H01e Lab Report

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Abstract

This study investigates the dynamics of a pendulum bob within an accelerated frame of reference. The measured eigenfrequency of the apparatus was $(0.65755 \pm 0.00987)Hz$, closely aligning with theoretical predictions of $(0.67003 \pm 0.00043)Hz$, yielding an absolute error of $(0.01248 \pm 0.00988)Hz$. The damping constant *delta* was determined to be $(0.002685 \pm 0.005883)Hz$, however with a large relative uncertainty of 2.191, primarily due to significant amplification of uncertainty from the methodology chosen. The calculated linear accelerations of the pendulum bob in the accelerated frame of reference for the y and z axes were consistent with the measured data when considering unaccounted for factors. However, discrepancies were noted in the acceleration results for these axes, attributed to errors in mathematical modeling.

1 Introduction

The aim of this experiment is to record the instantaneous angular velocity, acceleration and linear acceleration (acceleration without $\vec{g}$) of a pendulum bob in its accelerated frame of reference. With this data further analysis will be performed to calculate the measured and theoretical oscillation frequency of the pendulum bob, it's damping constant and finally a comparison of the measured acceleration and linear acceleration with a computed recreation from measured angular velocity. The significance of this experiment lies in using the measured parameters to describe the simple pendulum's motion and behaviour.

When discussing a pendulum there is often two cases of interest. The mathematical(or ideal) pendulum in which additional forces such as friction are ignored, and the real pendulum where friction forces give rise to damping constant that brings the system towards a point of equilibrium over time [1].

2 Theoretical Basis

2.1 Ideal Pendulum

The equation of motion of for an ideal pendulum is of the form [2]:

$$\ddot{\theta} + \frac{mgl}{J_p}\sin(\theta) = \ddot{\theta} + \omega_0^2 \sin(\theta) = 0 \tag{1}$$

Where:

$$\begin{aligned} \theta &= \text{Pendulum Angle}(rads) \\ \omega_0 &= \text{Initial Angular acceleration}(rads/s) \\ m &= \text{Mass}(Kg) \\ g &= \text{Gravity}(m/s^2\ ) \\ l &= \text{Pendulum arm}(m\ ) \\ J_p &= \text{Pendulum Bob moment of inertia } (Kgm^2\ ) \end{aligned}$$

Jp is defined as:

$$J_p = \frac{1}{12}m(h_b^2 + d_b^2) + m(l + \frac{d_b}{2})^2 \tag{2}$$

Where:

$$\begin{aligned} h_b &= \text{Bob height}(m) \\ d_b &= \text{Bob depth } (m\ ) \end{aligned}$$

Through Newton's second law and the small angles approximation the solution of Eq.1 is:

$$\theta = \theta_0 \cos(\omega_0 t + \beta) \tag{3}$$

Where:

$$\theta_0 = \text{Initial pendulum angle}(rads)$$

That describes the position of the pendulum bob at any point in time. Eqs.1 and 3 along with the systems initial conditions can be used to further determine characteristics of the pendulum bob's motion such as oscillation eigenfrequency. [1][2]:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgl}{J_p}} \tag{4}$$

2.2 Real Pendulum

When considering frictional forces and the damping of the pendulum over time Eq.1 now becomes[2]:

$$\ddot{\theta} + 2\delta\dot{\theta} + \omega_0^2\theta = 0 \tag{5}$$

Where:

$$\delta = \text{Damping Constant}(Hz)$$

The addition of the damping term leads to a solution of:

$$\theta = \theta_0 e^{-\delta t}\cos(2\pi f_d t + \beta) \tag{6}$$

With f_d begin the eigenfrequency of a real pendulum subject to damping [2].

$$f_d = \frac{\sqrt{\omega_0^2 - \delta^2}}{2\pi} \tag{7}$$

2.2.1 Transformation into an accelerated frame of reference

An objective of the this experiment was to calculate and compare the acceleration and linear acceleration of the pendulum bob with measured results. As such a change of reference from a stationary point of reference to the Pendulum bob's accelerated frame was required [2].

$$\vec{a} = \vec{g} + \vec{a_{cf}} + \vec{a_\alpha} = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - \vec{\alpha} \times \vec{r} \tag{8}$$

Where:

