Coordinate Systems

Spherical ($\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$)

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan(\sqrt{x^2 + y^2}/z) \\ \phi = \arctan(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical ($\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$)

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \arctan(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Vector Derivatives

Cartesian ($d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$, $dV = dx\,dy\,dz$)

Gradient: $\nabla f = \partial_x f\,\hat{\mathbf{x}} + \partial_y f\,\hat{\mathbf{y}} + \partial_z f\,\hat{\mathbf{z}}$

Divergence: $\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$

Curl: $$\nabla \times \mathbf{F} = \begin{cases} \partial_y F_z - \partial_z F_y & \text{in } \hat{\mathbf{x}} \\ \partial_z F_x - \partial_x F_z & \text{in } \hat{\mathbf{y}} \\ \partial_x F_y - \partial_y F_x & \text{in } \hat{\mathbf{z}} \end{cases}$$

Laplacian: $\nabla^2 f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$

Spherical ($d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$)

Gradient: $$\nabla f = \partial_r f\,\hat{\mathbf{r}} + \frac{1}{r}\partial_\theta f\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\partial_\phi f\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\partial_r(r^2 F_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\partial_\phi F_\phi$$

Curl: $$\nabla \times \mathbf{F} = \begin{cases} \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta\,F_\phi) - \partial_\phi F_\theta\right] & \text{in } \hat{\mathbf{r}} \\ \frac{1}{r}\left[\frac{1}{\sin\theta}\partial_\phi F_r - \partial_r(rF_\phi)\right] & \text{in } \hat{\boldsymbol{\theta}} \\ \frac{1}{r}\left[\partial_r(rF_\theta) - \partial_\theta F_r\right] & \text{in } \hat{\boldsymbol{\phi}} \end{cases}$$

Laplacian: $$\nabla^2 f = \frac{1}{r^2}\partial_r\left(r^2\partial_r f\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta f\right) + \frac{\partial_\phi^2 f}{r^2\sin^2\theta}$$

Cylindrical ($d\mathbf{l} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$, $dV = \rho\,d\rho\,d\phi\,dz$)

Gradient: $$\nabla f = \partial_\rho f\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\partial_\phi f\,\hat{\boldsymbol{\phi}} + \partial_z f\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\partial_\rho(\rho F_\rho) + \frac{1}{\rho}\partial_\phi F_\phi + \partial_z F_z$$

Curl: $$\nabla \times \mathbf{F} = \begin{cases} \frac{1}{\rho}\partial_\phi F_z - \partial_z F_\phi & \text{in } \hat{\boldsymbol{\rho}} \\ \partial_z F_\rho - \partial_\rho F_z & \text{in } \hat{\boldsymbol{\phi}} \\ \frac{1}{\rho}\left[\partial_\rho(\rho F_\phi) - \partial_\phi F_\rho\right] & \text{in } \hat{\mathbf{z}} \end{cases}$$

Laplacian: $$\nabla^2 f = \frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho f\right) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$$

Vector Identities Products

$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$\nabla(fg) = f\nabla g + g\nabla f$

$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f$

$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla f$

$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \times (\nabla f) = 0$

$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

$\int_a^b \nabla f \cdot d\mathbf{r} = f(b) - f(a)$ $\quad \iiint_V (\nabla \cdot \mathbf{F})\,dV = \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS$ $\quad \iint_\Sigma (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r}$

$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$

Trigonometric Identities ($\alpha, \beta, \theta \in \mathbb{R},\ z, a, b \in \mathbb{C}$)

$e^{i\theta} = \cos\theta + i\sin\theta \quad \Re e^{i\theta} = \cos\theta \quad \Im e^{i\theta} = \sin\theta$

$\csc z = 1/\sin z \quad \sec z = 1/\cos z \quad \cot z = 1/\tan z$

$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2 z = \sec^2 z \quad 1 + \cot^2 z = \csc^2 z$

$2i\sin z = e^{iz} - e^{-iz} \quad 2\cos z = e^{iz} + e^{-iz} \quad \cos 2z = \cos^2 z - \sin^2 z$

$\sin(iz) = i\sinh z \quad \cos(iz) = \cosh z \quad \cosh^2 z - \sinh^2 z = 1$

$\sin z = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta \quad \cos z = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

$\sin(-z) = -\sin z \quad \cos(-z) = +\cos z \quad \tan(-z) = -\tan z$

$\sin(\pi - z) = +\sin z \quad \cos(\pi - z) = -\cos z \quad \tan(\pi - z) = -\tan z$

$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b \quad \cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$

$2\cos a\cos b = \cos(a+b) + \cos(a-b) \quad 2\sin a\sin b = \cos(a-b) - \cos(a+b)$

$2\sin a\cos b = \sin(a+b) + \sin(a-b) \quad \operatorname{sinc} z = \sin z/z \quad \operatorname{sinc} 0 := 1$

$\langle\cos^2 x\rangle = \langle\sin^2 x\rangle = 1/2 \quad \langle\cos x\rangle = \langle\sin x\rangle = \sqrt{2}/2$

$\operatorname{arsinh} z = \ln(z + \sqrt{z^2 + 1})\ \forall z \quad \operatorname{arcosh} z = \ln(z + \sqrt{z^2 - 1})\ \forall z \geq 1$

$2\operatorname{arctanh} z = \ln(1+z) - \ln(1-z),\ \forall |z| < 1$

in $\mathbb{R}$: $\log\alpha + \log\beta = \log(\alpha\beta) \quad \log\alpha - \log\beta = \log(\alpha/\beta) \quad \alpha\log\beta = \log(\beta^\alpha)$

Gamma Function ($\gamma \equiv$ Euler-Mascheroni constant, $z \in \mathbb{C} \setminus \mathbb{Z}^-$, $n \in \mathbb{N}$)

$\psi(z) = \psi^{(0)}(z) \equiv$ digamma, $\psi^{(m)}(z) \equiv$ polygamma function, $\mathrm{B}(z_1, z_2) \equiv$ beta function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,\ \Re(z) > 0 \quad \Gamma(1+z) = z\,\Gamma(z) \quad \Gamma(n) = (n-1)!$$

$$\Gamma(1-z)\Gamma(z) = \pi/\sin\pi z \quad \Gamma(1-z) = -z\Gamma(-z) \quad \overline{\Gamma(z)} = \Gamma(\overline{z}) \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(z)\Gamma(z + \tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \quad 1/\Gamma(-n) = 1/\Gamma(0) = 0 \quad \Gamma(1) = 0! = 1$$

$$\Gamma(z - m) = (-1)^{m-1}\ \Gamma(-z)\Gamma(1+z)/\Gamma(m+1-z) \quad \psi(z) = \Gamma'(z)/\Gamma(z)$$

$$\psi^{(m)}(z) = \frac{d^m}{dz^m}\psi(z) = \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z) \quad \psi(z) = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}}\right)dt,\ \Re(z) > 0$$

$$\psi(z+1) = \int_0^1 \frac{1 - t^z}{1 - t}dt - \gamma \quad \psi(n+1) = H_n - \gamma \quad H_n = \sum_{k=1}^n \frac{1}{k}$$

$$\mathrm{B}(z_1, z_2) = \Gamma(z_1)\Gamma(z_2)/\Gamma(z_1 + z_2) \quad \mathrm{B}(z_1, z_2) = \mathrm{B}(z_2, z_1) \quad \mathrm{B}(1, x) = 1/x$$

$$\mathrm{B}(x, 1-x) = \pi/\sin\pi x \quad \mathrm{B}\left(\frac{z_1+1}{2}, \frac{z_2+1}{2}\right) = 2\int_0^{\pi/2}\sin^{z_1}\theta\cos^{z_2}\theta\,d\theta$$

$$\mathrm{B}(z_1 + 1, z_2) = \mathrm{B}(z_1, z_2)\frac{z_1}{z_1 + z_2} \quad \mathrm{B}(z, z) = \frac{1}{z}\int_0^{\pi/2}\frac{d\theta}{(\sqrt[z]{\sin\theta} + \sqrt[z]{\cos\theta})^{2z}},\ z \neq 1$$

Taylor Series ($\alpha \in \mathbb{R}$, $z \in \mathbb{C} \cap \mathrm{Dom}_f$, $s \in \mathbb{C}$)

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \ldots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n + \ldots$$

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots \quad \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots \quad \frac{1}{1-z} = 1 + z + z^2 + \ldots$$

$$(1+z)^s = 1 + sz + \frac{s(s-1)}{2!}z^2 + \ldots \quad \sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \quad \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \quad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$$

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots \quad \tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \ldots \quad (\text{both for } |z| < \frac{\pi}{2})$$

$$\arcsin z = z + \frac{z^3}{6} + \frac{3z^5}{40} + \ldots \quad \operatorname{arsinh} z = z - \frac{z^3}{6} + \frac{3z^5}{40} - \ldots \quad (\text{both for } |z| < 1)$$

$$\arccos z = \frac{\pi}{2} - \arcsin z \quad \operatorname{arcosh} z = (-1)^{\left\lfloor\frac{\arg z}{2\pi}\right\rfloor}\left(\frac{i\pi}{2} - iz - \frac{iz^3}{6} - \frac{3iz^5}{40} - \ldots\right)$$

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots \quad \operatorname{artanh} z = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots,\ \text{both for} |z| < 1$$

Symbols ($i, j, n, \{a_n\} \in \mathbb{N}$)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \qquad \epsilon_{a_1 a_2 \cdots a_n} = \begin{cases} +1 & \text{if even permutation of } (1, 2, \cdots, n), \\ -1 & \text{if odd permutation of } (1, 2, \cdots, n), \\ 0 & \text{otherwise (repeated indexes).} \end{cases}$$

Prefixes (SI units) Å (not SI) 10^{-10}

Q (quetta)	10^{30}	T (tera)	10^{12}	d (deci)	10^{-1}	f (femto)	10^{-15}
R (ronna)	10^{27}	G (giga)	10^{9}	c (centi)	10^{-2}	a (atto)	10^{-18}
Y (yotta)	10^{24}	M (mega)	10^{6}	m (mili)	10^{-3}	z (zepto)	10^{-21}
Z (zetta)	10^{21}	k (kilo)	10^{3}	μ (micro)	10^{-6}	y (yocto)	10^{-24}
E (exa)	10^{18}	h (hecto)	10^{2}	n (nano)	10^{-9}	r (ronto)	10^{-27}
P (peta)	10^{15}	da (deca)	10^{1}	p (pico)	10^{-12}	q (quecto)	10^{-30}

Integrals ($n \in \mathbb{N}_0$, $m \in \mathbb{Z}$, $\alpha, \beta, \gamma, \delta, \mu, \nu, \sigma, r \in \mathbb{R}$, $x \in \mathbb{R} \cap \mathrm{Dom}_f$, $a, b, z \in \mathbb{C}$)

+C omitted. Avoid division by 0. Most results can be extended to $\mathbb{C}$.

