GETTING TO KNOW THE CARDINALS

NOTES ON AKIHIRO KANAMORI'S *The Higher Inifite*

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0 Preliminaries

Definition 0.1. For $X \subseteq \mathrm{On}$, γ is a *limit point of* X iff $\bigcup(X \cap \gamma) = \gamma > 0$

Notice that a limit point is necessarily a limit ordinal: if $\gamma = \alpha + 1$ then $\bigcup \gamma = \alpha$ and we have $\bigcup(X \cap \gamma) \subseteq \bigcup \gamma = \alpha < \gamma$.

Definition 0.2. C is *closed unbounded in* δ iff C is an unbounded subset of δ containing all its limit points less than δ.

These sets are also called *club* sets. Here C is unbounded in δ iff for all $x \in \delta$ there exists $y \in C$ such that $x \leq y$ (the alternative would have $x < y$). Equivalently, $\sup(C) = \sup(\delta)$. Also equivalently, C is cofinal in δ.

Definition 0.3. For regular $\nu < \delta$, C is *ν-closed unbounded in* δ iff C is an unbounded subset of δ containing all its limit points less than δ of cofinality ν.

Definition 0.4. For limit ordinals δ, S is *stationary in* δ iff $S \subseteq \delta$ and $S \cap C \neq \emptyset$ for any C closed unbounded in δ.

Definition 0.5. If $\langle X_\alpha \mid \alpha < \delta \rangle \in {}^{\delta}\mathcal{P}(\delta)$, then its diagonal intersection is $\{\xi < \delta \mid \xi \in \bigcap_{\alpha<\xi} X_\alpha\}$, denoted $\triangle_{\alpha<\delta} X_\alpha$.

Definition 0.6. For $X \subseteq \mathrm{On}$ and $f : X \to \mathrm{On}$, f is *regressive* iff $f(\alpha) < \alpha$ for every $\alpha \in X - \{\emptyset\}$.

Proposition 0.7. *Suppose that $\lambda > \omega$ is regular.*

(a) If $\gamma < \lambda$ and $\langle C_\alpha \mid \alpha < \gamma \rangle$ is a sequence of sets closed unbounded in λ, then $\bigcap_{\alpha<\gamma} C_\alpha$ is closed unbounded in λ.

(b) If $\langle C_\alpha \mid \alpha < \lambda \rangle$ is a sequence of sets closed unbounded in λ, then its diagonal intersection $\triangle_{\alpha<\lambda} C_\alpha$ is closed unbounded in λ.

(c) (Fodor) *If S is stationary in λ and $f : S \to \lambda$ is regressive, then there is an $\alpha < \lambda$ such that $f^{-1}(\{\alpha\})$ is stationary in λ.*

(d) If $\nu < \lambda$ is regular, $S \subseteq \{\xi < \lambda \mid \mathrm{cf}(\xi) = \nu\}$ is stationary in λ, and C is ν-closed unbounded in λ, then $S \cap C \neq \emptyset$.

Proof. (a) Let $C = \bigcap_{\alpha<\gamma} C_\alpha$. First, we show C is closed. Notice for all sets X and ordinals β, $\sup(\beta \cap X) \leq \beta$. Suppose $\sup(\delta \cap C) = \bigcup \delta \cap C = \delta$. Then for all $\alpha < \gamma$, $(\delta \cap C) \subseteq (\delta \cap C_\alpha)$. Thus $\delta = \sup(\delta \cap C) \leq \sup(\delta \cap C_\alpha) \leq \delta$. Hence $\sup(\delta \cap C_\alpha) = \delta$. Hence δ is a limit point and since each C_α is closed, $\delta \in C_\alpha$ for all $\alpha < \gamma$. Thus $\delta \in C$. Hence C is closed.

Now we show C is unbounded. Fix $\beta < \lambda$. Define $b_0 = \beta$. Given $b_n < \lambda$ for $n < \omega$, for each α choose $b_{\alpha,n} \in C_\alpha$ such that $b_{\alpha,n} \geq b_n$. Define $b_{n+1} = \sup\{b_{\alpha,n} \mid \alpha < \gamma\}$. Since λ is regular and $\gamma < \lambda$, $b_{n+1} < \lambda$. Thus, by unboundedness of each C_α, we can define the sequence $\{b_n \mid n < \omega\}$. For each α, $b_0 \leq b_{\alpha,0} \leq b_1 \leq b_{\alpha,1} \leq \cdots$, thus $\delta = \sup\{b_n \mid n < \omega\} = \sup\{b_{\alpha,n} \mid n < \omega\}$. Furthermore, since $\lambda > \omega$ regular, $\delta < \lambda$. If $\delta = b_{\alpha,n}$ for some n then for all $m + 1 > n$ and for all $\alpha < \gamma$, $\delta \leq b_{\alpha,m} \leq b_m \leq \delta$. Hence for all α, $\delta \in C_\alpha$. Thus we have $\delta \in C$ with $\delta \geq \beta$. Otherwise, δ is a limit ordinal and, for all α, we have $\delta = \sup(\delta \cap \{b_{\alpha,n} \mid n < \omega\}) \leq \sup(\delta \cap C_\alpha) \leq \delta$. Hence δ is a limit point less than λ for each C_α and, by closedness, $\delta \in C_\alpha$. Hence we have $\delta \in C$ with $\delta \geq \beta$. Thus C is unbounded. Thus C is closed unbounded in λ.

(b) $\square$