Coordinate Systems

Spherical ($\theta \in [0,\pi]$, $\phi \in [0, 2\pi)$)

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan(\sqrt{x^2+y^2}/z) \\ \phi = \arctan(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical ($\rho \in [0,\infty)$, $\phi \in [0, 2\pi)$)

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \arctan(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

Vector Derivatives

Cartesian ($d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$, $dV = dx\,dy\,dz$)

Gradient: $\nabla f = \partial_x f\,\hat{\mathbf{x}} + \partial_y f\,\hat{\mathbf{y}} + \partial_z f\,\hat{\mathbf{z}}$

Divergence: $\nabla\cdot\mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$

Curl: $\nabla\times\mathbf{F} = \begin{cases} \partial_y F_z - \partial_z F_y & \text{in } \hat{\mathbf{x}} \\ \partial_z F_x - \partial_x F_z & \text{in } \hat{\mathbf{y}} \\ \partial_x F_y - \partial_y F_x & \text{in } \hat{\mathbf{z}} \end{cases}$

Laplacian: $\nabla^2 f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$

Spherical ($d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$)

Gradient: $\nabla f = \partial_r f\,\hat{\mathbf{r}} + \frac{1}{r}\partial_\theta f\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\partial_\phi f\,\hat{\boldsymbol{\phi}}$

Divergence: $\nabla\cdot\mathbf{F} = \frac{1}{r^2}\partial_r(r^2 F_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\partial_\phi F_\phi$

Curl: $\nabla\times\mathbf{F} = \begin{cases} \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta\,F_\phi) - \partial_\phi F_\theta\right] & \text{in } \hat{\mathbf{r}} \\ \frac{1}{r}\left[\frac{1}{\sin\theta}\partial_\phi F_r - \partial_r(rF_\phi)\right] & \text{in } \hat{\boldsymbol{\theta}} \\ \frac{1}{r}\left[\partial_r(rF_\theta) - \partial_\theta F_r\right] & \text{in } \hat{\boldsymbol{\phi}} \end{cases}$

Laplacian: $\nabla^2 f = \frac{1}{r^2}\partial_r\left(r^2\partial_r f\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta f\right) + \frac{\partial_\phi^2 f}{r^2\sin^2\theta}$

Cylindrical ($d\mathbf{l} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$, $dV = \rho\,d\rho\,d\phi\,dz$)

Gradient: $\nabla f = \partial_\rho f\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\partial_\phi f\,\hat{\boldsymbol{\phi}} + \partial_z f\,\hat{\mathbf{z}}$

Divergence: $\nabla\cdot\mathbf{F} = \frac{1}{\rho}\partial_\rho(\rho F_\rho) + \frac{1}{\rho}\partial_\phi F_\phi + \partial_z F_z$

Curl: $\nabla\times\mathbf{F} = \begin{cases} \frac{1}{\rho}\partial_\phi F_z - \partial_z F_\phi & \text{in } \hat{\boldsymbol{\rho}} \\ \partial_z F_\rho - \partial_\rho F_z & \text{in } \hat{\boldsymbol{\phi}} \\ \frac{1}{\rho}\left[\partial_\rho(\rho F_\phi) - \partial_\phi F_\rho\right] & \text{in } \hat{\mathbf{z}} \end{cases}$

Laplacian: $\nabla^2 f = \frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho f\right) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$

Vector Identities Products

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$\nabla(fg) = f\nabla g + g\nabla f$

$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot\nabla f$

$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times\nabla f$

$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

$\nabla\cdot(\nabla\times\mathbf{A}) = \nabla\times(\nabla f) = 0$

$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

$\int_a^b \nabla f\cdot d\mathbf{r} = f(b) - f(a)$ $\quad\iiint_V (\nabla\cdot\mathbf{F})\,dV = \oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ $\quad\iint_\Sigma (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \oint_{\partial\Sigma}\mathbf{F}\cdot d\mathbf{r}$

$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)$

Trigonometric Identities ($m\in\mathbb{Z}$, $\alpha,\beta,\theta\in\mathbb{R}$, $z,a,b\in\mathbb{C}$)

$e^{i\theta} = \cos\theta + i\sin\theta \quad \Re e^{i\theta} = \cos\theta \quad \Im e^{i\theta} = \sin\theta$

$\csc z = 1/\sin z \quad \sec z = 1/\cos z \quad \cot z = 1/\tan z$

$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2 z = \sec^2 z \quad 1 + \cot^2 z = \csc^2 z \quad \cosh^2 z - \sinh^2 z = 1$

$2i\sin z = e^{iz} - e^{-iz} \quad 2\cos z = e^{iz} + e^{-iz} \quad \cos 2z = \cos^2 z - \sin^2 z$

$\sin(iz) = i\sinh z \quad \sinh(iz) = i\sin z \quad \arcsin(iz) = i\,\mathrm{arsinh}\,z \quad \mathrm{arsinh}(iz) = i\arcsin z$

$\cos(iz) = \cosh z \quad \cosh(iz) = \cos z \quad \arccos(iz) = \frac{\pi}{2} - i\,\mathrm{arsinh}\,z \quad \mathrm{arcosh}(iz) = \frac{i\pi}{2} - \arcsin z$

$\sin(\pi m) = 0 \quad \cos(\pi m + \frac{\pi}{2}) = 0 \quad \sin(2\pi m + \frac{\pi}{2}) = 1 \quad \cos(2\pi m) = 1$

$\sin z = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta \quad \cos z = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

$\sin(z\pm\pi) = -\sin z \quad \cos(z\pm\pi) = -\cos z \quad \sin(\pi\pm z) = \mp\sin z \quad \cos(\pi\pm z) = -\cos z$

$\sin(z\pm\frac{\pi}{2}) = \pm\cos z \quad \cos(z\pm\frac{\pi}{2}) = \mp\sin z \quad \sin(\frac{\pi}{2}\pm z) = \cos z \quad \cos(\frac{\pi}{2}\pm z) = \mp\sin z$

$\sin(a\pm b) = \sin a\cos b \pm \cos a\sin b \quad \cos(a\pm b) = \cos a\cos b \mp \sin a\sin b$

$2\cos a\cos b = \cos(a+b) + \cos(a-b) \quad 2\sin a\sin b = \cos(a-b) - \cos(a+b)$

$2\sin a\cos b = \sin(a+b) + \sin(a-b) \quad \mathrm{sinc}\,z = \sin z/z \quad \mathrm{sinc}\,0 := 1$

in $\mathbb{R}$: $\log\alpha + \log\beta = \log(\alpha\beta) \quad \log\alpha - \log\beta = \log(\alpha/\beta) \quad \alpha\log\beta = \log(\beta^\alpha)$

$\mathrm{arsinh}\,z = \ln(z + \sqrt{z^2+1})\ \forall z \quad \mathrm{arcosh}\,z = \ln(z + \sqrt{z^2-1})\ \forall z \geq 1 \quad 2\,\mathrm{arctanh}(z) = \ln(1+z) - \ln(1-z)\ \forall |z| < 1$

Gamma Function ($\gamma \equiv$ Euler-Mascheroni constant, $z\in\mathbb{C}\setminus\mathbb{Z}^-$, $n\in\mathbb{N}$)

$\psi(z) = \psi^{(0)}(z) \equiv$ digamma, $\psi^{(m)}(z) \equiv$ polygamma function, $\mathrm{B}(z_1,z_2) \equiv$ beta function

$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,\ \Re(z) > 0 \quad \Gamma(1+z) = z\,\Gamma(z) \quad \Gamma(n) = (n-1)!$

$\Gamma(1-z)\Gamma(z) = \pi/\sin\pi z \quad \Gamma(1-z) = -z\Gamma(-z) \quad \overline{\Gamma(z)} = \Gamma(\overline{z}) \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \quad 1/\Gamma(-n) = 1/\Gamma(0) = 0 \quad \Gamma(1) = 0! = 1$

$\Gamma(z-m) = (-1)^{m-1}\,\Gamma(-z)\Gamma(1+z)/\Gamma(m+1-z) \quad \psi(z) = \Gamma'(z)/\Gamma(z)$

$\psi^{(m)}(z) = \frac{d^m}{dz^m}\psi(z) = \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z) \quad \psi(z) = \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt,\ \Re(z) > 0$

$\psi(z+1) = \int_0^1 \frac{1-t^z}{1-t}dt - \gamma \quad \psi(n+1) = H_n - \gamma \quad H_n = \sum_{k=1}^n \frac{1}{k}$

$\mathrm{B}(z_1,z_2) = \Gamma(z_1)\Gamma(z_2)/\Gamma(z_1+z_2) \quad \mathrm{B}(z_1,z_2) = \mathrm{B}(z_2,z_1) \quad \mathrm{B}(1,x) = 1/x$

$\mathrm{B}(x,1-x) = \pi/\sin\pi x \quad \mathrm{B}\left(\frac{z_1+1}{2},\frac{z_2+1}{2}\right) = 2\int_0^{\pi/2}\sin^{z_1}\theta\cos^{z_2}\theta\,d\theta$

$\mathrm{B}(z_1+1,z_2) = \mathrm{B}(z_1,z_2)\frac{z_1}{z_1+z_2} \quad \mathrm{B}(z,z) = \frac{1}{z}\int_0^{\pi/2}\frac{d\theta}{(\sqrt[z]{\sin\theta} + \sqrt[z]{\cos\theta})^{2z}},\ z\neq 1$

Taylor Series ($\alpha\in\mathbb{R}$, $z\in\mathbb{C}\cap\mathrm{Dom}_f$, $s\in\mathbb{C}$)

$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \ldots + \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n + \ldots$

$e^z = 1 + z + \frac{z^2}{2!} + \ldots \quad \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots \quad \frac{1}{1-z} = 1 + z + z^2 + \ldots$

$(1+z)^s = 1 + sz + \frac{s(s-1)}{2!}z^2 + \ldots \quad \sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \ldots$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \quad \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots \quad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$

$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots \quad \tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \ldots$ (both for $|z| < \frac{\pi}{2}$)

$\arcsin z = z + \frac{z^3}{6} + \frac{3z^5}{40} + \ldots \quad \mathrm{arsinh}\,z = z - \frac{z^3}{6} + \frac{3z^5}{40} - \ldots$ (both for $|z| < 1$)

$\arccos z = \frac{\pi}{2} - \arcsin z \quad \mathrm{arcosh}\,z = (-1)^{\left\lfloor\frac{\arg z}{2\pi}\right\rfloor}\left(\frac{i\pi}{2} - iz - \frac{iz^3}{6} - \frac{3iz^5}{40} - \ldots\right)$

$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots \quad \mathrm{artanh}\,z = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots$, both for $|z| < 1$

Symbols ($i, j, n, \{a_n\}\in\mathbb{N}$)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \qquad \epsilon_{a_1 a_2\cdots a_n} = \begin{cases} +1 & \text{if even permutation of } (1,2,\cdots,n), \\ -1 & \text{if odd permutation of } (1,2,\cdots,n), \\ 0 & \text{otherwise (repeated indexes).} \end{cases}$$

Prefixes (SI units)

Å (not SI) 10^{-10}

Q (quetta)	10^{30}	T (tera)	10^{12}	d (deci)	10^{-1}	f (femto)	10^{-15}
R (ronna)	10^{27}	G (giga)	10^{9}	c (centi)	10^{-2}	a (atto)	10^{-18}
Y (yotta)	10^{24}	M (mega)	10^{6}	m (mili)	10^{-3}	z (zepto)	10^{-21}
Z (zetta)	10^{21}	k (kilo)	10^{3}	μ (micro)	10^{-6}	y (yocto)	10^{-24}
E (exa)	10^{18}	h (hecto)	10^{2}	n (nano)	10^{-9}	r (ronto)	10^{-27}
P (peta)	10^{15}	da (deca)	10^{1}	p (pico)	10^{-12}	q (quecto)	10^{-30}

Integrals ($n\in\mathbb{N}_0$, $m\in\mathbb{Z}$, $\alpha,\beta,\gamma,\delta,\mu,\nu,\sigma,r\in\mathbb{R}$, $x\in\mathbb{R}\cap\mathrm{Dom}_f$, $a,b,z\in\mathbb{C}$)

+C omitted. Avoid division by 0. Most results can be extended to $\mathbb{C}$.

