Homework 5

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Problem 1.

Part A.

This problem can be written as:

$$Gm = d + e \implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

Part B.

The only overdetermined parameter is m_3. The underdetermined parameters are m_1, and m_2.

Part C.

To determine m_3, we use the following equation:

$$Gm = d \implies \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} m_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

with the solution used being $m = (G^T G)^{-1} G^T d$ to minimize the squared error. The solution found was $m_3 = 1.5$

Part D.

To determine m_1 and m_2, we use the following equation:

$$Gm = d \implies \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

with the solution used being $m = G^T (GG^T)^{-1} d$ to minimize the model vector length. The solution found was $m_1 = m_2 = 0.5$.

Part E.

Knowing that $m = \begin{bmatrix} 0.5 & 0.5 & 1.5 \end{bmatrix}$, we can find the generalized inverse G^g as:

$$md^T (dd^T)^{-1} = G^{-g}$$

Since for this particular example, $\det(dd^T) = 0$ and as such is not invertable, I add a small dampening factor so that it can be inverted:

$$md^T(dd^T + \epsilon I)^{-1} = G^{-g}$$

The resultant generalized inverse is:

$$G^{-g} = \begin{bmatrix} 0.083 & 0.167 & 0.083 \\ 0.083 & 0.167 & 0.083 \\ 0.250 & 0.5 & 0.250 \end{bmatrix}$$

Part F.

The data variance was found as:

$$\sigma^2 = (Gm - d)^T(Gm - d)/1 = 0.5$$

Part G.

The covariance of the parameters is found as:

$$\mathrm{cov}(m) = G^{-g}\mathrm{d}G^{-gT} = \sigma^2 G^{-g}G^{-gT}$$

As discussed in Menke Ch. 7. The result is:

$$\mathrm{cov}(m) = \begin{bmatrix} 0.021 & 0.021 & 0.062 \\ 0.021 & 0.021 & 0.062 \\ 0.062 & 0.062 & 0.187 \end{bmatrix}$$

Part H.

The data resolution matrix is:

$$N = GG^{-g} = \begin{bmatrix} 0.167 & 0.333 & 0.167 \\ 0.250 & 0.500 & 0.250 \\ 0.250 & 0.500 & 0.250 \end{bmatrix}$$

Part I.

The model resolution matrix is:

$$R = G^{-g}G = \begin{bmatrix} 0.083 & 0.083 & 0.250 \\ 0.083 & 0.083 & 0.250 \\ 0.250 & 0.250 & 0.750 \end{bmatrix}$$

Part J.

We can find the null model vectors using:

$$Gm = 0 \implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{n1} \\ m_{n2} \\ m_{n3} \end{bmatrix} = 0$$

Which results in the following the set of equations:

$$m_1 + m_2 = 0$$
$$m_3 = 0$$

So the possible null vectors are:

$$m_{null} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Part K.

We can find the null model vectors as:

$$gm = 0 \implies \begin{bmatrix} g_{n1} & g_{n2} & g_{n3} \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 1.5 \end{bmatrix} = 0$$

Which results in the equation:

$$0.5g_{n1} + 0.5g_{n2} + 1.5g_{n3}$$

I find the null vectors by setting one variable to 0 and solving for the other two, and by finding solutions such that two of the variables add to the other one. The resulting null vectors are:

$$g_{null} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & -1 \end{bmatrix}, \begin{bmatrix} -1.5 & -1.5 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 4 & -1 \end{bmatrix}$$

Problem 2.

We now consider a new weighting matrix C, which is a diagonal matrix of the predetermined scaling factors. So our equation is now:

$$GC^{-1}Cm = d = G'm' \implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

We find the pseudo-inverse G'^{-g} to get $m' = G'^{-g}d$, from which we easily obtain m:

$$m' = Cm = \begin{bmatrix} 0.8 \\ 0.4 \\ 4.5 \end{bmatrix}$$
$$m = \begin{bmatrix} 0.8 \\ 0.2 \\ 1.5 \end{bmatrix}$$

Problem 3.

