Problem Set II: *Quantitative Macroeconomics*

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1 Computing Transitions in a Representative Agent Economy

Consider the following closed optimal growth economy populated by a large number of identical infinitely lived households that maximize:

$$\mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \tag{1}$$

Over the consumption and leisure $u(c_t) = ln(c_t)$**, subject to:**

$$c_t + i_t = y_t \tag{2}$$

$$y_t = k_t^{1-\theta}(zh_t)^{\theta} \tag{3}$$

$$i_t = k_{t+1} - (1-\delta)k_t \tag{4}$$

Set labor share to $\theta = 0.67$**. Also, to start with, set** $h_t = .31$ **for all t. Population does not grow.**

a) Compute the steady-state. Choose z to match an annual capital-output ratio of 4, and an investment-output ratio of .25.

The maximization problem can be written as follows:

$$\max_{c_t, k_{t+1}} \quad \mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}$$
$$\text{s.t.} \quad c_t + k_{t+1} = k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t$$

The Lagrangian associated to this problem:

$$\mathcal{L}(c_t, k_{t+1}, \lambda_t) = \mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} - \lambda_t(c_t + k_{t+1} - k_t^{1-\theta}(zh_t)^{\theta} - (1-\delta)k_t)$$

We can solve for the Euler equation combining the FOC's of this problem:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \rightarrow \beta^t u'(c_t) = \lambda_t \tag{5}$$

$$\frac{\partial \mathcal{L}}{k_{t+1}} = 0 \rightarrow \lambda_t = \lambda_{t+1}((1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta)) \tag{6}$$

Using 5 in periods t and t+1 in 6, we get the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})((1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta)) \tag{7}$$

Imposing the steady state in 7 we solve for k_{ss}:

$$k_{ss} = hz \left[\frac{\beta(1-\theta)}{1-\beta(1-\delta)}\right]^{\frac{1}{\theta}} \tag{8}$$

We know that the ratio capital to output is 4, therefore we can normalize the output equal to 1 in the steady state, so that $k^*_{ss} = 4$. We also know that the ratio investment to output is equal to 0.25, therefore $i^*_{ss} = 0.25$. Putting this together we can get the depreciation rate, z and the consumption in the steady state:

$$\begin{aligned} i_{ss} = k_{ss} - k_{ss} + \delta k_{ss} \rightarrow \delta = 0.0625 \\ c_{ss} = y_{ss} - i_{ss} = 0.75 \\ y_{ss} = k_{ss}^{1-\theta}(zh)^{\theta} \rightarrow z = \left(\frac{y}{k_{ss}^{1-\theta}h_{ss}^{\theta}}\right)^{\frac{1}{\theta}} = 1.629 \end{aligned}$$

Now working a bit with the Euler equation in the steady state we can get the discount factor:

$$\frac{1}{\beta} = (1-\theta)k_{ss}^{-\theta}(zh_{ss})^{\theta} + 1 - \delta \rightarrow \beta \approx 0.98$$

c) Compute the transition from the first to the second steady state and report the time-path for savings, consumption, labor and output.

To solve for the transition in Python we are going to use the Euler equation in terms of capital, from the resource constraint we get an expression for consumption in terms of capital:

$$\begin{aligned} u'(c_t) = \beta u'(c_{t+1})((1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta)) \\ c_t = k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t - k_{t+1} \end{aligned}$$

With log utility the Euler equation becomes:

$$c_{t+1} = \beta c_t((1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta))$$

Substituting the consumptions, I get the equation that I use to solve in Python:

$$k_{t+1}^{1-\theta}(zh_{t+1})^{\theta} + (1-\delta)k_{t+1} - k_{t+2} = \beta(k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t - k_{t+1})((1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta))$$

We can see the results in graphs (1,3,5 and 2), since we have a Cobb-Douglas production function, the inputs are complements, so when z increment, capital is going to increase also, the dynamic of capital consumption and output is very similar all three converge increasing to the steady state. Savings increase a lot in the first period and then converge decreasing to the new steady state.

d) Unexpected shocks. Let the agents believe productivity z_t doubles once and for all periods. However, after 10 periods, surprise the economy by cutting the productivity z_t back to its original value. Compute the transition for savings, consumption, labor and output.