Basic

$$\int(x+\alpha)^r dx = \frac{(x+\alpha)^{r+1}}{r+1} \quad \int x(x+\alpha)^r dx = \frac{(x+\alpha)^{r+1}(rx+x-\alpha)}{(r+1)(r+2)} \quad \int a^x dx = \frac{a^x}{\ln a} \quad \int u\,dv = uv - \int v\,du$$

Rational

$$\int\frac{dx}{\alpha x+\beta} = \frac{1}{\alpha}\ln|\alpha x + \beta| \quad \int\frac{dx}{x^2+\alpha^2} = \frac{1}{\alpha}\arctan\frac{x}{\alpha} \quad \int\frac{dx}{x^2-\alpha^2} = \frac{1}{2\alpha}\ln\left|\frac{x-\alpha}{x+\alpha}\right| \quad \int\frac{dx}{\alpha^2-x^2} = \frac{1}{2\alpha}\ln\left|\frac{\alpha+x}{\alpha-x}\right|$$

$$\int\frac{dx}{\alpha x^2+\beta x+\gamma} = \frac{2}{\sqrt{(4\alpha\gamma-\beta^2)}}\arctan\frac{2\alpha x+\beta}{\sqrt{(4\alpha\gamma-\beta^2)}} \quad \int\frac{dx}{(x+\alpha)(x+\beta)} = \frac{1}{\beta-\alpha}\ln\left|\frac{\alpha+x}{\beta+x}\right|$$

Roots

$$\int\sqrt{(x^2+\alpha^2)} = \frac{1}{2}\left[x\sqrt{(x^2+\alpha^2)} + \alpha^2\operatorname{arsinh}\frac{x}{|\alpha|}\right] \quad \int\sqrt{(x^2-\alpha^2)} = \frac{1}{2}\left[x\sqrt{(x^2-\alpha^2)} - \alpha^2\ln|\sqrt{(x^2-\alpha^2)} + x|\right]$$

$$\int\frac{dx}{\sqrt{(x^2+\alpha^2)}} = \operatorname{arsinh}\frac{x}{|\alpha|} \quad \int\frac{dx}{\sqrt{(-x^2+\alpha^2)}} = \arcsin\frac{x}{|\alpha|} \quad \int\frac{dx}{\sqrt{(x^2-\alpha^2)}} = \ln|\sqrt{(x^2-\alpha^2)} + x|$$

$$\int\frac{dx}{x\sqrt{(x^2+\alpha^2)}} = -\frac{1}{\alpha}\operatorname{arsinh}\frac{\alpha}{|x|} \quad \int\frac{dx}{x\sqrt{(-x^2+\alpha^2)}} = -\frac{1}{\alpha}\ln\frac{|\sqrt{(-x^2+\alpha^2)}+\alpha|}{|x|} \quad \int\frac{dx}{x\sqrt{(x^2-\alpha^2)}} = \frac{1}{\alpha}\arctan\frac{\sqrt{(x^2-\alpha^2)}}{\alpha}$$

$$\int\frac{x}{\sqrt{(x^2\pm\alpha^2)}}dx = \sqrt{(x^2\pm\alpha^2)} \quad \int\frac{x}{\sqrt{(-x^2+\alpha^2)}}dx = -\sqrt{(-x^2+\alpha^2)}$$

$$\int\frac{dx}{(x^2\pm\alpha^2)^{3/2}} = \frac{\pm x}{\alpha^2\sqrt{(x^2\pm\alpha^2)}} \quad \int\frac{dx}{(-x^2+\alpha^2)^{3/2}} = \frac{x}{\alpha\sqrt{(-x^2+\alpha^2)}}$$

$$\int\frac{x}{(x^2\pm\alpha^2)^{3/2}} = \frac{-1}{\sqrt{(x^2\pm\alpha^2)}} \quad \int\frac{x}{(x^2\pm\alpha^2)^{3/2}} = \frac{-1}{\sqrt{(x^2\pm\alpha^2)}}$$

Trigonometric ($\mu, \nu > 0,\ \chi \equiv x\alpha,\ \gamma \equiv \alpha + \beta,\ \delta \equiv \alpha - \beta$)

$$\int\sin x\,dx = -\cos x \quad \int\cos x\,dx = \sin x \quad \int\frac{dx}{\sin^2 x} = -\cot x \quad \int\frac{dx}{\cos^2 x} = \tan x \quad \int\frac{dx}{\tan^2 x} = -\cot x - x$$

$$\int\sinh x\,dx = \cosh x \quad \int\cosh x\,dx = \sinh x \quad \int\frac{dx}{\sinh^2 x} = -\coth x \quad \int\frac{dx}{\cosh^2 x} = \tanh x \quad \int\frac{dx}{\tanh^2 x} = -\coth x + x$$

$$\int\tan x\,dx = -\ln|\cos x| \quad \int\tan^2 x\,dx = \tan x - x \quad \int\tanh x\,dx = \ln\cosh x \quad \int\tanh^2 x\,dx = -\tanh x + x$$

$$\int\frac{dx}{\sin x} = -\ln\left|\frac{1}{\sin x} + \frac{1}{\tan x}\right| \quad \int\frac{dx}{\cos x} = \ln\left|\frac{1}{\cos x} + \tan x\right| \quad \int\frac{dx}{\tan x} = \ln|\sin x|$$

$$\int\sin^n\alpha x\,dx = -\frac{\sin^{n-1}\chi\,\cos\chi}{n\alpha} + \frac{n-1}{n}\int\sin^{n-2}\alpha x\,dx \quad \int\cos^n\alpha x\,dx = +\frac{\cos^{n-1}\chi\,\sin\chi}{n\alpha} + \frac{n-1}{n}\int\cos^{n-2}\alpha x\,dx$$

$$\int\sin\alpha x\sin\beta x\,dx = -\frac{\sin\gamma x}{2\gamma} + \frac{\sin\delta x}{2\delta} \quad \int\cos\alpha x\cos\beta x\,dx = +\frac{\sin\gamma x}{2\gamma} + \frac{\sin\delta x}{2\delta} \quad \int\sin\alpha x\cos\beta x\,dx = -\frac{\cos\gamma x}{2\gamma} - \frac{\cos\delta x}{2\delta}$$

$$\int x\sin\alpha x\,dx = \frac{\sin\chi}{\alpha^2} - \frac{x\cos\chi}{\alpha} \quad \int x\cos\alpha x\,dx = \frac{\cos\chi}{\alpha^2} + \frac{x\sin\chi}{\alpha} \quad \int x\,{\sin^2 \atop \cos^2}\,\alpha x\,dx = \mp\frac{2\chi\,\sin 2\chi + \cos 2\chi \mp 2\chi^2}{8\alpha^2}$$

$$\int x\,{\sin\alpha x\sin\beta x \atop \cos\alpha x\cos\beta x}\,dx = \mp\frac{x\sin\gamma x}{2\gamma} \mp \frac{\cos\gamma x}{2\gamma^2} + \frac{x\sin\delta x}{2\delta} + \frac{\cos\delta x}{2\delta^2} \quad \int x\sin\alpha x\cos\beta x\,dx = -\frac{x\cos\gamma x}{2\gamma} + \frac{\sin\delta x}{2\delta^2} - \frac{x\cos\delta x}{2\delta} + \frac{\sin\gamma x}{2\gamma^2}$$

Definite integrals ($m!! = m(m-2)(m-4)\ldots$, $\ -1!! = 0!! = 1!! = 1$)

$$\int_0^{\pi/2}\sin^\mu x\,dx = \int_0^{\pi/2}\cos^\mu x\,dx = \frac{1}{2}\mathrm{B}\left(\frac{\mu+1}{2}, \frac{1}{2}\right) = \frac{(n-1)!!}{n!!}\cdot\begin{cases} \frac{\pi}{2} & \text{if } \mu = n \text{ even} \\ 1 & \text{if } \mu = n \text{ odd} \end{cases} \quad \int_{-1}^{+1}{\sin(m\pi x)\sin(\tilde{m}\pi x) \atop \cos(m\pi x)\cos(\tilde{m}\pi x)}\,dx = \delta_{m,\tilde{m}}$$

$$\int_0^\pi {\sin x\,dx = 2 \atop \cos x\,dx = 0} \quad \int_0^\pi {\sin(m\pi x)\sin(\tilde{m}\pi x) \atop \cos(m\pi x)\cos(\tilde{m}\pi x)}\,dx = \frac{\pi}{2}\delta_{m,\tilde{m}} \quad \int_0^\pi \sin(mx)\cos(\tilde{m}x)dx = \begin{cases} 0 & \text{if } m+\tilde{m} \text{ even} \\ \frac{2m}{m^2-\tilde{m}^2} & \text{if } m+\tilde{m} \text{ odd} \end{cases}$$

$$\int_0^\pi \sin^n x\cos^{\tilde{n}} x\,dx = 0\ \forall\,\tilde{n} \text{ odd} \quad \int_0^{\pi\mu}{\sin^2 \alpha x \atop \cos^2 \alpha x}\,dx = \frac{1}{4\alpha}\left[2\pi\alpha\mu \mp \sin(2\pi\alpha\mu)\right] \overset{\text{if } \mu = n}{=} \frac{\pi n}{\alpha} \quad \int_0^\pi {\sin^3 x\,dx = \frac{4}{3} \atop \cos^3 x\,dx = 0}$$

$$\int_0^{2\pi}{\sin x \atop \cos x}\,dx = 0 \quad \int_0^{2\pi}\sin x\cos x\,dx = 0 \quad \int_0^{2\pi}\sin^n x\cos^{\tilde{n}} x\,dx = 0 \text{ if } n, \tilde{n} \text{ not both even} \quad \int_0^{2\pi}{\sin^3 x \atop \cos^3 x}\,dx = 0$$

$$\int_0^{2\pi}(1 - \cos x)^n\sin nx\,dx = 0 \quad \int_0^{2\pi}(1 - \cos x)^n\sin nx\,dx = (-1)^n\frac{\pi}{2^{n-1}}$$

