T2 - RC Circuit Analysis

Integrated Master in Physics Engineering

João Lehodey (96538), Jorge Silva (96545), Pedro Monteiro (93156)

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Contents

1 Introduction

The objective of this laboratory assignment is to study a RC circuit. The circuit can be seen in Figure 1.

In Section 2, a theoretical analysis of the circuit is presented. In Section 3, the circuit is analysed by simulation, and the results are compared to the theoretical results obtained in Section 2. The conclusions of this study are outlined in Section 4.

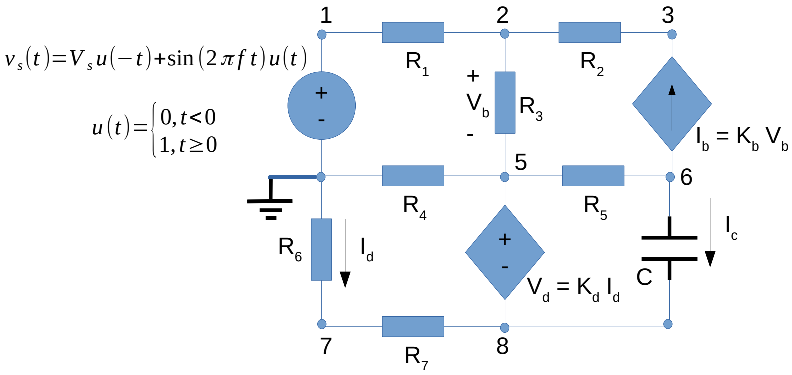

Figure 1: Circuit to be studied

2 Theoretical Analysis

In this section, the circuit shown in Figure 1 is analysed.

2.1 Values To Be Used

These are the values generated using datagen python program and will be used for the following calculations and simulations

R1	1013.59377072000Ohm
R2	2048.80282079000Ohm
R3	3012.28190136000Ohm
R5	3010.85004116000Ohm
R6	2001.10363671000Ohm
R7	1014.19251479000Ohm
Vs	5.02942659139V
C	0.00000101555F
Kb	7.26021477749
Kd	8.01430228621

Table 1: Initial Conditions

2.2 Step 1: Stationary analysis

For $t < 0$ we have that $v_s = V_s = C^{te}$, and also $i_c = \frac{d}{dt}u_c = 0$ (the capacitor acts as an open circuit), therefore the circuit is equivalent to the one shown in figure (2).

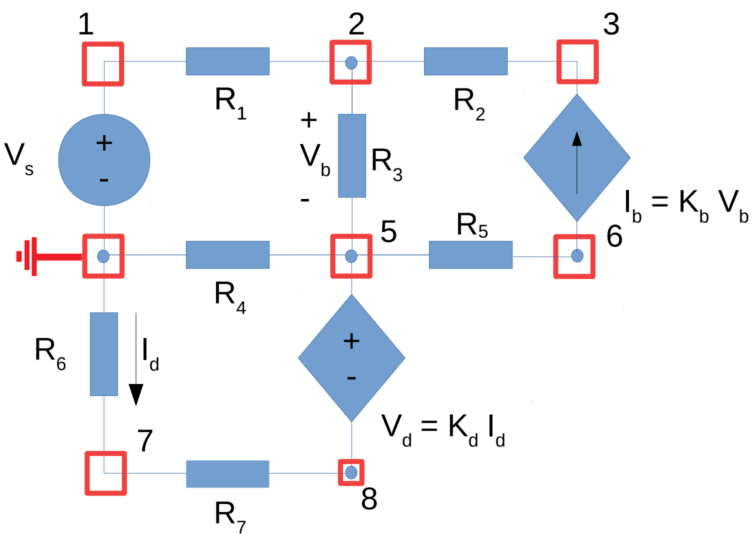

Figure 2: Circuit used to determine all the branch variables for $t < 0$.