The first ten periods are exactly the same as before, then in period 10, I introduce the productivity shock z decrease to the first value, the results are in graphs (6, 8, 10 and 7) after the shock the capital returns to the first steady state in a slower way than it was increasing. Consumption has a big fall in period 11, and then decrease slowly converging to the first steady state, the output follows a similar path. Savings decrease a lot in period 11, going below the first steady state and then converge increasing.

e) Bonus Question: Labor Choice Allow for elastic labor supply.

$$\mathbf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t u(c_t, 1-h_t)\right\} \tag{9}$$

$$c_t + i_t = y_t \tag{10}$$
$$y_t = k_t^{1-\theta}(zh_t)^{\theta} \tag{11}$$
$$i_t = k_{t+1} - (1-\delta)k_t \tag{12}$$

The maximization problem can be written as follows:

$$\begin{aligned} \max_{c_t, k_{t+1}, h_t} \quad & \mathbf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t u(c_t, 1-h_t)\right\} \\ \text{s.t.} \quad & c_t + k_{t+1} = k_t^{1-\theta}(zh_t)^{\theta} + (1-\delta)k_t \end{aligned}$$

The Lagrangian associated to this problem:

$$\mathcal{L}(c_t, k_{t+1}, h_t, \lambda_t) = \mathbf{E}_0\left\{\sum_{t=0}^{\infty}\beta^t u(c_t, 1-h_t)\right\} - \lambda_t(c_t + k_{t+1} - k_t^{1-\theta}(zh_t)^{\theta} - (1-\delta)k_t)$$

We can solve for the Euler equation combining the FOC's of this problem:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \rightarrow \beta^t u'(c_t) = \lambda_t$$
$$\frac{\partial \mathcal{L}}{k_{t+1}} = 0 \rightarrow \lambda_t = \lambda_{t+1}((1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta))$$
$$\frac{\partial \mathcal{L}}{h_t} = 0 \rightarrow \beta^t \kappa h_t^{\frac{1}{\nu}} = \lambda_t(\theta k_t^{1-\theta} z^{\theta} h_t^{\theta-1})$$

Using the FOC w.r.t to consumption in periods t and t+1 in the FOC w.r.t capital in t+1, we get the Euler equation:

$$c_{t+1} = \beta c_t((1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta))$$

Using the FOC's with respect to c_t and h_t:

$$h_t = \left(\frac{z^{\theta}\theta k_t^{1-\theta}}{c_t\kappa}\right)^{\frac{1}{1/\nu+(1-\theta)}}$$

I couldn't solve with Python this exercise, I tried to proceed as in the previous exercise but, I was not able to get consumption and labor in terms of capital, so I couldn't solve in the same way.

2 Solve the optimal COVID-19 lockdown model posed in the slides.

First I'm going to derive the equations of the slides analytically.

The maximization problem that the social planner face is the following:

$$\begin{aligned}
\max_{H_f,H_{nf}} \quad & Y(H_f,H_{nf}) - \kappa_f H_f - \kappa_{nf}H_{nf} - \omega D \\
\text{s.t.} \quad & H_f + H_{nf} \leq N \\
& Y(H_f,H_{nf}) = (A_f H_f^{\frac{\rho-1}{\rho}} + c(TW)A_f H_{nf}^{\frac{\rho-1}{\rho}})^{\frac{\rho}{\rho-1}} \\
& D = (1-\gamma)\beta(HC)\frac{i_0 H_f^2}{N}
\end{aligned}$$

The Lagrangian associated to this problem:

$$\begin{aligned}
\mathcal{L}(H_f,H_{nf},\lambda) = (A_f H_f^{\frac{\rho-1}{\rho}} + c(TW)A_f H_{nf}^{\frac{\rho-1}{\rho}})^{\frac{\rho}{\rho-1}}) - \kappa_f H_f \\
-\kappa_{nf}H_{nf} - \omega\left((1-\gamma)\beta(HC)\frac{i_0 H_f^2}{N}\right) - \lambda(H_f + H_{nf} - N)
\end{aligned}$$

