NEWTON POLYTOPES AND TORIC VARIETIES FOR PERIODIC GRAPH OPERATORS

A Dissertation

by

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ABSTRACT

We investigate the relationship between a periodic graph operator and its complex Bloch variety by studying the combinatorics and geometry of the Newton polytope of its characteristic polynomial. When this polytope is only homothetically indecomposable, we show that the irreducibility of the complex Bloch variety is preserved after a change of its period lattice. Associated to the Newton polytope of this characteristic polynomial is a normal toric variety. We compactify the complex Bloch variety in this toric variety to study its asymptotics. When the Newton polytope is full, we give a spectral-theoretic interpretation of these asymptotics.

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Chapter 5 is published work with Matthew Faust in [1], and Chapter 6 is based on an upcoming work with Matthew Faust, Stephen Shipman, and Frank Sottile in [2].

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NOMENCLATURE

Γ	A periodic graph
$\circ$	Composition of functions
$\mathbb{C}^n$	The n-dimensional complex affine space
X	A vector space or a variety
$\mathbb{C}^\times$	The algebraic torus $\mathbb{C} \setminus \{0\}$
$\mathbb{R}^n$	The n-dimensional vector space of real numbers
$x = x_1, \ldots, x_n$	Indeterminates or vectors of indeterminates
$\mathbb{C}[x]$	The ring of polynomials with coefficients in $\mathbb{C}$ in n variables
$\mathbb{C}[z^\pm]$	The ring of Laurent polynomials with coefficients in $\mathbb{C}$ in d variables
$\mathbb{T}$	The complex unit circle
$\mathbb{Z}^d$	The d-dimensional lattice of integer vectors
$\mathbb{N}^d$	The d-dimensional vectors of natural numbers
$\mathbb{P}^n$	The n-dimensional complex projective space
V	A function on the vertices of a periodic graph
c	A function on the edges of a periodic graph or a labeling
$\mathcal{V}$	The set of vertices of a periodic graph
$\mathcal{E}$	The set of edges of a periodic graph
Δ	The discrete Laplacian, also known as the graph Laplacian
e	The vector $(0, \ldots, 0, 1) \in \mathbb{Z}^{d+1}$
$\mathcal{A}$	A finite set
$\mathrm{conv}(\mathcal{A})$	The convex hull of a finite set $\mathcal{A}$

$Q\mathbb{Z}$	The free full rank subgroup $\bigoplus_{i=1}^{d} q_i\mathbb{Z}$ of $\mathbb{Z}^d$ where $q_i \in \mathbb{N}$
Q	An integer vector, a vector of natural numbers, or the convex hull of $\mathcal{A} \cup \{\mathbf{e}\}$
P	A lattice polytope
F	A face of a polytope
f_F	A facial polynomial of a polynomial f with respect to the face F of a polytope
$\mathrm{Newt}\, f$	The Newton polytope of a polynomial f
u, v	Vertices of a polytope or a graph
$\mathbb{C}[X]$	The coordinate ring of an affine variety
R	A commutative ring with 1
$\mathrm{spec}(\mathsf{R})$	The spectrum of a ring R
$\det(L_c(z))$	The determinant of a matrix $L_c(z)$
$\langle \cdot, \cdot \rangle$	An inner product
$\lVert \cdot \rVert$	A norm on a vector space
$\ell^2(\mathbb{Z})$	The Hilbert space of square-summable functions on $\mathbb{Z}$
$L^2(\mathbb{T})$	The Hilbert space of square-integrable functions on $\mathbb{T}$
W	A fundamental domain
$\sigma(L)$	The spectrum of an operator L
$\rho(L)$	The resolvent set of an operator L
L	An operator on a Hilbert space or a module homomorphism
L^*	The adjoint of an operator L
$\mathscr{F}$	The Fourier or Floquet transform or a sheaf of modules
$\hat{f}$	The Fourier or Floquet transform of a function f
D_c	A dispersion polynomial with labeling c
$X_{\mathcal{A}}$	An affine toric variety

N	A finitely generated free abelian group
M	The dual group of a finitely generated free abelian group N
$\mathbb{N}\mathcal{A}$	The monoid generated by a finite set $\mathcal{A}$
$\mathbb{C}[\mathbb{N}\mathcal{A}]$	The monoid algebra of the monoid $\mathbb{N}\mathcal{A}$
σ	A finitely generated submonoid of N or a cone in N
σ°	The relative interior of a cone σ
$\sigma^\vee$	The polar of a finitely generated submonoid σ
M_σ	The lineality space of a cone $\sigma^\vee \subseteq M$
V_σ	The affine toric variety associated to a cone σ
$\mathcal{Q}_\sigma$	A module of quasi-periodic functions associated to a cone σ
$\widetilde{\mathcal{Q}}_\sigma$	A sheaf of quasi-periodic functions on the toric variety V_σ
τ	A face of a cone
Σ	A fan
$\|\Sigma\|$	The support of a fan Σ
X_Σ	The toric variety associated to a fan Σ
Σ_P	The inner normal fan of a polytope P
$\widetilde{\mathcal{Q}}_\Sigma$	A sheaf of quasi-periodic functions on the toric variety X_Σ
M_F	The lineality space of a cone associated to the face F of a polytope
V_F	The affine toric variety associated to the face F of a polytope
$\mathscr{O}$	The structure sheaf of a toric variety
B	The base of the Newton polytope of a dispersion polynomial
$\mathrm{in}_G L_c(z)$	The facial matrix of the Floquet matrix $L_c(z)$ corresponding to a face G of the polytope Q such that $F = \|W\|G$ is a face of the full Newton polytope $P = \|W\|Q$
Γ_F	The facial graph of a periodic graph Γ corresponding to a face F of the Newton polytope of a periodic graph operator on quasi-periodic functions associated to the graph Γ

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

In the tight binding model of solid state physics, the Schrödinger operator (Laplacian plus potential) models the wave function of an electron [4]. The spectrum of this operator is a closed subset of $\mathbb{R}$. On a periodic medium, and after a Floquet transform, the spectrum is revealed to be the projection to $\mathbb{R}$ of an *analytic hypersurface* in $\mathbb{T}^d \times \mathbb{R}$, where $\mathbb{T}$ is the complex unit circle and d is the ambient dimension.

We discretize the setting to study a periodic graph operator L acting on the Hilbert space $\ell^2(\Gamma)$ of square-summable functions on the vertices $\mathcal{V}$ of a $\mathbb{Z}^d$-periodic graph Γ. As the action of $\mathbb{Z}^d$ commutes with the operator L, we may apply the Floquet transform to $\ell^2(\Gamma)$, which reveals further the structure of the spectrum of L. The Floquet transform is a linear isometry between $\ell^2(\Gamma)$ and the Hilbert space $L^2(\mathbb{T}^d)^{|W|}$ of square-integrable functions on $\mathbb{T}^d$, where $W \subset \mathcal{V}$ is a fundamental domain for the action of $\mathbb{Z}^d$ on Γ. The action of L on $L^2(\mathbb{T}^d)^{|W|}$ is multiplication by a $|W| \times |W|$ matrix $L(z)$ of Laurent polynomials in $z \in \mathbb{T}^d$.

For a fixed $z \in \mathbb{T}^d$, the matrix $L(z)$ is Hermitian and it has $|W|$ real eigenvalues $\lambda_1(z) \leq \cdots \leq \lambda_{|W|}(z)$. For $i \in \{1, \ldots, |W|\}$, the function $\lambda_i : \mathbb{T}^d \to \mathbb{R}$ is the i-th *band function*. This function is continuous and piece-wise analytic on $\mathbb{T}^d$; its range is the i-th *spectral band*. The spectral bands may overlap, or leave spaces between them known as *spectral gaps*. Developing an understanding of this band-gap structure is useful in applications involving nano-materials, topological insulators, and photonic crystals [5].

The union of the spectral bands is the spectrum $\sigma(L)$ of the operator L. This set may also be described as the projection to $\mathbb{R}$ of the set of points $(z, \lambda) \in \mathbb{T}^d \times \mathbb{R}$ such that there exists a nonzero function $\psi \in L^2(\mathbb{T}^d)^{|W|}$ solving the eigenvalue problem (also known as *spectral* problem) $L(z)\psi = \lambda\psi$. This set is known as the *dispersion relation*, and it may also be expressed as the set

$$\mathcal{B}_L(\mathbb{R}) := \Big\{(z, \lambda) \in \mathbb{T}^d \times \mathbb{R} \mid \det\big(L(z) - \lambda I_{|W|}\big) = 0\Big\},$$

a real algebraic variety known as the (real) *Bloch variety*. The spectrum $\sigma(L)$ is the image of $\mathcal{B}_L(\mathbb{R})$ under the projection map $\pi : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$. For $\lambda \in \sigma(L)$, the fiber $\mathcal{F}_{L,\lambda}(\mathbb{R}) := \pi^{-1}(\lambda) \cap \mathcal{B}_L(\mathbb{R})$ is a real algebraic variety known as the (real) *Fermi variety* at energy level λ.

By allowing $z \in (\mathbb{C}^\times)^d$, the matrix $L(z)$ is no longer selfadjoint for $z \in (\mathbb{C}^\times)^d \setminus \mathbb{T}^d$ and its spectrum $\sigma(L)$ is no longer real; however, we gain several things. The (complex) *Bloch variety* is the algebraic hypersurface $\mathcal{B}_L := \mathit{Var}(D)$ in $(\mathbb{C}^\times)^d \times \mathbb{C}$ given by the vanishing of the characteristic polynomial $D := \det(L(z) - \lambda I_{|W|})$ called the *dispersion polynomial* of L. For $\lambda \in \sigma(L)$, the fiber $\mathcal{F}_{L,\lambda} := \pi^{-1}(\lambda) \cap \mathcal{B}_L$ is the (complex) *Fermi variety* at energy λ; it is a hypersurface in $(\mathbb{C}^\times)^d$. Therefore, we may use effective methods of algebraic geometry and related areas to study these varieties and their relationship to periodic graph operators and their spectra.

In [6], Gieseker, Knörrer, and Trubowitz used the geometry of complex Bloch and Fermi varieties to address questions about a particular family of periodic graph operators. Their results relied on a compactification of the Bloch variety by embedding its ambient space $(\mathbb{C}^\times)^d \times \mathbb{C}$ in a projective space and subsequent blowups. This compactification is highly singular and difficult to understand. Bättig, motivated by an idea of Mumford [7], constructed an equivalent but intrinsic compactification of the Bloch variety in a suitable toric variety, whose structure is understood by the combinatorial data of a fan [8].

We simplify Bättig's construction using polytopes. Following [9], the domain of the dispersion polynomial D has a natural compactification in the normal toric variety X_Σ associated to the fan Σ of the Newton polytope P of D, which is the convex hull of the exponent vectors of D. The Zariski closure of $\mathcal{B}_L$ in X_Σ determines a compactification of this Bloch variety. In this compactification, the natural directions at infinity of $\mathcal{B}_L$ are recorded by the faces of P. The structure of the Newton polytope P was recently utilized in [3] by Faust and Sottile to give a bound for the number of complex critical points of the Bloch variety; this bound is attained when the critical point equations of the Bloch variety have no solutions on the points added in the compactification.

We investigate the geometry of the Newton polytope P of the dispersion polynomial D of a periodic graph operator L and how it informs the geometry of complex Bloch varieties. Chapter 2 provides a background on the spectral theory of periodic graph operators and Chapter 3 discusses the algebra and geometry necessary for this study. In Chapter 4, we give the necessary background on polytopes and toric varieties. Chapter 5 builds on the theory of homothetically indecomposable polytopes [10] to give criteria when the dispersion polynomial D is irreducible. The material of Chapter 5 is based on published work with Matthew Faust in [1]. In Chapter 6, we compactify the complex Bloch variety $\mathcal{B}_L$ in the toric variety X_Σ and show that when the Newton polytope of D is *full*, we may associate to a particular face of the polytope P a periodic, labeled, directed graph whose operator has Bloch variety equal to the intersection of the compactified Bloch variety and the orbit corresponding to that face. Chapter 6 is based on work with Matthew Faust, Stephen Shipman, and Frank Sottile in the upcoming article [2]. Chapter 7 gives a summary of this dissertation and concluding remarks.

We write $\mathbb{N}$ for the set of natural numbers ($0 \in \mathbb{N}$), $\mathbb{Z}$ for the set of integers, $\mathbb{Q}$ for the set of rational numbers, $\mathbb{R}$ for the set of real numbers, and $\mathbb{C}$ for the set of complex numbers.

2. SPECTRAL THEORY OF PERIODIC GRAPH OPERATORS

This dissertation studies mathematical structures arising in the spectral theory of periodic graph operators. These operators belong to the larger class of bounded linear operators. We recall standard results of bounded linear operators and their spectra. For a thorough treatment of the basics of spectral theory, see [11, Chapter 1] and [12].

2.1 Spectral Theory of Bounded Linear Operators

A *metric* on a complex vector space X is a function $\mathfrak{d} : X \times X \to \mathbb{R}$ such that for $f, g, h \in X$:

(i) $\mathfrak{d}(f, g) \geq 0$ and $\mathfrak{d}(f, g) = 0$ if and only if $f = g$,

(ii) $\mathfrak{d}(f, g) = \mathfrak{d}(f, g)$, and

(iii) $\mathfrak{d}(f, h) \leq \mathfrak{d}(f, g) + \mathfrak{d}(g, h)$.

The pair $(X, \mathfrak{d})$ is a *metric space*. If $(X, \mathfrak{d})$ is a metric space, a sequence $\{f_n\}_{n=1}^{\infty} \subset X$ *converges* to $f \in X$ if $\lim\limits_{n\to\infty} \mathfrak{d}(f, f_n) = 0$. A sequence $\{f_n\}_{n=1}^{\infty}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a nonnegative integer N such that $\mathfrak{d}(f_n, f_m) \leq \varepsilon$ for all $n, m \geq N$. If every Cauchy sequence has a limit, then $(X, \mathfrak{d})$ is *complete* with respect to the metric $\mathfrak{d}$.

A *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that for $f, g, h \in X$ and $\alpha \in \mathbb{C}$:

(i) $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$,

(ii) $\|\alpha f\| = |\alpha| \cdot \|f\|$, and

(iii) $\|f + g\| \leq \|f\| + \|g\|$,

where $|\alpha|$ is the modulus of α. The pair $(X, \|\cdot\|)$ is a *normed space*. A norm on X induces a metric $\mathfrak{d}_{\|\cdot\|}$ on X by $\mathfrak{d}_{\|\cdot\|}(f, g) := \|f - g\|$. A normed space $(X, \|\cdot\|)$ is *complete* if it is complete with respect to the metric $\mathfrak{d}_{\|\cdot\|}$.

Example 2.1.1. Let $\ell^2(\mathbb{Z}) := \{f : \mathbb{Z} \to \mathbb{C} \mid \sum_{n\in\mathbb{Z}} |f(n)|^2 < \infty\}$ be the vector space of square-summable functions on $\mathbb{Z}$. Define a norm on $\ell^2(\mathbb{Z})$ by $\|f\|_{\ell^2} := \Big(\sum_{n\in\mathbb{Z}} f(n)\overline{f(n)}\Big)^{1/2}$ for any $f \in \ell^2(\mathbb{Z})$. The pair $(\ell^2(\mathbb{Z}), \|\cdot\|_{\ell^2})$ is a complete normed space. $\diamond$

A subset W of a normed space X is ***dense*** if for each $f \in X$ there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset W$ such that $\lim_{n\to\infty} f_n = f$. It is ***countable*** if there exists a bijection $W \to \mathbb{Z}$. The normed space X is ***separable*** if it contains a countable dense subset.

Let $X = (X, \|\cdot\|_X)$ and $Y = (Y, \|\cdot\|_Y)$ be normed spaces. A (***linear***) ***operator*** from X to Y is a pair $(D(L), L)$, where $D(L)$ is a subspace of X, called the ***domain*** of L, and L is a linear map from $D(L)$ to Y. The ***kernel*** of L is the subspace $\ker(L) := \{f \in D(L) \mid Lf = 0\}$.

The operator $(D(L), L)$ is ***bounded*** if $D(L) = X$ and its norm, given by

$$\|L\| := \sup\big\{\|Lf\|_Y \mid f \in X \text{ and } \|f\|_X = 1\big\},$$

is finite. Denote a bounded operator $(D(L), L)$ from X to Y by $L : X \to Y$. We will also use this notation for the linear map $L : X \to Y$. A bounded operator $L : X \to X$ is an ***operator L on X***.

Example 2.1.2. The ***discrete Laplacian*** Δ on $\ell^2(\mathbb{Z})$ is defined by $(\Delta f)(n) := f(n{+}1) + f(n{-}1)$ for all $f \in \ell^2(\mathbb{Z})$. Then we obtain

$$\begin{aligned}\|\Delta f\|^2 &= \sum_{n\in\mathbb{Z}} (f(n{-}1) + f(n{+}1))\overline{(f(n{-}1) + f(n{+}1))} \\ &= \sum_{n\in\mathbb{Z}} |f(n{-}1) + f(n{+}1)|^2.\end{aligned}$$

By Minkowski's inequality*,

$$\Big(\sum_{n\in\mathbb{Z}} (|f(n{-}1) + f(n{+}1)|^2\Big)^{1/2} \le \Big(\sum_{n\in\mathbb{Z}} |f(n{-}1)|^2\Big)^{1/2} + \Big(\sum_{n\in\mathbb{Z}} |f(n{+}1)|^2\Big)^{1/2} = 2\|f\|.$$

Thus $\|\Delta f\| \le 2\|f\|$, showing that the discrete Laplacian on $\ell^2(\mathbb{Z})$ is bounded. $\diamond$

*If $g, h \in \ell^2(\mathbb{Z})$, then $\Big(\sum_{n\in\mathbb{Z}} |g(n) + h(n)|^2\Big)^{1/2} \le \Big(\sum_{n\in\mathbb{Z}} |g(n)|^2\Big)^{1/2} + \Big(\sum_{n\in\mathbb{Z}} |h(n)|^2\Big)^{1/2}$.

Let X, Y, and Z be normed spaces and let $L : X \to Y$ and $K : Y \to Z$ be bounded operators. The composition $KL : X \to Z$ is a bounded operator. If $\alpha \in \mathbb{C}$, then the operator $\alpha L : X \to Y$ defined by $(\alpha L)f := \alpha \cdot (Lf)$ is also bounded. We denote by I the identity operator on any vector space. Thus, the *identity* operator I on X is defined by $If := f$ for all $f \in X$. The operator $L : X \to Y$ is *invertible* if there exists an operator $L' : Y \to X$ such that $LL' = I$ and $L'L = I$. If such an operator L' exists, it is unique and it is the *inverse* of L and denoted by L^{-1}. If $M : X \to Y$ is a bounded operator, then the operator $L + M : X \to Y$ defined by $(L + M)f := Lf + Mf$ is also a bounded operator.

Let L be a bounded operator on X. The set

$$\rho(L) := \{\lambda \in \mathbb{C} \mid L - \lambda I : X \to X \text{ is bijective and its inverse is bounded}\}$$

is the *resolvent set* of L. Its complement $\sigma(L) := \mathbb{C} \setminus \rho(L)$ is the *spectrum* of L. A complex number $\lambda \in \sigma(L)$ is an *eigenvalue* of L if there exists a nonzero function $f \in X$ such that $Lf = \lambda f$. If λ is an eigenvalue of L, then f is an *eigenfunction* of λ. Thus, λ is an eigenvalue of L if and only if $\ker(L - \lambda I) = \{f \in X \mid (L - \lambda I)f = 0\} \neq \{0\}$. The set of eigenvalues of L is the *discrete spectrum* of L and its complement in $\sigma(L)$ is the *essential spectrum* of L.

Example 2.1.3. Suppose the discrete Laplacian Δ on $\ell^2(\mathbb{Z})$ has an eigenvalue $\lambda \in \mathbb{C}$. Then there exists a nonzero function $f \in \ell^2(\mathbb{Z})$ such that $\Delta f = \lambda f$. Then $\Delta f(n) = f(n+1) + f(n-1) = \lambda f(n)$ is equivalent to the recurrence relation $f(n) - \lambda f(n-1) + f(n-2) = 0$. Its characteristic polynomial is $x^2 - \lambda x + 1$, which has solutions $\frac{1}{2}(\lambda \pm \sqrt{\lambda^2 - 4})$. Let $r := \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4})$ and $s := \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4})$. Then $f(n) = \alpha r^n + \beta s^n$ for $\alpha, \beta \in \mathbb{C}$. Observe that $r = s^{-1}$.

The discriminant of $x^2 - \lambda x + 1$ is $\lambda^2 - 4$. If $\lambda^2 - 4 > 0$, then r and s are real and $r \neq s$. If $\alpha \neq 0$, then $\lim\limits_{n\to\infty} |f(n)| = \infty$. If $\beta \neq 0$, then $\lim\limits_{n\to-\infty} |f(n)| = \infty$. Thus, $f \notin \ell^2(\mathbb{Z})$.

If $\lambda^2 - 4 \leq 0$, then parametrize λ by $\lambda = 2\cos\theta$:

$$r = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}) = \frac{1}{2}(2\cos\theta - \sqrt{2\cos\theta)^2 - 4}) = e^{i\theta}, \text{ and}$$
$$s = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4}) = \frac{1}{2}(2\cos\theta + \sqrt{2\cos\theta)^2 - 4}) = e^{-i\theta}.$$

Then $f(n) = \alpha e^{in\theta} + \beta e^{-in\theta}$. Thus f is constant or oscillating as n goes to $\pm\infty$. Hence, $f \notin \ell^2(\mathbb{Z})$ and the discrete spectrum of Δ is empty. $\diamond$

An *inner product* on a vector space X is a function $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ such that for $f, g, h \in X$ and $\alpha, \beta \in \mathbb{C}$:

(i) $\langle \alpha f + \beta g, h\rangle = \overline{\alpha}\langle f, g\rangle + \overline{\beta}\langle g, h\rangle$,

(ii) $\langle f, \alpha g + \beta h\rangle = \alpha\langle f, g\rangle + \beta\langle g, h\rangle$,

(iii) $\langle f, f\rangle > 0$ (where $f \neq 0$), and

(iv) $\langle f, g\rangle = \overline{\langle g, h\rangle}$,

where $\alpha \mapsto \overline{\alpha}$ denotes complex conjugation. The pair $(X, \langle\cdot,\cdot\rangle)$ is an *inner product space*. An inner product on X induces a norm $\|\cdot\|_{\langle\cdot,\cdot\rangle}$ on X by $\|x\|_{\langle\cdot,\cdot\rangle} := \langle x, x\rangle^{1/2}$. The inner product space $(X, \langle\cdot,\cdot\rangle)$ is *complete* if it is complete with respect to the metric $\mathfrak{d}_{\|\cdot\|_{\langle\cdot,\cdot\rangle}}$, in which case $(X, \langle\cdot,\cdot\rangle)$ is a *Hilbert space*.

Example 2.1.4. For $f, g \in \ell^2(\mathbb{Z})$, define $\langle f, g\rangle_{\ell^2} := \sum_{n\in\mathbb{Z}} f(n)\overline{g(n)}$. Then $\langle\cdot,\cdot\rangle_{\ell^2}$ is an inner product on $\ell^2(\mathbb{Z})$ and it induces the norm $\|\cdot\|_{\ell^2}$ defined in Example 2.1.1. The inner product space $(\ell^2(\mathbb{Z}), \langle\cdot,\cdot\rangle_{\ell^2})$ is complete, making it into a Hilbert space [11, Theorem 1.2.7]. Since $\mathbb{Z}$ is countable, it follows that $\ell^2(\mathbb{Z})$ is separable. $\diamond$

Let $H := (H, \langle\cdot,\cdot\rangle)$ be a separable complex Hilbert space and let L be a bounded operator on H. The *adjoint* of L is the operator L^* on H such that for $f, g \in H$, $\langle L^* f, g\rangle = \langle f, Lg\rangle$. The operator L^* is bounded [11, Proposition 1.4.5]. If $L^* = L$, then L is *selfadjoint*.

Example 2.1.5. Let Δ be the discrete Laplacian on $\ell^2(\mathbb{Z})$. For $f, g \in \ell^2(\mathbb{Z})$,

$$\begin{aligned}\langle \Delta f, g\rangle &= \sum_{n\in\mathbb{Z}} (f(n+1) + f(n-1))\overline{g(n)} \\ &= \sum_{n\in\mathbb{Z}} f(n)(\overline{g(n+1)} + \overline{g(n-1)}) = \langle f, \Delta g\rangle,\end{aligned}$$

showing that Δ on $\ell^2(\mathbb{Z})$ is selfadjoint. $\diamond$

Proposition 2.1.6. [11, Proposition 1.4.7] . Let H be a separable complex Hilbert space, and let L be a bounded operator on H. If L is selfadjoint, then $\sigma(L)$ is a compact subset of $\mathbb{R}$.

Let L be a bounded operator on a separable complex Hilbert space H, and let $\lambda \in \mathbb{C}$. A sequence $\{w_n\}_{n=0}^{\infty} \subset H$ is a *Weyl sequence* for the operator L *at* λ if for each $n \in \mathbb{N}$, $\|w_n\| = 1$ and $\lim_{n\to\infty} \|(L - \lambda I)w_n\| = 0$.

Theorem 2.1.7. [11, Theorem 1.4.20]. Let H be a separable complex Hilbert space and let L be a bounded operator on H. If $\{w_n\}_{n=0}^{\infty} \subset H$ is a Weyl sequence for the operator L at $\lambda \in \mathbb{C}$, then $\lambda \in \sigma(L)$. If L is selfadjoint and $\lambda \in \sigma(L)$, then there exists a Weyl sequence for L at λ.

Let U be a bounded operator on H and let U^* be its adjoint. Then U is *unitary* if $U^* = U^{-1}$. Assume U is unitary. If L is a bounded operator on H, then the operator $L - \lambda I$ is invertible if and only if $U^*(L - \lambda I)U$ is invertible. It follows that $\sigma(L) = \sigma(U^*LU)$. More generally, if $G := (G, \langle\cdot,\cdot\rangle_G)$ is a separable complex Hilbert space, a bounded operator $\mathfrak{U} : G \to H$ is *unitary* if it is invertible and $\langle \mathfrak{U}f, \mathfrak{U}g\rangle_H = \langle f, g\rangle_G$ for all $f, g \in G$. If such $\mathfrak{U}$ exists, then $\mathfrak{U}^{-1} = \mathfrak{U}^*$ and G and H are isomorphic as Hilbert spaces. Consequently, $\sigma(L) = \sigma(\mathfrak{U}L\mathfrak{U}^*)$.

Example 2.1.8. The (*compact*) *torus* is the group $\mathbb{T} := \{z \in \mathbb{C}^\times \mid |z| = 1\}$. Consider the vector space $L^2(\mathbb{T}) := \left\{f : \mathbb{T} \to \mathbb{C} \mid \int_{\mathbb{T}} |f(z)|^2 dz < \infty\right\}$ of square-integrable functions on the torus $\mathbb{T}$, where dz is the Haar measure on $\mathbb{T}$ (hence $\int_{\mathbb{T}} dz = 1$). For $f, g \in L^2(\mathbb{T})$, define $\langle f, g\rangle_{L^2} := \int_{\mathbb{T}} f(z)\overline{g(z)}dz$. Then $\langle\cdot,\cdot\rangle_{L^2}$ is an inner product on $L^2(\mathbb{T})$ and the pair $(L^2(\mathbb{T}), \langle\cdot,\cdot\rangle_{L_2})$ is a Hilbert space.

