

Coordinate Systems
Spherical ($\theta \in [0,\pi], \phi \in [0,2\pi)$)

$$\begin{cases} x=r\sin\theta\cos\phi \\ y=r\sin\theta\sin\phi \\ z=r\cos\theta \end{cases}$$
$$\begin{cases} \hat{\mathbf{x}}=\sin\theta\cos\phi\,\hat{\mathbf{r}}+\cos\theta\cos\phi\,\hat{\boldsymbol{\theta}}-\sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}}=\sin\theta\sin\phi\,\hat{\mathbf{r}}+\cos\theta\sin\phi\,\hat{\boldsymbol{\theta}}+\cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}}=\cos\theta\,\hat{\mathbf{r}}-\sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$
$$\begin{cases} r=\sqrt{x^2+y^2+z^2} \\ \theta=\arctan(\sqrt{x^2+y^2}/z) \\ \phi=\arctan(y/x) \end{cases}$$
$$\begin{cases} \hat{\mathbf{r}}=\sin\theta\cos\phi\,\hat{\mathbf{x}}+\sin\theta\sin\phi\,\hat{\mathbf{y}}+\cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}}=\cos\theta\cos\phi\,\hat{\mathbf{x}}+\cos\theta\sin\phi\,\hat{\mathbf{y}}-\sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}}=-\sin\phi\,\hat{\mathbf{x}}+\cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical ($\rho \in [0,\infty), \phi \in [0,2\pi)$)

$$\begin{cases} x=\rho\cos\phi \\ y=\rho\sin\phi \\ z=z \end{cases}$$
$$\begin{cases} \hat{\mathbf{x}}=\cos\phi\,\hat{\boldsymbol{\rho}}-\sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}}=\sin\phi\,\hat{\boldsymbol{\rho}}+\cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}}=\hat{\mathbf{z}} \end{cases}$$
$$\begin{cases} \rho=\sqrt{x^2+y^2} \\ \phi=\arctan(y/x) \\ z=z \end{cases}$$
$$\begin{cases} \hat{\boldsymbol{\rho}}=\cos\phi\,\hat{\mathbf{x}}+\sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}}=-\sin\phi\,\hat{\mathbf{x}}+\cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}}=\hat{\mathbf{z}} \end{cases}$$

Vector Derivatives
Cartesian ($d\mathbf{l}=dx\,\hat{\mathbf{x}}+dy\,\hat{\mathbf{y}}+dz\,\hat{\mathbf{z}}, dV=dx\,dy\,dz$)

Gradient: $\boldsymbol{\nabla} f = \partial_x f\,\hat{\mathbf{x}} + \partial_y f\,\hat{\mathbf{y}} + \partial_z f\,\hat{\mathbf{z}}$

Divergence: $\boldsymbol{\nabla} \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$

Curl: $\boldsymbol{\nabla} \times \mathbf{F} = \begin{cases} \partial_y F_z - \partial_z F_y & \text{in } \hat{\mathbf{x}} \\ \partial_z F_x - \partial_x F_z & \text{in } \hat{\mathbf{y}} \\ \partial_x F_y - \partial_y F_x & \text{in } \hat{\mathbf{z}} \end{cases}$

Laplacian: $\nabla^2 f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$

Spherical ($d\mathbf{l}=dr\,\hat{\mathbf{r}}+r\,d\theta\,\hat{\boldsymbol{\theta}}+r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}, dV=r^2\sin\theta\,dr\,d\theta\,d\phi$)

Gradient: $\boldsymbol{\nabla} f = \partial_r f\,\hat{\mathbf{r}} + \frac{1}{r}\partial_\theta f\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\partial_\phi f\,\hat{\boldsymbol{\phi}}$

Divergence: $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r^2}\partial_r(r^2F_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\partial_\phi F_\phi$

Curl: $\boldsymbol{\nabla} \times \mathbf{F} = \begin{cases} \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta\,F_\phi) - \partial_\phi F_\theta\right] & \text{in } \hat{\mathbf{r}} \\ \frac{1}{r}\left[\frac{1}{\sin\theta}\partial_\phi F_r - \partial_r(rF_\phi)\right] & \text{in } \hat{\boldsymbol{\theta}} \\ \frac{1}{r}\left[\partial_r(rF_\theta) - \partial_\theta F_r\right] & \text{in } \hat{\boldsymbol{\phi}} \end{cases}$

Laplacian: $\nabla^2 f = \frac{1}{r^2}\partial_r\left(r^2\partial_r f\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta f\right) + \frac{\partial_\phi^2 f}{r^2\sin^2\theta}$

Cylindrical ($d\mathbf{l}=d\rho\,\hat{\boldsymbol{\rho}}+\rho\,d\phi\,\hat{\boldsymbol{\phi}}+dz\,\hat{\mathbf{z}}, dV=\rho\,d\rho\,d\phi\,dz$)

Gradient: $\boldsymbol{\nabla} f = \partial_\rho f\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\partial_\phi f\,\hat{\boldsymbol{\phi}} + \partial_z f\,\hat{\mathbf{z}}$

Divergence: $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{\rho}\partial_\rho(\rho F_\rho) + \frac{1}{\rho}\partial_\phi F_\phi + \partial_z F_z$

Curl: $\boldsymbol{\nabla} \times \mathbf{F} = \begin{cases} \frac{1}{\rho}\partial_\phi F_z - \partial_z F_\phi & \text{in } \hat{\boldsymbol{\rho}} \\ \partial_z F_\rho - \partial_\rho F_z & \text{in } \hat{\boldsymbol{\phi}} \\ \frac{1}{\rho}\left[\partial_\rho(\rho F_\phi) - \partial_\phi F_\rho\right] & \text{in } \hat{\mathbf{z}} \end{cases}$

Laplacian: $\nabla^2 f = \frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho f\right) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$

Vector Identities Products
 $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$
 $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$
 $\nabla(fg) = f\nabla g + g\nabla f$
 $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$
 $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f$
 $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
 $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla f$
 $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$
 $\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \times (\nabla \cdot \mathbf{A}) = 0$
 $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$
 $\int_a^b \nabla f \cdot d\mathbf{r} = f(b) - f(a) \quad \iiint_V (\nabla \cdot \mathbf{F})dV = \oint\!\!\!\oint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS \quad \iint_\Sigma (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{r}$
 $\frac{d}{dx} f_{h(x)}^{g(x)} f(t) dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$

Trigonometric Identities ($m \in \mathbb{Z}, \alpha, \beta, \theta \in \mathbb{R}, z, a, b \in \mathbb{C}$)
 $e^{i\theta} = \cos\theta + i\sin\theta \quad \Re e^{i\theta} = \cos\theta \quad \Im e^{i\theta} = \sin\theta$
 $\csc z = 1/\sin z \quad \sec z = 1/\cos z \quad \cot z = 1/\tan z$
 $\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2 z = \sec^2 z \quad 1 + \cot^2 z = \csc^2 z \quad \cosh^2 z - \sinh^2 z = 1$
 $2i\sin z = e^{iz} - e^{-iz} \quad 2\cos z = e^{iz} + e^{-iz} \quad \cos 2z = \cos^2 z - \sin^2 z$
 $\sin(iz) = i\sinh z \quad \sinh(iz) = i\sin z \quad \arcsin(iz) = i\operatorname{arsinh} z \quad \operatorname{arsinh}(iz) = i\arcsin z$
 $\cos(iz) = \cosh z \quad \cosh(iz) = \cos z \quad \arccos(iz) = \frac{\pi}{2} - i\operatorname{arsinh} z \quad \operatorname{arcosh}(iz) = \frac{i\pi}{2} - \arcsin z$
 $\sin(\pi m) = 0 \quad \cos(\pi m + \frac{\pi}{2}) = 0 \quad \sin(2\pi m + \frac{\pi}{2}) = 1 \quad \cos(2\pi m) = 1$
 $\sin z = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta \quad \cos z = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$
 $\sin(z\pm\pi) = -\sin z \quad \cos(z\pm\pi) = -\cos z \quad \sin(\pi\pm z) = \mp\sin z \quad \cos(\pi\pm z) = -\cos z$
 $\sin(z\pm\frac{\pi}{2}) = \pm\cos z \quad \cos(z\pm\frac{\pi}{2}) = \mp\sin z \quad \sin(\frac{\pi}{2}\pm z) = \cos z \quad \cos(\frac{\pi}{2}\pm z) = \mp\sin z$
 $\sin(a\pm b) = \sin a\cos b \pm \cos a\sin b \quad \cos(a\pm b) = \cos a\cos b \mp \sin a\sin b$
 $2\cos a\cos b = \cos(a+b) + \cos(a-b) \quad 2\sin a\sin b = \cos(a-b) - \cos(a+b)$
 $2\sin a\cos b = \sin(a+b) + \sin(a-b) \quad \operatorname{sinc} z = \sin z/z \quad \operatorname{sinc} 0 := 1$
 $\text{in } \mathbb{R} : \log\alpha + \log\beta = \log(\alpha\beta) \quad \log\alpha - \log\beta = \log(\alpha/\beta) \quad \alpha\log\beta = \log(\beta^\alpha)$
 $\operatorname{arsinh} z = \ln(z + \sqrt{z^2+1}) \forall z \quad \operatorname{arcosh} z = \ln(z + \sqrt{z^2-1}) \forall z \geq 1 \quad 2\operatorname{artanh}(z) = \ln(1+z) - \ln(1-z) \forall |z| < 1$
Gamma Function ($\gamma \equiv$ Euler-Mascheroni constant, $z \in \mathbb{C} \setminus \mathbb{Z}^-, n \in \mathbb{N}$)
 $\psi(z) = \psi^{(0)}(z) \equiv$ digamma, $\psi^{(m)}(z) \equiv$ polygamma function, $B(z_1, z_2) \equiv$ beta function
 $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \Re(z) > 0 \quad \Gamma(1+z) = z\,\Gamma(z) \quad \Gamma(n) = (n-1)!$
 $\Gamma(1-z)\Gamma(z) = \pi/\sin\pi z \quad \Gamma(1-z) = -z\Gamma(-z) \quad \overline{\Gamma(z)} = \Gamma(\overline{z}) \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$
 $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \quad 1/\Gamma(-n) = 1/\Gamma(0) = 0 \quad \Gamma(1) = 0! = 1$
 $\Gamma(z-m) = (-1)^{m-1}\Gamma(-z)\Gamma(1+z)/\Gamma(m+1-z) \quad \psi(z) = \Gamma'(z)/\Gamma(z)$
 $\psi^{(m)}(z) = \frac{d^m}{dz^m}\psi(z) = \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z) \quad \psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt, \Re(z) > 0$
 $\psi(z+1) = \int_0^1 \frac{1-t^z}{1-t} dt - \gamma \quad \psi(n+1) = H_n - \gamma \quad H_n = \sum_{k=1}^n \frac{1}{k}$
 $B(z_1, z_2) = \Gamma(z_1)\Gamma(z_2)/\Gamma(z_1+z_2) \quad B(z_1, z_2) = B(z_2, z_1) \quad B(1, x) = 1/x$
 $B(x, 1-x) = \pi/\sin\pi x \quad B\left(\frac{z_1+1}{2}, \frac{z_2+1}{2}\right) = 2\int_0^{\pi/2} \sin^{z_1}\theta \cos^{z_2}\theta d\theta$
 $B(z_1+1, z_2) = B(z_1, z_2) \frac{z_1}{z_1+z_2} \quad B(z, z) = \frac{1}{z}\int_0^{\pi/2} \frac{d\theta}{(\sqrt[2]{\sin\theta} + \sqrt[2]{\cos\theta})^2 z}, z \neq 1$