Using nodal analysis we get the following system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & -C_1 - C_2 - C_3 & C_2 & C_3 & 0 & 0 & 0 \\ 0 & C_2 + K_b & -C_2 & -K_b & 0 & 0 & 0 \\ 0 & -K_b & 0 & C_5 + K_b & -C_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -C_6 - C_7 & C_7 \\ 0 & 0 & 0 & 1 & 0 & K_d C_6 & -1 \\ 0 & C_3 & 0 & -C_3 - C_4 - C_5 & C_5 & C_7 & -C_7 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_5 \\ e_6 \\ e_8 \\ e_7 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the circuit we reach the following results:

V(1)	5.02942659139V
V(2)	3.17059656393V
V(3)	-0.58687557146V
V(5)	3.17084917137V
V(6)	8.69270077110V
V(7)	2.10994450352V
V(8)	3.17929937492V

Table 2: Results of Nodal Analysis using Octave

2.3 Step 2: Determining the equivalent resistance.

To get the equation for the capacitor voltage, v_c, we need to determine the time constant $\tau = R_{eq}C$, where C is the capacitance and R_{eq} is the resistance measured from the terminals of the capacitor, v_6, v_8, however due to the circuit geometry and the presence of dependent sources we can't simplify the circuit directly. To calculate R_{eq}, we cancel out the internal sources in the circuit (V_s), and apply a known voltage, V_x, in the terminals v_6, v_8, then we simply calculate the ratio between the voltage from the source and the current that passes through it, I_x. To calculate I_x we, again, perform the nodal analysis, but on the modified circuit in figure (3). Note that $V_x = v_6 - v_8$, where v_6 and v_8 were the nodes voltages obtained in step 1.

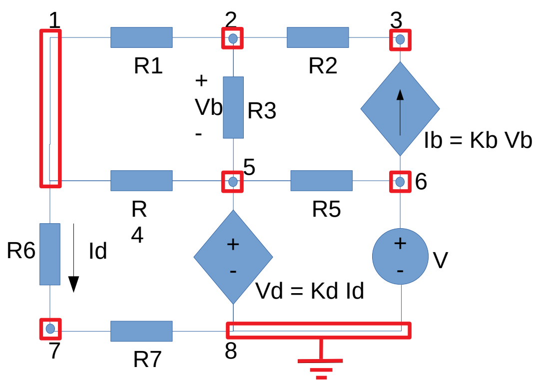

Figure 3: Circuit configuration for measuring the equivalent resistance, R_{eq}, from terminals 6 and 8.

Using nodal analysis we get the following system of equations:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ C_6K_d & 0 & 0 & -1 & 0 & -C_6K_d \\ -C_1-C_4-C_6 & C_1 & 0 & C_4 & 0 & C_6 \\ 0 & C_2+K_b & -C_2 & -K_b & 0 & 0 \\ C_1 & -C_1-C_2-C_3 & C2 & C3 & 0 & 0 \\ C_6 & 0 & 0 & 0 & 0 & -C_6-C_7 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = \begin{bmatrix} V_x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The current I_x is then given by the equation:

$$I_x = C_5(e_6 - e_5) \tag{1}$$

The results are the folowing:

V(1)	0.00000000000V
V(2)	0.00000000000V
V(3)	0.00000000000V
V(5)	-0.00000000000V
V(6)	5.51340139618V
V(7)	-0.00000000000V

Table 3: Results of Nodal Analysis using Octave

Req	3010.85004116000 Ohm

Table 4: Equivalent resistance

2.4 Step 3: Natural solution

The natural solution for a capacitor is given by the following equation:

$$v_{c_n}(t) = Ve^{-\frac{1}{RC}t} \tag{2}$$

where V is a constant and R is the resistance measured from the terminals of the capacitor. To determine V, we note that the voltage of a capacitor must always be continuous (or else there would be an infinite power in the capacitor in the instance of the discontinuity), therefore if $v_{6_n}(0^-) = v_{6n}(0^+)$ and $v_{6_n}(0^-) = V_x$, then $V = V_x$. Therefore the natural solution for the capacitor is given by the equation (**??**).

$$v_{6_n}(t) = V_x e^{-\frac{1}{RC}t}, \tag{3}$$

for $t \geq 0$.