I'm going to focus in the case of $\lambda = 0$ as in the slides, I'm going to solve the FOC's with $\lambda = 0$:

$$\frac{\partial\mathcal{L}}{\partial H_f} = 0 \to (A_f H_f^{\frac{\rho-1}{\rho}} + c(TW)A_f H_{nf}^{\frac{\rho-1}{\rho}})^{\frac{1}{\rho-1}} A_f H_f^{\frac{-1}{\rho}} = \kappa_f + 2\omega(1-\gamma)\beta(HC)\frac{i_0 H_f}{N} \tag{13}$$

$$\frac{\partial\mathcal{L}}{H_{nf}} = 0 \to (A_f H_f^{\frac{\rho-1}{\rho}} + c(TW)A_f H_{nf}^{\frac{\rho-1}{\rho}})^{\frac{1}{\rho-1}} c(TW)A_f H_{nf}^{\frac{-1}{\rho}} = \kappa_{nf} \tag{14}$$

a) Show your results for a continuum of combinations of the $\beta \in [0;1]$ parameter (vertical axis) and the $c(TW) \in [0;1]$ parameter (hztal axis). That is, plot for each pair of β and c(TW) the optimal allocations of H, H_f , $H_n f$, H_f/H, output, welfare, amount of infections and deaths. Note that if H = N there is no lockdown, so pay attention to the potential non-binding constraint $H < N$. Discuss your results.

The results are provided in graphs (11, 12, 13, 18,17, 16, 14 and 15), the graphs have to be interpreted in the following way, recall that $\beta(HC)$ is the conditional infection rate that depends on the human contact, and c(TW) is a parameter of productivity of the telework, for example graph 11 that indicates the hours at job place, the variable H_f takes high values, when c(TW) is close to zero, for small values of the productivity of the telework the hours working at the job place are higher, however in graph 12, that describe the hours working at home, the variable H_{nf} takes high values when the productivity of telework is high. We can interpret all the graphs is the same way, graphs (14 and 15) that indicates the number of infections and the number of deaths, takes high values when c(TW) takes values closer to 0, that is the telework is not productive and when the $\beta(HC)$ is high, that is when the human contact is high. Finally graphs (16 and 17) describe the output and the welfare, which are high when the telework is very productive.

What happens to your results when you increase (decrease) ρ or ω?

Increment in ρ from 1.1 to 10.

An increase the elasticity of substitution means that the social planner now can more easily substitute hours worked at the job place for hours worked at home. The most remarkable thing with respect to the previous situation is that the hours worked at the job place now the social planner only send people to work at home when the productivity of telework is really high, therefore the hours worked at home decrease with respect to the situation in a. The results for all variables are provided in graphs (19,20,2926,24,25,22 and 23).

Increment in ω from 22 to 100.

For higher ω the social planner care more about the number of deaths, in the graphs we observe that the planner send more individuals to work home, the number of hours teleworking increase but not much and therefore the number of hours at the job place decrease. The number of infections and deaths also decrease with respect to the situation in a.