Taylor Series ($\alpha \in \mathbb{R}, z \in \mathbb{C} \cap \operatorname{Dom} f, s \in \mathbb{C}$)

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \ldots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n + \ldots$$
$$e^z = 1 + z + \frac{z^2}{2!} + \ldots \quad \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots \quad \frac{1}{1-z} = 1 + z + z^2 + \ldots$$
$$(1+z)^s = 1 + sz + \frac{s(s-1)}{2!}z^2 + \ldots \quad \sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \quad \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \quad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$$
$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots \quad \tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \ldots \quad (\text{both for } |z| < \frac{\pi}{2})$$
$$\arcsin z = z + \frac{z^3}{6} + \frac{3z^5}{40} + \ldots \quad \operatorname{arsinh} z = z - \frac{z^3}{6} + \frac{3z^5}{40} - \ldots \quad (\text{both for } |z| < 1)$$
$$\arccos z = \frac{\pi}{2} - \arcsin z \quad \operatorname{arcosh} z = (-1)\left\lfloor \frac{\arg z}{2\pi} \right\rfloor \left(\frac{i\pi}{2} - iz - \frac{iz^3}{6} - \frac{3iz^5}{40} - \ldots \right)$$
$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots \quad \operatorname{artanh} z = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots, \text{ both for } |z| < 1$$

Symbols ($i, j, n, \{a_n\} \in \mathbb{N}$)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{otherwise.} \end{cases} \quad \epsilon_{a_1 a_2 \cdots a_n} = \begin{cases} +1 & \text{if even permutation of } (1, 2, \ldots, n), \\ -1 & \text{if odd permutation of } (1, 2, \ldots, n), \\ 0 & \text{otherwise (repeated indexes).} \end{cases}$$

Prefixes (SI units)		Â (not SI)	10 ^{−10}
Q (quetta)	10 ³⁰	T (tera)	10 ¹²
R (ronna)	10 ²⁷	G (giga)	10 ⁹
Y (yotta)	10 ²⁴	M (mega)	10 ⁶
Z (zetta)	10 ²¹	k (kilo)	10 ³
E (exa)	10 ¹⁸	h (hecto)	10 ²
P (peta)	10 ¹⁵	da (deca)	10 ¹
		d (deci)	10 ^{−1}
		c (centi)	10 ^{−2}
		m (mili)	10 ^{−3}
		μ (micro)	10 ^{−6}
		n (nano)	10 ^{−9}
		p (pico)	10 ^{−12}
		f (femto)	10 ^{−15}
		a (atto)	10 ^{−18}
		z (zepto)	10 ^{−21}
		y (yocto)	10 ^{−24}
		r (ronto)	10 ^{−27}
		q (quecto)	10 ^{−30}

Integrals ($n \in \mathbb{N}_0, m \in \mathbb{Z}, \alpha, \beta, \gamma, \delta, \mu, \nu, \sigma, \tau \in \mathbb{R}, x \in \mathbb{R} \cap \operatorname{Dom} f, a, b, z \in \mathbb{C}$)
+C omitted. Avoid division by 0. Most results can be extended to C.

Basic
$$\int (x+\alpha)^r dx = \frac{(x+\alpha)^{r+1}}{r+1} \quad \int x(x+\alpha)^r dx = \frac{(x+\alpha)^{r+1}(rx+x-\alpha)}{(r+1)(r+2)} \quad \int a^x dx = \frac{a^x}{\ln a} \quad \int u\,dv = uv - \int v\,du$$

Rational
$$\int \frac{dx}{\alpha x + \beta} = \frac{1}{\alpha} \ln|\alpha x + \beta| \quad \int \frac{dx}{x^2 + \alpha^2} = \frac{1}{\alpha} \arctan \frac{x}{\alpha} \quad \int \frac{dx}{x^2 - \alpha^2} = \frac{1}{2\alpha} \ln \left| \frac{x-\alpha}{x+\alpha} \right| \quad \int \frac{dx}{\alpha^2 - x^2} = \frac{1}{2\alpha} \ln \left| \frac{\alpha+x}{\alpha-x} \right|$$
$$\int \frac{dx}{\alpha x^2 + \beta x + \gamma} = \frac{2}{\sqrt{(4\alpha\gamma - \beta^2)}} \arctan \frac{2\alpha x + \beta}{\sqrt{(4\alpha\gamma - \beta^2)}} \quad \int \frac{dx}{(x+\alpha)(x+\beta)} = \frac{1}{\beta - \alpha} \ln \left| \frac{\alpha+x}{\beta+x} \right|$$

Roots
$$\int \sqrt{x^2 + \alpha^2} = \frac{1}{2} [x\sqrt{x^2 + \alpha^2} + \alpha^2 \operatorname{arsinh} \frac{x}{|\alpha|}] \quad \int \sqrt{x^2 - \alpha^2} = \frac{1}{2} [x\sqrt{x^2 - \alpha^2} - \alpha^2 \ln|\sqrt{x^2 - \alpha^2} + x|]$$
$$\int \frac{dx}{\sqrt{x^2 + \alpha^2}} = \operatorname{arsinh} \frac{x}{|\alpha|} \quad \int \frac{dx}{\sqrt{-x^2 + \alpha^2}} = \arcsin \frac{x}{|\alpha|} \quad \int \frac{dx}{\sqrt{x^2 - \alpha^2}} = \ln|\sqrt{x^2 - \alpha^2} + x|$$
$$\int \frac{dx}{x\sqrt{x^2 + \alpha^2}} = -\frac{1}{\alpha} \operatorname{arsinh} \frac{\alpha}{|x|} \quad \int \frac{dx}{x\sqrt{-x^2 + \alpha^2}} = -\frac{1}{\alpha} \ln \left| \frac{\sqrt{-x^2 + \alpha^2} + \alpha}{|x|} \right| \quad \int \frac{dx}{x\sqrt{x^2 - \alpha^2}} = \frac{1}{\alpha} \arctan \frac{\sqrt{x^2 - \alpha^2}}{\alpha}$$
$$\int \frac{x}{\sqrt{(x^2 \pm \alpha^2)}^3} dx = \sqrt{x^2 \pm \alpha^2} \quad \int \frac{x}{\sqrt{-(x^2 + \alpha^2)}} dx = -\sqrt{-(x^2 + \alpha^2)}$$
$$\int \frac{dx}{(x^2 \pm \alpha^2)^{3/2}} = \frac{\pm x}{\alpha^2 \sqrt{x^2 \pm \alpha^2}} \quad \int \frac{dx}{(-x^2 + \alpha^2)^{3/2}} = -\sqrt{\frac{x}{-x^2 + \alpha^2}}$$
$$\int \frac{x}{(x^2 \pm \alpha^2)^{3/2}} = \frac{-1}{\sqrt{x^2 \pm \alpha^2}} \quad \int \frac{x}{(x^2 \pm \alpha^2)^{3/2}} = \frac{-1}{\sqrt{x^2 \pm \alpha^2}}$$