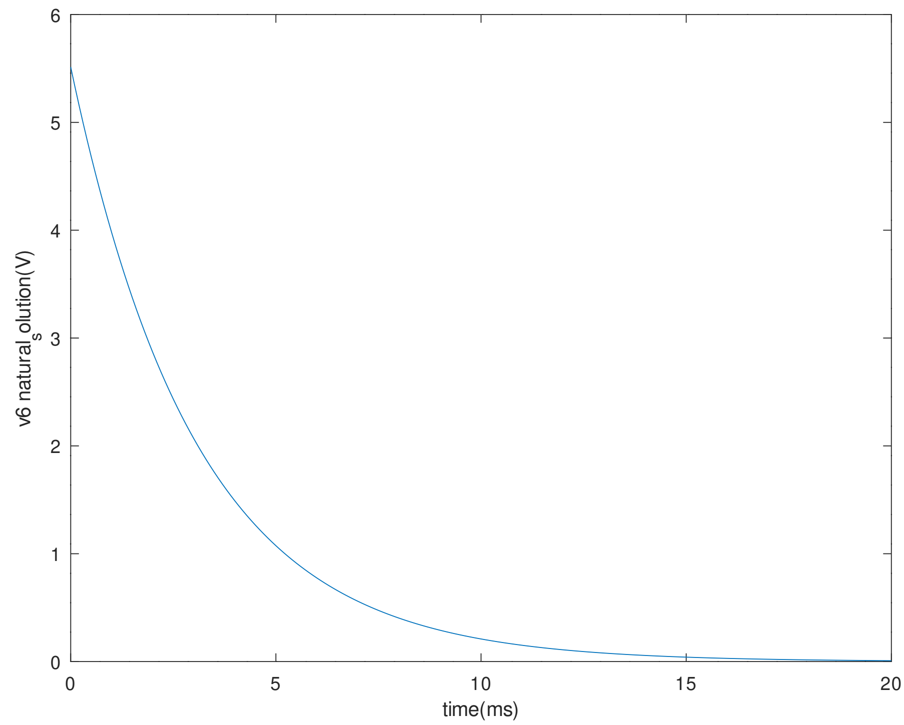

Figure 4: Ntural solution plot

2.5 Step 4: Forced Solution

A phaser is a complex number that carries information about the phase and amplitude of a sinusoidal voltage or current. This mathematical construct is particularly useful for studying the steady state solutions of a circuit, in which every current and voltage oscillates with the same known frequency. For example, our voltage source $V_s(t) = Acos(wt - \phi) = Re\{Ae^{j(wt-\phi)}\}$. Because the frequency is the same for every voltage and current in the circuit in steady state, we can ignore w. Its phaser representation therefore is $\tilde{V_s} = Ae^{-j\phi}$.

Using phasers, we applied the node method to determine the voltage of each node in steady-state. We defined the node connected to R_6, R_4 and the independent voltage source as ground. We obtained the following equations:

$$\tilde{V_1} = e^{j\frac{\pi}{2}}$$

$$\frac{\tilde{V_1} - \tilde{V_2}}{R_1} + \frac{\tilde{V_5} - \tilde{V_2}}{R_3} + \frac{\tilde{V_3} - \tilde{V_2}}{R_2} = 0$$

$$\frac{\tilde{V_2} - \tilde{V_3}}{R_2} + K_b(\tilde{V_2} - \tilde{V_5}) = 0$$

$$\frac{-\tilde{V_7}}{R_6} + \frac{\tilde{V_8} - \tilde{V_7}}{R_7} = 0$$

$$\frac{\tilde{V_8} - \tilde{V_6}}{Z_c} + \frac{\tilde{V_5} - \tilde{V_6}}{R_5} - K_b(\tilde{V_2} - \tilde{V_5}) = 0$$

$$\frac{\tilde{V_2} - \tilde{V_5}}{R_3} - \frac{\tilde{V_5}}{R_4} + \frac{\tilde{V_6} - \tilde{V_5}}{R_5} + \frac{\tilde{V_7} - \tilde{V_8}}{R_7} + \frac{\tilde{V_6} - \tilde{V_8}}{Z_c} = 0$$

$$\tilde{V_8} + K_d(-\frac{\tilde{V_7}}{R_6}) - \tilde{V_5} = 0$$

In matrix form, the system looks like this:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & -(C_1+C_2+C_3) & C_2 & C_3 & 0 & 0 & 0 \\ 0 & C_2+K_b & -C_2 & -K_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -C_6-C_7 & C_7 \\ 0 & -K_b & 0 & C_5+K_b & -A_c-C_5 & 0 & A_c \\ 0 & C_3 & 0 & -C_3-C_4-C_5 & C_5+A_c & C_7 & -C_7-A_c \\ 0 & 0 & 0 & 1 & 0 & K_d*C_6 & -1 \end{bmatrix} \begin{bmatrix} \tilde{V_1} \\ \tilde{V_2} \\ \tilde{V_3} \\ \tilde{V_5} \\ \tilde{V_6} \\ \tilde{V_7} \\ \tilde{V_8} \end{bmatrix} = \begin{bmatrix} e^{j\frac{\pi}{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the system, we get yhe following results:

A(1)	0+1i
A(2)	-7.0078e-16+0.63041i
A(3)	-1.1776e-15-0.11669i
A(5)	0.056906+0.6351i
A(6)	-4.6629e-16+0.41952i
A(7)	-7.0262e-16+0.63214i

Table 5: Complex Amplitude with GND connected to Vs as in inicial diagram

A(1)	7.0262e-16+0.36786i
A(2)	1.8354e-18-0.0017304i
A(3)	-4.7501e-16-0.74883i
A(5)	0.056906+0.002962i
A(6)	2.3633e-16-0.21262i
A(7)	0

Table 6: Complex Amplitude with GND as node 8

2.6 Step 5: Total solution

To convert a phaser to a real time function, we simply use the "equivalence":

$$Acos(wt-\phi) \leftrightarrow Ae^{-j\phi}$$

It is known that

$$V_{(node)}(t) = V_{forced(node)}(t) + V_{natural(node)}(t)$$

Before performing this simple computation, we must note that, for determining the natural solution, we defined node 8 as being ground; whereas, for determining the forced solution, we defined the node connected to R_6, R_4 and the independent voltage source as ground. To solve this inconsistency, we simply subtract from every node phaser (as calculated in the previous section) the value of $\tilde{V_8}$, so as to make node 8 the ground node on both situations. Having done this, we get:

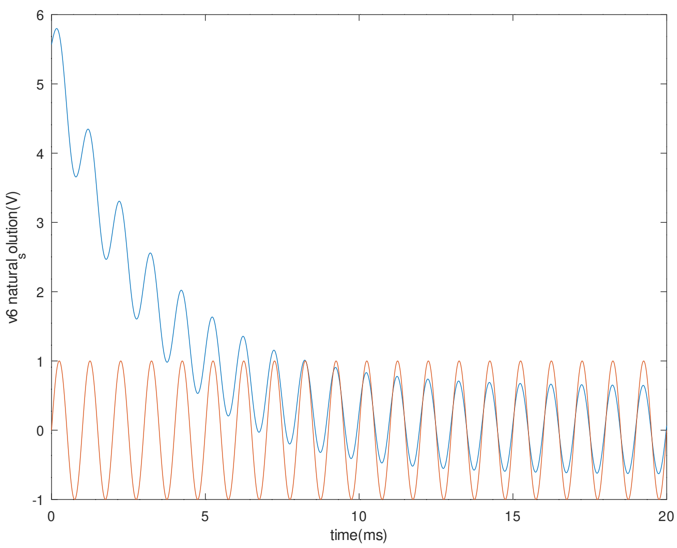

Figure 5: Total solution

2.7 Step 6: Frequency response

To determine the frequency response of the circuit we have to simplify it, this can be achieved by determining the Thevening equivalent circuit from the terminals v_6 and v_8, as shown in figure (**??**).

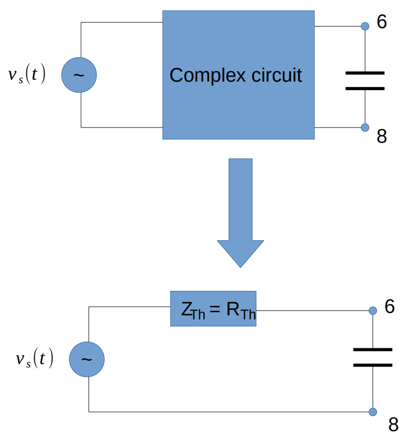

Figure 6: Determining the transfer function, $\bar{T}(w)$, by resorting to circuit simplification, using the Thevenin equivalent(seen from the terminals of the capacitor).