Basic

$\int(x+\alpha)^r dx = \frac{(x+\alpha)^{r+1}}{r+1} \quad \int x(x+\alpha)^r dx = \frac{(x+\alpha)^{r+1}(rx+x-\alpha)}{(r+1)(r+2)} \quad \int a^x dx = \frac{a^x}{\ln a} \quad \int u\,dv = uv - \int v\,du$

Rational

$\int\frac{dx}{\alpha x+\beta} = \frac{1}{\alpha}\ln|\alpha x+\beta| \quad \int\frac{dx}{x^2+\alpha^2} = \frac{1}{\alpha}\arctan\frac{x}{\alpha} \quad \int\frac{dx}{x^2-\alpha^2} = \frac{1}{2\alpha}\ln|\frac{x-\alpha}{x+\alpha}| \quad \int\frac{dx}{\alpha^2-x^2} = \frac{1}{2\alpha}\ln|\frac{\alpha+x}{\alpha-x}|$

$\int\frac{dx}{\alpha x^2+\beta x+\gamma} = \frac{2}{\sqrt{(4\alpha\gamma-\beta^2)}}\arctan\frac{2\alpha x+\beta}{\sqrt{(4\alpha\gamma-\beta^2)}} \quad \int\frac{dx}{(x+\alpha)(x+\beta)} = \frac{1}{\beta-\alpha}\ln|\frac{\alpha+x}{\beta+x}|$

Roots

$\int\sqrt{(x^2+\alpha^2)} = \frac{1}{2}[x\sqrt{(x^2+\alpha^2)} + \alpha^2\,\mathrm{arsinh}\frac{x}{|\alpha|}] \quad \int\sqrt{(x^2-\alpha^2)} = \frac{1}{2}[x\sqrt{(x^2-\alpha^2)} - \alpha^2\ln|\sqrt{(x^2-\alpha^2)}+x|]$

$\int\frac{dx}{\sqrt{(x^2+\alpha^2)}} = \mathrm{arsinh}\frac{x}{|\alpha|} \quad \int\frac{dx}{\sqrt{(-x^2+\alpha^2)}} = \arcsin\frac{x}{|\alpha|} \quad \int\frac{dx}{\sqrt{(x^2-\alpha^2)}} = \ln|\sqrt{(x^2-\alpha^2)}+x|$

$\int\frac{dx}{x\sqrt{(x^2+\alpha^2)}} = -\frac{1}{\alpha}\,\mathrm{arsinh}\frac{\alpha}{|x|} \quad \int\frac{dx}{x\sqrt{(-x^2+\alpha^2)}} = -\frac{1}{\alpha}\ln\frac{|\sqrt{(-x^2+\alpha^2)}+\alpha|}{|x|} \quad \int\frac{dx}{x\sqrt{(x^2-\alpha^2)}} = \frac{1}{\alpha}\arctan\frac{\sqrt{(x^2-\alpha^2)}}{\alpha}$

$\int\frac{x}{\sqrt{(x^2\pm\alpha^2)}}dx = \sqrt{(x^2\pm\alpha^2)} \quad \int\frac{x}{\sqrt{(-x^2+\alpha^2)}}dx = -\sqrt{(-x^2+\alpha^2)}$

$\int\frac{dx}{(x^2\pm\alpha^2)^{3/2}} = \frac{\pm x}{\alpha^2\sqrt{(x^2\pm\alpha^2)}} \quad \int\frac{dx}{(-x^2+\alpha^2)^{3/2}} = \frac{x}{\alpha\sqrt{(-x^2+\alpha^2)}}$

$\int\frac{x}{(x^2\pm\alpha^2)^{3/2}} = \frac{-1}{\sqrt{(x^2\pm\alpha^2)}} \quad \int\frac{x}{(x^2\pm\alpha^2)^{3/2}} = \frac{-1}{\sqrt{(x^2\pm\alpha^2)}}$

Trigonometric ($\mu,\nu > 0$, $\chi\equiv x\alpha$, $\gamma\equiv\alpha+\beta$, $\delta\equiv\alpha-\beta$)

$\int\sin x\,dx = -\cos x \quad \int\cos x\,dx = \sin x \quad \int\frac{dx}{\sin^2 x} = -\cot x \quad \int\frac{dx}{\cos^2 x} = \tan x \quad \int\frac{dx}{\tan^2 x} = -\cot x - x$

$\int\sinh x\,dx = \cosh x \quad \int\cosh x\,dx = \sinh x \quad \int\frac{dx}{\sinh^2 x} = -\coth x \quad \int\frac{dx}{\cosh^2 x} = \tanh x \quad \int\frac{dx}{\tanh^2 x} = -\coth x + x$

$\int\tan x\,dx = -\ln|\cos x| \quad \int\tan^2 x\,dx = \tan x - x \quad \int\tanh x\,dx = \ln\cosh x \quad \int\tanh^2 x\,dx = -\tanh x + x$

$\int\frac{dx}{\sin x} = -\ln|\frac{1}{\sin x} + \frac{1}{\tan x}| \quad \int\frac{dx}{\cos x} = \ln|\frac{1}{\cos x} + \tan x| \quad \int\frac{dx}{\tan x} = \ln|\sin x|$

$\int\sin^n\alpha x\,dx = -\frac{\sin^{n-1}\chi\,\cos\chi}{n\alpha} + \frac{n-1}{n}\int\sin^{n-2}\alpha x\,dx \quad \int\cos^n\alpha x\,dx = +\frac{\cos^{n-1}\chi\,\sin\chi}{n\alpha} + \frac{n-1}{n}\int\cos^{n-2}\alpha x\,dx$

$\int\sin\alpha x\sin\beta x\,dx = -\frac{\sin\gamma x}{2\gamma} + \frac{\sin\delta x}{2\delta} \quad \int\cos\alpha x\cos\beta x\,dx = +\frac{\sin\gamma x}{2\gamma} + \frac{\sin\delta x}{2\delta} \quad \int\sin\alpha x\cos\beta x\,dx = -\frac{\cos\gamma x}{2\gamma} - \frac{\cos\delta x}{2\delta}$

$\int x\sin\alpha x\,dx = \frac{\sin\chi}{\alpha^2} - \frac{x\cos\chi}{\alpha} \quad \int x\cos\alpha x\,dx = \frac{\cos\chi}{\alpha^2} + \frac{x\sin\chi}{\alpha} \quad \int x\genfrac{}{}{0pt}{}{\sin^2}{\cos^2}\alpha x\,dx = \mp\frac{2\chi\sin 2\chi + \cos 2\chi \mp 2\chi^2}{8\alpha^2}$

$\int x\genfrac{}{}{0pt}{}{\sin\alpha x\sin\beta x}{\cos\alpha x\cos\beta x}dx = \mp\frac{x\sin\gamma x}{2\gamma} \mp \frac{\cos\gamma x}{2\gamma^2} + \frac{x\sin\delta x}{2\delta} + \frac{\cos\delta x}{2\delta^2} \quad \int x\sin\alpha x\cos\beta x\,dx = -\frac{x\cos\gamma x}{2\gamma} + \frac{\sin\delta x}{2\delta^2} - \frac{x\cos\delta x}{2\delta} + \frac{\sin\gamma x}{2\gamma^2}$

Definite integrals ($m!! = m(m-2)(m-4)\ldots$, $-1!! = 0!! = 1!! = 1$)

$\int_0^{\pi/2}\sin^\mu x\,dx = \int_0^{\pi/2}\cos^\mu x\,dx = \frac{1}{2}\mathrm{B}(\frac{\mu+1}{2},\frac{1}{2}) = \frac{(n-1)!!}{n!!}\cdot\begin{cases}\frac{\pi}{2} & \text{if } \mu = n \text{ even} \\ 1 & \text{if } \mu = n \text{ odd}\end{cases} \quad \int_{-1}^{+1}\genfrac{}{}{0pt}{}{\sin(m\pi x)\sin(\tilde{m}\pi x)}{\cos(m\pi x)\cos(\tilde{m}\pi x)}dx = \delta_{m,\tilde{m}}$

$\int_0^\alpha\genfrac{}{}{0pt}{}{\sin(\frac{m\pi x}{\alpha})\sin(\frac{\tilde{m}\pi x}{\alpha})}{\cos(\frac{m\pi x}{\alpha})\cos(\frac{\tilde{m}\pi x}{\alpha})}dx = \frac{\alpha}{2}\delta_{m,\tilde{m}}$ also valid $m = 1/2$ $\quad\int_0^\alpha\sin(\frac{m\pi x}{\alpha})\cos(\frac{\tilde{m}\pi x}{\alpha})dx = \begin{cases} 0 & \text{if } m+\tilde{m} \text{ even} \\ \frac{2m\alpha}{m^2-\tilde{m}^2}\frac{1}{\pi} & \text{if } m+\tilde{m} \text{ odd}\end{cases} \quad \int_0^\pi\genfrac{}{}{0pt}{}{\sin x\,dx = 2}{\cos x\,dx = 0}$

$\int_0^\pi\sin^n x\cos^{\tilde{n}} x\,dx = 0\ \forall\,\tilde{n}$ odd $\quad\int_0^{\pi\mu}\genfrac{}{}{0pt}{}{\sin^2\alpha x}{\cos^2\alpha x}dx = \frac{1}{4\alpha}[2\pi\alpha\mu \mp \sin(2\pi\alpha\mu)] \overset{\text{if } \mu = n}{=} \frac{\pi n}{\alpha} \quad \int_0^\pi\genfrac{}{}{0pt}{}{\sin^3 x\,dx = \frac{4}{3}}{\cos^3 x\,dx = 0}$

$\int_0^{2\pi}\genfrac{}{}{0pt}{}{\sin x}{\cos x}dx = 0 \quad \int_0^{2\pi}\sin x\cos x\,dx = 0 \quad \int_0^{2\pi}\sin^n x\cos^{\tilde{n}} x\,dx = 0$ if $n,\tilde{n}$ not both even $\quad\int_0^{2\pi}\genfrac{}{}{0pt}{}{\sin^3 x}{\cos^3 x}dx = 0$

$\int_0^{2\pi}(1-\cos x)^n\sin nx\,dx = 0 \quad \int_0^{2\pi}(1-\cos x)^n\cos nx\,dx = (-1)^n\frac{\pi}{2^{n-1}}$

Parity $\quad$ *Even* : $f_e(-x) = f_e(x)$ $\int_{-\alpha}^{+\alpha}f_e(x)\,dx = 2\int_0^\alpha f_e(x)\,dx$ $\quad$ *Odd* : $f_o(-x) = -f_o(x)$ $\int_{-\alpha}^{+\alpha}f_o(x)\,dx = 0$

$f_e : \cos x,\ \cosh x,\ x^{2n},\ e^{-x^2},\ |x|,\ \delta_{ij},\ \delta(x),\ \mathbb{R},\ 1/f_e,\ f_o',\ f_e\pm f_e,\ f_e\cdot f_e,\ f_o\cdot f_o,\ \mathcal{F}\{f_e(x)\}(\xi), \ldots$

$f_o : \sin x,\ \sinh x,\ x^{2n+1},\ \tan x,\ \mathrm{erf}\,x,\ \mathrm{sign}\,x,\ \ln(\frac{1+x}{1-x}),\ 1/f_o,\ f_e',\ f_o\pm f_o,\ f_e\cdot f_o,\ \mathcal{F}\{f_o(x)\}(\xi), \ldots$

Log/Exp ($r\neq -1$)

$\int x^r\ln x\,dx = x^{r+1}\left(\frac{\ln x}{r+1} - \frac{1}{(r+1)^2}\right) \quad \int(\ln x)^n\,dx = (-1)^n n!\,x\sum_{k=0}^n\frac{(-\ln x)^k}{k!} \quad \int\frac{dx}{(e^{-x/\alpha}+1)} = \alpha\ln(e^{x/\alpha}+1)$