Trigonometric ($\mu, \nu > 0, \chi \equiv x\alpha, \gamma \equiv \alpha + \beta, \delta \equiv \alpha - \beta$)
$$\int \sin x\,dx = -\cos x \quad \int \cos x\,dx = \sin x \quad \int \frac{dx}{\sin^4 x} = -\cot x \quad \int \frac{dx}{\cos^4 x} = \tan x \quad \int \frac{dx}{\tan^4 x} = -\cot x - x$$
$$\int \sinh x\,dx = \cosh x \quad \int \cosh x\,dx = \sinh x \quad \int \frac{dx}{\sinh^4 x} = -\coth x \quad \int \frac{dx}{\cosh^4 x} = \tanh x \quad \int \frac{dx}{\tanh^4 x} = -\coth x + x$$
$$\int \tan x\,dx = -\ln|\cos x| \quad \int \tan^2 x\,dx = \tan x - x \quad \int \tanh x\,dx = \ln \cosh x \quad \int \tanh^2 x\,dx = -\tanh x + x$$
$$\int \frac{dx}{\sin x} = -\ln \left| \frac{1}{\sin x} + \frac{1}{\tan x} \right| \quad \int \frac{dx}{\cos x} = \ln \left| \frac{1}{\cos x} + \tan x \right| \quad \int \frac{dx}{\tan x} = \ln|\sin x|$$
$$\int \sin^n \alpha x\,dx = -\frac{\sin^{n-1} \chi \cos \chi}{n\alpha} + \frac{n-1}{n} \int \sin^{n-2} \alpha x\,dx \quad \int \cos^n \alpha x\,dx = \frac{\cos^{n-1} \chi \sin \chi}{n\alpha} + \frac{n-1}{n} \int \cos^{n-2} \alpha x\,dx$$
$$\int \sin \alpha x \sin \beta x\,dx = -\frac{\sin \gamma x}{2\gamma} + \frac{\sin \delta x}{2\delta} \quad \int \cos \alpha x \cos \beta x\,dx = +\frac{\sin \gamma x}{2\gamma} + \frac{\sin \delta x}{2\delta} \quad \int \sin \alpha x \cos \beta x\,dx = -\frac{\cos \gamma x}{2\gamma} - \frac{\cos \delta x}{2\delta}$$
$$\int x \sin \alpha x\,dx = \frac{\sin \chi}{\alpha^2} - \frac{x \cos \chi}{\alpha} \quad \int x \cos \alpha x\,dx = \frac{\cos \chi}{\alpha^2} + \frac{x \sin \chi}{\alpha} \quad \int x \sin^2 \alpha x\,dx = \mp \frac{2x \sin 2\chi + \cos 2\chi \mp 2\chi^2}{8\alpha^2}$$
$$\int x \sin \alpha x \sin \beta x\,dx = \mp \frac{x \sin \gamma x}{2\gamma} \mp \frac{\cos \gamma x + x \sin \delta x}{2\gamma^2} + \frac{\cos \delta x}{2\delta^2} \quad \int x \sin \alpha x \cos \beta x\,dx = -\frac{x \cos \gamma x}{2\gamma} + \frac{\sin \delta x}{2\delta^2} - \frac{x \cos \delta x}{2\delta} + \frac{\sin \gamma x}{2\gamma^2}$$

Definite integrals ($m!!! = m(m-2)(m-4)\ldots, -1!!! = 0!!! = 1!!! = 1$)
$$\int_0^{\pi/2} \sin^\mu x\,dx = \int_0^1 2^{\cos^\mu x}\,dx = \frac{1}{2} B\left(\frac{\mu+1}{2}, \frac{1}{2}\right) = \frac{(n-1)!!!}{n!!!} \cdot \begin{cases} \frac{\pi}{2} & \text{if } \mu = n \text{ even} \\ \frac{1}{1} & \text{if } \mu = n \text{ odd} \end{cases} \quad \int_{-1}^{+1} \frac{\sin(m\pi x) \sin(\tilde{m}\pi x)}{\cos(m\pi x) \cos(\tilde{m}\pi x)} dx = \delta_{m, \tilde{m}}$$
$$\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{m\pi x}{\alpha}\right) \sin\left(\frac{\tilde{m}\pi x}{\alpha}\right)}{\cos\left(\frac{m\pi x}{\alpha}\right) \cos\left(\frac{\tilde{m}\pi x}{\alpha}\right)} dx = \frac{\pi}{2} \delta_{m, \tilde{m}} \quad \int_0^{\frac{\pi}{2}} \sin\left(\frac{m\pi x}{\alpha}\right) \cos\left(\frac{\tilde{m}\pi x}{\alpha}\right) dx = \begin{cases} 0 & \text{if } m+\tilde{m} \text{ even} \\ \frac{2}{m^2-\tilde{m}^2} \frac{1}{\pi} & \text{if } m+\tilde{m} \text{ odd} \end{cases} \quad \int_0^{\pi} \sin x\,dx = 2 \quad \int_0^{\pi} \cos x\,dx = 0$$
$$\int_0^{\pi} \sin^n x \cos^n x\,dx = 0 \, \forall \, \tilde{n} \text{ odd} \quad \int_0^{\pi} \mu \frac{\sin^2 \alpha x}{\cos^2 \alpha x} dx = \frac{1}{\alpha} [2\pi\alpha\mu \mp \sin(2\pi\alpha\mu)] \text{ if } \mu = \pm \frac{\pi n}{\alpha} \quad \int_0^{\pi} \frac{\sin^3 x}{\cos^3 x} dx = \frac{4}{3}$$
$$\int_0^{2\pi} \frac{\sin x}{\cos x} dx = 0 \quad \int_0^{2\pi} \sin x \cos x\,dx = 0 \quad \int_0^{2\pi} \sin^n x \cos^{\tilde{n}} x\,dx = 0 \text{ if } n, \tilde{n} \text{ not both even} \quad \int_0^{2\pi} \frac{\sin^3 x}{\cos^3 x} dx = 0$$
$$\int_0^{\pi} (1-\cos x)^n \sin nx\,dx = 0 \quad \int_0^{\pi} 2^n (1-\cos x)^n \cos nx\,dx = (-1)^n \frac{\pi}{2^{n+1}}$$

Parity
$$\text{Even} : f_e(-x) = f_e(x) \quad \text{sym w.r.t. Y-axis} \quad \text{Odd} : f_o(-x) = -f_o(x) \quad \text{sym w.r.t. (0,0)}$$
$$\int_{-\alpha}^{+\alpha} f_e(x)\,dx = 2 \int_0^{\alpha} f_e(x)\,dx \quad \int_{-\alpha}^{+\alpha} f_o(x)\,dx = 0$$
$$f_e : \cos x, \cosh x, x^{2n}, e^{-x^2}, |x|, \delta_{ij}, \delta(x), \mathbb{R}, 1/f_e, f'_o, f_e \pm f_e, f_e \cdot f_e, f_o \cdot f_o, \mathcal{F}\{f_e(x)\}(\xi), \ldots$$
$$f_o : \sin x, \sinh x, x^{2n+1}, \tan x, \operatorname{erf} x, \operatorname{sign} x, \ln\left(\frac{1+x}{1-x}\right), 1/f_o, f'_e, f_o \pm f_o, f_e \cdot f_o, \mathcal{F}\{f_o(x)\}(\xi), \ldots$$

Log/Exp ($r \neq -1$)
$$\int x^r \ln x\,dx = x^{r+1} \left(\frac{\ln x}{r+1} - \frac{1}{(r+1)^2} \right) \quad \int (\ln x)^n\,dx = (-1)^n n! x \sum_{k=0}^n \frac{(-\ln x)^k}{k!} \quad \int \frac{dx}{(e^{-x}/\alpha + 1)} = \alpha \ln(e^x/\alpha + 1)$$
$$\int x e^{\alpha x^2} dx = \frac{e^{\alpha x^2}}{2\alpha} \quad \int x^n e^{\alpha x} dx = e^{\alpha x} \sum_{k=0}^n \frac{n!(-1)^k x^{n-k}}{(n-k)! \, \alpha^{k+1}} \quad \int \frac{e^{\alpha x}}{x^r} dx = \frac{1}{n-1} \left(-\frac{e^{\alpha x}}{x^{n-1}} + \alpha \int \frac{e^{\alpha x}}{x^{n-1}} dx \right)$$

Definite integrals ($r-1, \alpha > 0 \quad \gamma \equiv$ Euler-Mascheroni constant)
$$\int_0^\infty x^r e^{-\alpha x^2} dx = \frac{\Gamma\left(\frac{r+1}{2}\right)}{2\alpha^{\frac{r+1}{2}}} = \begin{cases} \frac{(2n-1)!!!}{2^{n+1}\alpha^n} \sqrt{\frac{\pi}{\alpha}} & \text{if } r = 2n \\ \frac{n!}{2\alpha^{n+1}\Gamma} & \text{if } r = 2n+1 \end{cases} \quad \int_0^\infty e^{-x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$$
$$\int_0^\infty x^r e^{-ax} dx = \frac{\Gamma(r+1)}{a^{r+1}} \text{ if } r = n \quad \frac{n!}{a^{n+1}\Gamma} \text{ if } (r>-1, \Re(a)>0) \quad \int_0^\infty \sqrt{x} e^{-x} dx = \frac{\sqrt{\pi}}{2} \quad \int_0^\infty \frac{x-1}{e^x-1} dx = \frac{\pi^2}{6}$$
$$\int_0^\infty e^{-ax^b} dx = a^{-1/b} \Gamma\left(\frac{1}{b}+1\right) \quad \int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}} \quad \int_0^{2\pi} e^{i(m-\tilde{m})\phi} d\phi = 2\pi \delta_{m, \tilde{m}}$$
$$\int_0^\infty e^{-\alpha x} \sin(\beta x) dx = \frac{\beta}{\alpha^2 + \beta^2} \quad \int_0^\infty e^{-\alpha x} \cos(\beta x) dx = \frac{\alpha}{\alpha^2 + \beta^2} \quad \int_0^\infty \frac{\ln x}{x} dx = \int_1^\infty \left(\frac{1}{x} - \frac{1}{x} \right) dx = -\gamma$$

Error function integrals ($\varphi = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \mu \equiv$ mean, $\sigma^2 \equiv$ variance)
$$\operatorname{erf}(\pm\infty) = \pm 1 \quad i \operatorname{erfi}(z) = \operatorname{erf}(iz)$$
$$\frac{2}{\sqrt{\pi}} \int_0^{\tilde{x}} e^{-t^2} dt = \operatorname{erf}(z) \quad \int \varphi dx = \frac{1}{2} \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) \quad \int \sqrt{x} e^{ax} dx = \frac{\sqrt{x} e^{ax}}{a} - \frac{\sqrt{\pi} \operatorname{erfi}(\sqrt{ax/\pi})}{2a^{3/2}}$$

Linear Algebra $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{L}, \mathbf{U}, \mathbf{I}, \mathbf{P} \in \mathcal{M}(\mathbb{K}))$
Matrices (Generalizable to arbitrary linear operators)