Parity $\quad$ *Even* : $f_e(-x) = f_e(x)$ sym w.r.t Y-axis $\quad$ *Odd* : $f_o(-x) = -f_o(x)$ sym w.r.t (0,0)

$$\int_{-\alpha}^{+\alpha} f_e(x)\,dx = 2\int_0^\alpha f_e(x)\,dx \qquad \int_{-\alpha}^{+\alpha} f_o(x)\,dx = 0$$

f_e : $\cos x,\ \cosh x,\ x^{2n},\ e^{-x^2},\ |x|,\ \delta_{ij},\ \delta(x),\ \mathbb{R},\ 1/f_e,\ f_o',\ f_e \pm f_e,\ f_e \cdot f_e,\ f_o \cdot f_o,\ \mathcal{F}\{f_e(x)\}(\xi), \ldots$

f_o : $\sin x,\ \sinh x,\ x^{2n+1},\ \tan x,\ \operatorname{erf} x,\ \operatorname{sign} x,\ \ln\left(\frac{1+x}{1-x}\right),\ 1/f_o,\ f_e',\ f_o \pm f_o,\ f_e \cdot f_o,\ \mathcal{F}\{f_o(x)\}(\xi), \ldots$

Log/Exp ($\tilde{n} \neq -1$)

$$\int x^r\ln x\,dx = x^{r+1}\left(\frac{\ln x}{r+1} - \frac{1}{(r+1)^2}\right) \quad \int\ln^n x\,dx = x\ln^n x - n\int\ln^{n-1} x\,dx \quad \int\frac{dx}{(e^{-x/\alpha}+1)} = \alpha\ln(e^{x/\alpha}+1)$$

$$\int xe^{\alpha x^2}dx = \frac{e^{\alpha x^2}}{2\alpha} \quad \int x^n e^{\alpha x}dx = \frac{x^n e^{\alpha x}}{\alpha} - \frac{n}{\alpha}\int x^{n-1}e^{\alpha x}dx \quad \int\frac{e^{\alpha x}}{x^n}dx = \frac{1}{n-1}\left(-\frac{e^{\alpha x}}{x^{n-1}} + \alpha\int\frac{e^{\alpha x}}{x^{n-1}}dx\right) \quad \int$$

$$\int(\ln x)^n\,dx = (-1)^n\,n!\,x\sum_{k=0}^n\frac{(-\ln x)^k}{k!} \quad x^{\tilde{n}+1}\left(\frac{\ln x}{\tilde{n}+1} - \frac{1}{(\tilde{n}+1)^2}\right)$$

Definite integrals ($r - 1$, $\alpha > 0$ $\ \gamma \equiv$Euler-Mascheroni constant)

$$\int_0^\infty x^r e^{-\alpha x^2}dx = \frac{\Gamma\left(\frac{r+1}{2}\right)}{2\alpha^{\frac{r+1}{2}}} = \begin{cases} \frac{(2n-1)!!}{2^{n+1}\,\alpha^n}\sqrt{\frac{\pi}{\alpha}} & \text{if } r = 2n \\ \frac{n!}{2\,\alpha^{n+1}} & \text{if } r = 2n+1 \end{cases} \qquad \int_0^\infty e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad \int_0^\infty x^2 e^{-\alpha x^2}dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_0^\infty x^r e^{-ax}dx = \frac{\Gamma(r+1)}{a^{r+1}} \overset{\text{if } r = n}{=} \frac{n!}{a^{n+1}}\ (r > -1, \Re(a) > 0) \quad \int_0^\infty\sqrt{x}e^{-x}dx = \frac{\sqrt{\pi}}{2} \quad \int_0^\infty\frac{x}{e^x - 1}dx = \frac{\pi^2}{6}$$

$$\int_0^\infty e^{-ax^b}dx = a^{-1/b}\,\Gamma\left(\frac{1}{b} + 1\right) \quad \int_{-\infty}^{+\infty}e^{-\alpha x^2 + \beta x}dx = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}} \quad \int_0^{2\pi}e^{i(m-\tilde{m})\phi}d\phi = 2\pi\delta_{m,\tilde{m}}$$

$$\int_0^\infty e^{-\alpha x}\sin(\beta x)dx = \frac{\beta}{\alpha^2 + b^2} \quad \int_0^\infty e^{-\alpha x}\cos(\beta x)dx = \frac{\alpha}{\alpha^2 + \beta^2} \quad \int_0^\infty\frac{\ln x}{e^x}dx = \int_1^\infty\left(\frac{1}{x} - \frac{1}{\lfloor x\rfloor}\right)dx = -\gamma$$

Error function integrals ($\varphi = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\mu \equiv$mean, $\sigma^2 \equiv$variance) $\quad \operatorname{erf}(\pm\infty) = \pm 1 \quad i\,\operatorname{erfi}(z) = \operatorname{erf}(iz)$

$$\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = \operatorname{erf}(z) \quad \int\varphi\,dx = \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right) \quad \int\sqrt{x}\,e^{ax}dx = \frac{\sqrt{x}\,e^{ax}}{a} - \frac{\sqrt{\pi}\ \operatorname{erfi}(\sqrt{a}\sqrt{x})}{2a^{3/2}}$$

Linear Algebra $(n, m, i, j, k, l \in \mathbb{N}_0,\ \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{L}, \mathbf{U}, \mathbf{I}, \mathbf{P} \in \mathcal{M}(\mathbb{K}))$

Matrices (Generalizable to arbitrary linear operators)

$\mathbf{A}_{m\times n}$ matrix with m rows and n columns; m and n dimensions of $\mathbf{A}$

$\mathbf{A}=(a)_{ij}\ \ \mathbf{A}^{\mathrm{T}}=(a)_{ji}\equiv$ transpose of $\mathbf{A}$ $\ \mathbf{A}_{n\times n}\equiv$ square matrix

$\mathbf{D}=\mathbf{D}_{n\times n}:\ i\neq j\ \forall i,j \Rightarrow d_{ij}=0,\ \mathbf{D}=\mathrm{diag}(d_1,\cdots,d_n)\equiv$ diagonal matrix

$\mathbf{L}=\mathbf{L}_{n\times n}:\ l_{ij}=0\ \forall i<j,\ \mathbf{L}\equiv$ lower triangular matrix

$\mathbf{U}=\mathbf{U}_{n\times n}:\ u_{ij}=0\ \forall i>j,\ \mathbf{U}\equiv$ upper triangular matrix

$\mathbf{I}=\mathbf{I}_n=\mathrm{diag}(1,\cdots,1)\equiv$ identity matrix $\ (\mathbf{I}_n)_{ij}=\delta_{ij}$

$\mathbf{A}_{n\times n}\equiv$ Invertible $\Leftrightarrow \exists\, \mathbf{B}_{n\times n}\mid \mathbf{AB}=\mathbf{BA}=\mathbf{I}_n,\ \mathbf{B}=\mathbf{A}^{-1}\equiv$ inverse of $\mathbf{A}$

$\mathbf{A}_{n\times n}\equiv$ singular matrix $\Leftrightarrow \mathbf{A}$ not invertible $\Leftrightarrow \det \mathbf{A}=0$

Let $\mathbf{A}_{m\times n},\ 0<k\le m,n:$ minor of degree k of $\mathbf{A}$ is the determinant of a matrix obtained from $\mathbf{A}$ by deleting $m-k$ rows and $n-k$ columns

Let $\mathbf{A}_{n\times n},\ \mathbf{A}_{ij}$ submatrix, by deleting row i and column j from $\mathbf{A}$,

$c_{ij}=(-1)^{i+j}\cdot\det\mathbf{A}_{ij}\ \ \mathbf{C}=(c)_{ij}\equiv$ cofactor matrix

$\mathrm{adj}\,\mathbf{A}=\mathbf{C}^{\mathrm{T}}\equiv$ adjugate matrix of $\mathbf{A}$ $\ \mathbf{A}^{-1}=\mathrm{adj}\,\mathbf{A}/\det\mathbf{A}$

$\mathbf{A}=\mathbf{A}^{\mathrm{T}}\Leftrightarrow\mathbf{A}$ symmetric matrix $\ \mathbf{A}=-\mathbf{A}^{\mathrm{T}}\Leftrightarrow\mathbf{A}$ anti-symmetric matrix

$\mathbf{A}^{\dagger}=(\overline{\mathbf{A}})^{\mathrm{T}}=\overline{\mathbf{A}^{\mathrm{T}}}\equiv$ conjugate transpose or Hermitian transpose of $\mathbf{A}$

$\mathbf{A}=\mathbf{A}^{\dagger}\Leftrightarrow\mathbf{A}$ Hermitian matrix $\ \mathbf{A}=-\mathbf{A}^{\dagger}\Leftrightarrow\mathbf{A}$ anti-Hermitian matrix

$\mathbf{A}^{\dagger}\mathbf{A}=\mathbf{A}\mathbf{A}^{\dagger}\Leftrightarrow\mathbf{A}$ normal matrix $\ \mathbf{A}^{\dagger}=\mathbf{A}^{-1}\Leftrightarrow\mathbf{A}$ unitary matrix

$\det\mathbf{A}_{n\times n}=|\mathbf{A}|=\sum_{i=1}^{n}a_{ij}c_{ij}=\sum_{j=1}^{n}a_{ij}c_{ij}\ \ \det\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$

$\mathrm{tr}\,\mathbf{A}_{n\times n}=\sum_{i=1}^{n}a_{ii}\ \ \ \mathrm{rank}\,\mathbf{A}:=\dim(\mathrm{img}\,\mathbf{A}_{m\times n})\le\min\{m,n\}$

rank of $\mathbf{A}$: number of linearly independent columns (or rows) of $\mathbf{A}$