$\int xe^{\alpha x^2}dx = \frac{e^{\alpha x^2}}{2\alpha} \quad \int x^n e^{\alpha x}dx = e^{\alpha x}\sum_{k=0}^n\frac{n!\,(-1)^k}{(n-k)!}\frac{x^{n-k}}{\alpha^{k+1}} \quad \int\frac{e^{\alpha x}}{x^n}dx = \frac{1}{n-1}\left(-\frac{e^{\alpha x}}{x^{n-1}} + \alpha\int\frac{e^{\alpha x}}{x^{n-1}}dx\right)$

Definite integrals ($r - 1$, $\alpha > 0$ $\gamma\equiv$Euler-Mascheroni constant)

$\int_0^\infty x^r e^{-\alpha x^2}dx = \frac{\Gamma\left(\frac{r+1}{2}\right)}{2\alpha^{\frac{r+1}{2}}} = \begin{cases}\frac{(2n-1)!!}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}} & \text{if } r = 2n \\ \frac{n!}{2\,\alpha^{n+1}} & \text{if } r = 2n+1\end{cases} \quad \int_0^\infty e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad \int_0^\infty x^2 e^{-\alpha x^2}dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$

$\int_0^\infty x^r e^{-\alpha x}dx = \frac{\Gamma(r+1)}{\alpha^{r+1}} \overset{\text{if } r = n}{=} \frac{n!}{\alpha^{n+1}}$ $(r > -1, \Re(a) > 0)$ $\quad\int_0^\infty\sqrt{x}e^{-x}dx = \frac{\sqrt{\pi}}{2} \quad \int_0^\infty\frac{x}{e^x-1}dx = \frac{\pi^2}{6}$

$\int_0^\infty e^{-ax^b}dx = a^{-1/b}\Gamma\left(\frac{1}{b}+1\right) \quad \int_{-\infty}^{+\infty}e^{-\alpha x^2+\beta x}dx = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\beta^2}{4\alpha}} \quad \int_0^{2\pi}e^{i(m-\tilde{m})\phi}d\phi = 2\pi\delta_{m,\tilde{m}}$

$\int_0^\infty e^{-\alpha x}\sin(\beta x)dx = \frac{\beta}{\alpha^2+b^2} \quad \int_0^\infty e^{-\alpha x}\cos(\beta x)dx = \frac{\alpha}{\alpha^2+\beta^2} \quad \int_0^\infty\frac{\ln x}{e^x}dx = \int_1^\infty\left(\frac{1}{x} - \frac{1}{\lfloor x\rfloor}\right)dx = -\gamma$

$\int_0^\infty xe^{-\alpha x}\sin(\beta x)dx = \frac{2\alpha\beta}{(\alpha^2+\beta^2)^2} \quad \int_0^\infty xe^{-\alpha x}\cos(\beta x)dx = \frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)^2} \quad \int_\Omega e^{iqr\cos\theta}d\Omega = 4\pi\frac{\sin(qr)}{qr}$

Error function integrals ($\varphi = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $\mu\equiv$mean, $\sigma^2\equiv$variance) $\quad\mathrm{erf}(\pm\infty) = \pm 1 \quad i\,\mathrm{erfi}(z) = \mathrm{erf}(iz)$

$\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = \mathrm{erf}(z) \quad \int\varphi\,dx = \frac{1}{2}\mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right) \quad \int\sqrt{x}\,e^{ax}dx = \frac{\sqrt{x}\,e^{ax}}{a} - \frac{\sqrt{\pi}\,\mathrm{erfi}(\sqrt{a}\sqrt{x})}{2a^{3/2}}$

Miscellaneous

$\frac{d|x|}{dx} = \mathrm{sgn}(x) \quad \frac{d^2|x|}{d^2x} = \delta(x) \quad \langle\sin x\rangle = \langle\cos x\rangle = \sqrt{2}/2 \quad \langle\sin^2 x\rangle = \langle\cos^2 x\rangle = 1/2$

Linear Algebra ($n, m, i, j, k, l \in \mathbb{N}_0$, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{L}, \mathbf{U}, \mathbf{I}, \mathbf{P} \in \mathcal{M}(\mathbb{K})$)

Matrices (Generalizable to arbitrary linear operators)

$\mathbf{A}_{m\times n}$ matrix with m rows and n columns; m and n dimensions of $\mathbf{A}$

$\mathbf{A} = (a)_{ij}$ $\mathbf{A}^{\mathrm{T}} = (a)_{ji} \equiv$ transpose of $\mathbf{A}$ $\mathbf{A}_{n\times n} \equiv$ square matrix

$\mathbf{D} = \mathbf{D}_{n\times n}: \ i \neq j \ \forall i, j \Rightarrow d_{ij} = 0$, $\mathbf{D} = \mathrm{diag}(d_1, \cdots, d_n) \equiv$ diagonal matrix

$\mathbf{L} = \mathbf{L}_{n\times n}: \ l_{ij} = 0 \ \forall i < j$, $\mathbf{L} \equiv$ lower triangular matrix

$\mathbf{U} = \mathbf{U}_{n\times n}: \ u_{ij} = 0 \ \forall i > j$, $\mathbf{U} \equiv$ upper triangular matrix

$\mathbf{I} = \mathbf{I}_n = \mathrm{diag}(1, \cdots, 1) \equiv$ identity matrix $(\mathbf{I}_n)_{ij} = \delta_{ij}$

$\mathbf{A}_{n\times n} \equiv$ Invertible $\Leftrightarrow \exists\, \mathbf{B}_{n\times n} \mid \mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, $\mathbf{B} = \mathbf{A}^{-1} \equiv$ inverse of $\mathbf{A}$

$\mathbf{A}_{n\times n} \equiv$ singular matrix $\Leftrightarrow$ $\mathbf{A}$ not invertible $\Leftrightarrow \det \mathbf{A} = 0$

Let $\mathbf{A}_{m\times n}$, $0 < k \leq m, n$: minor of degree k of $\mathbf{A}$ is the determinant of a matrix obtained from $\mathbf{A}$ by deleting $m-k$ rows and $n-k$ columns

Let $\mathbf{A}_{n\times n}$, $\mathbf{A}_{ij}$ submatrix, by deleting row i and column j from $\mathbf{A}$,

$c_{ij} = (-1)^{i+j} \cdot \det \mathbf{A}_{ij}$ $\mathbf{C} = (c)_{ij} \equiv$ cofactor matrix

$\mathrm{adj}\, \mathbf{A} = \mathbf{C}^{\mathrm{T}} \equiv$ adjugate matrix of $\mathbf{A}$ $\mathbf{A}^{-1} = \mathrm{adj}\, \mathbf{A} / \det \mathbf{A}$

$\mathbf{A} = \mathbf{A}^{\mathrm{T}} \Leftrightarrow \mathbf{A}$ symmetric matrix $\mathbf{A} = -\mathbf{A}^{\mathrm{T}} \Leftrightarrow \mathbf{A}$ anti-symmetric matrix

$\mathbf{A}^{\dagger} = (\overline{\mathbf{A}})^{\mathrm{T}} = \overline{\mathbf{A}^{\mathrm{T}}} \equiv$ conjugate transpose or Hermitian transpose of $\mathbf{A}$

$\mathbf{A} = \mathbf{A}^{\dagger} \Leftrightarrow \mathbf{A}$ Hermitian matrix $\mathbf{A} = -\mathbf{A}^{\dagger} \Leftrightarrow \mathbf{A}$ anti-Hermitian matrix

$\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}\mathbf{A}^{\dagger} \Leftrightarrow \mathbf{A}$ normal matrix $\mathbf{A}^{\dagger} = \mathbf{A}^{-1} \Leftrightarrow \mathbf{A}$ unitary matrix

$\det \mathbf{A}_{n\times n} = |\mathbf{A}| = \sum_{i=1}^{n} a_{ij} c_{ij} = \sum_{j=1}^{n} a_{ij} c_{ij}$ $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\mathrm{tr}\, \mathbf{A}_{n\times n} = \sum_{i=1}^{n} a_{ii}$ $\mathrm{rank}\, \mathbf{A} := \dim(\mathrm{img}\, \mathbf{A}_{m\times n}) \leq \min\{m, n\}$

rank of $\mathbf{A}$: number of linearly independent columns (or rows) of $\mathbf{A}$

$\ker \mathbf{A} = \{\mathbf{x} \in \mathbb{K}^n \mid \mathbf{Ax} = \mathbf{0}\}$ $\ker \mathbf{A} + \mathrm{rank}\, \mathbf{A} = n$, $\mathbf{A}_{m\times n}$

$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} \equiv$ commutator $[\mathbf{A}, \mathbf{B}] = 0 \Leftrightarrow \mathbf{A}, \mathbf{B}$ commute

$\{\mathbf{A}, \mathbf{B}\} = \mathbf{AB} + \mathbf{BA} \equiv$ anticommutator $2\mathbf{AB} = [\mathbf{A}, \mathbf{B}] + \{\mathbf{A}, \mathbf{B}\}$

Let $\mathbf{A}_{n\times n}$, $\mathbf{v}_{n\times 1} \neq \mathbf{0}$, $\lambda \in \mathbb{K}$, $\mathbf{Av} = \lambda\mathbf{v}$: $\mathbf{v} \equiv$ eigenvector, $\lambda \equiv$ eigenvalue

$p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \{\lambda_k\}$ $(\mathbf{A} - \lambda_k \mathbf{I})\mathbf{v}_k = \mathbf{0} \Rightarrow \{\mathbf{v}_k\}$

$\mu_{\mathbf{A}}(\lambda_k) \equiv$ algebraic multiplicity: $\max\{l \mid p(\lambda) = (\lambda - \lambda_k)^l \cdot q(\lambda),\ q(\lambda_k) \neq 0\}$

$\gamma_{\mathbf{A}} = \dim \ker(\mathbf{A} - \lambda_k \mathbf{I}) \equiv$ geometric multiplicity $1 \leq \gamma_{\mathbf{A}}(\lambda_k) \leq \mu_{\mathbf{A}}(\lambda_k)$

$\gamma_{\mathbf{A}}(\lambda_k) = \mu_{\mathbf{A}}(\lambda_k) \ \forall k \Leftrightarrow \exists \mathcal{B}' = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \equiv$ eigenbasis $\Rightarrow \exists \mathbf{P} \mid \mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$

$\mathbf{P} = {}_{\mathcal{B}'}\mathbf{P}_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}\to\mathcal{B}'} \equiv$ change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ ${}_{\mathcal{B}'}D_{\mathcal{B}'} = {}_{\mathcal{B}}A_{\mathcal{B}}$

$\mathbf{P} = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ $\mathbf{D} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ $\mathbf{A} \sim \mathbf{D} \Rightarrow |\mathbf{A}| = |\mathbf{D}|$, $\mathrm{tr}\, \mathbf{A} = \mathrm{tr}\, \mathbf{D}$

Properties ($\theta \in \mathbb{R}$, $\eta, \nu, \omega, \tau \in \mathbb{C}$, $\vec{u}, \vec{v} \in \mathbb{C}^n$)

$\mathbf{A}(\nu + \omega) = \nu\mathbf{A} + \omega\mathbf{A}$ $\tau(\mathbf{A} + \mathbf{B}) = \tau\mathbf{A} + \tau\mathbf{B}$ $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{CB}$ $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ $\mathbf{AB} \not\equiv \mathbf{BA}$

Let v, w arbitrary column vectors, j^{th} column of $\mathbf{A}$ $a_j = \nu \cdot v + \omega \cdot w$:

$|\mathbf{A}| = \nu \cdot |a_1, \cdots, a_{j-1}, v, a_{j+1}, \cdots, a_n| + \omega \cdot |a_1, \cdots, a_{j-1}, w, a_{j+1}, \cdots, a_n|$