Appendix: Graphs

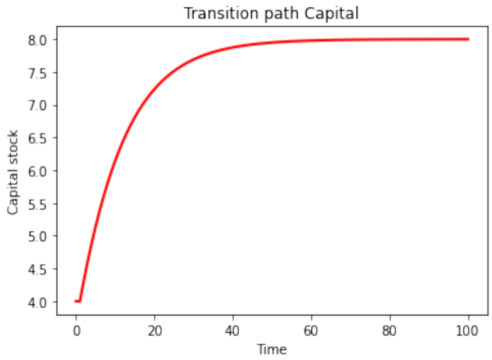

Figure 1: Transition Capital c

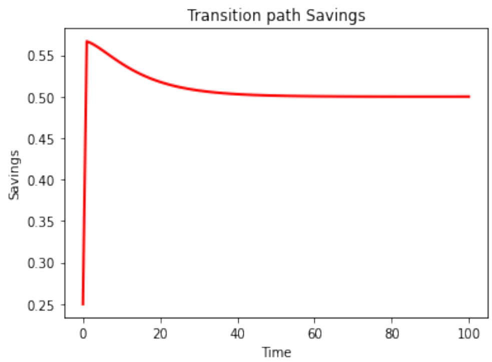

Figure 2: Transition Savings c

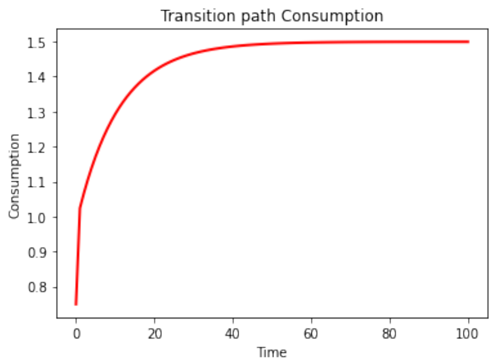

Figure 3: Transition Consumption c

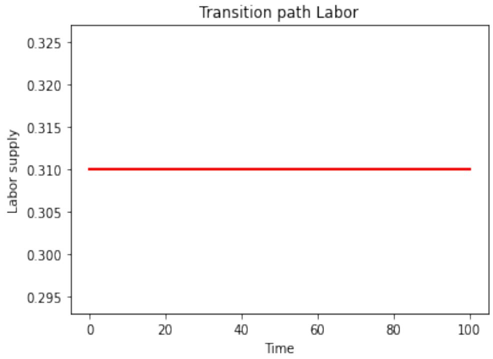

Figure 4: Transition Labor c

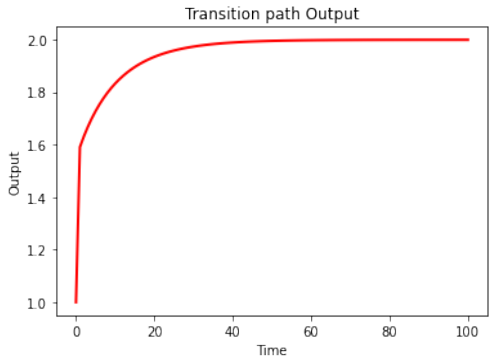

Figure 5: Transition Output c

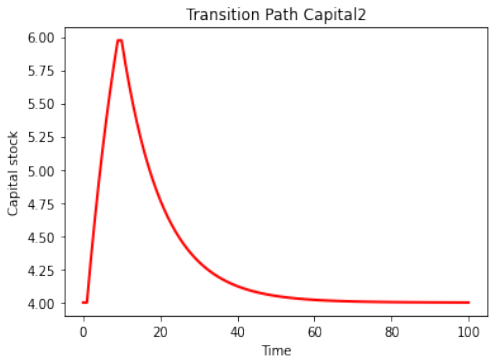

Figure 6: Transition Capital d

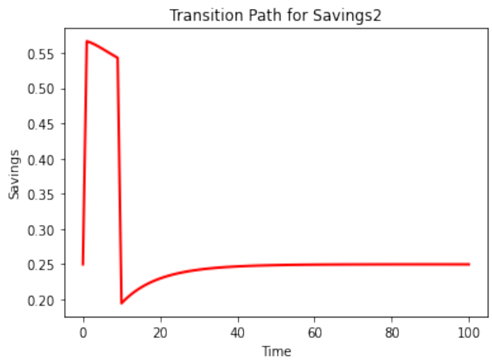