$\ker\mathbf{A}=\{\mathbf{x}\in\mathbb{K}^n\mid\mathbf{Ax}=\mathbf{0}\}\ \ \ker\mathbf{A}+\mathrm{rank}\,\mathbf{A}=n,\ \mathbf{A}_{m\times n}$

$[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA}\equiv$ commutator $\ [\mathbf{A},\mathbf{B}]=0\Leftrightarrow\mathbf{A},\mathbf{B}$ commute

$\{\mathbf{A},\mathbf{B}\}=\mathbf{AB}+\mathbf{BA}\equiv$ anticommutator $\ 2\mathbf{AB}=[\mathbf{A},\mathbf{B}]+\{\mathbf{A},\mathbf{B}\}$

Let $\mathbf{A}_{n\times n},\ \mathbf{v}_{n\times1}\neq\mathbf{0},\ \lambda\in\mathbb{K},\ \mathbf{Av}=\lambda\mathbf{v}:\mathbf{v}\equiv$ eigenvector, $\lambda\equiv$ eigenvalue

$p(\lambda)=|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\{\lambda_k\}\ \ (\mathbf{A}-\lambda_k\mathbf{I})\mathbf{v}_k=\mathbf{0}\Rightarrow\{\mathbf{v}_k\}$

$\mu_{\mathbf{A}}(\lambda_k)\equiv$ algebraic multiplicity: $\max\{l\mid p(\lambda)=(\lambda-\lambda_k)^l\cdot q(\lambda),\ q(\lambda_k)\neq0\}$

$\gamma_{\mathbf{A}}=\dim\ker(\mathbf{A}-\lambda_k\mathbf{I})\equiv$ geometric multiplicity $\ 1\le\gamma_{\mathbf{A}}(\lambda_k)\le\mu_{\mathbf{A}}(\lambda_k)$

$\gamma_{\mathbf{A}}(\lambda_k)=\mu_{\mathbf{A}}(\lambda_k)\ \forall k\Leftrightarrow\exists\mathcal{B}'=\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}\equiv$ eigenbasis $\Rightarrow\exists\mathbf{P}\mid\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$

$\mathbf{P}={}_{\mathcal{B}'}\mathbf{P}_{\mathcal{B}}=\mathbf{P}_{\mathcal{B}\to\mathcal{B}'}\equiv$ change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ $\ {}_{\mathcal{B}'}D_{\mathcal{B}'}={}_{\mathcal{B}}A_{\mathcal{B}}$

$\mathbf{P}=[\mathbf{v}_1\cdots\mathbf{v}_n]\ \ \mathbf{D}=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)\ \ \mathbf{A}\sim\mathbf{D}\Rightarrow|\mathbf{A}|=|\mathbf{D}|,\ \mathrm{tr}\,\mathbf{A}=\mathrm{tr}\,\mathbf{D}$

Properties $\ (\theta\in\mathbb{R},\ \eta,\nu,\omega,\tau\in\mathbb{C},\ \ \vec{u},\vec{v}\in\mathbb{C}^n)$

$\mathbf{A}(\nu+\omega)=\nu\mathbf{A}+\omega\mathbf{A}\ \ \tau(\mathbf{A}+\mathbf{B})=\tau\mathbf{A}+\tau\mathbf{B}\ \ \mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$

$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{CB}\ \ (\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}\ \ \mathbf{AB}\not\equiv\mathbf{BA}$

Let v,w arbitrary column vectors, j^{th} column of $\mathbf{A}$ $a_j=\nu\cdot v+\omega\cdot w$:

$|\mathbf{A}|=\nu\cdot|a_1,\cdots,a_{j-1},v,a_{j+1},\cdots,a_n|+\omega\cdot|a_1,\cdots,a_{j-1},w,a_{j+1},\cdots,a_n|$

$|a_1,\cdots,u,\cdots,u,\cdots,a_n|=0\ \ |\mathbf{A}_\sigma|=\mathrm{sign}(\sigma)\cdot|\mathbf{A}|,\ \sigma\equiv$ permutation

$|\tau\mathbf{A}|=\tau^n|\mathbf{A}|\ \ |\mathbf{A}|^{\mathrm{T}}=|\mathbf{A}^{\mathrm{T}}|\ \ |\mathbf{A}|^{\dagger}=|\mathbf{A}^{\dagger}|\ \ \overline{|\mathbf{A}|}=|\overline{\mathbf{A}}|\ \ |\mathbf{A}|^{-1}=|\mathbf{A}^{-1}|$

$\overline{\overline{\mathbf{A}}}=\mathbf{A}\ \ |\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|\ \ |\mathbf{U}|=e^{i\theta}\ \ |\mathbf{U}|=1\Rightarrow\mathbf{U}\in SU(n)\ \ |\mathbf{A}|=\prod_{k=1}^{n}\lambda_k$

$\mathrm{tr}(\tau\mathbf{A})=\tau\,\mathrm{tr}\,\mathbf{A}\ \ \mathrm{tr}\,\mathbf{A}=\mathrm{tr}\,\mathbf{A}^{\mathrm{T}}\ \ \mathrm{tr}\,\mathbf{A}^{\dagger}=\mathrm{tr}\,\overline{\mathbf{A}}=\overline{\mathrm{tr}\,\mathbf{A}}\ \ \mathrm{tr}(\mathbf{A}+\mathbf{B})=\mathrm{tr}\,\mathbf{A}+\mathrm{tr}\,\mathbf{B}$

$\mathrm{tr}(\mathbf{AB})=\mathrm{tr}(\mathbf{BA})\Rightarrow\mathrm{tr}[\mathbf{A},\mathbf{B}]=0\ \ \mathrm{tr}\,\mathbf{A}=\sum_{k=1}^{n}\lambda_k$

$\mathrm{tr}(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n)=\mathrm{tr}(\mathbf{A}_n\mathbf{A}_1\cdots\mathbf{A}_{n-1})=\cdots=\mathrm{tr}(\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_n\mathbf{A}_1)$

$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}=\mathbf{A}\ \ (\mathbf{A}+\mathbf{B})^{\mathrm{T}}=\mathbf{A}^{\mathrm{T}}+\mathbf{B}^{\mathrm{T}}\ \ (\eta\mathbf{A})^{\mathrm{T}}=\eta\mathbf{A}^{\mathrm{T}}\ \ (\mathbf{AB})^{\mathrm{T}}=(\mathbf{BA})^{\mathrm{T}}$

$(\mathbf{A}^{-1})^{\mathrm{T}}=(\mathbf{A}^{\mathrm{T}})^{-1}\ \ \ \mathrm{rg}\,\mathbf{A}=\mathrm{rg}\,\mathbf{A}^{\mathrm{T}}\ \ (\mathbf{A}^{-1})^{\dagger}=(\mathbf{A}^{\dagger})^{-1}\ \ \ \mathrm{rg}\,\mathbf{A}=\mathrm{rg}\,\mathbf{A}^{\dagger}$

$(\mathbf{A}^{\dagger})^{\dagger}=\mathbf{A}\ \ (\mathbf{A}+\mathbf{B})^{\dagger}=\mathbf{A}^{\dagger}+\mathbf{B}^{\dagger}\ \ (\eta\mathbf{A})^{\dagger}=\overline{\eta}\mathbf{A}^{\dagger}\ \ (\mathbf{AB})^{\dagger}=(\mathbf{BA})^{\dagger}$

$\vec{u}\cdot\vec{v}=\langle\mathbf{u},\mathbf{v}\rangle=\mathbf{u}^{\dagger}\mathbf{v}\ \ \ \|\mathbf{u}\|:=\sqrt{\langle\mathbf{u},\mathbf{u}\rangle}\ \ \ |\langle\mathbf{u},\mathbf{v}\rangle|\le\|\mathbf{u}\|\|\mathbf{v}\|\ \ \ \|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\|$

$(\mathbf{A}^{-1})^{-1}=\mathbf{A}\ \ (\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}\ \ (\eta\mathbf{A})^{-1}=\mathbf{A}^{-1}/\eta\ \ \mathbf{D}^{-1}=\mathrm{diag}(1/d_i)$

$[\mathbf{A},\mathbf{B}]=-[\mathbf{B},\mathbf{A}]\ \ [\mathbf{A},\mathbf{B}+\mathbf{C}]=[\mathbf{A},\mathbf{B}]+[\mathbf{A},\mathbf{C}]\ \ [\mathbf{A},\mathbf{A}]=[\mathbf{A},\mathbf{A}^n]=0$

$[\mathbf{A},\mathbf{BC}]=[\mathbf{A},\mathbf{B}]\mathbf{C}+\mathbf{B}[\mathbf{A},\mathbf{C}]\ \ [\mathbf{A},[\mathbf{B},\mathbf{C}]]+[\mathbf{B},[\mathbf{C},\mathbf{A}]]+[\mathbf{C},[\mathbf{A},\mathbf{B}]]=0$

$[\mathbf{A},\mathbf{B}]^{\dagger}=[\mathbf{B}^{\dagger},\mathbf{A}^{\dagger}]\ \ \mathbf{A}=\mathbf{A}^{\dagger}\Rightarrow\lambda_{\mathbf{A}}\in\mathbb{R}\ \ \mathbf{A}=-\mathbf{A}^{\dagger}\Rightarrow\lambda_{\mathbf{A}}\in i\mathbb{R}$

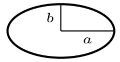
if $\mathbf{A}=\mathbf{A}^{\dagger},\mathbf{B}=\mathbf{B}^{\dagger}$: $\ i[\mathbf{A},\mathbf{B}]=(i[\mathbf{A},\mathbf{B}])^{\dagger},\ \ \{\mathbf{A},\mathbf{B}\}=\{\mathbf{A},\mathbf{B}\}^{\dagger}$

if $\mathbf{A}=\mathbf{A}^{\dagger},\mathbf{B}=\mathbf{B}^{\dagger}$, and $[\mathbf{A},\mathbf{B}]=0$: $\ \mathbf{AB}=(\mathbf{AB})^{\mathrm{T}}$