$\mathbf{A}_{m \times n}$ matrix with m rows and n columns; m and n dimensions of \mathbf{A}

$\mathbf{A}=(a)_{ij}$ $\mathbf{A}^{\mathsf{T}}=(a)_{ji}$ \equiv transpose of \mathbf{A} $\mathbf{A}_{n \times n} \equiv$ square matrix

$\mathbf{D}=\mathbf{D}_{n \times n} \colon i \neq j \, \forall i, j \Rightarrow d_{ij}=0$, $\mathbf{D}=\text{diag}(d_1, \cdots, d_n) \equiv$ diagonal matrix

$\mathbf{L}=\mathbf{L}_{n \times n} \colon l_{ij}=0 \, \forall i < j$, $\mathbf{L} \equiv$ lower triangular matrix

$\mathbf{U}=\mathbf{U}_{n \times n} \colon u_{ij}=0 \, \forall i > j$, $\mathbf{U} \equiv$ upper triangular matrix

$\mathbf{I}=\mathbf{I}_n=\text{diag}(1, \cdots, 1) \equiv$ identity matrix $(\mathbf{I}_n)_{ij}=\delta_{ij}$

$\mathbf{A}_{n \times n} \equiv$ Invertible $\Leftrightarrow \exists \mathbf{B}_{n \times n} \mid \mathbf{AB}=\mathbf{BA}=\mathbf{I}_n$, $\mathbf{B}=\mathbf{A}^{-1} \equiv$ inverse of \mathbf{A}

$\mathbf{A}_{n \times n} \equiv$ singular matrix $\Leftrightarrow \mathbf{A}$ not invertible $\Leftrightarrow \det \mathbf{A}=0$

Let $\mathbf{A}_{m \times n}$, $0 < k \leq m, n$: minor of degree k of \mathbf{A} is the determinant of a matrix obtained from \mathbf{A} by deleting $m-k$ rows and $n-k$ columns

Let $\mathbf{A}_{n \times n}$, \mathbf{A}_{ij} submatrix, by deleting row i and column j from \mathbf{A} ,

$c_{ij}=(-1)^{i+j} \cdot \det \mathbf{A}_{ij}$ $\mathbf{C}=(c)_{ij} \equiv$ cofactor matrix

$\text{adj } \mathbf{A}=\mathbf{C}^{\mathsf{T}} \equiv$ adjugate matrix of \mathbf{A} $\mathbf{A}^{-1}=\text{adj } \mathbf{A}/\det \mathbf{A}$

$\mathbf{A}=\mathbf{A}^{\mathsf{T}} \Leftrightarrow \mathbf{A}$ symmetric matrix $\mathbf{A}=-\mathbf{A}^{\mathsf{T}} \Leftrightarrow \mathbf{A}$ anti-symmetric matrix

$\mathbf{A}^{\dagger}=(\overline{\mathbf{A}})^{\mathsf{T}}=\overline{\mathbf{A}^{\mathsf{T}}} \equiv$ conjugate transpose or Hermitian transpose of \mathbf{A}

$\mathbf{A}=\mathbf{A}^{\dagger} \Leftrightarrow \mathbf{A}$ Hermitian matrix $\mathbf{A}=-\mathbf{A}^{\dagger} \Leftrightarrow \mathbf{A}$ anti-Hermitian matrix

$\mathbf{A}^{\dagger} \mathbf{A}=\mathbf{AA}^{\dagger} \Leftrightarrow \mathbf{A}$ normal matrix $\mathbf{A}^{\dagger}=\mathbf{A}^{-1} \Leftrightarrow \mathbf{A}$ unitary matrix

$\det \mathbf{A}_{n \times n}=|\mathbf{A}|=\sum_{i=1}^n a_{ij} c_{ij}=\sum_{j=1}^n a_{ij} c_{ij}$ $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$

$\text{tr } \mathbf{A}_{n \times n}=\sum_{i=1}^n a_{ii}$ $\text{rank } \mathbf{A} \mathrel{\mathop:}= \dim(\text{img } \mathbf{A}_{m \times n}) \leq \min\{m, n\}$

rank of \mathbf{A} : number of linearly independent columns (or rows) of \mathbf{A}

$\ker \mathbf{A}=\{\mathbf{x} \in \mathbb{K}^n \mid \mathbf{Ax}=\mathbf{0}\}$ $\ker \mathbf{A}+\text{rank } \mathbf{A}=n$, $\mathbf{A}_{m \times n}$

$[\mathbf{A}, \mathbf{B}]=\mathbf{AB}-\mathbf{BA} \equiv$ commutator $[\mathbf{A}, \mathbf{B}]=0 \Rightarrow \mathbf{A}, \mathbf{B}$ commute

$\{\mathbf{A}, \mathbf{B}\}=\mathbf{AB}+\mathbf{BA} \equiv$ anticommutator $2\mathbf{AB}=[\mathbf{A}, \mathbf{B}]+\{\mathbf{A}, \mathbf{B}\}$

Let $\mathbf{A}_{n \times n}$, $\mathbf{v}_{n \times 1} \neq \mathbf{0}$, $\lambda \in \mathbb{K}$, $\mathbf{Av}=\lambda \mathbf{v} \colon \mathbf{v} \equiv$ eigenvector, $\lambda \equiv$ eigenvalue

$p(\lambda)=|\mathbf{A}-\lambda \mathbf{I}|=0 \Rightarrow \{\lambda_k\}$ $(\mathbf{A}-\lambda_k \mathbf{I})\mathbf{v}_k=\mathbf{0} \Rightarrow \{\mathbf{v}_k\}$

$\mu_{\mathbf{A}}(\lambda_k) \equiv$ algebraic multiplicity: $\max\{l \mid p(\lambda)=(\lambda-\lambda_k)^l \cdot q(\lambda), \, q(\lambda_k) \neq 0\}$

$\gamma_{\mathbf{A}}=\dim \ker(\mathbf{A}-\lambda_k \mathbf{I}) \equiv$ geometric multiplicity $1 \leq \gamma_{\mathbf{A}}(\lambda_k) \leq \mu_{\mathbf{A}}(\lambda_k)$

$\gamma_{\mathbf{A}}(\lambda_k)=\mu_{\mathbf{A}}(\lambda_k) \, \forall k \Leftrightarrow \exists \mathcal{B}'=\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \equiv$ eigenbasis $\Rightarrow \exists \mathbf{P} \mid \mathbf{P}^{-1} \mathbf{AP}=\mathbf{D}$

$\mathbf{P}=\mathcal{B}'\mathcal{P}\mathcal{B}=\mathbf{P}_{\mathcal{B}} \Rightarrow \mathcal{B}' \equiv$ change of basis matrix from \mathcal{B} to \mathcal{B}' $\mathcal{B}'^t D \mathcal{B}'=\mathcal{B}^t \mathbf{A} \mathcal{B}$

$\mathbf{P}=[\mathbf{v}_1 \cdots \mathbf{v}_n]$ $\mathbf{D}=\text{diag}(\lambda_1, \cdots, \lambda_n)$ $\mathbf{A} \sim \mathbf{D} \Rightarrow |\mathbf{A}|=|\mathbf{D}|$, $\text{tr } \mathbf{A}=\text{tr } \mathbf{D}$

Properties $(\theta \in \mathbb{R}, \, \eta, \nu, \omega, \tau \in \mathbb{C}, \, \vec{u}, \vec{v} \in \mathbb{C}^n)$

$\mathbf{A}(\nu+\omega)=\nu \mathbf{A}+\omega \mathbf{A}$ $\tau(\mathbf{A}+\mathbf{B})=\tau \mathbf{A}+\tau \mathbf{B}$ $\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$

$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{CB}$ $(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}$ $\mathbf{AB} \neq \mathbf{BA}$

Let v, w arbitrary column vectors, j^{th} column of \mathbf{A} $a_j=\nu \cdot v+\omega \cdot w$:

$|\mathbf{A}|=\nu \cdot |a_1, \cdots, a_{j-1}, v, a_{j+1}, \cdots, a_n|+\omega \cdot |a_1, \cdots, a_{j-1}, w, a_{j+1}, \cdots, a_n|$

$|a_1, \cdots, u, \cdots, u, \cdots, a_n|=0$ $|\mathbf{A}_{\sigma}|=\text{sign}(\sigma) \cdot |\mathbf{A}|$, $\sigma \equiv$ permutation

$|\tau \mathbf{A}|=\tau^n |\mathbf{A}|$ $|\mathbf{A}|^{\mathsf{T}}=|\mathbf{A}^{\mathsf{T}}|$ $|\mathbf{A}|^{\dagger}=|\mathbf{A}^{\dagger}|$ $|\overline{\mathbf{A}}|=|\overline{\mathbf{A}}|$ $|\mathbf{A}|^{-1}=|\mathbf{A}^{-1}|$

$\overline{\overline{\mathbf{A}}}=\mathbf{A}$ $|\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|$ $|\mathbf{U}|=e^{i\theta}$ $|\mathbf{U}|=1 \Rightarrow \mathbf{U} \in SU(n)$ $|\mathbf{A}|=\prod_{k=1}^n \lambda_k$

$\text{tr}(\tau \mathbf{A})=\tau \text{tr } \mathbf{A}$ $\text{tr } \mathbf{A}=\text{tr } \mathbf{A}^{\mathsf{T}}$ $\text{tr } \mathbf{A}^{\dagger}=\text{tr } \overline{\mathbf{A}}=\overline{\text{tr } \mathbf{A}}$ $\text{tr}(\mathbf{A}+\mathbf{B})=\text{tr } \mathbf{A}+\text{tr } \mathbf{B}$

$\text{tr}(\mathbf{AB})=\text{tr}(\mathbf{BA}) \Rightarrow \text{tr}[\mathbf{A}, \mathbf{B}]=0$ $\text{tr } \mathbf{A}=\sum_{k=1}^n \lambda_k$