Consider the bounded operator $\mathscr{F}:\ell^2(\mathbb{Z})\to L^2(\mathbb{T})$ defined by $(\mathscr{F}(f))(z):=\sum\limits_{n\in\mathbb{Z}} f(n)z^{-n}$ for $f\in\ell^2(\mathbb{Z})$ and all $z\in\mathbb{T}$. This operator is the *Fourier transform* from $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{T})$. We write $\hat{f}$ for the Fourier transform $\mathscr{F}(f)$ of a function $f\in\ell^2(\mathbb{Z})$. The adjoint $\mathscr{F}^*:L^2(\mathbb{T})\to\ell^2(\mathbb{Z})$ of the Fourier transform $\mathscr{F}:\ell^2(\mathbb{Z})\to L^2(\mathbb{T})$ is given by $(\mathscr{F}^*(g))(n):=\int\limits_{\mathbb{T}} g(z)z^n dz$ for $g\in L^2(\mathbb{T})$ and all $n\in\mathbb{Z}$. The Fourier transform is unitary. Let $f,g\in\ell^2(\mathbb{Z})$ and let $\hat{f},\hat{g}\in L^2(\mathbb{T})$ be the Fourier transform of f and g, respectively. Then

$$\begin{aligned}\langle\hat{f},\hat{g}\rangle_{L^2} &= \int\limits_{\mathbb{T}}\hat{f}(z)\overline{\hat{g}(z)}dz = \int\limits_{\mathbb{T}}\sum_{n\in\mathbb{Z}} f(n)z^{-n}\cdot\overline{\sum_{m\in\mathbb{Z}} g(m)z^{-m}}dz\\ &= \int\limits_{\mathbb{T}}\sum_{n\in\mathbb{Z}} f(n)z^{-n}\cdot\sum_{m\in\mathbb{Z}}\overline{g(m)}z^{m}dz.\end{aligned}$$

If $m\neq n$, then $\int\limits_{\mathbb{T}} f(n)z^{-n}\overline{g(m)}z^m dz=0$. Thus

$$\int\limits_{\mathbb{T}}\sum_{n\in\mathbb{Z}} f(n)z^{-n}\cdot\sum_{m\in\mathbb{Z}}\overline{g(m)}z^{m}dz = \int\limits_{\mathbb{T}}\sum_{n\in\mathbb{Z}} f(n)\overline{g(n)}dz = \sum_{n\in\mathbb{Z}} f(n)\overline{g(n)} = \langle f,g\rangle_{\ell^2}.$$

Let Δ be the discrete Laplacian on $\ell^2(\mathbb{Z})$. Since $\mathscr{F}$ is unitary, it follows that $\sigma(\Delta)=\sigma(\mathscr{F}\Delta\mathscr{F}^*)$.

Let $\lambda\in\mathbb{C}$. Consider how the operator $\mathscr{F}\Delta\mathscr{F}^*-\lambda I$ acts on $\hat{f}\in L^2(\mathbb{T})$. Fix $z_0\in\mathbb{T}$. Then

$$\begin{aligned}\big((\mathscr{F}\Delta\mathscr{F}^*-\lambda I)\hat{f}\big)(z_0) &= \Big(\mathscr{F}\int\limits_{\mathbb{T}}(\hat{f}(z)z^{n+1}+\hat{f}(z)z^{n-1})dz\Big)(z_0)-\lambda\hat{f}(z_0)\\ &= \big((z_0+z_0^{-1}-\lambda)\hat{f}\big)(z_0).\end{aligned}$$

It follows that for each $z\in\mathbb{T}$, $\mathscr{F}\Delta\mathscr{F}^*-\lambda I$ acts on $\hat{f}$ as multiplication by $z+z^{-1}-\lambda$. We will show this is a bijection if and only if $z+z^{-1}-\lambda\neq 0$. Since $z\in\mathbb{T}$, it follows that $z+z^{-1}=2\operatorname{Re}(z)$ ($\operatorname{Re}(z)$ is the real part of the complex number z). As $\operatorname{Re}(z)\in[-1,1]$, $\sigma(\Delta)=[-2,2]$. Since the discrete spectrum of Δ is the empty set (see Example 2.1.3), the essential spectrum of Δ is $[-2,2]$.

Suppose $\lambda\in\mathbb{C}$ satisfies $z+z^{-1}\neq\lambda$ for all $z\in\mathbb{T}$. Then the operator $\mathscr{G}:L^2(\mathbb{T})\to L^2(\mathbb{T})$, defined by $(\mathscr{G}\hat{g})(z):=\frac{1}{z+z^{-1}-\lambda}\hat{g}(z)$ for $\hat{g}\in L^2(\mathbb{T})$ and all $z\in\mathbb{T}$, is well-defined. This operator is the inverse of $\mathscr{F}\Delta\mathscr{F}^*-\lambda I$. It follows that $\lambda\in\rho(\Delta)$.

Conversely, suppose there exists $z_0 \in \mathbb{T}$ such that $z_0 + z_0^{-1} = \lambda$. Set $z_0 = e^{2\pi i\theta}$ for some θ. Consider the sequence $\{w_n\}_{n=0}^{\infty} \subset L^2(\mathbb{T})$, where for each $n \in \mathbb{N}$, $w_n(z) = \sqrt{n/2}$ if $z = e^{2\pi i\theta t}$ with $t \in [-1/n, 1/n]$ and $w_n = 0$ otherwise. For each $n \in \mathbb{N}$, $\|w_n\| = 1$ and

$$\lim_{n\to\infty} \|(\mathscr{F}\Delta\mathscr{F}^* - \lambda I)w_n\| = 0.$$

Then $\{w_n\}_{n=0}^{\infty}$ is a Weyl sequence for $\mathscr{F}\Delta\mathscr{F}^* - \lambda I$ at λ. By Theorem 2.1.7, $\lambda \in \sigma(\Delta)$. Thus $\lambda \in \sigma(\Delta)$ if and only if $z + z^{-1} = \lambda$ for some $z \in \mathbb{T}$. $\diamond$

2.2 Groups and Periodic Graphs

A *binary operation* on a nonempty set G is a function $*: G \times G \to G$ such that $(a, b) \mapsto a * b$.

Remark 2.2.1. Let $*$ be a binary operation on a nonempty set G and let $a, b \in G$. There are several notations for the image of (a, b) under $*$. In the multiplicative notation, $ab := a * b$ is called the *product* of a and b. In the additive notation, $a + b := a * b$ is called the *sum* of a and b. $\diamond$

The operation $*$ is *associative* if for a, b, and c in G, $a * (b * c) = (a * b) * c$. We denote the pair $(G, *)$ by G. We call G a *monoid* if there exists an element $e \in G$ such that for any $a \in G$, $a * e = a$ and $e * a = a$. This element is unique and is called the *identity* of G. A monoid G is called a *group* if for any $a \in G$, there exists an element $b \in G$ such that $a * b = e$ and $b * a = e$. This element is unique and is called the *inverse* of a. Let H be a nonempty subset of G that is closed under the binary operation $*$ on G. If H is itself a group under $*$, then the pair $(H, *)$ is called a *subgroup* of G, and denoted by H. The *index* of a subgroup H of G is the (cardinal) number $[G : H]$ of distinct left (equivalently, right) cosets of H in G. If $[G : H]$ is finite then H is a *finite-index* subgroup of G. A group A is *abelian* if its binary operation is commutative: for $a, b \in A$, $a * b = b * a$. We will use the additive notation for abelian groups. The identity of an abelian group A is denoted by 0, and the inverse of $a \in A$ is denoted by $-a$.

Given two monoids $G := (G, *)$ and $H := (H, \square)$, a *monoid homomorphism* of G into H is a function $f : G \to H$ such that $f(a * b) = f(a) \square f(b)$ for $a, b \in G$. If f is a bijection, then f is called a *monoid isomorphism*, and G and H are said to be *isomorphic* (denoted by

$G \cong H$). A monoid homomorphism $f : G \to G$ is called a *monoid endomorphism* of G. A monoid homomorphism (resp. monoid isomorphism, monoid endomorphism) between two groups is called a *group homomorphism* (resp. *group isomorphism*, *group endomorphism*).

Let $\mathcal{A}$ be an (nonempty) indexing set, and for each $\alpha \in \mathcal{A}$, let $G_\alpha := (G_\alpha, *_\alpha)$ be a group. The *direct product* of $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ is the set-theoretic product of the sets G_α, and denoted by $\prod_{\alpha\in\mathcal{A}} G_\alpha$. The set $\prod_{\alpha\in\mathcal{A}} G_\alpha$ admits a group structure under the componentwise operation $*$ derived from the operations $*_\alpha$: if $(a_\alpha)_{\alpha\in\mathcal{A}}, (b_\alpha)_{\alpha\in\mathcal{A}} \in \prod_{\alpha\in\mathcal{A}} G_\alpha$, then $(a_\alpha)_{\alpha\in\mathcal{A}} * (b_\alpha)_{\alpha\in\mathcal{A}} := (a_\alpha *_\alpha b_\alpha)_{\alpha\in\mathcal{A}}$. If e_α is the identity of G_α, then $(e_\alpha)_{\alpha\in\mathcal{A}}$ is the identity of $\prod_{\alpha\in\mathcal{A}} G_\alpha$. The inverse of $(a_\alpha)_{\alpha\in\mathcal{A}} \in \prod_{\alpha\in\mathcal{A}} G_\alpha$ is $(a_\alpha)^{-1}_{\alpha\in\mathcal{A}} := (a_\alpha^{-1})_{\alpha\in\mathcal{A}}$. If $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ is a family of groups, the *direct sum* of $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ is the subset $\bigoplus_{\alpha\in\mathcal{A}} A_\alpha$ of the direct product $\prod_{\alpha\in\mathcal{A}} A_\alpha$ consisting of elements $(a_\alpha)_{\alpha\in\mathcal{A}}$ with $a_\alpha \in A_\alpha$ such that $a_\alpha = 0$ for all but finitely many indices α. It follows that $\bigoplus_{\alpha\in\mathcal{A}} A_\alpha$ is a subgroup of $\prod_{\alpha\in\mathcal{A}} A_\alpha$.

Let G be a group and let B be a subset of G. Let $\{H_\alpha \mid \alpha \in \mathcal{A}\}$ be the family of all subgroups of G that contain B. The intersection $\langle B\rangle := \bigcap_{\alpha\in\mathcal{A}} H_\alpha$ is the *subgroup* of G *generated* by B. The elements of B are *generators* of the subgroup $\langle B\rangle$. If B is a finite subset of G and $G = \langle B\rangle$, then G is *finitely generated*.

Theorem 2.2.2. [13, Theorem I.2.8] If A is an abelian group and B is a nonempty subset of A, then the subgroup $\langle B\rangle$ of A consists of all linear combinations $n_1b_1+\cdots+n_kb_k$ for $n_1, \dots, n_k \in \mathbb{Z}$ and $b_1, \dots b_k \in B$.

A *basis* of an abelian group A is a subset B of A such that $A = \langle B\rangle$ and for distinct $b_1, \dots, b_k \in B$ and $n_1, \dots, n_k \in \mathbb{Z}$,

$$n_1b_1 + \cdots + n_kb_k = 0 \implies n_i = 0 \text{ for every } i \in \{1, \dots, k\}.$$

Theorem 2.2.3. [13, Theorem II.1.1] Let A be an abelian group. The following conditions are equivalent.

(i) A has a nonempty basis.

(ii) A is isomorphic to a direct sum of copies of the additive group $\mathbb{Z}$ of integers.

(iii) There is a nonempty set B and a function $\iota : B \to A$ with the following property: given an abelian group A' and a function $f : B \to A'$, there exists a unique group homomorphism $\overline{f} : A \to A'$ such that $\overline{f} \circ \iota = f$.

An abelian group A satisfying Theorem 2.2.3 is a *free abelian group* on the set B.

Theorem 2.2.4. [13, Theorem II.1.2] Any two bases of a free abelian group A have the same cardinality.

If B is a basis of a free abelian group A, then the cardinality $|B|$ is the *rank* of A. A *lattice* is a finitely generated free abelian group.

An (right) *action* of a group G on a set S is a function $\cdot : G \times S \to S$, where $(a, s) \mapsto a \cdot s$, such that for all $s \in S$, $e \cdot s = s$, and for $a, b \in G$, $(a * b) \cdot s = a \cdot (b \cdot s)$. An action of G on S is *free* if for any $s \in S$, the induced function $G \to S$, where $a \mapsto a \cdot s$, is injective. The *orbit* of $s \in S$ under the action of G is the set $G \cdot s := \{a \cdot s \mid a \in G\}$, and the set of orbits of S under G is denoted by S/G. An action of G on S is *cofinite* if S/G is a finite set.

For more on periodic graphs, see [3, 14, 15]. A *graph* is a pair $\Gamma := (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of *vertices* and where $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of *edges*. We write edges of Γ as $(u, v) \in \mathcal{E}$, where $u, v \in \mathcal{V}$ (we also use the notation $u \sim v$ for $(u, v) \in \mathcal{E}$). The graph Γ is *directed* if it has a function on $\mathcal{E}$ assigning to each edge an ordered pair of vertices. The graph Γ is *undirected* if $(u, v) \in \mathcal{E}$ if and only if $(v, u) \in \mathcal{E}$. We identify (u, v) with (v, u). A vertex u is *adjacent* to a vertex v if $(u, v) \in \mathcal{E}$, and (u, v) is *incident* to both u and v. If every vertex is adjacent to a finite number of vertices, then Γ is *locally finite*. A *loop* of Γ is an edge that connects a vertex to itself. If Γ is undirected and does not have loops, then Γ is *simple*.

Let G be a finitely generated free abelian group. A locally finite simple graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is *G-periodic* if it is equipped with a free cofinite action of G on $\mathcal{V}$ and $\mathcal{E}$. In this context, G is the *period lattice* of Γ. If Γ is G-periodic, a *fundamental domain* for the action of G on $\mathcal{V}$ is a finite set W of representatives of the orbits of G on $\mathcal{V}$.

It is useful (but not necessary) to consider the graph Γ immersed in $\mathbb{R}^d$ so that $\mathbb{Z}^d$ acts on Γ by translation: for $a \in \mathbb{Z}^d$ and $v \in \mathcal{V}$, $a+v \in \mathcal{V}$, and with edge incidences preserved: for $(u, v) \in \mathcal{E}$,

$a+(u,v) := (a+u, a+v) \in \mathcal{E}$. An edge incident on W has the form $(u, a+v)$ for some $u, v \in W$, and $a \in \mathbb{Z}^d$. The collection of $a \in \mathbb{Z}^d$ such that $(u, a+v) \in \mathcal{E}$ for some $u, v \in W$ is the *support* of Γ and denoted by $\mathcal{A}(\Gamma)$. Figure 2.1 illustrates two periodic graphs.

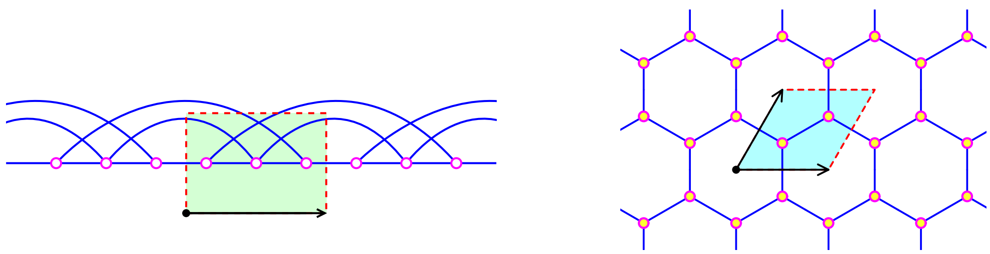

Figure 2.1: Two periodic graphs. The figure to the right is reprinted from [3, Figure 2].

The graph to the left is $\mathbb{Z}$-periodic with support $\{-1, 0, 1\}$, and the graph to the right, known as the *hexagonal lattice*, is $\mathbb{Z}^2$-periodic with support $\{(\pm 1, 0), (0, 0), (0, \pm 1)\}$.

Given $q_1, \ldots, q_d \in \mathbb{N}$, let $Q\mathbb{Z} := \bigoplus_{i=1}^{d} q_i\mathbb{Z}$, which is a finite-index subgroup of $\mathbb{Z}^d$. A $\mathbb{Z}^d$-periodic graph Γ carries the structure of a $Q\mathbb{Z}$-periodic graph. For a chosen fundamental domain W for the $\mathbb{Z}^d$-action on Γ, $Q\mathbb{Z}$ induces the fundamental domain W_Q, which is the union of the sets $a+W$ where $a \in \mathbb{Z}^d$ with $0 \leq a_i < q_i$ for $i = 1, \ldots, n$. The set W_Q is the *Q-expansion* of W. Figure 2.2 shows a $(3, 2)$-expansion of the fundamental domain W of the hexagonal lattice depicted in Figure 2.1.

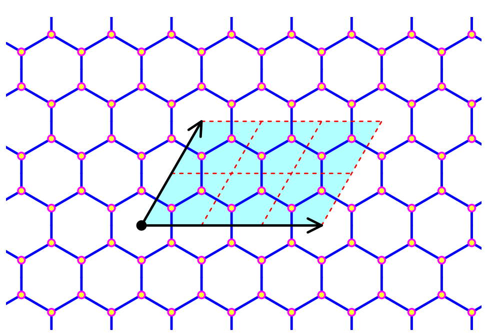

Figure 2.2: A periodic graph. Its period lattice is $(3, 2)\mathbb{Z}$. Reprinted from [1, Figure 2].

2.3 Operators on Periodic Graphs

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a $\mathbb{Z}^d$-periodic graph. A *labeling* (c, V) of Γ is a pair of functions $c : \mathcal{E} \to \mathbb{R}$ and $V : \mathcal{V} \to \mathbb{R}$ that are constant on orbits. For a given labeling $c := (c, V)$, the *periodic graph operator* L_c acts on complex-valued functions f on $\mathcal{V}$. For a function $f : \mathcal{V} \to \mathbb{C}$, the function $L_c f$ is defined by its value at $u \in \mathcal{V}$,

$$(L_c f)(u) := V(u)f(u) - \sum_{(u,v)\in\mathcal{E}} c_{(u,v)} f(v)$$

Example 2.3.1. If c is the constant function of value 1, then L_c is a *discrete periodic Schrödinger operator*. If V is the zero function, then L_c is a *Laplace-Beltrami* operator, and if in addition, c is the constant function of value 1, then L_c is the *graph Laplacian*. $\diamond$

Let $\ell^2(\Gamma) := \{f : \mathcal{V} \to \mathbb{C} \mid \sum_{u\in\mathcal{V}} |f(u)|^2 < \infty\}$ be the vector space of square summable functions on $\mathcal{V}$. Equipping $\ell^2(\Gamma)$ with the inner product $\langle f, g\rangle := \sum_{u\in\mathcal{V}} f(u)\overline{g(u)}$ gives it the structure of a Hilbert space.

Proposition 2.3.2. The operator L_c on $\ell^2(\Gamma)$ is bounded and selfadjoint.

Proof. Let $f \in \ell^2(\Gamma)$. As a generalization of Example 2.1.2, since V and c are constant on vertex and edge orbits, respectively, we obtain

$$\|L_c f\| \leq \Big(\sum_{u\in W} |V(u)| + \sum_{(v,w)\in\mathcal{E}_W} |c_{(v,w)}|\Big) \cdot \|f\|,$$

where $\mathcal{E}_W$ is a set of orbit representatives for the action of $\mathbb{Z}^d$ on $\mathcal{E}$. Thus, L_c is bounded. For any $f, g \in \ell^2(\Gamma)$,

$$\begin{aligned}\langle L_c f, g\rangle &= \sum_{u\in\mathcal{V}} \Big(V(u)f(u) - \sum_{(u,v)\in\mathcal{E}} c_{(u,v)} f(v)\Big)\overline{g(u)} \\ &= \sum_{u\in\mathcal{V}} \Big(V(u)f(u)\overline{g(u)} - \sum_{(u,v)\in\mathcal{E}} c_{(u,v)} f(v)\overline{g(u)}\Big).\end{aligned}$$

Relabeling vertices, we obtain

$$\sum_{u\in\mathcal{V}} f(u)\Big(V(u)\overline{g(u)} - \sum_{(u,v)\in\mathcal{E}} c_{(u,v)}\overline{g(v)}\Big) = \langle f, L_c g\rangle. \qquad \square$$

By Proposition 2.1.6, the spectrum $\sigma(L_c)$ is a compact subset of $\mathbb{R}$.

2.4 The Floquet Transform

For more, see [5, 15, 16]. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a $\mathbb{Z}^d$-periodic graph and let W be a fundamental domain for the action of $\mathbb{Z}^d$ on Γ. As the action of $\mathbb{Z}^d$ commutes with L_c, we may apply the Floquet transform to $\ell^2(\Gamma)$, which reveals more structure of $\sigma(L_c)$. For a function $f \in \ell^2(\Gamma)$, the *Floquet transform* of f is the function

$$\begin{aligned} \mathscr{F}(f) : \mathbb{T}^d \times \mathcal{V} &\longrightarrow \mathbb{C} \\ (z,u) &\longmapsto \sum_{a\in\mathbb{Z}^d} f(a+u) z^{-a} \end{aligned}$$

which satisfies $(\mathscr{F}(f))(z, b+u) = z^b(\mathscr{F}(f))(z,u)$ for $b \in \mathbb{Z}^d$. Thus, $\mathscr{F}(f)$ is determined by its values on W. If $f \in \ell^2(\Gamma)$, then

$$\sum_{u\in W} \int_{\mathbb{T}^d} |(\mathscr{F}(f))(z,u)dz|^2 < \infty,$$

showing $(\mathscr{F}(f))(z,u) \in L^2(\mathbb{T}^d)$ for each $u \in W$. By the Plancherel Theorem, the Floquet transform is a linear isometry between $\ell^2(\Gamma)$ and the Hilbert space $L^2(\mathbb{T}^d, \mathbb{C}^W)$ of square-integrable functions on $\mathbb{T}^d$, where $\mathbb{C}^W$ is the vector space of complex-valued functions on W. For $u \in W$, the function $e_u : W \to \mathbb{C}$ is given by

$$e_u(v) := \begin{cases} 1 & \text{if } v = u, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{e_u \mid u \in W\}$ forms a basis of $\mathbb{C}^W$ called the *standard basis* of $\mathbb{C}^W$. Given $f \in \ell^2(\Gamma)$, we will write $\hat{f}$ for the Floquet transform $\mathscr{F}(f)$ in $L^2(\mathbb{T}^d, \mathbb{C}^W)$.

Proposition 2.4.1. The Floquet transform $\mathscr{F} : \ell^2(\Gamma) \to L^2(\mathbb{T}^d, \mathbb{C}^W)$ is a unitary operator.

Proof. Let $f, g \in \ell^2(\Gamma)$ and write $\hat{f}, \hat{g}$ for their Floquet transforms in $L^2(\mathbb{T}^d, \mathbb{C}^W)$. Then

$$\langle \hat{f}, \hat{g} \rangle_{L^2} = \int\limits_{\mathbb{T}^d} \hat{f}(z,u)\overline{\hat{g}(z,w)}dz = \int\limits_{\mathbb{T}^d} \sum_{a\in\mathbb{Z}^d} f(a+u)z^{-a} \cdot \overline{\sum_{b\in\mathbb{Z}^d} g(b+u)z^{-b}}dz.$$

Since $\overline{z^{-1}} = z$, it follows that

$$\begin{aligned}\int\limits_{\mathbb{T}^d} \sum_{a\in\mathbb{Z}^d} f(a+u)z^{-a} \cdot \overline{\sum_{b\in\mathbb{Z}^d} g(b+u)z^{-b}}dz &= \int\limits_{\mathbb{T}^d} \sum_{a\in\mathbb{Z}^d} f(a+u)z^{-a} \cdot \overline{\sum_{b\in\mathbb{Z}^d} g(b+u)}z^{b}dz \\ &= \int\limits_{\mathbb{T}^d} \sum_{a\in\mathbb{Z}^d} f(a+u)z^{-a} \cdot \sum_{b\in\mathbb{Z}^d} \overline{g(b+u)}z^{b}dz.\end{aligned}$$

Similar to Example 2.1.8, observe that if $a \neq b$, then $\int\limits_{\mathbb{T}^d} f(a+u)z^{-a}\overline{g(b+u)}z^b dz = 0$. Then

$$\begin{aligned}\langle \hat{f}, \hat{g} \rangle_{L^2} &= \int\limits_{\mathbb{T}^d} \sum_{a\in\mathbb{Z}^d} f(a+u)z^{-a} \cdot \sum_{b\in\mathbb{Z}^d} \overline{g(b+u)}z^{b}dz \\ &= \sum_{a\in\mathbb{Z}^d} f(a+u)\overline{g(a+u)} = \langle f, g \rangle_{\ell^2}.\end{aligned}$$

□

Let $f \in \ell^2(\Gamma)$ with Floquet transform $\hat{f} \in L^2(\mathbb{T}^d, \mathbb{C}^W)$ and let $\mathscr{F}^*$ be the adjoint of $\mathscr{F}$. For $u \in W$, the function $\hat{f}(u)$ is a function of z. The action of $\mathscr{F} L_c \mathscr{F}^*$ on $\hat{f}$ is defined by the value of the function $\mathscr{F} L_c \mathscr{F}^* \hat{f}$ at $u \in W$:

$$(\mathscr{F} L_c \mathscr{F}^* \hat{f})(u) = V(u)\hat{f}(u) - \sum_{(u,a+v)\in\mathcal{E}} z^a c_{(u,a+v)}\hat{f}(v),$$

where $v \in W$ and $a \in \mathcal{A}(\Gamma)$. In using the standard basis for $\mathbb{C}^W$, the operator $\mathscr{F} L_c \mathscr{F}^*$ becomes multiplication by a $|W|\times|W|$ matrix $L_c(z)$ whose rows and columns are indexed by the vertices of W. Let $u, v \in W$. If $\delta_{u,v}$ is the Kronecker delta function, the matrix entry in position (u, v) is given by the finite sum

$$\delta_{u,v}V(u) - \sum_{(u,a+v)\in\mathcal{E}} c_{(u,a+v)} z^a. \tag{2.1}$$

This entry is a Laurent polynomial in $z \in \mathbb{T}^d$ with exponents coming from the support $\mathcal{A}(\Gamma)$. The matrix $L_c(z)$ is called the *Floquet matrix* of L_c. Let $a \in \mathcal{A}(\Gamma)$. Observe that $(u, a+v) \in \mathcal{E}$ if and only if $(-a+u, v) \in \mathcal{E}$, $c_{(u,a+v)} = c_{(-a+u,v)}$, and for $z \in \mathbb{T}^d$, $\overline{z^a} = z^{-a}$. It follows that for each $z \in \mathbb{T}^d$, this matrix is Hermitian since $L_c(z)^T = L_c(z^{-1}) = L(\overline{z}) = \overline{L(z)}$.

As a generalization of Example 2.1.8, we have the following proposition.

Proposition 2.4.2. Let $\lambda \in \mathbb{C}$. The operator

$$\mathscr{F} L_c \mathscr{F}^* - \lambda I : L^2(\mathbb{T}^d, \mathbb{C}^W) \to L^2(\mathbb{T}^d, \mathbb{C}^W)$$

is a bijection if and only if, for each $z \in \mathbb{T}^d$, $L_c(z) - \lambda I_{|W|}$ is a bijection. The spectrum $\sigma(L_c)$ is the union of the eigenvalues of the matrix $L_c(z)$ for each $z \in \mathbb{T}^d$.

Example 2.4.3. [3, Example 1.1] Let Γ be the hexagonal lattice from Figure 2.1. Figure 2.3 depicts a labeling in a neighborhood of a fundamental domain W for the action of $\mathbb{Z}^2$ on Γ. The set $W = \{u, v\}$ consists of two vertices. There are three edge orbits, with labels α, β, and γ.

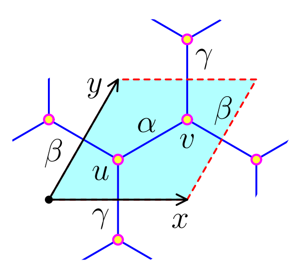

Figure 2.3: A labeling of the hexagonal lattice. Reprinted from [3, Figure 3].