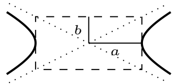
Conics $(\varepsilon,a,b,c,h,k,p,\ell\in\mathbb{R})$, $\varepsilon\equiv$ eccentricity, $c\equiv$ focal distance, $p\equiv$ focal parameter, $\ell\equiv$ semi-latus rectum, $a\equiv$ semi-major axis, $b\equiv$ semi-minor axis, $\ell=p\varepsilon$, $c=a\varepsilon$, $p+c=a/\varepsilon$, $(h,k)\equiv$ center, $(h,k)_{\text{parabola}}\equiv$ vertex

Vertical parabola: $(y-k)=\dfrac{1}{4p}(x-h)^2,\ \varepsilon=1$ $\ $ Circle: $(x-h)^2+(y-k)^2=a^2,\ \varepsilon=0$

Ellipse: $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1,\ \varepsilon=\sqrt{1-\left(\dfrac{b}{a}\right)^2}$



Hyperbola: $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1,\ \varepsilon=\sqrt{1+\left(\dfrac{b}{a}\right)^2}$



Complex Analysis $(\alpha,\beta,r,\theta,t,p,R\in\mathbb{R},\ z,w\in\mathbb{C},\ n,k\in\mathbb{N}_0,\ m\in\mathbb{N}_+,\ i^2=-1)$

p.v. $\equiv$ principal value $\ \ \gamma\equiv$ closed contour path positively oriented (anticlockwise)

$-\gamma\equiv\gamma$ with reverse orientation $\Rightarrow\int_{-\gamma}f(z)dz=-\int_\gamma f(z)dz$

$z=\alpha+i\beta=re^{i\theta}\ \ r=|z|=\sqrt{\alpha^2+\beta^2}\ \ \theta=\arctan(\beta/\alpha)\ \ \overline{z}=\alpha-i\beta\ \ z^{-1}=\dfrac{\overline{z}}{r^2}=\dfrac{1}{re^{i\theta}}$

$z\overline{z}=|z|^2\ \ z+\overline{z}=2\Re[z]\ \ z-\overline{z}=2i\Im[z]\ \ \ \sqrt[n]{z}=\sqrt[n]{r}\exp[i(\frac{\theta+2\pi k}{n})],\ k<n-1$

$z^w=e^{w\log z}\ \ \ \log z=\ln r+i(\theta\pm2\pi k)\xrightarrow{\text{p.v.}}\mathrm{Log}\,z=\ln r+i\theta,\ \theta\in(-\pi,\pi]$

$\mathrm{Log}\,e^z=z\Leftrightarrow\Im z\in(-\pi,\pi]\ \ \ \mathrm{Log}(zw)=\mathrm{Log}\,z+\mathrm{Log}\,w\pm i2\pi k$

$e^{\pm i2\pi n}=1\ \ e^{i\frac{\pi}{2}\pm i2\pi n}=i\ \ e^{i\pi\pm i2\pi n}=-1\ \ e^{i\frac{3\pi}{2}\pm i2\pi n}=-i$

$f(z)\Big|_{z_0}=\underbrace{\sum_{n=0}^{\infty}a_n(z-z_0)^n}_{\text{analytical part}}+\underbrace{\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}}_{\text{principal part}},\ a_{\pm n}=\dfrac{1}{2\pi i}\oint_\gamma\dfrac{f(z)}{(z-z_0)^{\pm n+1}}dz$

$f(z)$ complex differentiable at z_0 if $\exists f'(z_0)=\lim_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$

$f:U\subseteq\mathbb{C}\to\mathbb{C}$, U open set: f holomorphic on U if $\forall z_0\in U,\ \exists f'(z_0)$

f holomorphic at z_0 if f holomorphic on some neighborhood of z_0

$f(x+iy)=u(x,y)+iv(x,y)$ holomorphic $\Rightarrow u,v$ satisfy Cauchy-Riemann (C.R.)

C.R.: $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$ or $\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta}$ and $\dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}$

$\partial_x u,\partial_y u,\partial_x v,\partial_y v$ continuous and satisfy C.R. $\Rightarrow f$ holomorphic

$\forall f$ holomorphic: u,v harmonic on $\mathbb{R}^2\Leftrightarrow\nabla^2u=0,\nabla^2v=0$

$\forall f$ holomorphic and γ enclosing no holes: $\oint_\gamma f(z)dz=0$

$\oint_\gamma\dfrac{f(z)}{z-z_0}dz=2\pi if(z_0)$ and $\oint_\gamma\dfrac{f(z)}{(z-z_0)^{n+1}}dz=\dfrac{2\pi i}{n!}f^{(n)}(z_0)$

$\forall f$, if Γ continuously differentiable: $\int_\sigma f(z)dz=\int_a^b f(\Gamma(t))\cdot\Gamma'(t)$

$\ell(\Gamma)=\int_a^b|\Gamma'(t)|dt\equiv$ contour length $\ $ generally: $\Gamma(t)=\Gamma_R=z_0+Re^{it},\ \ell(\Gamma_R)=Rt_{\max}$

$\forall f$ holomorphic on U, except at a finite number of isolated singularities z_k:

$\oint_\gamma f(z)dz=2\pi i\sum_k\mathrm{Res}(f,z_k),\ \mathrm{Res}(f,z_k)\equiv$ residue of f at z_k

$\equiv$ coefficient c_{-1} of $(z-z_k)^{-1}$ in Laurent series of f around z_k

f holomorphic on U except at $a\in U\equiv f\in\mathcal{O}(U\setminus\{a\})$, possible isolated singularities:

- a removable singularity $\Leftrightarrow\exists g\in\mathcal{O}(U)\mid f(z)=g(z)\ \forall z\in U\setminus\{a\}$
- a pole $\Leftrightarrow\exists g\in\mathcal{O}(U),g(a)\neq0\mid f(z)=\frac{g(a)}{(z-a)^m}\ \forall z\in U\setminus\{a\};m\equiv$ pole order
- a essential singularity $\Leftrightarrow$ Laurent series principal part has ∞ terms

For poles z_j of order m: $\mathrm{Res}(f,z_j)=\dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}((z-z_j)^mf(z))\Big|_{z_j}$

$\nexists$ such formula for essential singularities

Estimation lemma: $f(z)\in\mathbb{C}$, continuous on Γ and $\exists\ M\in\mathbb{R}$ such that :

$|f(z)|\le M\ \forall z\in\Gamma\Rightarrow\left|\int_\Gamma f(z)\,dz\right|\le M\cdot l(\Gamma),\ M:=\sup_{z\in\Gamma}|f(z)|$

$\therefore$ if $|f(z)|\le\frac{C}{|z|^p},p>1;C_{R^+}\equiv\Gamma_R,\ t\in[0,\pi],\ z_0=0\Rightarrow\left|\int_{C_{R^+}}f(z)\,dz\right|\xrightarrow{R\to\infty}0$

Jordan's lemma: $f(z)=e^{i\alpha z}g(z)\in\mathbb{C},\alpha>0$, continuous on $C_{R^+}\Rightarrow$

$\left|\int_{C_{R^+}}f(z)\,dz\right|\le\dfrac{\pi}{\alpha}M_R,\ M_R:=\max_{\theta\in[0,\pi]}\left|g\left(Re^{i\theta}\right)\right|\ \therefore$ if $M_R\xrightarrow{R\to\infty}0\Rightarrow\int_{C_{R^+}}f(z)\,dz\xrightarrow{R\to\infty}0$

Analogous for $C_{R^-}\equiv\Gamma_R,\ t\in[\pi,2\pi],z_0=0$ when $\alpha<0$

Fourier Analysis $\ (\xi,x\in\mathbb{R})$

$\mathcal{F}\{f(x)\}(\xi)=\hat{f}(\xi)=\int_{-\infty}^{+\infty}f(x)e^{-i2\pi\xi x}dx\ \ \ \mathcal{F}^{-1}\{f(\xi)\}(x)=\int_{-\infty}^{+\infty}f(\xi)e^{+i2\pi\xi x}d\xi$

$f(x-x_0)\overset{\mathcal{F}}{\Longleftrightarrow}e^{-i2\pi x_0\xi}\hat{f}(\xi)\ \ \ e^{i2\pi\xi_0x}f(x)\overset{\mathcal{F}}{\Longleftrightarrow}\hat{f}(\xi-\xi_0)\ \ \ f(ax)\overset{\mathcal{F}}{\Longleftrightarrow}\dfrac{1}{|a|}\hat{f}\left(\dfrac{\xi}{a}\right)$

$f(x)\in\mathbb{R}\Rightarrow\hat{f}(-\xi)=\overline{\hat{f}(\xi)}\ \ \mathcal{F}^{-1}f(x)=\mathcal{F}(f(-x))\ \ \mathcal{F}(f(-x))=(\mathcal{F}f)(-x)\ \ \mathcal{F}^2f(x)=f(-x)$

Convolution

$(f*g)(x)=\int_{-\infty}^{+\infty}f(x-y)g(y)\,dy\ \ \ f*g=g*f\ \ \ (f*g)*h=f(g*h)$

$f*(g+h)=f*g+f*h\ \ \mathcal{F}(f*g)=(\mathcal{F}f)(\mathcal{F}g)\ \ \mathcal{F}(fg)=\mathcal{F}f*\mathcal{F}g$

Dirac Delta

$\delta(x)=\begin{cases}0, & x\neq0\\ \infty, & x=0\end{cases}\ \ \int_{-\infty}^{+\infty}\delta(x)\,dx=1\ \ \mathcal{F}\{\delta(x)\}(\xi)=1\Leftrightarrow\mathcal{F}^{-1}\{1\}(x)=\delta(x)$

$\delta(ax)=\dfrac{1}{|a|}\delta(x)\ \ \ h(x)\delta(x)=h(0)\delta(x)\ \ \ \delta(x)*f(x)=f(x)$

$\delta(x-a)*f(x)=f(x-a)\ \ \ \delta(x-a)*\delta(x-b)=\delta(x-(a+b))$

$\int_{-\infty}^{+\infty}\delta(x-a)\,f(x)\,dx=f(a)\ \ \ \int_{-\infty}^{+\infty}\delta(x)\,f(x+a)\,dx=f(a)$

Real integrals via residues

($\mathfrak{R}\equiv$ rational function, no essential singularities nor branch cuts inside contour)