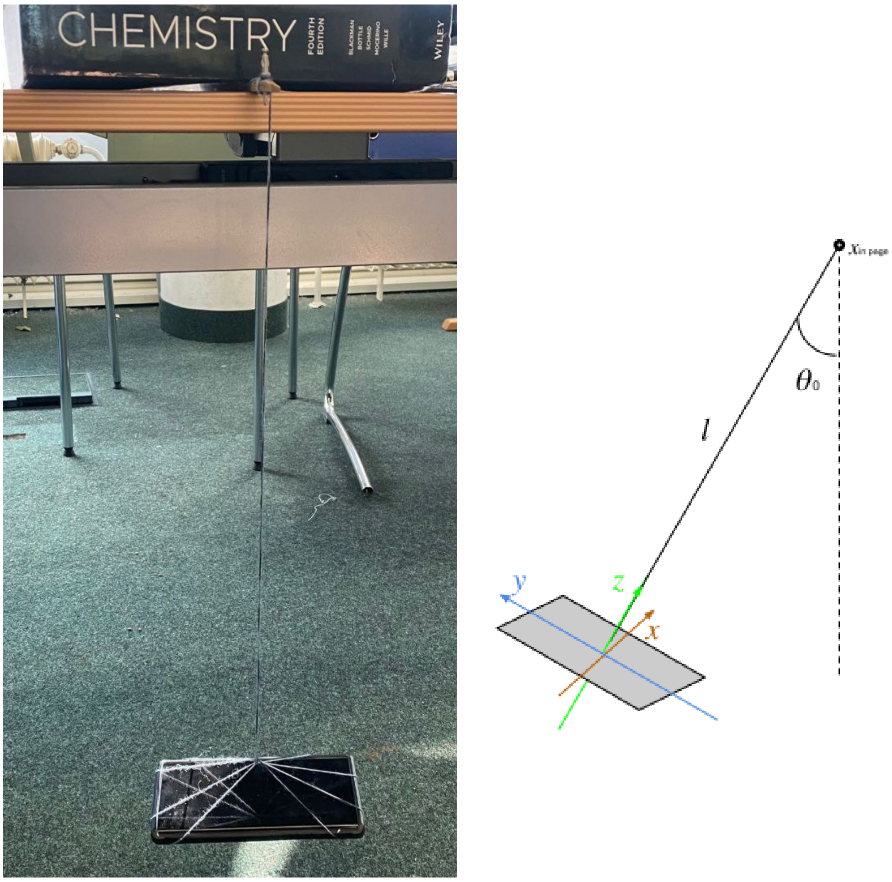
$$\begin{aligned}
\vec{a} &= \text{Bob acceleration}(m/s^2) \\
\vec{a_{cf}} &= \text{Centrifugal acceleration}(m/s^2) \\
\vec{a_\alpha} &= \text{Euler Acceleration}(m/s^2) \\
\vec{\omega} &= \text{Angular velocity of Bob}(rad/s^2) \\
\vec{\alpha} &= \text{Angular acceleration}(rads/s^2)
\end{aligned}$$

Eq. 8 takes into account the acceleration due to gravity $\vec{g}$, centrifugal acceleration $\vec{a_{cf}}$, and Euler acceleration $\vec{a_\alpha}$ and can be determined from $\vec{g}$, $\vec{\omega}$ and $\vec{\alpha}$. where $\vec{\alpha}$ can be determined using Eq. 6

3 Experimental Apparatus and Methods

3.1 Data Gathering - Task 1

Fig 1. shows the apparatus used to record the angular velocity, acceleration and linear acceleration of a pendulum bob, in motion. A smartphone of mass $(0.227 \pm 0.0005)kg$ was used in tandem with the PhyPox™ application and a string cradle with pendulum arm situated above the center of mass to instantaneously record data.



(a) Experimental Apparatus (b) Labeled Diagram

Figure 1: Experimental Setup

The apparatus has a pendulum arm length(suspension point to phone's center of mass) of $(0.540 \pm 0.0005)m$, the Pendulum bob was dropped at the a initial angle $(30 \pm 1)°$, and remained in motion for at least t=120s seconds over a total of 5 trials.

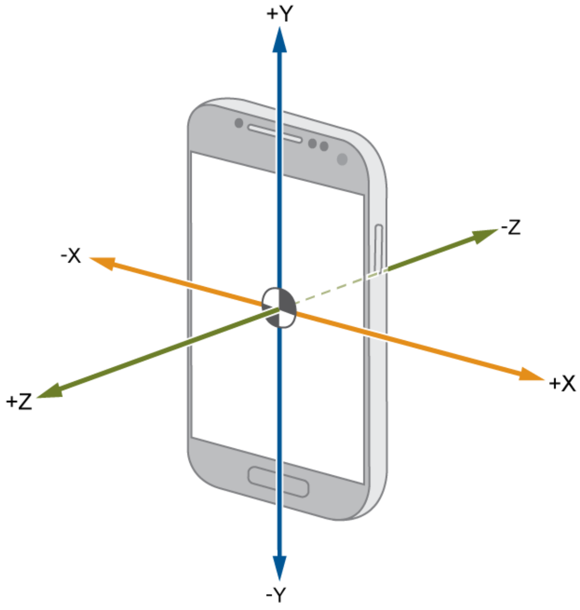


Figure 2: Smartphone Orientation

The Orientation of smartphone in accordance with the sensors used Fig 1.2 was such that the pendulum arm always remained situated through the z-axis and the oscillation of the phone was about the x-axis. Resulting in a $\vec{g}$ and calculated $\vec{r}$, $\vec{\omega}$ and $\vec{\alpha}$ of:

$$\vec{g} = \begin{pmatrix} 0 \\ g\sin(\theta) \\ g\cos(\theta) \end{pmatrix}, \vec{r} = \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix}, \quad \vec{\omega} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$$

3.2 Eigenfrequencies - Task 2

Using the $\vec{\omega_x}$ data collected in Section 3.1 a Fourier transform was performed as such a transformation highlights the dominant oscillation frequencies of the pendulum in motion. The dominant frequencies f_0^i of each trial were then averaged to determine f_0. This value was then compared with the theoretical eigenfrequency determined using Eq. 4.

It should also be noted that only one Trial, (Trial 5) was selected for further analysis in this paper to avoid redundancy of information and repetitive figures.

3.3 Damping Constant - Task 3

The damping constant δ was determined by taking a random sample point of $\omega_x(t)$, about the rotation axis Fig. 1(b) data collected. The sample point was used in the derivative of Eq. 6 in conjunction with Eq. 7 to infer the δ required for oscillation damping present.