$\text{tr}(\mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_n)=\text{tr}(\mathbf{A}_n \mathbf{A}_1 \cdots \mathbf{A}_{n-1})=\cdots=\text{tr}(\mathbf{A}_2 \mathbf{A}_3 \cdots \mathbf{A}_n \mathbf{A}_1)$

$(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}}=\mathbf{A}$ $(\mathbf{A}+\mathbf{B})^{\mathsf{T}}=\mathbf{A}^{\mathsf{T}}+\mathbf{B}^{\mathsf{T}}$ $(\eta \mathbf{A})^{\mathsf{T}}=\eta \mathbf{A}^{\mathsf{T}}$ $(\mathbf{AB})^{\mathsf{T}}=(\mathbf{BA})^{\mathsf{T}}$

$(\mathbf{A}^{-1})^{\mathsf{T}}=(\mathbf{A}^{\mathsf{T}})^{-1}$ $\text{rg } \mathbf{A}=\text{rg } \mathbf{A}^{\mathsf{T}}$ $(\mathbf{A}^{-1})^{\dagger}=(\mathbf{A}^{\dagger})^{-1}$ $\text{rg } \mathbf{A}=\text{rg } \mathbf{A}^{\dagger}$

$(\mathbf{A}^{\dagger})^{\dagger}=\mathbf{A}$ $(\mathbf{A}+\mathbf{B})^{\dagger}=\mathbf{A}^{\dagger}+\mathbf{B}^{\dagger}$ $(\eta \mathbf{A})^{\dagger}=\overline{\eta} \mathbf{A}^{\dagger}$ $(\mathbf{AB})^{\dagger}=(\mathbf{BA})^{\dagger}$

$\vec{u} \cdot \vec{v}=\langle \mathbf{u}, \mathbf{v} \rangle=\mathbf{u}^{\dagger} \mathbf{v}$ $\| \mathbf{u} \|:=\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \| \mathbf{u} \| \| \mathbf{v} \|$ $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$

$(\mathbf{A}^{-1})^{-1}=\mathbf{A}$ $(\mathbf{AB})^{-1}=\mathbf{B}^{-1} \mathbf{A}^{-1}$ $(\eta \mathbf{A})^{-1}=\mathbf{A}^{-1}/\eta$ $\mathbf{D}^{-1}=\text{diag}(1/d_i)$

$[\mathbf{A}, \mathbf{B}]=-\mathbf{[B, A]}$ $[\mathbf{A}, \mathbf{B}+\mathbf{C}]=[\mathbf{A}, \mathbf{B}]+[\mathbf{A}, \mathbf{C}]$ $[\mathbf{A}, \mathbf{A}]=[\mathbf{A}, \mathbf{A}^n]=0$

$[\mathbf{A}, \mathbf{BC}]=[\mathbf{A}, \mathbf{B}]\mathbf{C}+\mathbf{B}[\mathbf{A}, \mathbf{C}]$ $[\mathbf{A}, [\mathbf{B}, \mathbf{C}]]+[\mathbf{B}, [\mathbf{C}, \mathbf{A}]]+[\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = 0$

$[\mathbf{A}, \mathbf{B}]^{\dagger}=[\mathbf{B}^{\dagger}, \mathbf{A}^{\dagger}]$ $\mathbf{A}=\mathbf{A}^{\dagger} \Rightarrow \lambda_{\mathbf{A}} \in \mathbb{R}$ $\mathbf{A}=-\mathbf{A}^{\dagger} \Rightarrow \lambda_{\mathbf{A}} \in i\mathbb{R}$

if $\mathbf{A}=\mathbf{A}^{\dagger}, \mathbf{B}=\mathbf{B}^{\dagger}$: $i[\mathbf{A}, \mathbf{B}]=\left(i[\mathbf{A}, \mathbf{B}]\right)^{\dagger}$, $\{\mathbf{A}, \mathbf{B}\}=\{\mathbf{A}, \mathbf{B}\}^{\dagger}$

if $\mathbf{A}=\mathbf{A}^{\dagger}, \mathbf{B}=\mathbf{B}^{\dagger}$, and $[\mathbf{A}, \mathbf{B}]=0$: $\mathbf{AB}=(\mathbf{AB})^{\mathsf{T}}$

Conics $(\varepsilon, a, b, c, h, \ell \in \mathbb{R}, \, \varepsilon \equiv$ eccentricity, $c \equiv$ focal distance, $p \equiv$ focal parameter, $\ell \equiv$ semi-latus rectum, $a \equiv$ semi-major axis, $b \equiv$ semi-minor axis, $\ell=p\varepsilon$, $c=a\varepsilon$, $p+c=a/\varepsilon$, $(h, k) \equiv$ center, (h, k) parabola \equiv vertex

Vertical parabola: $(y-k)=\frac{1}{4p}(x-h)^2$, $\varepsilon=1$ Circle: $(x-h)^2+(y-k)^2=a^2$, $\varepsilon=0$

Ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, $\varepsilon=\sqrt{1-\left(\frac{b}{a}\right)^2}$

Hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, $\varepsilon=\sqrt{1+\left(\frac{b}{a}\right)^2}$

Complex Analysis $(\alpha, \beta, r, \theta, t, p, R \in \mathbb{R}, \, z, w \in \mathbb{C}, \, n, k \in \mathbb{N}_0, \, m \in \mathbb{N}_+, \, i^2=-1)$
p.v. \equiv principal value $\gamma \equiv$ closed contour path positively oriented (anticlockwise)
 $-\gamma \equiv \gamma$ with reverse orientation $\Rightarrow \int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$

$z=\alpha+i\beta=re^{i\theta}$ $r=|z|=\sqrt{\alpha^2+\beta^2}$ $\theta=\arctan(\beta/\alpha)$ $\overline{z}=\alpha-i\beta$ $z^{-1}=\frac{\overline{z}}{r^2}=\frac{1}{re^{i\theta}}$

$z\overline{z}=|z|^2$ $z+\overline{z}=2\Re[z]$ $z-\overline{z}=2i\Im[z]$ $\nabla \overline{z}=\nabla \overline{r} \exp[i(\frac{\theta+2\pi k}{n})]$, $k < n-1$

$z^w=e^{w \log z}$ $\log z=\ln r+i(\theta\pm 2\pi k)$ $\xrightarrow{\text{P.V.}} \text{Log } z=\ln r+i\theta$, $\theta \in (-\pi, \pi]$

$\text{Log } e^z=z \Leftrightarrow \Im z \in (-\pi, \pi]$ $\text{Log}(zw)=\text{Log } z+\text{Log } w \pm i2\pi k$

$e \pm i2\pi n=1$ $e^{i\frac{\pi}{2} \pm i2\pi n}=i$ $e^{i\pi \pm i2\pi n}=-1$ $e^{i\frac{3\pi}{2} \pm i2\pi n}=-i$

$f(z)\Big|_{z_0}=\sum_{n=0}^{\infty} \underbrace{a_n(z-z_0)^n}_{\text{analytical part}}+\sum_{n=1}^{\infty} \underbrace{a_{-n}(z-z_0)^{-n}}_{\text{principal part}}$, $a_{\pm n}=\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{\pm n+1}} dz$

$f(z)$ complex differentiable at z_0 if $\exists f'(z_0)=z \lim_{z \rightarrow z_0} \frac{f(z)-f(z_0)}{z-z_0}$

$f \colon U \subseteq \mathbb{C} \rightarrow U$, U open set: f holomorphic on U if $\forall z_0 \in U$, $\exists f'(z_0)$

f holomorphic at z_0 if f holomorphic on some neighborhood of z_0

$f(x+iy)=u(x, y)+iv(x, y)$ holomorphic $\Rightarrow u, v$ satisfy Cauchy-Riemann (C.R.)

C.R.: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ or $\frac{\partial u}{\partial r}=\frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r}=-\frac{1}{r} \frac{\partial u}{\partial \theta}$

$\partial_x u, \partial_y u, \partial_x v, \partial_y v$ continuous and satisfy C.R. $\Rightarrow f$ holomorphic

$\forall f$ holomorphic: u, v harmonic on $\mathbb{R}^2 \Rightarrow \nabla^2 u=0, \nabla^2 v=0$

$\forall f$ holomorphic and γ enclosing no holes: $\oint_{\gamma} f(z) dz=0$

$\oint_{\gamma} \frac{f(z)}{z-z_0} dz=2\pi i f(z_0)$ and $\oint_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} dz=\frac{2\pi i}{n!} f^{(n)}(z_0)$

$\forall f$, if Γ continuously differentiable: $\int_{\sigma} f(z) dz=\int_a^b f(\Gamma(t)) \cdot \Gamma'(t)$

$\ell(\Gamma)=\int_a^b |\Gamma'(t)| dt \equiv$ contour length generally: $\Gamma(t)=f_R= z_0+R e^{it}$, $\ell(\Gamma_R)=R t_{\max}$

$\forall f$ holomorphic on U , except at a finite number of isolated singularities z_k :

$\oint_{\gamma} f(z) dz=2\pi i \sum_k \text{Res}(f, z_k)$, $\text{Res}(f, z_k) \equiv$ residue of f at z_k

\equiv coefficient c_{-1} of $(z-z_k)^{-1}$ in Laurent series of f around z_k

f holomorphic on U except at $a \in U \equiv f \in \mathcal{O}(U \setminus \{a\})$, possible isolated singularities:

- a removable singularity $\Leftrightarrow \exists g \in \mathcal{O}(U) \mid f(z)=g(z) \, \forall z \in U \setminus \{a\}$
- a pole $\Leftrightarrow \exists g \in \mathcal{O}(U)$, $g(a) \neq 0 \mid f(z)=\frac{g(a)}{(z-a)^m} \, \forall z \in U \setminus \{a\}$; $m \equiv$ pole order
- a essential singularity \Leftrightarrow Laurent series principal part has ∞ terms