$|a_1, \cdots, u, \cdots, u, \cdots, a_n| = 0$ $|\mathbf{A}_\sigma| = \mathrm{sign}(\sigma) \cdot |\mathbf{A}|$, $\sigma \equiv$ permutation

$|\tau\mathbf{A}| = \tau^n |\mathbf{A}|$ $|\mathbf{A}|^{\mathrm{T}} = |\mathbf{A}^{\mathrm{T}}|$ $|\mathbf{A}|^{\dagger} = |\mathbf{A}^{\dagger}|$ $\overline{|\mathbf{A}|} = |\overline{\mathbf{A}}|$ $|\mathbf{A}|^{-1} = |\mathbf{A}^{-1}|$

$\overline{\overline{\mathbf{A}}} = \mathbf{A}$ $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ $|\mathbf{U}| = e^{i\theta}$ $|\mathbf{U}| = 1 \Rightarrow \mathbf{U} \in SU(n)$ $|\mathbf{A}| = \prod_{k=1}^{n} \lambda_k$

$\mathrm{tr}(\tau\mathbf{A}) = \tau\, \mathrm{tr}\, \mathbf{A}$ $\mathrm{tr}\, \mathbf{A} = \mathrm{tr}\, \mathbf{A}^{\mathrm{T}}$ $\mathrm{tr}\, \mathbf{A}^{\dagger} = \mathrm{tr}\, \overline{\mathbf{A}} = \overline{\mathrm{tr}\, \mathbf{A}}$ $\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}\, \mathbf{A} + \mathrm{tr}\, \mathbf{B}$

$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}) \Rightarrow \mathrm{tr}[\mathbf{A}, \mathbf{B}] = 0$ $\mathrm{tr}\, \mathbf{A} = \sum_{k=1}^{n} \lambda_k$

$\mathrm{tr}(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n) = \mathrm{tr}(\mathbf{A}_n\mathbf{A}_1 \cdots \mathbf{A}_{n-1}) = \cdots = \mathrm{tr}(\mathbf{A}_2\mathbf{A}_3 \cdots \mathbf{A}_n\mathbf{A}_1)$

$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$ $(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$ $(\eta\mathbf{A})^{\mathrm{T}} = \eta\mathbf{A}^{\mathrm{T}}$ $(\mathbf{AB})^{\mathrm{T}} = (\mathbf{BA})^{\mathrm{T}}$

$(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$ $\mathrm{rg}\, \mathbf{A} = \mathrm{rg}\, \mathbf{A}^{\mathrm{T}}$ $(\mathbf{A}^{-1})^{\dagger} = (\mathbf{A}^{\dagger})^{-1}$ $\mathrm{rg}\, \mathbf{A} = \mathrm{rg}\, \mathbf{A}^{\dagger}$

$(\mathbf{A}^{\dagger})^{\dagger} = \mathbf{A}$ $(\mathbf{A} + \mathbf{B})^{\dagger} = \mathbf{A}^{\dagger} + \mathbf{B}^{\dagger}$ $(\eta\mathbf{A})^{\dagger} = \overline{\eta}\mathbf{A}^{\dagger}$ $(\mathbf{AB})^{\dagger} = (\mathbf{BA})^{\dagger}$

$\vec{u} \cdot \vec{v} = \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\dagger}\mathbf{v}$ $\|\mathbf{u}\| := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ $(\eta\mathbf{A})^{-1} = \mathbf{A}^{-1}/\eta$ $\mathbf{D}^{-1} = \mathrm{diag}(1/d_i)$

$[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]$ $[\mathbf{A}, \mathbf{B} + \mathbf{C}] = [\mathbf{A}, \mathbf{B}] + [\mathbf{A}, \mathbf{C}]$ $[\mathbf{A}, \mathbf{A}] = [\mathbf{A}, \mathbf{A}^n] = 0$

$[\mathbf{A}, \mathbf{BC}] = [\mathbf{A}, \mathbf{B}]\mathbf{C} + \mathbf{B}[\mathbf{A}, \mathbf{C}]$ $[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = 0$

$[\mathbf{A}, \mathbf{B}]^{\dagger} = [\mathbf{B}^{\dagger}, \mathbf{A}^{\dagger}]$ $\mathbf{A} = \mathbf{A}^{\dagger} \Rightarrow \lambda_{\mathbf{A}} \in \mathbb{R}$ $\mathbf{A} = -\mathbf{A}^{\dagger} \Rightarrow \lambda_{\mathbf{A}} \in i\mathbb{R}$

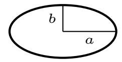
if $\mathbf{A} = \mathbf{A}^{\dagger}, \mathbf{B} = \mathbf{B}^{\dagger}$: $i[\mathbf{A}, \mathbf{B}] = (i[\mathbf{A}, \mathbf{B}])^{\dagger}$, $\{\mathbf{A}, \mathbf{B}\} = \{\mathbf{A}, \mathbf{B}\}^{\dagger}$

if $\mathbf{A} = \mathbf{A}^{\dagger}, \mathbf{B} = \mathbf{B}^{\dagger}$, and $[\mathbf{A}, \mathbf{B}] = 0$: $\mathbf{AB} = (\mathbf{AB})^{\mathrm{T}}$

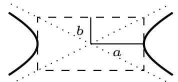
Conics ($\varepsilon, a, b, c, h, k, p, \ell \in \mathbb{R}$), $\varepsilon \equiv$ eccentricity, $c \equiv$ focal distance, $p \equiv$ focal parameter, $\ell \equiv$ semi-latus rectum, $a \equiv$ semi-major axis, $b \equiv$ semi-minor axis, $\ell = p\varepsilon$, $c = a\varepsilon$, $p + c = a/\varepsilon$, $(h, k) \equiv$ center, $(h, k)_{\text{parabola}} \equiv$ vertex

Vertical parabola: $(y-k) = \dfrac{1}{4p}(x-h)^2$, $\varepsilon = 1$ Circle: $(x-h)^2 + (y-k)^2 = a^2$, $\varepsilon = 0$

Ellipse: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, $\varepsilon = \sqrt{1 - \left(\dfrac{b}{a}\right)^2}$



Hyperbola: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, $\varepsilon = \sqrt{1 + \left(\dfrac{b}{a}\right)^2}$



Complex Analysis ($\alpha, \beta, r, \theta, t, p, R \in \mathbb{R}$, $z, w \in \mathbb{C}$, $n, k \in \mathbb{N}_0$, $m \in \mathbb{N}_+$, $i^2 = -1$)
p.v. $\equiv$ principal value $\gamma \equiv$ closed contour path positively oriented (anticlockwise)
$-\gamma \equiv \gamma$ with reverse orientation $\Rightarrow \int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$

$z = \alpha + i\beta = re^{i\theta}$ $r = |z| = \sqrt{\alpha^2 + \beta^2}$ $\theta = \arctan(\beta/\alpha)$ $\overline{z} = \alpha - i\beta$ $z^{-1} = \dfrac{\overline{z}}{r^2} = \dfrac{1}{re^{i\theta}}$

$z\overline{z} = |z|^2$ $z + \overline{z} = 2\Re[z]$ $z - \overline{z} = 2i\Im[z]$ $\sqrt[n]{z} = \sqrt[n]{r}\exp[i(\frac{\theta + 2\pi k}{n})]$, $k < n-1$

$z^w = e^{w \log z}$ $\log z = \ln r + i(\theta \pm 2\pi k) \xrightarrow{\text{p.v.}} \mathrm{Log}\, z = \ln r + i\theta$, $\theta \in (-\pi, \pi]$

$\mathrm{Log}\, e^z = z \Leftrightarrow \Im z \in (-\pi, \pi]$ $\mathrm{Log}(zw) = \mathrm{Log}\, z + \mathrm{Log}\, w \pm i2\pi k$

$e^{\pm i2\pi n} = 1$ $e^{i\frac{\pi}{2} \pm i2\pi n} = i$ $e^{i\pi \pm i2\pi n} = -1$ $e^{i\frac{3\pi}{2} \pm i2\pi n} = -i$

$$f(z)\Big|_{z_0} = \underbrace{\sum_{n=0}^{\infty} a_n(z-z_0)^n}_{\text{analytical part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}}_{\text{principal part}}, \quad a_{\pm n} = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(z)}{(z-z_0)^{\pm n+1}}\, dz$$

$f(z)$ complex differentiable at z_0 if $\exists f'(z_0) = \lim\limits_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$

$f: U \subseteq \mathbb{C} \to \mathbb{C}$, U open set: f holomorphic on U if $\forall z_0 \in U$, $\exists f'(z_0)$
f holomorphic at z_0 if f holomorphic on some neighborhood of z_0
$f(x+iy) = u(x, y) + iv(x, y)$ holomorphic $\Rightarrow u, v$ satisfy Cauchy-Riemann (C.R.)

C.R.: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ or $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$\partial_x u, \partial_y u, \partial_x v, \partial_y v$ continuous and satisfy C.R. $\Rightarrow f$ holomorphic

$\forall f$ holomorphic: u, v harmonic on $\mathbb{R}^2 \Leftrightarrow \nabla^2 u = 0, \nabla^2 v = 0$

$\forall f$ holomorphic and γ enclosing no holes: $\oint_{\gamma} f(z)dz = 0$

$\oint_{\gamma} \dfrac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$ and $\oint_{\gamma} \dfrac{f(z)}{(z-z_0)^{n+1}}\, dz = \dfrac{2\pi i}{n!} f^{(n)}(z_0)$

$\forall f$, if Γ continuously differentiable: $\int_{\sigma} f(z)dz = \int_a^b f(\Gamma(t)) \cdot \Gamma'(t)$

$\ell(\Gamma) = \int_a^b |\Gamma'(t)|dt \equiv$ contour length generally: $\Gamma(t) = \Gamma_R = z_0 + Re^{it}$, $\ell(\Gamma_R) = Rt_{\max}$

$\forall f$ holomorphic on U, except at a finite number of isolated singularities z_k:

$\oint_{\gamma} f(z)dz = 2\pi i \sum_k \mathrm{Res}(f, z_k)$, $\mathrm{Res}(f, z_k) \equiv$ residue of f at z_k

$\equiv$ coefficient c_{-1} of $(z - z_k)^{-1}$ in Laurent series of f around z_k

f holomorphic on U except at $a \in U \equiv f \in \mathcal{O}(U \setminus \{a\})$, possible isolated singularities:

- a removable singularity $\Leftrightarrow \exists g \in \mathcal{O}(U) \mid f(z) = g(z) \ \forall z \in U \setminus \{a\}$
- a pole $\Leftrightarrow \exists g \in \mathcal{O}(U), g(a) \neq 0 \mid f(z) = \frac{g(a)}{(z-a)^m} \ \forall z \in U \setminus \{a\}$; $m \equiv$ pole order
- a essential singularity $\Leftrightarrow$ Laurent series principal part has ∞ terms

For poles z_j of order m: $\mathrm{Res}(f, z_j) = \dfrac{1}{(m-1)!} \dfrac{d^{m-1}}{dz^{m-1}}((z - z_j)^m f(z))\Bigg|_{z_j}$

$\nexists$ such formula for essential singularities

Estimation lemma: $f(z) \in \mathbb{C}$, continuous on Γ and $\exists\ M \in \mathbb{R}$ such that :

$|f(z)| \leq M \ \forall z \in \Gamma \Rightarrow \left|\int_{\Gamma} f(z)\, dz\right| \leq M \cdot l(\Gamma)$, $M := \sup\limits_{z\in\Gamma} |f(z)|$

$\therefore$ if $|f(z)| \leq \frac{C}{|z|^p}$, $p > 1$; $C_{R^+} \equiv \Gamma_R$, $t \in [0, \pi]$, $z_0 = 0 \Rightarrow \left|\int_{C_{R^+}} f(z)\, dz\right| \xrightarrow{R\to\infty} 0$

Jordan's lemma: $f(z) = e^{i\alpha z} g(z) \in \mathbb{C}, \alpha > 0$, continuous on $C_{R^+} \Rightarrow$