Figure 7: Transition Savings d

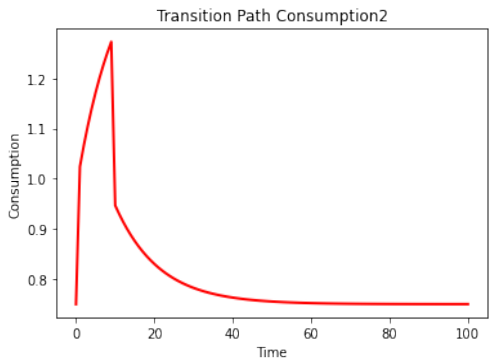

Figure 8: Transition Consumption d

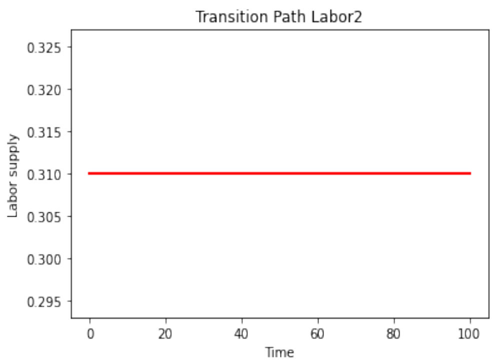

Figure 9: Transition Labor d

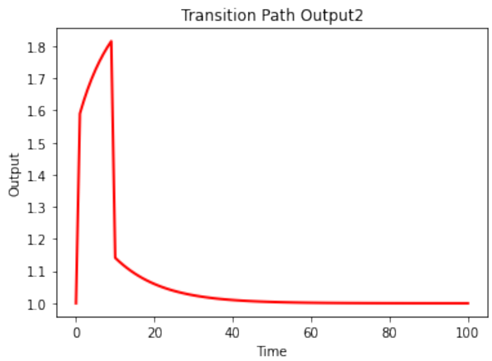

Figure 10: Transition Output d

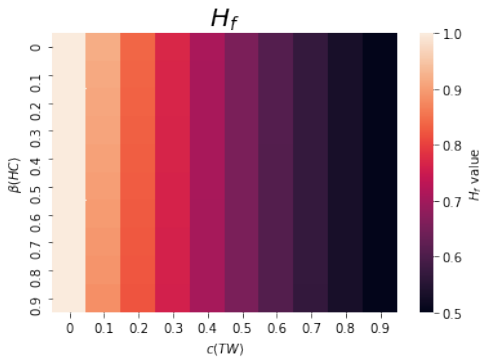

Figure 11: Hours at job place A

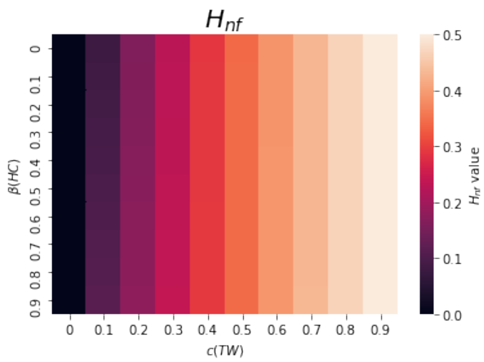

Figure 12: Hours Telework A

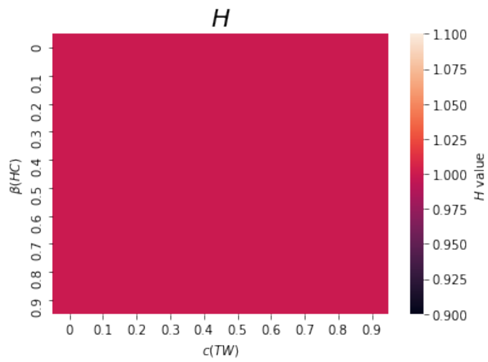

Figure 13: Aggregate Hours