The equivalent impedance Z_{Th} has already been determined, its value is equal to the equivalent resistance R_{eq}, viewed from the terminals 6 and 8. To determine the value for U_{Th} we use the same system as obtained in step 1 of the theoretical analysis, but with v_s, set as a variable instead of a constant. Solving the the system, we get the voltages v_6 and v_8 as a function of v_s, and as expected, the voltage drop $v_6 - v_8$ is proportional to v_s.
With the circuit simplified (see figure (**??**)), we can relate the source phasor $\bar{U}_s$ with output phasor $\bar{U}_c$, with the folowing equations:

$$\bar{I}_c = \frac{\alpha \bar{U}_s}{R_{Th} + \frac{1}{jwC}} \iff \bar{U}_c = \frac{\frac{1}{jwC}}{R_{Th} + \frac{1}{jwC}} \alpha \bar{U}_s, \tag{4}$$

where $\alpha \bar{U}_s = \bar{U}_{Th}$. Therefore we get:

$$\bar{T}(w) = \frac{\bar{U}_s}{\bar{U}_c} = \frac{\alpha}{j\tau\omega + 1} \tag{5}$$

, where $\tau = R_{Th}C$.

3 Simulation Analysis

3.1 Operating Point Analysis

3.1.1 First task

Table 7 shows ——————————

Name	Value [A or V]
@c[i]	0.000000e+00
@gb[i]	-1.83398e-03
@r1[i]	1.833900e-03
@r2[i]	1.833984e-03
@r3[i]	-8.38592e-08
@r4[i]	-7.79510e-04
@r5[i]	-1.83398e-03
@r6[i]	-1.05439e-03
@r7[i]	-1.05439e-03
v(1)	5.029427e+00
v(2)	3.170597e+00
v(3)	-5.86876e-01
v(5)	3.170849e+00
v(6)	8.692701e+00
v(7)	2.109945e+00
v(8)	3.179299e+00
na	0.000000e+00

Table 7: Operating point. Variables v(i) are of type *voltage* and expressed in Volt; other variables are of type *current* and expressed in Ampere

The results are the same as the ones obtained using Octave. The subject will be further developed in Section 4.

3.1.2 Second task

Table 8 ———————————————

Name	Value [A or V]
@gb[i]	0.000000e+00
@r1[i]	0.000000e+00
@r2[i]	0.000000e+00
@r3[i]	0.000000e+00
@r4[i]	0.000000e+00
@r5[i]	-1.83118e-03
@r6[i]	0.000000e+00
@r7[i]	0.000000e+00
v(1)	0.000000e+00
v(2)	0.000000e+00
v(3)	0.000000e+00
v(5)	0.000000e+00
v(6)	5.513401e+00
v(7)	0.000000e+00
v(8)	0.000000e+00
na	0.000000e+00

Table 8: Operating point. Variables v(i) are of type *voltage* and expressed in Volt; other variables are of type *current* and expressed in Ampere