For poles z_j of order m : $\text{Res}(f, z_j)=\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z-z_j)^m f(z)) \Big|_{z_j}$

\nexists such formula for essential singularities

Estimation lemma: $f(z) \in \mathbb{C}$, continuous on Γ and $\exists M \in \mathbb{R}$ such that :

$|f(z)| \leq M \, \forall z \in \Gamma \Rightarrow \left| \int_{\Gamma} f(z) dz \right| \leq M \cdot l(\Gamma)$, $M:=\sup_{z \in \Gamma} |f(z)|$

\therefore if $|f(z)| \leq \frac{C}{|z|^p}$, $p > 1$; $C_{R+} \equiv \Gamma_R$, $t \in [0, \pi]$, $z_0=0 \Rightarrow \left| \int_{C_{R+}} f(z) dz \right| \xrightarrow{R \rightarrow \infty} 0$

Jordan's lemma: $f(z)=e^{i\alpha z} g(z) \in \mathbb{C}$, $\alpha > 0$, continuous on $C_{R+} \Rightarrow$

$\left| \int_{C_{R+}} f(z) dz \right| \leq \frac{\pi}{\alpha} M_R$, $M_R:=\max_{\theta \in [0, \pi]} \left| g\left(Re^{i\theta}\right) \right|$ \therefore if $M_R \xrightarrow{R \rightarrow \infty} 0 \Rightarrow \int_{C_{R+}} f(z) dz \xrightarrow{R \rightarrow \infty} 0$

Analogous for $C_R^{-} \equiv \Gamma_R$, $t \in [\pi, 2\pi]$, $z_0=0$ when $\alpha < 0$

Fourier Analysis $(\xi, x \in \mathbb{R})$

$\mathcal{F}\{f(x)\}(\xi)=f(\xi)=\int_{-\infty}^{+\infty} f(x) e^{-i2\pi \xi x} dx$ $\mathcal{F}^{-1}\{f(\xi)\}(x)=\int_{-\infty}^{+\infty} f(\xi) e^{+i2\pi \xi x} d\xi$

$f(x-x_0) \xLeftrightarrow{\mathcal{F}} e^{-i2\pi x_0 \xi} f(\xi)$ $e^{i2\pi \xi_0 x} f(x) \xLeftrightarrow{\mathcal{F}} f(\xi-\xi_0)$ $f(ax) \xLeftrightarrow{\mathcal{F}} \frac{1}{|a|} f\left(\frac{\xi}{a}\right)$

$f(x) \in \mathbb{R} \Rightarrow \hat{f}(-\xi)=\overline{\hat{f}(\xi)}$ $\mathcal{F}^{-1}f(x)=\mathcal{F}(f(-x))$ $\mathcal{F}(f(-x))=(\mathcal{F}f)(-x)$ $\mathcal{F}^2 f(x)=f(-x)$

Convolution

$(f * g)(x)=\int_{-\infty}^{+\infty} f(x-y) g(y) dy$ $f * g=g * f$ $(f * g) * h=f(g * h)$

$f * (g+h)=f * g+f * h$ $\mathcal{F}(f * g)=(\mathcal{F}f)(\mathcal{F}g)$ $\mathcal{F}(fg)=\mathcal{F}f * \mathcal{F}g$

Dirac Delta

$\delta(x)=\begin{cases} 0, & x \neq 0 \\ \infty, & x=0 \end{cases}$ $\int_{-\infty}^{+\infty} \delta(x) dx=1$ $\mathcal{F}\{\delta(x)\}(\xi)=1 \Leftrightarrow \mathcal{F}^{-1}\{1\}(x)=\delta(x)$

$\delta(ax)=\frac{1}{|a|} \delta(x)$ $h(x) \delta(x)=h(0) \delta(x)$ $\delta(x) * f(x)=f(x)$

$\delta(x-a) * f(x)=f(x-a)$ $\delta(x-a) * \delta(x-b)=\delta(x-(a+b))$

$\int_{-\infty}^{+\infty} \delta(x-a) f(x) dx=f(a)$ $\int_{-\infty}^{+\infty} \delta(x) f(x+a) dx=f(a)$

Real integrals via residues

($\Re \equiv$ rational function, no essential singularities nor branch cuts inside contour)

Rational trigonometric integrals (c.v.: $z=e^{i\theta}$)

\Re with no singularities on contour \equiv unit circle: $x^2+y^2=1$

$\int_0^{2\pi} \Re(\cos \theta, \sin \theta) d\theta=2\pi \sum_{|z_k|<1} \text{Res}\left[\frac{1}{z} \Re\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right), z_k\right] \in \mathbb{R}$

Rational improper integrals

\Re with no poles in \mathbb{R} , $\lim_{x \rightarrow \infty} x \Re(x)=0$, contour \equiv upper/lower half-plane

$\int_{-\infty}^{+\infty} \Re(x) dx=2\pi i \sum_{\Im[z_k]>0} \text{Res}\left[\Re(z), z_k\right]=-2\pi i \sum_{\Im[z_k]<0} \text{Res}\left[\Re(z), z_k\right] \in \mathbb{R}$

Cauchy principal value

f continuous on $[a, b] \in \mathbb{R}$ except at isolated poles $\{x_k\}$, $m=1$

$\oint_a^b f(x) dx \equiv \mathbf{P.V.} \cdot \oint_a^b f(x) dx \equiv \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_a^{x_1-\varepsilon} f(x) dx + \int_{x_1+\varepsilon}^{x_2-\varepsilon} f(x) dx + \cdots + \int_{x_n+\varepsilon}^b f(x) dx \right\}$

Improper integrals with principal value

f holomorphic, except at isolated poles $\{z_k\}$, $m=1$; $\lim_{|z| \rightarrow \infty} z f(z)=0$ on $\Im z > 0 \Rightarrow$

$\oint_{-\infty}^{+\infty} f(x) dx=2\pi i \sum_{\Im[z_k]>0} \text{Res}\left[f(z), z_k\right] + \pi i \sum_{\Im[z_k]=0} \text{Res}\left[f(z), z_k\right] \in \mathbb{R}$ Analogous for $\Im z < 0$, just multiply by (-1)

Semi-improper integrals with principal value

f holomorphic, except at isolated singularities $\{z_k\} \notin \mathbb{R}^+$, and except at

isolated poles $\{x_k\} \in \mathbb{R}^+$, $m=1$; $\lim_{|z| \rightarrow \infty} z f(z) \neq \infty$; \lim_{z

Tensors (generalizable to \mathbb{R}^n)
Definition and Operations Vectors can expressed in different bases: $\{e_1, e_2\}, \{e_{1'}, e_{2'}\}, \dots$

$$\vec{A} = A^1 e_1 + A^2 e_2 = (e_1, e_2)(A^1, A^2)^T = A^{1'} e_{1'} + A^{2'} e_{2'} = (e_{1'}, e_{2'})(A^{1'}, A^{2'})^T$$

Einstein convention: summation over repeated indices (up - down)
inverse: primed \leftrightarrow unprimed, transpose: upper \leftrightarrow lower

$$M = (M_{ij}^{i'}) = \begin{pmatrix} M_{11}^{1'} & M_{12}^{1'} \\ M_{21}^{1'} & M_{22}^{1'} \end{pmatrix} \quad (M^{-1})^T = (M_{i'j}') = \begin{pmatrix} M_{1'}^{1'} & M_{2'}^{1'} \\ M_{1'}^{2'} & M_{2'}^{2'} \end{pmatrix} \quad M_{i'}^j M_k^{i'} = \delta_k^j$$

Change of basis: $A^{i'} = M_{j'}^{i'} A^j, \quad e_{i'} = M_{ij'}^{i'} e_j, \quad \det M \neq 0$

Covariant v^i : transform against basis vectors $\{e_i\}$, with $M_{j'}^{i'}$

Covariant w_i : transform with basis vectors $\{e_i\}$, with $M_{i'}^j$

Dot product via metric: $g_{ij} = e_i \cdot e_j \quad g = g^T \quad g^{-1}$ \Rightarrow raises indices

$$\vec{A} \cdot \vec{B} = A^1 B^1 g_{11} + A^1 B^2 g_{12} + A^2 B^1 g_{21} + A^2 B^2 g_{22} = A^i g_{ij} B^j = \vec{A}^T g \vec{B} \quad \|\vec{A}\| = \sqrt{\vec{A} \cdot \vec{A}}$$

Coordinate metrics in flat euclidean metric:

$g_{\text{cartesian}} = \delta_{ij} = \mathbb{I}_n \quad g_{\text{spherical}} = \text{diag}(1, r^2, r^2 \sin^2 \theta) \quad g_{\text{cylindrical}} = \text{diag}(1, \rho^2, 1)$

Inverses: $g_{\text{cart}}^{-1} = \delta_{ij} \quad g_{\text{sph}}^{-1} = \text{diag}(1, 1/r^2, 1/r^2 \sin^2 \theta) \quad g_{\text{cyl}}^{-1} = \text{diag}(1, 1/\rho^2, 1)$

Dual Basis $\{e^1, e^2\}$ dual to $\{e_1, e_2\} \quad e^i \cdot e_j = \delta_j^i$

Relation with metric: $e^i = g^{ij} e_j \quad g^{ij} \equiv$ inverse of the metric

$$\vec{A} = A^i e_i = A_i g^{ij} e_j \quad \textbf{Index lowering: } A_i = g_{ij} A^j \quad \textbf{Index raising: } A^i = g^{ij} A_j$$

Metric under change of basis: $g_{i'j'} = M_{i'}^i M_{j'}^j g_{ij} \Leftrightarrow g' = (M^{-1})^T g M^{-1}$

Dot product is invariant under change of basis

Tensor: Any object that transforms as: $T_{i'j'}^i = M_{i'}^i M_{j'}^j T_{ij}$ is a tensor

Tensor product properties: $(\vec{A}, \vec{B}, \vec{C}$ vectors, $\lambda \in \mathbb{R}, V, V \otimes V$ vector spaces)

- $(\lambda \vec{A}) \otimes \vec{B} = \lambda(\vec{A} \otimes \vec{B})$
- $\vec{A} \otimes (\lambda \vec{B}) = \lambda(\vec{A} \otimes \vec{B})$
- $\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$
- $(\vec{A} + \vec{B}) \otimes \vec{C} = \vec{A} \otimes \vec{C} + \vec{B} \otimes \vec{C}$
- $\vec{A} \otimes (\vec{B} + \vec{C}) = \vec{A} \otimes \vec{B} + \vec{A} \otimes \vec{C}$
- $(\vec{A} \otimes \vec{B})(\vec{C} \otimes \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D})$

Bases of tensor product space $V \otimes V : \{e_i \otimes e^j\}, \{e^i \otimes e_j\}, \{e^i \otimes e^j\}, \{e_i \otimes e_j\}$

Equivalent definition of tensor: Element of $V \otimes V$ formed as a linear combination of the basis elements: $\mathcal{T} = T_{11} e^1 \otimes e^1 + T_{12} e^1 \otimes e^2 + T_{21} e^2 \otimes e^1 + T_{22} e^2 \otimes e^2$

In compact and general notation: $\mathcal{T} = T_{ij} e^i \otimes e^j$ (generalizable to the other bases).