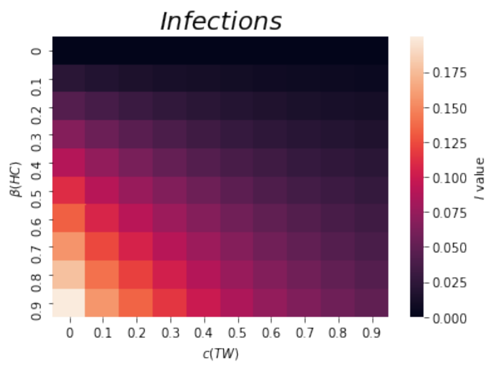

Figure 14: Infections A

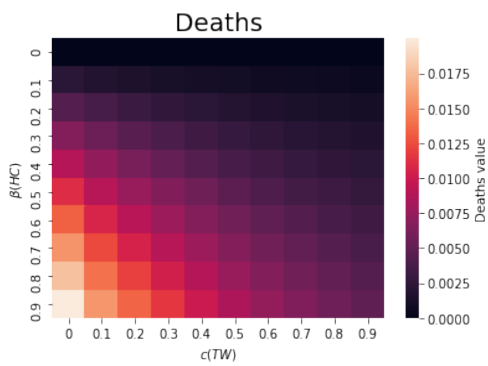

Figure 15: Deaths A

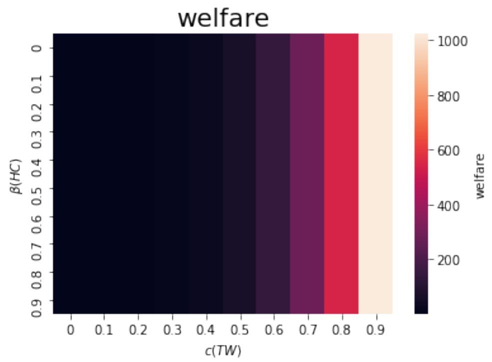

Figure 16: Welfare A

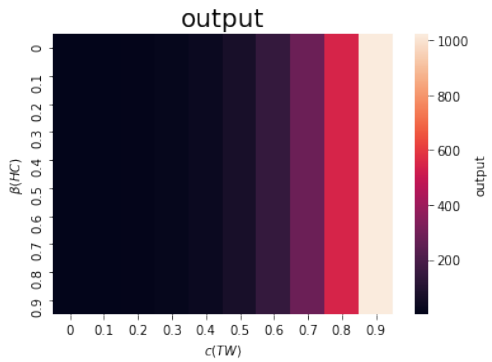

Figure 17: Output a

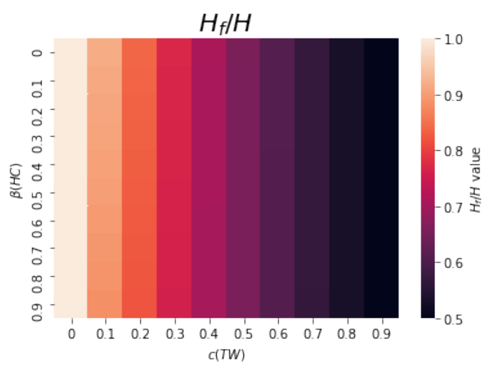

Figure 18: Proportion of Hours Worked at job place a

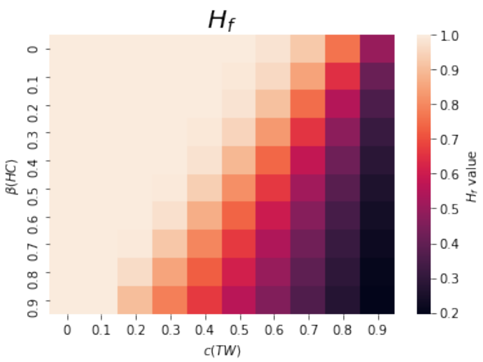

Figure 19: Hours at job place b

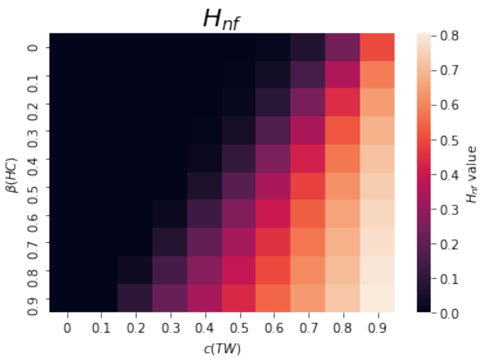

Figure 20: Hours Telework B