3.4 Measured and calculated accelerations - Task 4

Through use of Eq. 8 and the calculated vectors found in Section 3.1 the acceleration $\vec{a}$ and linear acceleration $\vec{a_l}$ vectors were determined to be:

$$\vec{a} = \begin{pmatrix} 0 \\ g\sin(\theta) + \alpha l \\ g\cos(\theta) + \omega^2 l \end{pmatrix}, \tag{9}$$

$$\vec{a_l} = \begin{pmatrix} 0 \\ \alpha l \\ \omega^2 l \end{pmatrix}, \tag{10}$$

4 Results and Analysis

4.1 Data Gathering - Task 1

For all Trials to minimize the interference and noise from the dropping pendulum bob the data was trimmed to 35 seconds into the pendulums motion, while also considering initial variables.

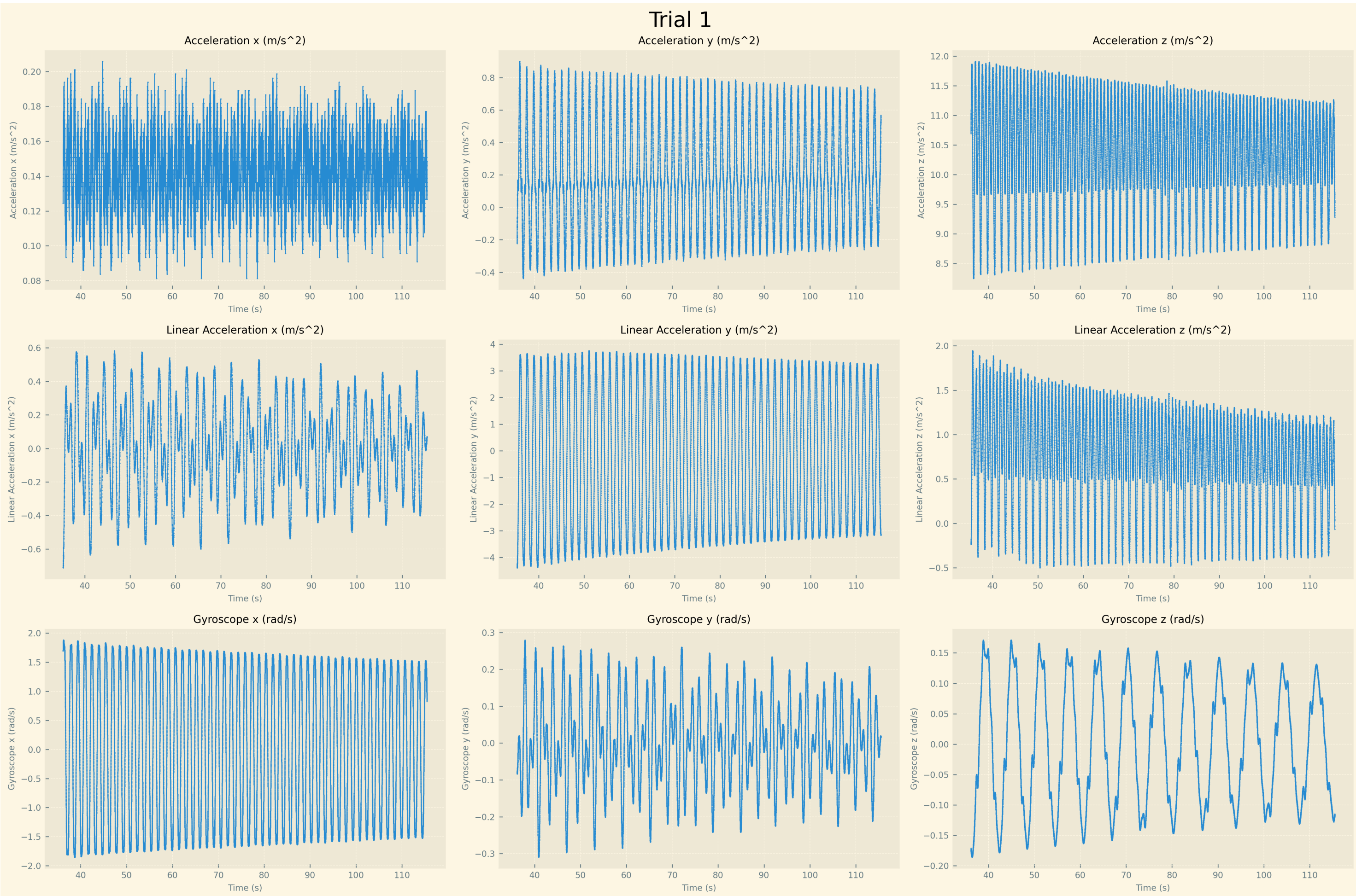


Figure 3: Raw Data - Trial 1

