum_{m=1}^{M} \left(y_i - \sum_{m=1}^{q} a_m B_m(x)\right)\right) x_m\right)^2$$

projecting the gaussian's n dimensional parameters to d dimensions as

$$y = f(x) + N(0, \sigma^2)^n,$$

$$E(f(x) - \hat{f}(x))^{2n}) \simeq Var(N(0, \sigma^2)^n)$$

$$E(f(x) - \hat{f}(x)^2)^n = n\,E(f(x) - \hat{f}(x)^2) = M\sigma^2$$

$$\boxed{E(f(x) - \hat{f}(x))^2 = M\sigma^2/n}$$

Computer exercise 2

Using samples 1 to 10 of $\omega_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \omega_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \omega_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\phi\left(\frac{(x - x_i)}{h}\right) \propto e^{-\frac{(x - x_i)^t (x - x_i)}{2h^2}}$$

$$= \frac{1}{2\pi^{d/2} h_n^d} e^{-x^2/2h_n^2} \qquad p_n(x) = \frac{1}{n} \sum_{i=1}^{n} \delta(x - x_i)$$

Using MatLab's Xtrain function

at $h=1$,

$$\begin{bmatrix} (0.50) \\ (1.0) \\ (0.0) \end{bmatrix} = \omega_2, \quad \begin{bmatrix} .31 \\ 1.51 \\ -.5 \end{bmatrix} = \omega_2, \quad \begin{bmatrix} -.3 \\ .44 \\ -.1 \end{bmatrix} = \omega_2$$

at $h=.1$,

$$\begin{bmatrix} 0.50 \\ 1.0 \\ 0.0 \end{bmatrix} = \omega_2, \quad \begin{bmatrix} .31 \\ 1.51 \\ -.5 \end{bmatrix} = \omega_2 \quad \begin{bmatrix} .3 \\ .44 \\ -.1 \end{bmatrix} = \omega_2$$