Part A.

For the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

we can find the eigenvalues as:

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$
$$\therefore \lambda = -1, 3$$

The resulting eigenvectors are then (after being normalized):

$$\lambda = -1 :$$
$$(A - \lambda I)v = \begin{bmatrix} 1+1 & 2 \\ 2 & 1+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda = 3 :$$
$$(A - \lambda I)v = \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} -v_1 + v_2 \\ v_1 - v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Part B.

Writing the matrix in the eigenvector decomposition:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = P\Lambda P^T = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Part C.

Code that calculates the inverse for both the matrix A and its eigenvector decomposition is included. They are found to be the same.

$$M^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} = (P\Lambda P^T)^{-1}$$

Problem 4.

Part A.

For the matrix:

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

we can find the eigenvalues as:

$$\det\Big(\begin{bmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix} \Big) = (4-\lambda)(1-\lambda) - 4 = 4 - 5\lambda + \lambda^2 - 4 = \lambda(\lambda - 5)$$
$$\therefore \lambda = 0, 5$$

The resulting eigenvectors are then (after being normalized):

$$\lambda = 0 :$$
$$(A - \lambda I)v = \begin{bmatrix} 4+0 & -2 \\ -2 & 1+0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4v_1 - 2v_2 \\ -2v_1 + v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$v = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\lambda = 5 :$$
$$(A - \lambda I)v = \begin{bmatrix} 1-5 & -2 \\ -2 & 1-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} -4v_1 - 2v_2 \\ -2v_1 - 4v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$v = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Part B.

For the matrix:

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

we can find the eigenvalues as:

$$\det\Big(\begin{bmatrix} 0-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \Big) = -\lambda \det\Big(\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \Big) - \det\Big(\begin{bmatrix} 1 & 0 \\ 0 & 3-\lambda \end{bmatrix} \Big) = -\lambda^3 + 5\lambda^2 - 5\lambda - 3 = 0$$
$$\therefore \lambda = 1 - \sqrt{2}, 1 + \sqrt{2}, 3$$

The resulting eigenvectors are then (without normalizing):

$$\lambda = 1 - \sqrt{2}:$$

$$(B - \lambda I)v = B = \begin{bmatrix} 0 - 1 + \sqrt{2} & 1 & 0 \\ 1 & 2 - 1 + \sqrt{2} & 0 \\ 0 & 0 & 3 - 1 + \sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (-1 + \sqrt{2})v_1 + v_2 \\ v_1 + (1 + \sqrt{2})v_2 \\ (2 + \sqrt{2})v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} -1 - \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1 + \sqrt{2}:$$

$$(B - \lambda I)v = B = \begin{bmatrix} 0 - 1 - \sqrt{2} & 1 & 0 \\ 1 & 2 - 1 - \sqrt{2} & 0 \\ 0 & 0 & 3 - 1 - \sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} (-1 - \sqrt{2})v_1 + v_2 \\ v_1 + (1 - \sqrt{2})v_2 \\ (2 - \sqrt{2})v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} \sqrt{2} - 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 3:$$

$$(B - \lambda I)v = B = \begin{bmatrix} 0 - 3 & 1 & 0 \\ 1 & 2 - 3 & 0 \\ 0 & 0 & 3 - 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -3v_1 + v_2 \\ v_1 - v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Part C.