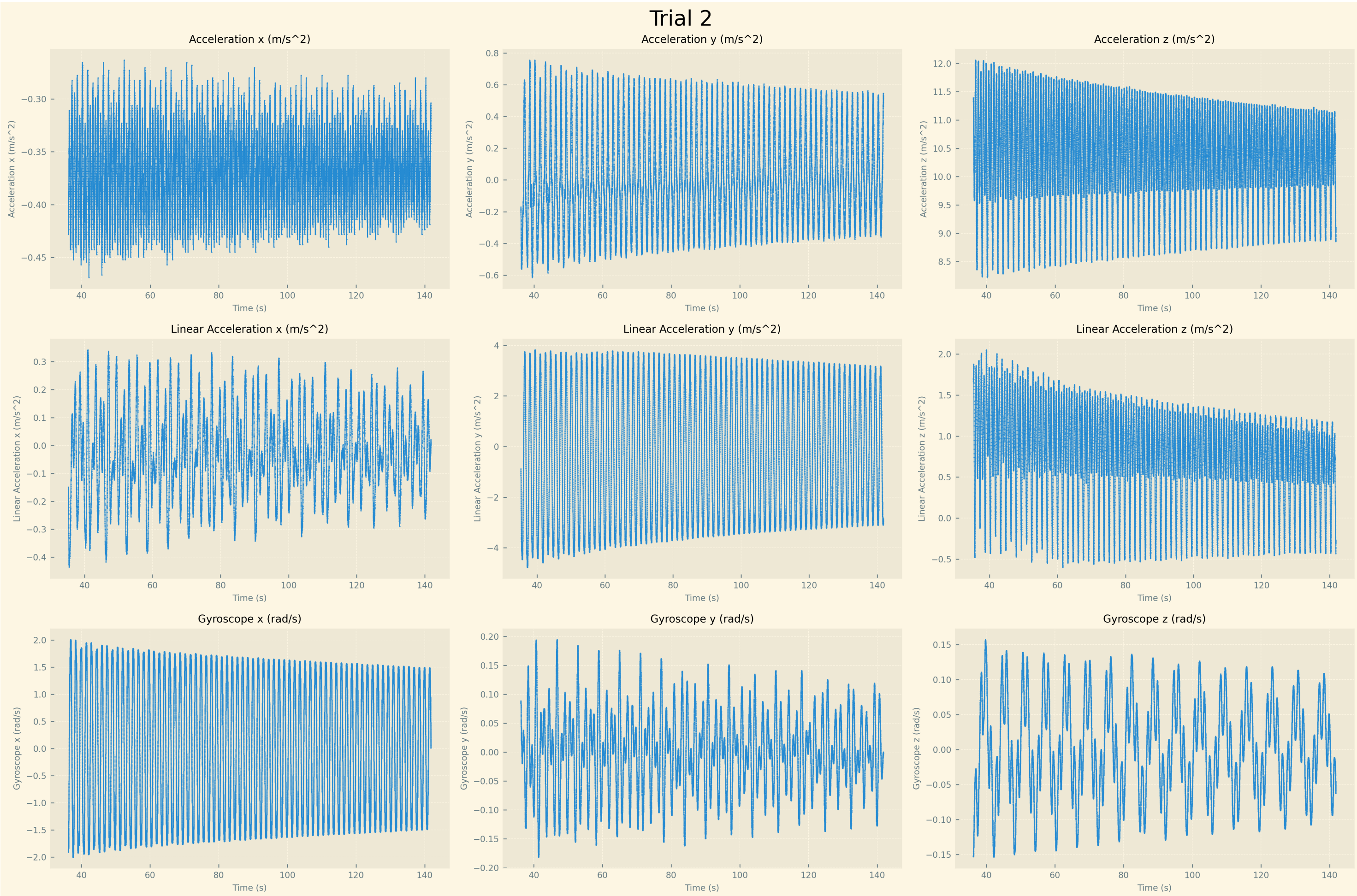


Figure 4: Raw Data - Trial 2

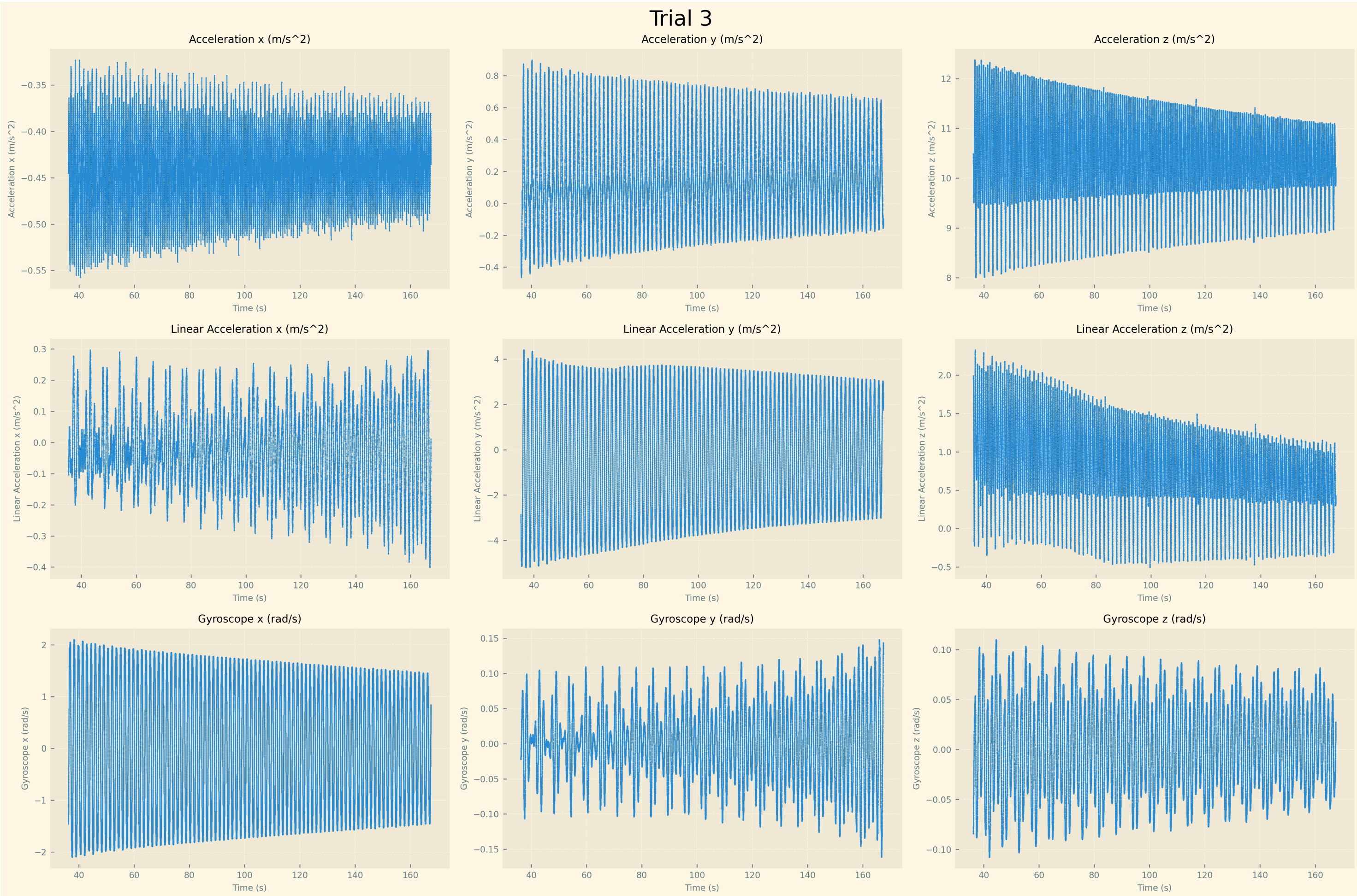


Figure 5: Raw Data - Trial 3

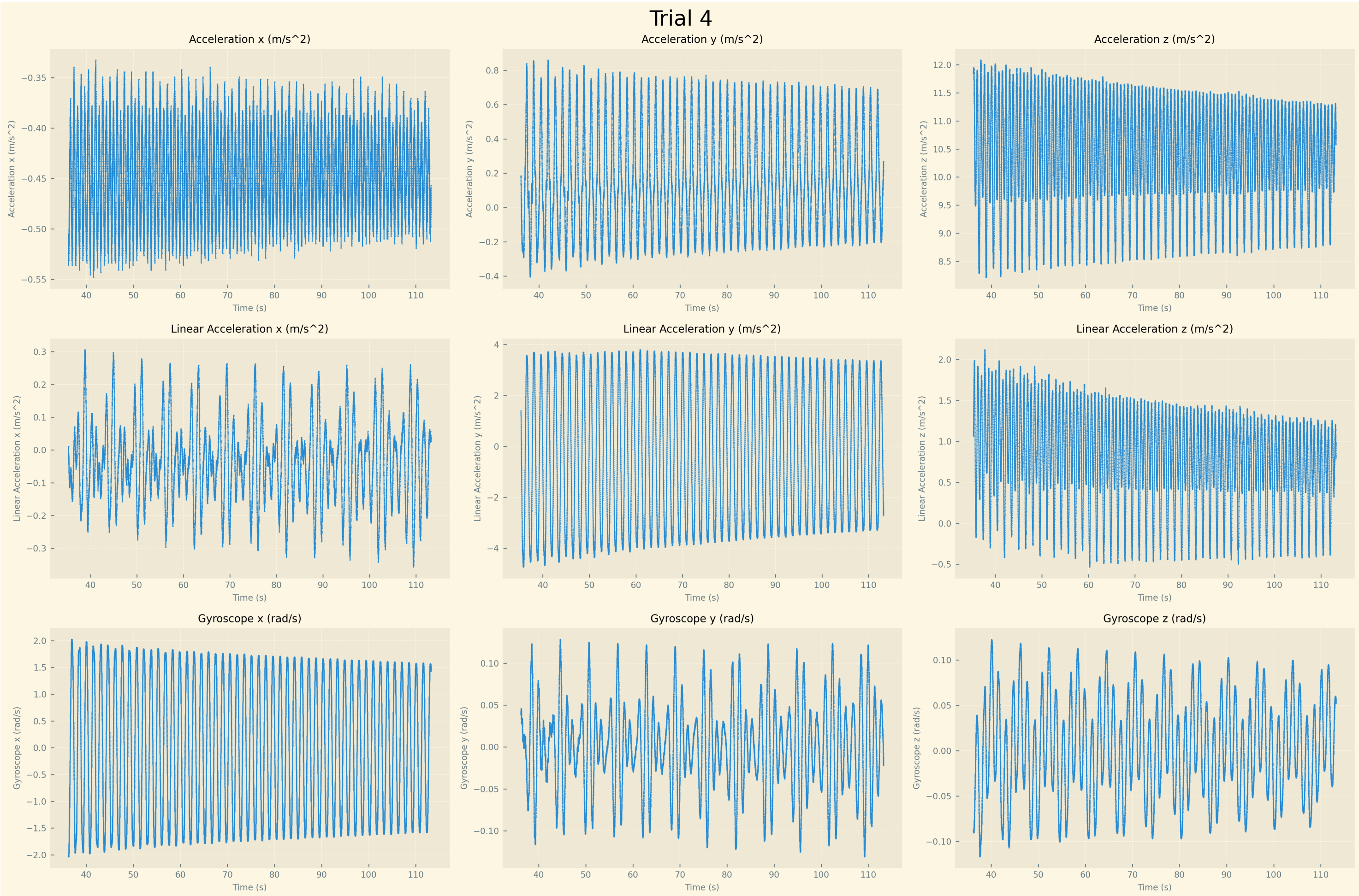


Figure 6: Raw Data - Trial 4

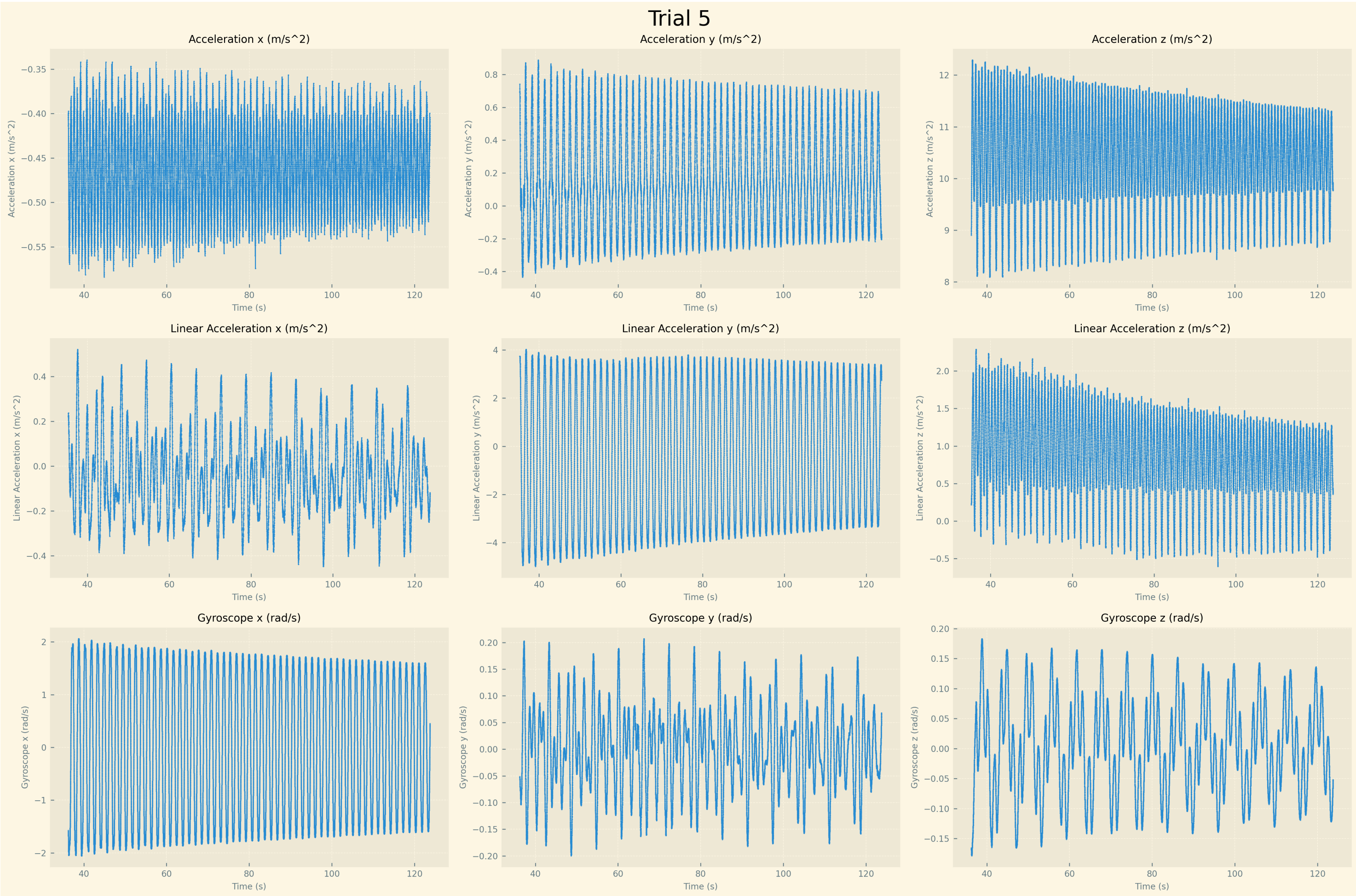


Figure 7: Raw Data - Trial 5

Across all Trials the graphs of note are Gyroscope $\hat{x}$, Acceleration and Linear Acceleration(a,a_l respectively) in $\hat{y}$ and $\hat{z}$ as the Gyroscope $\hat{x}$ models the angular velocity about the pendulum's rotational plane, a_y and a_z are the acceleration components of this rotation, with a_{ly} and $4a_{lz}$ being the same without gravity considered.