Let L be the operator with potential V and with edge orbits labeled by α, β, and γ as in Figure 2.3.

Let $(x, y) \in \mathbb{T}^2$. Then the operator $\mathscr{F} L \mathscr{F}^*$ acts on $\hat{f} \in L^2(\mathbb{T}^2, \mathbb{C}^2)$ by

$$
\begin{aligned}
(\mathscr{F} L \mathscr{F}^* \hat{f})(u) &= V(u)\hat{f}(u) - \alpha \hat{f}(v) - \beta x^{-1}\hat{f}(v) - \gamma y^{-1}\hat{f}(v), \\
(\mathscr{F} L \mathscr{F}^* \hat{f})(v) &= V(v)\hat{f}(v) - \alpha \hat{f}(u) - \beta x \hat{f}(u) - \gamma y \hat{f}(u).
\end{aligned}
$$

Collecting coefficients from $\hat{f}(u)$ and $\hat{f}(v)$, the operator $\mathscr{F} L \mathscr{F}^*$ becomes multiplication by the matrix

$$
L(x, y) = \begin{pmatrix} V(u) & -\alpha - \beta x^{-1} - \gamma y^{-1} \\ -\alpha - \beta x - \gamma y & V(v) \end{pmatrix}.
$$

For $(x, y) \in \mathbb{T}^2$, $L(x, y)^T = L(x^{-1}, y^{-1}) = L(\overline{x}, \overline{y}) = \overline{L(x, y)}$, so the matrix $L(x, y)$ is Hermitian, showing that the operator L is selfadjoint. $\diamond$

2.5 Bloch Varieties

This section is adapted from [1, Section 1.3]. For $z \in \mathbb{T}^d$, the matrix $L_c(z)$ is Hermitian. Thus, it has $|W|$ real eigenvalues $\lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_{|W|}(z)$. For $j \in \{1, \ldots, |W|\}$, the function $\lambda_j : \mathbb{T}^d \to \mathbb{R}$ is the *j-th band function*, and its graph $\{(z, \lambda_j(z)) \mid z \in \mathbb{T}^d\}$ is the *j-th branch*. The image of λ_j is the *j-th spectral band*.

The *dispersion polynomial* of $L_c(z)$ is the characteristic polynomial $D_c := \det(L_c(z) - \lambda I_{|W|})$, and the (*real*) *Bloch variety* of the operator L_c is the hypersurface

$$
\mathcal{B}_{L_c}(\mathbb{R}) = \mathit{Var}(D_c) := \{(z, \lambda) \in \mathbb{T}^d \times \mathbb{R} \mid D_c(z, \lambda) = 0\}.
$$

The image of the Bloch variety under the projection to $\mathbb{R}$ is the spectrum $\sigma(L_c)$, and the projection is a function λ on the Bloch variety.

Example 2.5.1. Let L be the operator from Example 2.4.3. Figure 2.4 shows the Bloch variety of the operator L with zero potential and edge orbits labeled by $\alpha = 6, \beta = 2$, and $\gamma = 3$.

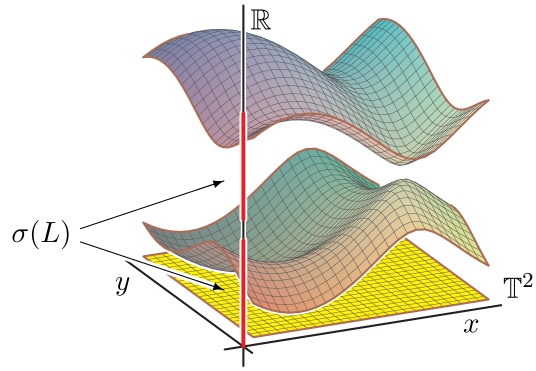

Figure 2.4: A Bloch variety for the hexagonal lattice. Reprinted from [3, Figure 4].

The spectrum $\sigma(L)$ is the union of two spectral bands. $\diamond$

To use strong methods from classical algebraic geometry, it is natural to allow complex parameters $c : \mathcal{E} \to \mathbb{C}$ and $V : \mathcal{V} \to \mathbb{C}$ and variables $z \in (\mathbb{C}^\times)^d$ and $\lambda \in \mathbb{C}$. With complex parameters and variables, the matrix $L_c(z)$ is no longer Hermitian, but it retains the property $L_c(z)^T = L_c(z^{-1})$. The (*complex*) *Bloch variety* of the operator L_c is the hypersurface

$$\mathcal{B}_L = Var(D_c) := \left\{(z, \lambda) \in (\mathbb{C}^\times)^d \times \mathbb{C} \mid D_c(z, \lambda) = 0\right\}.$$

In the complex Bloch variety, we cannot distinguish the branches λ_j. Thus, we will consider projection to the last coordinate as a function λ on the complex Bloch variety. A *critical point* of the function λ on the complex Bloch variety of the operator L_c is a point where the gradients in $(\mathbb{C}^\times)^d \times \mathbb{C}$ of λ and D_c are linearly dependent. The value of the function λ on such a critical point is a *critical value* of λ.

3. ALGEBRAIC GEOMETRY

As we saw in Section 2.5, the spectrum of a periodic graph operator is the image of a projection of the *Bloch variety* of the operator, which is an affine algebraic variety. Thus, we outline the algebraic geometry relevant to the study of Bloch varieties. For more background in algebraic geometry, see any of [9, 17, 18, 19].

3.1 Ideals and Varieties

A *monomial* in the variables $x_1, \dots, x_n$ is the product $x^a := x_1^{a_1} \cdots x_n^{a_n}$, where $a_1, \dots, a_n$ are non-negative integers. Given a monomial x^a, its (*total*) *degree* is the sum $\sum_{i=1}^{n} a_i$. A *polynomial* f in the variables $x_1, \dots, x_n$ is a linear combination of monomials

$$f = \sum_{a \in \mathbb{N}^n} c_a x^a,$$

where $c_a \in \mathbb{C}$ is a *coefficient* and all but finitely many coefficients are 0. The product $c_a x^a$ of a coefficient c_a and a monomial x^a is a *term*. The *support* of a polynomial f is the set $\mathcal{A}(f) \subset \mathbb{N}^n$ of exponent vectors that appear in f with a nonzero coefficient. The set of polynomials in the variables $x_1, \dots, x_n$ is denoted by $\mathbb{C}[x_1, \dots, x_n]$ and it is a ring under addition and multiplication. A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is *irreducible* if it is not constant and whenever there exist polynomials $g, h \in \mathbb{C}[x_1, \dots, x_n]$ with $f = gh$, either g or h is constant.

The set of n-tuples $z = (z_1, \dots, z_n)$ of complex numbers is (complex) *affine n-space* and it is denoted by $\mathbb{C}^n$. A polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ defines a function $f : \mathbb{C}^n \to \mathbb{C}$. Thus, for a subset $S \subseteq \mathbb{C}[x_1, \dots, x_n]$ of polynomials, the set

$$\mathit{Var}(S) := \{z \in \mathbb{C}^n \mid f(z) = 0 \text{ for } f \in S\}$$

is the *affine (algebraic) variety* defined by S. If S consists of a single polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, then $\mathit{Var}(S) = \mathit{Var}(f)$ is a *hypersurface*. If S consists of finitely many polynomials $f_1, \dots, f_m$, then we write $Var(f_1, \dots, f_m)$ for $Var(S)$. If X and Y are affine varieties and $X \subseteq Y$, then X is a *subvariety* of Y.

The product $X \times Y$ of two affine varieties X and Y is an affine variety. If $X \subseteq \mathbb{C}^m$ is defined by $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_m]$ and $Y \subseteq \mathbb{C}^n$ is defined by $g_1, \dots, g_s \in \mathbb{C}[y_1, \dots, y_n]$, then $X \times Y \subseteq \mathbb{C}^m \times \mathbb{C}^n = \mathbb{C}^{m+n}$ is defined by $f_1, \dots, f_r, g_1, \dots, g_s \in \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$.

Example 3.1.1. Affine n-space $\mathbb{C}^n = Var(0)$ and the empty set $\emptyset = Var(1)$ are affine varieties. $\diamond$

Example 3.1.2. The set $\mathrm{Mat}_{m\times n}(\mathbb{C})$ of $m\times n$ matrices may be identified with $\mathbb{C}^{mn}$. $\diamond$

Example 3.1.3. The set $\mathrm{GL}_m(\mathbb{C})$ of matrices $M \in \mathrm{Mat}_{m\times m}(\mathbb{C})$ with $\det M \neq 0$ may be identified with the set $\{(t, M) \in \mathbb{C} \times \mathrm{Mat}_{m\times m}(\mathbb{C}) \mid t \det M = 1\}$, making it into a hypersurface in $\mathbb{C} \times \mathrm{Mat}_{m\times m}(\mathbb{C})$. If $M \in \mathrm{GL}_m(\mathbb{C})$ and $N \in \mathrm{GL}_m(\mathbb{C})$, then $\det(MN) = (\det M)\cdot(\det N) \neq 0$. Let $M \in \mathrm{GL}_m(\mathbb{C})$. Since $\det M \neq 0$, M has an inverse M^{-1}, whose determinant is nonzero. Thus, $M^{-1} \in \mathrm{GL}_m(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$ is a group known as the *general linear group*. If $m = 1$, then this is the group of invertible elements of $\mathbb{C}$ and denoted by $\mathbb{C}^\times$. The group of invertible diagonal $m \times m$ matrices with complex entries is the *algebraic m-torus* $(\mathbb{C}^\times)^m$. $\diamond$

Example 3.1.4. Let Γ be a $\mathbb{Z}^d$-periodic graph and let L_c be a periodic graph operator on $\ell^2(\Gamma)$. Let D_c be the dispersion polynomial of L_c. The complex Bloch variety $\mathcal{B}_{L_c}$ is the hypersurface

$$Var(D_c) = \{(z, \lambda) \in (\mathbb{C}^\times)^d \times \mathbb{C} \mid D_c(z, \lambda) = 0\}.$$

Thus, $\mathcal{B}_{L_c}$ is an affine variety. $\diamond$

For a subset $Z \subseteq \mathbb{C}^n$, consider the set $I(Z) := \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(z) = 0 \text{ for all } z \in Z\}$ of polynomials that vanish on Z. If $Y \subseteq Z$ and $f \in I(Z)$, then $f(z) = 0$ for all $z \in Z$. In particular, $f(z) = 0$ for $z \in Y$. This means $f \in I(Y)$.

Let $S \subseteq T \subseteq \mathbb{C}[x_1, \dots, x_n]$. If $z \in Var(T)$, then $f(z) = 0$ for all $f \in T$. In particular, $g(z) = 0$ for all $g \in S \subseteq T$.

Thus, we may consider Var and I as the inclusion-reversing maps

$$\{\text{subsets of } \mathbb{C}[x_1, \ldots, x_n]\} \underset{I}{\overset{Var}{\rightleftarrows}} \{\text{subsets of } \mathbb{C}^n\}. \tag{3.1}$$

We will refine this correspondence to give a dictionary between structures in algebra and geometry.

An *ideal* of $\mathbb{C}[x_1, \ldots, x_n]$ is a subset $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ that is closed under addition and multiplication. Let $Z \subseteq \mathbb{C}^n$. If $f, g \in I(Z)$ and $h \in \mathbb{C}[x_1, \ldots, x_n]$, then $f(z) + h(z)g(z) = 0 + h(z) \cdot 0 = 0$ for all $z \in Z$. This implies $f + gh \in I(Z)$ so the set $I(Z)$ is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. This is the *ideal of Z*.

Let $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a subset of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. The *ideal $\langle S \rangle$* is the smallest ideal of $\mathbb{C}[x_1, \ldots, x_n]$ containing S. Explicitly, $\langle S \rangle$ is the set of polynomials $h_1 f_1 + \cdots + h_m f_m$ where $h_i \in \mathbb{C}[x_1, \ldots, x_n]$ and $f_i \in S$ for $i = 1, \ldots, m$.

Lemma 3.1.5. For any subset $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$, $Var(S) = Var(\langle S \rangle)$.

Proof. By definition, $S \subseteq \langle S \rangle$, so $Var(\langle S \rangle) \subseteq Var(S)$. Conversely, let $g \in \langle S \rangle$. Then there exist $h_1, \ldots, h_m \in \mathbb{C}[x_1, \ldots, x_n]$ and $f_1, \ldots, f_m \in S$ such that $g = h_1 f_1 + \cdots + h_m f_m$. For $z \in Var(S)$, $g(z) = h_1(z) f_1(z) + \cdots + h_m(z) f_m(z) = h_1 \cdot 0 + \cdots + h_m \cdot 0 = 0$, so $Var(S) \subset Var(\langle S \rangle)$. $\square$

An affine variety is often defined by infinitely many polynomials. Hilbert's Basis Theorem tells us that we only need finitely many of them.

Theorem 3.1.6 (Hilbert's Basis Theorem [20])**.** Every ideal I of $\mathbb{C}[x_1, \ldots, x_n]$ is finitely generated.

Let $X = Var(I) \subseteq \mathbb{C}^n$ be an affine variety defined by an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. By Hilbert's Basis Theorem, there exist $f_1, \ldots, f_m \in I$ that generate I. Thus, $I = \langle f_1, \ldots, f_m \rangle$ and by Lemma 3.1.5, $Var(I) = Var(f_1, \ldots, f_m)$.

Lemma 3.1.7. Let $Z \subseteq \mathbb{C}^n$. Then $I(Var(I(Z))) = I(Z)$ and $Var(I(Z))$ is the smallest variety containing Z.

Proof. Let $X = \mathit{Var}(I(Z))$ be the affine variety defined by the ideal $I(Z)$ and let $f \in I(Z)$. By definition of $I(X)$, $f(z) = 0$ for all $z \in X$. Thus $f \in I(X)$. As $Z \subseteq X$, it follows that $I(X) \subseteq I(Z)$. Thus, $I(Z) = I(X)$.

Suppose that $Y \subseteq \mathbb{C}^n$ is an affine variety with $Z \subseteq Y \subseteq X$. Then $I(X) \subseteq I(Y) \subseteq I(Z) = I(X)$ so $I(X) = I(Y)$. Then $X = Y$. Thus, X is the smallest variety containing Z. $\square$

Correspondence (3.1) refines to ideals of $\mathbb{C}[x_1, \ldots, x_n]$ and subvarieties of $\mathbb{C}^n$.

$$\left\{\text{ideals of } \mathbb{C}[x_1, \ldots, x_n]\right\} \underset{I}{\overset{\mathit{Var}}{\rightleftarrows}} \left\{\text{subvarieties of } \mathbb{C}^n\right\}. \tag{3.2}$$

This correspondence is not a bijection since the function Var is not surjective and the function I is not injective. For example, $I(\mathit{Var}(x^2)) = \langle x \rangle$. We refine Correspondence (3.2) by restricting the domain of Var to radical ideals. An ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is *radical* if whenever $f^r \in I$ for some positive integer r, then $f \in I$. The *radical* of an ideal I is the set $\sqrt{I} := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^m \in I \text{ for some positive integer } m\}$, and it is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$.

Lemma 3.1.8. If $Z \subseteq \mathbb{C}^n$, then $I(Z)$ is a radical ideal of $\mathbb{C}[x_1, \ldots, x_n]$. If I is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$, then $\mathit{Var}(I) = \mathit{Var}(\sqrt{I})$.

Proof. Suppose $z \in Z$ and $f^r \in I(Z)$ for some positive integer r. Then $f^r(z) = 0$ if and only if $f(z) = 0$. Thus, as $z \in Z$ was arbitrary, $f \in I(Z)$ so $I(Z)$ is radical.

Since $I \subseteq \sqrt{I}$, $\mathit{Var}(\sqrt{I}) \subseteq \mathit{Var}(I)$. Let $z \in \mathit{Var}(I)$ and let $f \in \sqrt{I}$. Then $f^r \in I$ for some positive integer r. This means $f^r(z) = 0$ which implies $f(z) = 0$. Thus, $z \in \mathit{Var}(\sqrt{I})$. $\square$

To restrict Correspondence (3.2) to a bijection, we need Hilbert's Nullstellensatz.

Theorem 3.1.9 (Hilbert's Nullstellensatz [21])**.** Suppose I is an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Then $I(\mathit{Var}(I)) = \sqrt{I}$.

A consequence of Hilbert's Nullstellensatz is the inclusion reversing correspondence

$$\left\{\text{radical ideals of } \mathbb{C}[x_1, \ldots, x_n]\right\} \underset{I}{\overset{\mathit{Var}}{\rightleftarrows}} \left\{\text{subvarieties of } \mathbb{C}^n\right\}. \tag{3.3}$$

between radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$ and subvarieties of $\mathbb{C}^n$. The functions Var and I are inverses of each other, making this correspondence into a bijection.

3.2 Topology

To discuss geometric properties of affine varieties in $\mathbb{C}^n$, it is convenient to give $\mathbb{C}^n$ a topology. A *topology* on $\mathbb{C}^n$ is a collection of subsets of $\mathbb{C}^n$ known as *open sets*, such that

(1) the empty set and $\mathbb{C}^n$ are open sets,

(2) an arbitrary union of open sets is open, and

(3) a finite intersection of open sets is open.

In this context, the set $\mathbb{C}^n$ with a topology is a *topological space*.

A topology on $\mathbb{C}^n$ may be specified by selecting a collection of subsets of $\mathbb{C}^n$ known as *basic open sets* and defining this topology to be the smallest collection of subsets containing basic open sets and that satisfy the properties (1) – (3) for open sets.

A *closed set* is the complement of an open set. Thus, a topology on $\mathbb{C}^n$ may also be described in terms of closed sets. Both the empty set and $\mathbb{C}^n$ are closed sets because they are the complements of one another. The intersection of an arbitrary collection of closed sets is closed, and the finite union of two closed sets is closed.

Let $I, J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be ideals. The *sum* of I and J is the set $I + J := \{f + g \mid f \in I, g \in I\}$, and it is the ideal $\langle I, J\rangle$ generated by $I \cup J$. The *product* of I and J is the ideal $I \cdot J := \langle fg \mid f \in I, g \in J\rangle \subset I \cap J$.

Lemma 3.2.1. [22, Remark 1.3 5,6] Let $I, J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be ideals. Then

(a) $Var(I + J) = Var(I) \cap Var(J)$, and

(b) $Var(I \cdot J) = Var(I \cap J) = Var(I) \cup Var(J)$.

By Lemma 3.2.1, a finite union of affine varieties is an affine variety, and together with the Hilbert Basis Theorem, the intersection of an arbitrary collection of affine varieties is an affine variety. Thus, affine varieties in $\mathbb{C}^n$ are closed sets of a topology on $\mathbb{C}^n$ known as the *Zariski topology* on $\mathbb{C}^n$. An affine variety in $\mathbb{C}^n$ is a *Zariski closed set*. A *Zariski open set* is the complement of a

Zariski closed set. By Lemma 3.1.7, if $Z \subseteq \mathbb{C}^n$, the *Zariski closure* of Z is the smallest affine variety $\overline{Z} \subseteq \mathbb{C}^n$ containing Z. The Zariski topology on a subvariety X of $\mathbb{C}^n$ is the subspace topology inherited from the Zariski topology on $\mathbb{C}^n$. A subset $Z \subseteq X$ is *Zariski dense* in X if $\overline{Z} = X$.

Affine subvarieties of $\mathbb{C}^n$ may also be equipped with the subspace topology inherited from the *Euclidean topology* on $\mathbb{C}^n$; thus, we may compare the Zariski topology on $\mathbb{C}^n$ with the *Euclidean* topology on $\mathbb{C}^n$. In the Euclidean topology on $\mathbb{C}^n$, the basic open sets are given by a collection of balls. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, a *ball* is a set $B_r(z) := \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid \sum |x_i - z_i|^2 < r\}$ for $r \geq 0$. On the other hand, in the Zariski topology, basic open sets are complements of hypersurfaces. Thus, Zariski open sets (resp. Zariski closed sets) are open sets in the Euclidean topology (resp. closed sets in the Euclidean topology), but the converse is not true.

An affine variety $X \subseteq \mathbb{C}^n$ is *reducible* if it is the union of two proper affine varieties $Y, Z \subset \mathbb{C}^n$. If X is not reducible, then it is *irreducible*. An ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is *prime* if $fg \in I$ implies $f \in I$ or $g \in I$. The geometric notion of irreducibility of an affine variety $X \subseteq \mathbb{C}^n$ may be interpreted as an algebraic property of $I(X)$.

Theorem 3.2.2. [18, Proposition 5.3] An affine variety $X \subseteq \mathbb{C}^n$ is irreducible if and only if $I(X)$ is a prime ideal.

The *dimension* of an affine variety $X \subseteq \mathbb{C}^n$ is the largest integer m such that there exists a strictly decreasing chain $X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_m \supsetneq \emptyset$ of irreducible subvarieties X_i of X. If X has dimension m, then its *codimension* is $n - m$.

3.3 Regular Maps

A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ defines a function $f : \mathbb{C}^n \to \mathbb{C}$ by evaluation at points in $\mathbb{C}^n$. If $X \subseteq \mathbb{C}^n$ is an affine variety, $f \in \mathbb{C}[x_1, \ldots, x_n]$ restricts to a *regular function* $X \to \mathbb{C}$. The set of regular functions on X has the structure of a ring given by multiplication of regular functions. This is the *coordinate ring* of X and it is denoted by $\mathbb{C}[X]$. The restriction of functions $f : \mathbb{C}^n \to \mathbb{C}$ to regular functions on X defines a surjective ring homomorphism $\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[X]$ and the kernel of this homomorphism is the ideal $I(X)$. It follows that $\mathbb{C}[X] \simeq \mathbb{C}[x_1, \ldots, x_n]/I(X)$.

A *$\mathbb{C}$-algebra* is a ring containing $\mathbb{C}$ as a subring, and it follows that it is equipped with the structure of a complex vector space. Let $X \subseteq \mathbb{C}^n$ be an affine variety. The cooridnate ring $\mathbb{C}[X] \simeq \mathbb{C}[x_1, \ldots, x_n]/I(X)$ is a $\mathbb{C}$-algebra and it has the structure of a complex vector space given by addition on $\mathbb{C}[X]$ and scalar multiplication by complex numbers.

Let $\varphi_1, \ldots, \varphi_s \in \mathbb{C}[X]$ be regular functions on an affine variety $X \subseteq \mathbb{C}^n$, and let $Y \subseteq \mathbb{C}^s$ be an affine variety. A function $\varphi : X \to Y$ such that $x \mapsto (\varphi_1(x), \ldots, \varphi_s(x))$ is a *regular map*. A regular map φ is an *isomorphism* if it is bijective and its inverse is a regular map. In this case, the affine varieties X and Y are *isomorphic*. A regular map $\varphi : X \to Y$ induces a homomorphism $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ of $\mathbb{C}$-algebras given by $f \mapsto f \circ \varphi$.

3.4 Projective Varieties

The set of one-dimensional subspaces of $\mathbb{C}^{n+1}$ is (*complex*) *projective n-space* and is denoted by $\mathbb{P}^n$. A point $\ell \in \mathbb{P}^n$ may be represented by the coordinates $[x_0, \ldots, x_n]$ of a nonzero vector in the subpace $\ell \subset \mathbb{C}^{n+1}$. If $r \in \mathbb{C}^\times$, then $r \cdot [x_0, \ldots, x_n] = [rx_0, \ldots, rx_n]$. This gives an equivalence relation $\sim$ on the set $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ of nonzero vectors in $\mathbb{C}^{n+1}$ (where $\mathbf{0}$ is the zero vector in $\mathbb{C}^{n+1}$) and $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\})/\sim$. The coordinates $[x_0, \ldots, x_n]$ are *homogeneous coordinates*.

For each $i = 0, \ldots, n$, let U_i be the set of points $\ell \in \mathbb{P}^n$ whose i-th coordinate is nonzero. Divide by this i-th coordinate to obtain a representative of ℓ of the form $[x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n]$. Thus $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$. The sets U_i are coordinate charts for $\mathbb{P}^n$ as a manifold.

Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial of degree d and denote by f_k the sum of the terms of f of degree k. The polynomial f_k is the *k-th homogeneous component of f*. If $[x_0, \ldots, x_n]$ and $[rx_0, \ldots, rx_n]$ are representatives of a point $\ell \in \mathbb{P}^n$, then

$$f(rx_0, \ldots, rx_n) = r^d f_d(x_0, \ldots, x_n) + \cdots + r f_1(x_0, \ldots, x_n) + f_0(x_0, \ldots, x_n).$$

Observe that $f(\ell)$ is a well-defined number only if f is constant. A polynomial f is *homogeneous* of degree d if $f = f_d$. Homogeneous polynomials are also known as *forms*. The set of homogeneous polynomials of degree d is denoted by $\mathbb{C}[x_0, \ldots, x_n]_d$. The set

$$Var(f_1, \ldots, f_s) := \left\{\ell \in \mathbb{P}^n \mid f_i \text{ is homogeneous and } f_i(\ell) = 0 \text{ for } i = 1, \ldots, s\right\}$$

defines a *projective variety*. An ideal I of $\mathbb{C}[x_0, \ldots, x_n]$ is *homogeneous* if $f \in I$ implies that all the homogeneous components of f are in I. It follows that projective varieties are given by homogeneous ideals. If $Z \subseteq \mathbb{P}^n$ is a subset of projective space, the set $I(Z) := \{f \in \mathbb{C}[x_0, \ldots, x_n] \mid f(\ell) = 0 \text{ for all } \ell \in Z\}$ is the *ideal of Z*. This is a homogeneous ideal.

4. POLYTOPES AND TORIC VARIETIES

In Chapter 5 we study the geometry of the Newton polytope of the dispersion polynomial of a periodic graph operator and give criteria when the dispersion polynomial is irreducible. To study the asymptotics of the complex Bloch variety of a periodic graph operator, in Chapter 6 we compactify this Bloch variety in the toric variety associated to the fan of its Newton polytope. Here, we outline the background on polytopes and toric varieties necessary for our study of complex Bloch varieties. For more on polytopes, see [23, 24], and for a thorough treatment of toric varieties, see any of [9, 23, 25, 26, 27].

4.1 Polytopes

Let $\mathcal{A} \subset \mathbb{R}^n$ be a finite set. The sum $\sum\limits_{a\in\mathcal{A}} \lambda_a a$, where $\lambda_a \geq 0$ and $\sum\limits_{a\in\mathcal{A}} \lambda_a = 1$, is a *convex combination* of points $a \in \mathcal{A}$. The *convex hull* of $\mathcal{A}$ is the set

$$\mathrm{conv}(\mathcal{A}) := \left\{ \sum_{a\in\mathcal{A}} \lambda_a a \mid \sum_{a\in\mathcal{A}} \lambda_a = 1 \text{ and } \lambda_a \geq 0 \text{ for all } a \in \mathcal{A} \right\}$$

of all convex combinations of points $a \in \mathcal{A}$. The convex hull of a finite set of points is a *polytope*. If $\mathcal{A} \subset \mathbb{Z}^n$, then $\mathrm{conv}(\mathcal{A})$ is a *lattice polytope*.

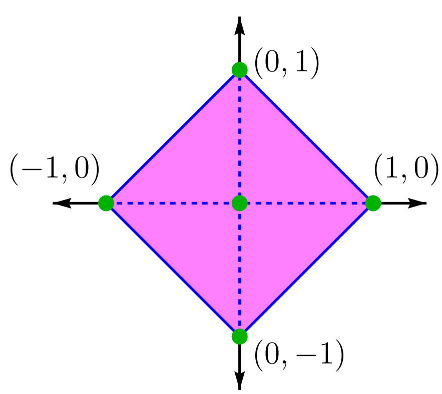

Figure 4.1: The two-dimensional cross polytope.

