Quantitative Macroeconomics
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Homework 4, due Monday Dec 21

I A simple wealth model

Consider the sequential problem of a household that maximizes over streams of future consumption,

$$\max_{\{c_t\}_t^T} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$

where $\beta = \frac{1}{1+\rho} \in (0,1)$. Labor supply is inelastic and normalized to one. The budget constraint for this household at period t is

$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t$$

where r_t is exogenously given. The individual faces a stochastic endowment process of efficiency units of labor $\{y_t\}_t^T$ with $y_t \in Y = \{y_1, y_2, ..., y_N\}$. This endowment process is Marlov with $\pi(y'|y)$ denoteing the probability that tomorrow's endowment is y' if today's endowment is y, i.e. $\sum_{y'} \pi(y'|y) = 1$.

Preferences. Consider 2 utility functions:

1. Quadratic utility:
$$u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$$
where we can set $\bar{c}$ high enough, say 100 times the maximal incoem, to avoid saturation of any consumer.

2. CRRA utility,
$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \tag{1}$$
with $\sigma > 0$.

Make your program code flexible for a applying both of these utility functions under different parameter values for $\bar{c}$ and σ (what value of σ yields the log utility?).

Factor prices and subjective discount rate. Assume $r = 4\% < \rho = 6\%$ (hence, in the presence of certainty equivalence, will agents like increasing or declining consumption profiles?). Normalize $w = 1$. Make your program code flexible in order to do potential sensitivity on these parameters

Income process. Assume a 2-state process (make your program code flexible for a larger state space with cardinality N). Let

$$Y = \{1 - \sigma_y, 1 + \sigma_y\}$$

and the transition income matrix

$$\begin{pmatrix} \frac{1+\gamma}{2} & \frac{1-\gamma}{2} \\ \frac{1-\gamma}{2} & \frac{1+\gamma}{2} \end{pmatrix}$$

This way, the variance of the income process and its persistence are respectively:

$$Var(y) = \sigma_y^2$$

and

$$Corr(y', y) = \frac{Cov(y', y)}{\sqrt{Var(y')}\sqrt{Var(y')}} = \gamma$$

Borrowing constraints. Consider two cases:

1. The natural borrowing constraint $a_{t+1} \geq -\overline{A} = -\frac{1+r}{r}\, y_{min}$. [1] [2] If the time horizon is finite, use the constraint that agents cannot die in debt, that is, $a_{T+1} \geq 0$. This implies borrowing constraints for all ages of the form:

$$a_{t+1} \geq -y_{min} \sum_{s=0}^{T-(t+1)} (1+r)^{-s}$$

2. $a_{t+1} \geq 0$, preventing borrowing altogether.

II Solving the ABHI Model

II.1 The recursive formulation

Formulate the problem of the agent recursively, i.e. write down Bellmans equation and derive the stochastic Euler equation.

II.2 The infinitely-lived households economy

For $T = \infty$ write a computer program that computes the value function $v(a, y)$ and the decision rules $a'(a, y)$ and $c(a, y)$ for a given choice of the utility function as well as a given parameterization of the income process. Solve this economy using both, discrete and continuous methods to approximate functions v, a' and c. That is, first, discretize state space

$$A \times Y = \{(a, y) : a \in A \text{ and } y \in Y\}$$

[1] This constraint ensures that an agent that has borrowed up to the maximal amount and has the worst income shock can repay the interest on her loans and still have non-negative consumption.

[2] Also, to avoid c being too small, we may reset $\overline{A} = \overline{A} - \epsilon$ where ϵ is a small number, say 1% of average income; if it does not work, increase ϵ.

where $A = \{a_1, ..., a_n\}$ and $Y = \{y_1, y_2\}$ with $a_1 = -A$ and use value function or euler equation methods. Make sure that you define the grid wide enough, i.e. for the last point in the grid a_n it should be the case that $a'(a_n, y2) < a_n$. Second, use a continuous method of your choice—I suggest you to start with piecewise linear splines (there will be potential kinks in the decision rules). Also include in your program a subroutine that simulates paths of consumption and asset holdings for the first 45 periods of an agents life, starting from arbitrary asset position $a_0 \in A$ and $y_0 \in Y$.