Rational trigonometric integrals (**c.v.**: $z=e^{i\theta}$)

$\mathfrak{R}$ with no singularities on contour $\equiv$ unit circle: $x^2+y^2=1$

$\int_0^{2\pi}\mathfrak{R}(\cos\theta,\sin\theta)d\theta=2\pi\sum_{|z_k|<1}\mathrm{Res}\left[\dfrac{1}{z}\mathfrak{R}\left(\dfrac{z+z^{-1}}{2},\dfrac{z-z^{-1}}{2i}\right),z_k\right]\in\mathbb{R}$

Rational improper integrals

$\mathfrak{R}$ with no poles in $\mathbb{R}$, $\lim_{x\to\infty}x\,\mathfrak{R}(x)=0$, contour $\equiv$ upper/lower half-plane

$\int_{-\infty}^{+\infty}\mathfrak{R}(x)dx=2\pi i\sum_{\Im[z_k]>0}\mathrm{Res}\,[\mathfrak{R}(z),z_k]=-2\pi i\sum_{\Im[z_k]<0}\mathrm{Res}\,[\mathfrak{R}(z),z_k]\in\mathbb{R}$

Cauchy principal value

f continuous on $[a,b]\in\mathbb{R}$ except at isolated poles $\{x_k\}$, $m=1$

$\int_a^b f(x)\,dx\equiv\mathbf{P.V.}\int_a^b f(x)\,dx\equiv\lim_{\varepsilon\to0^+}\left\{\int_a^{x_1-\varepsilon}f(x)\,dx+\int_{x_1+\varepsilon}^{x_2-\varepsilon}f(x)\,dx+\cdots+\int_{x_n+\varepsilon}^{b}f(x)\,dx\right\}$

Improper integrals with principal value

f holomorphic, except at isolated poles $\{z_k\}$, $m=1$; $\lim_{|z|\to\infty}z\,f(z)=0$ on $\Im z>0\Rightarrow$

$\int_{-\infty}^{+\infty}f(x)\,dx=2\pi i\sum_{\Im[z_k]>0}\mathrm{Res}\,[f(z),z_k]+\pi i\sum_{\Im[z_k]=0}\mathrm{Res}\,[f(z),z_k]\in\mathbb{R}$ Analogous for $\Im z<0$, just multiply by (-1)

Semi-improper integrals with principal value

f holomorphic, except at isolated singularities $\{z_k\}\not\subset\mathbb{R}^+$, and except at isolated poles $\{x_k\}\in\mathbb{R}^+,m=1$; $\lim_{|z|\to\infty}z\,f(z)\neq\infty$; $\lim_{z\to0}f(z)\neq\infty\Rightarrow\forall\alpha\in(0,1)$:

$\int_0^\infty\dfrac{f(x)}{x^\alpha}dx=\dfrac{2\pi i}{1-e^{-2\pi i\alpha}}\sum_k\mathrm{Res}\left[\dfrac{f(z)}{z^\alpha},z_k\right]+\dfrac{\pi i(1+e^{-2\pi i\alpha})}{1-e^{-2\pi i\alpha}}\sum_k\mathrm{Res}\left[\dfrac{f(z)}{z^\alpha},x_k\right]\in\mathbb{R}$

Fourier-like integrals

f holomorphic on $\Im z\gtrless0$, except at isolated sigularities $\{z_k\}$, and with isolated poles $\{x_k\}\in\mathbb{R}$, $m=1$; $\lim_{|z|\to\infty,\Im z\gtrless0}f(z)=0\Rightarrow$

$\int_{-\infty}^{+\infty}f(x)e^{\pm ikx}\,dx=\pm2\pi i\sum_{\Im z_k\gtrless0}\mathrm{Res}\left[f(z)e^{\pm ikz},z_k\right]\pm\pi i\sum_k\mathrm{Res}\left[f(z)e^{\pm ikz},x_k\right]$

Trick $\ \int_{-\infty}^{+\infty}\cos x\,dx=\Re\left[\int_{-\infty}^{+\infty}e^{ix}\,dx\right]\ \ \ \int_{-\infty}^{+\infty}\sin x\,dx=\Im\left[\int_{-\infty}^{+\infty}e^{ix}\,dx\right]$

Pauli Matrices σ $(\mathrm{tr}\,\sigma_j=0,\ \det\sigma_j=-1,\ \sigma_j^2=\mathbf{I}_2,\ \sigma_j=\sigma_j^{\dagger}=\sigma_j^{-1})$

$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\ \ \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\ \ \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ \ \sigma_j=\begin{pmatrix}\delta_{j3} & \delta_{j1}-i\,\delta_{j2}\\ \delta_{j1}+i\,\delta_{j2} & -\delta_{j3}\end{pmatrix}$

$\sigma_j\sigma_k=\delta_{jk}+i\epsilon_{jkl}\sigma_l\ \ \ [\sigma_j,\sigma_k]=2i\epsilon_{jkl}\sigma_l\ \ \ \{\sigma_j,\sigma_k\}=2\delta_{jk}\mathbf{I}_2\ \ \ i\sigma_1\sigma_2\sigma_3=-\mathbf{I}_2$

$\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)=\sigma_1\hat{x}_1+\sigma_2\hat{x}_2+\sigma_3\hat{x}_3\ \ \ \vec{a}\cdot\vec{\sigma}=\begin{pmatrix}a_3 & a_1-ia_2\\ a_1+ia_2 & -a_3\end{pmatrix}$

$\det(\vec{a}\cdot\vec{\sigma})=-|\vec{a}|^2\ \ \ \frac{1}{2}\,\mathrm{tr}\,((\vec{a}\cdot\vec{\sigma})\vec{\sigma})=\vec{a}\ \ \ \lambda^{\text{eigen}}_{(\vec{a}\cdot\vec{\sigma})}=\pm|\vec{a}|\ \ \ [\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}]=2i(\vec{a}\times\vec{b})\cdot\sigma$

Tensors (generalizable to $\mathbb{R}^n$)

Definition and Operations Vectors can expressed in different bases: $\{e_1, e_2\}, \{e_{1'}, e_{2'}\}, \ldots$

$$\vec{A} = A^1 e_1 + A^2 e_2 = (e_1, e_2)(A^1, A^2)^{\mathrm{T}} = A^{1'} e_{1'} + A^{2'} e_{2'} = (e_{1'}, e_{2'})(A^{1'}, \ A^{2'})^{\mathrm{T}}$$

Einstein convention: summation over repeated indices (up - down)

inverse: primed $\leftrightarrow$ unprimed, transpose: upper $\leftrightarrow$ lower

$$M = (M^{i'}_j) = \begin{pmatrix} M^{1'}_1 & M^{2'}_1 \\ M^{1'}_2 & M^{2'}_2 \end{pmatrix} \quad (M^{-1})^{\mathrm{T}} = (M^j_{i'}) = \begin{pmatrix} M^1_{1'} & M^2_{1'} \\ M^1_{2'} & M^2_{2'} \end{pmatrix} \quad M^j_{i'} M^{i'}_k = \delta^j_k$$

Change of basis: $A^{i'} = M^{i'}_j A^j$, $e_{i'} = M^j_{i'} e_j$, $\det M \neq 0$

Covariant v^i: transform against basis vectors $\{e_i\}$, with $M^{i'}_j$

Covariant w_i: transform with basis vectors $\{e_i\}$, with $M^j_{i'}$

Dot product via metric: $g_{ij} = e_i \cdot e_j \quad g = g^{\mathrm{T}} \quad g^{-1} \to$ raises indices

$$\vec{A} \cdot \vec{B} = A^1 B^1 g_{11} + A^1 B^2 g_{12} + A^2 B^1 g_{21} + A^2 B^2 g_{22} = A^i g_{ij} B^j = \vec{A}^{\mathrm{T}} g \vec{B} \quad \|\vec{A}\| = \sqrt{\vec{A} \cdot \vec{A}}$$

Coordinate metrics in flat euclidean metric:

$g_{\text{cartesian}} = \delta_{ij} = \mathbb{I}_n \quad g_{\text{spherical}} = \operatorname{diag}(1, r^2, r^2 \sin^2\theta) \quad g_{\text{cylindrical}} = \operatorname{diag}(1, \rho^2, 1)$

Inverses: $g^{-1}_{\text{cart}} = \delta_{ij} \quad g^{-1}_{\text{sph}} = \operatorname{diag}(1, 1/r^2, 1/r^2 sin^2\theta) \quad g^{-1}_{\text{cyl}} = \operatorname{diag}(1, 1/\rho^2, 1)$

Dual Basis $\{e^1, e^2\}$ dual to $\{e_1, e_2\}$ $\quad e^i \cdot e_j = \delta^i_j$

Relation with metric: $e^i = g^{ij} e_j \quad g^{ij} \equiv$ inverse of the metric

$\vec{A} = A^i e_i = A_i g^{ij} e_j$ **Index lowering:** $A_i = g_{ij} A^j$ **Index raising:** $A^i = g^{ij} A_j$

Metric under change of basis: $g_{i'j'} = M^i_{i'} M^j_{j'} g_{ij} \Leftrightarrow g' = (M^{-1})^T g M^{-1}$

Dot product is invariant under change of basis

Tensor: Any object that transforms as: $T'_{i'j'} = M^i_{i'} M^j_{j'} T_{ij}$ is a tensor

Tensor product properties: ($\vec{A}, \vec{B}, \vec{C}$ vectors, $\lambda \in \mathbb{R}$, $V, V \otimes V$ vector spaces)

1.	$(\lambda\vec{A}) \otimes \vec{B} = \lambda(\vec{A} \otimes \vec{B})$	4.	$(\vec{A} + \vec{B}) \otimes \vec{C} = \vec{A} \otimes \vec{C} + \vec{B} \otimes \vec{C}$
2.	$\vec{A} \otimes (\lambda\vec{B}) = \lambda(\vec{A} \otimes \vec{B})$	5.	$\vec{A} \otimes (\vec{B} + \vec{C}) = \vec{A} \otimes \vec{B} + \vec{A} \otimes \vec{C}$
3.	$\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$	6.	$(\vec{A} \otimes \vec{B})(\vec{C} \otimes \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D})$

Bases of tensor product space $V \otimes V$: $\{e_i \otimes e^j\}, \ \{e^i \otimes e_j\}, \ \{e^i \otimes e^j\}, \ \{e_i \otimes e_j\}$

Equivalent definition of tensor: Element of $V \otimes V$ formed as a linear combination of the basis elements: $\mathcal{T} = T_{11}\, e^1 \otimes e^1 + T_{12}\, e^1 \otimes e^2 + T_{21}\, e^2 \otimes e^1 + T_{22}\, e^2 \otimes e^2$

In compact and general notation: $\mathcal{T} = T_{ij}\, e^i \otimes e^j$ (generalizable to the other bases).