3.1.3 Third task

Figure 9 ———————————————

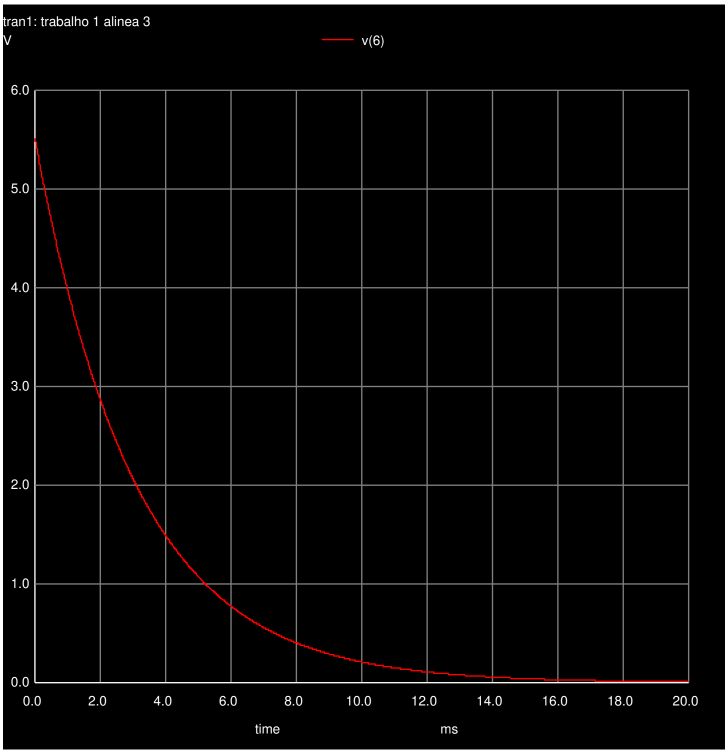

Figure 7: Transient output voltage of node 6

3.1.4 Fourth task

Figure **??**

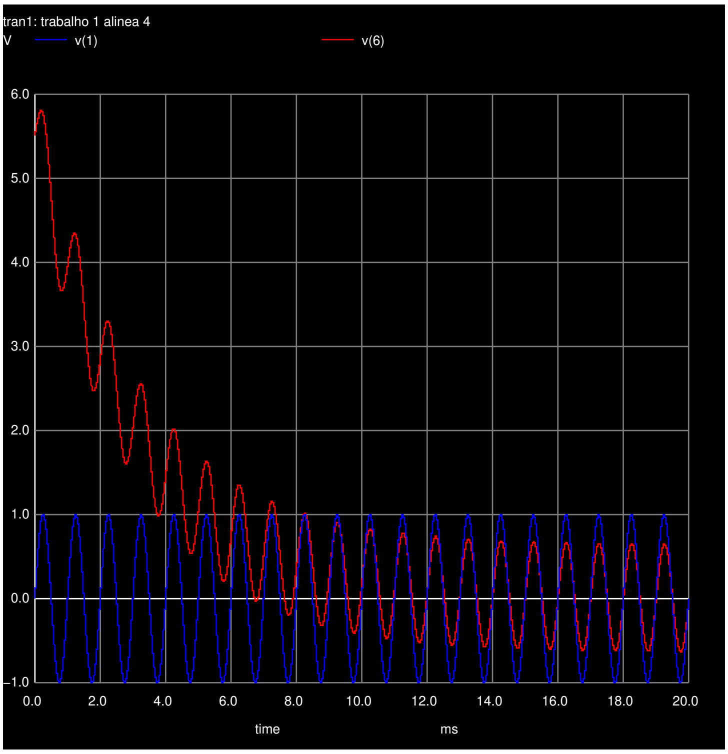

Figure 8: ——————-

3.1.5 Fifth task

Figure **??** ———————————

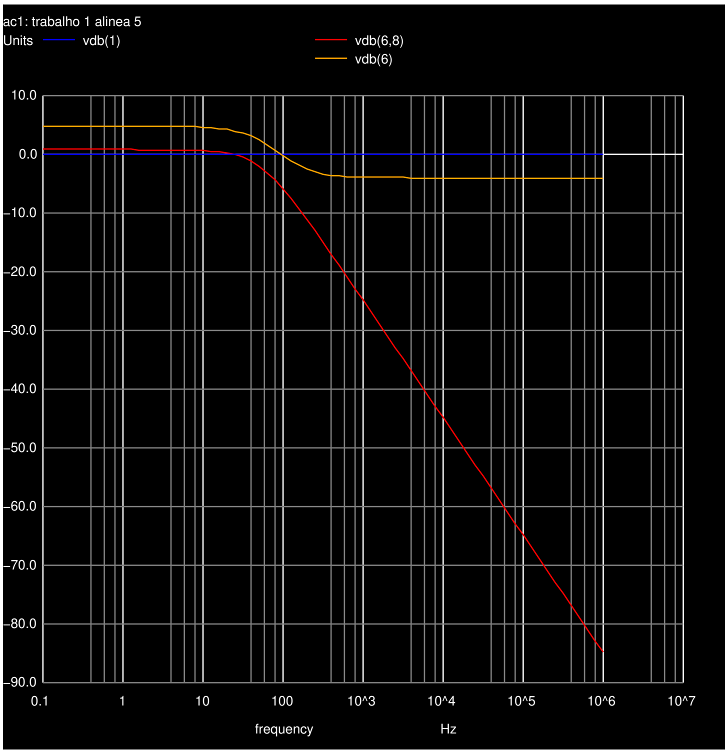

Figure 9: ——————-

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4 Conclusion

In this laboratory assignment the objective of analysing a simple circuit has been achieved. The analysis of the circuit was done both theoretically, using the Ocatve maths tool to solve the systems of equations obtained by using the mesh and node analysis, and by circuit simulation, using Ngspice. The simulation results perfectly matched the theoretical results. For example, if we look at the results tables for ngspice and octave we may note that the following relations hold:

$$y_1 = -r1[i]$$

$$y_2 = r2[i]$$

$$y_3 = r6[i] = r7[i]$$

$$C_i = 1/r[i]$$

which is to be expected. The reason for this perfect match is that the analysed circuit is linear, so that any method for analysing consists in solving a linear system of equations.