The remaining graphs a_x,a_{lx} and Gyroscope in $\hat{y}$, $\hat{z}$ can be utilized to describe additional factors that influenced the Pendulum Bob's motion, such as human error in dropping the Bob, and the torsional noise subsequently generated.

For all Trials there is a consistent discrepancy across a_z and a_{lz} in that every second trough of the periodic motion is significantly reduced. Also the equilibrium point of these components a_z,a_{lz} are $10.3m/s^2$ and $0.5m/s^2$ respectively. This result does not agree with conceptional knowledge such an experiment, it is widely accepted that the these acceleration components should oscillate around 9.81 for a_z and 0 for a_{lz}. These discrepancies suggest additional factors in the z-component impacted the measurement results. the most probable factor is a spring force generated from the rotational force on the string cradle, that unlike other conventional apparatus' is not a stiff pendulum and generating a oscillating z acceleration, this may also explain some of the noise present in the the remaining graphs.

For Trials 2,3,4 the equilibrium for a_x is $-0.40m/s^2$ and for Trial 1 it is $0.15m/s^2$ which is a significant difference from the expected equilibrium point of $0m/s^2$. Suggesting an systematic error in the apparatus that is predicted to be the result torsional restoring force in the string cradle, introducing an either $(+)$ or $(-)$ x-acceleration depending on the geometry of the particular trial.