$\left|\int_{C_{R^+}} f(z)\, dz\right| \leq \dfrac{\pi}{\alpha} M_R$, $M_R := \max\limits_{\theta\in[0,\pi]} \left|g\left(Re^{i\theta}\right)\right|$ $\therefore$ if $M_R \xrightarrow{R\to\infty} 0 \Rightarrow \int_{C_{R^+}} f(z)\, dz \xrightarrow{R\to\infty} 0$

Analogous for $C_{R^-} \equiv \Gamma_R$, $t \in [\pi, 2\pi]$, $z_0 = 0$ when $\alpha < 0$

Fourier Analysis ($\xi, x \in \mathbb{R}$)

$\mathcal{F}\{f(x)\}(\xi) = \hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-i2\pi\xi x}\, dx$ $\mathcal{F}^{-1}\{f(\xi)\}(x) = \int_{-\infty}^{+\infty} f(\xi)e^{+i2\pi\xi x}\, d\xi$

$f(x - x_0) \overset{\mathcal{F}}{\Longleftrightarrow} e^{-i2\pi x_0 \xi}\hat{f}(\xi)$ $e^{i2\pi\xi_0 x} f(x) \overset{\mathcal{F}}{\Longleftrightarrow} \hat{f}(\xi - \xi_0)$ $f(ax) \overset{\mathcal{F}}{\Longleftrightarrow} \dfrac{1}{|a|}\hat{f}\left(\dfrac{\xi}{a}\right)$

$f(x) \in \mathbb{R} \Rightarrow \hat{f}(-\xi) = \overline{\hat{f}(\xi)}$ $\mathcal{F}^{-1}f(x) = \mathcal{F}(f(-x))$ $\mathcal{F}(f(-x)) = (\mathcal{F}f)(-x)$ $\mathcal{F}^2 f(x) = f(-x)$

Convolution

$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)\, dy$ $f * g = g * f$ $(f * g) * h = f(g * h)$

$f * (g + h) = f * g + f * h$ $\mathcal{F}(f * g) = (\mathcal{F}f)(\mathcal{F}g)$ $\mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$

Dirac Delta

$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ $\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$ $\mathcal{F}\{\delta(x)\}(\xi) = 1 \Leftrightarrow \mathcal{F}^{-1}\{1\}(x) = \delta(x)$

$\delta(ax) = \dfrac{1}{|a|}\delta(x)$ $h(x)\delta(x) = h(0)\delta(x)$ $\delta(x) * f(x) = f(x)$

$\delta(x-a) * f(x) = f(x-a)$ $\delta(x-a) * \delta(x-b) = \delta(x - (a+b))$

$\int_{-\infty}^{+\infty} \delta(x-a)\, f(x)\, dx = f(a)$ $\int_{-\infty}^{+\infty} \delta(x)\, f(x+a)\, dx = f(a)$

Real integrals via residues
($\mathfrak{R} \equiv$ rational function, no essential singularities nor branch cuts inside contour)

Rational trigonometric integrals (**c.v.**: $z = e^{i\theta}$)

$\mathfrak{R}$ with no singularities on contour $\equiv$ unit circle: $x^2 + y^2 = 1$

$\int_0^{2\pi} \mathfrak{R}(\cos\theta, \sin\theta)d\theta = 2\pi \sum\limits_{|z_k|<1} \mathrm{Res}\left[\dfrac{1}{z}\mathfrak{R}\left(\dfrac{z + z^{-1}}{2}, \dfrac{z - z^{-1}}{2i}\right), z_k\right] \in \mathbb{R}$

Rational improper integrals

$\mathfrak{R}$ with no poles in $\mathbb{R}$, $\lim\limits_{x\to\infty} x\,\mathfrak{R}(x) = 0$, contour $\equiv$ upper/lower half-plane

$\int_{-\infty}^{+\infty} \mathfrak{R}(x)dx = 2\pi i \sum\limits_{\Im[z_k]>0} \mathrm{Res}\left[\mathfrak{R}(z), z_k\right] = -2\pi i \sum\limits_{\Im[z_k]<0} \mathrm{Res}\left[\mathfrak{R}(z), z_k\right] \in \mathbb{R}$

Cauchy principal value

f continuous on $[a, b] \in \mathbb{R}$ except at isolated poles $\{x_k\}$, $m = 1$

$\int_a^b f(x)\, dx \equiv \mathbf{P.V.} \int_a^b f(x)\, dx \equiv \lim\limits_{\varepsilon\to 0^+} \left\{ \int_a^{x_1 - \varepsilon} f(x)\, dx + \int_{x_1 + \varepsilon}^{x_2 - \varepsilon} f(x)\, dx + \cdots + \int_{x_n + \varepsilon}^{b} f(x)\, dx \right\}$

Improper integrals with principal value

f holomorphic, except at isolated poles $\{z_k\}$, $m = 1$; $\lim\limits_{|z|\to\infty} z\, f(z) = 0$ on $\Im z > 0 \Rightarrow$

$\int_{-\infty}^{+\infty} f(x)\, dx = 2\pi i \sum\limits_{\Im[z_k]>0} \mathrm{Res}\left[f(z), z_k\right] + \pi i \sum\limits_{\Im[z_k]=0} \mathrm{Res}\left[f(z), z_k\right] \in \mathbb{R}$ Analogous for $\Im z < 0$, just multiply by (-1)

Semi-improper integrals with principal value

f holomorphic, except at isolated singularities $\{z_k\} \not\subset \mathbb{R}^+$, and except at isolated poles $\{x_k\} \in \mathbb{R}^+$, $m = 1$; $\lim\limits_{|z|\to\infty} z\, f(z) \neq \infty$; $\lim\limits_{z\to 0} f(z) \neq \infty \Rightarrow \forall\alpha \in (0, 1)$:

$\int_0^{\infty} \dfrac{f(x)}{x^\alpha}\, dx = \dfrac{2\pi i}{1 - e^{-2\pi i\alpha}} \sum\limits_k \mathrm{Res}\left[\dfrac{f(z)}{z^\alpha}, z_k\right] + \dfrac{\pi i(1 + e^{-2\pi i\alpha})}{1 - e^{-2\pi i\alpha}} \sum\limits_k \mathrm{Res}\left[\dfrac{f(z)}{z^\alpha}, x_k\right] \in \mathbb{R}$

Fourier-like integrals

f holomorphic on $\Im z \gtrless 0$, except at isolated sigularities $\{z_k\}$, and with isolated poles $\{x_k\} \in \mathbb{R}$, $m = 1$; $\lim\limits_{|z|\to\infty, \Im z \gtrless 0} f(z) = 0 \Rightarrow$

$\int_{-\infty}^{+\infty} f(x)e^{\pm ikx}\, dx = \pm 2\pi i \sum\limits_{\Im z_k \gtrless 0} \mathrm{Res}\left[f(z)e^{\pm ikz}, z_k\right] \pm \pi i \sum\limits_k \mathrm{Res}\left[f(z)e^{\pm ikz}, x_k\right]$

Trick $\int_{-\infty}^{+\infty} \cos x\, dx = \Re\left[\int_{-\infty}^{+\infty} e^{ix}\, dx\right]$ $\int_{-\infty}^{+\infty} \sin x\, dx = \Im\left[\int_{-\infty}^{+\infty} e^{ix}\, dx\right]$

Pauli Matrices σ ($\mathrm{tr}\, \sigma_j = 0$, $\det \sigma_j = -1$, $\sigma_j^2 = \mathbf{I}_2$, $\sigma_j = \sigma_j^{\dagger} = \sigma_j^{-1}$)

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma_j = \begin{pmatrix} \delta_{j3} & \delta_{j1} - i\,\delta_{j2} \\ \delta_{j1} + i\,\delta_{j2} & -\delta_{j3} \end{pmatrix}$

$\sigma_j\sigma_k = \delta_{jk} + i\epsilon_{jkl}\sigma_l$ $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$ $\{\sigma_j, \sigma_k\} = 2\delta_{jk}\mathbf{I}_2$ $i\sigma_1\sigma_2\sigma_3 = -\mathbf{I}_2$

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = \sigma_1\hat{x}_1 + \sigma_2\hat{x}_2 + \sigma_3\hat{x}_3$ $\vec{a} \cdot \vec{\sigma} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}$

$\det(\vec{a} \cdot \vec{\sigma}) = -|\vec{a}|^2$ $\frac{1}{2}\,\mathrm{tr}\,((\vec{a} \cdot \vec{\sigma})\vec{\sigma}) = \vec{a}$ $\lambda^{\text{eigen}}_{(\vec{a}\cdot\vec{\sigma})} = \pm|\vec{a}|$ $[\vec{a} \cdot \vec{\sigma}, \vec{b} \cdot \vec{\sigma}] = 2i(\vec{a} \times \vec{b}) \cdot \sigma$

Study of a function in $\mathbb{R}$ ($f : \text{Dom}_f \subseteq \mathbb{R} \to \mathbb{R},\ x \mapsto f(x)$)

1. Find its domain (where the function is defined).
2. Analyse its simmetry.
3. Obtain the intersections with the axis: x-axis: $f(x) = 0$, y-axis: $f(0)$.
4. Compute $f'(x)$ and find the critical points: $f'(x) = 0$ or $\nexists f'(x)$.
5. Intervals of monotonicity: $f'(x) > 0 \to$ increasing $f'(x) < 0 \to$ decreasing.
6. Compute $f''(x)$ and find the inflection points: $f''(x) = 0$ or $\nexists f''(x)$.
7. Intervals of concavity: $f''(x) > 0 \to$ convex ($\cup$) $f''(x < 0) \to$ concave ($\cap$).
8. Local maxima and minima (two methods):
   - Analyse the monotonicity of $f(x)$ to the left and right of the critical point:
     $f : \searrow$ to $\nearrow \Rightarrow$ local minimum $f : \nearrow$ to $\searrow \Rightarrow$ local maximum
     If both sides have the same monotonicity $\Rightarrow$ inflection point
   - Evaluate $f''(x)$ at the critical point x_0:
     $f''(x_0) > 0 \Rightarrow$ local minimum $f''(x_0) < 0 \Rightarrow$ local maximum
     If $f'(x_0) = f''(x_0) = \cdots = f^{(k-1)}(x_0) = 0,\ f^{(k)}(x_0) \neq 0$:
     ⋆ If k is even: same criteria as with f''
     ⋆ If k is odd: $x_0 \Rightarrow$ inflection point

9.Asymptotes:
   - Vertical: Compute the left-hand and right-hand limits at discontinuities.
   - Horizontal: $\lim_{x\to\pm\infty} f(x) = L < \infty \Rightarrow$ horizontal asymptote $y = L$.
   - Oblique: if $\lim_{x\to\pm\infty} f(x) = L = \pm\infty \Rightarrow$ compute $m = \lim_{x\to\infty} f(x)/x \Rightarrow$
     if $m \neq 0$ compute $\lim_{x\to\pm\infty} n = [f(x) - mx] \Rightarrow$ oblique asymptote: $y = mx + n$.

EXTRA. Tangent line at $x = a$: $y - b = m(x - a)$ $\quad b = f(a) \quad m = f'(a)$

Optimization of scalar fields in $\mathbb{R}^n$ ($f : \text{Dom}_f \subseteq \mathbb{R}^n \to \mathbb{R},\ \mathbf{x} \mapsto f(\mathbf{x})$)

1. Find the domain and check for openness/compactness.
2. Compute the gradient vector: $\nabla f(\mathbf{x}) = \left(\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}\right)$.
3. Find critical points (stationary points): $\nabla f(\mathbf{x}) = \mathbf{0}$.
4. Compute the Hessian matrix: $H_f(\mathbf{x})_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$.
5. Classification of a critical point $\mathbf{x_0}$ using the eigenvalues $\{\lambda_i\}$ of $H_f(\mathbf{x_0})$:
   - All $\lambda_i > 0 \Rightarrow H_f$ positive definite $\Rightarrow \mathbf{x_0}$ is a **local minimum**.
   - All $\lambda_i < 0 \Rightarrow H_f$ negative definite $\Rightarrow \mathbf{x_0}$ is a **local maximum**.
   - Mixed signs in $\lambda_i \Rightarrow H_f$ indefinite $\Rightarrow \mathbf{x_0}$ is a **saddle point**.
   - Some $\lambda_i = 0 \Rightarrow$ The second-order test is **inconclusive**.
6. Sylvester's Criterion (for $n = 2$): $\Delta_1 = f_{xx},\ \Delta_2 = \det(H_f)$.
   $\Delta_2 > 0$ and $\Delta_1 > 0 \Rightarrow$ local minimum.
   $\Delta_2 > 0$ and $\Delta_1 < 0 \Rightarrow$ local maximum.
   $\Delta_2 < 0 \Rightarrow$ saddle point.
   $\Delta_2 = 0 \Rightarrow$ inconclusive.