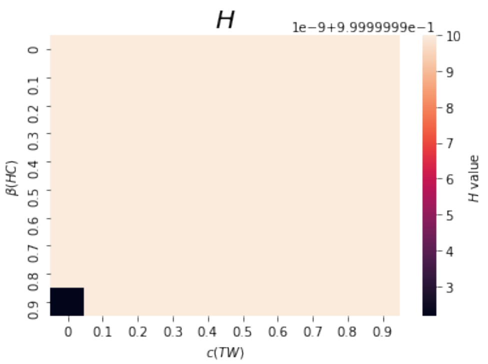

Figure 21: Aggregate Hours B

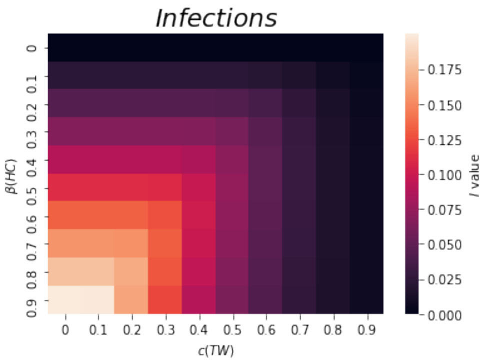

Figure 22: Infections B

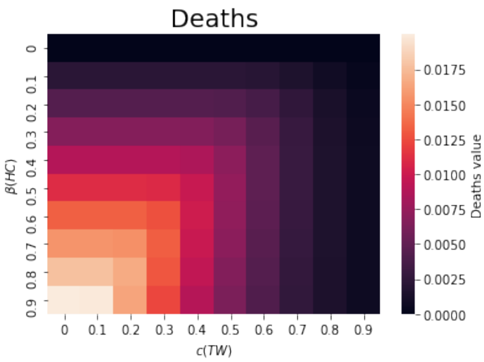

Figure 23: Deaths B

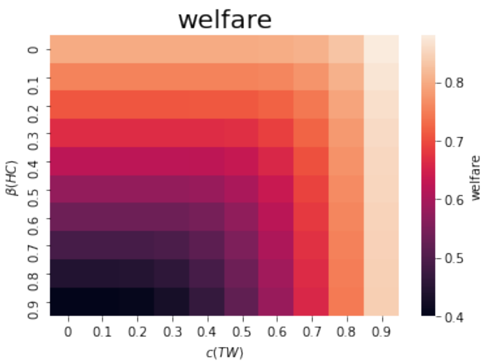

Figure 24: Welfare B

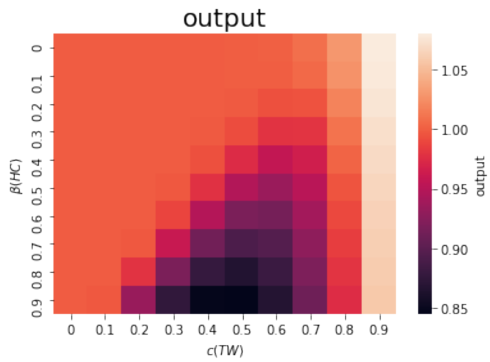

Figure 25: Output B

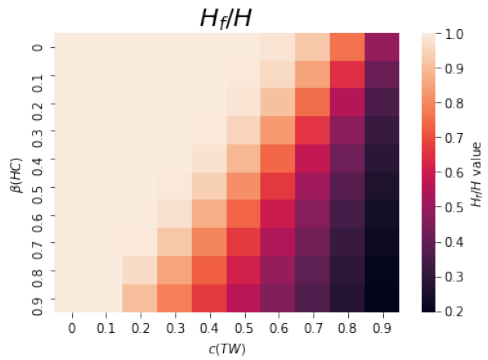

Figure 26: Proportion of Hours Worked at job place B

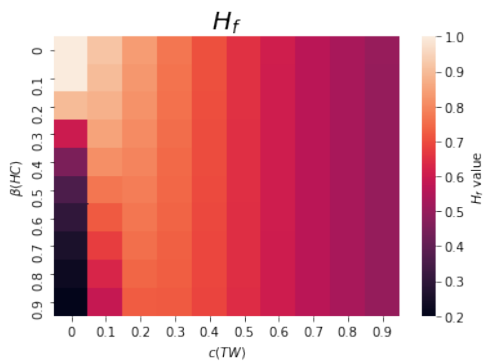

Figure 27: Hours at job place C