A tensor of type (r, s) has r contravariant and s covariant indexes.

$$\mathcal{T} \cdot \mathcal{V} = T_{ij} V_{kl} g^{ik} V^{jl} = T_{ij} V^{ij} \qquad T_{ljk} = g_{il} T^i{}_{jk} \qquad T^i{}_j{}^l = g^{kl} T^i{}_{jk}$$

Symmetric: $S_{\alpha\beta} = S_{\beta\alpha} \quad S^{\alpha\beta} = S^{\beta\alpha} \Rightarrow 2S^{\alpha\beta} T_{\alpha\beta} = S^{\alpha\beta}(T_{\alpha\beta} + T_{\beta\alpha})$

Antisymmetric: $A_{\alpha\beta} = -A_{\beta\alpha} \quad A^{\alpha\beta} = -A^{\beta\alpha} \Rightarrow 2A^{\alpha\beta} T_{\alpha\beta} = A^{\alpha\beta}(T_{\alpha\beta} - T_{\beta\alpha})$

Tensor Extension to a Manifold

Manifold $\mathcal{M}$: a surface (or hypersurface) embedded in a higher-dimensional space, Cartesian or Lorentzian. Before we were on the tangent plane to the manifold $T_P\mathcal{M}$. The tangent bundle of $\mathcal{M}$ is $\bigcup_{P \in \mathcal{M}} T_P\mathcal{M}$ and it has double the dimension of $\mathcal{M}$.

1. We need the expression for the coordinate change: $x^{i'} = x^{i'}(x^1, \ldots, x^n)$

This function can be understood as a parametrization over the manifold.

It allows tensors to be consistently defined over the whole manifold.

2. Compute the Jacobian matrix of the transformation and its inverse:

$$M = M^{i'}_j = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \cdots & \frac{\partial x^{1'}}{\partial x^n} \\ \cdots & \ddots & \cdots \\ \frac{\partial x^{n'}}{\partial x^1} & \cdots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \quad M^{-1} = M^i_{j'} = \begin{pmatrix} \frac{\partial x^1}{\partial x^{1'}} & \cdots & \frac{\partial x^1}{\partial x^{n'}} \\ \cdots & \ddots & \cdots \\ \frac{\partial x^n}{\partial x^{1'}} & \cdots & \frac{\partial x^n}{\partial x^{n'}} \end{pmatrix}$$

Change of coordinate matrices behave as a change of basis matrices.

3. We can construct the basis vectors as before: $e_{i'} = M^j_{i'} e_j$. In this way, each vector $e_{i'}$ moves in the direction of change of $x^{i'}$, and is constant in $x^j \ \forall j \neq i$.

NOTE: When computing the basis vectors, use $M^j_{i'} = (M^{-1})^{\mathrm{T}}$, not $M^{i'}_j = M$.

How to obtain the metric? We need to parametrize the surface by embedding it in a Cartesian space of higher dimension. This space has coordinates X^i

1. We parametrize the surface: $X^i = X^i(x^j)$.

2. The tangent vectors to the surface will be: $e_i = \dfrac{\partial X^i}{\partial x^i} e_{X^i}$

3. By the very definition of the metric: $g_{ij} = e_i \cdot e_j \quad e_{X^i} \cdot e_{X^j} = \delta_{X^i X^j}$

ODEs ($\alpha, \beta, c \in \mathbb{R}$, $\lambda \in \mathbb{C} \Leftrightarrow \lambda = \alpha + i\beta$, $y' = \frac{dy}{dx}$, sol $\equiv$ solution, const $\equiv$ constant)

First Order Equations

Separable $y' = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx$

Linear $y' + a(x)y = r(x) \Rightarrow y(x) = \left[\int r(x)\, e^{\int a(x)\, dx}\, dx + C\right] e^{-\int a(x)\, dx}$

Exact $M(x, y)\, dx + N(x, y)\, dy = 0$; if $\partial_y M = \partial_x N \Rightarrow \exists f(x, y) \equiv$ const, with: $\partial_x f = M \quad \partial_y f = N$ (Solve for f)

Non-Exact $M(x, y)\, dx + N(x, y)\, dy \neq 0$; if $\begin{cases} \frac{\partial_y M - \partial_x N}{N} = g(x) \Rightarrow \mu = e^{\int g\, dx} \\ \frac{\partial_x N - \partial_y M}{M} = h(y) \Rightarrow \mu = e^{\int h\, dy} \end{cases} \Rightarrow$

$\Rightarrow \mu[M(x, y)\, dx + N(x, y)\, dy] = 0 \Rightarrow$ Exact

Bernoulli $y' + a(x)y = r(x)y^n \to$ c.v. $z := y^{1-n} \Rightarrow$ Linear

Important Concepts

Linear: $y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = r(x)$

$r(x) = 0 \forall x \Rightarrow$ Homogeneous (homo) $\quad r(x) \not\equiv 0 \Rightarrow$ Inhomogeneous (inhomo)

$$\{y_i(x)\}_1^n \text{ Linearly Independent (LI)} \Leftrightarrow \mathcal{W}(\{y_i(x)\}) := \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

$\{y_i(x)\}$ LI sols of homo ODE $\Rightarrow y_h = c_1 y_1 + \cdots + c_n y_n$

$y_p \equiv$ particular sol of inhomo ODE $\quad y = y_h + y_p \equiv$ general sol of the ODE

Higher Order Linear ODEs

Constant Coefficients (for homo sol) $\{a_i\}_1^n \equiv$ const $\Rightarrow$ Let $y_p = e^{\lambda x}$, substitute $\Rightarrow$

$\Rightarrow$ solve for $\{\lambda_i\}$: $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$

- $\{\lambda_i\} \in \mathbb{R}$, no repetition: $y_h = C_1 e^{\lambda_1 x} + \cdots + C_n e^{\lambda_n x}$
- $\{\lambda_i\} \in \mathbb{R}$, k multiplicity: $y_h = (C_1 + C_2 x + \cdots + C_k x^{k-1})e^{\lambda_1 x} + \cdots + C_n e^{\lambda_n x}$
- $\{\lambda_i\} \in \mathbb{C}$, k multiplicity : $y_h = e^{\alpha x}\Big[(A_1 + A_2 x + \cdots + A_k x^{k-1})\cos(\beta x) +$ $+ (B_1 + B_2 x + \cdots + B_k x^{k-1})\sin(\beta x)\Big] + \cdots + C_n e^{\lambda_n x}$

Undetermined coefficients method (for inhomo sol) $\{a_i\}_1^n \equiv$ const $\Rightarrow$

$\Rightarrow$ Let y_p be the function shown in the table, substitute, and find the consts

$r(x)$	Roots	Form of y_p
$P_m(x)$	1. 0 is not a root 2. 0 is a root of multiplicity s	$Q_m(x)$ $x^s Q_m(x)$
$P_m(x)e^{\alpha x}$	1. α is not a root 2. α is a root of multiplicity s	$Q_m(x)e^{\alpha x}$ $x^s Q_m(x)e^{\alpha x}$
$P_m(x)\cos\beta x + T_n(x)\sin\beta x$	1. $\pm i\beta$ are not roots 2. $\pm i\beta$ are roots of multiplicity s	$Q_k(x)\cos\beta x + R_k(x)\sin\beta x$ $x^s(Q_k(x)\cos\beta x + R_k(x)\sin\beta x)$
$e^{\alpha x}(P_m(x)\cos\beta x + T_n(x)\sin\beta x)$	1. $\alpha \pm i\beta$ are not roots 2. $\alpha \pm i\beta$ are roots of multiplicity s	$(Q_k(x)\cos\beta x + R_k(x)\sin\beta x)e^{\alpha x}$ $x^s(Q_k(x)\cos\beta x + R_k(x)\sin\beta x)e^{\alpha x}$

$m, n, k \equiv$ degree of polynomes $\quad k = \max\{m, n\}$

$Q(x), R(x)$ must have all the terms: *i.e.* $Q_m(x) = A_1 + A_2 x + \cdots + A_{n+1}x^n$

Variation of parameters (for inhomo sol, $r(x)$ not in table) $\{a_i\}_1^n \equiv$ const $\Rightarrow$

$\Rightarrow$ Let: $y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$; $\{y_i\}$ LI sols of homo

$$\text{Impose: } \begin{cases} u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0 \\ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0 \\ \cdots \\ u_1' y_1^{(n-2)} + \cdots + u_n' y_n^{(n-2)} = 0 \\ u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = r(x) \end{cases} \Rightarrow \text{(system of } n \text{ equations)}$$

$u_i'(x) = \dfrac{\mathcal{W}(x)}{\mathcal{W}_i(x)} \quad \mathcal{W}_i(x) \equiv \mathcal{W}(x)$ with i-th column: $(0, 0, \cdots, r(x))^{\mathrm{T}} \quad u_i(x) = \int u_i'\, dx$

Euler Equation $x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = 0$

c.v. $x = e^t \Rightarrow y(x) = u(t)$, then $x\dfrac{d}{dx} \to \dfrac{d}{dt} \Rightarrow$ Transformed to const coeff eq in t :

$$y = u(t), \ \frac{dy}{dx} = \frac{1}{x}\frac{du}{dt}, \ \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right), \cdots \Rightarrow \text{Solve in } t, \text{ then } y(x) = u(\ln x)$$

Alternative: $y_h = x^\lambda$, substitute $x^n[\lambda(\lambda - 1)\cdots(\lambda - n + 1)] + \cdots + a_n = 0$