Constrained Optimization: Lagrange Multipliers ($f, g_i : \mathbb{R}^n \to \mathbb{R}$)

1. Combine objective f and m constraints $g_i(\mathbf{x}) = c_i$ using multipliers λ_i :
$$\mathcal{L}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i (g_i(\mathbf{x}) - c_i)$$
2. Solve the system of equations: $\begin{cases} \frac{\partial \mathcal{L}}{\partial x_j} = 0 \Rightarrow \nabla f(\mathbf{x}) = \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}) \\ \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \Rightarrow g_i(\mathbf{x}) = c_i \end{cases}$
3. Solve for $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ to find critical points P_j.
4. Evaluate $\bar{H}$ at each P_j. Size is $(m + n) \times (m + n)$:
$$\bar{H}(\mathbf{x}, \boldsymbol{\lambda}) = \left(\begin{array}{c|c} \mathbf{0}_{m\times m} & \mathbf{J}_g(\mathbf{x}) \\ \hline \mathbf{J}_g(\mathbf{x})^T & \mathbf{H_x}(\mathcal{L}) \end{array}\right) \qquad J_g(\mathbf{x}) = \left(\frac{\partial g_i}{\partial x_j}\right) \qquad H_{\mathbf{x}}\mathcal{L} = \left(\frac{\partial^2 \mathcal{L}}{\partial x_i \partial x_j}\right)$$
5. Check the last $n - m$ minors Δ_k of $\bar{H}$, from $k = 2m + 1$ up to $k = n + m$.
   - All checked minors have the same sign as $(-1)^m \Rightarrow P_j$ **local minimum**.
   - Minors alternate in sign, starting with $(-1)^{m+1} \Rightarrow P_j$ **local maximum**.

   – If $n = 2, m = 1$: let $\Delta_3 = \det(\bar{H}) \implies \Delta_3 > 0 \Rightarrow$ max. $\Delta_3 < 0 \Rightarrow$ min.
6. Compare $f(P_j)$ and check boundaries (if the domain is not compact).

Properties of Products

$$\prod_{i=1}^{n} c = c^n \qquad \prod_{i=1}^{n} (k \cdot a_i) = k^n \prod_{i=1}^{n} a_i \qquad \prod_{i=m}^{n} (a_i \cdot b_i) = \left(\prod_{i=m}^{n} a_i\right)\left(\prod_{i=m}^{n} b_i\right)$$
$$\prod_{i=m}^{n} \frac{a_i}{b_i} = \frac{\prod_{i=m}^n a_i}{\prod_{i=m}^n b_i} \qquad \ln\left(\prod_{i=m}^{n} a_i\right) = \sum_{i=m}^{n} \ln(a_i) \qquad \prod_{i=m}^{n} \frac{a_i}{a_{i-1}} = \frac{a_n}{a_{m-1}}$$
$$\prod_{i=m}^{n} a_i = \prod_{j=m+k}^{n+k} a_{j-k} \qquad \prod_{i=1}^{n} i = n! \qquad \prod_{i=1}^{\infty} (1 + a_i) < \infty \iff \sum_{i=1}^{\infty} a_i < \infty$$

Properties of Summations

$$\sum_{i=m}^{n} (k \cdot a_i) = k \sum_{i=m}^{n} a_i. \qquad \sum_{i=1}^{n} c = n \cdot c. \qquad \sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i.$$
$$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k} \qquad \sum_{i=m}^{n} a_i = \sum_{i=m}^{p} a_i + \sum_{i=p+1}^{n} a_i$$
$$\sum_{i=m}^{n} (a_i - a_{i-1}) = a_n - a_{m-1}. \qquad \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}.$$
- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. • $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. • $\displaystyle\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}$.

Applications of Integration

Arc length of a curve

For the curve $y = f(x)$ over the interval $[a, b]$: $L = \displaystyle\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

For the curve $x = f(y)$ over the interval $[c, d]$: $L = \displaystyle\int_c^d \sqrt{1 + [f'(y)]^2}\, dy$

For the curve $(x(t), y(t))$ for the interval $t \in [a, b]$: $L = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dy$

Area between Curves

For the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$, where $f(x) \geq g(x)$:
$$A = \int_a^b [f(x) - g(x)]\, dx$$
For the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$, where $f(y) \geq g(y)$:
$$A = \int_c^d [f(y) - g(y)]\, dy$$

Surface of Revolution

For a surface formed by rotating the curve $y = f(x)$ between the interval $[a, b]$, around the line $y = K$:
$$S = 2\pi \int_a^b |f(x) - K| \sqrt{1 + [f'(x)]^2}\, dx$$
For a surface formed by rotating the curve $x = f(y)$ between the interval $[c, d]$, around the line $x = K$:
$$S = 2\pi \int_a^b |x - K| \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$
For a surface obtained by rotating the curve $(x(t), y(t))$ for the interval $t \in [a, b]$, around the line $y = K$:
$$S = 2\pi \int_a^b |y(t) - K| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
For a surface obtained by rotating the curve $(x(t), y(t))$ for the interval $t \in [c, d]$, around the line $x = K$:
$$S = 2\pi \int_c^d |x(t) - K| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Volume of solid of revolution

For a solid formed by rotating the region between the curves of $f(x)$ and $g(x)$, defined between the interval $[a, b]$, around the line $y = K$:
$$V = \pi \int_a^b ([f(x) - K]^2 - [g(x) - K]^2) dx$$
For a solid formed by rotating the region between the curves of $f(x)$ and $g(x)$, defined between the interval $[a, b]$, around the line $x = K$:
$$V = 2\pi \int_a^b (x - K)[f(x) - g(x)] dx$$

Gram Schmidt orthonormalization (V inner product space, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ L.I. vectors)

$$\mathbf{u}_1 = \mathbf{v}_1 \qquad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$
$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 \qquad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$$
$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 \qquad \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}$$
$$\ldots$$
$$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k-1} \frac{\langle \mathbf{v}_k, \mathbf{u}_j\rangle}{\langle \mathbf{u}_j, \mathbf{u}_j\rangle}\mathbf{u}_j \qquad \mathbf{e}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}$$

- $\mathbb{R}^n : \langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y} = \sum_{i=1}^n x_i y_i$ • $L^2 : \langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx$ • Matrices: $\langle A, B\rangle = \text{Tr}(A^\dagger B)$

Combinatorics

Permutations (Order matters, all n elements are used)

Arrangements of n distinct elements: $P_n = n!$

Arrangements of n distinct elements in a closed loop: $PC_n = (n - 1)!$

Arrangements of n elements with repetition (element i repeated n_i times):
$$PR_n^{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!}, \quad \text{with} \sum_{i=1}^{k} n_i = n$$

Variations (Order matters, k elements selected from n)

Arrangements of k distinct elements from n: $V_{n,k} = \dfrac{n!}{(n - k)!}$

Arrangements of k elements from n, repetition allowed: $VR_{n,k} = n^k$

Combinations (Order does **not** matter, k elements selected from n)

Selection of k distinct elements from n: $C_{n,k} = \dbinom{n}{k} = \dfrac{n!}{k!(n - k)!}$

Selection of k elements from n, repetition allowed: $CR_{n,k} = \dbinom{n + k - 1}{k}$

Binomial Properties

$$\binom{n}{k} = \binom{n}{n - k} \qquad \binom{n}{k} = \binom{n - 1}{k} + \binom{n - 1}{k - 1} \qquad \sum_{i=k}^{n} \binom{i}{k} = \binom{n + 1}{k + 1}$$
$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k - j} = \binom{m + n}{k}$$
$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad \ln n! \approx n \ln n - n \qquad !n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx \frac{n!}{e}$$

Special Functions & Orthogonal Polynomials

Legendre Polynomials

Differential Equation: $(1 - x^2)y'' - 2xy' + l(l + 1)y = 0, \quad x \in [-1, 1]$

Rodrigues' Formula: $P_l(x) = \dfrac{1}{2^l l!}\dfrac{d^l}{dx^l}(x^2 - 1)^l$

Generating Function: $(1 - 2xt + t^2)^{-1/2} = \displaystyle\sum_{l=0}^{\infty} P_l(x) t^l$

Associated Legendre: $P_l^m(x) = (-1)^m (1 - x^2)^{m/2} \dfrac{d^m}{dx^m} P_l(x)$

Hermite Polynomials

Differential Equation: $y'' - 2xy' + 2ny = 0, \quad x \in (-\infty, \infty)$

Rodrigues' Formula: $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n} e^{-x^2}$

Generating Function: $e^{2xt - t^2} = \displaystyle\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$

Orthogonality: $\displaystyle\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{nm}$

Associated Laguerre Polynomials

Differential Equation: $xy'' + (k + 1 - x)y' + ny = 0, \quad x \in [0, \infty)$

Rodrigues' Formula: $L_n^k(x) = \dfrac{e^x x^{-k}}{n!}\dfrac{d^n}{dx^n}(e^{-x} x^{n+k})$

Normalization: $\displaystyle\int_0^{\infty} e^{-x} x^k L_n^k(x) L_m^k(x) dx = \frac{(n + k)!}{n!}\delta_{nm}$

Spherical Harmonics

$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{2l + 1}{4\pi}\frac{(l - m)!}{(l + m)!}} P_l^m(\cos\theta) e^{im\phi}$$

Orthogonality: $\displaystyle\int_0^{2\pi}\int_0^{\pi} Y_l^m Y_{l'}^{m'*} \sin\theta\, d\theta\, d\phi = \delta_{ll'}\delta_{mm'}$

Parity: $Y_l^m(\pi - \theta, \phi + \pi) = (-1)^l Y_l^m(\theta, \phi)$

Bessel Functions

Bessel Equation: $x^2 y'' + xy' + (x^2 - n^2)y = 0$

Generating Function: $e^{\frac{x}{2}(t - t^{-1})} = \displaystyle\sum_{n=-\infty}^{\infty} J_n(x) t^n$

Orthogonality on $[0, a]$: $\displaystyle\int_0^a J_n(x_{ni}\frac{\rho}{a}) J_n(x_{nj}\frac{\rho}{a}) \rho\, d\rho = \frac{a^2}{2}[J_{n+1}(x_{ni})]^2 \delta_{ij}$

Polynomial	Interval	Weight $w(x)$	Normalization
Legendre	$[-1, 1]$	1	$2/(2n + 1)$
Hermite	$(-\infty, \infty)$	e^{-x^2}	$\sqrt{\pi} 2^n n!$
Laguerre	$[0, \infty)$	$x^k e^{-x}$	$(n + k)!/n!$

Tensors (generalizable to $\mathbb{R}^n$)

Definition and Operations Vectors can expressed in different bases: $\{e_1, e_2\}, \{e_{1'}, e_{2'}\}, \dots$

$$\vec{A} = A^1 e_1 + A^2 e_2 = (e_1, e_2)(A^1, A^2)^{\mathrm{T}} = A^{1'} e_{1'} + A^{2'} e_{2'} = (e_{1'}, e_{2'})(A^{1'}, A^{2'})^{\mathrm{T}}$$

