Homework 7

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Problem 1.

A routine to generate the data matrix for different Fourier series was made, such that:

$$G^{(m)} = \begin{bmatrix} \sin(x_0) & \cos(x_0) & \sin(2x_0) & \cos(2x_0) & ... & \sin(mx_0) & \cos(mx_0) \\ \sin(x_1) & \cos(x_1) & \sin(2x_1) & \cos(2x_1) & ... & \sin(mx_1) & \cos(mx_1) \\ & & & \vdots & & & \\ \sin(x_N) & \cos(x_N) & \sin(2x_N) & \cos(2x_N) & ... & \sin(mx_N) & \cos(mx_N) \end{bmatrix}$$

Data matrices for m and $m+2$ parameters are generated, fitted using LSF and compared using the F-test. This is done repeatedly until they are found not found to be significantly different according to the F-test (since the models will be the same if $c = d = 0$). After a single iteration, comparing the models with $m = 2$ and $m = 4$ resulted in a p-val of 1.70×10^{-9}, which is well below the confidence level of $\alpha = 0.05$. Since the models are not found to be significantly different, the parameters $c = d = 0$.

Problem 2.

From the given tomography null vector, the following statements must be true:

- a) The average of the cells in row 2 cannot be estimated.
- b) The average value of all cells cannot be estimated.
- c) The differences in values of cells within row 2 cannot be estimated.

Problem 3.

Only the first equation has to be linearized, which takes the form:

$$x + y = \ln 10$$

A routine to perform damped LSF was implemented. The equation used was:

$$Gm = d \implies \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \ln 10 \\ 10 \\ 10 \end{bmatrix}$$

with the model parameters m being found as:

$$\begin{aligned} m &= (G^T G + \epsilon I)^{-1} G^T d \\ &= \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.076 \\ 3.076 \end{bmatrix} \end{aligned}$$

Problem 5.

The equation solved for was:

$$Gm = d \implies \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} m \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Part A.

The least squares solution is:

$$\begin{aligned} m &= (G^T G)^{-1} G^T d \\ m &= 3 \end{aligned}$$

Part B.

The data variance was found to be:

$$\begin{aligned} \sigma^2 &= (d - g(m))^T (d - g(m))/(5-1) \\ &= 2.5 \end{aligned}$$

which is used to find the model variance:

$$\begin{aligned} \mathrm{cov}(m) &= (G^T G)^{-1} G^T G (G^T G)^{-1} \sigma^2 \\ &= 0.5 \end{aligned}$$

Part C.

The data resolution matrix is:

$$\begin{aligned} N &= G(G^T G)^{-1} G^T \\ &= \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{aligned}$$

Part D.

The null data vector is:

$$\begin{bmatrix} 0 \end{bmatrix} = (G^T G)^{-1} G^T d_0 = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

$$\begin{aligned} &\implies 0.2(a + b + c + d + e) = 0 \\ &\implies a + b + c + d + e = 0 \end{aligned}$$

Any multiple of the null data vector will also be a null vector, so let $a = 1$:

$$b + c + d + e = -1$$
$$\implies b = c = d = e = -0.25$$
$$\therefore d_0 = \begin{bmatrix} 1 \\ -0.25 \\ -0.25 \\ -0.25 \\ -0.25 \end{bmatrix}$$

Part E.

Since this problem is overdetermined, there is no null model vector.

Problem 6.

The equation solved for was:

$$Gm = d \implies \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Part A.

The minimum length solution is:

$$m = G^T(GG^T)^{-1}d$$
$$m = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

Part B.

The data variance cannot be estimated, so it is set to $\sigma^2 = 1$ to find the model variance:

$$\mathrm{cov}(m) = G^T(GG^T)^{-1}(GG^T)^{-1}G\sigma^2$$
$$= \begin{bmatrix} 0.25 & 0.25 & 0 & 0 \\ 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.25 & 0.25 \end{bmatrix}$$

Part C.

The model resolution matrix is:

$$N = G^T(GG^T)^{-1}G$$
$$= \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Part D.

The null model vector is:

$$Gm_0 = 0$$
$$\implies \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} -b \\ -d \end{bmatrix}$$

Any multiple of the model null vector will also be a null vector, so let $a = 1$:

$$\therefore m_0 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Part E.

The null data vector is:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = G^T(GG^T)^{-1}d_0 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0.5 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$
$$\implies \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0.5 \begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix}$$

This can only be true if $a = b = 0$, so there is no data null vector.

Problem 7.

Part A.

The solution that minimizes the model length:

$$m \propto G^T(GG^T)^{-1}$$

$$G^T(GG^T)^{-1} = \begin{bmatrix} l_1 \\ l_2 \\ . \\ . \\ . \\ l_m \end{bmatrix} (l_1^2 + l_2^2 + ... + l_m^2)^{-1}$$

$$\therefore m \propto \frac{1}{l_1^2 + l_2^2 + ... + l_m^2} \begin{bmatrix} l_1 \\ l_2 \\ . \\ . \\ . \\ l_m \end{bmatrix}$$

Part B.

One potential weighing matrix could be from the uncertainty in the gravity measurements, which would be independent from the layer thicknesses chosen:

$$W = \begin{bmatrix} 1/\sqrt{\sigma_d^{(1)}} & 0 & . \; . \; . & 0 \\ 0 & 1/\sqrt{\sigma_d^{(2)}} & . \; . \; . & 0 \\ & . & & \\ & . & & \\ & . & & \\ 0 & 0 & . \; . \; . & 1/\sqrt{\sigma_d^{(n)}} \end{bmatrix}$$