For the matrix:

$$C = \begin{bmatrix} -2 & 6 \\ 6 & -2 \end{bmatrix}$$

we can find the eigenvalues as:

$$\det\left(\begin{bmatrix} -2 - \lambda & 6 \\ 6 & -2 - \lambda \end{bmatrix} \right) = (-2 - \lambda)^2 - 36 = 4 - 4\lambda + \lambda^2 - 36 = \lambda^2 - 4\lambda - 32 = (\lambda + 9)(\lambda - 4)$$

$$\therefore \lambda = -8, 4$$

The resulting eigenvectors are then (after being normalized):

$$\lambda = -8:$$

$$(C - \lambda I)v = \begin{bmatrix} -2 + 8 & 6 \\ 6 & -2 + 8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 6v_1 + 6v_2 \\ 6v_1 + 6v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 4:$$

$$(C - \lambda I)v = \begin{bmatrix} -2-4 & 6 \\ 6 & -2-4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} -6v_1 + 6v_2 \\ 6v_1 - 6v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 5.

Part A.

This is an underdetermined problem. So from the given SVD matrices, we must select U_p, S_p and V_p (choose $p = 3$ columns since there are 3 non-zero singular values). So the generalized inverse can be found as:

$$G^{-g} = V_p S_p^{-1} U_p^T = \begin{bmatrix} 0.65 & -0.5 & 0.27 \\ 0.27 & -0.5 & = 0.65 \\ 0.65 & 0.5 & 0.27 \\ 0.27 & 0.5 & -0.65 \end{bmatrix} \begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 1.4 & 0 \\ 0 & 0 & 0.77 \end{bmatrix} \begin{bmatrix} 0.5 & -0.71 & -0.5 \\ 0.5 & 0.71 & -0.5 \\ 0.71 & 0 & 0.71 \end{bmatrix}^T = \begin{bmatrix} 0.259 & -0.248 & 0.505 \\ 0.751 & 0.244 & -0.493 \\ -0.248 & 0.259 & 0.505 \\ 0.244 & 0.751 & -0.493 \end{bmatrix}$$

The estimated parameters m are then:

$$m = G^{-g} d = \begin{bmatrix} 0.259 \\ 0.750 \\ -0.248 \\ 0.244 \end{bmatrix}$$

Part B.

Since $U_p = U$, the null data vector is the 0 vector itself:

$$g_{null} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

Part C.

The column in V not considered for V_p is the null model vector:

$$m_{null} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

Problem 7.

Part D.

A 4-point deconvolution is found using the given equation using standard Least Squares Fit. The resulting deconvolution filter is:

$$f^{-1} = \begin{bmatrix} 0.432 \\ -0.163 \\ 0.098 \\ 0.084 \end{bmatrix}$$

Part E.

Figure 1 shows the source signal, the fitted filter and the convolution between these two with a delta pulse overlayed. A nice large pulse is seen at first, with following smaller pulses afterwards. It is not perfect due to the filter size and the simple fitting routine. Increasing the number of points to consider, as well as including a dampened solution perhaps would create a better fit.