A tensor of type (r, s) has r contravariant and s covariant indexes.

$$\mathcal{T} \cdot \mathcal{V} = T_{ij} V_{kl} g^{ik} V^{jl} = T_{ij} V^{ij} \quad T_{ijk} = g_{il} T^i{}_{jk} \quad T^i{}_{j}{}^l = g^{kl} T^i{}_{jk}$$

Symmetric: $S_{\alpha\beta} = S_{\beta\alpha} \quad S^{\alpha\beta} = S^{\beta\alpha} \Rightarrow 2S^{\alpha\beta} T_{\alpha\beta} = S^{\alpha\beta} (T_{\alpha\beta} + T_{\beta\alpha})$

Antisymmetric: $A_{\alpha\beta} = -A_{\beta\alpha} \quad A^{\alpha\beta} = -A^{\beta\alpha} \Rightarrow 2A^{\alpha\beta} T_{\alpha\beta} = A^{\alpha\beta} (T_{\alpha\beta} - T_{\beta\alpha})$

Tensor Extension to a Manifold

Manifold \mathcal{M} : a surface (or hypersurface) embedded in a higher-dimensional space, Cartesian or Lorentzian. Before we were on the tangent plane to the manifold $T_P \mathcal{M}$. The tangent bundle of \mathcal{M} is $\bigcup_{P \in \mathcal{M}} T_P \mathcal{M}$ and it has double the dimension of \mathcal{M} .

1. We need the expression for the coordinate change: $x^{i'} = x^i(x^1, \dots, x^n)$
This function can be understood as a parametrization over the manifold.
It allows tensors to be consistently defined over the whole manifold.

2. Compute the Jacobian matrix of the transformation and its inverse:

$$M = M_{j'}^{i'} = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^{n'}}{\partial x^1} & \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \quad M^{-1} = M_{j'}^i = \begin{pmatrix} \frac{\partial x^1}{\partial x^{1'}} & \dots & \frac{\partial x^1}{\partial x^{n'}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial x^{1'}} & \dots & \frac{\partial x^n}{\partial x^{n'}} \end{pmatrix}$$

Change of coordinate matrices behave as a change of basis matrices.

3. We can construct the basis vectors as before: $e_{i'} = M_{ij'}^{i'} e_j$. In this way, each vector $e_{i'}$ moves in the direction of change of $x^{i'}$, and is constant in $x^j \forall j \neq i$.

NOTE: When computing the basis vectors, use $M_{i'}^j = (M^{-1})^T$, not $M_{j'}^{i'} = M$.

How to obtain the metric? We need to parametrize the surface by embedding it in a Cartesian space of higher dimension. This space has coordinates X^i

1. We parametrize the surface: $X^i = X^i(x^j)$.

2. The tangent vectors to the surface will be: $e_i = \frac{\partial X^i}{\partial x^i} e_{X^i}$

3. By the very definition of the metric: $g_{ij} = e_i \cdot e_j \quad e_{X^i} \cdot e_{X^j} = \delta_{X^i X^j}$

ODEs $(\alpha, \beta, c \in \mathbb{R}, \lambda \in \mathbb{C} \Leftrightarrow \lambda = \alpha + i\beta, y' \equiv \frac{dy}{dx}, \text{sol} \equiv \text{solution, const} \equiv \text{const})$

First Order Equations

Separable $y' = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x) dx$

Linear $y' + a(x)y = r(x) \Rightarrow y(x) = \left[\int r(x)e^{\int a(x) dx} dx + C \right] e^{-\int a(x) dx}$

Exact $M(x, y) dx + N(x, y) dy = 0$; if $\partial_y M = \partial_x N \Rightarrow \exists f(x, y) \equiv \text{const}$, with:
 $\partial_x f = M \quad \partial_y f = N$ (Solve for f)

Non-Exact $M(x, y) dx + N(x, y) dy \neq 0$; if $\begin{cases} \frac{\partial_y M - \partial_x N}{N} = g(x) \Rightarrow \mu = e^{\int g dx} \\ \frac{\partial_x N - \partial_y M}{M} = h(y) \Rightarrow \mu = e^{\int h dy} \end{cases} \Rightarrow$

$\Rightarrow \mu[M(x, y) dx + N(x, y) dy] = 0 \Rightarrow \text{Exact}$

Bernoulli $y' + a(x)y = r(x)y^n \rightarrow \text{c.v. } z := y^{1-n} \Rightarrow \text{Linear}$

Important Concepts

Linear: $y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = r(x)$

$r(x) = 0 \forall x \Rightarrow$ Homogeneous (homo) $r(x) \neq 0 \Rightarrow$ Inhomogeneous (inhomo)

$\{y_i(x)\}_1^n$ Linearly Independent (LI) $\Leftrightarrow \mathcal{W}(\{y_i(x)\}) := \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0$

$\{y_i(x)\}$ LI sols of homo ODE $\Rightarrow y_h = c_1 y_1 + \dots + c_n y_n$

$y_p \equiv$ particular sol of inhomo ODE $y = y_h + y_p \equiv$ general sol of the ODE

Higher Order Linear ODEs

Constant Coefficients (for homo sol) $\{a_i\}_1^n \equiv \text{const} \Rightarrow$ Let $y_p = e^{\lambda x}$, substitute \Rightarrow solve for $\{\lambda_i\}$: $\lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0$

- $\{\lambda_i\} \in \mathbb{R}$, no repetition: $y_h = C_1 e^{\lambda_1 x} + \dots + C_n e^{\lambda_n x}$
- $\{\lambda_i\} \in \mathbb{R}$, k multiplicity: $y_h = (C_1 + C_2 x + \dots + C_k x^{k-1}) e^{\lambda_1 x} + \dots + C_n e^{\lambda_n x}$
- $\{\lambda_i\} \in \mathbb{C}$, k multiplicity : $y_h = e^{\alpha x} [(A_1 + A_2 x + \dots + A_k x^{k-1}) \cos(\beta x) + (B_1 + B_2 x + \dots + B_k x^{k-1}) \sin(\beta x)] + \dots + C_n e^{\lambda_n x}$

Undetermined coefficients method (for inhomo sol) $\{a_i\}_1^n \equiv \text{const} \Rightarrow$

\Rightarrow Let y_p be the function shown in the table, substitute, and find the constns

$r(x)$	Roots	Form of y_p
$P_m(x)$	1. 0 is not a root 2. 0 is a root of multiplicity s	$Q_m(x)$ $x^s Q_m(x)$
$P_m(x)e^{\alpha x}$	1. α is not a root 2. α is a root of multiplicity s	$Q_m(x)e^{\alpha x}$ $x^s Q_m(x)e^{\alpha x}$
$P_m(x) \cos \beta x + T_n(x) \sin \beta x$	1. $\pm i\beta$ are not roots 2. $\pm i\beta$ are roots of multiplicity s	$Q_k(x) \cos \beta x + R_k(x) \sin \beta x$ $x^s [Q_k(x) \cos \beta x + R_k(x) \sin \beta x]$
$e^{\alpha x} (P_m(x) \cos \beta x + T_n(x) \sin \beta x)$	1. $\alpha \pm i\beta$ are not roots 2. $\alpha \pm i\beta$ are roots of multiplicity s	$(Q_k(x) \cos \beta x + R_k(x) \sin \beta x) e^{\alpha x}$ $x^s [Q_k(x) \cos \beta x + R_k(x) \sin \beta x] e^{\alpha x}$

$m, n, k \equiv$ degree of polynomes $k = \max\{m, n\}$

$Q(x), R(x)$ must have all the terms: i.e. $Q_m(x) = A_1 + A_2 x + \dots + A_{n+1} x^n$

Variation of parameters (for inhomo sol, $r(x)$ not in table) $\{a_i\}_1^n \equiv \text{const} \Rightarrow$

\Rightarrow Let: $y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \dots + u_n(x)y_n(x)$; $\{y_i\}$ LI sols of homo

Impose: $\begin{cases} u_1' y_1 + u_2' y_2 + \dots + u_n' y_n = 0 \\ u_1' y_1' + u_2' y_2' + \dots + u_n' y_n' = 0 \\ \vdots \\ u_1^{(n-2)} y_1 + \dots + u_n^{(n-2)} y_n = 0 \\ u_1^{(n-1)} y_1 + \dots + u_n^{(n-1)} y_n = r(x) \end{cases} \Rightarrow \text{(system of } n \text{ equations)}$

$u_i'(x) = \frac{W(x)}{W_i(x)} \quad W_i(x) \equiv W(x)$ with i-th column: $(0, 0, \dots, r(x))^T \quad u_i(x) = \int u_i' dx$