Einstein convention: summation over repeated indices (up - down)

inverse: primed $\leftrightarrow$ unprimed, transpose: upper $\leftrightarrow$ lower

$$M = (M^{i'}_j) = \begin{pmatrix} M^{1'}_1 & M^{2'}_1 \\ M^{1'}_2 & M^{2'}_2 \end{pmatrix} \quad (M^{-1})^{\mathrm{T}} = (M^{j}_{i'}) = \begin{pmatrix} M^{1}_{1'} & M^{2}_{1'} \\ M^{1}_{2'} & M^{2}_{2'} \end{pmatrix} \quad M^j_{i'} M^{i'}_k = \delta^j_k$$

Change of basis: $A^{i'} = M^{i'}_j A^j$, $e_{i'} = M^j_{i'} e_j$, $\det M \neq 0$

Covariant v^i: transform against basis vectors $\{e_i\}$, with $M^{i'}_j$

Covariant w_i: transform with basis vectors $\{e_i\}$, with $M^j_{i'}$

Dot product via metric: $g_{ij} = e_i \cdot e_j \quad g = g^{\mathrm{T}} \quad g^{-1} \to$ raises indices

$$\vec{A} \cdot \vec{B} = A^1 B^1 g_{11} + A^1 B^2 g_{12} + A^2 B^1 g_{21} + A^2 B^2 g_{22} = A^i g_{ij} B^j = \vec{A}^{\mathrm{T}} g \vec{B} \quad \|\vec{A}\| = \sqrt{\vec{A} \cdot \vec{A}}$$

Coordinate metrics in flat euclidean metric:

$g_{\text{cartesian}} = \delta_{ij} = \mathbb{I}_n \quad g_{\text{spherical}} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta) \quad g_{\text{cylindrical}} = \mathrm{diag}(1, \rho^2, 1)$

Inverses: $g^{-1}_{\text{cart}} = \delta_{ij} \quad g^{-1}_{\text{sph}} = \mathrm{diag}(1, 1/r^2, 1/r^2 sin^2\theta) \quad g^{-1}_{\text{cyl}} = \mathrm{diag}(1, 1/\rho^2, 1)$

Dual Basis $\{e^1, e^2\}$ dual to $\{e_1, e_2\}$ $\quad e^i \cdot e_j = \delta^i_j$

Relation with metric: $e^i = g^{ij} e_j \quad g^{ij} \equiv$ inverse of the metric

$\vec{A} = A^i e_i = A_i g^{ij} e_j$ **Index lowering:** $A_i = g_{ij} A^j$ **Index raising:** $A^i = g^{ij} A_j$

Metric under change of basis: $g_{i'j'} = M^i_{i'} M^j_{j'} g_{ij} \Leftrightarrow g' = (M^{-1})^T g M^{-1}$

Dot product is invariant under change of basis

Tensor: Any object that transforms as: $T'_{i'j'} = M^i_{i'} M^j_{j'} T_{ij}$ is a tensor

Tensor product properties: ($\vec{A}, \vec{B}, \vec{C}$ vectors, $\lambda \in \mathbb{R}$, V, $V \otimes V$ vector spaces)

1. $(\lambda\vec{A}) \otimes \vec{B} = \lambda(\vec{A} \otimes \vec{B})$
2. $\vec{A} \otimes (\lambda\vec{B}) = \lambda(\vec{A} \otimes \vec{B})$
3. $\vec{A} \otimes \vec{B} \neq \vec{B} \otimes \vec{A}$
4. $(\vec{A} + \vec{B}) \otimes \vec{C} = \vec{A} \otimes \vec{C} + \vec{B} \otimes \vec{C}$
5. $\vec{A} \otimes (\vec{B} + \vec{C}) = \vec{A} \otimes \vec{B} + \vec{A} \otimes \vec{C}$
6. $(\vec{A} \otimes \vec{B})(\vec{C} \otimes \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D})$

Bases of tensor product space $V \otimes V$: $\{e_i \otimes e^j\}$, $\{e^i \otimes e_j\}$, $\{e^i \otimes e^j\}$, $\{e_i \otimes e_j\}$

Equivalent definition of tensor: Element of $V \otimes V$ formed as a linear combination of the basis elements: $\mathcal{T} = T_{11}\, e^1 \otimes e^1 + T_{12}\, e^1 \otimes e^2 + T_{21}\, e^2 \otimes e^1 + T_{22}\, e^2 \otimes e^2$

In compact and general notation: $\mathcal{T} = T_{ij}\, e^i \otimes e^j$ (generalizable to the other bases).

A tensor of type (r, s) has r contravariant and s covariant indexes.

$\mathcal{T} \cdot \mathcal{V} = T_{ij} V_{kl} g^{ik} V^{jl} = T_{ij} V^{ij} \qquad T_{ljk} = g_{il} T^i{}_{jk} \qquad T^i{}_j{}^l = g^{kl} T^i{}_{jk}$

Symmetric: $S_{\alpha\beta} = S_{\beta\alpha} \quad S^{\alpha\beta} = S^{\beta\alpha} \Rightarrow 2S^{\alpha\beta} T_{\alpha\beta} = S^{\alpha\beta}(T_{\alpha\beta} + T_{\beta\alpha})$

Antisymmetric: $A_{\alpha\beta} = -A_{\beta\alpha} \quad A^{\alpha\beta} = -A^{\beta\alpha} \Rightarrow 2A^{\alpha\beta} T_{\alpha\beta} = A^{\alpha\beta}(T_{\alpha\beta} - T_{\beta\alpha})$

Tensor Extension to a Manifold

Manifold $\mathcal{M}$: a surface (or hypersurface) embedded in a higher-dimensional space, Cartesian or Lorentzian. Before we were on the tangent plane to the manifold $T_P\mathcal{M}$. The tangent bundle of $\mathcal{M}$ is $\bigcup_{P \in \mathcal{M}} T_P\mathcal{M}$ and it has double the dimension of $\mathcal{M}$.

1. We need the expression for the coordinate change: $x^{i'} = x^{i'}(x^1, \dots, x^n)$

This function can be understood as a parametrization over the manifold.

It allows tensors to be consistently defined over the whole manifold.

2. Compute the Jacobian matrix of the transformation and its inverse:

$$M = M^{i'}_j = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \cdots & \frac{\partial x^{1'}}{\partial x^n} \\ \cdots & \ddots & \cdots \\ \frac{\partial x^{n'}}{\partial x^1} & \cdots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \quad M^{-1} = M^i_{j'} = \begin{pmatrix} \frac{\partial x^1}{\partial x^{1'}} & \cdots & \frac{\partial x^1}{\partial x^{n'}} \\ \cdots & \ddots & \cdots \\ \frac{\partial x^n}{\partial x^{1'}} & \cdots & \frac{\partial x^n}{\partial x^{n'}} \end{pmatrix}$$

Change of coordinate matrices behave as a change of basis matrices.

3. We can construct the basis vectors as before: $e_{i'} = M^j_{i'} e_j$. In this way, each vector $e_{i'}$ moves in the direction of change of $x^{i'}$, and is constant in $x^j\ \forall j \neq i$.

NOTE: When computing the basis vectors, use $M^j_{i'} = (M^{-1})^{\mathrm{T}}$, not $M^{i'}_j = M$.

How to obtain the metric? We need to parametrize the surface by embedding it in a Cartesian space of higher dimension. This space has coordinates X^i

1. We parametrize the surface: $X^i = X^i(x^j)$.

2. The tangent vectors to the surface will be: $e_i = \frac{\partial X^i}{\partial x^i} e_{X^i}$

3. By the very definition of the metric: $g_{ij} = e_i \cdot e_j \quad e_{X^i} \cdot e_{X^j} = \delta_{X^i X^j}$

ODEs ($\alpha, \beta, c \in \mathbb{R}$, $\lambda \in \mathbb{C} \Leftrightarrow \lambda = \alpha + i\beta$, $y' = \frac{dy}{dx}$, sol $\equiv$ solution, const $\equiv$ constant)

First Order Equations

Separable $y' = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx$

Linear $y' + a(x)y = r(x) \Rightarrow y(x) = \left[\int r(x)\, e^{\int a(x)\, dx}\, dx + C\right] e^{-\int a(x)\, dx}$

Exact $M(x, y)\, dx + N(x, y)\, dy = 0$; if $\partial_y M = \partial_x N \Rightarrow \exists f(x, y) \equiv$ const, with:

$\partial_x f = M \quad \partial_y f = N$ (Solve for f)

Non-Exact $M(x, y)\, dx + N(x, y)\, dy \neq 0$; if $\begin{cases} \frac{\partial_y M - \partial_x N}{N} = g(x) \Rightarrow \mu = e^{\int g\, dx} \\ \frac{\partial_x N - \partial_y M}{M} = h(y) \Rightarrow \mu = e^{\int h\, dy} \end{cases} \Rightarrow$

$\Rightarrow \mu[M(x, y)\, dx + N(x, y)\, dy] = 0 \Rightarrow$ Exact

Bernoulli $y' + a(x)y = r(x)y^n \to$ c.v. $z := y^{1-n} \Rightarrow$ Linear

Important Concepts

Linear: $y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = r(x)$

$r(x) = 0 \forall x \Rightarrow$ Homogeneous (homo) $\ r(x) \not\equiv 0 \Rightarrow$ Inhomogeneous (inhomo)

$\{y_i(x)\}_1^n$ Linearly Independent (LI) $\Leftrightarrow \mathcal{W}(\{y_i(x)\}) := \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$

$\{y_i(x)\}$ LI sols of homo ODE $\Rightarrow y_h = c_1 y_1 + \cdots + c_n y_n$

$y_p \equiv$ particular sol of inhomo ODE $\ y = y_h + y_p \equiv$ general sol of the ODE

Higher Order Linear ODEs

Constant Coefficients (for homo sol) $\{a_i\}_1^n \equiv$ const $\Rightarrow$ Let $y_h = e^{\lambda x}$, substitute $\Rightarrow$

$\Rightarrow$ solve for $\{\lambda_i\}$: $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$

- $\{\lambda_i\} \in \mathbb{R}$, no repetition: $y_h = C_1 e^{\lambda_1 x} + \cdots + C_n e^{\lambda_n x}$
- $\{\lambda_i\} \in \mathbb{R}$, k multiplicity: $y_h = (C_1 + C_2 x + \cdots + C_k x^{k-1})e^{\lambda_1 x} + \cdots + C_n e^{\lambda_n x}$
- $\{\lambda_i\} \in \mathbb{C}$, k multiplicity : $y_h = e^{\alpha x}\Big[(A_1 + A_2 x + \cdots + A_k x^{k-1})\cos(\beta x) +$

$+ (B_1 + B_2 x + \cdots + B_k x^{k-1})\sin(\beta x)\Big] + \cdots + C_n e^{\lambda_n x}$

Undetermined coefficients method (for inhomo sol) $\{a_i\}_1^n \equiv$ const $\Rightarrow$

$\Rightarrow$ Let y_p be the function shown in the table, substitute, and find the consts

$r(x)$	Roots of homo	Form of y_p
$P_m(x)$	1. 0 is not a root 2. 0 is a root of multiplicity s	$Q_m(x)$ $x^s Q_m(x)$
$P_m(x)e^{\alpha x}$	1. α is not a root 2. α is a root of multiplicity s	$Q_m(x)e^{\alpha x}$ $x^s Q_m(x)e^{\alpha x}$
$P_m(x)\cos\beta x + T_n(x)\sin\beta x$	1. $\pm i\beta$ are not roots 2. $\pm i\beta$ are roots of multiplicity s	$Q_k(x)\cos\beta x + R_k(x)\sin\beta x$ $x^s(Q_k(x)\cos\beta x + R_k(x)\sin\beta x)$
$e^{\alpha x}(P_m(x)\cos\beta x + T_n(x)\sin\beta x)$	1. $\alpha \pm i\beta$ are not roots 2. $\alpha \pm i\beta$ are roots of multiplicity s	$(Q_k(x)\cos\beta x + R_k(x)\sin\beta x)e^{\alpha x}$ $x^s(Q_k(x)\cos\beta x + R_k(x)\sin\beta x)e^{\alpha x}$