Example 4.1.1. The polytope $P = \operatorname{conv}\{(\pm 1, 0), (0, \pm 1)\} \subset \mathbb{R}^2$ depicted in Figure 4.1 is the two-dimensional *cross polytope*. $\diamond$

Let V be a linear subspace of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. A translate $x + V := \{x + v \in \mathbb{R}^n \mid v \in V\}$ of V by x is an *affine subspace*. The dimension of $x + V$ is the dimension of V. The *affine span* of a set $X \subset \mathbb{R}^n$ is the intersection $\operatorname{Aff}(X)$ of all affine subspaces containing it. The *dimension* of a polytope is the dimension of its affine span.

Proposition 4.1.2. [24, Theorem 2.15] A polytope is closed and bounded.

Let $w \in \mathbb{R}^n$ and let $P \subset \mathbb{R}^n$ be a polytope. Since P is closed and bounded, the dot product $c \mapsto w \cdot c$, as a function on $\mathbb{R}^n$ restricted to P, has a minimum value $h_P(w) := \min\{w \cdot c \mid c \in P\}$. The set $P_w := \{p \in P \mid w \cdot p = h_P(w)\}$ is the *face* of P *exposed* by w and the vector w is an *inner normal* of P_w. A face of P is also a polytope since $P_w = \operatorname{conv}\big\{a \in \mathcal{A} \mid w \cdot a = h_P(w)\big\}$. A face of P of codimension n is a *vertex*, a face of codimension $n - 1$ is an *edge*, and a face of P of codimension 1 is a *facet*.

The *Minkowski sum* of two polytopes $P, Q \subset \mathbb{R}^n$,

$$P + Q := \{p + q \in \mathbb{R}^n \mid p \in P, q \in Q\},$$

is a polytope. If $\lambda > 0$, the *dilation* $\lambda P := \{\lambda p \in \mathbb{R}^n \mid p \in P\}$ is also a polytope.

A polytope $P \subset \mathbb{R}^n$ is *indecomposable* if whenever there exist polytopes $Q, R \subset \mathbb{R}^n$ with $P = Q + R$, then Q or R is a point in $\mathbb{Z}^n$. Otherwise, P is *decomposable*. A lattice polytope Q is *homothetic* to P if there exist a point $a \in \mathbb{Z}^n$ and a positive rational number r such that $P = a + rQ$. A lattice polytope P is *only homothetically decomposable* if whenever there exist polytopes $Q, R \subset \mathbb{R}^n$ with $P = Q + R$, then Q and R are homothetic to P.

4.2 Newton Polytopes

A *Laurent monomial* in the variables $z_1, \ldots, z_n$ is the product $z^a := z_1^{a_1} \cdots z_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. A *Laurent polynomial* f in the variables $z_1, \ldots, z_n$ is a linear combination of

Laurent monomials

$$f = \sum_{a \in \mathbb{Z}^n} c_a z^a,$$

where $c_a \in \mathbb{C}$ is a *coefficient* and all but finitely many coefficients are 0. The *support* of the Laurent polynomial f is $\mathcal{A}(f) := \{a \in \mathbb{Z}^n \mid c_a \neq 0\}$. The product $c_a z^a$ of a coefficient c_a and a monomial z^a is a *term*.

The set of Laurent polynomials in the variables $z_1, \ldots, z_n$ is denoted by $\mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$, and it is a ring under addition and multiplication. A Laurent polynomial $f \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$ is *irreducible* if it is not a term, and whenever there exist polynomials $g, h \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$ with $f = gh$, either g or h is a term.

Let $f \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$ be a Laurent polynomial with support $\mathcal{A}(f) \subset \mathbb{Z}^n$. The *Newton polytope* of f is the convex hull $\mathrm{Newt}\, f := \mathrm{conv}(\mathcal{A}(f)) \subset \mathbb{R}^n$ of its support. Since the support of f lies in $\mathbb{Z}^n$, $\mathrm{Newt}\, f$ is a lattice polytope. For a face F of the polytope $\mathrm{Newt}\, f$, the *facial polynomial* f_F of f is the sum of the terms of f whose exponent vectors lie in $\mathcal{A}(f) \cap F$. A monomial whose exponent vector is a vertex of $\mathrm{Newt}\, f$ is an *extreme monomial* of f.

Given two Laurent polynomials f and g, the Newton polytope of their product fg is the Minkowski sum $\mathrm{Newt}\, fg = \mathrm{Newt}\, f + \mathrm{Newt}\, g$. A Laurent polynomial $f \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$ is *only homothetically reducible* if it is not a term and whenever there exist $g, h \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$ with $f = gh$, then $\mathrm{Newt}\, g$ and $\mathrm{Newt}\, h$ are homothetic to $\mathrm{Newt}\, f$.

Example 4.2.1. Let H be a hyperplane of $\mathbb{R}^n$, let $\mathcal{A} \subset H$ be a finite set, and let v be a point in $\mathbb{R}^n \smallsetminus H$. The convex hull of $\mathcal{A} \cup \{v\}$ is a *pyramid* with *apex* v. A face of $\mathrm{conv}(\mathcal{A} \cup \{v\})$ is *apical* if it contains the apex, v. Pyramids are only homothetically decomposable [28].

A Laurent polynomial in $\mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]$ of the form $g(x, y, z) = z^a + f(x, y)$, where $a \neq 0$ and $f \in \mathbb{C}[x^{\pm}, y^{\pm}] \smallsetminus \{0\}$, is only homothetically reducible. If $a = 1$, $\mathrm{Newt}\, g$ is indecomposable. $\diamond$

4.3 Toric Varieties from Monomial Maps

Recall from Example 3.1.3 that the algebraic torus $(\mathbb{C}^{\times})^n$ is the group of invertible diagonal $n \times n$ matrices over $\mathbb{C}$. The free abelian group $\mathbb{Z}^n$ of rank n is isomorphic to the group

$\mathrm{Hom}_g(\mathbb{C}^\times, (\mathbb{C}^\times)^n)$ of *cocharacters* from $\mathbb{C}^\times$ to $(\mathbb{C}^\times)^n$. The group $\mathbb{Z}^n$ is also isomorphic to the group $\mathrm{Hom}_g((\mathbb{C}^\times)^n, \mathbb{C}^\times)$ of *characters*. The coordinate ring of $(\mathbb{C}^\times)^n$ is the ring $\mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ of Laurent polynomials.

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of column vectors of an integer matrix with n rows. Writing $\mathbb{C}^\mathcal{A}$ for the complex vector space of functions from $\mathcal{A}$ to $\mathbb{C}$ with coordinates $(z_a \mid a \in \mathcal{A})$, we use the set $\mathcal{A}$ to embed $(\mathbb{C}^\times)^n$ into $\mathbb{C}^\mathcal{A}$ through the map

$$\begin{aligned} \varphi_\mathcal{A} : (\mathbb{C}^\times)^n &\longrightarrow (\mathbb{C}^\times)^\mathcal{A} \subset \mathbb{C}^\mathcal{A} \\ x &\longmapsto (x^a \mid a \in \mathcal{A}). \end{aligned}$$

The Zariski closure of the image $\varphi_\mathcal{A}((\mathbb{C}^\times)^n)$ defines an *affine toric variety $X_\mathcal{A}$*. The ideal of $X_\mathcal{A}$, denoted by $I_\mathcal{A}$, is the kernel of the map

$$\begin{aligned} \varphi^*_\mathcal{A} : \mathbb{C}[z_a \mid a \in \mathcal{A}] &\longrightarrow \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \\ z_a &\longmapsto x^a. \end{aligned} \tag{4.1}$$

The exponent of a monomial $z^u \in \mathbb{C}[z_a \mid a \in \mathcal{A}]$ is $u = (u_a \mid a \in \mathcal{A}) \in \mathbb{N}^\mathcal{A}$. For $u \in \mathbb{N}^\mathcal{A}$, let $\mathcal{A}u := \sum_{a\in\mathcal{A}} a u_a$. Note that $\ker \varphi^*_\mathcal{A}$ contains the set of binomials $\{z^u - z^v \mid \mathcal{A}u = \mathcal{A}v\}$.

Theorem 4.3.1. [25, Theorem 1.2] The ideal $I_\mathcal{A}$ is a prime ideal, and as a complex vector space, it is spanned by the set $\{z^u - z^v \mid \mathcal{A}u = \mathcal{A}v\}$.

Proof. The image of the map $\varphi^*_\mathcal{A}$ is the subalgebra in $\mathbb{C}[x^\pm]$ generated by the set $\{x^a \mid a \in \mathcal{A}\}$. Since $\mathbb{C}[x^\pm]$ is an integral domain, so is the image of $\varphi^*_\mathcal{A}$. Hence, the ideal $I_\mathcal{A} = \ker \varphi^*_\mathcal{A}$ is prime.

Let $\prec$ be a term order on $\mathbb{C}[z_a \mid a \in \mathcal{A}]$. For $f \in I_\mathcal{A}$, write

$$f = c_u z^u + \sum_{v \prec u} c_v z^v,$$

where $c_u \neq 0$, so that $\mathrm{in}_\prec(f) = c_u z^u$ is the initial term of f. It follows that $0 = \varphi^*_\mathcal{A}(f) = c_u x^{\mathcal{A}u} + \sum_{v\prec u} c_v x^{\mathcal{A}v}$. Then there exists $v \prec u$ such that $\mathcal{A}u = \mathcal{A}v$ (otherwise $c_u x^{\mathcal{A}u}$ is not canceled in $\varphi^*_{\mathcal{A}}(f)$ and $\varphi^*_{\mathcal{A}}(f) \neq 0$).

Suppose the leading term of f is $\prec$-minimal in the initial ideal $\mathrm{in}_\prec(I_{\mathcal{A}}) := \{\mathrm{in}_\prec(g) \mid g \in I_{\mathcal{A}}\}$. Set $\overline{f} := f - c_u(z^u - z^v)$ and observe that $\varphi^*_{\mathcal{A}}(\overline{f}) = 0$ and $\mathrm{in}_\prec(\overline{f}) \prec \mathrm{in}(f)$. Since the leading term of f is $\prec$-minimal in $\mathrm{in}_\prec(I_{\mathcal{A}})$, it follows that $\overline{f} = 0$ and f is a scalar multiple of a binomial of the form

$$z^u - z^v, \qquad \mathcal{A}u = \mathcal{A}v. \tag{4.2}$$

where $u, v \in \mathbb{N}^{\mathcal{A}}$.

Suppose now that $\mathrm{in}_\prec(f)$ is not $\prec$-minimal in $\mathrm{in}_\prec(I_{\mathcal{A}})$ and every polynomial in $\mathrm{in}_\prec(I_{\mathcal{A}})$, all of whose terms are $\prec$-less than $\mathrm{in}_\prec(f)$, is a linear combination of binomials of the form (4.2). Then $\overline{f}$ is a linear combination of binomials of the form (4.2), which implies that f is as well. $\square$

Recall that a monoid is a nonempty set with an associative binary operation and an identity element (see Section 2.2). Given a finite set $\mathcal{A} \subset \mathbb{Z}^n$, $\mathbb{N}\mathcal{A}$ is the submonoid of $\mathbb{Z}^n$ generated by $\mathcal{A}$. The *monoid algebra* of $\mathcal{A} \subset \mathbb{Z}^n$ is the set $\mathbb{C}[\mathbb{N}\mathcal{A}]$ of complex-linear combinations of elements of $\mathbb{N}\mathcal{A}$. We identify $\mathbb{C}[\mathbb{N}\mathcal{A}]$ with the set of Laurent polynomials whose exponents are from $\mathbb{N}\mathcal{A}$.

Corollary 4.3.2. [25, Corollary 1.3] Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set. The coordinate ring of the affine toric variety $X_{\mathcal{A}}$ is $\mathbb{C}[\mathbb{N}\mathcal{A}]$.

By the proof of Theorem 4.3.1, the coordinate ring of $X_{\mathcal{A}}$ is $\mathbb{C}[\mathbb{N}\mathcal{A}] \simeq \mathbb{C}[x^a \mid a \in \mathcal{A}]$. It follows by the standard algebra-geometry dictionary that $X_{\mathcal{A}} = \operatorname{spec} \mathbb{C}[\mathbb{N}\mathcal{A}]$.

4.4 Toric Varieties from Fans

We follow [27, 29] to construct an abstract toric variety by gluing affine toric varieties along common open subsets. These affine toric varieties and the gluing are recorded by the data of a fan.

Let N be a finitely generated free abelian group of rank n and let $M := \mathrm{Hom}(N, \mathbb{Z})$ be its dual group. Let σ be a finitely generated submonoid of N. The *polar* of σ,

$$\sigma^\vee := \{u \in M \mid u(v) \geq 0 \text{ for all } v \in \sigma\},$$

is a finitely generated submonoid of $M = \mathrm{Hom}(N, \mathbb{Z})$. A *cone* σ is a finitely generated submonoid that is *saturated*, which means $(\sigma^\vee)^\vee = \sigma$. The *dimension* of a cone is the rank of the subgroup it generates. A one-dimensional cone is a *ray*.

Example 4.4.1. If $\mathcal{A} = \{(1,1), (1,0), (1,-1)\} \subset \mathbb{Z}^2$, the monoids $\mathbb{N}\mathcal{A}$ and $\mathbb{Z}\mathcal{A}$ are cones. $\diamond$

A *face* τ of a cone σ is a submonoid $\tau = \{v \in \sigma \mid u(v) = 0\}$, for some $u \in \sigma^\vee$. A face τ of σ such that $\tau \neq \sigma$ is a *proper face*.

Lemma 4.4.2. [9, Lemma 1.2.6] Let σ be a cone. Then

(a) every face of σ is a cone,

(b) the intersection of any two faces of σ is a face of σ, and

(c) a face of a face of σ is a face of σ.

Let σ be a cone in N. As in Section 4.3, the monoid algebra of $\sigma^\vee \subseteq M$ is the set $\mathbb{C}[\sigma^\vee]$ of complex-linear combinations of elements of $\sigma^\vee$.

Example 4.4.3. The Laurent polynomial ring $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ is a monoid algebra. $\diamond$

Given a cone σ in N, we associate to σ the affine variety $V_\sigma := \operatorname{spec} \mathbb{C}[\sigma^\vee]$. If τ is a face of σ, then $\sigma^\vee \subset \tau^\vee$ induces the inclusion $V_\tau \subset V_\sigma$ as $\mathbb{C}[\sigma^\vee]$ and $\mathbb{C}[\tau^\vee]$ have the same function field and $\mathbb{C}[\sigma^\vee] \subset \mathbb{C}[\tau^\vee]$.

Let 0 be the identity of N. A cone σ is *pointed* if $\{0\}$ is a face of σ, in which case $\sigma^\vee$ generates M. A *fan* Σ in N is a finite collection of pointed cones in N where

(a) any face of a cone in Σ is a cone in Σ, and

(b) the intersection of any two cones in Σ is also a common face of each.

The *support* of a fan Σ is the set-theoretic union $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subseteq N$. A fan Σ is *complete* if $|\Sigma| = N$.

Let Σ be a fan in N. We obtain an algebraic variety X_Σ from the collection $\{V_\sigma \mid \sigma \in \Sigma\}$ by gluing along the inclusions induced by $V_\tau \subset V_\sigma$ whenever τ, σ are cones in Σ and τ is a face of σ.

Since a pointed cone contains $\{0\}$ as a face, the algebraic torus $V_0 = \operatorname{spec} \mathbb{C}[M]$ is contained in V_σ for every $\sigma \in \Sigma$ and the gluing is torus-equivariant. It follows that V_0 acts on the variety X_Σ, showing X_Σ is a toric variety. The variety X_Σ is the toric variety associated to the fan Σ.

Let σ be a cone in N. The *lineality space* of $\sigma^\vee$ is the group $M_\sigma := \sigma^\vee \cap (-\sigma^\vee) \subseteq M$. The affine variety V_σ has a distinguished point x_σ which corresponds to the maximal ideal of $\mathbb{C}[\sigma^\vee]$ that is the kernel of the map $\mathbb{C}[\sigma^\vee] \to \mathbb{C}$, where

$$\sigma^\vee \ni u \longmapsto \begin{cases} 1 & u \in M_\sigma \\ 0 & \text{otherwise.} \end{cases}$$

The *orbit* $\mathcal{O}_\sigma$ of the point x_σ is a torus orbit in V_σ.

Let $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$. Let P be a polytope in $M_\mathbb{R}$. For $v, w \in N$, $v \sim w$ if and only if $P_v = P_w$, which means v and w expose the same face of P. Let F be a face of P. The set $\sigma_F := \{u \in N \mid F \subset P_u\}$ is a pointed cone in N. The collection $\Sigma_P := \{\sigma_F \subset N \mid F \text{ is a face of } P\}$ is the (*inner normal*) *fan* to the polytope P. Each cone in Σ_P corresponds to a unique face of P as follows. Given a face F of P, the *relative interior* σ_F° of the cone $\sigma_F \in \Sigma_P$ is the set-theoretic difference of σ_F with the union of the cones $\sigma_G \in \Sigma_P$ for faces G of P that contain F. The face F is the face of P exposed by any $w \in \sigma_F^\circ$. If F is a face of P, the lineality space $M_F := M_{\sigma_F}$ is the linear span $a - b \in M$, where $a, b \in F \cap M$ (see [25, Section 3.1]).

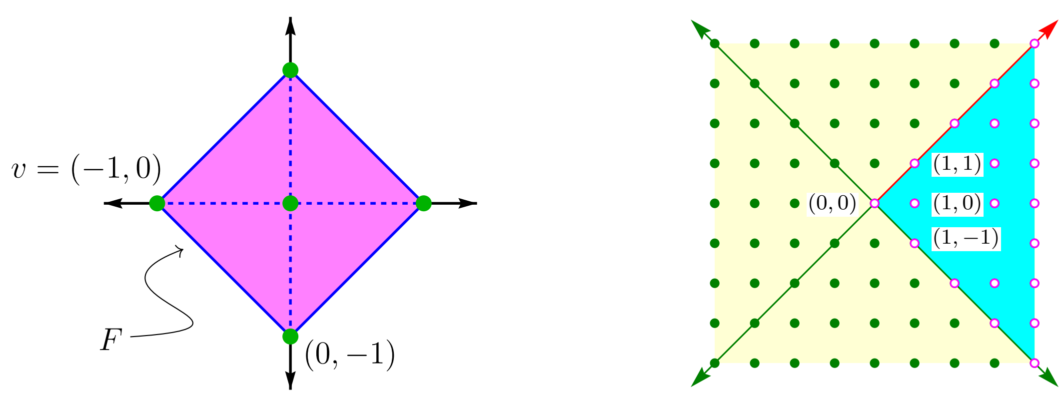

Figure 4.2: The two-dimensional cross polytope and and its fan.

Example 4.4.4. Figure 4.2 shows the two-dimensional cross polytope and its fan in $\mathbb{Z}^2$. The cone corresponding to the vertex $v = (-1, 0)$ is the cone generated by $(1, 1), (1, 0)$, and $(1, -1)$. The cone corresponding to the edge F of P is the ray generated by $(1, 1)$. $\diamond$

Since each element in N exposes a face of P, the fan Σ_P is complete. The next theorem shows us that the toric variety X_{Σ_P} associated to the fan Σ_P is compact.

Theorem 4.4.5. [9, Theorem 3.4.1] Let Σ be a fan in N and let X_Σ be its associated toric variety, Then X_Σ is compact if and only if Σ is complete.

4.5 Sheaves on Toric Varieties from Fans

In Chapter 6 we construct a particular class of sheaves on a toric variety constructed from a fan. We do not need sheaves in complete generality. This section is based on [23, VII.1].

Let R be a (commutative) ring with 1. A (left) *R-module* is an abelian group $\mathcal{M} := (\mathcal{M}, +)$ together with a function

$$\mathsf{R} \times \mathcal{M} \to \mathcal{M}, \qquad (r, a) \mapsto ra$$

such that for all $r, s \in \mathsf{R}$ and $a, b \in \mathcal{M}$,

(i) $r(a + b) = ra + rb$,

(ii) $(r + s)a = ra + sa$,

(iii) $r(sa) = (rs)a$, and

(iv) $\mathrm{Id}_{\mathsf{R}}\, a = a$.

If R is a field, then an R-module is a vector space.

Let R' be a ring that contains R as a subring. Let $\mathcal{M}$ be an R-module and let $\mathcal{M}'$ be an R'-module. An *R-module homomorphism $\varphi : \mathcal{M} \to \mathcal{M}'$* is a function such that for all $r, s \in \mathsf{R}$ and $a, b \in \mathcal{M}$, $\varphi(ra + sb) = r\varphi(a) + s\varphi(b)$. If $\mathsf{R} = \mathsf{R}'$ and φ is bijective, then φ is an *R-module isomorphism*.

Example 4.5.1. Every abelian group G is a $\mathbb{Z}$-module. $\diamond$

Let X be a toric variety. A *sheaf* of rational functions $\mathscr{F}$ on X consists of the following data:

(a) for each (Zariski) open subset $U \subseteq X$, $\mathscr{F}(U)$ is a ring of rational functions on U;

(b) for every inclusion $V \subseteq U$ of open subsets of X, $\rho_{V,U}$ is the inclusion $\mathscr{F}(U) \hookrightarrow \mathscr{F}(V)$;

such that $\mathscr{F}(\emptyset) = 0$, $\rho_{U,U} = \mathrm{Id}_{\mathscr{F}(U)}$, and for any inclusion $W \subseteq V \subseteq U$, $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$.

Example 4.5.2. Let Σ be the (inner normal) fan of a polytope and let X_Σ be the toric variety associated to Σ. For each open subset $U \subseteq X_\Sigma$, let $\mathscr{O}(U)$ be the ring of regular functions on U. For each open subset $V \subseteq U$ and $f \in \mathscr{O}(U)$, the restriction $f|_V$ of f to V is a regular function on V. Moreover, since a rational function which is regular locally is regular, it follows that $\mathscr{O}$ is a sheaf of rational functions on X_Σ. This sheaf is the *structure sheaf* of X_Σ. $\diamond$

Proposition 4.5.3. [23, Theorem 1.8] Let Σ be a fan in a finitely generated free abelian group N, let $M = \mathrm{Hom}(N, \mathbb{Z})$, and let X_Σ be the associated toric variety of the fan Σ.

(a) If $\sigma \in \Sigma$, then $\mathscr{O}(V_\sigma) = \mathbb{C}[\sigma^\vee]$. In particular, $\mathscr{O}(V_0) = \mathbb{C}[M]$

(b) If Σ is complete, then $\mathscr{O}(X_\Sigma) = \mathbb{C}$.

Let Σ be a fan and let X_Σ be its associated toric variety. Let $\mathscr{F}$ be a sheaf of rational functions on X_Σ such that for each V_σ, the set $\mathscr{F}(V_\sigma)$ is an $\mathscr{O}(V_\sigma)$-module of rational functions on V_σ. Then $\mathscr{F}$ is a *sheaf of $\mathscr{O}$-modules* on X_Σ. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules on X_Σ. For an open subset $U \subseteq X_\Sigma$, an element in $\mathscr{F}(U)$ is a *section* of $\mathscr{F}$ over U. A *global section* of $\mathscr{F}$ is an element of the $\mathscr{O}(X_\Sigma)$-module $\mathscr{F}(X_\Sigma)$.

5. IRREDUCIBILITY*

We study the (ir)reducibility of Bloch varieties upon a change of their period lattice. This line of work dates back to the 1980s, with a focus on the discrete periodic Schrödinger operator on $\ell^2(\mathbb{Z}^d)$. For this operator, irreducibility of the Bloch variety was proven in [8] for $d = 2$.

We use discrete geometry to study when the irreducibility of the dispersion polynomial of a periodic graph operator is preserved for a potential that is periodic with respect to the sublattice $Q\mathbb{Z}$ (see Section 2.2). For a $Q\mathbb{Z}$-periodic potential, we show that if enough of the facial polynomials of the corresponding dispersion polynomial D_Q are also facial polynomials of a $\mathbb{Z}^d$-periodic potential and the corresponding facial polynomials of D are irreducible, then D_Q factors "only homothetically" (Corollary 5.3.12). If this condition is met, then D_Q is irreducible if it has an irreducible facial polynomial (Corollary 5.2.5).

There is an overlap between our methods and those in [30], which were inspired by the work [31]. We translate these works to the language of discrete geometry. This allows us to use the theory of indecomposability for lattice polytopes in the study of irreducibility of the Bloch varieties. This enables us to apply our results to study the irreducibility of a larger class of dispersion polynomials; for example, those that arise from many-vertex models like the hexagonal lattice, as opposed to the single-vertex models such as the square lattice, that are the subjects of [30].

Section 5.1 introduces the background on changing the period lattice of the potential of a periodic graph operator. Section 5.2 uses classical results on the decomposability of polytopes to obtain analogous results for a class of Laurent polynomials. For a $\mathbb{Z}^d$-periodic potential, Section 5.3 discusses sufficient conditions when the irreducibility of the dispersion polynomial is preserved after changing the period lattice which will enable us to discuss irreducibility of the dispersion polynomial for more general potentials. Section 5.4 provides various applications of these results.

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This chapter is based on published work with Matthew Faust in [1].

5.1 Changing the Period Lattice of the Potential

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a $\mathbb{Z}^d$-periodic graph with fundamental domain W, where $m := |W|$, and let L be a periodic graph operator on $\ell^2(\Gamma)$. Fix $Q := (q_1, \ldots, q_d) \in \mathbb{N}^d$ and let $|Q| := \prod_{i=1}^{d} q_i$. We wish to study the dispersion polynomial D of L with labeling (V_Q, E), where $E : \mathcal{E} \to \mathbb{C}$ is a $\mathbb{Z}^d$-periodic edge labeling and $V_Q : \mathcal{V} \to \mathbb{C}$ is a $Q\mathbb{Z}^d$-periodic potential, rather than a $\mathbb{Z}^d$-periodic potential. We denote operator L with labeling (V_Q, E) by L_Q.

As $Q\mathbb{Z}$ is a free subgroup of $\mathbb{Z}^d$ of rank m, Γ is also a $Q\mathbb{Z}$-periodic graph. Thus, W_Q is a fundamental domain for the action of $Q\mathbb{Z}$ on $\mathcal{V}$ (see Section 2.2). The Floquet matrix of L_Q with respect to the $Q\mathbb{Z}$-periodic graph Γ with fundamental domain W_Q is denoted by $L_Q(z)$. Since $|W_Q| = |Q|m$, the matrix $L_Q(z)$ is a $|Q|m \times |Q|m$ matrix of Laurent polynomials.

We discuss an alternative representative of $L_Q(z)$ that comes from a change of basis and after a change of variables. Consider the surjective group homomorphism

$$\begin{aligned} \phi : \quad (\mathbb{C}^\times)^d &\longrightarrow (\mathbb{C}^\times)^d \\ (z_1, \ldots, z_d) &\longmapsto (z_1^{q_1}, \ldots, z_d^{q_d}), \end{aligned} \tag{5.1}$$

with *kernel group* $\mathcal{U}_Q := \prod_{i=1}^{d} \mathcal{U}_{q_i}$, where $\mathcal{U}_{q_i}$ is the multiplicative group of q_i-th roots of unity.