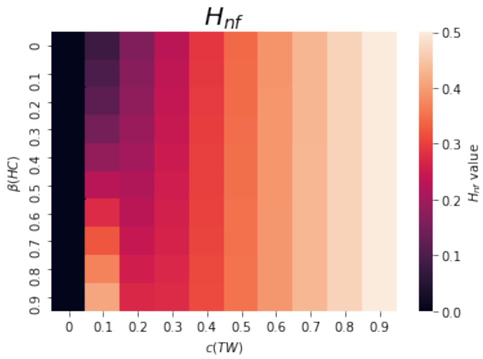

Figure 28: Hours Telework C

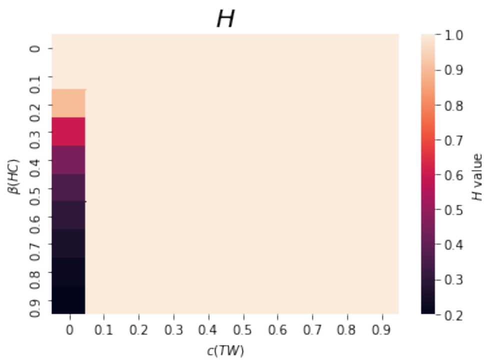

Figure 29: Aggregate Hours C

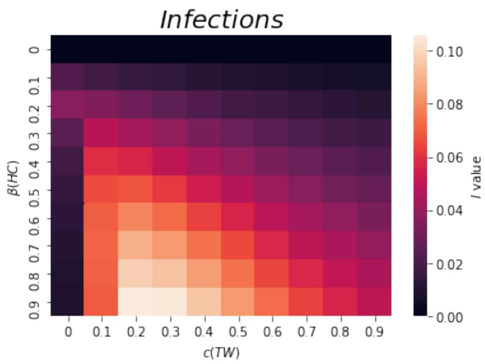

Figure 30: Infections C

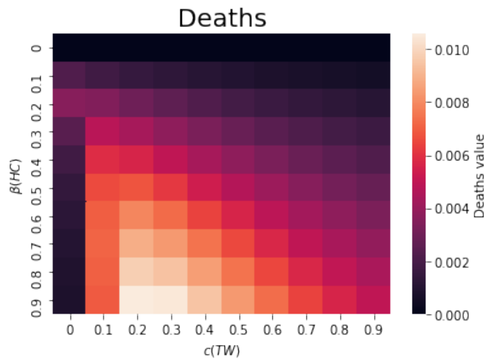

Figure 31: Deaths C

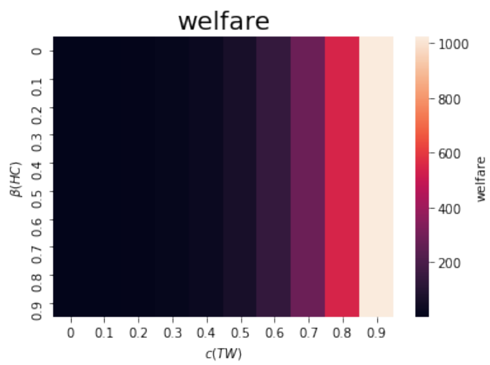

Figure 32: Welfare C

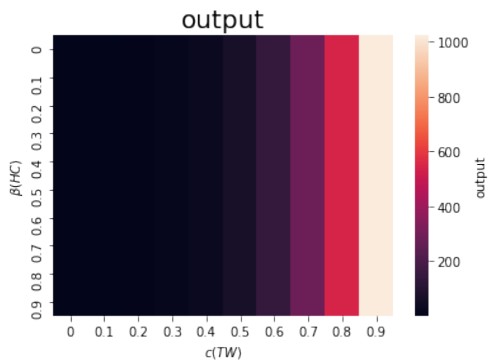

Figure 33: Output C

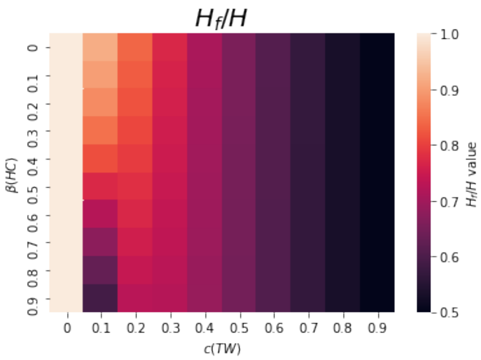

Figure 34: Proportion of Hours Worked at job place C