Euler Equation $x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_n y = 0$

c.v. $x = e^t \Rightarrow y(x) = u(t)$, then $x \frac{d}{dx} \rightarrow \frac{d}{dt} \Rightarrow$ Transformed to const coeff eq in t :

$y = u(t), \quad \frac{dy}{dx} = \frac{1}{x} \frac{du}{dt}, \quad \frac{d^2 y}{dx^2} = \frac{1}{x^2} \left(\frac{d^2 u}{dt^2} - \frac{du}{dt} \right), \dots \Rightarrow$ Solve in t , then $y(x) = u(\ln x)$

Alternative: $y_h = x^\lambda$, substitute $x^n [\lambda(\lambda - 1) \dots (\lambda - n + 1)] + \dots + a_n = 0$

- $\{\lambda_i\} \in \mathbb{R}$, no repetition: $y_h = C_1 x^{\lambda_1} + \dots + C_n x^{\lambda_n}$
- $\{\lambda_i\} \in \mathbb{R}$, k multiplicity: $y_h = (C_1 + C_2 \ln x + \dots + C_k (\ln x)^{k-1}) x^{\lambda_1} + \dots + C_n x^{\lambda_n}$
- $\{\lambda_i\} \in \mathbb{C}$, k multiplicity: $y_h = e^{\alpha x} [(A_1 + A_2 \ln x + \dots + A_k (\ln x)^{k-1}) \cos(\beta \ln x) + (B_1 + B_2 \ln x + \dots + B_k (\ln x)^{k-1}) \sin(\beta \ln x)] + \dots + C_n x^{\lambda_n}$

Systems of First-Order Linear ODEs $e^{Ax} = I + Ax + (Ax)^2/2! + (Ax)^3/3! + \dots$

(homo) $\vec{y}' = A\vec{y} \Rightarrow \vec{y}_h(x) = e^{Ax} \vec{c} \quad A_{n \times n}$ const; if diagonalizable: $A = PDP^{-1} \Rightarrow$
 $\Rightarrow e^{Ax} = P e^{Dx} P^{-1}$ with $e^{Dx} = \text{diag}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$

(inhomo) $\vec{y}' = A\vec{y} + \vec{r}(x) \Rightarrow \vec{y}_p(x) = e^{Ax} \int e^{-Ax} \vec{r}(x) dx \Rightarrow \vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x)$

Quaternions $\mathbb{H} (\alpha, \beta, \gamma, \delta, \lambda, \mu \in \mathbb{R}, \{1, i, j, k\}$ basis of $\mathbb{H}, q, p \in \mathbb{H})$

$q = \alpha + \beta i + \gamma j + \delta k \quad \Re[q] = \alpha \equiv \text{real part} \quad \Im[q] = \beta i + \gamma j + \delta k \equiv \text{vector part}$

$\vec{q} = \alpha - \beta i - \gamma j - \delta k \quad \|\vec{q}\|^2 = q\vec{q} = \vec{q}q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \quad q^{-1} = \frac{1}{\|\vec{q}\|^2} \vec{q}$

$\mathbf{U}_q = \frac{q}{\|q\|} \equiv$ versor of $q, \quad \|\mathbf{U}_q\| = 1 \Rightarrow \mathbf{U}_q \equiv$ unit quaternion $\alpha q = q\alpha$

$\lambda(\alpha_1 + \beta_1 i + \gamma_1 j + \delta_1 k) + \mu(\alpha_2 + \beta_2 i + \gamma_2 j + \delta_2 k) =$
 $= (\lambda\alpha_1 + \mu\alpha_2) + (\lambda\beta_1 + \mu\beta_2)i + (\lambda\gamma_1 + \mu\gamma_2)j + (\lambda\delta_1 + \mu\delta_2)k$

$i1 = 1i = i \quad j1 = 1j = j \quad k1 = 1k = k \quad i^2 = j^2 = k^2 = -1$

$ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j \quad ijk = -1$

$q = (r, \vec{v}), q \in \mathbb{H}, r = \Re[q], \vec{v} = \Im[q] \quad (r_1, \vec{v}_1) + (r_2, \vec{v}_2) = (r_1 + r_2, \vec{v}_1 + \vec{v}_2)$
 $(r_1, \vec{v}_1)(r_2, \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2, r_1 \vec{v}_2 + r_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2) \quad \|\vec{p}q\| = \|\vec{p}\| \|q\|$

$\vec{p}\vec{q} = \vec{q}\vec{p} \quad \vec{q} = -\frac{1}{2}(q + iqi + jqj + kqk) \quad \Re[q] = \frac{1}{2}(q + \vec{q}) \quad \Im[q] = \frac{1}{2}(q - \vec{q})$

Matrix representation: $\{1, i, j, k\} \mapsto \{\mathbf{I}, \sigma_1, \sigma_2, \sigma_3\}, \quad \sigma \equiv$ Pauli matrices

$q = \begin{bmatrix} \alpha + \beta i & \gamma + \delta i \\ -\gamma + \delta i & \alpha - \beta i \end{bmatrix} = \alpha \mathbf{I} + \beta i \sigma_3 + \gamma i \sigma_2 + \delta i \sigma_1 \quad \|q\|^2 = \det q \quad \Re[q] = \frac{1}{2} \text{tr } q \quad \vec{q} = q^\dagger$

Quantity	SI Unit	Quantity	SI Unit
Length	m	Mass	kg
Time	s	Temperature	K
Electric current	A	Amount of substance	mol
Luminous intensity	cd	Force	N=kg·m/s ²
Pressure	Pa=kg/(m·s ²)	Energy	J=kg·m ² /s ²
Power	W=kg·m ² /s ³	Electric charge	C=A·s
Voltage	V=kg·m ² /(A·s ³)	Resistance	(Ω)=kg·m ² /(A ² ·s ³)
Capacitance	F=A ² ·s ⁴ /(kg·m ²)	Magnetic flux	Wb=kg·m ² /(A·s ²)
Mag. flux density	T=kg/(A·s ²)	Inductance	H=kg·m ² /(A ² ·s ²)
Frequency	Hz=1/s	Radioactivity	Bq=1/s
Absorbed dose	Gy=m ² /s ²	Dose equivalent	Sv=m ² /s ²
Catalytic activity	kat=mol/s	Angular velocity	rad/s
Angular acceleration	rad/s ²	Dynamic viscosity	Pa·s=kg/(m·s)
Thermal conductivity	W/m·K=kg·m/(s ³ ·K)	Spec. heat capacity	J/kg·K=m ² /(s ² ·K)
Entropy	J/K=kg·m ² /(s ² ·K)	Heat flux density	W/m ² =kg/s ³
Luminance	cd/m ²	Illuminance	lx=cd·sr/m ²
Surface tension	N/m=kg/s ²	Moment of inertia	kg·m ²
Momentum	kg·m/s	Impulse	N·s=kg·m/s

Quantity	Symbol	Value	Unit
speed of light in vacuum	c	299 792 458	m s^{-1}
constant of gravitation	G	6.67430×10^{-11}	$\text{m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Planck constant	h	$6.62607015 \times 10^{-34}$	J Hz ⁻¹
reduced Planck constant	\hbar	$1.054571817 \times 10^{-34}$	J s
elementary charge	e	$1.602176634 \times 10^{-19}$	C
vacuum magnetic permeability	$\mu_0 = 4\pi\alpha h/e^2 c$	$1.25663706127 \times 10^{-6}$	N A ⁻²
vacuum electric permittivity	$\epsilon_0 = 1/\mu_0 c^2$	$8.8541878128 \times 10^{-12}$	F m ⁻¹
vacuum impedance	$Z_0 = \mu_0 c$	376.73031346177	Ω
Josephson constant	$K_J = 2e/h$	$483\,597.8484 \times 10^9$	Hz V ⁻¹
von Klitzing constant	$R_K = 2\pi h/e^2$	25 812.80745	Ω
magnetic flux quantum	$\Phi_0 = 2\pi h/2e$	$2.067833848 \times 10^{-15}$	Wb
conductance quantum	$G_0 = 2e^2/2\pi h$	$7.748091729 \times 10^{-5}$	S
inverse conductance quantum	G_0^{-1}	12 906.40372	Ω
electron mass	m_e	$9.1093837139 \times 10^{-31}$	kg
proton mass	m_p	$1.67262192595 \times 10^{-27}$	kg
proton-electron mass ratio	m_p/m_e	1836.152673426	—
fine-structure constant	$\alpha = e^2/4\pi\epsilon_0 \hbar c$	$7.2973525643 \times 10^{-3}$	—
inverse fine-structure	α^{-1}	137.035999177	—
Bohr Radius	$a_0 = \hbar/m_e c \alpha$	$5.29177210544 \times 10^{-11}$	m
classical electron radius	$r_e = \alpha^2 a_0$	$2.8179403205 \times 10^{-15}$	m
Bohr Magneton	$\mu_B = e\hbar/2m_e$	$9.274010657 \times 10^{-24}$	J T ⁻¹
Nuclear Magneton	$\mu_N = e\hbar/2mp$	$5.0507837393 \times 10^{-27}$	J T ⁻¹
Rydberg frequency	$cR_\infty = \frac{\alpha^2 m_e c^2}{2\hbar}$	$3.28984196025 \times 10^{15}$	Hz
Hartree energy	$E_h = \alpha^2 \hbar c R_\infty$	$4.35974472221 \times 10^{-18}$	J
Boltzmann constant	k_B	1.380649×10^{-23}	J K ⁻¹
Stefan–Boltzmann constant	$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$	$5.670374419 \times 10^{-8}$	W m ⁻² K ⁻⁴
Avogadro constant	N_A	$6.02214076 \times 10^{23}$	mol ⁻¹
molar gas constant	$R = N_A k_B$	8.314462618	J mol ⁻¹ K ⁻¹
Faraday constant	$F = N_A e$	96 485.33212	C mol ⁻¹