Fix $z \in \mathbb{T}^d$ and $u \in W$. For each $\rho \in \mathcal{U}_Q$, define the function $e_{\rho,u} : \mathcal{V} \to \mathbb{C}$ such that for $v \in \mathcal{V}$,

$$e_{\rho,u}(v) := \begin{cases} (\rho z)^a := \prod_{i=1}^{d} (\rho_i z_i)^{a_i} & \text{if } v = a + u \text{ for } a = (a_1, \ldots, a_d) \in \mathbb{Z}^d, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for $a \in \mathbb{Z}^d$ and $v \in \mathcal{V}$ with $v \neq u$,

$$\begin{aligned} e_{\rho,u}(a + u) &= (\rho z)^a = (\rho z)^a e_{\rho,u}(u), \text{ and} \\ e_{\rho,u}(a + v) &= 0 = (\rho z)^a e_{\rho,u}(v). \end{aligned} \tag{5.2}$$

By [6, Lemma 2.2], the set of functions $\{e_{\rho,u} \mid \rho \in \mathcal{U}_Q, u \in W\}$ forms a basis for functions $\psi : \mathcal{V} \to \mathbb{C}$ satisfying

$$\psi(Q_i + v) = z_i^{q_i} \psi(v), \quad \text{for } i = 1, \ldots, d.$$

Here, $Q_i := q_i \varepsilon_i$, where ε_i is the i-th standard basis vector of $\mathbb{R}^d$. The set $\{e_{\rho,u} \mid \rho \in \mathcal{U}_Q, u \in W\}$ also forms a basis for *Floquet functions* with *Floquet multiplier* z^Q with respect to the $Q\mathbb{Z}$-action (see [1, Remark 1.2]). Thus, we obtain a new matrix representation for $L_Q(z_1^{q_1}, \ldots, z_d^{q_d})$ in the basis $\{e_{\rho,u} \mid \rho \in \mathcal{U}_Q, u \in W\}$. For each $\rho \in \mathcal{U}_Q$, the weighted discrete Laplacian Δ_E in this basis is defined by

$$(\Delta_E e_{\rho,u})(u) := - \sum_{(u,a+v) \in \mathcal{E}} E_{(u,a+v)} (\rho z)^a e_{\rho,u}(v),$$

where $u, v \in W$. The Floquet matrix of Δ_E in the basis $\{e_{\rho,u} \mid \rho \in \mathcal{U}_Q, u \in W\}$ is a block-diagonal matrix given by $|Q| \times |Q|$ blocks, indexed by $\mathcal{U}_Q \times \mathcal{U}_Q$, of $m \times m$ matrices, indexed by $W \times W$:

$$\left(\hat{\Delta}_E(z)\right)_{\rho,\rho'} := \delta_{\rho,\rho'} \cdot \Delta_E(\rho z),$$

where the submatrix $\Delta_E(\rho z)$ represents the discrete Laplacian Δ_E with Floquet multiplier ρz (with respect to the $\mathbb{Z}^d$-action). The Floquet matrix of Δ_E in the basis $\{e_{\rho,u} \mid \rho \in \mathcal{U}_Q, u \in W\}$ is denoted by $\hat{\Delta}_E(z)$.

To discuss the potential V in this new basis, we will take a discrete Fourier transform. For each $\mu \in \mathcal{U}_Q$ and $a + v \in W_Q$, the discrete Fourier transform of the potential V is

$$(V e_{\mu,u})(a+v) = V(a+v) e_{\mu,u}(a+v) = \sum_{\rho \in \mathcal{U}_Q} \hat{V}_{\rho,\mu}(v) e_{\rho,u}(a+v) = \sum_{\rho \in \mathcal{U}_Q} \hat{V}_{\rho,\mu}(v) \rho^a e_{\rho,u}(v),$$

where $\hat{V}_{\rho,\mu}(v)$ is the Fourier coefficient of V on the orbit of $v \in W$ (see also [32, Equation 4.5]). To obtain a matrix multiplication operator in this basis, we solve for the coefficients and get

$$\hat{V}_{\rho,\mu}(v) e_{\rho,u}(v) = \frac{e_{\mu,u}(v)}{|Q|} \sum_{a+v \in W_Q} V(a+v)(\mu\rho^{-1})^a.$$

Let $\hat{V}$ be the matrix representation of V in the basis $\{e_{\rho,u} \mid \rho \in \mathcal{U}_Q, u \in W\}$; that is, $\hat{V}$ acts on the basis function $e_{\rho,u}(v)$ by

$$\big(\hat{V} e_{\rho,u}\big)(v) = \sum_{\mu \in \mathcal{U}_Q} \hat{V}_{\rho,\mu}(v) e_{\rho,u}(v) = \sum_{\mu \in \mathcal{U}_Q} \frac{e_{\mu,u}(v)}{|Q|} \sum_{a+v \in W_Q} V(a+v)(\mu\rho^{-1})^a \quad \big(= V e_{\rho,u}(v)\big).$$

This is a $|Q| \times |Q|$ block matrix with $m \times m$ entries, indexed the same as $\hat{\Delta}_E(z)$. Each $\hat{V}_{\rho,\mu}$ is an $m \times m$ diagonal matrix such that $(\hat{V}_{\rho,\mu})_{u,u} = \hat{V}_{\rho,\mu}(u)$.

Remark 5.1.1. If V is also $\mathbb{Z}^d$-periodic, then

$$\begin{aligned}
\hat{V}_{\rho,\mu}(v) e_{\rho,u}(v) &= \frac{1}{|Q|} \sum_{a+v \in W_Q} V(a+v) e_{\mu,u}(v)(\mu\rho^{-1})^a \\
&= \frac{1}{|Q|} \sum_{a+v \in W_Q} V(v) e_{\mu,u}(v)(\mu\rho^{-1})^a \\
&= \frac{V(v) e_{\mu,u}(v)}{|Q|} \sum_{a+v \in W_Q} (\mu\rho^{-1})^a.
\end{aligned}$$

Thus, $\hat{V}_{\rho,\mu}(v) = V(v)$ when $\rho = \mu$ and is 0 otherwise. That is, $\hat{V}$ is a diagonal matrix. $\diamond$

The $m|Q| \times m|Q|$ matrix $L_Q(z^Q)$ with respect to the basis $\{e_{\rho,u} \mid \rho \in \mathcal{U}_Q, u \in W\}$ is

$$\hat{L}_Q(z) = \hat{V} + \hat{\Delta}_E(z).$$

Let $D_Q(z, \lambda) = \det(L_Q(z) - \lambda I)$ and $\hat{D}_Q(z, \lambda) = \det(\hat{L}_Q(z) - \lambda I)$. As $\hat{L}_Q(z, \lambda)$ is $L_Q(z^Q)$ after a change of basis,

$$D_Q(z^Q, \lambda) = \det(L_Q(z^Q) - \lambda I) = \det(\hat{L}_Q(z) - \lambda I) = \hat{D}_Q(z, \lambda).$$

Example 5.1.2. Let us continue Example 2.4.3. When we view the hexagonal lattice as $\mathbb{Z}^2$-periodic, as in the case of Figure 2.1, with a $\mathbb{Z}^2$-periodic potential V, we get the Floquet matrix

$$L(x,y) = \begin{pmatrix} V(u) & -\alpha - \beta x^{-1} - \gamma y^{-1} \\ -\alpha - \beta x - \gamma y & V(v) \end{pmatrix}.$$

Let $Q = (2,1)$ and let V_Q be a $Q\mathbb{Z}$-periodic potential, then $L_Q(x,y)$ is given by the matrix

$$\begin{pmatrix} V_Q(u) & -\alpha - \gamma y^{-1} & 0 & -\beta x^{-1} \\ -\alpha - \gamma y & V_Q(v) & -\beta & 0 \\ 0 & -\beta & V_Q((1,0)+u) & -\alpha - \gamma y^{-1} \\ -\beta x & 0 & -\alpha - \gamma y & V_Q((1,0)+v) \end{pmatrix}.$$

If V satisfies $V(u) = \frac{V_Q(u)+V_Q((1,0)+u)}{2}$ and $V(v) = \frac{V_Q(v)+V_Q((1,0)+v)}{2}$, then $\hat{L}_Q(x,y)$ is a 2×2 block matrix with each entry a 2×2 matrix. Explicitly,

$$\hat{L}_Q(x,y) = \begin{pmatrix} L(x,y) & (\hat{V}_Q)_{1,-1} \\ (\hat{V}_Q)_{-1,1} & L(-x,y) \end{pmatrix}, \text{ where}$$

$$(\hat{V}_Q)_{1,-1} = (\hat{V}_Q)_{-1,1} = \begin{pmatrix} \frac{V_Q(u)-V_Q((1,0)+u)}{2} & 0 \\ 0 & \frac{V_Q(v)-V_Q((1,0)+v)}{2} \end{pmatrix}.$$

$\diamond$

5.2 Only Homothetic Polynomials

Only homothetic decomposability was considered in [10, 33], where it is shown that if enough faces of a polytope are only homothetically decomposable, the polytope is only homothetically decomposable. We prove an similar result for only homothetically reducible Laurent polynomials.

If f, g, and h are Laurent polynomials such that $f = gh$ and F is a face of $\mathrm{Newt}\, f$, we write $f_F = g_F h_F$ as the factorization of the facial polynomial f_F into g_F and h_F. Recall that there exists an inner normal $w \in \mathbb{R}^d$ that exposes F (and is such that $f_F = f_w$) and $g_F = g_w$ and $h_F = h_w$.

Remark 5.2.1. If f is only homothetically reducible, then whenever $f = gh$, there exists $r, t \in \mathbb{Q}$ such that $r \operatorname{Newt} f = \operatorname{Newt} g$ and $t \operatorname{Newt} f = \operatorname{Newt} h$. By the definition of only homothetic irreducibility, there exists a_g and a_h in $\mathbb{Z}^d$ so that $a_g + r \operatorname{Newt} f = \operatorname{Newt} g$ and $a_h + t \operatorname{Newt} f = \operatorname{Newt} h$. It follows that $(a_g + r \operatorname{Newt} f) + (a_h + t \operatorname{Newt} f) = \operatorname{Newt} f$ and $a_g + a_h = 0$. Thus, there exists $g' = z^{a_h} g$ and $h' = z^{a_g} h$ such that $r \operatorname{Newt} f = \operatorname{Newt} g'$ and $t \operatorname{Newt} f = \operatorname{Newt} h'$. ◇

Lemma 5.2.2. Let f, g, and h be Laurent polynomials and suppose that $f = gh$. Let F_1 and F_2 be faces of $\operatorname{Newt} f$ with $\dim F_1 \cap F_2 \geq 1$ whose corresponding facial polynomials, f_{F_1} and f_{F_2}, are only homothetically reducible. If $\operatorname{Newt} g_{F_1} = r \operatorname{Newt} f_{F_1}$ and $\operatorname{Newt} h_{F_1} = t \operatorname{Newt} f_{F_1}$ for some pair $r, t \in \mathbb{Q}$, then $\operatorname{Newt} g_{F_2} = r \operatorname{Newt} f_{F_2}$ and $\operatorname{Newt} h_{F_2} = t \operatorname{Newt} f_{F_2}$.

Proof. Since $f = gh$, $f_{F_1} = g_{F_1} h_{F_1}$. As f_{F_1} is only homothetic reducible we have that $r \operatorname{Newt} f_{F_1} = \operatorname{Newt} g_{F_1}$ and $t \operatorname{Newt} f_{F_1} = \operatorname{Newt} h_{F_1}$ for some $r, t \in \mathbb{Q}$. Let $F' = F_1 \cap F_2$. As $F' \subset F_1$, it follows that $\operatorname{Newt} g_{F'} = r \operatorname{Newt} f_{F'}$ and $\operatorname{Newt} h_{F'} = t \operatorname{Newt} f_{F'}$. The polynomial f_{F_2} is only homothetically reducible and must agree with its restriction to F'; it follows that $r \operatorname{Newt} f_{F_2} = \operatorname{Newt} g_{F_2}$ and $t \operatorname{Newt} f_{F_2} = \operatorname{Newt} h_{F_2}$. □

A *strong chain of faces* of a polytope P is a sequence of faces $F_1, \ldots, F_n$ of P *of length* n such that for each i, $\dim F_i \cap F_{i+1} \geq 1$.

Example 5.2.3. Adjacent triangular facets of a 3-dimensional pyramid share an edge, and thus give a strong chain of faces of length 2. ◇

Theorem 5.2.4. Let f, g, and h be Laurent polynomials such that $f = gh$. If for each pair (a, b) of distinct vertices of $\operatorname{Newt} f$ there is a strong chain of faces $F_1, \ldots, F_n$ such that $a \in F_1$, $b \in F_n$, and for each F_i, the corresponding facial polynomial f_{F_i} is only homothetically reducible, then f is only homothetically reducible.

Proof. By Lemma 5.2.2, there exist a pair of rational numbers $r, t \in \mathbb{Q}$ such that $r \operatorname{Newt} f_{F_i} = \operatorname{Newt} g_{F_i}$ and $t \operatorname{Newt} f_{F_i} = \operatorname{Newt} h_{F_i}$ for all $i = 1, \ldots, n$. As $a \in F_1$ and $b \in F_n$, $r \operatorname{Newt} f_a = \operatorname{Newt} g_a$, $t \operatorname{Newt} f_a = \operatorname{Newt} h_a$, $r \operatorname{Newt} f_b = \operatorname{Newt} g_b$, and $t \operatorname{Newt} f_b = \operatorname{Newt} h_b$. This is the case for all vertex pairs (a, b) of $\operatorname{Newt} f$. In particular, we may fix a and let b vary over the other vertices. As any vertex of $\operatorname{Newt} f$ must come from the Minkowski sum of a pair of vertices u, v where $u \in \operatorname{Newt} g$ and $v \in \operatorname{Newt} h$, and any vertex u of $\operatorname{Newt} g$ or v of $\operatorname{Newt} h$ must be a summand for some vertex of $\operatorname{Newt} f$, it follows that $r \operatorname{Newt} f = \operatorname{Newt} g$ and $t \operatorname{Newt} f = \operatorname{Newt} h$. $\square$

Corollary 5.2.5. Suppose that f is only homothetically reducible. If there is a face F of $\operatorname{Newt} f$ of $\dim F \geq 1$ such that f_F is irreducible, then f is irreducible.

Proof. Suppose that f is only homothetically reducible. Let F be a face of $\operatorname{Newt} f$ such that $f|_F$ is irreducible. Suppose g, h are Laurent polynomials such that $f = gh$. As f is only homothetically reducible, there exists $r, s \in \mathbb{Q}$ such that $r \operatorname{Newt} f = \operatorname{Newt} g$ and $t \operatorname{Newt} f = \operatorname{Newt} h$. Thus, for any face F' of $\operatorname{Newt} f$, $r \operatorname{Newt} f_{F'} = \operatorname{Newt} g_{F'}$ and $t \operatorname{Newt} f_{F'} = \operatorname{Newt} h_{F'}$. Notice that f_F is irreducible and therefore, one of $g|_F$ or h_F is a monomial. As one of h_F or g_F must be a monomial (which by Remark 5.2.1 we assume to be the constant monomial), either t or r is zero. $\square$

5.3 Expanded Dispersion Polynomials

Let Γ be a $\mathbb{Z}^d$-periodic graph with fundamental domain W, let $L(z)$ be the Floquet matrix of a periodic graph operator L with a $\mathbb{Z}^d$-periodic labeling (V, E) (that is, both E and V are $\mathbb{Z}^d$-periodic), and let $D(z, \lambda) := \det(L(z) - \lambda I)$ be its dispersion polynomial.

Fix $Q = (q_1, \dots, q_d) \in \mathbb{N}^d$, and consider Γ as a $Q\mathbb{Z}$-periodic graph with fundamental domain W_Q. For a $Q\mathbb{Z}$-periodic potential V_Q, let $L_Q(z)$ be the Floquet matrix of L_Q acting on the $Q\mathbb{Z}$-periodic graph Γ with fundamental domain W_Q and with the labeling (V_Q, E). Let $\hat{L}_Q(z)$ be the matrix obtained from the Floquet matrix $L_Q(z^Q)$ after the change of basis (5.2), and let $\hat{V}$ be the matrix representing V_Q after the change of basis in Section 5.1.

Recall that $D_Q(z, \lambda) = \det(L_Q(z) - \lambda I)$, $\hat{D}_Q(z, \lambda) = \det(\hat{L}_Q(z) - \lambda I)$, $D_Q(z^Q, \lambda) = \hat{D}_Q(z, \lambda)$, $|Q| := \prod_{i=1}^{d} q_i$, and $\mathcal{U}_Q := \prod_{i=1}^{d} \mathcal{U}_{q_i}$, where $\mathcal{U}_{q_i}$ is the multiplicative group of q_i-th roots of unity. We will write D, D_Q, and $\hat{D}_Q$ in place of $D(z, \lambda), D_Q(z, \lambda)$, and $\hat{D}_Q(z, \lambda)$ respectively.

We seek conditions on D and Q which imply that if $V_Q = V$ then D_Q is irreducible. In this case, the $Q\mathbb{Z}$-periodic potential V_Q is also $\mathbb{Z}^d$-periodic.

Suppose that $V_Q = V$. By Remark 5.1.1, $\hat{V}$ is given by a diagonal matrix when the potential V_Q is $\mathbb{Z}^d$-periodic. It follows that $\hat{D}_Q(z,\lambda)$ may be expressed in terms of D as

$$\hat{D}_Q(z,\lambda) := \det(\hat{L}_Q(z) - \lambda I) = \prod_{\mu\in\mathcal{U}_Q} \det(L(\mu z) - \lambda I) = \prod_{\mu\in\mathcal{U}_Q} D(\mu z,\lambda). \tag{5.3}$$

Due to this expression, $\operatorname{Newt}\hat{D}_Q = |Q|\operatorname{Newt} D$. As $D_Q(z^Q,\lambda) = \hat{D}_Q(z,\lambda)$, $\operatorname{Newt} D_Q$ is the polytope obtained after multiplying the i-th coordinate of each point of $\operatorname{Newt} D$ by $\frac{|Q|}{q_i}$. That is, $(a_1,\ldots,a_d,a_{d+1})$ is a vertex of $\operatorname{Newt} D$ if and only if $\left(\frac{|Q|a_1}{q_1},\ldots,\frac{|Q|a_d}{q_d}, |Q|a_{d+1}\right)$ is a vertex of $\operatorname{Newt} D_Q$. Therefore, $w = (w_1,\ldots,w_{d+1}) \in \mathbb{Z}^d$ exposes a face of $\operatorname{Newt} D$ if and only if $w' = (q_1w_1,\ldots,q_dw_d,w_{d+1})$ exposes a face of $\operatorname{Newt} D_Q$. We call $\operatorname{Newt} D_Q$ a *contracted Q-dilation* of $\operatorname{Newt} D$ (a contracted Q-dilation is a $\left(\frac{|Q|}{q_1},\ldots,\frac{|Q|}{q_d},|Q|\right)$-dilation). We will often write $(D_Q)_w$ for $(D_Q)_{w'}$; similarly, if F is the face of $\operatorname{Newt} D_Q$ exposed by w', we will write D_F for the corresponding facial polynomial of D and vice versa.

Lemma 5.3.1. Let V_Q be the $\mathbb{Z}^d$-periodic potential V and suppose that D is only homothetically reducible. Then D_Q is only homothetically reducible.

Proof. Suppose D is only homothetically reducible and $D_Q = g(z,\lambda)h(z,\lambda)$, where $g(z,\lambda)$ is not a monomial. As $D_Q(z^Q,\lambda) = \hat{D}_Q(z,\lambda)$, it suffices to show $\operatorname{Newt} g(z^Q,\lambda)$ is homothetic to $\operatorname{Newt}\hat{D}_Q$. By Remark 5.2.1, as D is only homothetically reducible, if $f_1,\ldots,f_l$ are its irreducible factors, then there exist $r_1,\ldots,r_l \in \mathbb{Q}$ such that $\operatorname{Newt} f_i = r_i \operatorname{Newt} D$. As V_Q is $\mathbb{Z}^d$-periodic, it follows that $\operatorname{Newt} f_i = \frac{r_i}{|Q|}\operatorname{Newt}\hat{D}_Q$.

By Equation (5.3), $g(z^Q,\lambda)h(z^Q,\lambda) = \hat{D}_Q = \prod\limits_{\mu\in\mathcal{U}_Q} D(\mu z,\lambda)$. Thus there exists an integer s with $0 < s \leq l|Q|$ such that $g(z^Q,\lambda) = \prod\limits_{i=1}^{s}\kappa_i(z,\lambda)$, where each $\kappa_i(z,\lambda) = f_j(\mu z,\lambda)$ for some $j \in [l]$ and $\mu \in \mathcal{U}_Q$ (noting that each $f_j(\mu z,\lambda)$ is an irreducible factor of $D(z^Q,\lambda)$). If $\kappa_i(z,\lambda) = f_j(\mu z,\lambda)$ then let $\chi_i = r_j$. Thus $\operatorname{Newt} g(z^Q,\lambda) = \frac{\sum_{i=1}^s \chi_i}{|Q|}\operatorname{Newt}\hat{D}_Q$, and thus we have $\operatorname{Newt} g(z,\lambda) = \frac{\sum_{i=1}^s \chi_i}{|Q|}\operatorname{Newt} D_Q$. □

Remark 5.3.2. Lemma 5.3.1 extends to facial polynomials. For a $\mathbb{Z}^d$-periodic potential, if D_w is only homothetically reducible, so is $(D_Q)_w$. Thus the results of this section extend to $(D_Q)_w$. $\diamond$

The following lemma is considered folklore and will provide us motivation. For $A \in \mathbb{N}^d$, let $Q/A := (\frac{q_1}{a_1}, \dots, \frac{q_d}{a_d})$, and write $A \mid Q$ if $a_i \mid q_i$ for all $i = 1, \dots, d$.

Lemma 5.3.3. Suppose $A = (a_1, \dots, a_d) \in \mathbb{N}^d$ such that $A \mid Q$ and let V_Q be an $A\mathbb{Z}$-periodic potential. If D_Q is irreducible, then D_A is irreducible.

Proof. By way of contradiction, suppose that D_A is reducible, that is, $D_A = f(z, \lambda)g(z, \lambda)$. The fundamental domain W_Q is a Q/A-expansion of W_A, hence

$$D_Q(z_1^{\frac{q_1}{a_1}}, \dots, z_d^{\frac{q_d}{a_d}}, \lambda) = \prod_{\mu \in \mathcal{U}_{Q/A}} D_A(\mu z, \lambda) = \prod_{\mu \in \mathcal{U}_{Q/A}} f(\mu z, \lambda)g(\mu z, \lambda).$$

By Lemma 3.1 of [30], there exist f' and g' such that

$$f'(z_1^{\frac{q_1}{a_1}}, \dots, z_d^{\frac{q_d}{a_d}}, \lambda) = \prod_{\mu \in \mathcal{U}_{Q/A}} f(\mu z, \lambda) \quad \text{and} \quad g'(z_1^{\frac{q_1}{a_1}}, \dots, z_d^{\frac{q_d}{a_d}}, \lambda) = \prod_{\mu \in \mathcal{U}_{Q/A}} g(\mu z, \lambda).$$

Therefore $D_Q(z_1, \dots, z_d, \lambda) = f'(z_1, \dots, z_d, \lambda)g'(z_1, \dots, z_d, \lambda)$. $\square$

By Lemma 5.3.3, if D_Q is irreducible and $A \mid Q$, then D_A is irreducible. For the remaining section, we assume D is irreducible for the $\mathbb{Z}^d$-periodic potential V and that $V_Q = V$. Let $\sigma = \{\sigma_1, \dots, \sigma_k\} \in \binom{[d]}{k}$ be a k-element subset of the set $[d] := \{1, 2, \dots, d\}$. Let $\bar{\sigma} = [d] \smallsetminus \sigma$ be the complement of σ in $[d]$. Define $\sigma \odot Q = (\sigma \odot q_1, \sigma \odot q_2, \dots, \sigma \odot q_d)$, where $\sigma \odot q_i = q_i$ if $i \in \sigma$, and $\sigma \odot q_j = 1$ if $j \notin \sigma$. Let $D_{\sigma \odot Q}$ be the dispersion polynomial given by the periodic graph operator L, with the $\mathbb{Z}^d$-periodic (and therefore $(\sigma \odot Q)\mathbb{Z}$-periodic) labeling (V_Q, E) associated to the $(\sigma \odot Q)\mathbb{Z}$-periodic graph Γ with fundamental domain given by the expansion $W_{\sigma \odot Q}$ of W. This notation will allows us study irreducibility of the dispersion polynomial as we incrementally expand coordinate-wise from $D = D_{(1,\dots,1)}$ to D_Q. Lemma 5.3.3 suggests this approach, as we know that if D_Q is irreducible, then for any $k < d$, we must have that any $D_{\sigma \odot Q}$ is irreducible for all $\sigma \in \binom{[d]}{k}$. Indeed, this leads us to the following theorem.

Theorem 5.3.4. Fix a positive integer $k < d$ and suppose that $D_{\sigma\odot Q}$ is irreducible for all $\sigma \in \binom{[d]}{k}$. If no $k+1$ coordinates of Q share a common factor, then D_Q is irreducible.

Proof. Assume that no $k+1$ coordinates of Q share a common factor. Suppose there exist polynomials g, h, with g not a monomial, such that

$$D_Q = g(z,\lambda)h(z,\lambda).$$

Reordering, if necessary, we may assume that $\sigma = [k]$. As W_Q is an expansion of $W_{\sigma\odot Q}$,

$$D_Q(z^{\bar{\sigma}\odot Q},\lambda) = \prod_{\gamma\in\mathcal{U}_{\bar{\sigma}\odot Q}} D_{\sigma\odot Q}(z_1,\dots,z_k,\gamma_1 z_{k+1},\dots,\gamma_{d-k}z_d,\lambda).$$

As $D_{\sigma\odot Q}$ is irreducible, there exist $\gamma^1,\dots,\gamma^s \in \mathcal{U}_{\bar{\sigma}\odot Q}$ for some $s \geq 1$ such that

$$g(z^{\bar{\sigma}\odot Q},\lambda) = \prod_{i=1}^{s} D_{\sigma\odot Q}(z_1,\dots,z_k,\gamma_1^i z_{k+1},\dots,\gamma_{d-k}^i z_d,\lambda).$$

Expand this so that

$$g(z^{Q},\lambda) = \prod_{i=1}^{s}\prod_{\mu\in\mathcal{U}_Q} D(\mu_1 z_1,\dots,\mu_k z_k,\gamma_1^i z_{k+1},\dots,\gamma_{d-k}^i z_d,\lambda)$$

can be written as a product of $S := s\prod_{i=1}^{k} q_i$ irreducible polynomials. As σ is arbitrary (that is, the same argument holds for any $\sigma \in \binom{[d]}{k}$ after reordering coordinates), the product $q_{\sigma_1}\cdots q_{\sigma_k}$ divides S for all $\sigma \in \binom{[d]}{k}$. By our assumption, no $k+1$ coordinates of Q share a common factor. Therefore, if p^a is a prime power that divides $|Q|$, there exists $\sigma \in \binom{[d]}{k}$ such that $p^a \mid q_{\sigma_1}\cdots q_{\sigma_k}$, and thus $p^a \mid S$. As S is at most $|Q|$, it follows that $S = |Q|$, and so h must be a monomial. $\square$

To apply Theorem 5.3.4, we need to find conditions that imply $D_{\sigma\odot Q}$ is irreducible for all $|\sigma| \geq 1$. Rather than depending strictly on Q, these conditions examine the reducibility of D_Q in relation to the interplay between Q and the support of D. We begin this discussion with a remark.