- $\{\lambda_i\} \in \mathbb{R}$, no repetition: $y_h = C_1 x^{\lambda_1} + \cdots + C_n x^{\lambda_n}$
- $\{\lambda_i\} \in \mathbb{R}$, k multiplicity: $y_h = (C_1 + C_2 \ln x + \cdots + C_k(\ln x)^{k-1})x^{\lambda_1} + \cdots + C_n x^{\lambda_n}$
- $\{\lambda_i\} \in \mathbb{C}$, k multiplicity: $y_h = x^\alpha\Big[\left(A_1 + A_2 \ln x + \cdots + A_k(\ln x)^{k-1}\right)\cos(\beta \ln x) +$ $+ \left(B_1 + B_2 \ln x + \cdots + B_k(\ln x)^{k-1}\right)\sin(\beta \ln x)\Big] + \cdots + C_n x^{\lambda_n}$

Systems of First-Order Linear ODEs $\quad e^{Ax} = I + Ax + (Ax)^2/2! + (Ax)^3/3! + \ldots$

(homo) $\vec{y}' = A\vec{y} \Rightarrow \vec{y}_h(x) = e^{Ax}\vec{c} \quad A_{n \times n}$ const; if diagonalizable: $A = PDP^{-1} \Rightarrow$

$\Rightarrow e^{Ax} = Pe^{Dx}P^{-1}$ with $e^{Dx} = \operatorname{diag}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$

(inhomo) $\vec{y}' = A\vec{y} + \vec{r}(x) \Rightarrow \vec{y}_p(x) = e^{Ax}\int e^{-Ax}\vec{r}(x)\, dx \Rightarrow \vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x)$

Quaternions $\mathbb{H}$ ($\alpha, \beta, \gamma, \delta, \lambda, \mu \in \mathbb{R}$, $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ basis of $\mathbb{H}$, $q, p \in \mathbb{H}$)

$q = \alpha + \beta\,\mathbf{i} + \gamma\,\mathbf{j} + \delta\,\mathbf{k} \quad \Re[q] = \alpha \equiv$ real part $\quad \Im[q] = \beta\,\mathbf{i} + \gamma\,\mathbf{j} + \delta\,\mathbf{k} \equiv$ vector part

$\overline{q} = \alpha - \beta\,\mathbf{i} - \gamma\,\mathbf{j} - \delta\,\mathbf{k} \quad \|q\|^2 = q\overline{q} = \overline{q}q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \quad q^{-1} = \dfrac{1}{\|q\|^2}\overline{q}$

$\mathbf{U}_q = \dfrac{q}{\|q\|} \equiv$ versor of q, $\|\mathbf{U}_q\| = 1 \Rightarrow \mathbf{U}_q \equiv$ unit quaternion $\quad \alpha q = q\alpha$

$$\lambda(\alpha_1 + \beta_1\,\mathbf{i} + \gamma_1\,\mathbf{j} + \delta_1\,\mathbf{k}) + \mu(\alpha_2 + \beta_2\,\mathbf{i} + \gamma_2\,\mathbf{j} + \delta_2\,\mathbf{k}) = \\ = (\lambda\alpha_1 + \mu\alpha_2) + (\lambda\beta_1 + \mu\beta_2)]\mathbf{i} + (\lambda\gamma_1 + \mu\gamma_2)\mathbf{j} + (\lambda\delta_1 + \mu\delta_2)\mathbf{k}$$

$\mathbf{i}1 = 1\mathbf{i} = \mathbf{i} \quad \mathbf{j}1 = 1\mathbf{j} = \mathbf{j} \quad \mathbf{k}1 = 1\mathbf{k} = \mathbf{k} \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$

$\mathbf{ij} = -\mathbf{ji} = \mathbf{k} \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \quad \mathbf{ijk} = -1$

$q = (r, \vec{v})$, $q \in \mathbb{H}$, $r = \Re[q]$, $\vec{v} = \Im[q] \quad (r_1, \vec{v}_1) + (r_2, \vec{v}_2) = (r_1 + r_2, \vec{v}_1 + \vec{v}_2)$

$(r_1, \vec{v}_1)(r_2, \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2, r_1\vec{v_2} + r_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2) \quad \|pq\| = \|p\|\|q\|$

$\overline{pq} = \overline{q}\,\overline{p} \quad \overline{q} = -\frac{1}{2}(q + \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k}) \quad \Re[q] = \frac{1}{2}(q + \overline{q}) \quad \Im[q] = \frac{1}{2}(q - \overline{q})$

Matrix representation: $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \mapsto \{\mathbf{I}, \sigma_1, \sigma_2, \sigma_3\}$, $\sigma \equiv$ Pauli matrices

$$q = \begin{bmatrix} \alpha + \beta\,\mathbf{i} & \gamma + \delta\,\mathbf{i} \\ -\gamma + \delta\,\mathbf{i} & \alpha - \beta\,\mathbf{i} \end{bmatrix} = \alpha\mathbf{I} + \beta i\sigma_3 + \gamma i\sigma_2 + \delta i\sigma_1 \quad \|q\|^2 = \det q \quad \Re[q] = \tfrac{1}{2}\operatorname{tr} q \quad \overline{q} = q^\dagger$$

Quantity	**SI Unit**	**Quantity**	SI Unit
Length	m	Mass	kg
Time	s	Temperature	K
Electric current	A	Amount of substance	mol
Luminous intensity	cd	Force	N=kg·m/s^2
Pressure	Pa=kg/(m·s^2)	Energy	J=kg·m^2/s^2
Power	W=kg·m^2/s^3	Electric charge	C=A·s
Voltage	V=kg·m^2/(A·s^3)	Resistance	(Ω)=kg·m^2/(A^2·s^3)
Capacitance	F=A^2·s^4/(kg·m^2)	Magnetic flux	Wb=kg·m^2/(A·s^2)
Mag. flux density	T=kg/(A·s^2)	Inductance	H=kg·m^2/(A^2·s^2)
Frequency	Hz=1/s	Radioactivity	Bq=1/s
Absorbed dose	Gy=m^2/s^2	Dose equivalent	Sv=m^2/s^2
Catalytic activity	kat=mol/s	Angular velocity	rad/s
Angular acceleration	rad/s^2	Dynamic viscosity	Pa·s=kg/(m·s)
Thermal conductivity	W/m·K=kg·m/(s^3·K)	Spec. heat capacity	J/kg·K=m^2/(s^2·K)
Entropy	J/K=kg·m^2/(s^2·K)	Heat flux density	W/m^2=kg/s^3
Luminance	cd/m^2	Illuminance	lx=cd·sr/m^2
Surface tension	N/m=kg/s^2	Moment of inertia	kg·m^2
Momentum	kg·m/s	Impulse	N·s=kg·m/s

Quantity	**Symbol**	**Value**	**Unit**
speed of light in vacuum	c	299 792 458	m s^{-1}
constant of gravitation	G	6.67430×10^{-11}	m^3 kg^{-1} s^{-2}
Planck constant	h	$6.62607015 \times 10^{-34}$	J Hz^{-1}
reduced Planck constant	$\hbar$	$1.054571817 \times 10^{-34}$	J s
elementary charge	e	$1.602176634 \times 10^{-19}$	C
vacuum magnetic permeability	$\mu_0 = 4\pi\alpha\hbar/e^2c$	$1.25663706127 \times 10^{-6}$	N A^{-2}
vacuum electric permittivity	$\varepsilon_0 = 1/\mu_0 c^2$	$8.8541878128 \times 10^{-12}$	F m^{-1}
vacuum impedance	$Z_0 = \mu_0 c$	376.73031346177	Ω
Josephson constant	$K_J = 2e/h$	$483\,597.8484 \times 10^9$	Hz V^{-1}
von Klitzing constant	$R_K = 2\pi\hbar/e^2$	25 812.80745	Ω
magnetic flux quantum	$\Phi_0 = 2\pi\hbar/2e$	$2.067833848 \times 10^{-15}$	Wb
conductance quantum	$G_0 = 2e^2/2\pi\hbar$	$7.748091729 \times 10^{-5}$	S
inverse conductance quantum	G_0^{-1}	12 906.40372	Ω
electron mass	m_e	$9.1093837139 \times 10^{-31}$	kg
proton mass	m_p	$1.67262192595 \times 10^{-27}$	kg
proton-electron mass ratio	m_p/m_e	1836.152673426	—
fine-structure constant	$\alpha = e^2/4\pi\varepsilon_0\hbar c$	$7.2973525643 \times 10^{-3}$	—
inverse fine-structure	α^{-1}	137.035999177	—
Bohr Radius	$a_0 = \hbar/m_e c\alpha$	$5.29177210544 \times 10^{-11}$	m
classical electron radius	$r_e = \alpha^2 a_0$	$2.8179403205 \times 10^{-15}$	m
Bohr Magneton	$\mu_B = e\hbar/2m_e$	$9.2740100657 \times 10^{-24}$	J T^{-1}
Nuclear Magneton	$\mu_N = e\hbar/2m_p$	$5.0507837393 \times 10^{-27}$	J T^{-1}
Rydberg frequency	$cR_\infty = \frac{\alpha^2 m_e c^2}{2h}$	$3.28984196025 \times 10^{15}$	Hz
Hartree energy	$E_h = \alpha^2 hcR_\infty$	$4.3597447221 \times 10^{-18}$	J
Boltzmann constant	k_B	1.380649×10^{-23}	J K^{-1}
Stefan–Boltzmann constant	$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$	$5.670374419 \times 10^{-8}$	W m^{-2} K^{-4}
Avogadro constant	N_A	$6.02214076 \times 10^{23}$	mol^{-1}
molar gas constant	$R = N_A k_B$	8.314462618	J mol^{-1} K^{-1}
Faraday constant	$F = N_A e$	96 485.33212	C mol^{-1}
Non-SI units			
h-bar c	$\hbar c$	197.3269804	eV nm=MeV fm
electron volt	eV	$1.602176634 \times 10^{-19}$	J
atomic mass unit	u	$1.66053906892 \times 10^{-27}$	kg
atomic mass unit	u	931.49410242	MeV c^{-2}
Fermi coupling constant	$G_F^0 = G_F/(\hbar c)^3$	1.1663787×10^{-5}	GeV^{-2}

Integral Calculator Wolfram Alpha Phase Plane Plotter CODATA values