Homework 6

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(Dated: April 29, 2020)

Course: GPHY 560

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Problem 2.

A black-and-white image of composer Duke Ellington was used. A SVD decomposition routine from the Numpy library in Python was used. Figure 1 shows the resulting images. The image seems to be very well represented within the first 10 singular values.

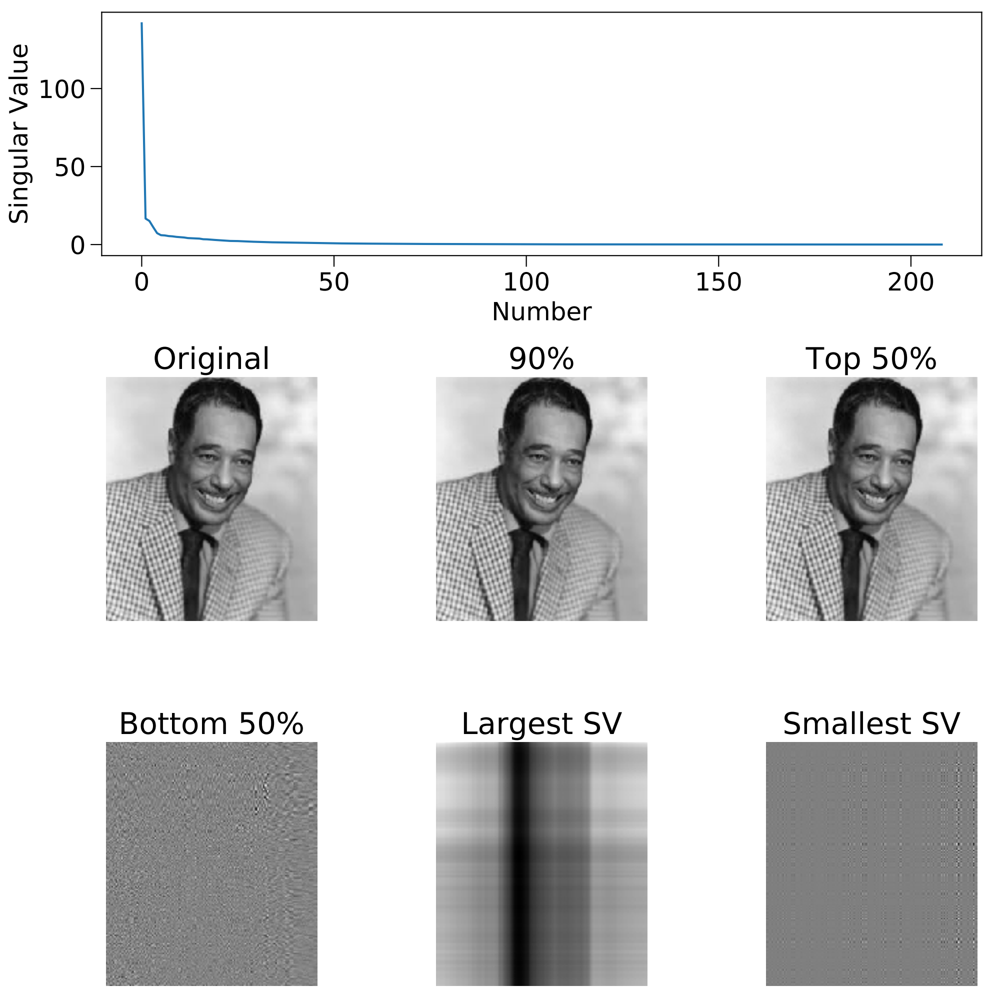

FIG. 1. Results of Problem 2.

Problem 3.

Part A.

We can linearize the equation $d = 1/x$ by substituting for the variable x:

$$d = x'$$

with

$$x' = \frac{1}{x} \tag{0.1}$$

The equation used to fit is then:

$$Gm = d \implies \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} x' \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

A LSF fit for x' determines $x' = 2.3333$, which results in $x = 0.4286$.

Part B.

Expanding the equation in a Taylor series uses the Jacobian matrix instead of the data matrix to fit for the model. The Jacobian for this problem is:

$$J_i = \begin{bmatrix} -1/x_i^2 \\ -1/x_i^2 \\ -1/x_i^2 \end{bmatrix}$$

And x is found iteratively as:

$$x_{i+1} = x_i + (J_i^T J_i)^{-1} J_i^T (d - 1/x_i)$$

with the fit converging to $x = 0.4286$ after 10 iterations.