Remark 5.3.5. We study how $D(z,\lambda)$ relates to $D(\mu z,\lambda)$ for $\mu \in \mathcal{U}_Q$. In particular, we consider if there exists a $\mu \in \mathcal{U}_Q$ such that $D(\mu z,\lambda)$ is given by $D(z,\lambda)$ up to multiplication by some constant. Since $D(z,\lambda)$ has a term that is constant as a polynomial in z (a term that is a constant or a power of λ), we may always assume that if such a μ exists, then $D(\mu z,\lambda) = D(z,\lambda)$. $\diamond$

Before stating these conditions in generality, we begin by building some intuition by studying the case when $d = 1$. Suppose that $\sigma = \{1\}$, $z = z_1$, and that $q = q_1 > 1$. As $D(z,\lambda)$ is irreducible,

$$D_q(z^q,\lambda) = \prod_{\mu\in\mathcal{U}_q} D(\mu z,\lambda).$$

If $D_q(z,\lambda) = gh$, where g and h are not monomials, then there exist $\mu_1,\ldots,\mu_s \in \mathcal{U}_q$, where $1 \le s < q$, such that

$$g(z^q,\lambda) = \prod_{i=1}^{s} D(\mu_i z,\lambda).$$

As $s < q$, there exists $\mu' \in \mathcal{U}_q$ such that $\mu'\mu_1 \notin \{\mu_1,\ldots,\mu_s\}$; indeed, such a μ' must exist otherwise $s = q$ and $D_q(z,\lambda)$ is irreducible as then h must be a monomial. As multiplying z by elements of $\mathcal{U}_q$ does not change $g(z^q,\lambda)$, we have

$$g(z^q,\lambda) = g((\mu' z)^q,\lambda) = \prod_{i=1}^{s} D(\mu'\mu_i z,\lambda).$$

As each $D(\mu z,\lambda)$ is irreducible, there is a $j \in \{1,\ldots,s\}$ such that $D(\mu'\mu_1 z,\lambda) = D(\mu_j z,\lambda)$ (see Remark 5.3.5). As $\mu'\mu_1 \ne \mu_j$, we have that $\hat{\mu} = \mu'\mu_1(\mu_j)^{-1}$ is not 1, and thus we have a nontrivial element $\hat{\mu} \in \mathcal{U}_q$ satisfying $D(\hat{\mu}z,\lambda) = D(z,\lambda)$. Since $D(\hat{\mu}z,\lambda) = D(z,\lambda)$, if $\upsilon(z,\lambda)$ is a monomial term of $D(z,\lambda)$, then $\upsilon(\hat{\mu}z,\lambda) = \upsilon(z,\lambda)$. Thus if $D_q(z,\lambda)$ is reducible, then ord($\hat{\mu}$), the order of $\hat{\mu}$, must divide the exponent of z in any term $\upsilon(z,\lambda)$ of $D(z,\lambda)$.

Let b' be the greatest common divisor of the finite set of integers $\{r \mid (r,t) \in \mathcal{A}(D(z,\lambda)\}$. Since $\hat{\mu}$ fixes the terms of D, ord($\hat{\mu}$) divides b'. Since ord($\hat{\mu}$) divides $q = |\mathcal{U}_q|$, it follows that $\gcd(q,b') \ne 1$. Thus if $D(z,\lambda)$ is irreducible and $\gcd(q,b') = 1$, we obtain a contradiction. Thus $D_q(z,\lambda)$ is irreducible (as our assumption that $D_q(z,\lambda)$ is reducible implies that $\gcd(q,b') \ne 1$).

Indeed, if $\gcd(q, b') = 1$, then $\gcd(\text{ord}(\hat{\mu}), b') = 1$. By Euclid's algorithm and the definition of b', there must exist $z^{r_1}\lambda^{t_1}, \ldots, z^{r_l}\lambda^{t_l}$ as monomials, or their inverses, that appear as a term with a nonzero coefficient in $D(z, \lambda)$ with $\sum r_i = b'$. Therefore, as $\hat{\mu}^{b'} \neq 1$,

$$\hat{\mu}^{b'} z^{r_1+\cdots+r_l}\lambda^{t_1\ldots t_l} \neq z^{r_1+\cdots+r_l}\lambda^{t_1\ldots t_l}, \tag{5.4}$$

and so we cannot have $(\hat{\mu}z)^{r_i}\lambda^{t_i} = z^{r_i}\lambda^{t_i}$ for all $i \in [l]$. This contradicts the assumption that $D_q(z, \lambda)$ is reducible (if it were, $\hat{\mu}$ would fix the terms of $D(z, \lambda)$).

To state the more general case, we first need to introduce a definition that will allow us to identify the values ord$(\hat{\mu})$ can take for $D_{\sigma\odot Q}$ to be reducible.

Definition 5.3.6. Let $\sigma \in \binom{[n]}{k}$ for some $k \in [n]$ and let $j \in \sigma$. Let B be the collection of b such that there is a vector in the integer span of $\mathcal{A}(D)$ that is b in the j-th coordinate and 0 for every other coordinate $i \in \sigma$. The set B is an ideal of $\mathbb{Z}$ and is therefore principal. Define $\mathit{Div}_{j,\sigma}(D)$ to be the principal generator of B (the greatest common divisor of the elements in B). ◇

If $Q = q_1$, then $\text{Div}_{1,\{1\}}(D) = b'$ (where b' is from the discussion of the one-dimensional case). In general, $D_{\sigma\odot Q}$ can factor only if ord$(\hat{\mu})$ divides $\text{Div}_{1,\sigma}(D)$ (as otherwise the same situation as (5.4) arises).

Example 5.3.7. Consider the polynomial $f(z_1, z_2, \lambda) = z_1^2 z_2^2 + \lambda z_1^4 + \lambda^3$. Suppose there is a $\mu_1 \in \mathbb{T}$ such that $f(\mu_1 z_1, z_2, \lambda) = cf(z_1, z_2, \lambda)$ for some $c \in \mathbb{C}$. As every term must be fixed under $z_1 \to \mu_1 z_1$, $c = 1$ because λ^3 is invariant with respect to this change of variables. By definition, $\text{Div}_{1,\{1\}}(f(z_1, z_2, \lambda)) = 2$. Thus $\mu_1^2 = 1$, that is $\mu_1 = \pm 1$. This agrees with the fact that $\mu_1^2 z_1^2 z_2^2 = z_1^2 z_2^2$.

In this case, $\text{Div}_{1,\{1,2\}}(f(z_1, z_2, \lambda)) = 4$. Given μ_1 and μ_2 in $\mathbb{T}$, where $f(\mu_1 z_1, \mu_2 z_2, \lambda) = cf(z_1, z_2, \lambda)$, then $c = 1$. As λz_1^4 is independent of μ_2, the order of μ_1 must divide 4.

Finally consider $h(z_1, z_2, \lambda) = z_1^{-3} z_2^2 + z_1^2 z_2^{-1} + \lambda$. Assume μ_1 and μ_2 are in $\mathbb{T}$ such that $h(\mu_1 z_1, \mu_2 z_2, \lambda) = ch(z_1, z_2, \lambda)$. Again, $c = 1$. We have $(z_1^{-3} z_2^2)(z_1^2 z_2^{-1})^2 = z_1$. Therefore, we find that $\text{Div}_{1,\{1,2\}}(h(z_1, z_2, \lambda)) = 1$. As $\mu_1^{-3} z_1^{-3} \mu_2^2 z_2^2 = z_1^{-3} z_2^2$ and $\mu_1^2 z_1^2 \mu_2^{-1} z_2^{-1} = z_1^2 z_2^{-1}$, it follows

$z_1 = (\mu_1^{-3} z_1^{-3} \mu_2^2 z_2^2)(\mu_1^2 z_1^2 \mu_2^{-1} z_2^{-1})^2 = \mu_1 z_1$. We conclude that $\mu_1 = 1$. $\diamond$

Remark 5.3.8. If $\sigma' \subseteq \sigma$ then $\mathrm{Div}_{j,\sigma'}(D)$ divides $\mathrm{Div}_{j,\sigma}(D)$. $\diamond$

We now state the general case. Recall Remark 5.3.5; that is, we assume that if there exists a $\mu \in \mathcal{U}_Q$ such that $D(\mu z, \lambda) = cD(z, \lambda)$, then $c = 1$.

Lemma 5.3.9. Let V_Q be a $\mathbb{Z}^d$-periodic potential. Suppose that there exists $\sigma' \in \binom{[d]}{k-1}$, where $1 \leq k \leq d$, such that $D_{\sigma' \odot Q}$ is irreducible. Let $\sigma = i \cup \sigma'$ for some $i \notin \sigma'$. If q_i is coprime to $b = \mathrm{Div}_{i,\sigma}(D)$, then $D_{\sigma \odot Q}$ is irreducible.

Proof. After reordering we may assume that $i = 1$ and $\sigma = \{1, 2, \ldots, k\}$. By way of contradiction, suppose that $D_{\sigma \odot Q}$ is reducible with factor g that is not a monomial, but $\gcd(q_1, b) = 1$. Let $\sigma' = \sigma \setminus \{1\}$. Then

$$D_{\sigma \odot Q}(z_1^{q_1}, z_2, \ldots, z_d, \lambda) = \prod_{\mu \in \mathcal{U}_{q_1}} D_{\sigma' \odot Q}(\mu z_1, z_2, \ldots, z_d, \lambda).$$

As each $D_{\sigma' \odot Q}(\mu z_1, z_2, \ldots, z_d, \lambda)$ is irreducible, g must have the following factorization,

$$g(z_1^{q_1}, z_2, \ldots, z_d, \lambda) = \prod_{i=1}^{s} D_{\sigma' \odot Q}(\mu_i z_1, z_2, \ldots, z_d, \lambda),$$

where $\mu_i \in U_{q_1}$ and $1 \leq s < q_1$; that is, a nonempty proper subset of the irreducible factors of $D_{\sigma \odot Q}(z_1^{q_1}, z_2, \ldots, z_d, \lambda)$ must appear as the irreducible factors of $g(z_1^{q_1}, z_2, \ldots, z_d, \lambda)$. As $s < q_1$, there exists $\mu' \in \mathcal{U}_{q_1}$ such that $\mu' \mu_1 = \hat{\mu} \notin \{\mu_1, \mu_2, \ldots, \mu_s\}$. Notice we have the following two factorizations,

$$g(z_1^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda) = \prod_{i=1}^{s} \prod_{\gamma \in \mathcal{U}_{(1,q_2,\ldots,q_k)}} D(\mu_i z_1, \gamma_2 z_2, \ldots, \gamma_k z_k, z_{k+1}, \ldots, z_d, \lambda),$$

$$g((\mu' z_1)^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda) = \prod_{i=1}^{s} \prod_{\gamma \in \mathcal{U}_{(1,q_2,\ldots,q_k)}} D(\mu' \mu_i z_1, \gamma_2 z_2, \ldots, \gamma_k z_k, z_{k+1}, \ldots, z_d, \lambda).$$

As each $D(\mu z, \lambda)$ is irreducible and

$$g(z_1^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda) = g((\mu' z_1)^{q_1}, \ldots, z_k^{q_k}, z_{k+1}, \ldots, z_d, \lambda),$$

these two factorizations are the same. For $\hat{\mu}$ and a given $\gamma \in \mathcal{U}_{(1,q_2,\ldots,q_k)}$, there exists $\mu_l \in \{\mu_1, \mu_2, \ldots \mu_s\}$ and $\gamma' \in \mathcal{U}_{(1,q_2,\ldots,q_k)}$ with

$$D(\hat{\mu} z_1, \gamma_2 z_2, \ldots, \gamma_k z_k, z_{k+1}, \ldots, z_d, \lambda) = D(\mu_l z_1, \gamma_2' z_2, \ldots, \gamma_k' z_k, z_{k+1}, \ldots, z_d, \lambda).$$

Let $\tilde{\mu} = \mu_l(\hat{\mu})^{-1}$. Notice that, as $\hat{\mu} \notin \{\mu_1, \ldots, \mu_s\}$, $\tilde{\mu} \neq 1$. In particular, $\operatorname{ord}(\tilde{\mu})$ is an integer greater than 1 that divides q_1. If $z_2, \ldots, z_k$ do not appear in a monomial $\upsilon(z_1, z_{k+1}, \ldots z_d, \lambda)$ with support in the integral span of $\mathcal{A}(D)$ then $\upsilon(\tilde{\mu} z_1, z_{k+1}, \ldots z_d, \lambda) = \upsilon(z_1, z_{k+1}, \ldots z_d, \lambda)$. Since $\gcd(q_1, b) = 1$, by the definition of $\mathrm{Div}_{1,\{1,2,\ldots,k\}}(D)$ $(= b)$, there exists a term $\upsilon(z_1, z_{k+1}, \ldots z_d, \lambda)$ of D which is not fixed by $\tilde{u}$, a contradiction. $\square$

Remark 5.3.10. From the proof of Lemma 5.3.9 we can recover a version of [30, Lemma 3.4]. In particular, suppose that for all $\mu \in \mathcal{U}_Q$ one has $D(\mu z, \lambda) \neq D(z, \lambda)$ (using the assumption that D has a term that is constant as a polynomial in z). This condition essentially encapsulates condition (A2) of [30], which is an assumption of [30, Lemma 3.4]. Under this condition, notice that if $g|D_Q$ and g is not a monomial, then we must have that $D(\mu z, \lambda)|g(z^Q, \lambda)$ for every $\mu \in \mathcal{U}_Q$; but then $g(z^Q, \lambda) = \prod_{\mu \in \mathcal{U}_Q} D(\mu z, \lambda) = D_Q(z^Q, \lambda)$. Thus Lemma 5.3.9 essentially gives us conditions for when (A2) holds when expanding the fundamental domain along a single coordinate axis (allowing us to apply the argument of [30, Lemma 3.4]).

More generally, suppose that there exists $\sigma' \subsetneq \sigma$ such that $D_{\sigma' \odot Q}$ is irreducible and contains a constant term as a polynomial in z. Without loss of generality, let $\sigma = \{1, \ldots, l\} \cup \sigma'$ where $\{1, \ldots, l\} \subseteq \overline{\sigma'}$, and let $\mathcal{U}_{Q'} = \mathcal{U}_{\{1,\ldots,l\} \odot Q}$. If

$$D_{\sigma' \odot Q}(z_1, \ldots, z_d, \lambda) \neq D_{\sigma' \odot Q}(\mu_1 z_1, \ldots, \mu_l z_l, z_{l+1}, \ldots, z_d, \lambda) \text{ for any } \mu \in \mathcal{U}_{Q'},$$

then $D_{\sigma\odot Q}$ is irreducible. We avoid further discussions of this general criteria, as our goal is to present practically verifiable conditions on D that enable us to conclude irreducibility for D_Q. $\diamond$

We will use the following corollary often in the examples of Section 5.4.

Corollary 5.3.11. Let V_Q be the $\mathbb{Z}^d$-periodic potential V, and suppose that D is irreducible. If there exist terms $z_1^{a_1}, \ldots, z_d^{a_d}$ with nonzero coefficients in D, then D_Q is irreducible for all Q such that $\gcd(q_i, a_i) = 1$ for all i.

Proof. No matter our choice of i and $\sigma \subseteq [d]$, $\mathrm{Div}_{i,\sigma}(D)|a_i$ and thus $\gcd(q_i, \mathrm{Div}_{i,\sigma}(D)) = 1$. Starting with the fact that $D = D_{\{\}\odot Q}$ is irreducible and applying Lemma 5.3.9, at each step, as $\gcd(q_i, \mathrm{Div}_{i,\sigma}(D)) = 1$, $D_{\sigma\odot Q}$ is irreducible for each $\sigma \subseteq [d]$. $\square$

A facial polynomial $(D_Q)_F$ is *$\mathbb{Z}^d$-periodic* if there exists a $\mathbb{Z}^d$-periodic potential V' corresponding to a dispersion polynomial D'_Q such that $(D_Q)_F = (D'_Q)_F$. Suppose that for a $Q\mathbb{Z}$-periodic potential V_Q the facial polynomial $(D_Q)_F$ is $\mathbb{Z}^d$-periodic due to the existence of a $\mathbb{Z}^d$-periodic V'. By Remark 5.3.2, if D_F is only homothetically reducible for V', then $(D_Q)_F$ is only homothetically reducible. By Theorem 5.2.4, we obtain the following corollary.

Corollary 5.3.12. Suppose that for every facet F of $\mathrm{Newt}\, D_Q$, except possibly one, $(D_Q)_F$ is $\mathbb{Z}^d$-periodic via the existence of a $\mathbb{Z}^d$-periodic potential V_F. If, for each F, D_F is only homothetically reducible for V_F, then D_Q is only homothetically reducible.

5.4 Applications

We conclude with examples of periodic graph operators associated to various families of periodic graphs which have irreducible Bloch varieties. We assume that all edge labels are nonzero.

For a Laurent polynomial f and a face F of $\mathrm{Newt}\, f$, the facial polynomial f_F is *potential-independent* if the potential, treated as a finite vector of indeterminates, does not appear in the coefficients of f_F. If a facial polynomial $(D_Q)_F$ is potential-independent, then $(D_Q)_F$ is $\mathbb{Z}^d$-periodic via the zero potential. A face F of $\mathrm{Newt}\, D$ (and its facial polynomial D_F) is *apical* if F contains the apex of $\mathrm{Newt}\, D$, $(0, \ldots, 0, m)$.

Let S_m be the symmetric group on m elements, $L(z,\lambda) := L(z) - \lambda I$, and let $L_{i,j}(z,\lambda)$ be the (i,j) entry of $L(z,\lambda)$. For $w \in \mathbb{Z}^{d+1}$, we call $w \cdot (a,l)$ the *weight* of the term $z^a\lambda^l$ with respect to w. The monomial terms in $\tau L(z,\lambda) := \prod_{i=1}^{m} L_{i,\tau(i)}(z,\lambda)$ are said to be terms *produced* by τ. Notice that in this way, $D(z,\lambda) = \sum_{\tau\in S_m} \mathrm{sgn}(\tau)\tau L(z,\lambda)$. We say a permutation $\tau \in S_m$ *contributes* to terms of D_w if $\mathcal{A}(\tau L(z,\lambda)) \cap \mathcal{A}(D_w) \neq \emptyset$. We say τ is *nonzero* if $\tau L(z,\lambda) \neq 0$.

A $\mathbb{Z}^d$-periodic graph is a *1-vertex graph* if it has a single vertex orbit with respect to its $\mathbb{Z}^d$-action. For $d \geq 1$, let Γ be a 1-vertex $\mathbb{Z}^d$-periodic graph. Let L be a periodic graph operator associated to Γ. By [28], D is irreducible as $\mathrm{Newt}\, D$ is a pyramid of height 1. If $d > 1$, any apical facet of $\mathrm{Newt}\, D$ is also a pyramid of height 1 and thus has an irreducible facial polynomial.

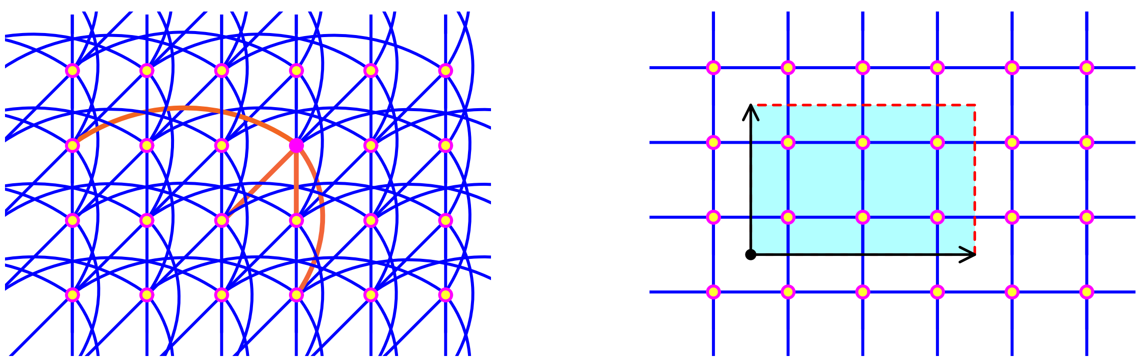

Figure 5.1: On the graph to the left, the orange edges are representatives of the edge orbits. On the graph to the right, the square lattice with a highlighted $(3,2)$-expansion of the fundamental domain is depicted. Reprinted from [1, Figure 6].

Suppose $Q \in \mathbb{N}^d$ and let V be a $Q\mathbb{Z}$-periodic potential. For any nonbase face F, $\hat{L}_Q(z,\lambda)$ has one contributing permutation (the identity permutation) through its main diagonal, with each entry in the diagonal contributing terms of the same negative weight. Thus the facial polynomial $(\hat{D}_Q)_F$, and therefore $(D_Q)_F$, is potential-independent. By Equation (5.3), $(D_Q)_F(z^Q,\lambda) = \prod_{\mu\in\mathcal{U}_Q} D_F(\mu z,\lambda)$. It follows that $\mathrm{Newt}\, D_Q$ is a contracted Q-dilation of $\mathrm{Newt}\, D$ and therefore a pyramid. Hence, D_Q is only homothetically reducible. To conclude that D_Q is irreducible, we must show that one facial polynomial is irreducible.

Example 5.4.1. Consider a $\mathbb{Z}^d$-periodic 1-vertex graph Γ, where $d \geq 1$, such that D has a facial polynomial D_F with the extreme monomials $z_1^{a_1}, \ldots, z_d^{a_d}$. Two examples of 1-vertex graphs with this property are shown in Figure 5.1. If q_i is coprime to a_i for each i, then by Corollary 5.3.11 $(D_Q)_F$ is irreducible. Thus, by Corollary 5.2.5, $D_Q(z, \lambda)$ is irreducible for all potentials.

Consider the left-hand graph of Figure 5.1. The polytope $\operatorname{Newt} D$ has a face with the extreme monomials z_1^3 and z_2^2. If q_1 is coprime to 3 and q_2 is coprime to 2 then $D_Q(z, \lambda)$ is irreducible. $\diamond$

Example 5.4.2. Let $d \geq 1$. Take any 1-vertex $\mathbb{Z}^d$-periodic graph with at least one edge. Pick an apical facet F of $\operatorname{Newt} D$. Notice that there must be some monomial z^a occurring as a term of D_F with a nonzero coefficient, for some $a \ (\neq \mathbf{0}) \in \mathbb{Z}^d$. Due to this, the collection of $Q = (q_1, \ldots, q_d) \in \mathbb{N}^d$ such that $D_{\{i\} \odot Q}$ is irreducible for all $i \in [d]$ is infinite. In particular, this set contains the $Q \in \mathbb{N}^d$ such that $\gcd(a_i, q_i) = 1$; as $\mathrm{Div}_{i,\{i\}}(D_F)$ must divide a_i, $D_{\{i\} \odot Q}$ is irreducible by Lemma 5.3.9. Moreover, we consider the infinite set of $Q \in \mathbb{N}^d$ such that $\gcd(a_i, q_i) = 1$ and the coordinates of Q are pairwise coprime. Given a Q in this infinite subset, we see that $(D_Q)_F$ is irreducible by Theorem 5.3.4. Thus D_Q is irreducible for all potentials. $\diamond$

Remark 5.4.3. The results of these last two examples overlap with the results and methods of [30]. In particular, if F is a facet that is not the base, then D_Q is irreducible if $D_F(\mu z, \lambda) \neq D_F(z, \lambda)$ for all $\mu \in \mathcal{U}_Q$ (this is what they refer to as condition (A2), see Remark 5.3.10). In [30] 1-vertex graphs were considered, and thus checking whether (A2) is satisfied is sufficient to conclude irreducibility of D_Q; as this condition implies irreducibility of the facial polynomial $(D_Q)_F$ and only homothetic reducibility immediately follows from the fact that the Newton polytope are pyramids (this is essentially [30, Lemma 3.6]).

A difference between these methods is that we only require that any facial polynomial be irreducible, whereas in [30] they always fix the face given by $w = (1, \ldots, 1, -1)$. For example, in [30] it is concluded that the dispersion polynomial obtained from the Schrödinger operator associated to the Harper lattice is irreducible for all $(q_1, q_2)\mathbb{Z}$-periodic potentials when q_1 and q_2 are coprime, but choosing another facial polynomial (for example, corresponding to $w = (-1, 0, -1)$) reveals that q_1 and q_2 do not need to be coprime. $\diamond$

We adapt the following setting from [3, Chapter 4]. Let Γ be a connected, $\mathbb{Z}^d$-periodic graph with fundamental domain W and support $\mathcal{A}(\Gamma)$. The graph Γ is *dense* if for all $a \in \mathcal{A}(\Gamma)$, there is an edge in Γ between each pair of vertices in the union of W and $a + W$.

Example 5.4.4. Consider a $\mathbb{Z}^d$-periodic dense graph Γ. As Γ is dense, by [3, Lemma 4.3], $\operatorname{Newt} D$ and its apical facets are pyramids for a generic labeling c. For $Q \in \mathbb{N}^d$, it is straightforward to deduce that any nonbase facial polynomial D_Q is potential-independent and thus only homothetically reducible (such as in Examples 5.4.1 and 5.4.2); in fact, any connected 1-vertex graph is dense). By Theorem 5.2.4, D_Q is only homothetically reducible. To show D_Q is irreducible for all potentials, it suffices to show that $(D_Q)_F$ is irreducible for some face F.

In dimensions 2 and 3, proved that each D_F is irreducible when F is not a vertex. By [3, Theorem 4.2], for a generically labeled dense $\mathbb{Z}^2$- or $\mathbb{Z}^3$-periodic graph, the zero-set of D_F is smooth (and $D|_F$ is square-free) for any nonbase face F. In particular, this implies that for every facet F, D_F is irreducible (see [34]). It follows that D_F is irreducible for any nonbase face F. As the nonbase facial polynomials are potential-independent, it follows that D_Q is irreducible for all potentials for infinitely many choices of Q (as in Example 5.4.2).

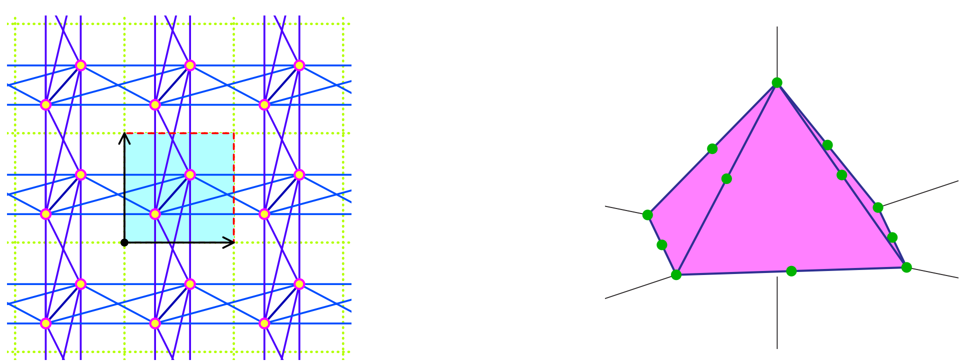

Figure 5.2: A periodic dense graph and its corresponding Newton polytope. Reprinted from [1, Figure 8].

Consider the $\mathbb{Z}^2$-periodic dense graph from [35] and depicted in Figure 5.2. The Floquet matrix of the periodic graph operator has entries:

$$L_{1,1}(z,\lambda) = \alpha + \beta_1(2 - z_1 - z_1^{-1}) + \beta_2 + \beta_3 + \gamma_1(2 - z_2 - z_2^{-1}) + \gamma_2 + \gamma_3 + V_1 - \lambda$$
$$L_{1,2}(z,\lambda) = -\alpha - \beta_2 z_1 - \beta_3 z_1^{-1} - \gamma_2 z_2 - \gamma_3 z_2^{-1}$$
$$L_{2,1}(z,\lambda) = -\alpha - \beta_2 z_1^{-1} - \beta_3 z_1 - \gamma_2 z_2^{-1} - \gamma_3 z_2$$
$$L_{2,2}(z,\lambda) = \alpha + \beta_4(2 - z_1 - z_1^{-1}) + \beta_2 + \beta_3 + \gamma_4(2 - z_2 - z_2^{-1}) + \gamma_2 + \gamma_3 + V_2 - \lambda.$$

Where $\alpha, \beta_i, \gamma_j$ are edge labels. Let F be the facet of $\mathrm{Newt}\, D$ exposed by $(-1,-1,-1)$, then D_F has exactly the monomial terms $\lambda^2, z_1^2, z_2^2, \lambda z_1, \lambda z_2$, and $z_1 z_2$. By [3, Theorem 4.2], D_F is irreducible. As λz_1, λz_2, and λ^2 are terms of D_F, $\mathrm{Div}_{1,\sigma}(D_F)$ and $\mathrm{Div}_{2,\sigma}(D_F)$ both equal 1 for $\sigma = \{1,2\}$. By Corollary 5.3.11, $(D_Q)_F$ is irreducible for any choice of q_1 and q_2, and by Corollary 5.2.5, D_Q is irreducible for all potentials. $\diamond$

6. TORIC COMPACTIFICATIONS OF PERIODIC GRAPH OPERATORS

In Chapter 5, we gave criteria for the (ir)reducibility of the dispersion polynomial of a periodic graph operator based on the geometry of its Newton polytope. Associated to this polytope is a fan Σ which corresponds to a normal toric variety X_Σ that contains the ambient space of the (complex) Bloch variety of this operator. In this chapter, we compactify this Bloch variety in X_Σ and study its asymptotics. Studying compactifications of Bloch varieties is not new (see [6, 8, 3, 1, 36]). Our contribution is to realize the compactification of the Bloch variety of a periodic graph operator as coming from an operator that extends to the boundary of its ambient toric variety.