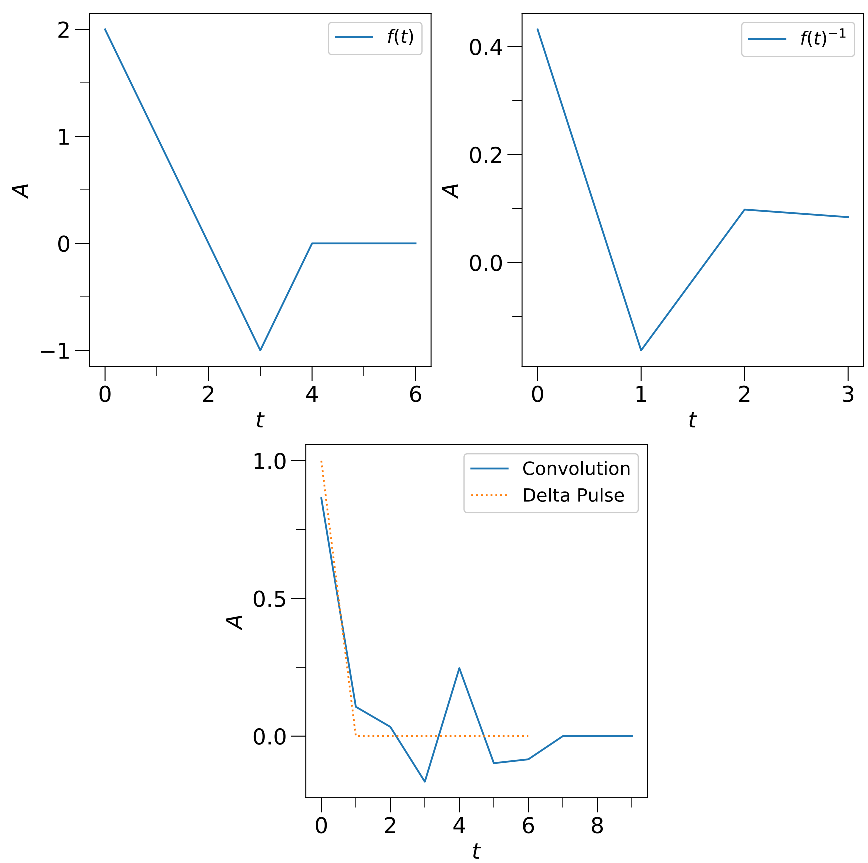

FIG. 1. Resulting plots for the function f (top left), the fitted 4-point filter f^{-1} (top right), and the resulting convolution $f * f^{-1}$ (bottom).

Part F.

With G being the data matrix from the source signal, and d the delta pulse vector:

$$\sigma^2 = (Gm - d)^T(Gm - d)/3 = 0.045$$

$$\text{cov}(m) = (G^TG)^{-1}G^TG(G^TG)^{-1}\sigma^2 = \begin{bmatrix} 0.010 & -0.004 & 0.002 & 0.002 \\ -0.004 & 0.011 & 0.005 & 0.002 \\ 0.002 & 0.005 & 0.011 & -0.004 \\ 0.002 & 0.002 & -0.004 & 0.010 \end{bmatrix}$$

Part G.

Figure 2 shows the fitted filter convolved with the new given signal. It is possible to appreciate the mentioned spikes, with a variability in intensity.

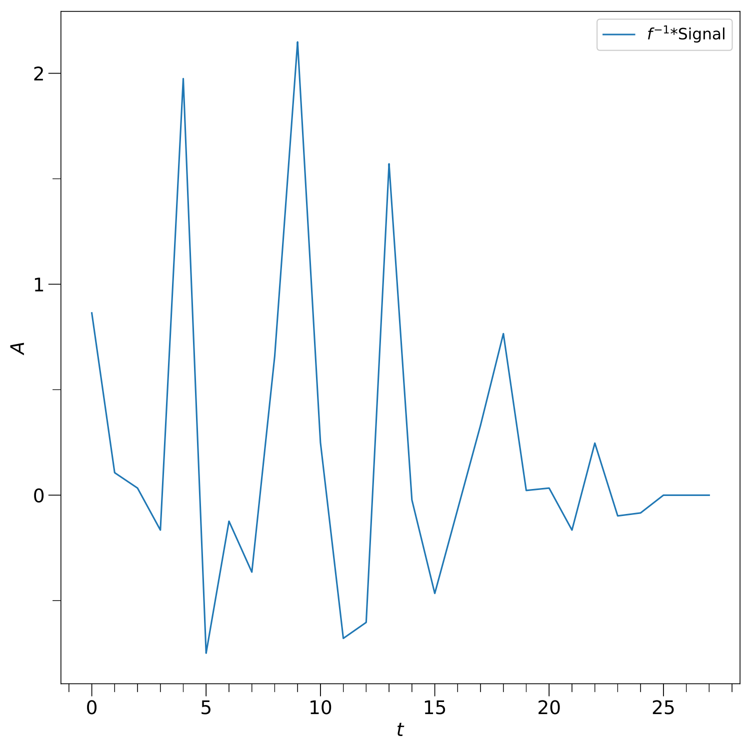

FIG. 2. Resulting time series after convolving the new given signal with the fitted filter f^{-1}.