4.2 Eigenfrequencies - Task 2

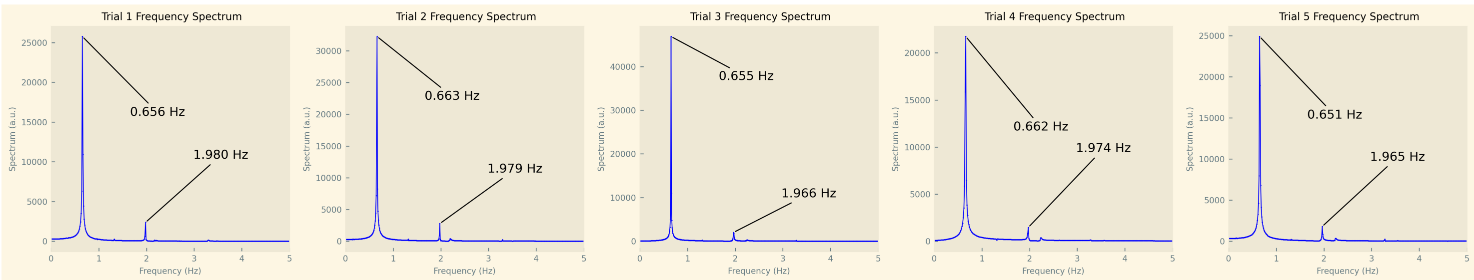


Figure 8: Frequency Spectrum's - All Trials

Theoretical f_0	$(0.67003 \pm 0.00043)Hz$
Measured f_0	$(0.65755 \pm 0.00987)Hz$
Absolute Error	$(0.01248 \pm 0.00988)Hz$

Table 1: Eigenfrequency Table

Across all 5 trials an average f_0 of $(0.65755 \pm 0.00987)Hz$ was found which strongly coincides with the theoretical value calculated at $(0.67003 \pm 0.00043)Hz$ and an absolute error between the two of $(0.01248 \pm 0.00988)Hz$. suggesting a strong agreement between experimental data collected about the $\hat{x}$-axis and the theoretical predictions.

4.3 Damping Constant - Task 3

Damping Constant δ	$(0.002685 \pm 0.005883)Hz$
δ Relative Uncertainty	2.191

Table 2: Damping Constant

The damping constant found $(0.002685 \pm 0.005883)Hz$ when used to compare to the Gyroscope $\hat{x}(\omega_x)$ that was collected Fig. 9 shows a largely similar damping factor as the pendulum oscillates. However the damping constants relative uncertainty percentage of 219.1% indicates a very low precision with a significantly large spread of measurements collected. subsequently the conclusions made with Fig. 9 are widely imprecise.