In Section 6.1 we give an algebro-geometric analogue of the Floquet transform of a function by introducing sheaves of *quasi-periodic* functions on the affine toric varieties corresponding to the cones in the fan Σ. In Section 6.2, we glue these sheaves together to form a sheaf on the toric variety X_Σ and show this sheaf is isomorphic to a trivial sheaf on X_Σ.

The Newton polytope of a dispersion polynomial has a distinguished facet known as its base. In Section 6.3, we define a periodic graph operator as an endomorphism on a sheaf of quasi-periodic functions on the affine toric variety corresponding to the base of its Newton polytope. The characteristic matrix of this operator is an endomorphism on a trivial sheaf on this affine toric variety, and the Bloch variety of this operator is revealed as the support of the kernel of this characteristic matrix. In Section 6.4, we compactify this Bloch variety in the toric variety X_Σ associated to the fan Σ of the Newton polytope of its dispersion polynomial. Section 6.5 shows that if this Newton polytope is *full*, we may associate to an apical, nonvertex face of the polytope a periodic, labeled, directed graph whose operator has Bloch variety equal to the intersection of the compactified Bloch variety and the orbit corresponding to that face.

This chapter is based on work with Matthew Faust, Stephen Shipman, and Frank Sottile in the upcoming article [2].

6.1 Sheaves of Quasi-Periodic Functions

We follow the description of cones and fans from [27, Section 2.1]. Let N be a finitely generated free abelian group and let $M := \mathrm{Hom}(N, \mathbb{Z})$ be its dual group. Let Σ be a fan in N. We study a particular class of sheaves on affine toric varieties associated to cones in Σ.

Given a cone $\sigma \in \Sigma$, its corresponding affine toric variety V_σ is $\operatorname{spec} \mathbb{C}[\sigma^\vee]$. Let $\mathscr{O}_{V_\sigma}$ be the structure sheaf on V_σ and set $\mathsf{R} := \mathscr{O}_{V_\sigma}(V_\sigma) = \mathbb{C}[\sigma^\vee]$. In order to give an algebro-geometric analogue to the Floquet transform of a function, we consider the quasi-coherent sheaf $\prod_{a \in M} \mathscr{O}_{V_\sigma}$. Since a quasi-coherent sheaf on an affine variety is determined by its module of global sections, it suffices to consider the R-module $\mathsf{R}^M := \prod_{a \in M} \mathsf{R}$. For each $a \in M$, the function

$$
\begin{aligned}
ev(a) : \mathsf{R}^M &\longrightarrow R \\
\psi &\longmapsto \psi(a)
\end{aligned}
$$

is a homomorphism of R-modules called the *evaluation* of ψ *at* a in M.

Let $0 \in M$ be the identity of the group M. The *lineality space* of the cone $\sigma^\vee \subseteq M$ is the group $M_\sigma := \sigma^\vee \cap (-\sigma^\vee)$. Consider a splitting $\beta : M/M_\sigma \to M$ of the exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & M_\sigma \arrow[r, hook, "\iota"'] & M \arrow[r, two heads, "\pi"'] & M/M_\sigma \arrow[r] \arrow[l, bend right, "\beta"'] & 0
\end{tikzcd}
\tag{6.1}
$$

where ι is the inclusion $M_\sigma \subset M$ and π is the projection onto M/M_σ. Then M is the internal direct sum $M_\sigma \oplus \beta(M/M_\sigma)$. Write $m \in \sigma^\vee$ as $z^m \in \mathsf{R} = \mathbb{C}[\sigma^\vee]$. A function $\psi \in \mathsf{R}^M$ is *quasi-periodic* with respect to the splitting β if for $a \in M, \ell \in M_\sigma$, and $m \in M/M_\sigma$, $\psi(a + \ell + \beta(m)) = z^\ell \psi(a)$. The set $\mathcal{Q}_\beta$ of such quasi-periodic functions is closed under addition and multiplication by elements in R, making it into an R-submodule of R^M.

If $\gamma : M/M_\sigma \to M$ is another splitting of the exact sequence (6.1), then the difference $\beta - \gamma$ is a homomorphism from M/M_σ to M_σ. Precomposing this with the projection $\pi : M \to M/M_\sigma$ gives the homomorphism $\rho : M \to M_\sigma$ defined by $\rho(a) := (\beta - \gamma)(\pi(a))$. In particular, for $\ell \in M_\sigma$,

$\rho(\ell) = 0$, and for $m \in M/M_\sigma$, $\rho(\beta(m)) = (\beta - \gamma)(m) = \rho(\gamma(m))$ as $\pi(\beta(m)) = \pi(\gamma(m)) = m$.

Let $\mathsf{R}^\times$ be the group of units of R. For $a \in M$, multiplication by $z^{\rho(a)} \in \mathsf{R}^\times$ is an isomorphism $\mathsf{R} \xrightarrow{\sim} \mathsf{R}$ of R-modules. This induces the R-module isomorphism $A_{\gamma\beta} : \mathsf{R}^M \xrightarrow{\sim} \mathsf{R}^M$ given by $(A_{\gamma\beta}\psi)(a) := z^{\rho(a)}\psi(a)$. Moreover, $A_{\gamma\beta}^{-1} = A_{\beta\gamma}$ since $(A_{\beta\gamma}\psi)(a) = z^{-\rho(a)}\psi(a)$ for $a \in M$.

Lemma 6.1.1. Let β and γ be splittings of the exact sequence (6.1) and let $\mathcal{Q}_\gamma$ be the R-module of quasi-periodic functions with respect to the splitting γ. Under the isomorphism $A_{\gamma\beta}$, the R-modules $\mathcal{Q}_\beta$ and $\mathcal{Q}_\gamma$ are isomorphic.

Proof. Let $\psi_\beta \in \mathcal{Q}_\beta$, $a \in M$, $\ell \in M_\sigma$, and $m \in M/M_\sigma$. Since ρ is a homomorphism, $\rho(\ell) = 0$, and $\rho(\gamma(m)) = (\beta - \gamma)(m)$,

$$\begin{aligned}
(A_{\gamma\beta}\psi_\beta)(a + \ell + \gamma(m)) &= z^{\rho(a)} \cdot z^{\rho(\ell)} \cdot z^{(\beta-\gamma)(m)}\psi_\beta(a + \ell + \gamma(m)) \\
&= z^{\rho(a)} \cdot z^{(\beta-\gamma)(m)}\psi_\beta(a + \ell + (\gamma - \beta)(m) + \beta(m)) \\
&= z^{\ell} \cdot z^{\rho(a)} \cdot z^{(\beta-\gamma)(m)+(\gamma-\beta)(m)}\psi_\beta(a + \beta(m)) = z^{\ell} A_{\gamma\beta}\psi_\beta(a)
\end{aligned}$$

showing that $A_{\gamma\beta}\psi_\beta \in \mathcal{Q}_\gamma$. Similarly, if $\psi_\gamma \in \mathcal{Q}_\gamma$, then $A_{\beta\gamma}\psi_\gamma \in \mathcal{Q}_\beta$. Since $A_{\gamma\beta}$ is an isomorphism, it follows that $\mathcal{Q}_\beta \simeq \mathcal{Q}_\gamma$. □

Proposition 6.1.2. Under the evaluation $ev(0) : \mathsf{R}^M \to \mathsf{R}$, the R-module $\mathcal{Q}_\beta$ is isomorphic to R.

Proof. Let $\psi_\beta \in \mathcal{Q}_\beta$ and $a \in M$. Then $\psi_\beta(a) = z^a\psi_\beta(0)$, showing $\psi_\beta \in \mathcal{Q}_\beta$ is determined by its value at 0. □

For the remainder of this chapter, we fix a splitting $\beta : M/M_\sigma \to M$ such that $M = M_\sigma \oplus \beta(M/M_\sigma)$ and denote $\mathcal{Q}_\beta$ by $\mathcal{Q}_\sigma$. Let $\widetilde{\mathcal{Q}}_\sigma$ be the sheaf on $V_\sigma = \operatorname{spec}\mathbb{C}[\sigma^\vee]$ associated to the R-module $\mathcal{Q}_\sigma$. This sheaf is the *sheaf of quasi-periodic functions* on V_σ.

6.2 Gluing Sheaves of Quasi-Periodic Functions

The toric variety X_Σ associated to the fan Σ is obtained from the collection $\{V_\sigma \mid \sigma \in \Sigma\}$ of affine toric varieties V_σ by gluing along the inclusions $V_\tau \subset V_\sigma$ whenever τ, σ are cones in Σ and τ is a face of σ. We show there is a sheaf $\widetilde{\mathcal{Q}}_\Sigma$ on X_Σ obtained from the collection $\{\widetilde{\mathcal{Q}}_\sigma \mid \sigma \in \Sigma\}$ of sheaves of quasi-periodic functions on affine toric varieties V_σ corresponding to cones σ in Σ. We also show this sheaf is isomorphic to the trivial sheaf $\mathscr{O}_{X_\Sigma}$.

Theorem 6.2.1. There exists a rank-one sheaf $\widetilde{\mathcal{Q}}_\Sigma$ on X_Σ such that for any cone σ in Σ, the restriction of $\mathcal{Q}_\Sigma$ to the affine toric variety V_σ is isomorphic to $\mathcal{Q}_\sigma$.

We give gluing data for the collection $\{\widetilde{\mathcal{Q}}_\sigma \mid \sigma \in \Sigma\}$. For the remainder of this section, let τ, σ be cones in Σ and assume τ is a face of σ. Let $\mathsf{R} = \mathbb{C}[\sigma^\vee]$ and $\mathsf{S} = \mathbb{C}[\tau^\vee]$ so that $\mathsf{R} \subseteq \mathsf{S}$. Let M_σ and M_τ be the lineality spaces of the cones $\sigma^\vee$ and $\tau^\vee$, respectively, and note that $M_\sigma \subset M_\tau$. Consider a splitting $\gamma : M_\tau/M_\sigma \to M_\tau$ of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_\sigma \arrow[r, hook, "\iota"'] & M_\tau \arrow[r, two heads, "\pi"'] & M_\tau/M_\sigma \arrow[l, bend right, "\gamma"'] \arrow[r] & 0 \end{tikzcd} \tag{6.2}$$

where ι is the inclusion $M_\sigma \subset M_\tau$ and π is the projection onto M_τ/M_σ. Then M_τ is the internal direct sum $M_\sigma \oplus \gamma(M_\tau/M_\sigma)$. By considering a splitting $\beta : M/M_\sigma \to M$ of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_\tau \arrow[r, hook, "\iota"'] & M \arrow[r, two heads, "\pi"'] & M/M_\tau \arrow[l, bend right, "\beta"'] \arrow[r] & 0, \end{tikzcd} \tag{6.3}$$

M is the internal direct sum $M_\tau \oplus \beta(M/M_\tau) = M_\sigma \oplus \gamma(M_\tau/M_\sigma) \oplus \beta(M/M_\tau)$.

Let $\pi_\gamma : M \to \gamma(M_\tau/M_\sigma)$ be the projection onto $\gamma(M_\tau/M_\sigma) \subset M_\tau$. For $a \in M$, $\ell \in M_\sigma$, $m' \in M_\tau/M_\sigma$ and $m \in M/M_\tau$, the map π_γ satisfies $\pi_\gamma(a + \ell + \gamma(m') + \beta(m)) = \pi_\gamma(a) + \gamma(m')$. Let $\mathsf{S}^\times$ be the group of units of S. For $a \in M$, multiplication by $z^{\pi_\gamma(a)} \in \mathsf{S}^\times$ is an isomorphism $\mathsf{S} \to \mathsf{S}$ of S-modules. Since $\mathsf{R}^M \otimes_{\mathsf{R}} \mathsf{S} \simeq \mathsf{S}^M$, multiplication by $z^{\pi_\gamma(a)}$ for all $a \in M$ induces an S-module homomorphism $g_{\tau\sigma} : \mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{S} \to \mathcal{Q}_\tau$ defined by $(g_{\tau\sigma}\psi_\sigma)(a) := z^{\pi_\gamma(a)}\psi_\sigma(a)$ for $a \in M$.

Lemma 6.2.2. Let $\gamma : M_\tau/M_\sigma \to M_\tau$ and $\beta : M/M_\tau \to M$ be splittings of the exact sequences (6.2) and (6.3), respectively. Then $g_{\tau\sigma}(\mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{S}) \subseteq \mathcal{Q}_\tau$

Proof. Let $\psi_\sigma \in \mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{S}$. Since M_τ is the internal direct sum $M_\sigma \oplus \gamma(M_\tau/M_\sigma)$, any element of M_τ may be expressed as a sum $\ell + \gamma(m')$ for $\ell \in M_\sigma$ and $m' \in M_\tau/M_\sigma$. Let $a \in M$, $\ell \in M_\sigma$,

$m' \in M_\tau/M_\sigma$ and $m \in M/M_\sigma$. As $\pi_\gamma(\ell + \beta(m)) = 0$,

$$g_{\tau\sigma}\psi_\sigma(a + \ell + \gamma(m') + \beta(m)) = z^{\pi_\gamma(a+\ell+\gamma(m')+\beta(m))}\psi_\sigma(a + \ell + \gamma(m') + \beta(m))$$
$$= z^{\pi_\gamma(a)+\gamma(m')} \cdot z^\ell \psi_\sigma(a) = z^{\ell+\gamma(m')} g_{\tau\sigma}\psi_\sigma(a),$$

showing that $g_{\tau\sigma}\psi_\tau \in \mathcal{Q}_\tau$. $\square$

To show that the homomorphisms $g_{\tau\sigma}$ are gluing data for a sheaf on X_Σ, we need to show they behave well under composition. Suppose η is a face of the cone τ and let $\mathsf{T} = \mathbb{C}[\eta^\vee]$. Let M_η be the lineality space of $\eta^\vee$. Observe that $M_\sigma \subset M_\tau \subset M_\eta$ (since τ is assumed to be a face of the cone σ). Consider a splitting $\gamma' : M_\eta/M_\sigma \to M_\eta$ of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_\tau \arrow[r, hook, "\iota"'] & M_\eta \arrow[r, two heads, "\pi"'] & M_\eta/M_\tau \arrow[l, bend right, "\gamma'"'] \arrow[r] & 0 \end{tikzcd} \tag{6.4}$$

where ι is the inclusion $M_\tau \subset M_\eta$ and π is the projection onto M_η/M_τ. Then M_η is the internal direct sum $M_\tau \oplus \gamma'(M_\eta/M_\tau)$. By considering the splitting $\gamma : M_\tau/M_\sigma \to M_\tau$ of the exact sequence (6.2), then M_η is the internal direct sum $M_\sigma \oplus \gamma(M_\tau/M_\sigma) \oplus \gamma'(M_\eta/M_\tau)$.

Similarly, we may consider a splitting $\widetilde{\gamma} : M_\eta/M_\sigma \to M_\eta$ of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_\sigma \arrow[r, hook, "\iota"'] & M_\eta \arrow[r, two heads, "\pi"'] & M_\eta/M_\sigma \arrow[l, bend right, "\widetilde{\gamma}"'] \arrow[r] & 0 \end{tikzcd} \tag{6.5}$$

where ι is the inclusion $M_\sigma \subset M_\eta$ and π is the projection onto M_η/M_σ. Then M_η is the internal direct sum $M_\sigma \oplus \widetilde{\gamma}(M_\eta/M_\sigma)$. It follows that $\gamma(M_\tau/M_\sigma) \oplus \gamma'(M_\eta/M_\tau) \simeq \widetilde{\gamma}(M_\eta/M_\sigma)$.

Consider a splitting $\beta' : M/M_\eta \to M$ of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_\eta \arrow[r, hook, "\iota"'] & M \arrow[r, two heads, "\pi"'] & M/M_\eta \arrow[l, bend right, "\beta'"'] \arrow[r] & 0 \end{tikzcd} \tag{6.6}$$

where ι is the inclusion $M_\eta \subset M$ and π is the projection onto M/M_η. It follows that

$$M = M_\eta \oplus \beta'(M/M_\eta) = M_\sigma \oplus \widetilde{\gamma}(M_\eta/M_\sigma) \oplus \beta'(M/M_\eta)$$
$$= M_\sigma \oplus \gamma(M_\tau/M_\sigma) \oplus \gamma'(M_\eta/M_\tau) \oplus \beta'(M/M_\eta).$$

Let $\pi_{\widetilde{\gamma}}$ and $\pi_{\gamma'}$ be the projection of M onto $\widetilde{\gamma}(M_\eta/M_\sigma)$ and $\gamma'(M_\eta/M_\tau)$, respectively. Then $\pi_{\widetilde{\gamma}} = \pi_\gamma + \pi_{\gamma'}$. Since η is a face of τ and σ, consider the induced homomorphisms $g_{\eta\tau} : \mathsf{S}^M \to \mathsf{T}^M$, $g_{\tau\sigma} : \mathsf{R}^M \to \mathsf{S}^M$, and $g_{\eta\sigma} : \mathsf{R}^M \to \mathsf{T}^M$.

Lemma 6.2.3. Let $\eta \subset \tau \subset \sigma$ and let $\gamma' : M_\eta/M_\tau \to M_\eta$ and $\widetilde{\gamma} : M_\eta/M_\sigma \to M_\eta$ be splittings of the exact sequences (6.4) and (6.5), respectively. Then $g_{\eta\sigma} = g_{\eta\tau} \circ g_{\tau\sigma}$.

Proof. Let $\psi_\sigma \in \mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{T}$. Since $\pi_\gamma + \pi_{\gamma'} = \pi_{\widetilde{\gamma}}$, we obtain

$$(g_{\eta\tau} \circ g_{\tau\sigma}\psi_\sigma)(a) = z^{\pi_{\gamma'}(a)} \cdot z^{\pi_\gamma(a)}\psi_\sigma(a) = z^{(\pi_{\gamma'}+\pi_\gamma)(a)}\psi_\sigma(a)$$
$$= z^{\pi_{\widetilde{\gamma}}(a)}\psi_\sigma(a) = g_{\eta\sigma}\psi_\sigma(a)$$

for all $a \in M$. Thus, $g_{\eta\sigma} = g_{\eta\tau} \circ g_{\tau\sigma}$. □

Suppose there exists $\tau' \in \Sigma$ such that $\eta \subset \tau' \subset \sigma$. By Lemma 6.2.3, $g_{\eta\tau'} \circ g_{\tau'\sigma} = g_{\eta\sigma} = g_{\eta\tau} \circ g_{\tau\sigma}$ so the homomorphisms $g_{\tau\sigma}$ behave well under composition.

Proof of Theorem 6.2.1. By Lemma 6.2.3, the collection $\{g_{\tau\sigma} \mid \tau, \sigma \in \Sigma, \tau \subset \sigma\}$ of homomorphisms is a gluing datum for a sheaf $\widetilde{\mathcal{Q}}_\Sigma$ on X_Σ. Since $ev(0) \circ g_{\tau\sigma} : \mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{S} \xrightarrow{\sim} \mathsf{S}$ is an isomorphism, the sheaves $\widetilde{\mathcal{Q}}_\Sigma$ and $\mathscr{O}_{X_\Sigma}$ are isomorphic. □

By Lemma 6.1.1 and Proposition 6.1.2, the sheaf $\widetilde{\mathcal{Q}}_\Sigma$ is unique up to unique isomorphism. We call $\widetilde{\mathcal{Q}}_\Sigma$ the *sheaf of quasi-periodic functions on X_Σ*.

Example 6.2.4. Let Σ be the fan depicted in Figure 6.1. Consider the cone σ generated by the vectors $(1,1), (1,0)$ and $(1,-1)$, and the ray τ generated by the vector $(1,1)$. Suppose $\mathbb{Z}^2$ is the

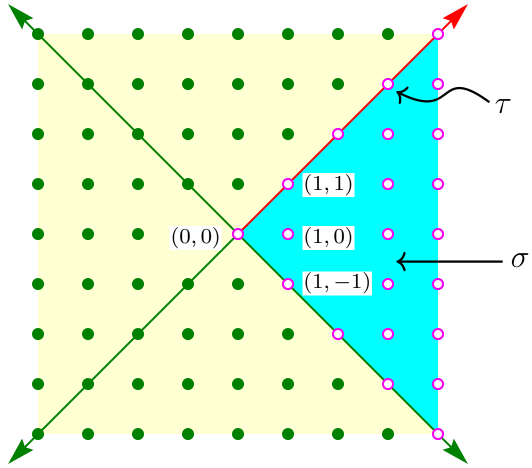

Figure 6.1: The fan of the two-dimensional cross polytope.

internal direct sum $\mathbb{Z}(1,-1) \oplus \mathbb{Z}(1,0)$. Set $\mathsf{R} = \mathbb{C}[xy, xy^{-1}, x]$ and $\mathsf{S} = \mathbb{C}[(xy^{-1})^{\pm}, x]$. Then

$$\mathcal{Q}_\sigma = \{\psi_\sigma \in \mathsf{R}^{\mathbb{Z}^2} \mid \psi_\sigma(a+r, b+s) = \psi_\sigma(a,b)\} \text{ and}$$
$$\mathcal{Q}_\tau = \{\psi_\tau \in \mathsf{S}^{\mathbb{Z}^2} \mid \psi_\tau(a+r+s, b-r) = (xy^{-1})^r \psi_\tau(a,b)\}.$$

We show how to glue $\widetilde{\mathcal{Q}}_\sigma$ and $\widetilde{\mathcal{Q}}_\tau$ along the intersection $V_\sigma \cap V_\tau = V_\tau$. Let $\pi(a,b) := (b,-b)$ be the projection of $\mathbb{Z}(1,-1) \oplus \mathbb{Z}(1,0)$ onto $\mathbb{Z}(1,-1)$. Then $g_{\tau\sigma} : \mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{S} \to \mathcal{Q}_\tau$ is given by

$$g_{\tau\sigma}\psi(a,b) := (xy)^{\pi(a,b)}\psi(a,b), \qquad \text{where } \pi(a+r+s, b-r) = \pi(a,b) + (r,-r).$$

Let $\psi_\sigma \in \mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{S}$. Since $\pi(r,0) = (0,0)$,

$$g_{\tau\sigma}\psi_\sigma(a+r+s, b-r) = (xy^{-1})^r g_{\tau\sigma}\psi_\sigma(a,b),$$

showing that $g_{\tau\sigma}\psi_\sigma \in \mathcal{Q}_\tau$ and $g_{\tau\sigma}(\mathcal{Q}_\sigma \otimes_{\mathsf{R}} \mathsf{S}) \subseteq \mathcal{Q}_\tau$. $\diamond$

Let W be a finite set and let $a \in M$. Taking the direct sum of $|W|$ copies of $\mathsf{R}^M = \mathbb{C}[\sigma^\vee]^M$, we obtain the R-module homomorphism

$$\bigoplus_{u\in W} ev(a) : \bigoplus_{u\in W} \mathsf{R}^M \to \bigoplus_{u\in W} \mathsf{R}, \tag{6.7}$$

which we identify with $ev(\bullet, a) : \mathsf{R}^{W\times M} \to \mathsf{R}^W$. We also identify the R-submodule $\bigoplus_{u\in W} \mathcal{Q}_\sigma$ with

$$\mathcal{Q}_\sigma^W := \Big\{\psi_\sigma \in \mathsf{R}^{W\times M} \mid \psi_\sigma(u, a{+}\ell{+}\beta(m)) = z^\ell \psi_\sigma(u,a) \text{ for } \ell \in M_\sigma \text{ and } m \in M/M_\sigma\Big\}.$$

By Proposition 6.1.2, we obtain the following corollary.

Corollary 6.2.5. For any cone σ in Σ, the R-module homomorphism $ev(\bullet, 0) : \mathcal{Q}_\sigma^W \to \mathsf{R}^W$ is an isomorphism.

Using Theorem 6.2.1 and Corollary 6.2.5, we obtain the sheaf $\widetilde{Q}_\Sigma^W := \bigoplus_{u\in W} \widetilde{Q}_\Sigma$ of quasi-periodic functions on X_Σ. We summarize the results of this section with the following theorem.

Theorem 6.2.6. For any cone σ in Σ, the restriction of $\widetilde{\mathcal{Q}}_\Sigma^W$ to the affine toric variety V_σ is isomorphic to $\widetilde{\mathcal{Q}}_\sigma^W$. The sheaf $\widetilde{\mathcal{Q}}_\Sigma^W$ is isomorphic to the trivial sheaf $\mathscr{O}_{X_\Sigma}^W$.

6.3 Periodic Graph Operators on Sheaves of Quasi-Periodic Functions

We use the tools developed in Sections 6.1 and 6.2 to define periodic graph operators as endomorphisms on sheaves of quasi-periodic functions. We define the Bloch variety of a periodic graph operator and show it is the support of a particular kernel sheaf.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a $\mathbb{Z}^d$-periodic graph and let $W \subset \mathcal{V}$ be a fundamental domain for the action of $\mathbb{Z}^d$ on Γ. We identify the set of vertices $\mathcal{V}$ with $W \times \mathbb{Z}^d$ as in [37, Section 2.2.1]. Consider the $\mathbb{C}[z^\pm]$-module

$$\mathcal{Q}_0^W := \Big\{\psi \in \mathbb{C}[z^\pm]^{W\times\mathbb{Z}^d} \mid \psi(u, a+\ell) = z^\ell \psi(u,a) \text{ for } u \in W \text{ and } a, \ell \in \mathbb{Z}^d\Big\}$$

of quasi-periodic functions on $(\mathbb{C}^\times)^d$. Fix a labeling $c := (c, V)$ of Γ. A *periodic graph operator* on $\mathcal{Q}_0^W$ is a $\mathbb{C}[z^\pm]$-module endomorphism $L_c : \mathcal{Q}_0^W \to \mathcal{Q}_0^W$ such that for $\psi \in \mathcal{Q}_0^W$, $L_c\psi$ is defined by its value at $(u,a) \in W \times \mathbb{Z}^d$,

$$(L_c\psi)(u,a) := V(u,a)\psi(u,a) - \sum_{(u,a)\sim(v,b)} c_{(u,a)\sim(v,b)}\psi(v,b),$$

where $(u,a)\sim(v,b)$ means $(u{+}a, v{+}b) \in \mathcal{E}$ (see Section 2.3).

Set $\mathbf{0} := (0,\ldots,0) \in \mathbb{Z}^d$. By Corollary 6.2.5, the induced action of L_c on $\mathbb{C}[z^{\pm}]^W$ is multiplication by the *Floquet matrix* $L_c(z)$ (see Section 2.1), and by extension of scalars, the induced action of $L_c - \lambda\,\mathrm{Id}_{\mathcal{Q}_0^W}$ on $\mathbb{C}[z^{\pm},\lambda]^W$ is multiplication by the characteristic matrix $L_c(z) - \lambda I_W$. The kernel of the endomorphism $L_c(z) - \lambda I_W$ on $\mathbb{C}[z^{\pm},\lambda]^W$ induces a sheaf $\mathscr{F}$ on $(\mathbb{C}^\times)^d \times \mathbb{C}$ called the *Floquet sheaf* of the operator $L_c(z) - \lambda I_W$. The support of this Floquet sheaf $\mathscr{F}$ is the *Bloch variety* $\mathcal{B}_{L_c}$ ([37, Section 4.2.4]). Equivalently, the Bloch variety $\mathcal{B}_{L_c}$ is the set $Var(D_c)$, where $D_c := \det(L_c(z) - \lambda I_W)$ is the *dispersion polynomial* of the operator L_c.