$m, n, k \equiv$ degree of polynomes $\ k = \max\{m, n\}$

$Q(x), R(x)$ must have all the terms: *i.e.* $Q_m(x) = A_1 + A_2 x + \cdots + A_{m+1}x^m$

Variation of parameters (for inhomo sol, $r(x)$ not in table) $\{a_i\}_1^n \equiv$ const $\Rightarrow$

$\Rightarrow$ Let: $y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$; $\{y_i\}$ LI sols of homo

Impose: $\begin{cases} u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0 \\ u_1'y_1' + u_2'y_2' + \cdots + u_n'y_n' = 0 \\ \cdots \\ u_1'y_1^{(n-2)} + \cdots + u_n'y_n^{(n-2)} = 0 \\ u_1'y_1^{(n-1)} + \cdots + u_n'y_n^{(n-1)} = r(x) \end{cases} \Rightarrow$ (system of n equations)

$u_i'(x) = \frac{\mathcal{W}(x)}{\mathcal{W}_i(x)} \quad \mathcal{W}_i(x) \equiv \mathcal{W}(x)$ with i-th column: $(0, 0, \cdots, r(x))^{\mathrm{T}} \quad u_i(x) = \int u_i'\, dx$

Euler Equation $x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = 0$

c.v. $x = e^t \Rightarrow y(x) = u(t)$, then $x\frac{d}{dx} \to \frac{d}{dt} \Rightarrow$ Transformed to const coeff eq in t :

$y = u(t)$, $\frac{dy}{dx} = \frac{1}{x}\frac{du}{dt}$, $\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right)$, $\cdots \Rightarrow$ Solve in t, then $y(x) = u(\ln x)$

Alternative: $y_h = x^\lambda$, substitute $x^n[\lambda(\lambda-1)\cdots(\lambda-n+1)] + \cdots + a_n = 0$

- $\{\lambda_i\} \in \mathbb{R}$, no repetition: $y_h = C_1 x^{\lambda_1} + \cdots + C_n x^{\lambda_n}$
- $\{\lambda_i\} \in \mathbb{R}$, k multiplicity: $y_h = (C_1 + C_2 \ln x + \cdots + C_k(\ln x)^{k-1})x^{\lambda_1} + \cdots + C_n x^{\lambda_n}$
- $\{\lambda_i\} \in \mathbb{C}$, k multiplicity: $y_h = x^\alpha\Big[\left(A_1 + A_2 \ln x + \cdots + A_k(\ln x)^{k-1}\right)\cos(\beta \ln x) +$

$+ \left(B_1 + B_2 \ln x + \cdots + B_k(\ln x)^{k-1}\right)\sin(\beta \ln x)\Big] + \cdots + C_n x^{\lambda_n}$

Systems of First-Order Linear ODEs $\quad e^{Ax} = I + Ax + (Ax)^2/2! + (Ax)^3/3! + \dots$

(homo) $\vec{y}' = A\vec{y} \Rightarrow \vec{y}_h(x) = e^{Ax}\vec{c}\ \ A_{n\times n}$ const; if diagonalizable: $A = PDP^{-1} \Rightarrow$

$\Rightarrow e^{Ax} = Pe^{Dx}P^{-1}$ with $e^{Dx} = \mathrm{diag}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$

(inhomo) $\vec{y}' = A\vec{y} + \vec{r}(x) \Rightarrow \vec{y}_p(x) = e^{Ax}\int e^{-Ax}\vec{r}(x)\, dx \Rightarrow \vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x)$

Quaternions $\mathbb{H}$ ($\alpha, \beta, \gamma, \delta, \lambda, \mu \in \mathbb{R}$, $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ basis of $\mathbb{H}$, $q, p \in \mathbb{H}$)

$q = \alpha + \beta\,\mathbf{i} + \gamma\,\mathbf{j} + \delta\,\mathbf{k} \quad \Re[q] = \alpha \equiv$ real part $\quad \Im[q] = \beta\,\mathbf{i} + \gamma\,\mathbf{j} + \delta\,\mathbf{k} \equiv$ vector part

$\overline{q} = \alpha - \beta\,\mathbf{i} - \gamma\,\mathbf{j} - \delta\,\mathbf{k} \quad \|q\|^2 = q\overline{q} = \overline{q}q = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \quad q^{-1} = \frac{1}{\|q\|^2}\overline{q}$

$\mathbf{U}_q = \frac{q}{\|q\|} \equiv$ versor of q, $\ \|\mathbf{U}_q\| = 1 \Rightarrow \mathbf{U}_q \equiv$ unit quaternion $\quad \alpha q = q\alpha$

$\lambda(\alpha_1 + \beta_1\,\mathbf{i} + \gamma_1\,\mathbf{j} + \delta_1\,\mathbf{k}) + \mu(\alpha_2 + \beta_2\,\mathbf{i} + \gamma_2\,\mathbf{j} + \delta_2\,\mathbf{k}) =$
$= (\lambda\alpha_1 + \mu\alpha_2) + (\lambda\beta_1 + \mu\beta_2)\mathbf{i} + (\lambda\gamma_1 + \mu\gamma_2)\mathbf{j} + (\lambda\delta_1 + \mu\delta_2)\mathbf{k}$

$\mathbf{i}1 = 1\mathbf{i} = \mathbf{i} \quad \mathbf{j}1 = 1\mathbf{j} = \mathbf{j} \quad \mathbf{k}1 = 1\mathbf{k} = \mathbf{k} \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$

$\mathbf{ij} = -\mathbf{ji} = \mathbf{k} \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \quad \mathbf{ijk} = -1$

$q = (r, \vec{v})$, $q \in \mathbb{H}$, $r = \Re[q]$, $\vec{v} = \Im[q] \quad (r_1, \vec{v}_1) + (r_2, \vec{v}_2) = (r_1 + r_2, \vec{v}_1 + \vec{v}_2)$

$(r_1, \vec{v}_1)(r_2, \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2, r_1\vec{v}_2 + r_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2) \quad \|pq\| = \|p\|\|q\|$

$\overline{pq} = \overline{q}\ \overline{p} \quad \overline{q} = -\frac{1}{2}(q + \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k}) \quad \Re[q] = \frac{1}{2}(q + \overline{q}) \quad \Im[q] = \frac{1}{2}(q - \overline{q})$

Matrix representation: $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\} \mapsto \{\mathbf{I}, \sigma_1, \sigma_2, \sigma_3\}$, $\sigma \equiv$ Pauli matrices

$q = \begin{bmatrix} \alpha + \beta\,\mathbf{i} & \gamma + \delta\,\mathbf{i} \\ -\gamma + \delta\,\mathbf{i} & \alpha - \beta\,\mathbf{i} \end{bmatrix} = \alpha\mathbf{I} + \beta i\sigma_3 + \gamma i\sigma_2 + \delta i\sigma_1 \quad \|q\|^2 = \det q \quad \Re[q] = \frac{1}{2}\operatorname{tr} q \quad \overline{q} = q^\dagger$

Quantity	SI Unit	Quantity	SI Unit
Length	m	Mass	kg
Time	s	Temperature	K
Electric current	A	Amount of substance	mol
Luminous intensity	cd	Force	N=kg·m/s^2
Pressure	Pa=kg/(m·s^2)	Energy	J=kg·m^2/s^2
Power	W=kg·m^2/s^3	Electric charge	C=A·s
Voltage	V=kg·m^2/(A·s^3)	Resistance	(Ω)=kg·m^2/(A^2·s^3)
Capacitance	F=A^2·s^4/(kg·m^2)	Magnetic flux	Wb=kg·m^2/(A·s^2)
Mag. flux density	T=kg/(A·s^2)	Inductance	H=kg·m^2/(A^2·s^2)
Frequency	Hz=1/s	Radioactivity	Bq=1/s
Absorbed dose	Gy=m^2/s^2	Dose equivalent	Sv=m^2/s^2
Catalytic activity	kat=mol/s	Angular velocity	rad/s
Angular acceleration	rad/s^2	Dynamic viscosity	Pa·s=kg/(m·s)
Thermal conductivity	W/m·K=kg·m/(s^3·K)	Spec. heat capacity	J/kg·K=m^2/(s^2·K)
Entropy	J/K=kg·m^2/(s^2·K)	Heat flux density	W/m^2=kg/s^3
Luminance	cd/m^2	Illuminance	lx=cd·sr/m^2
Surface tension	N/m=kg/s^2	Moment of inertia	kg·m^2
Momentum	kg·m/s	Impulse	N·s=kg·m/s

Quantity	Symbol	Value	Unit
speed of light in vacuum	c	299 792 458	m s^{-1}
constant of gravitation	G	6.67430×10^{-11}	m^3 kg^{-1} s^{-2}
Planck constant	h	$6.62607015 \times 10^{-34}$	J Hz^{-1}
reduced Planck constant	$\hbar$	$1.054571817 \times 10^{-34}$	J s
elementary charge	e	$1.602176634 \times 10^{-19}$	C
vacuum magnetic permeability	$\mu_0 = 4\pi\alpha\hbar/e^2c$	$1.25663706127 \times 10^{-6}$	N A^{-2}
vacuum electric permittivity	$\varepsilon_0 = 1/\mu_0 c^2$	$8.8541878128 \times 10^{-12}$	F m^{-1}
vacuum impedance	$Z_0 = \mu_0 c$	376.73031346177	Ω
Josephson constant	$K_J = 2e/h$	$483\,597.8484 \times 10^9$	Hz V^{-1}
von Klitzing constant	$R_K = 2\pi\hbar/e^2$	25 812.80745	Ω
magnetic flux quantum	$\Phi_0 = 2\pi\hbar/2e$	$2.067833848 \times 10^{-15}$	Wb
conductance quantum	$G_0 = 2e^2/2\pi\hbar$	$7.748091729 \times 10^{-5}$	S
inverse conductance quantum	G_0^{-1}	12 906.40372	Ω
electron mass	m_e	$9.1093837139 \times 10^{-31}$	kg
proton mass	m_p	$1.67262192595 \times 10^{-27}$	kg
proton-electron mass ratio	m_p/m_e	1836.152673426	—
fine-structure constant	$\alpha = e^2/4\pi\varepsilon_0\hbar c$	$7.2973525643 \times 10^{-3}$	—
inverse fine-structure	α^{-1}	137.035999177	—
Bohr Radius	$a_0 = \hbar/m_e c\alpha$	$5.29177210544 \times 10^{-11}$	m
classical electron radius	$r_e = \alpha^2 a_0$	$2.8179403205 \times 10^{-15}$	m
Bohr Magneton	$\mu_B = e\hbar/2m_e$	$9.2740100657 \times 10^{-24}$	J T^{-1}
Nuclear Magneton	$\mu_N = e\hbar/2m_p$	$5.0507837393 \times 10^{-27}$	J T^{-1}
Rydberg frequency	$cR_\infty = \frac{\alpha^2 m_e c}{2h}$	$3.28984196025 \times 10^{15}$	Hz
Hartree energy	$E_h = \alpha^2 hcR_\infty$	$4.35974472221 \times 10^{-18}$	J
Boltzmann constant	k_B	1.380649×10^{-23}	J K^{-1}
Stefan–Boltzmann constant	$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$	$5.670374419 \times 10^{-8}$	W m^{-2} K^{-4}
Avogadro constant	N_A	$6.02214076 \times 10^{23}$	mol^{-1}
molar gas constant	$R = N_A k_B$	8.314462618	J mol^{-1} K^{-1}
Faraday constant	$F = N_A e$	96 485.33212	C mol^{-1}
Non-SI units			
h-bar c	$\hbar c$	197.3269804	eV nm=MeV fm
electron volt	eV	$1.602176634 \times 10^{-19}$	J
atomic mass unit	u	$1.66053906892 \times 10^{-27}$	kg
atomic mass unit	u	931.49410242	MeV c^{-2}
Fermi coupling constant	$G_F^0 = G_F/(\hbar c)^3$	1.1663787×10^{-5}	GeV^{-2}

Integral Calculator Wolfram Alpha Phase Plane Plotter CODATA values