II.3 The life-cycle economy

Repeat the exercise for $T = 45$, where now we aim for a sequence of functions

$$\{v_t(a, y), a'_t(a, y), c_t(a, y)\}_{t=0}^{T=45}$$

Remember that here you can iterate backwards from $v_{t+1}(a, y) = 0$ for all $(a, y) \in A \times Y$..

II.4 Partial equilibrium

Use the code from above to answer the following: Let $\sigma = 2$ and $\bar{c} = 100$, and the borrowing constraint equal to the natural borrowing limit.

II.4.1 With certainty

Fist, let $\gamma = 0$ and $\sigma_y = 0$, that is, there is no uncertainty.

1. For $T = \infty$ plot the consumption function(s). On the x-axis should be a, on the y-axis $c(a, y_1)$ and $c(a, y_2)$ for both preference specifications. Also generate a time profile of consumption by choosing a0 as starting assets and by using the policy functions $c(a, y)$ and $a'(a, y)$.

2. Do the same as in the previous question, but now with $T = 45$. For the consumption function plots pick two ages, say plot $c_5(a, y)$ and $c_{40}(a, y)$.

II.4.2 With uncertainty

Now let $\gamma = 0$ and $\sigma_y = .1$.

1. Plot and compare the consumption functions (for each y plot $c(a, y)$ against a) under certainty equivalence (quadratic case) with the consumption function derived in the presence of a precautionary saving motive. Are the differences more pronounced for $T = \infty$ or $T = 45$ and why? How do they compare to what you found in the case of certainty..

2. Present and compare representative simulated time paths of consumption for the certainty equivalence and precautionary saving economy. On the x-axis should be time, on the y-axis the income shock and the consumption realization. You may limit yourself to the $T = 45$ case.

3. Increase prudence by increasing $\sigma = 2$ to $\sigma = 5$ and $\sigma = 20$. How much do your answers change and why?

4. Increase the variance of the income shock from $\sigma_y = .1$ to $\sigma_y = .5$. What happens to the consumption function in the certainty equivalence case (you should know the theoretical answer to that question). Also plot the new consumption functions for the precautionary savings case. Are the differences between certainty equivalence and precautionary savings consumption functions bigger or smaller now? Explain. Support your explanations with simulated time paths of consumption. Again limit yourself to $T = 45$. How much do your answers change and why?

5. Increase the persistence of the income shocks from $\gamma = 0$ to $\gamma = .95$ (keep $\sigma_y = .5$ as well as all other parameters constant). How much do your answers change and why?.

II.5 General equilibrium

II.5.1 The simple ABHI model

Use your programs from before to compute the general equilibrium. In particular, the algorithm goes like this:

(a) Guess an interest rate $r \in (\delta, \rho)$

(b) Use the first order conditions for the firm to deteremine $K(r)$ and $w(r)$.

(c) Solve the household problem for given r and $w(r)$: Here you will use your code of the previous exercises.

(d) Use the optimal decision rule $a'(a, y)$ together with the exogenous Markov chain π to find an invariant distribution Φ_r associated with $a'(a, y)$ and π.

(d) Compute

$$Ea(r) = \int a'(a, y) d\Phi_r$$

Again, if you have discretized the state space, the integral really is a sum.

(e) Compute

$$d(r) = K(r) - Ea(r)$$

If $d(r)$ is close enough to zero, you have found a stationary recursive equilibrium, if not, update your guess for r going back to (a).

Report the endogenous distribution of consumption, income and wealth and compare them to the corresponding data distributions reported in the Handbook Chapter by Krueger, Mitman and Perri. Compare also the joint distribution of consumption and wealth in the model and the data.

II.5.2 Solve Aiyagari (1994)

Use your programs to reproduce selected results on the size of precautionary savings from Aiyagari's (1994) Table 2. Note that this exercise uses the same code as the one you programmed for the previous section (with infinite horizon) except for adding more realistic features (e.g., a Markov chain of more than 2 state income shocks) and using some of Aiygari's choices for the parameters in utility funcitions, borrowing limits, etc.

Again, report the endogenous distribution of consumption, income and wealth and compare them to the corresponding data distributions reported in the Handbook Chapter by Krueger, Mitman and Perri. Compare also the joint distribution of consumption and wealth in the model and the data.