Lemma 6.3.1. The kernel of the operator $L_c(z) - \lambda I_W$ on $\mathbb{C}[z^{\pm},\lambda]^W$ equals the kernel of the operator $\alpha(L_c(z) - \lambda I_W)$ on $\mathbb{C}[z^{\pm},\lambda]^W$ for any unit $\alpha \in \mathbb{C}[z^{\pm}]$. It follows that the support of $L_c(z) - \lambda I_W$ equals the support of $\alpha(L_c(z) - \lambda I_W)$ for any unit $\alpha \in \mathbb{C}[z^{\pm}]$.

Let $\mathcal{A}(D_c) \subset \mathbb{Z}^{d+1}$ be support of D_c. The Newton polytope of D_c is $\mathrm{Newt}(D_c) := \mathrm{conv}(\mathcal{A}(D_c)) \subset \mathbb{R}^{d+1}$. Set $P := \mathrm{Newt}(D_c)$ and let Σ_P be the fan in $\mathbb{Z}^{d+1}$ associated to P. Recall from Section 4.4 that each cone $\sigma_F \in \Sigma_P$ corresponds to a unique face F of P. For $\mathbf{e} := (\mathbf{0}, 1) \in \mathbb{Z}^{d+1}$, the ray $\sigma_B := \mathbb{N}\mathbf{e}$ in Σ_P corresponds to the base $B = \mathrm{conv}(\mathcal{A}(\det L_c(z)))$ of P. Let $V_B \simeq (\mathbb{C}^\times)^d \times \mathbb{C}$ be the affine toric variety corresponding to the ray σ_B. The lineality space of $\sigma_B^\vee$ is $M_B = \mathbb{Z}^d \times \{0\}$. By extension of scalars, the $\mathbb{C}[z^{\pm}]$-module $\mathcal{Q}_0^W$ is isomorphic to the $\mathbb{C}[\sigma_B^\vee]$-module

$$\mathcal{Q}_B^W := \big\{\psi_B \in \mathbb{C}[\sigma_B^\vee]^{W\times\mathbb{Z}^{d+1}} \mid \psi_B(u, a + \ell + \beta(m)) = z^\ell \psi_B(u,a), \qquad \text{for } \ell \in M_B \text{ and } m \in \mathbb{Z}^{d+1}/M_B\big\}$$

of quasi-periodic functions on V_B. By Corollary 6.2.5, the induced action of $L_c - \lambda\,\mathrm{Id}_{\mathcal{Q}_B^W}$ on $\mathbb{C}[\sigma_B^\vee]^W \simeq \mathbb{C}[z^{\pm},\lambda]^W$ is multiplication by the characteristic matrix $L_c(z) - \lambda I_W$, and the support of $\mathscr{F}$ is the Bloch variety $\mathcal{B}_{L_c}$ in $V_B \simeq (\mathbb{C}^\times)^d \times \mathbb{C}$.

6.4 A Toric Compactification of the Bloch Variety

Let P be the Newton polytope of the dispersion polynomial D_c of a periodic graph operator L_c. As a complex algebraic hypersurface in V_B, the Bloch variety $\mathcal{B}_{L_c}$ is not compact. By Theorem 4.4.5, the toric variety X_{Σ_P} associated to P is compact. Thus, as $V_B \subset X_{\Sigma_P}$, the Zariski closure $\overline{\mathcal{B}_{L_c}}$ of $\mathcal{B}_{L_c}$ in X_{Σ_P} is a compactification of the Bloch variety $\mathcal{B}_{L_c}$. To study the points added in the compactification, we adapt the following terminology from [36].

Lemma 6.4.1. [36, Lemma 1.10] Let F be a face of the polytope P, and let M_F be the lineality space of the cone $\sigma_F^\vee \subseteq \mathbb{Z}^{d+1}$. We have canonical maps

$$\mathbb{C}[M_F] \xhookrightarrow{\iota} \mathbb{C}[\sigma_F^\vee] \xrightarrow{\pi} \mathbb{C}[M_F], \qquad \pi \circ \iota = I \tag{6.8}$$

that induce canonical maps of affine toric varieties

$$\mathcal{O}_F \xhookrightarrow{\pi^*} V_F \xrightarrow{\iota^*} \mathcal{O}_F,$$

where $\mathcal{O}_F$ denotes the orbit in V_F corresponding to the face F.

Let f be a polynomial with support $\mathcal{A}(f) \subset \mathbb{Z}^{d+1}$ and let F be a face of the polytope P. Recall from Section 4.2 that the facial polynomial f_F of f is the sum of the terms of f whose exponent vectors lie in $\mathcal{A}(f) \cap F$. Since we will encounter polynomials whose notation involves subscripts, we will also write $in_F f$ for the facial polynomial f_F.

Lemma 6.4.2. [36, Lemma 1.11] Let f be a polynomial with support $\mathcal{A}(f) \subset \mathbb{Z}^{d+1}$ and let $Var(f) \subset (\mathbb{C}^\times)^{d+1}$. Denote the Zariski closure of $Var(f)$ in X_Σ by $\overline{Var(f)}$. For any face F of the polytope P and $r \in F \cap \mathbb{Z}^{d+1}$,

(1) $z^{-r}f \in \mathbb{C}[\sigma_F^\vee]$ and $Var(z^{-r}f) \subset V_F$ is $\overline{Var(f)} \cap V_F$, and

(2) $z^{-r}f_F = \pi(z^{-r}f) \in \mathbb{C}[M_F]$ and $Var(z^{-r}f_F) \subset V_F$ is $\overline{Var(f)} \cap \mathcal{O}_F$ where the map $\pi : \mathbb{C}[\sigma_F^\vee] \twoheadrightarrow \mathbb{C}[M_F]$ is the canonical map in Lemma 6.4.1.

Proof. (1) The polynomial f defines $Var(f) \subset (\mathbb{C}^\times)^{d+1}$. Let $\langle f \rangle$ be the principal ideal of $\mathbb{C}[\mathbb{Z}^{d+1}]$ generated by f. The closure $\overline{Var(f)} \cap V_F$ in V_F is defined by $\langle f \rangle \cap \mathbb{C}[\sigma_F^\vee]$. Let αz^c be a term of f with $\alpha \in \mathbb{C}^\times$ and $c \in \mathcal{A}(f) \cap F$. Then the polynomial $z^{-c}f$ has α as a constant term and $z^{-c}f \in \mathbb{C}[\sigma_F^\vee]$ since $\mathcal{A}(f) - c \subseteq \sigma_F^\vee$. If $a \in \sigma_F^\vee$, then $z^a \in \mathbb{C}[\sigma_F^\vee]$ so that $z^a z^{-c} f \in \mathbb{C}[\sigma_F^\vee]$. Conversely, suppose $z^a z^{-c} f \in \mathbb{C}[\sigma_F^\vee]$ for some $a \in \mathbb{Z}^{d+1}$. Then αz^a is a term of $z^a z^{-c} f$, which implies $a \in \sigma_F^\vee$. Thus $\langle f \rangle \cap \mathbb{C}[\sigma_F^\vee]$ is the principal ideal of $\mathbb{C}[\sigma_F^\vee]$ generated by $z^{-c}f$. Let $r \in F \cap \mathbb{Z}^{d+1}$. Then $z^{-r}f = z^{c-r}z^{-c}f$. Since $c - r \in M_F$, z^{c-r} is invertible in $\mathbb{C}[\sigma_F^\vee]$ and $\langle z^{-r}f \rangle = \langle z^{-c}f \rangle$.

(2) Using the map π in (6.8), $\pi(z^{-r}f) = z^{-r}f_F$. □

Let F be a face of P and let $r \in F \cap \mathbb{Z}^{d+1}$. By Lemma 6.4.2, the polynomial $z^{-r}D_c$ is in the monoid algebra $\mathbb{C}[\sigma_F^\vee]$ and $z^{-r}\text{in}_F D_c$ is in $\mathbb{C}[M_F]$. As $D_c = \det(L_c(z) - \lambda I_W)$, we investigate when the endomorphism $L_c(z) - \lambda I_W$ of $\mathbb{C}[\sigma_B^\vee]^W$ extends to $\mathbb{C}[\sigma_F^\vee]^W$ such that, when the terms of its corresponding matrix restrict to $\mathbb{C}[M_F]$, it becomes an endomorphism of $\mathbb{C}[M_F]^W$ and the determinant of its corresponding matrix is nonzero. In Section 6.5, we show that if P is *full*, then $L_c(z) - \lambda I_W$ extends to an endomorphism of $\mathbb{C}[\sigma_F^\vee]^W$ in this way for particular faces F of the polytope P.

6.5 Full Newton Polytopes

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a $\mathbb{Z}^d$-periodic graph and let $W \subset \mathcal{V}$ be a set of orbit representatives of the free action of $\mathbb{Z}^d$ on Γ. Consider a periodic graph operator L_c on $\mathcal{Q}_B^W$ and let $L_c(z)$ be its Floquet matrix. Let S_W be the symmetric group on the set W. For $u, v \in W$, let $f_{u,v}$ denote the entry of the Floquet matrix $L_c(z)$ in the (u, v)-position (see Section 2.4). We may express the determinant of $L_c(z)$ as

$$\det L_c(z) = \sum_{\rho \in S_W} \text{sgn}(\rho) \prod_{u \in W} f_{u,\rho(u)}, \tag{6.9}$$

where $sgn(\rho)$ is the sign of the permutation ρ.

We adapt the following setting from [3, Chapter 4]. Since $\mathcal{A}(\Gamma)$ is the support of the graph Γ, each term in $L_c(z)$ has support a subset of $\text{conv}(\mathcal{A}(\Gamma))$ and each diagonal entry of the characteristic matrix $L_c(z) - \lambda I_W$ has support a subset of $\mathcal{A}(\Gamma) \cup \{\mathbf{e}\}$. Set $Q := \text{conv}(\mathcal{A}(\Gamma) \cup \{\mathbf{e}\})$.

Lemma 6.5.1. [3, Lemma 4.1] Let P be the Newton polytope of the dispersion polynomial D_c. Then P is a subpolytope of the dilation $|W|Q$.

Proof. Since

$$\mathrm{Newt}\Big(\prod_{u\in W}(f_{u,\rho(u)}-\lambda\delta_{u,\rho(u)})\Big)=\sum_{u\in W}\mathrm{Newt}(f_{u,\rho(u)}-\lambda\delta_{u,\rho(u)}),$$

for each $\rho\in S_W$, $\prod_{u\in W}(f_{u,\rho(u)}-\lambda\delta_{u,\rho(u)})$ has Newton polytope a subpolytope of $|W|Q$. $\square$

The Newton polytope $P=\mathrm{Newt}\,D_c$ is *full* if $P=|W|Q$. If P is full, for each face F of P there exists a unique face G of Q such that $F=|W|G$. Suppose the Newton polytope P is full. Let F be a face of P and let G be its corresponding face in Q. The *facial matrix* $in_G(L_c(z)-\lambda I_W)$ corresponding to the face G is the matrix of terms of $L_c(z)-\lambda I_W$ whose exponent vectors lie in G. We define the facial matrices $in_G L_c(z)$ and $in_G\lambda I_W$ corresponding to G analogously.

Lemma 6.5.2. Suppose $P=\mathrm{Newt}(D_c)$ is full. Let G be a face of Q such that $F=|W|G$ is a face of P. Then $\mathrm{in}_F\det L_c(z)=\det(\mathrm{in}_G L_c(z))$.

Proof. Since $\det L_c(z)=\sum_{\rho\in S_W}\mathrm{sgn}(\rho)\prod_{u\in W}f_{u,\rho(u)}$, we obtain

$$\mathrm{in}_F\big(\det L_c(z)\big)=\sum_{\rho\in S_W}\mathrm{sgn}(\sigma)\mathrm{in}_F\Big(\prod_{u\in W}f_{u,\rho(u)}\Big).$$

Thus, it suffices to show that for all $\rho\in S_W$,

$$\mathrm{in}_F\Big(\prod_{u\in W}f_{u,\rho(u)}\Big)=\prod_{u\in W}\Big(\mathrm{in}_G L_c(z)\Big)_{u,\rho(u)}.$$

Let $\rho\in S_W$. Each of the terms of the facial polynomial $\mathrm{in}_F\Big(\prod_{u\in W}f_{u,\rho(u)}\Big)$ has weight h_F and for each $u\in W$, each of the terms of the facial polynomial $\big(\mathrm{in}_G L_c(z)\big)_{u,\rho(u)}$ has weight h_G. Since P is full, $h_F=|W|h_G$. As each vertex of P has a corresponding term in $\det L_c(z)$ with a nonzero coefficient, the result follows. $\square$

Lemma 6.5.3. Let G be a nonvertex face of Q such that $F = |W|G$ is a face of P. The determinant of the facial matrix $\text{in}_G L_c(z)$ is nonzero.

Proof. Since G is not a vertex, the face F is not a vertex, so F contains a vertex s in the base of P. By Lemma 6.5.2, $\det(\text{in}_G L_c(z)) = \text{in}_F \det L_c(z)$ and this determinant is nonzero since $\text{in}_F(\det L_c(z))$ has a term whose support is s. $\square$

Recall from Section 4.2 that a face of a pyramid is *apical* if it contains the apex of the pyramid. Since the polytope $Q = \text{conv}(\mathcal{A}(\Gamma) \cup \{\mathbf{e}\})$ is a pyramid, an apical face of Q contains its apex, $\mathbf{e}$. If the Newton polytope P is full, then $P = |W|Q$ shows P is a pyramid with apex $|W|\mathbf{e}$.

Remark 6.5.4. Let G be an apical face of Q. Since $\mathbf{e} \in G$, $\text{in}_G(\lambda I_W) = \lambda I_W$. Thus, the determinant of the facial matrix $\text{in}_G(\lambda I_W)$ is nonzero. $\diamond$

Lemma 6.5.5. Suppose $P = \text{Newt}(D_c)$ is full. Let G be an apical face of Q such that $F = |W|G$ is a face of P. Then $\text{in}_F D_c = \det(\text{in}_G L_c(z) - \lambda I_W)$.

From now on, we assume the Newton polytope $P = \text{Newt}(D_c)$ is full. Let G be a nonvertex apical face of Q such that $F = |W|G$ is a face of P, and let $r \in G \cap \mathbb{Z}^{d+1}$. By Lemma 6.5.3, the determinant of the matrix $z^{-r}\text{in}_G L_c(z)$ is nonzero. Let ι be the inclusion $\mathbb{C}[M_F]^W \subset \mathbb{C}[\sigma_F^\vee]^W$ and π the projection of $\mathbb{C}[\sigma_F^\vee]^W$ onto $\mathbb{C}[M_F]^W$. This induces an endomorphism on $\mathbb{C}[M_F]^W$

$$\begin{tikzcd} \mathbb{C}[M_F]^W \arrow[rr] \arrow[d, hook, "\iota"'] & & \mathbb{C}[M_F]^W \\ \mathbb{C}[\sigma_F^\vee]^W \arrow[rr, "z^{-r}(L_c(z) - \lambda I_W)"] & & \mathbb{C}[\sigma_F^\vee]^W . \arrow[u, two heads, "\pi"'] \end{tikzcd} \tag{6.10}$$

given by the composition $\pi \circ (L_c(z) - \lambda I_W) \circ \iota$.

Lemma 6.5.6. Let G be a nonvertex apical face of Q such that $F = |W|G$ is a face of P, and let $r \in G \cap \mathbb{Z}^{d+1}$. The action of the endomorphism $\pi \circ (z^{-r}(L_c(z) - \lambda I_W)) \circ \iota$ on $\mathbb{C}[M_F]^W$ is given as multiplication by the matrix $z^{-r}(\text{in}_G L_c(z) - \lambda I_W)$.

Proof. In using the standard basis of $\mathbb{C}[M_F]^W$ (which is indexed by $u \in W$), the endomorphism $\pi \circ z^{-r}(L_c(z) - \lambda I_W) \circ \iota$ becomes multiplication by a $|W| \times |W|$ matrix whose rows and columns are indexed by vertices in W. Its (u, v)-position is given by

$$-\delta_{u,v} z^{-r}\lambda - \sum_{u \sim a+v} c_{u \sim a+v} z^{a-r} \tag{6.11}$$

where $a \in G \cap \mathbb{Z}^{d+1}$ and $\delta_{u,v}$ is the Kronecker symbol. This matrix is $z^{-r}(\mathrm{in}_G L_c(z) - \lambda I_W)$. $\square$

Remark 6.5.7. Let G be a nonvertex apical face of Q such that $F = |W|G$ is a face of P, and let $r \in G \cap \mathbb{Z}^{d+1}$. By Lemma 6.5.5, $\det(z^{-r}(\mathrm{in}_G L_c(z) - \lambda I_W))$ is nonzero. The support of the kernel of $z^{-r}(\mathrm{in}_G L_c(z) - \lambda I_W)$ is the set

$$\mathit{Var}(\det(z^{-r}(\mathrm{in}_G L_c(z) - \lambda I_W))).$$

If there exists $r' \in G$ with $r' \neq r$, then $z^{-r}(\mathrm{in}_G L_c(z) - \lambda I_W) = z^{r'-r} \cdot z^{-r'}(\mathrm{in}_G L_c(z) - \lambda I_W)$, where $z^{r'-r}$ is invertible in $\mathbb{C}[\sigma_F^\vee]$. Observe that the kernel of $z^{-r}(\mathrm{in}_G L_c(z) - \lambda I_W)$ is the kernel of $z^{-r'}(\mathrm{in}_G L_c(z) - \lambda I_W)$, so there is no loss in generality in choosing any $r \in G \cap \mathbb{Z}^{d+1}$. $\diamond$

Let G be a nonvertex apical face of Q such that $F = |W|G$ is a face of the polytope P. Let $r \in (G \cap \mathbb{Z}^{d+1}) \smallsetminus \{\mathbf{e}\}$ and let $\pi : \mathbb{Z}^{d+1} \to \mathbb{Z}^d$ be the projection onto the first d factors. By definition of the Floquet matrix $L_c(z)$, each edge of Γ has a weight in $\mathcal{A}(\Gamma)$. Remove each directed edge whose weight does not lie in G and add $-r$ to the weights in G. The weights in $-r + G$ correspond to a set of directed edges $\mathcal{E}_F$ for a graph $\Gamma_F := (\mathcal{V}, \mathcal{E}_F)$ called the *facial graph* of Γ with respect to r. The weights of the edges of this graph correspond to the operator $z^{-r}\mathrm{in}_G L_c(z)$.

Lemma 6.5.8. Let G be a nonvertex apical face of Q such that $F = |W|G$ is a face of P, and let $r \in (G \cap \mathbb{Z}^{d+1}) \smallsetminus \{\mathbf{e}\}$. Then the facial graph Γ_F with respect to $r \in G \cap \mathbb{Z}^{d+1}$ is $\mathbb{Z}^{\dim F - 1}$-periodic.

Proof. Since G is a nonvertex apical face of Q, so is F. Thus, the face $H := F \cap B$ of F and B is not empty. Since F is an apical face of the pyramid P and B is a facet of P, H is a facet of F.

Consider a splitting $\beta : M_F/M_H \to M_F$ of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_H \arrow[r, hook, "\iota"'] & M_F \arrow[r, two heads, "\pi"'] & M_F/M_H \arrow[r] \arrow[l, bend right, "\beta"'] & 0 \end{tikzcd}$$

where ι is the inclusion $M_H \subset M_F$ and π is the projection of M_F onto M_F/M_H. Then M_F is the internal direct sum $M_H \oplus \beta(M_F/M_H)$ and $\mathbb{C}[M_F] \simeq \mathbb{C}[M_H] \otimes_{\mathbb{C}} \mathbb{C}[\beta(M_F/M_H)]$. Since H is a face of B, the terms of $z^{-r}\text{in}_G L_c(z)$ lie in $\mathbb{C}[M_H]$, and since H is a face of F, M_H has rank $\dim H = \dim F - 1$. As the set $\mathcal{E}_F$ is obtained by removing edges of the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ whose corresponding weights are not in $\mathcal{A}(\Gamma)$, it follows that Γ_F is $\mathbb{Z}^{\dim F-1}$-periodic. □

Theorem 6.5.9. Suppose the Newton polytope P is full. Let G be a nonvertex apical face of Q such that $F = |W|G$ is a face of P, and let $r \in (G \cap \mathbb{Z}^{d+1}) \smallsetminus \{\mathbf{e}\}$. Then the facial graph Γ_F with respect to r is a $\mathbb{Z}^{\dim F-1}$-periodic, directed graph whose operator $z^{-r}\text{in}_G L_c(z)$ has Bloch variety $\overline{\mathcal{B}_{L_c}} \cap \mathcal{O}_F$.

Proof. It remains to show $\overline{\mathcal{B}_{L_c}} \cap \mathcal{O}_F = \mathit{Var}(\det(z^{-r}(\text{in}_G L_c(z) - \lambda I_W)))$. By Lemma 6.4.2,

$$\begin{aligned} \overline{\mathcal{B}_{L_c}} \cap \mathcal{O}_F &= \mathit{Var}(z^{-|W|r}\text{in}_F \det(L_c(z) - \lambda I_W)) \\ &= \mathit{Var}(z^{-|W|r} \det(\text{in}_G L_c(z) - \lambda I_W)) \\ &= \mathit{Var}(\det(z^{-r}(\text{in}_G L_c(z) - \lambda I_W))). \end{aligned}$$

where the second equality follows from Lemma 6.5.5. The theorem is proved. □

We end this chapter with an example of Theorem 6.5.9.

Example 6.5.10. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be the $\mathbb{Z}^2$-periodic graph depicted in the left side of Figure 6.2. It has two orbits of vertices and five orbits of edges. Let L be a periodic graph operator on Γ with u and v as values of the potential at the vertices u and v. Let a, b, c, d and e be the edge weights.

Let $\mathcal{A} \subset \mathbb{Z}^3$ be the support of the dispersion polynomial D. The Newton polytope $P = \text{conv}(\mathcal{A})$ is depicted in the right side of Figure 6.2.

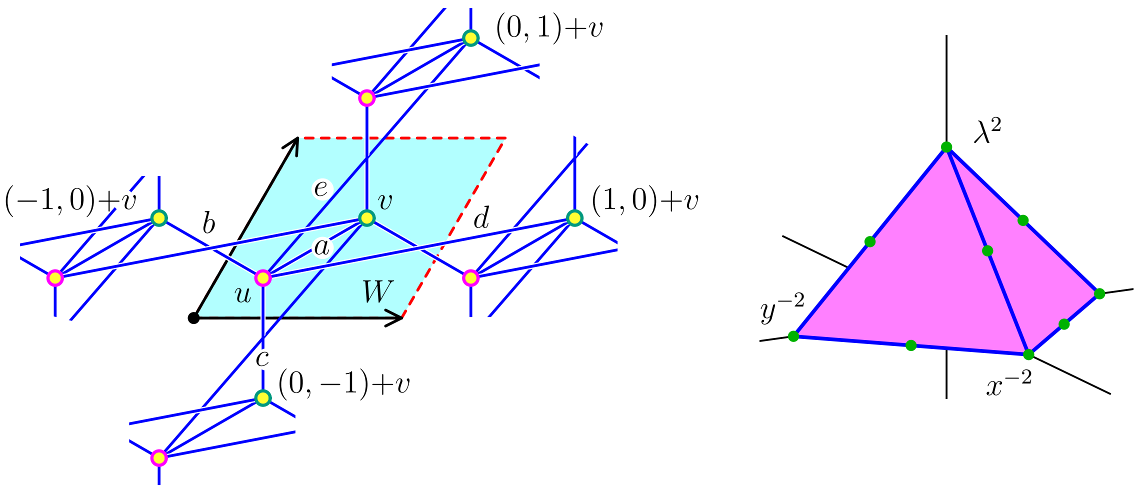

Figure 6.2: A periodic graph and its corresponding Newton polytope, which is full.

Let B be the base of the polytope P. The operator L is an endomorphism on the free $\mathbb{C}[\sigma_B^\vee]$-module

$$\mathcal{Q}_B^W \simeq \Big\{\psi_B \in \mathbb{C}[x^\pm, y^\pm, \lambda]^\mathcal{V} \mid \psi_B(w,(r,s,t)) = x^r y^s \psi_B(w,(0,0,0))\Big\}.$$

Its characteristic matrix is

$$L(x,y) - \lambda I_W = \begin{pmatrix} u{-}\lambda & -a{-}bx^{-1}{-}cy^{-1}{-}dx{-}ey \\ -a{-}bx{-}cy{-}dx^{-1}{-}ey^{-1} & v{-}\lambda \end{pmatrix},$$

and its dispersion polynomial $D = \det(L(x,y) - \lambda I_2)$ is

$$\lambda^2 - (u+v)\lambda + uv - (a^2+b^2+c^2+d^2+e^2) - (ab+ad)(x+x^{-1}) - (ac+ae)(y+y^{-1}) -$$
$$(bc+de)(xy^{-1}+x^{-1}y) - (be+cd)(xy+(xy)^{-1}) - bd(x^2+x^{-2}) - ce(y^2+y^{-2}).$$

Let F be the face of the polytope P exposed by the vector $(1,1,-1)$, and let G be the corresponding face of F in $Q = \operatorname{conv}(\mathcal{A}(\Gamma) \cup \{(0,0,1)\})$. By choosing $(-1,0,0) \in G$, the endomorphism

$L(x,y) - \lambda I_W$ extends to act on $\mathbb{C}[\sigma_F^\vee]^W$ as multiplication by the matrix

$$xL(x,y) - x\lambda I_2 = \begin{pmatrix} ux - x\lambda & -ax - b - cxy^{-1} - dx^2 - exy \\ -ax - bx^2 - cxy - d - exy^{-1} & vx - x\lambda \end{pmatrix}.$$

On $\mathbb{C}[M_F]^W$, the action of this endomorphism is multiplication by the matrix

$$x\mathrm{in}_G(L(x,y) - \lambda I_2) = \begin{pmatrix} -x\lambda & -b - cxy^{-1} \\ -d - exy^{-1} & -x\lambda \end{pmatrix}.$$

By Lemma 6.4.2, $x^2 D_F = (x\lambda)^2 + (be+cd)xy^{-1} - bd + ce(xy^{-1})^2 = \det(x\mathrm{in}_G(L_F(x,y) - \lambda I_2))$.

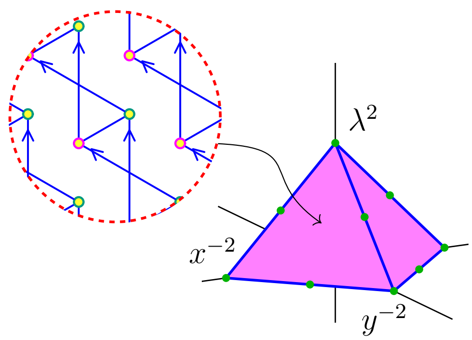

Figure 6.3: A facial graph associated to the facet of a full Newton polytope.

Figure 6.3 shows the corresponding $\mathbb{Z}$-periodic graph Γ_F. $\diamond$

7. SUMMARY

We investigated the relationship bewteen the Newton polytope of the dispersion polynomial of a periodic graph operator and its complex Bloch variety. In Chapter 5 we used this Newton polytope to give criteria when the irreducibility of this dispersion polynomial is preserved after a change in the period lattice. Chapter 5 is based on published work with Matthew Faust in [1]. In Chapter 6 we compactified this Bloch variety in the normal toric variety associated to the fan of this Newton polytope. We realized this compactification of the Bloch variety as coming from an operator that extends to the boundary of this toric variety. Chapter 6 is based on an upcoming article with Matthew Faust, Stephen Shipman, and Frank Sottile in [2].

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