Part C.

Both models converged to the same value of x, but a major difference in the variance. The linearized fit resulted in a variance of $\sigma_x^2 = 2.1111$ whereas the Taylor approximation resulted in a variance of $\sigma_x^2 = 0.4286$.

Problem 4.

Part A.

The Gaussian form can be linearized as:

$$\ln d = -\frac{1}{2s^2} x^2 + \ln A$$

so it is linear with respect to x^2. The equations used then are:

$$Gm = d \implies \begin{bmatrix} 36 & 1 \\ 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1/2s^2 \\ \ln A \end{bmatrix} = \begin{bmatrix} \ln 1.6 \\ \ln 3 \\ \ln 3 \end{bmatrix}$$

which through LSF results in the parameters $-1/2s^2 = -0.1964$ and $\ln A = 1.772$. It is easy to find then that $s^2 = 25.4530$ and $A = 3.2452$.

Part B.

The data variance can be found as:

$$\begin{aligned}\sigma^2 &= (d - Gm)^T(d - Gm)/(3-2)\\ &= 7.704 \times 10^{-32}\end{aligned}$$

It is important to note this is the data variance of $\ln d$, and not of d.

Part C.

The covariance matrix can be found as:

$$\begin{aligned}\mathrm{cov}(m) &= (G^TG)^{-1}G^TG(G^TG)^{-1}\sigma^2\\ &= \begin{bmatrix} 1.128 \times 10^{-34} & -1.655 \times 10^{-33} \\ -1.655 \times 10^{-33} & 4.995 \times 10^{-32}\end{bmatrix}\end{aligned}$$

It is important to note this is the covariance of the fitted parameters $-1/2s^2$ and $\ln A$, not the parameters s^2 and A of the Gaussian distribution.

Problem 5.

Part A.

We again fit for the Gaussian model using the Jacobian method instead. The Jacobian matrix for this model is:

$$J_i = \begin{bmatrix} -(-6/s_i^2)A_i\exp(6^2/2s_i^2) & \exp(6^2/2s_i^2) \\ -(-2/s_i^2)A_i\exp(2^2/2s_i^2) & \exp(2^2/2s_i^2) \\ -(6/s_i^2)A_i\exp(-6^2/2s_i^2) & \exp(-6^2/2s_i^2)\end{bmatrix}$$

and the parameters are found directly:

$$m = \begin{bmatrix} s^2 \\ A \end{bmatrix}$$

by iterating $m_{i+1} = m_i + (J_i^TJ_i)^{-1}J_i^T(d - g(m))$, converging to a solution. The resulting parameters of the Gaussian fit are $s^2 = 25.4529$ and $A = 3.2452$.

Part B.

The data variance can be found as:

$$\begin{aligned}\sigma^2 &= (d - g(m))^T(d - g(m))/(3-2)\\ &= 2.410 \times 10^{-11}\end{aligned}$$

Part C.

The covariance matrix can be found as:

$$\begin{aligned}\mathrm{cov}(m) &= (J^T J)^{-1}\sigma^2 \\ &= \begin{bmatrix} 1.023 \times 10^{-10} & -9.742 \times 10^{-12} \\ -9.742 \times 10^{-12} & 1.328 \times 10^{-11} \end{bmatrix}\end{aligned}$$

Problem 6.

The Jacobian method for LSF was used to locate the earthquake. The travel time can be found as:

$$t = t_0 + \sqrt{(x - x_0)^2 + (y - y_0)^2}/v$$

The parameters to be found are:

$$m = \begin{bmatrix} x_0 \\ y_0 \\ t_0 \end{bmatrix}$$

The Jacobian of this model is then:

$$J_i = \begin{bmatrix} \frac{-(0-x_0)}{dv} & \frac{-(0-y_0)}{dv} & 1 \\ \frac{-(10-x_0)}{dv} & \frac{-(10-y_0)}{dv} & 1 \\ \frac{-(10-x_0)}{dv} & \frac{-(0-y_0)}{dv} & 1 \\ \frac{-(0-x_0)}{dv} & \frac{-(10-y_0)}{dv} & 1 \end{bmatrix}$$

The resulting fit for the earthquake's origin time and coordinates from the iterative process is:

$$\begin{aligned} x_0 &= 7.9 \text{ km} \\ y_0 &= 6.31 \text{ km} \\ t_0 &= 1.28 \text{ s} \end{aligned}$$