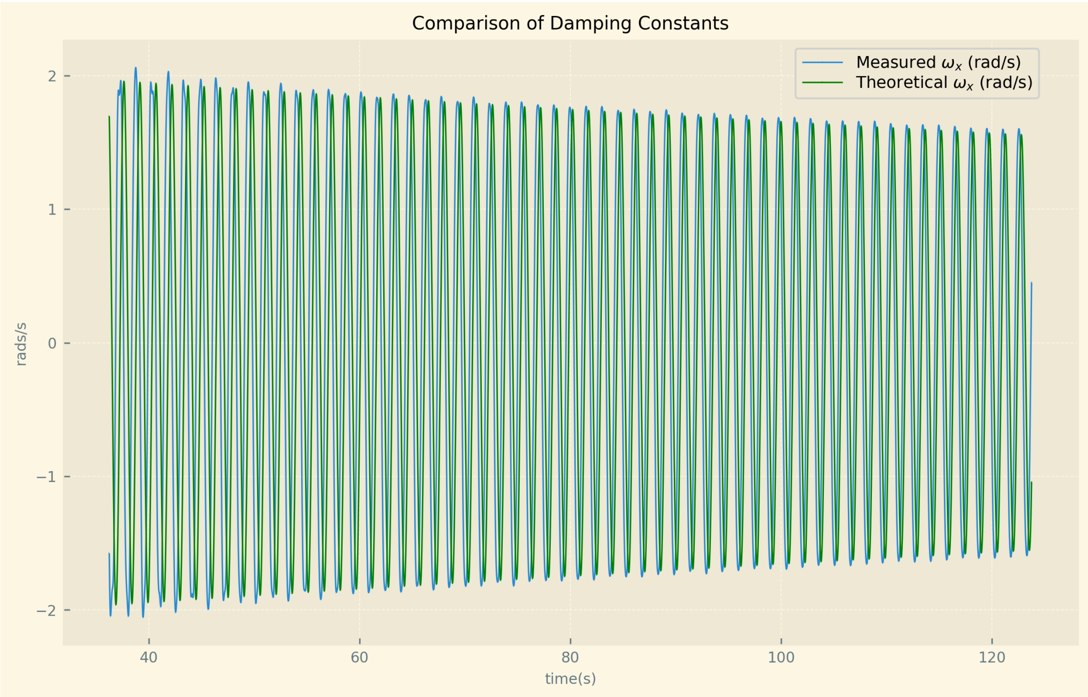


Figure 9: Comparison of ω_x with the derivative of Eq. 6

4.4 Measured and calculated accelerations - Task 4

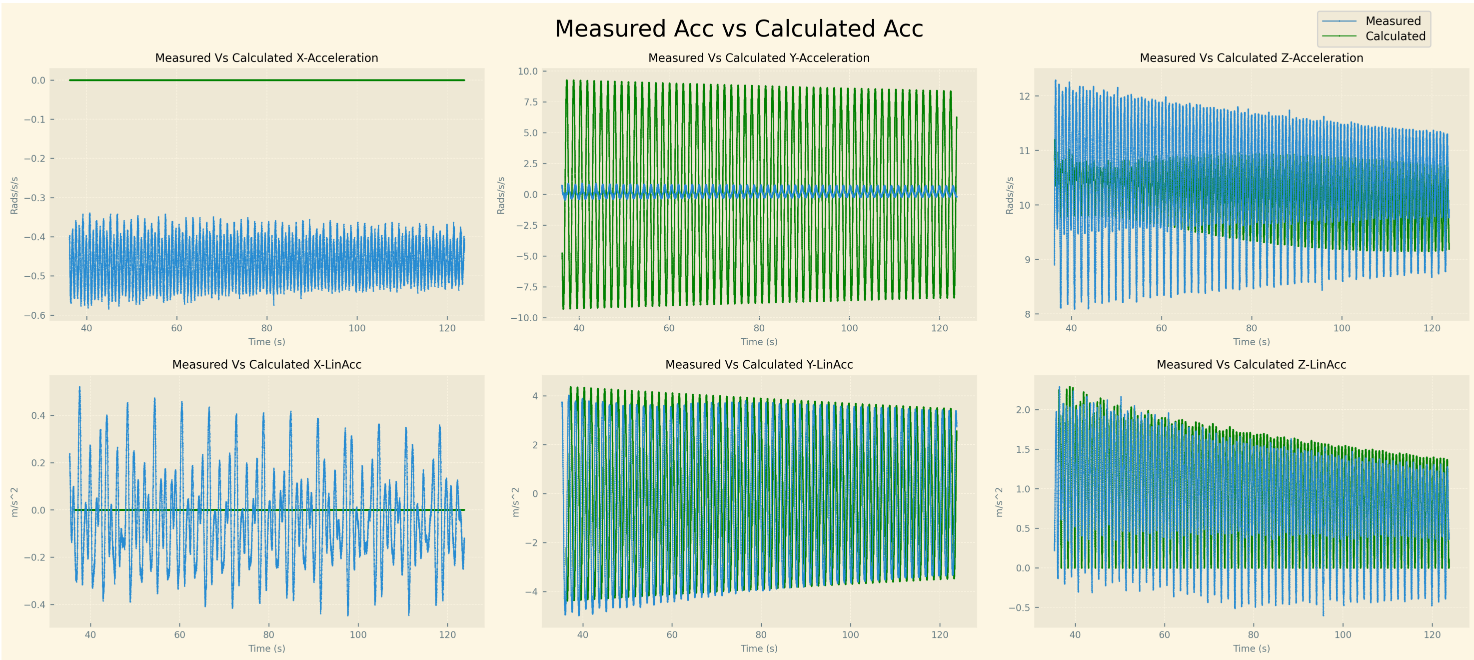


Figure 10: $\vec{a}$ and $\vec{a_l}$ Measured vs. Calculated

Predictably both measured accelerations in the x-direction is zero as our calculations assume only movement in the rotational x, y-plane, and the measured comparisons highlight the noise and additional factors that influenced the motion of the pendulum bob in the experiment.

Both Linear Accelerations in the y and z directions follow similar trends with their respective measured counterparts, with notable exception trough-skipping in the z-axis due to the spring force that arises from the string cradle throughout the pendulum bob's motion. These trends further support the δ found. However it is prudent to keep in mind that the relative uncertainty of δ of 2.191 greatly reduces the significance of this similarity in trends as Fig. 10 do not present the margin of error introduced by the damping constant. Furthermore this can be extended to the y-direction graphs as they also used δ for their calculated data.
Regarding the Accelerations in the y-direction there is a significant difference in the amplitude between the measured and calculated, such as difference has been attributed to a measured component that was not considered in its mathematical model, or a modeling error when calculating.

The calculated Acceleration in the z-direction is also vastly different from the measured results, both the magnitude of its amplitude and its envelope do not resemble measured results. It is unknown as to what factor influenced such as result from its variables; l, ω_x, g_z.

5 Discussion and Conclusion

The results of measuring a pendulum bob's accelerated frame of reference indicated the present of unaccounted for factors that of a spring force in z due rotational motion exerted on the string cradle, a torsional restoring force in x from the string cradle as well as additional noise generated from dropping the cradle by hand.

The measured eigenfrequency of the apparatus was measured to be $(0.65755 \pm 0.00987)Hz$ strongly agreeing with theoretical predictions of $(0.67003 \pm 0.00043)Hz$ with absolute error of $(0.01248 \pm 0.00988)Hz$. the damping constant δ for Trial 5 was determined to be $(0.002685 \pm 0.005883)Hz$ with a large relative uncertainty of 2.191, which arose from a inference from measured data that greatly amplified uncertainty.

The damping constant was then used in conjunction with initial variables to calculate the acceleration and linear acceleration of the pendulum bob in its accelerated frame of reference and compare these results with its measured counterpart. In particular the linear acceleration in y and z supported that was which was measured when taking into consideration the present additional factors. However the acceleration of these two axes did not coincide with measured data and had large discrepancies that was predicted to arise from errors in mathematical modeling.

For future experiments it is recommended to go with a rigid pendulum arm, and a more rigorous approach to determining the damping constant δ for the real pendulum.

References

[1] Y. D. Hugh and R. A. Freedman. Periodic motion. In *University with Modern Physics*, pages 457–484. Pearson, 15th edition, 2020.

[2] M. Ziese. *H10e Pendulum as an accelerated Frame of Reference*. Universität Leipzig, October 2024.